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Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstract. The workshop discussed the interactions between algebraic geometry and various areas of noncommutative algbebra including finite dimensional algebras, representation theory of algebras and noncommutative algebraic geometry. More than 45 mathematicians participated with a notable number of young mathematicians present.

Mathematics Subject Classification (2000): 14A22, 16K20, 16S38, 16P90.

Introduction by the Organisers

This meeting had over 45 participants from 11 countries (Australia, Belgium, Canada, France, Germany, Italy, Israel, Norway, Russia, UK and the US) and 26 lectures were presented during the five day period. The sponsorship of the European Union allowed the organizers to invite and secure the participation of a number of young investigators. Some of these young mathematicians presented thirty minute lectures. As always, there was a substantial amount of mathematical activity outside the formal lecture sessions.

This meeting explored the applications of ideas and techniques from algebraic geometry to noncommutative algebra . Several lecturers presented open problems to stimulate the interest of researchers in other areas. Areas covered include

- noncommutative projective algebraic geometry,
- Hopf algebras,
- combinatorial ring theory,
- symplectic reflection algebras,
- representation theory of quivers and preprojective algebras
- homological techniques and derived categories

The sweep of the meeting can be seen from de Jong's contribution that uses contemporary algebraic geometry to prove a theorem in the classical theory of finite dimensional division algebras to the works of Keller-Reiten and Ingalls on cluster algebras. Additionally, de Jong notes a result obtained during the workshop with van den Bergh. Looking to the future, Goodearl and Zelmanov propose a number challenging problems. Zelmanov discusses both an interesting Lie algebra example and a possible connection to an old problem of Kurosh.

The previous paragraph represents just a sampling of the scope and variety of the mathematics at the meeting. The abstracts following will give the whole story.

Workshop: Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstracts

Primitivity in twisted homogeneous coordinate rings

Jason Bell
(joint work with Dan Rogalski)

Let X be a projective variety over an uncountable algebraically closed field k and let σ be an automorphism of X. Given an invertible sheaf $\mathcal{L}$ over X, we can define the *twisted homogeneous coordinate ring*

$$B(X, \mathcal{L}, \sigma) \ = \ \bigoplus_{i=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}_n),$$

where $\mathcal{L}_n$ is the sheaf $\mathcal{L} \otimes \sigma^*(\mathcal{L}) \otimes \cdots \otimes (\sigma^{n-1}) * (\mathcal{L})$. We examine the question when such algebras are primitive. We prove in particular that primitivity for such rings can be reduced to a geometric condition. That is, we show that primitivity is equivalent in such rings to the property that the set of points $x \in X$ which fail to have a dense orbit under σ is not itself a dense subset of X. Furthermore, we show that if X has a dense subset Y such that every $y \in Y$ fails to have a dense orbit in X under σ then there is $f \in k(X)$ such that $f \circ \sigma = f$.

Using this fact, we are able to obtain the following result. Let P be a prime ideal of $B(X, \mathcal{L}, \sigma)$. Then the following are equivalent:

(1) P is primitive;
(2) $Q(B(X, \mathcal{L}, \sigma)/P)$, the quotient division algebra, has centre equal to k;
(3) $B(X, \mathcal{L}, \sigma)/P$ has only finitely many height one primes.

This can be regarded as a twisted-homogeneous coordinate ring analogue of the Dixmier-Moeglin equivalence for enveloping algebras of finite dimensional Lie algebras [2, §14.4.1]. It was previously known for connected finitely graded domains of GK dimension two [1]. This correspondence indicates that for such algebras the number of height one primes is either uncountable or finite; in particular, it is impossible to have a countably infinite number of height one primes. We thus pose the following question.

Question. Let k be an uncountable field and let A be a finitely generated Noetherian k-algebra of finite GK dimension. Suppose the number of height one primes of A is countable. Is it then necessarily finite?

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Deformed Preprojective Algebras and Calogero-Moser Spaces

Yuri Berest
(joint work with Oleg Chalykh)

We clarify the relation between the following objects: (a) the rank 1 torsion-free modules (ideals) over the first Weyl algebra $A_1(\mathbb{C})$; (b) simple modules over deformed preprojective algebras $\Pi_\lambda(Q)$ introduced by Crawley-Boevey and Holland [5]; and (c) simple modules over the rational Cherednik algebras $H_{0,c}(S_n)$ associated to symmetric groups [8]. The isomorphism classes of each type of these objects can be parametrized geometrically by the same space (namely, the Calogero-Moser algebraic varieties); however, no direct links (functors) between the corresponding module categories seem to be known. We construct such functors by translating our earlier results on A_∞-modules over A_1 (see [3]) to the more familiar language of quiver representations. We mention that the question of explaining the "mysterious bijection" between ideal classes of A_1 and simple modules of Cherednik algebras was first raised in [8] and emphasized further in [2] (see *loc. cit.*, Remark 1.1).

1. For each $n \geq 0$, let $\tilde{\mathcal{C}}_n$ be the space of linear maps

$$\{(\bar{X}, \bar{Y}, \bar{v}, \bar{w}) \, : \, \bar{X}, \bar{Y} \in \mathtt{End}(\mathbb{C}^n)\,,\ \bar{v} \in \mathtt{Hom}(\mathbb{C}, \mathbb{C}^n),\ \bar{w} \in \mathtt{Hom}(\mathbb{C}^n, \mathbb{C})\}\ ,$$

satisfying the equation $[\bar{X},\, \bar{Y}] + \mathtt{Id} = \bar{v}\,\bar{w}$. The group $\mathtt{GL}(n, \mathbb{C})$ acts on $\tilde{\mathcal{C}}_n$ in the natural way:

$$(\bar{X}, \bar{Y}, \bar{v}, \bar{w}) \mapsto (g\bar{X}g^{-1},\, g\bar{Y}g^{-1},\, g\bar{v},\, \bar{w}g^{-1}),\ g \in \mathtt{GL}_n(\mathbb{C})\ ,$$

and, following [9], we can define the n-th *Calogero-Moser space* $\mathcal{C}_n$ to be the quotient variety $\tilde{\mathcal{C}}_n//\mathtt{GL}_n(\mathbb{C})$ modulo this action. In fact, $\mathtt{GL}_n(\mathbb{C})$ acts freely on $\tilde{\mathcal{C}}_n$, and $\mathcal{C}_n$ turns out to be a smooth affine variety of dimension $2n$ (see [9]).

2. Let $Q = (I,\, Q)$ be a finite quiver with vertex set I and arrow set Q, and let $\bar{Q}$ be its double (i.e. the quiver obtained from Q by adding a reverse arrow a^* to each arrow $a \in Q$). Following [5], for each $\lambda = (\lambda_i) \in \mathbb{C}^I$, we define the *deformed preprojective algebra of weight* λ by

$$\Pi_\lambda(Q) := \mathbb{C}\bar{Q} \Big/ \left\langle \sum_{a \in Q} [a,\, a^*] - \sum_{i \in I} \lambda_i e_i \right\rangle\ .$$

Here $\mathbb{C}\bar{Q}$ denotes the path algebra of the double quiver $\bar{Q}$ and $e_i \in \mathbb{C}\bar{Q}$ stand for the orthogonal idempotents corresponding to the trivial paths in $\bar{Q}$.

We will be concerned with the following example. Let Q^{CM} be the quiver consisting of two vertices $\{0,\, 1\}$ and two arrows $v : 0 \to 1$ and $X : 1 \to 1$. Write $w := v^*$ and $Y := X^*$ for the opposite arrows in $\bar{Q}$. The algebra $\Pi_\lambda := \Pi_\lambda(Q^{\mathsf{CM}})$ is then generated by X, Y, v, w and the idempotents e_0 and e_1, which, apart from the standard path algebra relations, satisfy $[X,\, Y] + vw - \lambda_1 e_1 = 0$ and $wv + \lambda_0 e_0 = 0$.

3. Now, fix an integer $n \geq 0$ and let $\lambda = (-n, 1)$. Then, to each point of the variety $\tilde{\mathcal{C}}_n$ we can naturally associate a right module over Π_λ of dimension $n+1$. All such modules are simple and, as was originally observed by Crawley-Boevey [6], *all* simple modules of Π_λ having dimension vector $\alpha = (1, n)$ are of this form [7]. Thus, we can identify the space $\mathcal{C}_n$ with the representation variety $\texttt{Rep}(\Pi_\lambda, \alpha)//G(\alpha)$ parametrizing the isomorphism classes of simple Π_λ-modules of dimension $\alpha = (1, n)$.

On the other other hand, according to [4], the varieties $\mathcal{C}_n$ also parametrize the isomorphism classes of right ideals of the first Weyl algebra $A_1(\mathbb{C})$. Our aim is to relate the simple modules of Π_λ to ideals of A_1 in a natural (functorial) way. To this end, we will use the following simple, but crucial observation.

Lemma 1. $A_1(\mathbb{C})$ *is isomorphic to the quotient of* Π_λ *by the two-sided ideal generated by the idempotent* e_0 .

Indeed, combined with the canonical projection $\Pi_\lambda \twoheadrightarrow \Pi_\lambda/\langle e_0\rangle$, the algebra map $\mathbb{C}\langle x, y\rangle \to \Pi_\lambda$, $x \mapsto X$, $y \mapsto Y$, is an epimorphism with kernel containing $xy - yx - 1$. The induced map $A_1(\mathbb{C}) := \mathbb{C}\langle x, y\rangle/\langle xy - yx - 1\rangle \to \Pi_\lambda/\langle e_0\rangle$ is then an isomorphism of algebras, since A_1 is simple.

4. Let $\Pi := \Pi_\lambda$ (with $\lambda = (-n, 1)$ as above), $A := \Pi/\langle\, e_0\,\rangle \cong A_1(\mathbb{C})$, $B := e_0\,\Pi\, e_0$, and let $\texttt{Mod}(\Pi)$, $\texttt{Mod}(A)$ and $\texttt{Mod}(B)$ denote the corresponding categories of (right) modules. Then we have the following six functors

$$\texttt{Mod}(A) \;\underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{i_*}}}\; \texttt{Mod}(\Pi) \;\underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}}\; \texttt{Mod}(B)$$

satisfying the standard "recollement" conditions (see [1]). Briefly, i_* is the restriction functor corresponding to the canonical epimorphism $i : \Pi \to A$; it has both the right adjoint $i^! = \texttt{Hom}_\Pi(A, \text{—})$ and the left adjoint $i^* = \text{—} \otimes_\Pi A$, satisfying $i^!\, i_* \simeq i^* i_* \simeq \texttt{Id}_{\texttt{Mod}(A)}$. Next, the functor j^* is defined by $j^*(X) = X e_0$; it is exact and has also the right adjoint $j_* = \texttt{Hom}_B(\Pi e_0, \text{—})$ and the left adjoint $j_! = \text{—} \otimes_B e_0\Pi$, satisfying $j^* j_* \simeq j^* j_! \simeq \texttt{Id}_{\texttt{Mod}(B)}$. Moreover, we have $j^* i_* = 0$, $i^* j_! = i^! j_* = 0$, so $j^* : \texttt{Mod}(\Pi) \to \texttt{Mod}(B)$ identifies $\texttt{Mod}(B)$ as the quotient category of $\texttt{Mod}(\Pi)$ by the (strict) image of i_*, i. e. $\texttt{Mod}(B) \simeq \texttt{Mod}(\Pi)/\texttt{Mod}(A)$. Our main result is the following

Theorem 2. *The composition of functors*

$$F : \texttt{Mod}(\Pi) \xrightarrow{j^*} \texttt{Mod}(B) \xrightarrow{j_!} \texttt{Mod}(\Pi) \xrightarrow{i^!} \texttt{Mod}(A)$$

maps injectively *the set of isomorphism classes of simple* Π*-modules of dimension* $\alpha = (1, n)$ *into the set of isomorphism classes of rank* 1 *torsion-free modules over* A*. If we identify* $\mathcal{C}_n = \texttt{Rep}(\Pi_\lambda, \alpha)//G(\alpha)$ *as above, then the map induced by* F *agrees with the Calogero-Moser map* ω *constructed in* [4] *and* [3].

Remark. Using the results of [7], one can easily compute the dimension vectors of all simple modules of the algebra Π: these are (k, kn), where $k = 0, 1, 2, \ldots$.

Theorem 2, however, is only true for modules of dimension $(1, n)$, and it does not seem to extend to simple Π-modules of higher dimensions.

5. Let $H := H_{0,1}(S_n)$ be the rational Cherednik algebra associated to the symmetric group S_n and the parameters $t = 0$ and $c = 1$ (see [8], Section 4). Explicitly, H is generated by two polynomial subalgebras $\mathbb{C}[x_1, x_2, \dots, x_n]$, $\mathbb{C}[y_1, y_2, \dots, y_n]$ and the elementary transpositions $s_{ij} \in S_n$ subject to the relations $s_{ij}x_i = x_j s_{ij}$, $s_{ij}y_i = y_j s_{ij}$, $[y_i, x_j] = s_{ij}$ $(i \neq j)$, and $[y_k, x_k] = -\sum_{i\neq k} s_{ik}$. Write $e := \frac{1}{n!}\sum_{\sigma\in S_n} \sigma$ for the symmetrizing idempotent in $\mathbb{C}S_n \subset H$, and $U := eHe$ for the corresponding spherical algebra of H. It is known (see [8], Theorem 1.24) that H is Morita equivalent to U, the corresponding equivalence being $e : \mathtt{Mod}(H) \to \mathtt{Mod}(U)$, $N \mapsto Ne$.

Lemma 3. *The map* $\theta : w\,a(X, Y)\,v \mapsto \sum_{i=1}^{n} e\,a(x_i, y_i)\,e$ *defines an algebra homomorphism from* $B = e_0\Pi e_0$ *to the spherical algebra* $U = eHe$.

Write $\theta_* : \mathtt{Mod}(U) \to \mathtt{Mod}(B)$ for the restriction functor corresponding to θ. Then our second result can be formulated as follows.

Theorem 4. *The composition of functors*

$$G : \mathtt{Mod}(H) \xrightarrow{e} \mathtt{Mod}(U) \xrightarrow{\theta_*} \mathtt{Mod}(B) \xrightarrow{j_*} \mathtt{Mod}(\Pi)$$

maps the set of isomorphism classes of simple H-modules bijectively *onto the set of isomorphism classes of simple* Π*-modules of dimension* $\alpha = (1, n)$*. If we identify* $\mathcal{C}_n = \mathtt{Rep}(\Pi_\lambda, \alpha)//G(\alpha)$ *as above, then the map induced by* G *agrees with an explicit construction of* [8] *(see loc. cit., Section 11).*

Finally, combining Theorems 2 and 4 together, we get

Theorem 5. *The composition of functors*

$$\mathtt{Mod}(H) \xrightarrow{e} \mathtt{Mod}(U) \xrightarrow{\theta_*} \mathtt{Mod}(B) \xrightarrow{j_!} \mathtt{Mod}(\Pi) \xrightarrow{i^!} \mathtt{Mod}(A)$$

maps the set of isomorphism classes of simple H-modules injectively into the set of isomorphism classes of rank 1 *torsion-free modules over* A.

6. The above results can be extended to an arbitrary affine Dynkin quiver related (via the McKay correspondence) to a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$. In that case the Weyl algebra A_1 is replaced by the "quantized coordinate ring" $\mathcal{O}^\lambda(\Gamma)$ of the Kleinian singularity $\mathbb{C}^2//\Gamma$ (see [5]) and the rational Cherednik algebra H by the symplectic reflection algebra $H_{0,\tau}(\mathbf{\Gamma}_n)$ associated to the n-th wreath product $\mathbf{\Gamma}_n := S_n \ltimes (\Gamma \times \Gamma \times \ldots \times \Gamma)$ (see [8]).

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Graded Calabi Yau Algebras and Superpotentials

Raf Bocklandt

A finitely generated algebra A over $\mathbb{C}$ is called 3-Calabi Yau if the third shift in the bounded derived category of finite dimensional modules is a Serre functor. This means there exist natural isomorphisms

$$\mathsf{Hom}_{\mathcal{D}^b\mathsf{Rep}A}(M,N) \cong \mathsf{Hom}_{\mathcal{D}^b\mathsf{Rep}A}(N,M[3])^*,\ \forall M,N \in \mathcal{D}^b\mathsf{Rep}A,$$

where the $*$ indicates the complex dual. Following Van den Bergh and Reiten [4], this condition can be restated in terms of traces: there exist trace functions $\mathsf{Tr}_M : \mathsf{Hom}_{\mathcal{D}^b\mathsf{Rep}A}(M,M[n]) \to \mathbb{C}$ for every $M \in \mathcal{D}^b\mathsf{Rep}A$ such that

$$\forall f \in \mathsf{Hom}_{\mathcal{D}^b\mathsf{Rep}A}(M,N) : g \in \mathsf{Hom}_{\mathcal{D}^b\mathsf{Rep}A}(N,M[n]) : \mathsf{Tr}_M(g \circ f) = \mathsf{Tr}_N(f[n] \circ g).$$

Furthermore these pairings must be nondegenerate. Instead of working in the derived category one would like to translate these traces to traces on the $\mathsf{Ext}^n_A(M,M)$. This is indeed possible but the commutation relation above will get extra minus signs because the correct way to do this is using graded functors for the triangulated category (see [3],[5] and the appendix in [7])

$$\forall f \in \mathsf{Ext}^i_A(M,N) : g \in \mathsf{Ext}^{n-i}_A(N,M) : \mathsf{Tr}_M(g * f) = (-1)^{i(n-i)}\mathsf{Tr}_N(f * g).$$

The existence of these trace functions will enable us to give a characterization of graded 3-dimensional Calabi Yau algebras in terms of generators and relations.

In this talk we will consider the cases of graded quotients of path algebras, an extended version of our results can be found in [7]. Similar results in different settings have been obtained by Reiten and Iyama [6], Rouquier and Chuang and Ginzburg.

In order to state our result, we need to introduce superpotentials. Let $\mathbb{C}Q$ be the path algebra of a quiver Q and put a gradation on $\mathbb{C}Q$ using the length of the paths.

The vector space $\mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]$ will be called the space of superpotentials. It has as basis the set of cycles up to cyclic permutation of the arrows. We can embed this space into $\mathbb{C}Q$ by mapping a cycle onto the sum of all its possible cyclic permutations:

$$\sigma : \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q : a_1 \cdots a_n \mapsto \sum_i a_i \cdots a_n a_1 \cdots a_{i-1}.$$

Another convention we will use is the inverse of arrows: if $p := a_1 \cdots a_n$ is a path and b an arrow, then $pb^{-1} = a_1 \cdots a_{n-1}$ if $b = a_n$ and zero otherwise. Similarly one can define $b^{-1}p$. These newly defined maps can be combined to obtain a 'derivation'

$$\partial_a : \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q] \to \mathbb{C}Q : p \mapsto \sigma(p)a^{-1} = a^{-1}\sigma(p).$$

From now on A will denote the quotient algebra $\mathbb{C}Q/\mathcal{I}$ by a finitely generated graded ideal $\mathcal{I} \subset \mathbb{C}Q_{\geq 2}$. For each vertex $i \in \mathbb{C}Q$ we denote the standard projective A-module by $P_i := Ai$ and the basic simple A-module by $S_i = P_i/A_{\geq 1}P_i$.

Theorem 1. *If A is Calabi Yau of dimension 3 then*

(1) *there is a homogeneous superpotential $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ such that*

$$A \cong A_W := \mathbb{C}Q/(\partial_a W : a \in Q_1),$$

(2) *every arrow in Q is contained in a cycle of σW,*
(3) *every vertex in Q is the source of two arrows and the target of two arrows.*

Sketch of the proof. As the global dimension and the CY-dimension coincide one can construct projective resolutions of the S_i like this

$$P \stackrel{(f_r)}{\to} \bigoplus_{t(r)=i} P_{h(r)} \stackrel{(rb^{-1})}{\to} \bigoplus_{t(b)=i} P_{h(b)} \stackrel{(\cdot b)}{\to} P_i \to S_i.$$

Calculating dimensions of extension spaces $\mathsf{Ext}^k_A(S_i, S_j)$ leads to the conclusion that $P \cong P_i$ and the number of relations between i and j is the same as the number of arrows between j and i. Also the f_r must be of degree one, so we can consider them as arrows. This enables us to match each arrow with a corresponding relation: $r = r_a \iff f_r = a$. One can use the grading and the connectedness of Q to show that all relations must have the same degree. This implies that because resolutions above are complexes, we must have that

$$\sum_a a r_a b^{-1} = \sum_a g_{bc} r_c,$$

where the $g_{bc} \in \mathbb{C}$. These coefficients also appear in the calculation of the composition of $\mathsf{Ext}^2_A(S_i, S_j)$ with $\mathsf{Ext}^1_A(S_j, S_i)$ and using the commutativity property of the traces one can show that g_{bc} can be rescaled to the identity matrix. If we define $W = \sum_a a r_a = \sum_a r_a a \mod [\mathbb{C}Q, \mathbb{C}Q]$ then we see that W must be the superpotential which generates the relations. Finally the structural conditions on the quiver ensure that the resolutions can indeed be exact. □

The theorem only goes in one direction: not every superpotential gives rise to an algebra that is Calabi Yau. However for every such algebra A one can construct a complex of A-bimodules (see also [2])

$$C_W: \quad \bigoplus_{i\in Q_0} F_{ii} \stackrel{(\cdot\tau da\cdot)}{\to} \bigoplus_{a\in Q_1} F_{t(a)h(a)} \stackrel{(\cdot\partial^2_{ba}W\cdot)}{\to} \bigoplus_{b\in Q_1} F_{h(b)t(b)} \stackrel{(\cdot db\cdot)}{\to} \bigoplus_{i\in Q_0} F_{ii} \stackrel{m}{\to} A$$

where $F_{ij} := Ai \otimes jA$, $db = b\otimes 1 - 1\otimes b$ and if c is a cycle then

$$\partial^2_{ba}c = \sum_{\sigma(ap_1bp_2)=\sigma(c)} p_1 \otimes p_2.$$

This complex is selfdual in the sense that $\mathsf{Hom}_{A-A}(C_W, A\otimes A)[3] \cong C_W$ (use the inner bimodule structure on $A\otimes A$ to get a bimodule structure on $\mathsf{Hom}_{A-A}(C_W, A\otimes A)$). Using this fact and results of King and Butler [1] on the minimal resolutions of path algebras with relations one can prove:

Theorem 2. *An algebra A derived from a superpotential is* 3*-Calabi Yau if and only if C_W is exact. If this is the case then C_W is the minimal resolution of A as an A-bimodule.*

This fact has a nice interpretation for the classification of *good superpotentials* i.e. superpotentials with an algebra that is indeed Calabi Yau.

Corollary. *For a given quiver Q and a given dimension d, the subset of* $\mathsf{Sup}_d Q$ *of good superpotentials of degree d is either the empty set or almost everything (in the measure theoretic sense).*

This is because we can check whether C_W is exact separately for every degree. The subspace of good superpotentials is an intersection of a countable number of Zariski open sets. If one of these sets is empty we're in the first case and otherwise the complement of this set is a countable union of hypersurfaces, which has measure zero for the standard measure on $\mathbb{C}^n$.

Finally, to obtain a list of the degrees that have good superpotentials for a given quiver, it is possible to use Groebner basis techniques if the structure of the quiver is not too complex (e.g. two vertices or a ring of vertices with multiple arrows between consecutive vertices).

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Noetherian Hopf Algebras

Ken Brown
(joint work with James J. Zhang)

I briefly review progress in studying infinite dimensional noetherian Hopf algebras over the past 10 years, before describing recent work of Zhang and myself [1] on the existence and nature of dualizing complexes for noetherian Hopf algebras. Finally, I explain how the results apply to (i) the study of Hochschild (co)homology of Hopf algebras, and (ii) the nature of the antipode. I include various open questions.

1. **Hopf algebra background.** Throughout, k will denote an algebraically closed field and A will be a noetherian Hopf $k-$algebra. Thus, A is equipped with a comultiplication $\Delta : A \longrightarrow A \otimes A : a \mapsto \sum a_1 \otimes a_2$, an algebra homomorphism; a counit $\epsilon : A \longrightarrow k$, an algebra homomorphism; and an antipode $S : A \longrightarrow A$, an algebra antihomomorphism. Throughout,

we will assume that S is bijective.

Question 1 (Skryabin)**.** Is S bijective for all noetherian Hopf algebras A?

There is considerable evidence in support of a positive answer:

Proposition 2 (Skryabin, 2006, [7])**.** *For a noetherian Hopf algebra A, S is always injective. When A is semiprime or satisfies a polynomial identity, Question 1 has a positive answer.*

We very briefly recall some very basic properties of finite dimensional Hopf algebras, which we will aim to generalise in the sequel. Details can be found in [4], for example. Let F be a finite dimensional Hopf $k-$algebra. Then F is a Frobenius algebra, so, in particular, F is self-injective. Indeed, the injective cogenerator $\mathrm{Hom}_k(F, k) \cong F$ as both left and right $F-$modules, although not in general as a bimodule; to make this an isomorphism of bimodules, we have to twist the action on one side using a suitable algebra automorphism ν, to obtain

$$\mathrm{Hom}_k(F, k) \cong {}^{\nu}F^1.$$

The automorphism ν, which is clearly uniquely determined up to an inner automorphism, is called the *Nakayama automorphism* in the theory of Frobenius algebras.

The self-duality resulting from the Frobenius property of F leads to the concept of the *integrals* of F, which are fundamental to the study of the structure of F. Namely, the *left integral* $\int_F^\ell$ is the unique copy of the trivial left $F-$module ${}_\epsilon k$ occurring as a left ideal of F; uniqueness means that it is an ideal of F. Similarly, the *right integral* $\int_F^r$ is a one-dimensional ideal of F, trivial on the right; F is called *unimodular* if the two integrals are equal. Later, we'll generalise

Proposition 3 (Oberst-Schneider, 1973, [5]). *For a finite dimensional Hopf algebra F, the Nakayama automorphism ν is trivial if and only if F is unimodular and $S^2 = 1$.*

2. AS-Gorenstein Hopf algebras, and integrals. How should the above ideas generalise to an infinite dimensional Hopf algebra A? Recall the following well-known definition, which applies in fact to *any* augmented algebra.

Definition 4. The Hopf algebra A is *AS-Gorenstein* if

(1) the injective dimension of ${}_AA$ is finite, say d;
(2) $\mathrm{Ext}^i_A({}_Ak, {}_AA) = 0$, for $i < d$;
(3) $\dim_k \mathrm{Ext}^d_A({}_Ak, {}_AA) = 1$;
(4) the right-hand versions of the above also hold.

In this case, the right injective dimension of A is also d. If A is AS-Gorenstein with finite global dimension (which is then necessarily d), A is called *AS-regular.*

I suggest that the correct analogue of the Frobenius property of finite dimensional Hopf algebras may be:

Question 5 (Brown-Goodearl, 1997, [2]). Is every noetherian Hopf algebra AS-Gorenstein?

Question 5 remains open at the time of writing. Positive answers are known for

(1) $A = \mathcal{U}(\mathfrak{g})$, the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$, with $d = \dim_k \mathfrak{g}$;
(2) $A = \mathcal{U}_q(\mathfrak{g})$, the quantised enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$, with $d = \dim_k \mathfrak{g}$;
(3) $A = \mathcal{O}_q[G]$, the quantised function algebra of a semisimple group G, with $d = \dim G$;
(4) $A = kG$, the group algebra of a polycyclic-by-finite group G, with d the Hirsch number of G;
(5) A an affine noetherian algebra satisfying a polynomial identity, with d the Gel'fand-Kirillov (GK) dimension of A.

Details for the first four of these classes can be found in [1, Section 6]; the fifth class requires by far the most difficult proof; for that, see [10].

In view of the substantial number of cases where Question 5 has a positive answer, it now makes sense to begin the systematic study of noetherian AS-Gorenstein Hopf algebras. The first step is to extend the definition of integral. This was achieved by Lu, Wu and Zhang in 2005: they defined [3] the *left integral* of the AS-Gorenstein noetherian Hopf algebra A of injective dimension d to be the one-dimensional $A-A-$bimodule

$$\int_A^{\ell} := \mathrm{Ext}^d_A({}_Ak, {}_AA);$$

notice that, exactly as when $d = 0$, this is a trivial *left* module, but the right structure may *not* be trivial. The *right integral* $\int_A^r$ is defined analogously.

The integrals were used in [3] to prove an infinite-dimensional analogue of a famous result of Larson and Sweedler characterising when a finite dimensional Hopf algebra has finite global dimension, and to initiate the study of (infinite-dimensional) Hopf algebras of low GK-dimension. The latter project is still in its infancy, so we state here as a sample problem:

Problem 6. Describe all prime affine Hopf algebras of GK-dimension 1.

In this talk, however, I will describe another application of the integral.

3. **Dualising complexes.** For the moment, we take B to be an arbitrary noetherian $k-$algebra, and we write B^e for $B \otimes_k B^{op}$.

Definition 7. A complex $R \in \mathcal{D}^b(B^e - \mathrm{mod})$ is a *dualising complex* if

(1) ${}_BR$ and R_B have finite injective dimension;
(2) R is homologically finite;
(3) the canonical maps $B \longrightarrow \mathrm{RHom}_B({}_{|}R, {}_{|}R)$ and $B \longrightarrow \mathrm{RHom}_B(R_{|}, R_{|})$are isomorphisms in $\mathcal{D}(B^e - \mathrm{mod})$.

Moreover R is *rigid* if there is a canonical isomorphism $R \cong \mathrm{RHom}_{B^e}(B, R \otimes R^{op})$.

When B has a dualising complex the functors $\mathrm{RHom}_B(-, R)$ and $\mathrm{RHom}_{B^{op}}(-, R)$ give a duality between $\mathcal{D}^b(B - \mathrm{mod})$ and $\mathcal{D}^b(B^{op} - \mathrm{mod})$. To give one example which we'll generalise below in the case of Hopf algebras, if B is finite dimensional then $B^* = \mathrm{Hom}_k(B, k)$ is a rigid dualising complex for B. Dualising complexes were introduced and studied in a noncommutative context by Yekutieli [9]. To improve the functoriality and uniqueness properties, rigid complexes were introduced by Van den Bergh in 1997, in [8]. Using his work, we prove our main result, whose statement needs some notation. First, $-[d]$ denotes the shift operator on complexes, moving d places to the left. Second, given a 1-dimensional representation π of a Hopf algebra A, the *left winding automorphism* τ^ℓ_π of A is defined by $\tau^\ell_\pi(a) = \sum \pi(a_1)a_2$, for $a \in A$; there is an analogous definition of a right winding automorphism.

Theorem 8. [1] *Let A be a noetherian AS-Gorenstein Hopf algebra of injective dimension d.*

(1) *A has rigid dualising complex ${}^{\nu}A^1[d]$. Here, ν is (by definition) the Nakayama automorphism of A; it is unique up to an inner automorphism.*
(2) *$\nu = S^2\tau^\ell_\pi$, where π is the representation defined by the right action of A on the left integral.*

Note that part (2) of the theorem generalises Proposition 3. There are applications of this result to (twisted) Poincaré duality of Hochschild (co)homology, but rather than give details here, I end by noting an amusing application to the antipode. Given A as in the theorem, $(A, \Delta^{op}, S^{-1}, \epsilon)$ is also a Hopf algebra, to which the theorem can be applied. By Van den Bergh's uniqueness result [8]for rigid dualising complexes, the resulting Nakayama automorphism of A must equal the one in part (2) of the theorem, to within an inner automorphism. Equating the two answers, one obtains the

Corollary 9. *With the notation and hypotheses of the theorem,* $S^4 = \gamma \circ \tau_\pi^r \circ (\tau_\pi^\ell)^{-1}$, *for some inner automorphism* γ *of* A.

When A has finite dimension, this is a 1976 result of Radford [6], with an explicit γ. We therefore propose

Problem 10. Determine γ.

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Brauer groups of surfaces

A.J. de Jong
(joint work with J. Starr)

In this talk we explained some of the results and methods of the papers [1], [2] and [3]. The main result mentioned is the following.

Theorem 1. *Let* F *be the function field of an algebraic surface over an algebraically closed field* k. *Then every element in the Brauer group of* F *has period equal to its index.*

From this theorem we deduced in the talk the following consequence, which was pointed out to us by Michel van den Bergh in a conversation.

Proposition 2. *For ever integer* n, *there exists a constant* $B(n)$ *with the following property: For every function field* F *as in the theorem of characteristic prime to* n, *for every Brauer class* α *over* F *of order dividing* n, *the class* α *is a sum of at most* $B(n)$ *classes of cyclic algebras over* F.

In order to prove this we use the map from Milnor K-theory to the Brauer group

$$MS_n : K_2(F)/n \longrightarrow Br(F)[n],$$

whose definition depends on the choice of a primitive nth root of 1. In fact, the celebrated Merkurjev-Suslin theorem tells us this map is an isomorphism as soon as F has a primitive nth root of 1. However, our proof does not use this theorem and it actually gives a proof of the surjectivity of MS_n the case of function fields of surfaces as well. This abstract for the Oberwolfach workshop seemed like a good place to point this out.

We sketch quickly the proof of the proposition. There is a reduction to the case where n is prime which we omit. The proof in the prime order case uses the compatibility of MS_n with norms. Namely, for a finite field extension $F \subset F'$ the maps MS_n and MS'_n (defined using the same root of 1) are compatible with both the restriction maps

$$K_2(F) \to K_2(F') \text{ and } Br(F) \to Br(F')$$

and the norm maps

$$K_2(F') \to K_2(F) \text{ and } Br(F') \to Br(F).$$

Note that the composition $K_2(F) \to K_2(F') \to K_2(F)$ is equal to multiplication by the degree $[F' : F]$ of F' over F. For a nice writeup of the definition of norm maps in Milnor K-theory see the expository note [4].

Thus we start with an element α in $Br(F)$ of prime order n. Since the period is equal to the index by the theorem, we know there is an extension of prime degree n splitting α. Thus there is an extension F' of F, of degree $[F' : F]$ dividing $(n-1)!$ *prime* to n such that $\alpha|_{F'}$ is split by a cyclic extension. Thus there is a symbol $\{f, g\} \in K_2(F')$ which maps to $\alpha|_{F'}$. We conclude that $\alpha = MS_n(Norm\{f, g\})/[F' : F]$. The final step is to analyze how the norm map in Milnor K-theory is defined.

Suppose that L/K is a field extension of degree d and suppose that K contains an algebraically closed field. Reading carefully in the write up [4] it follows that one can write the norm of a symbol in $K_2(L)$ as a sum of at most $B_d \leq 3 + 2B_{d-1}$ symbols. Here B_d is defined the number of symbols needed for norms from separable algebra extensions of K of degree at most d. So it follows that $B_d \leq 3^d$. Note that this implies the bound $B(n)$ in the proposition that one obtains from this is $B(n) \leq 3^{(n-1)!}$.

Of course this bound is probably far from optimal. For example if $n = 2, 3$ then the optimal bound is 1 (i.e., everybody is cyclic in this case of period 2,3 over the function field of a surface). The author of this note is not sure whether the bound $B(n)$ of the proposition should be 1 always.

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Quantized coordinate rings and torus orbits of symplectic leaves

K. R. Goodearl

A major goal in the study of a quantized algebra A (as for other noncommutative algebras appearing in representation theory) is to determine the primitive ideal space (= primitive spectrum), $\operatorname{prim} A$. Further, there should be tight relations between the quantized and classical situations. Anticipated connections between a classical object and a quantization may be briefly summarized in the following way:

Conjecture. *The primitive ideal space of a generic quantized coordinate ring matches the space of symplectic leaves in the classical object.*

To make this a little more precise:

(a) The given formulation assumes that the classical object (an affine variety or group) is defined over $\mathbb{C}$, that the quantization is a standard one relative to a single parameter q which is not a root of unity, and that the symplectic leaves are taken with respect to the Poisson structure induced by the quantization process.

(b) The desired "match" should include as much structure as possible. In particular, it should be not just a bijection, but a homeomorphism with respect to the Zariski topologies, and it should be equivariant with respect to appropriate group actions.

(c) If the situation is one in which the symplectic leaves are algebraic (i.e., locally closed in the Zariski topology), then the symplectic leaves in the classical object correspond to the Poisson-primitive ideals in its coordinate ring. The conjecture can then be rephrased in terms of matching the primitive spectrum of the quantized coordinate ring with the Poisson-primitive spectrum of the classical coordinate ring. In that formulation, the desired relationships should also hold for the quantized and classical coordinate rings relative to any base field over which the objects are defined.

(d) There exist multiparameter quantizations in which the symplectic leaves are not algebraic and do not match the primitive ideals in the quantized coordinate ring. We conjecture that in such situations the difficulty resides in the differential geometry, not in the algebra, and that the reformulated conjecture as in (c) should still hold.

To fix a basic example, let $\mathcal{O}_q(SL_n(\mathbb{C}))$ be the generic standard quantized coordinate ring of $SL_n(\mathbb{C})$. Hodges and Levasseur [2, 3] developed a bijection

$$\operatorname{prim} \mathcal{O}_q(SL_n(\mathbb{C})) \longleftrightarrow \operatorname{symp} SL_n(\mathbb{C}).$$

These spaces have finite stratifications such that the above bijection restricts to homeomorphisms on each stratum, but it is not yet known (except for the easy case $n = 2$) whether the bijection as a whole is a homeomorphism. The usual maximal torus H of $SL_n(\mathbb{C})$ acts on $\mathcal{O}_q(SL_n(\mathbb{C}))$ by winding automorphisms and on $\operatorname{symp} SL_n(\mathbb{C})$ by left translation; with respect to these actions, the above bijection is H-equivariant.

A generic standard quantization of SL_n can be defined over any field k containing a nonzero scalar q which is not a root of unity. Modulo some technical work (to free some present proofs from dependence on $\mathbb{C}$), it appears that the above picture generalizes to an H-equivariant bijection

$$\operatorname{prim} \mathcal{O}_q(SL_n(k)) \longleftrightarrow \text{Poisson-prim}\, \mathcal{O}(SL_n(k)),$$

where H is now the diagonal subgroup of $SL_n(k)$.

In typical situations in which the conjecture (either over $\mathbb{C}$ or over a more general base field) has been studied, there is a natural action of a torus H, and there are only finitely many H-orbits in the given spaces. Thus, the H-orbits provide a useful framework to which to tie information.

In the SL_n example, the H-orbits of symplectic leaves in $SL_n(\mathbb{C})$ are exactly the double Bruhat cells. On the quantum side, the H-orbits in $\operatorname{prim} \mathcal{O}_q(SL_n(\mathbb{C}))$ correspond to the H-stable prime ideals in $\mathcal{O}_q(SL_n(\mathbb{C}))$. Although not stated this way in [2, 3], one can see (with hindsight, and investing results of Joseph [4, 5]) the following path underlying the work of Hodges and Levasseur. First, the double Bruhat cells in $SL_n(\mathbb{C})$ can be described by vanishing and nonvanishing of certain specified sets of minors. When these minors are changed to quantum minors in $\mathcal{O}_q(SL_n(\mathbb{C}))$, one obtains generating sets for the H-stable prime ideals and for Ore sets of elements regular modulo these ideals.

In addition to completing the picture for SL_n and other semisimple groups (especially, developing the conjectured homeomorphisms, and making the machinery work over arbitrary base fields), one would like to develop similar pictures for other quantizations. Moving beyond algebraic groups, probably the most fundamental case is that of quantum matrices:

Problem. Develop "all of the above" for $\mathcal{O}_q(M_n(\mathbb{C}))$ and $\mathcal{O}_q(M_n(k))$.

Our exegesis of Hodges and Levasseur's work highlights the problem of finding and describing torus orbits of symplectic leaves in $M_n(\mathbb{C})$ as an important step. As before, the Poisson structure comes from the quantization – $\mathcal{O}(M_n(\mathbb{C}))$ is viewed as the semiclassical limit of the family of $\mathcal{O}_q(M_n(\mathbb{C}))$'s (as q varies). Here, one needs the maximal torus $(\mathbb{C}^\times)^n \subset GL_n(\mathbb{C})$ to act by both left and right translation to get finitely many orbits, and so we use $H = (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$ as our acting torus. In recent work with Brown and Yakimov [1], several equivalent descriptions of the H-orbits of symplectic leaves in $M_n(\mathbb{C})$ were developed, including one description via the vanishing and nonvanishing of explicit sets of minors.

This leads to a precise conjecture identifying generating sets of quantum minors for the H-stable prime ideals in $\mathcal{O}_q(M_n(\mathbb{C}))$. It had already been shown by Launois

[6], in response to an earlier conjecture of Lenagan and myself, that all the H-stable prime ideals in $\mathcal{O}_q(M_n(\mathbb{C}))$ can be generated by sets of quantum minors. However, his methods do not explicitly describe these sets. He has also [7] verified a conjecture of Lenagan, McCammond and myself, namely, that the poset of H-stable prime ideals in $\mathcal{O}_q(M_n(\mathbb{C}))$ (with respect to inclusion) is isomorphic to the following sub-poset of S_{2n} (with respect to the Bruhat order):

$$\{\sigma \in S_{2n} \mid |\sigma(i) - i| \le n \text{ for } i = 1, \dots, 2n\}.$$

(This poset also appears in [1]; we showed that it is anti-isomorphic to the poset of H-orbits of symplectic leaves in $M_n(\mathbb{C})$, with respect to the relation of inclusions of closures.) However, as with the previous result, the isomorphism is not explicit. Work is ongoing to make both results explicit.

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Rational Cherednik algebras, q-Schur algebras and quiver varieties

Iain Gordon

This is a report on work-in-progress.

Let $G_n = \mathfrak{S}_n \wr \mu_\ell$, the complex reflection group of type $G(\ell, 1, n)$ with reflection representation $\mathfrak{h}$. We are interested in the representation theory of $A(\mathbf{h})$, the *rational Cherednik algebra* associated to G_n. This algebra is a non-commutative deformation of the smash product $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * G_n$ which depends on parameters $\mathbf{h} = (h, H_1, \dots, H_{\ell-1}) \in \mathbb{Q}^\ell$. (Typically the parameters are taken to be complex numbers, but much of the most interesting representation theory already occurs for the rational numbers.)

There is a triangular decomposition of $A(\mathbf{h})$ as a vector space

$$A(\mathbf{h}) \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}G_n \otimes \mathbb{C}[\mathfrak{h}^*].$$

Thus it can be expected that $A(\mathbf{h})$ shares many properties with enveloping algebras of semi-simple Lie algebras, or rather central quotients of enveloping algebras of semi-simple Lie algebras. From this point of view the parameter space $\mathbb{Q}^\ell$ should be compared with the central character space $\mathfrak{h}/W$ for a semi-simple Lie algebra.

Category $\mathcal{O}(\mathbf{h})$ is defined to be the category of finitely generated $A(\mathbf{h})$-modules on which $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently. It is a highest weight category whose simple objects are parametrised by the irreducible representations of G_n, and whose ordering on simples is defined by *Lusztig's c-function* which is the scalar by which a certain central element $z(\mathbf{h}) \in \mathbb{C}G_n$ acts: $E_1 > E_2$ if and only if $\chi_{E_1}(z(\mathbf{h})) - \chi_{E_2}(z(\mathbf{h})) \in \mathbb{N}$. It was proved by Ginzburg–Gan–Opdam–Rouquier that $\mathcal{O}(\mathbf{h})$ is a *highest weight cover* of the Ariki–Koike algebra $\mathcal{H}(\mathbf{q})$ of type G_n (where $\mathbf{q} = \exp(2\pi\mathbf{h}\sqrt{-1})$). This means that there is an exact functor

$$\mathsf{KZ} : \mathcal{O}(\mathbf{h}) \longrightarrow \mathcal{H}(\mathbf{q})\text{-mod}$$

which is fully faithful on projective objects. Of course, if we have parameters $\mathbf{h}$ and $\mathbf{h}'$ such that $\mathbf{h} - \mathbf{h}' \in \mathbb{Z}^\ell$ then the associated values $\mathbf{q}$ are equal. In [1], Rouquier proved that if in addition the ordering on irreducible representations of G_n does not change then $\mathcal{O}(\mathbf{h})$ and $\mathcal{O}(\mathbf{h}')$ are equivalent. Thus if we refine the order slightly to say that $E_1 > E_2$ if and only if $\chi_{E_1}(z(\mathbf{h})) > \chi_{E_2}(z(\mathbf{h}))$ then parameter space splits up into a finite number of chambers; in the interior of each chamber there is a well-defined total order on the irreducible representations of G_n. As remarked on by Rouquier, this order is actually a little too strong: sometimes there are is a more natural *partial* order to take on the irreducible representations, and often this itself is refined by several chambers. In the Lie theoretic analogue, associated to each point in $\mathfrak{h}/W$ there is an orbit of weights and these are given the dominance order.

To understand better this ordering on irreducible representations of G_n, we relate the representation theory of $A(\mathbf{h})$ to the geometry of resolutions of the orbit space $(\mathfrak{h} \oplus \mathfrak{h}^*)/G_n$. This singular space has many crepant=symplectic resolutions given by Nakajima quiver varieties $\mathcal{M}_\theta(n)$. These varietes depend on a stability parameter $\theta \in \mathbb{Q}^\ell$ and are moduli spaces of certain θ-polystable representations of the doubled quiver of Q_∞, a cyclic quiver with ℓ vertices together with one extra arrow attached to a single vertex of the cyclic quiver. Not all choices of $\theta \in \mathbb{Q}^\ell$ given a resolution of the orbit space: the space of stability parameters splits up into a finite number of chambers; in the interior of each chamber the stability parameter produces a resolution $\mathcal{M}_\theta(n)$ together with a tautological vector bundle on it and this is constant in the chamber; on the walls $\mathcal{M}_\theta(n)$ is singular.

Theorem 1 (G-Stafford, Boyarchenko, Musson, Vale). *There exists (for most* $\mathbf{h}$*) a* $\mathbb{Z}$*–algebra* $Z(\mathbf{h})$ *such that*

$$\begin{array}{ccc} \textit{Coh-}Z(\mathbf{h}) & \xrightarrow{gr} & (\textit{Coh-})\mathcal{M}_\theta(n) \\ \wr\downarrow & & \downarrow\pi \\ A(\mathbf{h})\textit{-mod} & \xrightarrow{gr} & (\textit{Coh-})(\mathfrak{h}\oplus\mathfrak{h}^*)/G_n. \end{array}$$

Here $\theta = (h + H_1 + \cdots + H_{\ell-1}, -H_1, \ldots, -H_{\ell-1})$.

From now on let $\theta = (h + H_1 + \cdots + H_{\ell-1}, -H_1, \ldots, -H_{\ell-1})$ and assume $\mathbf{h}$ is in the *interior* of a Rouquier chamber.

Theorem 2 (G)**.**

(1) $\pi_\theta : \mathcal{M}_\theta(n) \longrightarrow (\mathfrak{h} \oplus \mathfrak{h}^*)/G_n$ *is a symplectic resolution.*
(2) $\mathcal{L}_\theta^n = \pi^{-1}(y_1 = \cdots = y_n = 0)$ *is a Lagrangian subvariety whose components are in natural bijection with the irreducible representations of* G_n*.*
(3) *There is a natural ordering on components; it is refined by Rouquier's ordering.*
(4) *This ordering arises in Uglov's combinatorics on higher level Fock space (the charge is related to* θ*).*

The key to the proof of this is that there is a Morse function on $\mathcal{M}_\theta(n)$ whose critical points are labelled by certain representations called baby Verma modules which are parametrised by the irreducible representations of G_n. Moreover, the value of this Morse function at the critical points is just Lusztig's c-function.

Thus $\mathcal{M}_\theta(n)$ (for the given θ) can accept much of the combinatorics of $\mathcal{O}(\mathbf{h})$. This leads to a few questions.

- Are there more equivalences between different $\mathcal{O}$'s than Rouquier predicts? (And are all chambers related by derived equivalences?)
- Under the mappings of the first theorem, does the coherent sheaf on $\mathcal{M}_\theta(n)$ corresponding to $A(\mathbf{h})$ give a generalisation of the Procesi bundle?
- Is the combinatorics of $\mathcal{O}(\mathbf{h})$ related to Shoji's generalised Green functions?
- (Rouquier/Yvonne) Are the multiplicities in $\mathcal{O}(\mathbf{h})$ described by combinatorics of Uglov's higher level Fock spaces?
- Is there a Heisenberg action on $\oplus_{n\geq 0} H^{2n}(\mathcal{L}_\theta^n, \mathbb{Z})$?

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Strongly exceptional sequences of line bundles on toric varieties

Lutz Hille
(joint work with Markus Perling)

1. Strongly exceptional sequences

1. Motivation. We work over the field of complex numbers $\mathbb{C}$ and consider the n–dimensional projective space $\mathbb{P}^n$.

Theorem 1.1 ([B])**.** *Define*

$$\Phi = \mathrm{Hom}_{\mathbb{P}^n}(\oplus_{i=0}^n \mathcal{O}(i), -) : \mathrm{Coh}\mathbb{P}^n \to \mathrm{mod}A.$$

Then $\mathbb{R}\Phi : \mathcal{D}^b(\mathrm{Coh}\mathbb{P}^n) \to \mathcal{D}^b(\mathrm{mod}A)$ *is an equivalence of derived categories, where* $A = \mathrm{End}(\oplus_{i=0}^n \mathcal{O}(i))$*.*

We want to extend this to smooth projective toric varieties.

2. Exceptional Sequences. Let X be a smooth projective variety and assume $H^l(X;\mathcal{O}_X) = 0$ for all $l \neq 0$ (this holds in particular for toric varieties). We assume n is the rank of the Grothendieck group $K_0(X)$ (where we already assume it is finitely generated, is also satisfied for toric varieties). Let $\varepsilon = (\mathcal{L}_1, \dots, \mathcal{L}_n)$ be a sequence of line bundles on X.

We say ε is

(1) *complete* if $n = \operatorname{rank} K_0(X)$;
(2) *full* if $\langle \mathcal{L}_1, \dots, \mathcal{L}_n \rangle = \mathcal{D}^b(X)$;
(3) *exceptional* if $\operatorname{Ext}^l(\mathcal{L}_j, \mathcal{L}_i) = 0 \forall j > i$ and $l \neq 0$;
(4) *strongly exceptional* if $\operatorname{Ext}^l(\mathcal{L}_i, \mathcal{L}_j) = 0$ for all $l \neq 0$. In that case one can change the ordering so that $\operatorname{Hom}(\mathcal{L}_j, \mathcal{L}_i) = 0 \quad \forall j > i$.[1]

3. Conjectures. Let X be a smooth projective toric variety from now on!

1. (A. King [Ki]) Every such X has a full, strongly exceptional sequence.
2. On any toric Fano variety X there exists a helix: i.e., if we extend the sequence ε to an infinite sequence $S(\varepsilon) = \{\mathcal{L}_i\}$ with $\mathcal{L}_{i+nl} := \mathcal{L}_i \otimes \omega_X^{-l}$ then each length n subsequence $(L_i, L_{i+1}, \dots, L_{i+n-1})$ of $S(\varepsilon)$ is also strongly exceptional.
3. On X a full exceptional sequence of line bundles exists.

Kawamata [Ka] proved X always has a full exceptional sequence of *sheaves*.

4. Toric varieties. Write Σ_X for the fan of the toric variety X.

Write $n = \operatorname{rank} K_0(X)$, $d = \dim X$ and t=rank of the equivariant Picard group. Then $\operatorname{rank} \operatorname{Pic}(X) = t - d$ and there are T-invariant prime divisors $D_1, \dots, D_t$ that provide a basis for the equivariant Picard group. Moreover, the canonical divisor is $\omega_X^{-1} = \mathcal{O}(\sum_{i=1}^t D_i)$.

A fan consists of a set of cones $\sigma \in \Sigma_X = \Sigma$. The cones are in order reversing bijection with the orbits of X under the torus action (the cones are partially ordered by the face relation; the orbits by the closure relation). The zero cone $\{0\}$ corresponds to the dense orbit, the maximal cones correspond to the fix points. The torus–invariant prime divisors correspond to the rays in Σ (cones of dimension one). Each ray is of the form $\tau_i = \mathbb{R}_{\geq 0} v_i$ for a unique indivisible lattice point v_i (it is the first lattice point along the ray τ_i).

5. The graph of a toric variety. Write $P(\Sigma) = \cup_{\sigma\in\Sigma} \operatorname{conv}\{0, v_i \mid v_i \in \sigma\}$, the union of the convex hull of the polytope $P(\sigma)$ with vertices v_i for v_i in σ and the origin. Then X is a Fano variety if and only if $P(\Sigma)$ is strictly convex. In that case, one has a *polytope* $P(\Sigma)$ from which one can recover the fan.

Define a graph Γ_X as the 1–skeleton of the simplicial complex $(\Sigma \cap S^{d-1})$. The vertices are the D_is, and there is an edge between i and j if $\operatorname{codim}_{D_i} D_i \cap D_j = 1$. When $\dim X = 2$, this graph is $\widetilde{A}_{t-1}$ (and $t = n$). If $\dim X = 3$ this graph is a planar graph (is is embedded in the 2–sphere).

[1] Can do this because if $\mathcal{L}$ and $\mathcal{L}'$ are non-isomorphic bundles only one of $\operatorname{Hom}(\mathcal{L}, \mathcal{L}')$ and $\operatorname{Hom}(\mathcal{L}', \mathcal{L})$ can be non-zero—enoough to prove this when $\mathcal{L}' = \mathcal{O}$; think about $\mathcal{O} \to \mathcal{L} \to \mathcal{O}$

Example. The Hirzebruch surface $X = \mathbb{F}_r$: it has rays through $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,r)$, four maximal cones generated by pairs of two neighboured rays and the cone $\{0\}$. Then $n = 4 = t$. In this case one can classify all full (strongly) exceptional sequences (see [H]).

2. Results

1. Constructions. For simplicity will focus on line bundles of the form $\mathcal{L} = \mathcal{O}(\sum a_i D_i)$ $a_i \in \{0, \pm 1\}$ and a particular sequence of line bundles. Note that $\omega_X^{-1} = \mathcal{O}(\sum_{i=1}^t D_i)$.

Theorem 2.1. *Let* $\varepsilon = (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \dots, \mathcal{O}(D_1 + \cdots + D_{t-1}))$.

(1) *If* ε *exceptional* $\Rightarrow v_1, \dots, v_t, v_1$ *is a Hamiltonian cycle in* Γ_X.
(2) *If* $\dim X \leq 3$ *then we have if and only if in (1).*
(3) ε *is complete* $\Leftrightarrow \varepsilon$ *is full* $\Leftrightarrow t = n \Leftrightarrow X = \mathbb{P}^n$ *or* $\dim X = 2$.
(4) *If* X *Fano and* ε *exceptional, then* ε *is strongly exceptional.*

As a corollary we get some full (strongly) exceptional sequences, respectively helices, on toric surfaces:

Theorem 2.2. *Let* $\varepsilon = (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \dots, \mathcal{O}(D_1 + \cdots + D_{t-1}))$ *and* $\dim X = 2$.

(1) *Then* ε *is full exceptional if we take the cyclic orientation on the divisors*
(2) ε *is strongly exceptional iff* $D_i^2 \geq -1$ *for all* $i = 1, \dots, t-1$
(3) $S(\varepsilon)$ *is a helix iff* $D_i^2 \geq -1$ *for all* i.

2. Counterexample. Take X equal to $\mathbb{F}_2$ iteratively blown up three times as described by the following fan. The fan is the fan of $\mathbb{F}_2$ as above and has three additional rays through $(1,-1)$, $(2,-1)$, $(3,-1)$.

Theorem 2.3 ([HP])**.** *On the surface* X *there is no complete (of length 7) strongly exceptional sequence of line bundles.*

3. 3–dimensional toric Fano varieties.

Theorem 2.4. *Suppose* X *is Fano and* $\dim X = 3$. *Then a complete strongly exceptional sequence exists and can be completed to a helix.*

4. Maximal toric fano Varieties. Complete strongly exceptional sequences exist on the maximal Fanos (recent joint work with B. Nill).

5. Conclusions. Conjecture 1 is false even for some surfaces. For Conjecture 2 we have a proof in dimension less or equal to 3 and for maximal toric Fano varieties. We have also checked the conjecture for many examples in dimension 4. A possible counterexample to Conjecture 3 seems to be out of any computable range.

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Classifying graded finite dimensional representations

Birge Huisgen-Zimmermann

(joint work with E. Babson, S.O. Smalø, and R. Thomas)

The following is an excerpt of joint work with E. Babson and R. Thomas [1]; the last result is taken from [2].

Let A be a basic, positively graded, finite dimensional algebra over an algebraically closed field K, and J its radical, L its Loewy length. Primarily, we address classifiability of the d-dimensional graded representations of A, for a given positive integer d, through coarse or fine moduli spaces (Theorems A and B). The final result shows how such graded classification problems impinge on the ungraded representation theory of A, more specifically, on a better understanding of the irreducible components of varieties parametrizing the representations of dimension d.

Here are the basic results, the first addressing the local case, that is, the case of A-modules with fixed simple top.

Theorem A. *For any d and any simple A-module T, the d-dimensional graded A-modules with top T have a fine moduli space classifying them up to graded-isomorphism. This moduli space is a projective variety. It has a natural partition into closed subvarieties such that each of the projective varieties in this partition classifies the graded representations M with fixed radical layering*

$$\mathbb{S}(M) = (M/JM, JM/J^2M, \dots, J^LM/J^{L+1}M)$$

up to graded-isomorphism.

This sweeping classifiability of *graded* local modules stands in marked contrast to the ungraded situation, which has been addressed in [8, 10]: Namely, if P denotes the projective cover of T, the d-dimensional A-modules with fixed simple top T have a fine (or equivalently, a coarse) moduli space if and only if all submodules of JP which have codimension d in P are invariant under the endomorphisms of P. The latter is obviously a fairly stringent condition, as simple examples of low dimension already illustrate. The subcategory of graded objects, by contrast, shows sufficient additional rigidity to prevent existence of proper degenerations. On the side, we mention that geometric features of the arising moduli spaces – the number of irreducible components, for instance – become combinatorially accessible from a presentation of A in terms of quiver and relations; this direction is pursued in [8, 2].

The situation of a general top T is addressed by the following result.

Theorem B. *Let* $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ *be any sequence of* $L + 1$ *finite dimensional semisimple* A*-modules and* $\mathcal{G}(\mathbb{S})$ *the class of all* d*-dimensional graded* A*-modules* M *with* $\mathbb{S}(M) = \mathbb{S}$.

• *If the objects in* $\mathcal{G}(\mathbb{S})$ *which are generated in degree zero possess a coarse moduli space, classifying their graded-isomorphism classes, then all objects in* $\mathcal{G}(\mathbb{S})$ *are direct sums of local modules (= modules with simple tops).*

• *Conversely, suppose that all objects in* $\mathcal{G}(\mathbb{S})$ *are direct sums of local modules. Then* $\mathcal{G}(\mathbb{S})$ *has a "concretely describable" finite partition such that each of the resulting subclasses is classified by a fine moduli space, up to graded-isomorphism.*

Arbitrary projective varieties arise as moduli spaces in this context; examples can be found in [6] and, accessed through alternate methods, in [1]; these examples are projective completions of the affine examples constructed in [7, Section 6]. Our final theorem singles out a class of algebras which motivated our interest in moduli spaces of graded representations, in that the latter make a natural appearance in the description of the irreducible components of varieties of ungraded representations.

Before we specialize, we include a few general comments on such irreducible components. Letting $\mathbf{Mod}_d$ be the classical affine variety of all d-dimensional left A-modules and $\mathcal{C}$ an irreducible component, one aims at describing generic properties of the modules parametrized by $\mathcal{C}$ (that is, properties shared by all modules corresponding to the points in a nonempty open subset of $\mathcal{C}$). This line of inquiry was opened up by Kac and Schofield, and extended by Crawley-Boevey and Schröer via a different angle of approach (see [9], [11], [5]).

Since, generically, the modules M parametrized by $\mathcal{C}$ have fixed radical layering $\mathbb{S} = \mathbb{S}(M)$, one reduces this task to a, in many ways more manageable, subvariety of a Grassmannian variety; this variety, $\mathfrak{Grass}(\mathbb{S})$, was first introduced and studied by Bongartz and the author in [3, 4]. It provides an alternate parametrization of the modules with fixed radical layering $\mathbb{S}$, next to the parametrization by a suitable subvariety of the standard module variety $\mathbf{Mod}_d$. The relevant geometric information can be shifted back and forth between the Grassmannian and the affine settings.

In the final result cited, we specialize to *truncated path algebras*, i.e., to algebras of the form $A = KQ/I$, where KQ is the path algebra of a quiver Q and I the ideal generated by all paths of length $L + 1$. Such an algebra is endowed with a natural grading by path lengths; in other words, it coincides with the graded algebra associated to the radical filtration $A \supseteq J \supseteq J^2 \supseteq \cdots$. By Gr-$\mathfrak{Grass}(\mathbb{S})$, we denote the subvariety of $\mathfrak{Grass}(\mathbb{S})$ consisting of the points corresponding the graded modules with radical layering $\mathbb{S}$ which are generated in degree zero. Whenever these graded modules possess a moduli space (see above), this moduli space coincides with Gr-$\mathfrak{Grass}(\mathbb{S})$.

Theorem C. *Again, let* $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ *be a sequence of* $L+1$ *finite dimensional semisimple A-modules.*

(1) *The variety* Gr-$\mathfrak{Grass}(\mathbb{S})$ *is irreducible, projective, smooth, and rational.*

(2) *The ungraded counterpart, namely* $\mathfrak{Grass}(\mathbb{S})$*, is an iterated vector bundle over* Gr-$\mathfrak{Grass}(\mathbb{S})$.

Theorem C yields a detailed understanding of "the" generic module with radical layering $\mathbb{S}$; see [2].

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Clusters, wide subcategories, and noncrossing partitions

Colin Ingalls

(joint work with Hugh Thomas)

For a general hereditary algebra A, there is a well-known bijective correspondences between finitely generated torsion classes, partial tilting objects which are tilting on their support, and clusters. We show that there is another family in bijective correspondence: finitely generated, exact abelian, extension-closed subcategories of the representations of A. Subcategories which are exact abelian, and extension-closed are called *wide subcategories*, and have been studied by Hovey [Ho].

The motivation for our interest here comes from a topic in geometric group theory, the noncrossing partitions of a Coxeter group. (There is an order on a Coxeter group called absolute order, and the noncrossing partitions are those elements of the group in the interval between the identity element and a fixed choice of Coxeter element.) The connection between hereditary algebras and Coxeter groups is made by interpreting the quiver of A as a Coxeter diagram, and using the edge orientations to determine the choice of Coxeter element.

If A is of finite or tame type, we show that the finitely generated wide subcategories of rep A, ordered by inclusion, form a partially ordered set isomorphic to the noncrossing partitions.

In finite type, all wide subcategories are finitely generated, so it is a triviality that they form a lattice under inclusion. Thus, we recover the result that in finite type, the noncrossing partitions form a lattice. This was originally proved using the classification of finite reflection groups and a computer check for the exceptional groups, and the first case free proof was given in 2005 by Brady and Watt [BW]. We also recover the bijection between clusters and noncrossing partitions found by Reading in 2005 [Re].

Once we leave finite type, the noncrossing partitions do not form a lattice. However, we believe that the respresentation-theoretic perspective will still be helpful in better understanding these posets.

We conjecture that the bijection between noncrossing partitions and finitely generated wide subcategories extends to wild type.

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Acyclic Calabi-Yau categories are cluster categories

Bernhard Keller

(joint work with Idun Reiten)

Let k be a field and Q a finite quiver without oriented cycles. Let kQ be the path algebra of Q and $\operatorname{mod} kQ$ the category of k-finite-dimensional right kQ-modules. The cluster category $\mathcal{C}_Q$ was introduced in [1] (for general Q) and, independently, in [4] (for Q of type A_n). It is defined as the orbit category of the bounded derived category $\mathcal{D}^b(\operatorname{mod} kQ)$ under the action of the automorphism $\Sigma^{-1} \circ S^2$, where S is the suspension (=shift) functor of the derived category and Σ its Serre functor, characterized by the Serre duality formula

$$D\operatorname{Hom}(X, Y) = \operatorname{Hom}(Y, \Sigma X) \ ,$$

where D is the duality functor $\mathrm{Hom}_k(?, k)$. The motivation behind this definition was to find a 'categorification' of the cluster algebras introduced by Fomin-Zelevinsky in [6]. This program has been quite successful, *cf. e.g.* [3] [5] and the references given there. The cluster category has the following properties (explained below in more detail):

a) $\mathcal{C}_Q$ is a triangulated category. In fact, it is even an algebraic triangulated category, *i.e.* there is a triangle equivalence between $\mathcal{C}_Q$ and the stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$.
b) $\mathcal{C}_Q$ *is* Hom-*finite* (*i.e.* all its morphism spaces are finite-dimensional) and *Calabi-Yau of CY-dimension* 2. By this, one means that it admits a Serre functor Σ (which is induced by that of the derived category) and that there is an isomorphism of triangle functors between Σ and S^2. Note that this last property holds almost by definition of $\mathcal{C}_Q$.
c) If T_Q denotes the image of the free module kQ under the projection from the derived category to the cluster category, then T_Q is a *cluster-tilting object* in $\mathcal{C}_Q$, *i.e.* we have
   c1) $\mathrm{Hom}(T_Q, ST_Q) = 0$ and
   c2) for each object X, if we have $\mathrm{Hom}(T_Q, SX) = 0$, then X is in $\mathrm{add}(T_Q)$.
d) The endomorphism algebra of T_Q is isomorphic to kQ. In particular, its ordinary quiver does not admit oriented cycles.

These properties were proved in [1] except for a), which was proved in [9]. We say that a k-linear category is a 2-*Calabi-Yau category* if it satisfies a) and b). Our main result is that properties a) to d) characterize the cluster category if k is algebraically closed:

Theorem. *Suppose that k is algebraically closed. If $\mathcal{C}$ is an algebraic* 2*-Calabi-Yau category and admits a cluster-tilting object T such that the ordinary quiver Q of the endomorphism algebra of T does not contain oriented cycles, then there is a triangle equivalence from $\mathcal{C}_Q$ to $\mathcal{C}$ which takes the object T_Q to T.*

The theorem allows one to show that cluster categories, whose definition may seem artifical at first glance, do occur in nature: Let k be an algebraically closed field of characteristic 0, S the completed power series algebra $k[[X, Y, Z]]$ and G the cyclic group of order three acting linearly on S such that a generator of G multiplies the three variables by the same primitive third root of unity. It is not hard to show that the fixed point algebra $R = S^G$ is a Gorenstein complete local normal domain that has an isolated singularity. We consider the Frobenius category $\mathcal{E} = CM(R)$ of its maximal Cohen-Macaulay modules. By a theorem of Auslander's, the stable category $\mathcal{C} = \underline{\mathcal{E}}$ is 2-Calabi-Yau. Work of Iyama [8] shows that $T = S$ considered as an R-module is a cluster-tilting object in $\mathcal{C}$. Its ring of R-linear endomorphisms is isomorphic to the skew group algebra $S * G$ and its endomorphism ring in $\underline{\mathcal{E}}$ is the path algebra of the generalized Kronecker quiver Q with three arrows. Thus the hypotheses of the theorem are satisfied and we obtain a triangle equivalence between $\mathcal{C}_Q$ and $\underline{CM}(R)$. In particular, this allows us to compute the Auslander-Reiten quiver of $\underline{CM}(R)$. It also shows that Yoshino's classification of the rigid Cohen-Macaulay modules [12] in $\underline{CM}(R)$ is equivalent to the classification of the cluster-tilting objects in the cluster category of the generalized Kronecker quiver with three arrows and thus [5] to that of the cluster variables in the corresponding cluster algebra [11].

Assume that k is algebraically closed. Let $\mathcal{C}$ be a 2-Calabi-Yau category admitting a cluster-tilting object T. One can show that the number of pairwise indecomposable non isomorphic direct factors of T does not depend on the choice of T, *cf.* [10]. We call this number the *rank* of $\mathcal{C}$. We say that $\mathcal{C}$ is *acyclic* if it admits a cluster tilting object the quiver of whose endomorphism algebra does not have oriented cycles (or equivalently, if it is triangle equivalent to a cluster category).

Conjecture. *If $\mathcal{C}$ is of rank at most three and the quivers of the endomorphism algebras of its tilting objects admit neither loops nor 2-cycles, then it is acyclic.*

It was shown in [7] that if $\mathcal{C}$ is the stable module category $\underline{\mathrm{mod}}\,\Lambda(\Delta)$ of a preprojective algebra associated with a simply laced Dynkin diagram Δ, then the quivers of the endomorphism algebras of its cluster-tilting objects admit neither loops nor 2-cycles. It follows that this property also holds if $\mathcal{C}$ is constructed as a 'CY-subquotient' of $\underline{\mathrm{mod}}\,\Lambda(\Delta)$ (for the 'CY-subquotient' construction, *cf.* section 2 of [2] and section 5.4 of [5]). Thus the conjecture implies that any CY-subquotient of rank ≤ 3 of $\underline{\mathrm{mod}}\,\Lambda(\Delta)$ is acyclic. This holds indeed in all the examples we have checked.

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Moduli of sheaves from moduli of Kronecker modules

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(joint work with Luis Álvarez-Cónsul)

Let X be a projective scheme over an algebraically closed field and with structure sheaf $\mathcal{O}$. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X. The projectivity of X can be encoded by a group of automorphisms of $\mathrm{Coh}(X)$, denoted $E \mapsto E(n)$ for $n \in \mathbb{Z}$, given by tensoring n-times by an ample invertible sheaf $\mathcal{O}(1)$. The ampleness here implies in particular that the dimension of $H^0(E(n)) := \mathrm{Hom}(\mathcal{O}(-n), E)$ is, for $n \gg 0$, given by a polynomial $P_E(n)$, called the Hilbert polynomial of E.

Now, to X or $\mathrm{Coh}(X)$, we can associate the moduli spaces $M_X(P)$ of semistable coherent sheaves of given Hilbert polynomial P. The aim of our work [1] is to shed new light on the construction of the schemes $M_X(P)$, on why they are also projective and on their natural homogeneous coordinates, called 'theta functions'.

The simple answer to the question of why $M_X(P)$ is projective is that it is constructed by Geometric Invariant Theory, as developed by Mumford [2] for precisely such purposes. However, we wish to give a more natural answer, in terms of the functors

$$\Phi\colon \mathrm{Coh}(X) \to \mathrm{Mod}(A)\colon E \mapsto \mathrm{Hom}(T^\vee, E)$$

where $T = \mathcal{O}(n_0) \oplus \mathcal{O}(n_1)$ and $A = \mathrm{End}(T)$ for suitable $n_1 > n_0 \in \mathbb{Z}$. Concretely, the A-module $V = \Phi(E)$ is precisely specified by the H-Kronecker module

$$\phi_E \colon V_0 \otimes H \to V_1,$$

where $V_i = H^0(E(n_i))$ and $H = \mathrm{Hom}(\mathcal{O}(n_0), \mathcal{O}(n_1))$. Note that Φ has a left adjoint

$$\Phi^\vee\colon \mathrm{Mod}(A) \to \mathrm{Coh}(X)\colon V \mapsto T^\vee \otimes_A V.$$

More concretely, $\Phi^\vee(V)$ is the cokernel of the obvious map

$$V_0 \otimes H \otimes \mathcal{O}(-n_1) \longrightarrow V_0 \otimes \mathcal{O}(-n_0) \oplus V_1 \otimes \mathcal{O}(-n_1).$$

Note also that, for every dimension vector $v = (v_0, v_1)$, there is a naturally projective variety $M_H(v)$ which is the moduli space of semistable H-Kronecker modules of dimension vector v. Thus, our more natural answer to the question of why $M_X(P)$ is projective is the following.

Theorem 1. *For $n_1 \gg n_0 \gg 0$ (i.e. $\exists N_0\ \forall n_0 \geq N_0\ \exists N_1\ \forall n_1 \geq N_1$), the functor Φ gives a closed embedding $M_X(P) \hookrightarrow M_H(P(n_0), P(n_1))$.*

Observe that a special case of this, applied to the simple point sheaves $\mathcal{O}_x$, gives the usual embeddings $X \hookrightarrow \mathbb{P}(H^*) = M_H(1,1)$, which exhibit X itself as a projective scheme.

We remark that this provides a new interpretation of the known construction of $M_X(P)$ by Simpson [5]. We also note an important technical *caveat*: the embedding is scheme-theoretic except possibly at strictly semistable points in positive characteristic, when it may only be set-theoretic. However, the embedding is also functorial, so it should be a genuine embedding of 'non-commutative moduli spaces', in any sensible interpretation of that term.

Indeed, the functoriality of the embedding actually extends from semistable sheaves to regular sheaves, in the sense of Castelnuovo-Mumford. Recall that E is m-regular if $H^i(E(m-i)) = 0$ for $i \geq 1$. This implies in particular that $\dim H^0(E(n)) = P(n)$ for $n \geq m$.

Theorem 2. *If $\mathcal{O}(-n_0)$ is n_1-regular, then Φ is fully faithful on all n_0-regular sheaves E, i.e. the natural map $\varepsilon_E\colon T^\vee \otimes_A \operatorname{Hom}(T^\vee, E) \to E$ is an isomorphism.*

Note that the standard 'boundedness' lemma says that, given P, for $n \gg 0$, all semistable E of Hilbert polynomial P are n-regular. Now, to get from Theorem 2 to Theorem 1, we must compare the notions of semistability for sheaves and Kronecker modules.

Definition 3. An A-module V, i.e. a Kronecker module $\phi\colon V_0 \otimes H \to V_1$, is *semistable* iff, for all $V' \subset V$

$$\frac{\dim V_0'}{\dim V_1'} \leq \frac{\dim V_0}{\dim V_1}$$

The standard definition of semistability for sheaves involves the condition of purity: E is *pure* iff it has no proper subsheaves with lower dimensional support. Note that $\dim \operatorname{Supp} E = \deg P_E$.

Definition 4. A sheaf E is *semistable* iff E is pure and, for all $E' \subset E$,

$$\frac{P_{E'}(n)}{r_{E'}} \leq \frac{P_E(n)}{r_E} \quad \text{for } n \gg 0,$$

where r_E is the 'multiplicity' of E, i.e. the leading coefficient of P_E.

However, for our purposes, this definition has a crucial reformulation.

Lemma 5. *A sheaf E is semistable if and only if, for all $E' \subset E$,*

$$\frac{P_{E'}(n_0)}{P_{E'}(n_1)} \leq \frac{P_E(n_0)}{P_E(n_1)} \quad \text{for } n_1 \gg n_0 \gg 0.$$

Thus E is semistable if and only if for all $E' \subset E$, $\Phi(E')$ does not destabilise $\Phi(E)$ for $n_1 \gg n_0 \gg 0$. Note also that purity is now more clearly a consequence of semistability, rather than simply a necessary condition for the definition to make sense (cf. [3] for a similar conclusion).

Working from the reformulation in Lemma 5, one can then show that, as long as E is pure, such n_0, n_1 can be found uniformly, i.e. independent of E, and that the submodules of the form $\Phi(E')$ are the critical ones for determining the semistability of $\Phi(E)$. We do not know whether the purity assumption can be removed from this argument, but it seems quite possible that it cannot be. Thus we obtain the following uniform characterisation of semistability of sheaves.

Theorem 6. *Given P, for $n_1 \gg n_0 \gg 0$, any E of Hilbert polynomial P is semistable if and only if E is n_0-regular and pure, and $\Phi(E)$ is semistable.*

With further work, including Langton's method to prove the properness of $M_X(P)$, the forward implication (E semistable $\Rightarrow \Phi(E)$ semistable) of Theorem 6, together with Theorem 2, yields Theorem 1.

Finally, we turn to the question of the natural homogeneous coordinates. For $M_H(v)$, these come from the theory of determinantal semi-invariants of quivers, as formulated by Schofield & Van den Bergh [4]. Firstly, they show that an A-module V is semistable if and only if there is a map of projective modules

$$\gamma\colon P_1^{k_1} \to P_0^{k_0}$$

such that $\operatorname{Hom}(\gamma, V)\colon V_0^{k_0} \to V_1^{k_1}$ is an isomorphism, i.e. $\theta_\gamma(V) \neq 0$ where $\theta_\gamma(V) := \det \operatorname{Hom}(\gamma, V)$. Note the necessary restriction $k_1/k_0 = \dim V_0 / \dim V_1$, which leaves one degree of freedom, corresponding to the degree of θ_γ as a semi-invariant. Secondly, such 'theta functions' θ_γ actually span the graded ring $S^\bullet(v)$ of semi-invariants and hence give homogeneous coordinates for the projective embeddings of $M_H(v) := \operatorname{Proj} S^\bullet(v)$.

Now, the adjunction $\operatorname{Hom}(\gamma, \Phi(E) = \operatorname{Hom}(\Phi^\vee(\gamma), E)$ tells us that the restriction of such θ_γ to $M_X(P)$ is of the form $\theta_\delta(E) := \det \operatorname{Hom}(\delta, E)$ for

$$\delta\colon \mathcal{O}(-n_1)^{k_1} \to \mathcal{O}(-n_0)^{k_0},$$

with $k_1/k_0 = P(n_0)/P(n_1)$. Thus, firstly, the condition in Theorem 6 that $\Phi(E)$ is semistable may be replaced by the condition that $\theta_\delta(E) \neq 0$ for some such δ and, secondly, such theta functions yield projective embeddings of $M_X(P)$, except possibly at strictly semistable points in positive characteristic.

To conclude, one may roughly consider that the philosophy of [4] has been imported to $\operatorname{Coh}(X)$. Thus the invertible sheaves $\mathcal{O}(-n)$ for $n > m$ behave like indecomposable projective objects for m-regular sheaves E and the theta functions which detect semistability come from maps δ between projectives as above. However, technicalities such as the role of purity and the uniformity in the condition "$n_1 \gg n_0 \gg 0$" make things a little more complicated in this case.

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DG deformation theory of objects in homotopy and derived categories

Valery Lunts
(joint work with D. Orlov)

For an object E which is a DG module over a DG category $\mathcal{A}$, we develop its deformation theory in the corresponding homotopy and derived categories. These functors depend only on the (quasi-isomorphism class of the) DG algebra $\mathcal{C} = \operatorname{End}(E)$. As a "base" of our infinitesimal deformations we allow any artinian (noncommutative) DG algebra. The main result is the pro-representability of the derived deformation functor by the complete DG algebra $\hat{S} = (B\mathcal{C})^*$ — the linear dual of the bar construction $B\mathcal{C}$ of $\mathcal{C}$.

The main theorem is proved in conjunction with a result about the Fourier-Mukai transform from the derived category of $\hat{S}$-modules to that of $\mathcal{C}$-modules. Namely, it is proved that the bar complex $B\mathcal{C} \otimes \mathcal{C}$ defines a full and faithful functor on the subcategory generated by the $\hat{S}$-module k.

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Symplectic reflection algebras at $t = 0$ and deformed preprojective algebras

Maurizio Martino

Let X be an affine algebraic variety over the complex numbers with coordinate ring $\mathbb{C}[X]$. We say that X is *Poisson* if there exists a map $\{-,-\} : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}[X]$ satisfying:

i) $\mathbb{C}[X]$ equipped with the bracket $\{-,-\}$ is a Lie algebra over $\mathbb{C}$;
ii) for all $x, y, z \in \mathbb{C}[X]$, $\{x, yz\} = y\{x, z\} + \{x, y\}z$.

Suppose that X is Poisson. There is a natural stratification of X as follows. For each $x \in \mathbb{C}[X]$ one defines the vector field $\Xi_x = \{x, -\}$. For $p_1, p_2 \in X$ we say that $p_1 \sim p_2$ if there exists a piecewise holomorphic curve from p_1 to p_2 such that each smooth piece is the integral curve to Ξ_x for some x. This defines an equivalence relation on X and the equivalence classes are called *symplectic leaves*. It is a theorem going back to A.A.Kirillov [4] (see also [6]) that the symplectic leaves of X are symplectic manifolds which are Poisson embedded in X, and are maximal with respect to this property. In general, the calculation of symplectic leaves for an affine algebraic Poisson variety is a difficult problem. We consider two cases arising in context of noncommutative algebra.

Symplectic reflection algebras were introduced by Etingof and Ginzburg in [3], we recall briefly the definition. Let V be a finite dimensional complex vector space with symplectic form ω. Suppose that a finite subgroup of the symplectic group of V acts in such a way that V cannot be written as the direct sum of two proper symplectic G-stable subspaces. The set of symplectic reflections, S, consists of elements in G which fix a subspace of codimension two. For each $s \in S$ we define an alternating form, ω_s, on V whose radical is $\mathrm{Ker}(1-s)$ and which agrees with ω on $\mathrm{Im}(1-s)$. Let $TV * G$ be the skew group algebra of the tensor algebra, TV, with G. Then for $t \in \mathbb{C}$ and a class function $c : S \to \mathbb{C}$ the corresponding symplectic reflection algebra is

$$H_{t,c} = TV * G / \langle v \otimes w - w \otimes v - t\omega(v,w) - \sum_{s \in S} c(s)\omega_s(v,w)s : v, w \in V \rangle.$$

In the case that $t = 0$ the centre, Z_c, of $H_{0,c}$ is a Poisson algebra via a quantisation procedure and the symplectic leaves of $X_c := \mathrm{Spec} Z_c$ play an important role in the representation theory of $H_{0,c}$, see [1].

Deformed preprojective algebras were introduced in [2]. Let Q be a finite quiver with vertex set I. Let $\overline{Q}$ be the quiver obtained from Q by adding a reverse arrow a^* for each arrow a in Q. Denote its path algebra by $\mathbb{C}\overline{Q}$ and the trivial paths by e_i. For $\lambda \in \mathbb{C}^I$ the corresponding preprojective algebra is

$$\Pi_\lambda(Q) = \mathbb{C}\overline{Q} / \langle \sum_{a \in Q} (aa^* - a^*a) - \sum_{i \in I} \lambda_i e_i \rangle.$$

For a dimension vector $\alpha \in \mathbb{Z}^I_{\geq 0}$ the space parametrising all α-dimensional representations of $\Pi_\lambda(Q)$, $\mathrm{Rep}(\Pi_\lambda(Q), \alpha)$, is the fibre of a moment map for the symplectic space $\mathrm{Rep}(\overline{Q}, \alpha) := \sum_{a \in \overline{Q}} \mathrm{Mat}(\alpha_{\mathrm{head}(a)} \times \alpha_{\mathrm{tail}(a)}, \mathbb{C})$ with the action of $G(\alpha) := \prod_{i \in I} \mathrm{GL}(\alpha_i, \mathbb{C})$ by simultaneous conjugation. The affine quotient $\mathcal{N}(\lambda, \alpha) := \mathrm{Rep}(\overline{Q}, \alpha)//G(\alpha)$ is then a Marsden-Weinstein reduction so naturally carries the structure of a Poisson variety. On the level of representations, $\mathcal{N}(\lambda, \alpha)$ parametrises semisimple representations of $\Pi_\lambda(Q)$. Given a semisimple $\Pi_\lambda(Q)$-module, M, we can split it into its isotypic components $M = M_1^{n_1} \oplus \cdots \oplus M_t^{n_t}$ and then we say that M has representation type (defined only up to permutation of direct summands) equal to $(\beta_1, n_1; \ldots; \beta_t, n_t)$, where β_i is the dimension vector of M_i.

We can now state the main result which is contained in [5].

Theorem.

(1) *For any quiver Q, dimension vector α and parameter λ the symplectic leaves of $\mathcal{N}(\lambda, \alpha)$ are equal to the representation type strata.*

(2) *Let V be the symplectic vector space $(\mathbb{C}^2)^{\oplus n}$ acted on by the wreath product $S_n \wr \Gamma$ where Γ is a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$. The Poisson varieties X_c, corresponding to (V, G), are isomorphic to certain reductions $\mathcal{N}(\lambda, \alpha)$ and this isomorphism maps symplectic leaves to symplectic leaves.*

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Hopf algebra actions and stability of radicals of algebras

Susan Montgomery

Let A be an algebra over a field k, and let H be a finite-dmensional Hopf algebra acting on A (that is, A is an H-module algebra). In this talk we consider the question as to when the Jacobson radical $J(A)$, or the prime radical $P(A)$, is stable under the action of H. We also discuss the related question as to when the smash product $A\#H$ is semiprime.

Both questions have a long history. First, both radicals are trivially stable if $H = kG$, for G a group, since then G acts as automorphisms. In 1975, Fisher [5] looked at the *H-Jacobson radical*, although not much progress was made. Also in 1975, G. Bergman [1] asked whether $J(A)$ is a graded ideal, if A is graded by the group G and $|G|$ is a unit in A (the hypothesis on $|G|$ is necessary: consider $A = kG$ itself, where char $k = p > 0$ and p divides $|G|$).

Translated into Hopf algebra language, Bergman's question asks if $J(A)$ is stable under the action of the Hopf algebra $H = k^G$, the dual of kG; the hypothesis on $|G|$ says that H^* is semisimple. Bergman's question was answered positively in 1984 in [3], but there has been almost no progress on the general question until recently.

Second, in 1984 it was asked by Cohen and Fischman [2] whether $A\#H$ was semiprime, assuming that A was semiprime and H was semisimple. This question remains open, although there are many partial results in the literature, mostly with some additional assumptions about H or about its action on A.

In the last few years there has been progress on both questions, in particular when A itself is either Noetherian or a PI-algebra. First note that the prime radical question can be thought of as "dual" to the semiprimeness question:

Proposition 1. [11][12, Remark 5.8] *Let $\mathcal{C}$ be a class of algebras over k such that $\mathcal{C}$ is closed under finite extensions and under homomorphic images. Then the following are equivalent:*
(1) $P(A)$ is H-stable for all H-module algebras A in $\mathcal{C}$;
(2) for all H^*-semiprime H^*-module algebras A' in $\mathcal{C}$, $A'\#H^*$ is semiprime.

The proof is an elementary application of the Duality Theorem. More generally it is not difficult to see that a necessary condition for stability of $P(A)$, for all A, is that H is cosemisimple, and dually that a necessary condition for $A\#H$ to be semiprime for all H-semiprime algebras A, is that H is semisimple.

The first recent result on radicals is due to Linchenko [6]; he proved that if A has dimension $n < \infty$, and the antipode S has order 2, then $J(A)$ is H-stable if either k has char 0 or char p where $p > n$.

To simpilfy the statement of results, we will assume from now on that k has characteristic 0. It is known in this case that $S^2 = id \iff H$ is semisimple $\iff H$ is cosemisimple, by results of Larson and Radford.

Linchenko's result was extended in [7] by showing that for any algebra A, the "finite radical" $J_{fin}(A)$ is always H-stable, provided H is (co)semisimple. Here $J_{fin}(A)$ is the intersection of the annihilators of the finite-dimensional irreducible modules of A. The stability of $J_{fin}(A)$ was used in [7] to show stability of $J(A)$ for affine PI-algebras.

Now for affine PI-algebras, $P(A) = J(A)$. Looking at $P(A)$ instead of $J(A)$, the affine hypothesis can be removed:

Theorem 2. [8] *Let A be a PI-algebra and assume H is cosemisimple. Then $P(A)$ is H-stable.*

A similar result holds for Noetherian algebras:

Theorem 3. [14] *Let A be a Noetherian algebra and assume H is cosemisimple. Then $P(A)$ is H-stable.*

By the duality proposition 1, one immediately has:

Corollary 4. *Let A be either PI or Noetherian, and assume that A is H-semiprime and that H is semisimple. Then $A\#H$ is semiprime.*

The general case of Theorems 2 and 3, for arbitrary A, remains open.

The case of the Jacobson radical appears to be more difficult, and the stability of $J(A)$ is unknown even if A is either Noetherian or PI and H is (co)semisimple. Examples exist in characteristic 0 in which $P(A)$ is H-stable but $J(A)$ is not; however in these examples, either H is not finite-dimensional or is not semisimple. One result is known relating the two radical questions:

Theorem 5. [7] *Assume that $J(A)$ is H-stable for all H-module algebras A. Then $P(A)$ is H-stable for all H-module algebras A*

The proof of Theorem 5 works if the algebras A are all PI-algebras.

We close with another possible approach to these problems. We assume that the Hopf algebra H is an abelian extension, as follows. Let L be a factorizable group, that is $L = FG$, where F and G are subgroups of L with $F \cap G = (1)$; a basic example is $L = S_n = C_n S_{n-1}$, where we identify C_n with the subgroup generated by $(12 \cdots n)$. Then $\{F, G\}$ forms a *matched pair of groups* in the sense of Takeuchi, and we may construct a bicrossed product $H = k^G \#_\sigma^\tau kF$, where $\sigma : F \times F \to k^G$ and $\tau^* : G \times G \to k^F$ are 2-cocycles. Such Hopf algebras are classified by the group $\mathrm{OpExt}(F, G)$, consisting of equivalence classes of pairs $[\sigma, \tau]$; see Masuoka's survey [10].

Moreover, for any Hopf algebra H and a "dual cocycle" $\Omega \in H \otimes H$, one may form a new Hopf algebra H^Ω by twisting the comultiplication of H using Ω (this construction is due to Drinfeld). If H acts on A, one may also twist A to a new algebra A^{Ω} on which H^{Ω} acts.

Theorem 6. [12] *Let $H = (k^G \#_{\sigma}^{\tau} kF)^{\Omega}$, the twist of a bicrossed product. Let A be an H-semiprime H-module algebra. Then $A\#H$ is semiprime.*

Bicrossed products of groups are an important class of Hopf algebras, as they are closely related to the group-theoretical quasi-Hopf algebras studied in [15] [4][13]. By definition a quasi-Hopf algebra is group theoretical if its category of representations is a group theoretical category $\mathcal{C}(L, \omega, F, \alpha)$, where L is a finite group, $F \subset L$ is a subgroup, $\omega : G \times G \times G \to k^{\times}$ is a normalized 3-cocycle, and $\alpha : F \times F \to k^{\times}$ is a normalized 2-cocycle such that $\omega|_F = 1$ [15]. A major example is given by the quasi-Hopf version of the bicrossed products above, although in the quasi-Hopf case, G might not be a group and is replaced by a fixed set Q of coset representatives of F in L. It is shown in [13] that any group-theoretical Hopf algebra H is gauge-equivalent to some quasi-Hopf algebra $M = k^Q \#_{\sigma}^{\tau} kF$, where σ and τ are defined using ω. This means that $H = M^{\Gamma}$, a twist of M by $\Gamma \in H \otimes H$, although in this case the twisting element Γ does not have to be a cocycle.

It is an open question as to whether every semisimple Hopf algebra over $\mathbb{C}$ is group-theoretical [4]. However even if this question from [4] turns out to be true (and it looks very difficult), the semiprimeness problem would not automatically follow from Theorem 6, since its proof required that the twisting element be a cocycle. In fact twisting the corresponding algebra A by a Γ which is not a cocycle can give an algebra A^{Γ} which is not associative.

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(Non-)singular quiver moduli

Markus Reineke

1. Motivation

As an example of the interaction between Noncommutative Algebra and Algebraic Geometry, one may consider the interplay between the path algebra $\mathbf{C}Q$ of a quiver and a series of moduli spaces $(M_d(Q))_d$ for d-dimensional representations of Q. In the following, questions concerning the global geometry of such moduli will be considered, with emphasis on computation of topological invariants like Betti numbers. The background is the problem of classification of (or producing normal forms for) classes of quiver representations. Explicit knowledge of topological invariants can sometimes hint on such normal forms (see e.g. [6]), or at least give a measure for the complexity of this problem.

2. Notation

Let Q be a finite quiver with set of vertices Q_0 and set of arrows Q_1, with associated Euler form $\langle \_, \_ \rangle$. Let $d \in \mathbf{N}Q_0$ be a dimension vector, and let $\Theta \in (\mathbf{Q}Q_0)^*$ be a stability condition. The slope $\mu(X)$ of a non-zero complex representation X of Q is defined as $\mu(X) := \Theta(\underline{\dim} X)/\dim X$. The representation X is called stable (resp. semistable) if $\mu(U) < \mu(X)$ (resp. $\mu(U) \leq \mu(X)$) for all non-zero proper subrepresentations U of X. Note that in case $\Theta = 0$, the notion of stability reduces to simplicity.

By [3], there exists a complex algebraic variety $M_d^{st}(Q)$ whose points correspond to isomorphism classes of stable representations of Q of dimension vector d. Moreover, there exists a complex algebraic variety $M_d^{sst}(Q)$ whose points correspond to isomorphism classes of polystable representations of Q of dimension vector d, where a representation is called polystable if it is the direct sum of stable representations of the same slope.

Both moduli spaces are defined using Geometric Invariant Theory. Basic facts on their geometry are the following: the variety $M_d^{st}(Q)$ is always smooth, and it is an open subset of $M_d^{sst}(Q)$. The latter moduli admits a projective morphism to the affine variety $M_d^{ssimp}(Q)$, the moduli of semisimple representations of Q of dimension vector Q (defined with the aid of the stability $\Theta = 0$). In particular, if Q has no oriented cycles, $M_d^{ssimp}(Q)$ reduces to a single point, thus $M_d^{sst}(Q)$ is projective in this case.

We call d coprime for Θ if $\mu(e) \neq \mu(d)$ for all $0 < e < d$. In this case, we have $M_d^{st}(Q) = M_d^{sst}(Q)$ by definition. If, in addition, Q does not contain oriented cycles, we thus arrive at a smooth projective complex algebraic variety.

3. The coprime case

Theorem 1 ([5]). *Given (Q, d, Θ) as above, define*

$$P_d(q) := \sum_{d^*} (-1)^{s-1} q^{-\sum_{k \leq l} \langle d^l, d^k \rangle} \prod_{k=1}^{s} \prod_{i \in Q_0} \prod_{l=1}^{d_i^k} (1 - q^{-l})^{-1} \in \mathbf{Q}[[q]],$$

where the sum runs over all tuples $(d^1, \ldots, d^s)$ of non-zero dimension vectors such that $\sum_k d^k = d$ and $\mu(d^1 + \ldots + d^k) > \mu(d)$ for all $k < s$. If d is coprime for Θ, then the Poincare polynomial in rational singular cohomology of $M_d^{st}(Q)$ is given by

$$(q-1) \cdot P_d(q) = \sum_i \dim H^i(M_d^{st}(Q), \mathbf{Q}) q^{i/2}$$

(the fractional power on the right hand side is reasonable since the odd cohomology vanishes).

No explicit formula for the Euler characteristic of $M_d^{st}(Q)$ can be derived from this, since all summands in the definition of $P_d(q)$ have poles at $q = 1$. Moreover, no positive (combinatorial) formula for the Betti numbers can be derived from this.

Based on a string-theoretic argument by M. Douglas, the following conjecture on the asymptotic behaviour of the Euler characteristic has been found with the aid of computer experiments by T. Weist:

Conjecture 2. *Assume Θ generic. There exists a constant $C_Q \in \mathbf{R}$ such that for large d coprime to Θ, we have*

$$\log \chi(M_d^{st}(Q)) \approx C_Q \cdot \sqrt{\dim M_d^{st}(Q) - 1}.$$

4. The non-coprime case

For general d, one has the choice to study the non-projective moduli $M_d^{st}(Q)$, the singular moduli $M_d^{sst}(Q)$, or a closely related moduli, which is hopefully smooth and projective (over the affine base $M_d^{ssimp}(Q)$).

1. **$M_d^{st}(Q)$.** The following two results are proved in [8, 9]. The first is general, and basically means that the arithmetic geometry of the moduli $M_d^{st}(Q)$ is of a very special (simple) nature. The second is more special, only applying to moduli of simple representations, and gives a closed formula for the Euler characteristic in this case.

There exists an integral model $\mathcal{M}_d$ of $M_d^{st}(Q)$, i.e. a scheme over Spec$\mathbf{Z}$ whose base extension to Spec$\mathbf{C}$ is isomorphic to $M_d^{st}(Q)$.

Theorem 3. *For all (Q, d, Θ) as above, there exists a (recursively computable) polynomial $A_d(t) \in \mathbf{Z}[t]$ such that, for all finite fields k, the evaluation $A_d(|k|)$ equals the number of k-rational points of the reduction of $\mathcal{M}_d$ to k.*

The multiplicative group $\mathbf{C}^*$ acts naturally on $M_d^{simp}(Q)$ by scalar multiplication of the linear maps representing the arrows of Q in a representation. Formation of the quotient by this action yields a projectivization $\mathbf{P}M_d^{simp}(Q)$.

Theorem 4. *For all d, the Euler characteristic in cohomology with compact support of $\mathbf{P}M_d^{simp}(Q)$ equals the number of cyclic equivalence classes of primitive cycles in Q of dimension vector d.*

2. **Smooth models.** The results of this subsection will appear in [10]. Choose another dimension vector $n \in \mathbf{N}Q_0$ and consider the projective representation $P^n := \bigoplus_{i \in Q_0} P_i^{n_i}$.

There exists a complex algebraic variety $M_{d,n}^{\Theta}(Q)$ parametrizing pairs $(M, f : P^n \to M)$ consisting of a semistable representation M of Q of dimension vector d and a morphism f such that all proper subrepresentations containing the image of f have slope strictly smaller than the one of M. This moduli space arises naturally by framing (see e.g. [7]).

The moduli $M_{d,n}^{\Theta}(Q)$ is always smooth and admits a projective morphism $\pi : M_{d,n}^{\Theta}(Q) \to M_d^{sst}(Q)$, whose generic fibre is a projective space. In case d is coprime, this moduli can be identified with the total space $\mathbf{P}(\bigoplus_{i \in Q_0} \mathcal{V}_i^{n_i})$, where the $\mathcal{V}_i$ are the tautological bundles on the fine moduli $M_d^{sst}(Q)$.

The Betti numbers of $M_{d,n}^{\Theta}(Q)$ are given by the following identity of generating functions:

$$\sum_d (\sum_i \dim H^i(M_{d,n}^{\Theta}(Q), \mathbf{Q}) q^{i/2}) t^d = (\sum_d P_d(q) t^d)^{-1} \cdot (\sum_d q^{n \cdot d} P_d(q) t^d),$$

where the calculation has to be carried out formally in a skew polynomial ring with $t^e \cdot t^f = q^{-\langle e,f \rangle} t^{e+f}$.

In the special case $\Theta = 0$, there even exist positive formulas for the Betti numbers, in terms of numbers of multipartitions fulfilling certain inequalities.

Building on [1, 2, 4], the fibres of π can be described (locally, analytically) as moduli of the form $M_{d',n'}^{0,nilp}(Q')$, the superscript indicating nilpotent representations.

3. $M_d^{sst}(Q)$**.** Using the smooth models $\pi : M_{d,n}^{\Theta}(Q) \to M_d^{sst}(Q)$, it should be possible to compute the intersection Betti numbers of $M_d^{sst}(Q)$. Applying the decomposition theorem to the push-forward of the constant perverse sheaf on $M_{d,n}^{\Theta}(Q)$, the multiplicities of the resulting intersection cohomology complexes should be computable using the description of the fibres of π and the cohomology of the smooth models.

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Tilting modules over Calabi-Yau algebras

Idun Reiten
(joint work with Osamu Iyama)

Tilting theory is well developed for finite dimensional algebras, due to work of a large number of people. It has played an important role in this area, in particular after Happel discovered the connection with derived categories. The theory has had interesting applications within for example algebraic groups and in algebraic geometry. Here we deal with some classes of noetherian rings, and point out connections with cluster algebras and with noncommutative crepant resolutions. We refer to [1] and the references given there.

1. Calabi-Yau algebras

Let R be a commutative noetherian ring of Krull dimension d and Λ a module-finite R-algebra. We say that Λ is Calabi-Yau of dimension n ($n-\mathrm{CY}$ for short) if we have a functorial isomorphism $\mathrm{Hom}(X, Y[n]) \simeq D\,\mathrm{Hom}(Y, X)$ for all X, Y in the bounded derived category $\mathcal{D}^b(\mathrm{f.l}\,\Lambda)$ of finite length Λ-modules, where D denotes the Matlis dual.

We say that Λ is $n-\mathrm{CY}^-$ if the same formula holds for X in $\mathcal{D}^b(\mathrm{f.l}\,\Lambda)$ and Y in $\mathcal{K}^b(\mathrm{pr}\,\Lambda)$, the bounded complexes of finitely generated projective Λ-modules. Then if Λ is $n-\mathrm{CY}$, it is $n-\mathrm{CY}^-$ [Rickard]. There is the following characterization of these properties under more assumptions on R.

Theorem 1. *Let R be a local Gorenstein of dimension d and $R \subset \Lambda$. Then we have the following:*

(a) *Λ is $n-\mathrm{CY}$ or $n-\mathrm{CY}^-$ implies that $n = d$.*
(b) *Λ is $d-\mathrm{CY}^-$ if and only if Λ is symmetric R-order.*
(c) *Λ is $d-\mathrm{CY}$ if and only if Λ is symmetric order and* $\mathrm{gl.dim}\,\Lambda = d$.

Here Λ is a symmetric R-algebra if we have a two-sided Λ-isomorphism $\Lambda \simeq \mathrm{Hom}_R(\Lambda, R)$ and Λ is an R-order if Λ is a (maximal) Cohen-Macaulay R-module.

Note that if $\Lambda = R$ is local commutative then Λ is $d - \mathrm{CY}^-$ if and only if Λ is Gorenstein of dimension d, and Λ is $d - \mathrm{CY}$ if and only if Λ is regular of dimension d.

Examples of CY-algebras come from skew group rings $S \star G$ where the ring $S = \mathbb{C}[X_1, \cdots, X_d]$ and G is a finite subgroup of $SL(d, \mathbb{C})$.

2. Tilting theory

Assume that R is a complete local normal Gorenstein domain and $R \subset \Lambda$. Recall that a finitely generated Λ-module T is a classical tilting module if $\mathrm{pd}_\Lambda T \leq 1$, $\mathrm{Ext}^1_\Lambda(T,T) = 0$ and there is an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with T_i in $\mathrm{add}\, T$. The last condition can be replaced by saying that the number of nonisomorphic indecomposable summands of T is the number t of nonisomorphic simple Λ-modules. We shall mainly deal with tilting modules which are reflexive R-modules and with the case $d = 3$.

Let now $P_1 \coprod \cdots \coprod P_t$, where we assume that the P_i are indecomposable and pairwise nonisomorphic. Then for each $k = 1, \ldots, t$ there is a unique indecomposable Λ-module $P_k^* \not\simeq P_k$ such that $T = P/P_k \coprod P_k^*$ is a tilting module, and it is reflexive as an R-module. We write $\mu_k(\Lambda) = \Lambda'$, where $\Lambda' = \mathrm{End}_\Lambda(T)$. We can show that $\mu_k(\mu_k(\Lambda)) = \Lambda$. Continuing this way, we get a sequence $\Lambda, \Lambda', \ldots, \Lambda^{(r)}$ of $3 - \mathrm{CY}$-algebras such that we get from one algebra to the next one via a special tilting module. An interesting property is that we can get directly from Λ to $\Lambda^{(r)}$ using a tilting Λ-module. This is based upon the following result which we prove working in a more general context (see section 3).

Theorem 2. *Let Λ be a $3-\mathrm{CY}$, T_1, T_2 reflexive tilting modules, $\Gamma_i = \mathrm{End}_\Lambda(T_i)$ for $i = 1, 2$. Then $U = \mathrm{Hom}_\Lambda(T_1, T_2)$ is a reflexive tilting Γ_1-module and $\mathrm{End}_{\Gamma_1}(U) = \Gamma_2$.*

3. Connection with noncommutative crepant resolutions (NCCR) of Van den Bergh

The above theorem is proved by working in the more general setting of NCCR. Let R be a 3-dimensional normal Gorenstein domain and $R \subset \Lambda$.

We say that the Λ-module M gives a NCCR if

(i)M is reflexive (as R-module) and is a height one generator (that is M_P is a Λ_P-generator for all prime ideals P in R of height ≤ 1).

(ii)Γ_P is an R_P-order with $\mathrm{gl.dim}\, \Gamma_P$ =height P for P a maximal ideal in R, where $\Gamma = \mathrm{End}_\Lambda(M)$.

Then we have the following contribution to the solution of a conjecture of Van den Bergh.

Theorem 3. *Let Λ be as above, and assume that M_1 and M_2 are Λ-modules giving NCCR, and let $\Gamma_i = \mathrm{End}_\Lambda(M_i)$.Then $U = \mathrm{Hom}_\Lambda(M_1, M_2)$ is a reflexive tilting Γ_1-module with $\mathrm{End}_\Lambda(U) = \Gamma_2$, and hence Γ_1 and Γ_2 are derived equivalent.*

A crucial step in the proof is the following.

Lemma 4. *If R is a local ring and Λ is an isolated singularity, then* $\operatorname{depth}\operatorname{Hom}_\Lambda(M,N) \geq 3$ *implies that* $\operatorname{Ext}^1_\Lambda(M,N) = 0$.

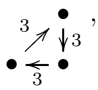
The following result gives the connection with section 2.

Theorem 5. *Let Λ be a $3-\mathrm{CY}$-algebra. Then the Λ-modules M giving an NCCR are exactly the reflexive tilting modules.*

4. Connection with cluster algebras

Let R be complete local Gorenstein ring with maximal ideal m such that $R/m = K$ is an algebraically closed field and Λ a module-finite R-algebra as before. We give a brief indication of how to use this theory to model some of the essential ingredients in the definition of the cluster algebras of Fomin-Zelevinsky.

Let Q be a finite quiver with vertices $1, \cdots, n$, and no loops or cycles of length 2. Associated with this is a cluster algebra $\mathcal{A}(Q)$ which is a subalgebra of the rational function field $\mathbb{Q}(X_1, \cdots, X_n)$. For each $i = 1, \cdots, n$ there is the Fomin-Zelevinsky mutation, which gives a new quiver $\mu_i(Q)$. If we start with the quiver of a $3-\mathrm{CY}$ algebra Λ with no loops or 2-cycles, we give an interpretation of $\mu_i(Q)$ as the quiver obtained from the quiver $\operatorname{End}_\Lambda(T)$, where T is an appropriate tilting module, after removing possible 2-cycles. It is a open problem if in fact 2-cycles can appear this way. If we start with the skew group ring $S \star G$, where $G = \langle \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}; \rho^3 = 1 \rangle$ then the quiver Q is



, and no 2-cycles will appear.

In such cases we get a better modelling, and can view reflexive tilting modules as analogs of cluster, using the results of section 2.

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Degeneration of modules and the construction of Prüfer modules

Claus Michael Ringel

Let Λ be an artin algebra (this means that Λ is a module-finite k-algebra, where k is an artinian commutative ring). Bautista-Peres [1] and Smalø [6] have recently shown the following: Let W, W' be Λ-modules of finite length with isomorphic top and isomorphic first syzygy modules. If W and W' have no self-extensions, then W and W' are isomorphic. This is well-known in case k is an algebraically closed field, but it is of interest to know such a result also for example for Λ being a finite ring. Actually, for k an algebraically closed field, the usual algebraic geometry arguments allow a stronger conclusion: If W has no self-extension, then W' is a degeneration of W (in the following sense: W' belongs to the closure of the orbit of W in the corresponding module variety). The first aim of the lecture was to show a corresponding result for general Λ, using the notion of a degeneration as introduced by Riedtmann-Zwara [7]: the module W' is said to be a *degeneration* of W provided there is an exact sequence of finite length modules of the form: $0 \to X \to X \oplus W \to W' \to 0$ (in case k is algebraically closed, the notions coincide, as Zwara [8] has shown).

Proposition 1. *Let U_0, U_1 be finite length modules, and $w, w' \colon U_0 \to U_1$ monomorphisms. Denote by W, W' the cokernels of w, w', respectively. If W has no self-extensions, then W' is a degeneration of W.*

Indeed, let us describe in which way one obtains a corresponding Riedtmann-Zwara sequence. In order to do so, let us deal with a slightly more general setting: Start with a pair of maps $w_0, v_0 \colon U_0 \to U_1$ between finite length modules, such that w_0 is a proper monomorphism with cokernel W. Forming inductively pushouts, we obtain a sequence of maps $w_i, v_i \colon U_i \to U_{i+1}$ with $i \geq 0$, such that all the maps w_i are monomorphisms with cokernel W (and such that $w_{i+1}v_i = v_{i+1}w_i$ for all i). We form the direct limit U_∞ of all the modules U_i with respect to the monomorphisms w_i (and we may assume that these maps w_i are inclusion maps), and consider also the module U_∞/U_0.

If we assume that W has no self-extensions, then U_∞/U_0 is an (infinite) direct sum of copies of W, and this implies that one of the inclusion maps w_i is a split monomorphism: thus U_{i+1} is isomorphic to $U_i \oplus W$. Now, if v_0 is also a monomorphism, say with cokernel W', then the inductive constuction of the module U_{i+1} yields an exact sequence $0 \to U_i \to U_{i+1} \to W' \to 0$. As we have seen, we can replace U_{i+1} by $U_i \oplus W$, thus we deal with a Riedtmann-Zwara sequence. This completes the proof of Proposition 1.

Let us return to the general setting of dealing with a pair of maps $w_0, v_0 \colon U_0 \to U_1$ between finite length modules, such that w is a proper monomorphism with cokernel W. The maps $v_i \colon U_i \to U_{i+1}$ yield a map $v_\infty : U_\infty \to U_\infty$ which maps U_0 into U_1 and which induces an isomorphism $\overline{v} \colon U_\infty/U_0 \to U_\infty/U_1$. If we compose the canonical projection $U_\infty/U_0 \to U_\infty/U_1$ with the inverse of $\overline{v}$, we obtain a locally nilpotent surjective endomorphism of U_∞/U_0 with kernel W. Let us call a module M a *Prüfer module with basis* W, provided there exists a locally nilpotent surjective endomorphism of M with kernel W of finite length; thus U_∞/U_0 is a Prüfer module with basis W.

A module M is said to be of *finite type* provided it is a direct sum of copies of a finite number of indecomposable modules of finite length (thus if and only if M is both endo-finite and pure-projective). Note that for the tower construction exhibited above, the module U_∞ is of finite type if and only if the Prüfer module U_∞/U_0 is of finite type. We are interested in Prüfer modules which are not of finite type, since there is the following result:

Proposition 2. *Let M be a Prüfer module which is not of finite type, and let I be an infinite set. Then the product module M^I has an indecomposable direct summand G which is of infinite length and endo-finite.*

Indecomposable infinite length modules which are endo-finite have been labelled *generic* modules by Crawley-Boevey [2]. He has shown that the existence of a generic module implies that there are infinitely many isomorphism classes of indecomposable finite-length modules of some fixed endo-length d (and actually the proof shows that there are infinitely many natural numbers d such that there are infinitely many isomorphism classes of indecomposable finite-length modules of endo-length d).

Proposition 2 is based on previous investigations of Krause [3], see also [4]: Let M be a Prüfer module, thus there is a surjective locally nilpotent endomorphism f with kernel of finite length; denote by $W[n]$ the kernel of f^n. Then M^I contains the union $U = \bigoplus_n W[n]^I$. This submodule is a direct sum of copies of M, and it is a direct summand of M^I, say $M^I = U \oplus U'$. The module U' is endo-finite, thus a direct sum of copies of finitely many indecomposable endo-finite modules. In case the latter modules all are of finite length, then one can show that M is of finite type. This then completes the proof of Proposition 2.

We want to use the tower construction in order to obtain a wealth of Prüfer modules. For this, one needs submodules $U_0 \subset U_1$ with additional homomorphisms (or even embeddings) $U_0 \to U_1$, and of special interest seems to be the take-off part of the category of all Λ-modules of finite length (as introduced in [5]).

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Classifying birationally commutative surfaces

D. Rogalski
(joint work with J.T. Stafford)

This report describes joint work with Toby Stafford on the classification of special kinds of noncommutative surfaces. The main theorem is a bit complicated to state, so (as in the half-hour talk presented at Oberwolfach), we will get right to the the statement of the theorem and then proceed with some comments and explanations. Throughout, k is an algebraically closed base field.

Theorem 1. *Let A be a connected $\mathbb{N}$-graded k-algebra (so $A = k \oplus A_1 \oplus A_2 \oplus \dots$) satisfying the following hypotheses:*

1. *A is a noetherian domain which is generated as an algebra in degree 1.*
2. *A is a* birationally commutative surface*. This means that the homogeneous quotient ring of A has the form $Q(A) = K[t^{\pm 1}; \sigma]$ where K is a field with* $\operatorname{tr.deg} K/k = 2$.
3. *The field automorphism $\sigma \in \operatorname{Aut}(K)$ is* geometric*. This means that there exists a projective surface Y with $k(Y) = K$ and an automorphism η of Y, such that η induces σ via pullback of rational functions.*

Then A is isomorphic in large degree to a naïve blowup $R(X, \mathcal{L}, \tau, Z)$, where X is a projective surface with $k(X) = K$, $\tau \in \operatorname{Aut}(X)$, $\mathcal{L}$ is a τ-ample invertible sheaf, and Z is a 0-dimensional subscheme of X.

The main success of the theorem is that the quite general class of algebras described by the hypotheses is now completely classified in terms of the geometric data of naïve blowups. Now let us describe the various parts of the theorem in more detail. First we make some comments about the list of hypotheses. Hypothesis (1) is very mild; our current understanding of noncommutative surfaces is very rough, and so the removal of any of these assumptions is beside the point at the moment. The point of Hypothesis (2) is to restrict to a class of rings in which the connection with algebraic geometry is especially strong. The method of the proof (which we will not otherwise touch on much) is to construct the needed surface X using the point modules for the ring A. Under hypothesis (2) $Q(A)_{\geq 0}$ is itself a K-point module for A, and in essence this is what makes the theorem work. Hypothesis (3) is the most curious one, and we will make some further comments about it below.

Now we explain the conclusion of the theorem further. The naïve blowups appearing there were first studied by Keeler, Stafford, and the author in [KRS]. Suppose one is given a projective scheme X, a subscheme Z of X, an invertible sheaf $\mathcal{L}$, and an automorphism τ of X. For each $n \geq 0$, set $\mathcal{L}_n = \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \dots \otimes (\sigma^{n-1})^* \mathcal{L}$. Let $\mathcal{I}$ be the sheaf defining the subscheme Z, and define also $\mathcal{I}_n = \mathcal{I} \cdot \sigma^* \mathcal{I} \cdot \dots \cdot (\sigma^{n-1})^* \mathcal{I}$. Then the graded ring $B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n)$ has a natural multiplication defined by $x * y = x \otimes (\sigma^m)^*(y)$ for $x \in B_m$, $y \in B_n$. B is called a *twisted homogeneous coordinate ring* and its properties were studied by Artin and Van den Bergh in [AV], and in greater detail by Keeler in [Ke].

The property of σ-ampleness is technical so we will not define it; when satisfied the ring B is noetherian and the noncommutative space $\operatorname{proj} -B$ associated to B is simply X itself. Finally, the naïve blowup $R(X, Z, \mathcal{L}, \sigma)$ is simply the subring $\bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n \otimes \mathcal{I}_n)$ of B. These rings R have many interesting and unusual properties. The definition mimics the definition of a commutative blowup of X at Z using Rees rings, but twisted by σ. The associated noncommutative space $\operatorname{proj} -R$ does not quite have the formal properties one expects of a blowup. Still, the point is that these rings are entirely geometrically defined and the study of their properties reduces to commutative algebraic geometry.

Example 2. One of the simplest examples of a naïve blowup is the following. Take $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}(1)$, Z the single reduced point $(1 : 1 : 1)$, and σ an automorphism given by a generic diagonal matrix in $\mathrm{PGL}_k(2)$. Then $B(X, \mathcal{L}, \sigma)$ has a presentation $k\langle x_1, x_2, x_3\rangle/\{x_i x_j - p_{ij} x_j x_i | i < j\}$ for some constants $p_{ij} \in k$ (depending on σ) which satisfy $p_{12}p_{23} = p_{13}$. Then $R(X, Z, \mathcal{L}, \sigma)$ is simply the subring of B generated by $x_1 - x_3$ and $x_2 - x_3$.

In the last part of the talk, we discussed the subtle property of geometricity appearing in the hypothesis (3) of the theorem in more detail. It is not at all obvious at first glance, for example, that non-geometric automorphisms of finintely generated field extensions of k even exist. To show that they do, we discussed the following example which was suggested to us by Michael Artin.

Let K/k be a field extension with $\operatorname{tr.deg} K/k = 2$, and let $\sigma \in \operatorname{Aut}_k(K)$. Let Y be any nonsingular surface with function field $k(Y) = K$, and consider the corresponding birational map $\sigma : Y \dashrightarrow Y$ (which we give the same name). If σ is geometric, then there also exists a surface Z with $k(Z) = K$ and an automorphism $\tau : Z \to Z$ corresponding to σ. Letting $f : Y \dashrightarrow Z$ be a compatible birational map, we have $f\sigma^n = \tau^n f$ for all n. Then it is easy to check using simple facts from the theory of nonsingular projective surfaces (as in [Hr, Chapter V]) that there is an bound N such that the birational map $\sigma^n : Y \dashrightarrow Y$ has at most N fundamental points for all $n \geq 0$.

Now let $K = k(u, v)$ be rational functions and $Y = \mathbb{P}^2$. Let $\sigma_1 : Y \dashrightarrow Y$ be the *Cremona transformation* defined by $(a : b : c) \mapsto (bc : ac : ab)$ which has three fundamental points $\{(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)\}$. Let $\sigma_2 \in PGL_k(2) = \operatorname{Aut}(Y)$ be chosen generically. Setting $\sigma = \sigma_2\sigma_1 : Y \dashrightarrow Y$, then one may check that σ^n has $3n$ fundamental points for each n (the idea is that σ_2 moves all of the previously contracted points "out of the way" before σ_1 is applied again). By the discussion above, σ cannot be geometric.

In fact, the author recently discovered that Diller and Favre have developed a rather complete theory of the dynamics of birational maps of surfaces over $\mathbb{C}$ [DF], which along the way answers many of the fundamental questions about geometricity in the surface case. Using their results, we have been able to obtain the following amusing ring-theoretic characterization (which is work in progress).

Theorem 3. *Let A be a connected graded birationally commutative surface, so $Q(A) \cong K[t^{\pm 1}; \sigma]$ with $\operatorname{tr.deg} K/k = 2$, and assume that A does not have exponential growth. Then if σ is geometric, then $\operatorname{GK} A = 3$ or $\operatorname{GK} A = 5$, while if σ is non-geometric then $\operatorname{GK} A = 4$.*

This theorem allows one to replace hypothesis (3) of the main theorem by a hypothesis that $\operatorname{GK} A = 3$ or $\operatorname{GK} A = 5$ if one wishes (since a noetherian graded ring cannot have exponential growth [SZ]).

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A combinatorial approach to the dual of Lusztig's semicanonical basis

Jan Schröer

(joint work with Christof Geiß, Bernard Leclerc)

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra of Dynkin type $\Delta \in \{\mathsf{A}_n(n \geq 1), \mathsf{D}_n(n \geq 4), \mathsf{E}_n(n = 6, 7, 8)\}$. By $\mathfrak{n}$ we denote a maximal nilpotent subalgebra of $\mathfrak{g}$, and let $\mathbb{C}[N]$ be the graded dual of the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. (Thus $\mathbb{C}[N]$ is the commutative algebra of polynomial functions on a Lie group N with Lie algebra $\mathfrak{n}$.)

Berenstein, Fomin and Zelevinsky have shown that $\mathbb{C}[N]$ can be equipped naturally with a cluster algebra structure. In [3] we "categorify" the cluster algebra $\mathbb{C}[N]$, i.e. we realize it inside the category of finite-dimensional modules over a preprojective algebra.

Let Λ be the preprojective algebra associated to a Dynkin quiver Q of Dynkin type Δ. This is the finite-dimensional associative algebra

$$\Lambda = \mathbb{C}\overline{Q}/\langle \sum_{a \in Q_1} [a, \bar{a}] \rangle,$$

where $\overline{Q}$ denotes the double of Q and Q_1 is the set of arrows of Q. We denote by I the set of vertices of Q, and by $\Lambda_{\mathbf{d}}$ the affine variety of Λ-modules with dimension vector $\mathbf{d} = (d_i)_{i \in I}$.

Let r be the number of positive roots of Q. A Λ-module T is *maximal rigid* if $\operatorname{Ext}^1_\Lambda(T, T) = 0$ and if $T = T_1 \oplus \cdots \oplus T_r$ with T_i indecomposable and $T_i \ncong T_j$ for all $i \neq j$. Without loss of generality, $T_{r-n+1}, \ldots, T_r$ are projective. By Γ_T we denote the quiver of the endomorphism algebra $\mathrm{End}_\Lambda(T)$. It's vertices are indexed by the T_i.

Let $B(T) = (t_{ij})_{1 \le i,j \le r}$ be the $r \times r$-matrix defined by

$$t_{ij} = (\text{number of arrows } j \to i \text{ in } \Gamma_T) - (\text{number of arrows } i \to j \text{ in } \Gamma_T).$$

The quiver Γ_T does not have 2-cycles, so at least one of the two summands in the definition of t_{ij} is zero. Define $B(T)^\circ = (t_{ij})$ to be the $r \times (r-n)$-matrix obtained from $B(T)$ by deleting the last n columns.

For $k \in [1, r-n]$ there is a short exact sequence

$$0 \to T_k \xrightarrow{f} \bigoplus_{t_{ik}>0} T_i^{t_{ik}} \to T_k^* \to 0$$

where f is a minimal left $\mathrm{add}(T/T_k)$-approximation of T_k (i.e. the map $\mathrm{Hom}_\Lambda(f, T)$ is surjective, and every morphism g with $gf = f$ is an isomorphism). Set

$$\mu_{T_k}(T) = T_k^* \oplus T/T_k.$$

Then $\mu_{T_k}(T)$ is again a maximal rigid module. In particular, T_k^* is indecomposable. We call $\mu_{T_k}(T)$ the *mutation of T in direction T_k*.

If $\widetilde{B} = (b_{ij})$ is any $r \times (r-n)$-matrix, then the *principal part* B of $\widetilde{B}$ is obtained from $\widetilde{B}$ by deleting the last n rows. The following definition is due to Fomin and Zelevinsky: Given some $k \in [1, r-n]$ define a new $r \times (r-n)$-matrix $\mu_k(\widetilde{B}) = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases}$$

where $i \in [1, r]$ and $j \in [1, r-n]$. One calls $\mu_k(\widetilde{B})$ a *mutation* of $\widetilde{B}$.

The quivers of the endomorphism algebras $\mathrm{End}_\Lambda(T)$ and $\mathrm{End}_\Lambda(\mu_{T_k}(T))$ are related via Fomin and Zelevinsky's mutation rule:

Theorem 1. *Let Λ be a preprojective algebra of Dynkin type Δ. For a maximal rigid Λ-module T as above and $k \in [1, r-n]$ we have*

$$B(\mu_{T_k}(T))^\circ = \mu_k(B(T)^\circ).$$

Lusztig [4] proved that the enveloping algebra $U(\mathfrak{n})$ is isomorphic to

$$\mathcal{C} = \bigoplus_{\mathbf{d} \in \mathbb{N}^I} \mathcal{C}(\mathbf{d}),$$

where the $\mathcal{C}(\mathbf{d})$ are certain vector spaces of $\mathrm{GL}_{\mathbf{d}}$-invariant constructible functions on the affine varieties $\Lambda_{\mathbf{d}}$. This yields a new basis $\mathcal{S}$ of $U(\mathfrak{n})$ indexed by the irreducible components of the varieties $\Lambda_{\mathbf{d}}$, called the *semicanonical basis* [4].

Let $\mathcal{C}^*$ be the graded dual of $\mathcal{C}$. A multiplication on $\mathcal{C}^*$ is defined via the natural comultiplication of the Hopf algebra $U(\mathfrak{n}) \cong \mathcal{C}$. One can identify $\mathcal{C}^*$ and $\mathbb{C}[N]$ in a natural way. In [1] we considered the basis $\mathcal{S}^*$ of $\mathcal{C}^*$ dual to the semicanonical basis of $\mathcal{C}$, and began to study its multiplicative properties.

For a Λ-module $x \in \Lambda_{\mathbf{d}}$ define the *evaluation form* $\delta_x \colon \mathcal{C} \to \mathbb{C}$ which maps a constructible function $f \in \mathcal{C}(\mathbf{d})$ to $f(x)$. Define

$$\langle x \rangle := \{y \in \Lambda_{\mathbf{d}} \mid \delta_y = \delta_x\}.$$

This is a constructible subset of $\Lambda_{\mathbf{d}}$. We can choose a *finite* set $R(\mathbf{d}) \subset \Lambda_{\mathbf{d}}$ such that

$$\Lambda_{\mathbf{d}} = \bigsqcup_{x \in R(\mathbf{d})} \langle x \rangle.$$

Each irreducible component Z of $\Lambda_{\mathbf{d}}$ has a unique stratum $\langle x \rangle \cap Z$ containing a dense open subset of Z, and the points of this stratum are called the *generic* points of Z. We can then reformulate the definition of $\mathcal{S}^*$ as follows: the element ρ_Z of $\mathcal{S}^*$ labelled by Z is equal to δ_x for a generic point x of Z.

A Λ-module x is called *rigid* if $\mathrm{Ext}^1_\Lambda(x,x) = 0$. If x is rigid, it is generic and δ_x is a dual semicanonical basis vector.

Using our "categorification" of the cluster algebra structure on $\mathbb{C}[N]$ we obtain the following result [3]:

Theorem 2. *Let*

$$T = T_1 \oplus \cdots \oplus T_r$$

be a maximal rigid Λ-module with $T_{r-n+1}, \dots, T_r$ projective. For $k \in [1, r-n]$ let $\mu_{T_k}(T) = T_k^* \oplus T/T_k$ be the the mutation of T in direction k. Then the following hold:

- *$\{\delta_{T_i} \mid 1 \le i \le r\}$ is a multiplicative subset of the dual semicanonical basis of $\mathbb{C}[N]$, i.e.*
  $$\delta_{T_1}^{n_1} \cdot \delta_{T_2}^{n_2} \cdots \delta_{T_r}^{n_r} = \delta_{T_1^{n_1} \oplus \cdots \oplus T_r^{n_r}}$$
  is again in the dual semicanonical basis for all $n_i \ge 0$;
- $$\delta_{T_k^*} = \frac{\prod_i \delta_{T_i}^{a_{ik}} + \prod_j \delta_{T_j}^{a_{kj}}}{\delta_{T_k}}$$
  where a_{ij} denotes the number of arrows from j to i in Γ_T.

Thus, starting with some nice maximal rigid module $T = T_1 \oplus \cdots \oplus T_r$, where all the δ_{T_i} are explicitely known (such modules T can be constructed), our results yield a combinatorial construction of numerous other dual semicanonical basis vectors.

The proof of the above theorem uses a general multiplication formula for evaluation forms, see [2].

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On some domains with small GK dimension

Agata Smoktunowicz

The structure of finitely graded domains with quadratic growth was described in [1]. Bell and Small showed that given a finitely graded Goldie non-PI domain of Gelfand-Kirillov (GK) dimension 2 over an algebraically closed field, the centralizer of a non-scalar element of this domain is an affine commutative domain of Gelfand-Kirillov dimension 1. They conjectured that the same holds in the ungraded case [2]. Small and Warfield [6] proved that if R is a finitely generated prime algebra over a field of Gelfand-Kirillov dimension 1 then the center of R is a finitely generated F–algebra of Gelfand-Kirillov dimension 1. Another important result is a theorem of Small, Stafford and Warfield [5] which says that if R is a finitely generated semiprime algebra of GK dimension 1, then the center of R is a Noetherian domain of GK dimension 1. A result of Zhang [12] says that if R is an affine domain with quotient division algebra Q and A is a commutative subalgebra of Q, then $\mathrm{GKdim}\, A \leq \mathrm{GKdim}\, R$ where $\mathrm{GKdim}\, R$ denotes the Gelfand-Kirillov dimension of R. Given these results it seems natural to look at centers in domains of Gelfand-Kirillov dimension 2. Smith and Zhang [7] showed that if R is a finitely generated non-PI domain with quotient division ring Q, then the GK dimension of the center of Q is at most $\mathrm{GKdim}\, R - 2$. Therefore, if R is a finitely generated non-PI F–algebra which is a domain with quadratic growth and Z is the center of R then GK dimension of Z is 0, hence Z is algebraic over F. The main results of this talk are the following.

Theorem 1 ([9])**.** *Let F be a field, and let R be a non-PI affine F-algebra (not necessarily graded) which is a domain with quadratic growth, and let $x \in R$ be transcendental over F. Then the centralizer C of x is a PI domain. Moreover, the quotient ring of C is a finite dimensional vector space over $F(x)$, the field of rational functions in the indeterminate x.*

As a corollary the following theorem may be stated.

Theorem 2 ([9])**.** *Let F be a field, and let R be an affine F–algebra (not necessarily graded) which is a domain with quadratic growth. If the center of R is infinitely generated F–algebra, then either R is PI or else R is algebraic over F.*

In the proofs of these results some special types of algebras appear. This is the motivation for prove the following theorem [8].

Theorem 3. *Let F be a field, m, c and let R be a K-algebra which is a domain of Gelfand-Kirillov dimension smaller than 3 and larger than 1. Let $x \in R$ be transcendental over F, and let x, y generate R. Assume that $\sum_{i,j\leq c} \alpha_{i,j} x^i y x^j = 0$, for some $\alpha_{i,j} \in F$ (and not all $\alpha_{i,j}$ equal 0). Then R is a homomorphic image of* *a finitely presented algebra with quadratic growth. Moreover, if K is a finite field then R satisfies a polynomial identity.*

In [1] Artin and Stafford described the structure of finitely graded domains with quadratic growth in terms of algebras associated with automorphisms of elliptic curves. They also proved that there are no finitely graded domains with GK dimension strictly between 2 and $\frac{11}{5}$. Artin, Stafford and Van den Bergh conjectured [1, 11] that a finitely graded domain cannot have Gelfand-Kirillov dimension strictly between 2 and 3. It was shown in [10] that this conjecture is true. In the talk we will give some ideas of the proof.

Theorem 4 ([10])**.** *Let K be a field, and let R be a finitely graded K-algebra which is a domain. Then R cannot have Gelfand-Kirillov dimension strictly between 2 and 3.*

Recall that, by Bergman's Gap Theorem, there are no algebras with GK dimension strictly between 1 and 2 [3], [4, p. 18].

We use the same terminology as in [1]. We call a graded K-algebra $R = \bigoplus_{n\geq 0} R_n$ *finitely graded* if it is a finitely generated algebra, and if R_0 is a finite dimensional vector space over K. Fix a finitely graded domain R with $\operatorname{GKdim} R < 3$. The graded ring of fractions $Q = Q(R)$ of R is the ring obtained by inverting homogeneous elements from R. It is described as a skew Laurent polynomial ring $D[z, z^{-1}; \sigma]$, in which σ is an automorphism of a division ring D, and multiplication is defined by $zd = d^\sigma z$.

In the proof of Theorem 4 the following theorem proved by Artin and Stafford in [1] is used.

Theorem 5 ([1])**.** *Let R be a finitely graded K-algebra and assume that R is an Ore domain with graded quotient division ring $Q(R) = D[z, z^{-1}; \sigma]$. If we have $\operatorname{GKdim} D < 2$, then $\operatorname{GKdim} R \leq 2$.*

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Serre functors, symmetric algebras and TQFT

Catharina Stroppel

In this talk we first recall the notion of a Serre functor and a characterisation of symmetric algebras via Serre functors. We describe the Serre functor for the bounded derived category $\mathcal{D}^b(\mathrm{Perv}_B(G/P))$ of B-equivariant sheaves on the flag variety G/P. The Serre functor can be thought of being the square of the Ringel duality functor. On the other hand we can build up the Serre functor for these categories from a braid group action via derived auto-equivalences. Our main result will be that for $G = SL(n, \mathbb{C})$ we also get braid group actions on certain $\mathcal{D}^b(\mathrm{Perv}_B(G/P))$, giving rise to a functorial invariant of oriented tangles and cobordisms. More precisely, we will describe a 3-dimensional TQFT with corners. We finish with a conjectural connection to Khovanov homology.

1. Serre functors

Let $\mathcal{C}$ be a $\mathbb{C}$-linear category with finite dimensional homomorphism spaces. A functor $S : \mathcal{C} \to \mathcal{C}$ is *a (right) Serre functor* if it is an auto-equivalence and

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathcal{C}}(M, SN)^*, \tag{1}$$

naturally in M and N. In general, a Serre functor does not need to exists, but if it exist, then it is unique up to isomorphism. Let A be a finite dimensional algebra over $\mathbb{C}$ and $A-\mathrm{mod}$ the category of finitely generated A-modules.

Example 2. Let $\mathcal{C}$ be the category of finite dimensional free A-modules. Assume S is a Serre functor then we have isomorphisms

$$A \cong \mathrm{Hom}_{\mathcal{C}}(A, A) \cong \mathrm{Hom}_{\mathcal{C}}(A, SA)^* \cong (SA)^*, \tag{3}$$

of vector spaces, even of A-bimodules by the naturality condition in (1). Hence $S \cong \mathrm{ID}$ iff $A \cong A^*$ as A-bimodule, hence if and only if A is a symmetric algebra.

The following theorem is due to Happel ([6]):

Theorem 4. $\mathcal{D}^b(A) = \mathcal{D}^b(A-\mathrm{mod})$ *has a Serre functor S if and only if A has finite global dimension. In this case* $S \cong \mathcal{L}(A^* \otimes_A \bullet)$.

Bondal and Kapranov ([4]) asked (with a conjectural answer) the following question: What is the Serre functor for $\mathcal{D}^b(\mathrm{Perv}_B(G/B))$, where $G = SL(n, \mathbb{C})$, B is the Borel subalgebra of all upper triangular matrices and $\mathrm{Perv}_B(G/B)$ denotes the category of B-equivariant perverse sheaves on the flag variety G/B. It is known that $\mathrm{Perv}_B(G/B) \cong A_n - \mathrm{mod}$ for some finite dimensional $\mathbb{C}$-algebra A_n of finite global dimension. In particular, Happel's theorem holds, and gives a description of the Serre functor. For example A_2 is given by the following quiver with relations

$$0 \underset{g}{\overset{f}{\leftrightarrows}} 1, \qquad g \circ f = 0$$

Since, however, the algebras A_n are not explicitly known for $n > 4$, Happel's description of the Serre functor is not satisfying. Instead we propose the following

Theorem 5 (joint with V. Mazorchuk, [9])**.** *For any $n \geq 2$, $1 \leq i \leq n-1$, there are right exact functors $C_i : A_n - \mathrm{mod} \to A_n - \mathrm{mod}$, such that*

(1) *the left derived functors $\mathcal{L}C_i$ define auto-equivalences of $\mathcal{D}^b(A_n - \mathrm{mod})$,*
(2) *these functors define a (weak) braid group action,*
(3) *if $w_o = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression of the longest element of S_n and $\mathcal{L}C_{w_0} = \mathcal{L}C_{s_{i_1}}\mathcal{L}C_{s_{i_2}}\cdots\mathcal{L}C_{s_{i_r}}$ the corresponding functor, then $(\mathcal{L}C_{w_0})^2$ is the Serre functor of $\mathcal{D}^b(A_n - \mathrm{mod})$.*

Example 6. In the example above we could take the idempotent e_1 to the vertex 1 and consider the $A := A_2$-bimodule $Ae_1 \otimes e_1A$. It defines an exact endofunctor θ_1 of $A - \mathrm{mod}$. There is a canonical map $Ae_1 \otimes e_1A \to A$ given by multiplication and dually $c : A \to Ae_1 \otimes e_1A$ defining a natural transformation $\mathrm{ID} \to \theta_1$. The functor $\mathcal{L}C_1$ is then nothing else then $Cone(\mathrm{ID} \to \theta_1)$. (If we identify $A - \mathrm{mod}$ with the principal block $\mathcal{O}_0(\mathfrak{sl}_2)$ of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathfrak{sl}_2$ then θ_1 is exactly the translation functor through the wall and the C_1 is Irving's shuffling functors.)

Remark 7.

(1) An independent, geometric proof is given in [2].
(2) There are at least two different ways to define these functors in question, either as so-called twisting functors (as for example studied in [1] and geometrically in [2]) or as Irving's shuffling functors (see e.g. [10]).
(3) One can show that $\mathcal{L}C_{w_0}$ is, up to isomorphism, independent of the chosen reduced expression of w_0. It maps projectives to tilting modules.
(4) Since a Serre functor commutes with auto-equivalences, it should commute with all $\mathcal{L}C_i$. Hence it makes perfectly sense that the Serre functor corresponds to the element w_0^2, the generator of the centre of the braid group.

2. Some symmetric algebras

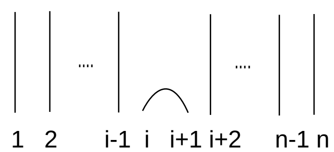
If we choose the functors to be given by Irving's shuffling functors then they even restrict to functors $C_i : A_n^P - \mathrm{mod} \to A_n^P - \mathrm{mod}$, where $A_n^P - \mathrm{mod} \cong \mathrm{Perv}_B(G/P)$. (This is not true for the functors studied in [2]). For G/P, we get the same result ([9]) as in Theorem 5, except that $S \cong (\mathcal{L}C_{w_0})^2[k]$ for some shift $[k]$. Using results of [7] one can show that in all cases the Serre functor is trivial when restricted to the additive category Add_n^P of projective-injective (and therefore also tilting) modules in $A_n^P - \mathrm{mod}$. In fact, the Serre functor can be described as a partial coapproximation with respect to projective injective modules (in the sense of [8]). Together with the characterisation of symmetric algebras in Example 2 we get the following theorem verifying a conjecture of M. Khovanov:

Theorem 8 (joint with V. Mazorchuk, [9])**.** *If T_n^P is a minimal generator of Add_n^P then $\mathrm{End}_{A_n}(T_n^P)$ is a symmetric algebra. Up to isomorphism it only depends on the partition of n defined by P, not on the composition or Young subgroup.*

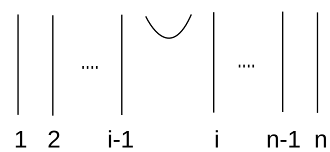
It is known ([3]) that the algebras A_n^P are Koszul, hence can be equipped with a $\mathbb{Z}$-grading which we will fix from now on and consider the category $A_n^P - \mathrm{mod}^{\mathbb{Z}}$ of finitely generated graded A_n^P-modules. the functors C_i have standard graded lifts which we denote by C_i as well. We define the category $\mathcal{C}_n := B_n - \mathrm{mod}^{\mathbb{Z}} = \oplus_{i=0^n} A_n^{P_i} - \mathrm{mod}^{\mathbb{Z}}$, where P_i is the parabolic subgroup corresponding to the Young subgroup $S_i \times S_{n-i}$ of S_n and set $\mathcal{C}_n := \mathbb{C} - \mathrm{mod}$ if $0 \leq n < 2$.

3. Tangle invariants and TQFT

One can show ([10]) that $\mathcal{D}^b(\mathcal{C}_n)$ categorifies the n-fold tensor product of the 2-dimensional representation of quantum $\mathfrak{sl}_2$, with commuting action of the Temperley-Lieb algebra. This is the space where the Jones polynomial of tangles can be defined. We want to enrich this to a functorial tangle invariant as follows: Let us first consider (n,n)-tangle diagrams. To the identity tangle diagram with n strands we associate the identity functor on $\mathcal{D}^b(\mathcal{C}_n)$. To the i-th right twisted curl we associate the functor $C_i\langle 1\rangle$ (where $\langle 1\rangle$ denotes a shift in the grading). To the i-th left twisted curl we associate its inverse functor. For the U-turns we assign the functors θ_i which are the translation through the i-th walls as mentioned in the example earlier. On the other hand, the θ_i are compositions of two other functors (roughly speaking translation to the wall and translation out of the wall) which we assign to the diagrams



and



Let $\mathcal{T}an$ denote the category of tangles, i.e. objects are the positive integers and morphisms are unframed tangle diagrams. Let $\mathcal{T}an^{or}$ be the 2-category of oriented tangles and cobordisms, i.e. objects are the positive integers, morphisms are unframed oriented tangles and 2-morphisms are diagrams of tangle cobordisms. For details see for example [5]. Let $\mathcal{F}unc$ denote the 2-category defined as follows: the objects are the bounded homotopy categories $\mathcal{D}^b_{per}(B_n - \mathrm{mod})$ of perfect complexes of graded B_n-modules. The 1-morphisms are roughly speaking functors between these categories. More precisely they are objects in the homotopy category of graded $B_n - B_m$-bimodules (considered as functors given by tensoring with the complexes of graded $B_n - B_m$-bimodules). The 2-morphisms are the natural transformations between the functors, but after forgetting the grading and only up to a multiplication with a homogeneous element of degree 0 of the centre of the source or image category. We get the following

Theorem 9 (see [11]). *There is a functor of 2-categories*

$$\Phi^{or} : \mathcal{T}an^{or} \to \mathcal{F}unc$$

which is given on objects by

$$n \in \mathbb{Z}_{>0} \mapsto \mathcal{D}^b_{per}(\mathrm{mod}^{\mathbb{Z}}\text{-}B_n),$$

and on elementary 1-morphisms by the assignments mentioned above such that

(1) *if t_1 and t_2 are 1-morphisms which differ by a sequence of Reidemeister moves then there is an isomorphism of functors $\Phi^{or}(t_1) \cong \Phi^{or}(t_2)$.*
(2) *if c_1 and c_2 are sequences of generating 2-morphisms which differ by a sequence of movie moves then $\Phi^{or}(c_1) = \Phi^{or}(c_2)$.*

Hence we constructed a 3-dimensional TQFT with corners. Under this functor the Euler characteristic of the cobordisms corresponds to the degrees of the natural transformations. For more details we refer to [10], [11].

4. The role of End^P_T

We associated in particular to each $(2m, 2n)$-tangle t a functor $\Phi^{or}(t)$. Recall that for each $A^{P_k}_{2n}$ we have a full projective tilting module $T^{P_k}_{2n}$. If we first restrict $\Phi^{or}(t)$ to a functor from $\mathbf{D}^b(A^{P_n}_{2n} - \mathrm{mod}^{\mathbb{Z}})$ to $\mathbf{D}^b(A^{P_m}_{2m} - \mathrm{mod}^{\mathbb{Z}})$, and then to perfect complexes of projective-injective-tilting modules we finally assign to each $(2m, 2n)$-tangle a functor which can be realized as tensoring with a complex $\check{X}(\Phi^{or}(t))$ of $(\mathrm{End}_{A_{2m}}(T^{P_m}_{2m}), \mathrm{End}_{A_{2n}}(T^{P_n}_{2n}))$-bimodules. We conjecture the following direct connection to Khovanov homology:

Conjecture 10 ([11]).

(1) *For any natural number m, there is an isomorphism of algebras p_m : $\mathrm{End}_{A_{2m}}(T^{P_m}_{2m}) \cong \mathcal{H}_m$, where $\mathcal{H}_m$ denotes Khovanov's algebra.*
(2) *The homological tangle invariant $t \mapsto \mathbf{H}^\bullet(\check{X}_{\Phi^{or}(t)})$ is Khovanov's invariant.*

This conjecture is illustrated for $n = m = 2$ in [11]. Work in progress connects End^P_T with the corresponding Springer fibre (thanks to Theorem 8).

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Noncommutative Poisson Geometry

Geert Van de Weyer

Double Poisson algebras were introduced by M. Van den Bergh in [1] as a generalization of classical Poisson geometry to the setting of noncommutative geometry. The key fact being that an algebra A equipped with a double Poisson bracket has a canonical Poisson structure on all its finite dimensional representation spaces $\mathsf{rep}_n(A)$. More specifically, a double Poisson algebra A is an associative unital algebra equipped with a linear map

$$\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$$

that is a derivation in its second argument for the outer A-bimodule structure on $A\otimes A$, where the outer action of A on $A\otimes A$ is defined as $a.a'\otimes a''.b := (aa')\otimes(a''b)$. Furthermore, we must have that $\{\!\{a,b\}\!\} = -\{\!\{b,a\}\!\}^o$ and that the double Jacobi identity holds for all $a, b, c \in A$:

$$\begin{aligned}\{\!\{a, \{\!\{b,c\}\!\}'\}\!\} \otimes \{\!\{b,c\}\!\}'' + \{\!\{c,a\}\!\}'' \otimes \{\!\{b, \{\!\{c,a\}\!\}'\}\!\} \\ + \{\!\{c, \{\!\{a,b\}\!\}'\}\!\}'' \otimes \{\!\{a,b\}\!\}'' \otimes \{\!\{c, \{\!\{a,b\}\!\}'\}\!\}' = 0,\end{aligned}$$

where we used Sweedler notation, that is $\{\!\{x,y\}\!\} = \sum \{\!\{x,y\}\!\}' \otimes \{\!\{x,y\}\!\}''$ for all $x, y \in A$. Such a map is called a *double Poisson bracket.*

A double Poisson bracket yields, for each n, a classical Poisson bracket on the coordinate ring $\mathbb{C}[\mathsf{rep}_n(A)]$ of the variety of n-dimensional representations of A through $\{a_{ij}, b_{k\ell}\} := \{\!\{a,b\}\!\}'_{kj}\{\!\{a,b\}\!\}''_{i\ell}$. This bracket restricts to a Poisson bracket on $\mathbb{C}[\mathsf{rep}_n(A)]^{\mathsf{GL}_n}$, the coordinate ring of the quotient variety $\mathsf{iss}_n(A)$ under the action of the natural symmetry group GL_n of $\mathsf{rep}_n(A)$.

We will study double Poisson brackets on a direct sum $S = M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$ of matrix algebras over $\mathbb{C}$. Because such algebras are formally smooth, we know from [1] that all double Poisson brackets are determined by double Poisson tensors. That is, elements of degree 2 in $\mathbb{D}S = T_S\mathbb{D}\mathrm{er}(S)$ where $\mathbb{D}\mathrm{er}(S) = \mathrm{Der}(S, S\otimes S)$ is the module of double derivations, i.e. the module of derivations from S to the S-bimodule $S\otimes S$, where the S-action on $S\otimes S$ is the outer action. $\mathbb{D}\mathrm{er}(S)$ is an S-bimodule through the inner action: $(s.\vartheta.t)(u) = \vartheta(u)'t \otimes s\vartheta(u)''$. A first important result is the explicit description of $\mathbb{D}\mathrm{er}(S)$ and $\mathbb{D}\mathrm{er}_T(S)$. Here, $\mathbb{D}\mathrm{er}_T(S)$ is the bimodule of T-linear double derivations with $T \subset S$ a subalgebra.

That is, double derivations that are identically zero on T. We have that

$$\mathbb{D}\mathrm{er}(S) \cong \bigoplus_{i=1}^{k} M_{d_i}(\mathbb{C})^{\oplus d_i^2-1} \oplus \bigoplus_{i\neq j} M_{d_i\times d_j}(\mathbb{C})^{\oplus d_i d_j}$$

as S-bimodules where S acts on the right hand side expression by matrix multiplication. If $T = M_{e_1}(\mathbb{C}) \oplus \ldots M_{e_\ell}(\mathbb{C})$ is a finite dimensional semi-simple subalgebra of S with Bratelli diagram with respect to S given by $(a_{ij})_{(i,j)=(1,1)}^{(k,\ell)}$, then

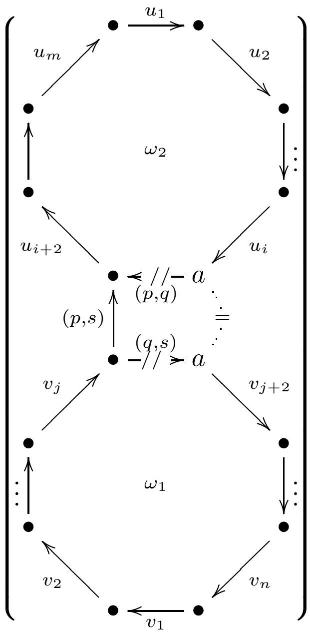
$$\mathbb{D}\mathrm{er}_T(S) \cong \bigoplus_{i=1}^{k} M_{d_i}(\mathbb{C})^{\oplus r_i} \oplus \bigoplus_{i\neq j} M_{d_i\times d_j}(\mathbb{C})^{\oplus r_{ij}}$$

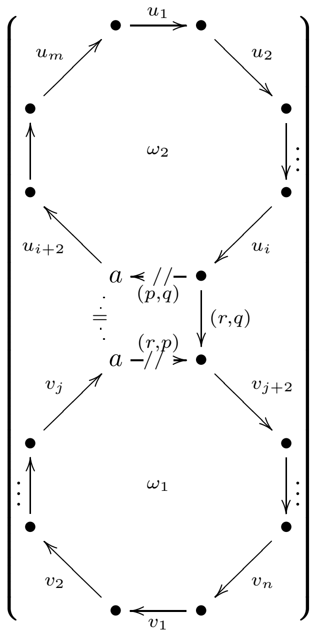
as S-bimodules, with $r_i = \sum_{u=1}^{l} a_{iu}^2 - 1$ and $r_{ij} = \sum_{u=1}^{l} a_{iu}a_{ju}$.

Using these two theorems, we are able to formulate an explicit description of the graded Lie algebra $\mathbb{D}S/[\mathbb{D}S,\mathbb{D}S][1]$, where the bracket on $\mathbb{D}S/[\mathbb{D}S,\mathbb{D}S][1]$ is the bracket associated to the double Schouten-Nijenhuis bracket on $\mathbb{D}S$. This description is formulated in terms of the *double derivation quiver* Q_S associated to S. Assign to S a quiver $\overline{Q}_S$ on k vertices with $d_i d_j$ arrows between each two vertices $i \neq j$ and $d_i^2 - 1$ loops in all vertices i, where the arrows are indexed by index sets $C_{ji} = \{1,\ldots,d_j\} \times \{1,\ldots,d_i\}$ if $i \neq j$ and $C_{ii} = \{1,\ldots,d_j\} \times \{1,\ldots,d_i\}\backslash\{(1,1)\}$. Then $\mathbb{D}S/[\mathbb{D}S,\mathbb{D}S][1]$ is isomorphic as a graded Lie algebra to $\mathbb{C}Q_S/[\mathbb{C}Q_S,\mathbb{C}Q_S]_{super}$, where the bracket on two words $\omega_1 = v_1\ldots v_n$ and $\omega_2 = u_1\ldots u_m$ in $\mathbb{C}\overline{Q}_S/[\mathbb{C}\overline{Q}_S,\mathbb{C}\overline{Q}_S]_{super}$ is given by

$$\sum_{a\in(\overline{Q}_S)_0} (-1)^{(i+j)(n-1)}A - (-1)^{(i+j+1)(n-1)}B$$

where

$A =$ 

$, B =$ 

This result can then be used to dermine all monomials of degree 2 in $\mathbb{D}S$ that yield nontrivial double Poisson structures on S. It can also be used to compute the first double Poisson-Lychnerowicz cohomology groups for S.

Although the representation varieties of finite dimensional semi-simple algebras are rather simple and the quotient varieties consist of a finite number of points, double Poisson structures on these algebras yield interesting noncommutative geometry as they can be extended to double Poisson structures on the free product of such algebras. For such a free product $S * T$, the quotient variety $\mathsf{iss}_n(S * T)$ is no longer trivial and double Poisson structures can yield nontrivial Poisson structures on this variety.

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Rigid Dualizing Complexes via Differential Graded Algebras

Amnon Yekutieli
(joint work with J.J. Zhang)

Rigid dualizing complexes were introduced by Van den Bergh in the context of noncommutative algebraic geometry, where they proved to be extremely useful. The advantage of rigidity is that it eliminates automorphisms, thus making dualizing complexes unique, and even functorial.

This talk is about rigid dualizing complexes over *commutative* K-algebras. If K is a field then the "noncommutative" results specialize to yield an enormous amount of information on rigid dualizing complexes and their variance properties. Indeed, one can recover most of the important features of Grothendieck duality (for affine schemes), including explicit formulas, with relatively little effort.

However we want to consider commutative algebras over any noetherian commutative base ring K. It turns out that this causes serious technical issues, due to lack of flatness. Even defining rigidity (i.e. writing Van den Bergh's rigidity equation) is a problem! Our solution was to use differential graded algebras. Thus, if A is a K-algebra which is not flat, we replace A with a quasi-isomorphic DG K-algebra $\tilde{A}$ which has suitable flatness properties, and use $\tilde{A}$ to formulate the rigidity equation for complexes of A-modules.

Actually, this method enables us to work with relative rigid complexes. Namely, given a homomorphism $A \to B$ between K-algebras, we can consider rigid complexes of B-modules relative to A. (This is nontrivial even when the base K is a field.) The theory of rigid complexes we developed is quite rich, and may be of independent interest in ring theory.

When our base ring K is regular (e.g. the ring of integers) we obtain a comprehensive theory of rigid dualizing complexes, once again producing most of the important features of Grothendieck duality for affine K-schemes.

Full details can be found in the papers [5, 6].

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Some ring theoretic problems inspired by combinatorial group theory

Efim Zelmanov

The talk focused on several open problems in Ring Theory. Here are some samples.

Problem 1. Does there exist an infinite dimensional quadratic nil algebra?

If not then the Kurosh Problem has positive solution for graded finitely presented algebras.

Let $p > 0$ be a prime number and F be a field of characteristic p. Consider the algebra of truncated polynomials in countably many variables

$$F[t_0, t_1, t_2, ...|{t_i}^p = 0, i \geq 0].$$

Let

$$\begin{aligned} \partial_i &= \partial/\partial t_i, \\ V_1 &= \partial_1 + {t_0}^{p-1}\partial_2 + {t_0}^{p-1}{t_1}^{p-1}\partial_3 \ t \ldots, \\ V_2 &= \partial_2 + {t_1}^{p-1}\partial_3 + {t_1}^{p-1}{t_2}^{p-1}\partial_4 \ t \ldots. \end{aligned}$$

Theorem 2 (Petrogradsky, Shestakov-Zelmanov)**.** *The Lie algebra L generated by V_1, V_2 is nil and* $1 < \mathrm{GKdim}\, L < 2$.

Problem 3. Is the associative algebra generated by V_1, V_2 nil?

Let U be the universal enveloping algebra of L, Z be the center of U, and $D = (Z - \{0\})^{-1}U$.

Problem 4. Is D an algebraic division algebra?

Artin-Schelter regular algebras of global dimension 4

James J. Zhang

(joint work with Diming Lu, John Palmieri, Quanshui Wu, Jun Zhang)

One of the main questions in noncommutative algebra and noncommutative algebraic geometry is the classification of Artin-Schelter regular algebras of global dimension four. A connected $\mathbb{N}$-graded algebra A is called Artin-Schelter regular if (a) $\mathrm{gldim} A = d < \infty$, (b) $\mathrm{Ext}^i_A(A/A_{\geq 1}, A) = 0$ for all $i \neq d$ and $\mathrm{Ext}^d_A(A/A_{\geq 1}, A) = A/A_{\geq 1}$, and (c) $\mathrm{GKdim} A < \infty$ [1]. The classification of Artin-Schelter regular algebras of global dimension three was finished in 1990s by Artin, Schelter, Tate, Van den Bergh and Stephenson. The associated projective schemes of Artin-Schelter regular algebras of global dimension three are quantum projective 2-spaces, denoted by $q\mathbb{P}^2$. Artin-Schelter regular algebras have been used more and more recently. For example, Artin-Schelter regular algebras of global dimension four are used to construct quantum projective 3-spaces, denoted by $q\mathbb{P}^3$s, and other noncommutative subspaces.

In this talk we introduce two new general methods in the study of Artin-Schelter regular algebras of global dimension four. One is the A_∞-algebra method and the other is the extension method. The A_∞-algebra was introduced by Stasheff in the study of topology. Roughly speaking, an A_∞-algebra E is a graded vector space equipped with a sequence of multiplications $m_n : E^{\otimes n} \to E$ of degree $2-n$. For $n \geq 3$, m_n are called higher multiplication, as m_2 plays the role of the usual multiplication and m_1 is the differential. A general principle of Keller says that when $E := \mathrm{Ext}^*_A(k,k)$ is the Ext-algebra of an Artin-Schelter regular A (where $k = A/A_{\geq 1}$), the multiplications m_n of the Ext-algebra E are determined by the degree n relations of A and vice versa. When A is not Koszul, E has non-trivial higher multiplications. In certain cases, we are able to classify all possible A_∞-structures in the Ext-algebra E. Therefore we can recover Artin-Schelter regular algebras by using Keller's general principle. By using the A_∞-algebra method we completely classify $\mathbb{N}^2$-graded Artin-Schelter regular algebras of global dimension four that are generated by two elements of degrees $(1,0)$ and $(0,1)$ into four families [2].

The extension method is a generalization of two classical constructions, namely, the Ore extension $A[y;\sigma,\delta]$ and the normal extension. In general the extension of two graded rings are not well-understood. Note that the Ore extension is a kind of construction by adding one element. We are mainly working on so-called double extensions, denoted by $A_P[y_1, y_2; \sigma, \delta, \omega]$, that can be obtained by adding two generators simultaneously. The double extension method is a simple and efficient way of producing many Artin-Schelter regular of global dimension four (or higher) and we also manage to prove some nice results about homological properties. There are many open questions about the double extensions and more general extensions. By using double extensions we classify all $\mathbb{N}^2$-graded regular algebras of global dimension four that are neither Ore extensions nor normal extensions [3].

Both the A_∞-algebra method and the extension method are useful for other classes of noncommutative algebras. This talk was based on some joint work with Lu, Palmieri and Wu [2] and with Zhang [3].

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