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Quadratic Forms and Linear Algebraic Groups

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Abstract. Topics discussed at the Oberwolfach workshop *Quadratic Forms and Linear Algebraic Groups*, held in June 2006, included besides the algebraic theory of quadratic and Hermitian forms and their Witt groups several aspects of the theory of linear algebraic groups and homogeneous varieties where geometric methods have proved successful in recent years, notably Chow motives and the essential and canonical dimensions of algebraic groups.

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Introduction by the Organisers

The workshop was organized by Detlev Hoffmann (Nottingham), Alexandr Merkurjev (Los Angeles), and Jean-Pierre Tignol (Louvain-la-Neuve), and was attended by 52 participants. Funding from the Marie Curie Programme of the European Union provided complementary travel support for young researchers, and it also allowed for the invitation of six PhD students in addition to established researchers.

The workshop followed a long and illustrious tradition of Oberwolfach meetings on quadratic forms initiated by M. Knebusch, A. Pfister and W. Scharlau in the 1970's. Initially, the topics ranged from the arithmetic theory of quadratic forms and lattices to real algebraic geometry. In the last decade, however, the algebraic theory of quadratic forms sustained a vigorous development of its own, under the influence of geometric methods, and new connections with linear algebraic groups over arbitrary fields appeared. Recently, striking new results, such as Voevodsky's proof of the Milnor conjecture, were obtained by an infusion of new techniques from motivic cohomology and algebraic topology.

The schedule of the meeting comprised 22 lectures of 45 minutes each, which presented recent progress and interesting new directions in various topics where the algebraic theory of quadratic forms, Galois cohomology, algebraic geometry and the theory of linear algebraic groups mutually stimulate each other, notably Witt groups of triangulated categories, Chow motives of homogeneous varieties, and the essential and canonical dimensions of algebraic groups. Some new connections with representation theory were also discussed.

Workshop: Quadratic Forms and Linear Algebraic Groups

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Abstracts

The problem of ruledness for quadrics, and an application to automorphism groups of affine quadrics

Burt Totaro

We determine the automorphism group for a large class of affine quadrics over a field, viewed as affine algebraic varieties. In particular, we show that the automorphism group of the n-sphere $\{x_0^2 + \cdots + x_n^2 = 1\}$ over the real numbers is just the orthogonal group $O(n+1)$ whenever n is a power of 2. It is not known whether the same is true for arbitrary n: this shows how little is known about affine algebraic geometry.

The result on spheres is similar to Wood's theorem that when n is a power of 2, every polynomial mapping from the real n-sphere to a lower-dimensional sphere is constant [4]. Wood's theorem is optimal at least in low dimensions, since the Hopf maps are nonconstant polynomial maps $S^3 \to S^2$, $S^7 \to S^4$, and $S^{15} \to S^8$.

For isotropic affine quadrics, we have the following easy observation.

Lemma 1. *Let $X = \{f = 0\}$ be an affine quadric in an n-dimensional vector space over a field k with $n \geq 3$. If the homogeneous part of degree 2 of f is isotropic, then the automorphism group of X is infinite-dimensional.*

For example, as is well known, the affine quadric surface $x_1x_2 + x_3^2 = 1$ has the automorphism

$$(x_1, x_2, x_3) \mapsto (x_1 - f(x_2)^2 x_2 - 2f(x_2)x_3, x_2, f(x_2)x_2 + x_3)$$

for any polynomial f in one variable. Likewise, Lemma 1 implies that the automorphism group of the n-sphere $\{x_0^2 + \cdots + x_n^2 = 1\}$ over the complex numbers is infinite-dimensional for every $n \geq 2$.

By contrast, we show that the automorphism group is finite-dimensional for many affine quadrics such that the homogenous part of degree 2 is anisotropic. Recall that the first Witt index of an anisotropic quadratic form q is the Witt index of q over the function field of the projective quadric associated to q. Informally speaking, over a sufficiently large field, most anisotropic forms of any given dimension have first Witt index equal to 1 (the smallest possible).

Theorem 2. *Let q be an anisotropic quadratic form over a field k of characteristic not 2. Suppose that q has first Witt index $i_1(q)$ equal to 1. Then, for any $c \neq 0$ in k, the automorphism group of the affine quadric $\{q = c\}$ is equal to the orthogonal group $O(q)$.*

The proof uses Karpenko's theorem that an anisotropic projective quadric with first Witt index 1 over a field k is not ruled ([2], Theorem 6.4), together with some general arguments of birational geometry. Here we say that a variety X over a field k is ruled if there is a variety Y over k such that X is birational to $Y \times \mathbb{P}^1$ over k.

Note that it is perfectly possible for an anisotropic projective quadric to be ruled. For example, as Knebusch found ([3], pp. 73–74), an r-fold Pfister form with $r \geq 2$ over any field is ruled (that is, the corresponding projective quadric is ruled). Explicitly, for $r = 2$, the Pfister quadric surface $Q = \{x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 = 0\}$ is ruled over its subconic $C = \{y_0^2 - ay_1^2 - by_2^2 = 0\}$, since we can define a birational equivalence from $C \times \mathbb{P}^1$ to Q by

$$[y_0, y_1, y_2], [u, v] \mapsto [uy_0 + avy_1, uy_1 + vy_0, uy_2, vy_2].$$

By Hoffmann, every anisotropic form with dimension $2^a + 1$ over any field has first Witt index 1 [1]. In the case of the real numbers, Theorem 2 then implies that for n a power of 2, the automorphism group of the n-sphere as an affine algebraic variety over the real numbers is just the orthogonal group.

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Motivic decompositions of anisotropic projective homogeneous varieties I

Kirill Zainoulline

Let F be a field. For a smooth projective variety X over F consider its *Chow motive* $\mathcal{M}(X)$. One of the basic problems arising in the theory of Chow motives is How to express $\mathcal{M}(X)$ as a direct sum of indecomposable objects?

In what follows we explain the machinery of proving such results for Chow motives of projective homogeneous varieties. The main motivation for our work was the result of Karpenko where he gave a 'shortened' construction of a Rost motive for a norm quadric. The exposition below can be viewed as an attempt to generalize and apply the arguments of Karpenko to any projective homogeneous variety.

From now on let G be an *anisotropic* simple algebraic group. Let $\mathcal{M}(G)$ be the category of Chow motives of projective homogeneous G-varieties introduced and studied by Chernousov and Merkurjev. Recall that a projective homogeneous G-variety X can be viewed as the quotient variety G/P, where P is a parabolic subgroup of G. Depending on the 'size' of P the problem of providing the motivic decomposition of X can be subdivided into three steps

(1) To express the motive of a general flag G/P in terms of motives of minimal flags G/P_{max} (here P_{max} is a maximal parabolic subgroup of G). For a wide class of groups and parabolic subgroups it was done in the joint paper with B. Calmès, V. Petrov and N. Semenov [1].

(2) To express the motive of a minimal flag G/P_{max} in terms of motives of 'simplest' flags G/P_{smp}. For instance, to express the motive of a generalized Severi-Brauer variety in terms of motives of usual Severi-Brauer varieties. This is the work in progress with A. Vishik [4]. Certain partial results in this direction can be also extracted from [5].
(3) The most complicated and interesting step is to express the motive of a simplest flag as a direct sum of indecomposable (usually non-geometric) objects (like Rost motives, etc.). In this case we provide a general algorithm which covers all known examples (Pfister quadrics, maximal orthogonal Grassmannians, G_2-varieties) as well as provides new ones (F_4-varieties, E_6-varieties, E_8-varieties [3], [2]).

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Motivic decompositions of anisotropic projective homogeneous varieties II

Nikita Semenov

In the present note we study projective homogeneous varieties (twisted flag varieties) and their decompositions in the category of Chow motives. The history of this problem starts with a celebrated result of Rost [Ro98] devoted to a motivic decomposition of a Pfister quadric. In the present text we concentrate ourself on projective G-homogeneous varieties, where the group G is of exceptional type.

Let k be any field. From now on we work in the category of Chow motives introduced in [Ma68]. First we state the following theorem:

Theorem 1 (S. Nikolenko, N. Semenov, K. Zainoulline)**.** *Let X be an anisotropic variety over a field k such that over a cubic field extension k' of k it becomes isomorphic to the projective homogeneous variety G/P, where G is a split simple group of type* F_4 *and P its maximal parabolic subgroup of type P_1 or P_4 (enumeration of roots follows Bourbaki).*

Then the Chow motive of X (with $\mathbb{Z}$-coefficients) is isomorphic to the direct sum of twisted copies of an indecomposable motive R

$$\mathcal{M}(X) \cong \bigoplus_{i=0}^{7} R(i) \qquad (1)$$

which has the property that over k' it becomes isomorphic to the direct sum of twisted Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.

For the proof of this theorem we refer the reader to [NSZ]. The main steps of the proof include the structure of the Picard group of X, splitting properties of groups of type F_4, and Rost nilpotence theorem for projective homogeneous varieties.

Next we consider the group of zero-cycles of twisted flag varieties of types F_4 and E_6.

Theorem 2 (V. Petrov, N. Semenov, K. Zainoulline)**.** *Let k be a field of characteristic different from $2, 3$. Let G be an exceptional simple algebraic group over k of type F_4 or ${}^1E_6$ with trivial Tits algebras and X a projective G-homogeneous variety over k. Then the degree map $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is injective.*

The proof of this theorem can be found in [PSZa], where we use the ideas of Krashen [Kr05] and Popov [Po05].

The next theorem provides an interesting application of the previous two results.

Theorem 3. *Assume $\operatorname{char} k \neq 2, 3$. Then in the notation of Theorem 1 the group $\mathrm{CH}_2(R)$ contains 3-torsion.*

In the proof of this theorem the ideas of Karpenko and Merkurjev [KM02] as well as two previous results play a crucial role.

Finally, we formulate the following theorem:

Theorem 4 (V. Petrov, N. Semenov, K. Zainoulline)**.** *Let G be a group of type E_8, which splits by a field extension k'/k of degree 5. Let X be a projective G-homogeneous variety of parabolic subgroups of type P_8. Then*

(1)
$$\mathcal{M}(X) \simeq \bigoplus_{j=0}^{1} \bigoplus_{i=0}^{23} R(i + 10j) \mod 5,$$
where for any field extension K/k the motive R is decomposable over K iff X_K has a 0-cycle of degree 1. Moreover, over any such K it becomes isomorphic to the direct sum of twisted Lefschetz motives:
$$R_K \simeq \bigoplus_{i=0}^{4} (\mathbb{Z}/5)(6i)$$

(2) *There exists a motive $\widetilde{R}$ with $\mathbb{Z}$-coefficients such that $\widetilde{R} \mod 5 \simeq R$. This motive is a generalized Rost motive with $\mathbb{Z}$-coefficients for a symbol in $(\mathrm{K}_3^M/5)(k)$ given by the Rost invariant of G.*

In fact, there exists a general method that allows to decompose a motive of any generically cellular variety, which splits by a field extension of a prime degree; see [PSZb] (note that the varieties from Theorems 1 and 4 are so). This method covers all cases known so far in the literature.

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Chow motives of flag varieties

Baptiste Calmès

(joint work with V. Petrov, N. Semenov and K. Zainoulline)

Introduction

Chow groups have proved to be useful in understanding properties of algebraic objects such as quadratic forms, central simple algebras (with or without involution). For example, Karpenko [4] relates the fact that a division algebra decomposes with the p-torsion in the codimension 2 Chow group of its corresponding Severi-Brauer variety, and thus proves that indecomposable algebras of arbitrarily large p-primary degree exist.

Chow motives are a convenient category in which computations can be understood naturally.

To the algebraic objects cited above correspond projective homogeneous varieties, which are (in the split case) quotients of a semi-simple algebraic group by a parabolic subgroup. Their Chow groups are described in a very combinatorial way in the split case, using root systems, Weyl groups, Dynkin diagrams, etc. We can take advantage of this combinatorial description for computations. It is because a lot of information has been extracted from the Chow groups of such varieties that the systematic study of their motives is certainly interesting.

1. Chow Motives

Let us just recall briefly that the category of Chow motives $\mathcal{M}(F)$ over a field F is constructed in three steps. First, one constructs the category of correspondences, in which the objects are smooth projective varieties over F and the morphisms are elements of the Chow group of the product of the source and the target. Then, one formally adds the kernels of projectors (pseudo-abelianization) and inverses the Lefschetz motive (complement of a point in $\mathbf{P}^1$). There is a realization functor from $\mathcal{M}(F)$ to the Abelian groups which shows that motivic decompositions induce decompositions of Chow groups.

2. Decompositions

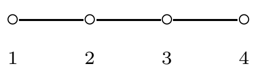
A natural question about a variety X is whether its motive decomposes as a direct sum of smaller objects, which implies that all the information contained in the Chow groups can be expressed in terms of these smaller objects. We will restrict our study to the case of projective homogeneous varieties. It is trivial that when the variety has a rational point, we can "cut out" a corresponding direct factor. A systematic combinatorial study description of this phenomenon is done in [2]. We are interested in cases where there is no rational point. Then, the motive might be irreducible (*e.g.* the Severi-Brauer variety of a division algebra) or not (*e.g.* an anisotropic Pfister quadric, which decomposes as shifted copies of a "Rost motive").

2.1. **Higher flag varieties.** In the following, we will partly explain the situation when the variety we consider is a "higher flag variety".

Let G be a split connected semi-simple group. We choose a maximal split torus T and a Borel subgroup B containing T. Let Σ be the set of roots of G, Π the set of simple roots relatively to B. If S is a subset of Π, we denote by P_S the standard parabolic subgroup generated by B and the subgroups $U_{-\alpha}$ for $\alpha \in S$. Every parabolic subgroup of G is a conjugate of a standard parabolic subgroup.

By definition, if $S \subset S'$, we have $P_S \subset P_{S'}$, and so the variety $X_S = G/P_S$ naturally maps to $X_{S'} = G/P_{S'}$.

Let us look at the example of $SL_5 = SL(V)$, which is a group of type A_4, with Dynkin diagram



If we fix a subset of this group, we obtain a variety X_S whose points are flags $V_1 \subset \cdots \subset V_i \subset \cdots \subset V$ for $i \notin S$ where the dimension of V_i is exactly i. If we take a larger $S' \supset S$, the map from X_S to $X_{S'}$ just "forgets" a part of the flag.

If we choose $S = \{3, 4\}$, $S_1 = \{1, 3, 4\}$ and $S_2 = \{2, 3, 4\}$, we obtain the varieties and maps

$$\begin{array}{ccc} & X_S = \{\text{flags } V_1 \subset V_2 \subset V\} & \\ \swarrow & & \searrow \\ X_{S_1} = \mathbf{P}^1 & & X_{S_2} = \mathrm{Gr}(2,5) \end{array}$$

If we twist this whole construction by a cocycle, we obtain all projective homogeneous varieties (we keep the same notation, in particular for X_S). For example, the generalized Severi-Brauer varieties are forms of projective homogeneous varieties under SL_n. They parametrize flags of ideals of fixed reduced dimension in a central simple algebra A. Of course, if A is a division algebra, they have no rational point. For D a division algebra of degree 5, the above diagram becomes

$$\begin{array}{ccc} & \mathrm{SB}(1,2,D) & \\ \swarrow & & \searrow \\ \mathrm{SB}(D) & & \mathrm{SB}(2,5) \end{array}$$

2.2. **Motivic decompositions.** We now explain several cases where the motive of X_S (eventually without point) can be expressed in terms of the motives of $X_{S'}$ for S' containing S. Although for simplicity, we only give here results for groups of type A_n, the same type of results hold for groups whose Dynkin diagram does not branch (A_n, B_n, C_n, G_2 and F_4). The standard parabolic subgroups corresponding to subsets S where exactly one simple root is missing are maximal (proper) parabolic subgroups with respect to inclusion.

In the case $G = \mathrm{PGL}_1(A)$, where A is a central simple algebra of degree $n+1$, $n > 0$, the set of F-points of a projective G-homogeneous variety can be identified with the set of flags of (right) ideals

$$SB(d_1, \ldots, d_k, A) = \{I_1 \subset I_2 \subset \ldots \subset I_k \subset A\}$$

of fixed reduced dimensions $1 \le d_1 < d_2 < \ldots < d_k \le n$. Observe that this variety is a twisted form of G'/P, where $G' = \mathrm{PGL}_{n+1}$ (or SL_{n+1}) and P is the standard parabolic subgroup corresponding to the simple roots on the Dynkin diagram, numbered by d_i. We define $\delta_i = d_{i+1} - d_i$ (assuming $d_0 = 0$ and $d_{k+1} = n$) for $i = 0 \ldots k$.

Theorem 1. *Suppose that* $\gcd(\mathrm{ind}(A), d_1, \ldots, \hat{d}_m, \ldots, d_k) = 1$, *then*

$$\mathrm{SB}(d_1, \ldots, d_k, A) \simeq \bigoplus_{\lambda} \mathrm{SB}(d_1, \ldots, \hat{d}_m, \ldots, d_k)[\delta_m \delta_{m-1} - |\lambda|],$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\delta_{m-1}})$ *of* $|\lambda|$ *such that* $\delta_m \ge \lambda_1 \ge \ldots \ge \lambda_{\delta_{m-1}} \ge 0$.

For the proof of this theorem, we refer the reader to [1]. The main tools are the Grassmann bundle theorem and the fact that the Brauer group of a regular Noetherian scheme injects in the Brauer group of its function field.

Example 2. *The variety* $\mathrm{SB}(1,2,D)$ *for a division algebra* D *of degree* 5 *corresponds to the parabolic* P_S *with* $S=\{3,4\}$. *We get* $d_1=1$, $d_2=2$, $\delta_0=1$, $\delta_1=1$, $\delta_2=3$. *By removing the dimension* 1 *part of the flag, we get*

$$\mathrm{SB}(1,2,D)\simeq \mathrm{SB}(2,D)\oplus \mathrm{SB}(2,D)[1]$$

and by removing the dimension 2 *part of the flag, we get*

$$\mathrm{SB}(1,2,D)\simeq \bigoplus_{i=0}^{3}\mathrm{SB}(D)[i].$$

There are a few noteworthy consequences of the theorem.

Corollary 3. *The motive of the variety of complete flags (corresponding to the Borel* $P_\emptyset$*) decomposes as*

$$\mathrm{SB}(1,\dots,n,A)\simeq \bigoplus_{i=0}^{n(n-1)/2}\mathrm{SB}(A)[i]^{\oplus a_i},$$

where a_i *are the coefficients of the polynomial* $\varphi_n(z)=\sum_i a_i z^i=\prod_{k=2}^{n}\frac{z^k-1}{z-1}$.

Corollary 4. *The motive of the "incidence variety"* $\mathrm{SB}(1,n,A)$ *is isomorphic to*

$$\mathrm{SB}(1,n,A)\simeq \bigoplus_{i=0}^{n-1}\mathrm{SB}(A)[i].$$

3. Further decomposition of a special case

Although the variety $\mathrm{SB}(2,D)$ for a division algebra D of degree 5 corresponds to a maximal parabolic ($S=\{1,3,4\}$), its motive can still be decomposed. It is not too difficult to show that

$$\mathrm{SB}(2,D)\simeq \mathrm{SB}(D')\oplus \mathrm{SB}(D')[2]$$

where D' is an algebra Brauer equivalent to $D^{\otimes 2}$.

Using the two different decompositions of $\mathrm{SB}(1,2,D)$ seen above, we get an isomorphism in the category of motives:

$$\bigoplus_{i=0}^{3}\mathrm{SB}(D)[i]\simeq \bigoplus_{i=0}^{3}\mathrm{SB}(D')[i]$$

Since $\mathrm{SB}(D)$ and $\mathrm{SB}(D')$ are both irreducible and not isomorphic in the category of motives (by a theorem of Karpenko [5]), this provides a counter example to the Krull-Schmidt theorem (unicity of the decomposition into irreducible factors) in the category of Motives under the action of $PGL(A)$, which is a simple group. Such a counter example had already been given in [3] but only under the action of a product of two simple groups.

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The oriented Chow ring and applications

Jean Fasel

Let A be a commutative noetherian ring of Krull dimension d and P a projective A-module of rank r. One can ask the following question: does P admit a free factor of rank one? Serre proved a long time ago that the answer is always positive when $r > d$. So in fact the first interesting case is when P is projective of rank equal to the dimension of A. Suppose now that X is an integral smooth scheme over a field k of characteristic not 2. To deal with the above question, Barge and Morel introduced the oriented Chow groups $\widetilde{CH}^j(X)$ of X (see [BM]) combining classical Chow groups and quadratic forms and associated to each vector bundle E of rank n an Euler class $\tilde{c}_n(E)$ in $\widetilde{CH}^n(X)$. Morel proved recently that, for $n = 2$ or $n \geq 4$, and for $X = \mathrm{Spec}(A)$ we have $\tilde{c}_n(P) = 0$ if and only if $P \simeq Q \oplus A$ (see [Mo], or [Fa] for the case $n = 2$).

The oriented Chow groups satisfy good functorial properties ([Fa]) and there are natural homomorphisms $\widetilde{CH}^j(X) \to CH^j(X)$ and $\widetilde{CH}^j(X) \to H^j(X, I^j)$ where the latter denotes some cohomology group of the Gersten-Witt complex filtered by the powers of the fundamental ideal. We have a pull-back morphism $f^* : \widetilde{CH}^j(X) \to \widetilde{CH}^j(Y)$ associated to each flat morphism $f : Y \to X$ and a push-forward morphism $g_* : \widetilde{CH}^j(Y, L) \to \widetilde{CH}^{j+r}(X)$ associated to each proper morphism $g : Y \to X$, where $r = \dim X - \dim Y$ and L is a suitable line bundle over Y. Using these properties and the deformation to the normal cone, it is possible to define for any embedding of smooth schemes $i : Y \to X$ a Gysin homomorphism $i^! : \widetilde{CH}^j(X) \to \widetilde{CH}^j(Y)$ having the meaning of intersecting with Y ([Fa2]). This homomorphism is then used to give a ring structure on the total oriented Chow group $\bigoplus_i \widetilde{CH}^i(X)$ of X. It turns out that $\widetilde{CH}^*(\_)$ is a contravariant functor from the category of smooth schemes over $\mathrm{Spec}(k)$ to the category of rings and there is a natural transformation $\widetilde{CH}^*(\_) \to CH^*(\_)$ ([Fa2]). The construction yields however a surprise: $\bigoplus_i \widetilde{CH}^i(X)$ is in general not commutative (not even graded commutative).

The next step in the theory is to compute the oriented Chow ring of a projective bundle $\mathbb{P}(E)$. Using the product, we would then get new characteristic classes for E. This is the object of [Fa3].

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Triangular Witt groups in modular representation theory

Paul Balmer

We discuss an example where Triangular Witt Groups [1] could possibly appear outside Algebraic Geometry, where they have initially been used. For this, we consider a triangulated category with duality, kG-stab, called the *stable category* of kG-modules, which is of interest in Modular Representation Theory and which is defined as the additive quotient of the category of finitely generated kG-modules by that of projective kG-modules. See more in [3] or [5]. Using [1], [2], [4] and [5], we prove that:

$$\begin{aligned}
W^{-1}(kG\text{-stab}) &= \mathrm{Ker}\big(W(kG\text{-proj}) \to W(kG\text{-mod})\big)\\
W^{0}(kG\text{-stab}) &= \mathrm{Coker}\big(W(kG\text{-proj}) \to W(kG\text{-mod})\big)\\
W^{1}(kG\text{-stab}) &= \mathrm{Ker}\big(W^{-}(kG\text{-proj}) \to W^{-}(kG\text{-mod})\big)\\
W^{2}(kG\text{-stab}) &= \mathrm{Coker}\big(W^{-}(kG\text{-proj}) \to W^{-}(kG\text{-mod})\big)
\end{aligned}$$

where the right-hand W is the usual Witt group of exact (here additive or abelian) categories with duality, and where W^{-} stands for the usual Witt group of skew-symmetric forms. These are kernels and cokernels of what could be called a Cartan homomorphism for Witt groups.

Finally, we prove that the Witt groups of kG-stab vanish for G a p-nilpotent group (for instance a p-group), where $p = \mathrm{char}(k)$. We ask if these groups vanish for any G. If not, one can consider Witt groups of kG-stab with various supports in its projective support variety, as well as dualities twisted by various endo-trivial representations.

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Witt transfers and dévissage

Jens Hornbostel
(joint work with B. Calmès)

The ultimate goal of this project is to compute Witt groups of (twisted) flag varieties, e.g. quadrics, Brauer-Severi varieties etc.

Let k be a field of characteristic different from 2. Throughout this note, we only consider separated regular noetherian schemes of finite Krull dimension. In the first part, we define transfer morphisms for Witt groups with respect to proper morphisms. We establish a base change theorem and a projection formula. In the second part, we prove a dévissage theorem for Witt groups and discuss further applications. Details are provided in the preprint [4].

1. Transfer morphisms for Witt groups

By work of Levine-Morel, Panin and others, it is known that for orientable cohomology theories on schemes (that is those satisfying the projective bundle theorem, e. g. Chow groups, K-theory and algebraic cobordism) one has transfer morphisms (also called push-forwards or norm morphisms) satisfying certain standard properties. Witt groups are not orientable, so the construction of transfers has to be more complicated. We refer the reader to Balmer's papers for the definition and basic properties of triangular Witt groups. Recall [1], [2] that for any triangulated category with duality $(T, *, \varpi)$, he defines a $\mathbf{Z}$-graded Witt group $W^*(T, *)$ which is 4-periodic, and there is a canonical isomorphism between $W^0(D^b(Vect(X)), *)$ and the classical Witt group $W(X)$, where $* = Hom(\ \ , O_X)$ is the standard duality. (When replacing O_X by some line bundle L we will write $W^*(X, L)$.) These triangular Witt groups are contravariant. When one wants to establish covariant functoriality, it is (similar as for algebraic K-theory) necessary to work with coherent sheaves rather than only vector bundles. Coherent Witt groups have been introduced by Gille [5], and we work with coherent Witt groups from now on.

Coherent Witt groups are still covariantly functorial with respect to flat maps: Let $f : X \to Y$ be a flat morphism, let L (resp. M) be a line bundle over X (resp. Y) and let $\phi : f^*M \to L$ be an isomorphism of line bundles. This defines a pull-back $(f, \phi)^* : W^i(Y, M) \to W^i(X, L)$.

The construction of the transfer morphisms involves the theory of duality of Grothendieck-Verdier-Hartshorne-Deligne (see [7], [10]). In particular, there is a right adjoint $f^!$ to the functor $\mathrm{R}f_* : D^b(X) \to D^b(Y)$, which formally induces a morphism of functors $\alpha : \mathrm{R}f_* \mathbf{R}\underline{Hom}(\_, f^! L) \to \mathbf{R}\underline{Hom}(\mathrm{R}f_*(\_), L)$. This is used to establish the following.

Theorem 1. *The functor* $\mathrm{R}f_*$ *induces a morphism* $f_* : \mathrm{W}^i(X, f^! L) \to \mathrm{W}^i(Y, L)$.

Recall that a "functor" between two triangulated categories with duality (A, D_A, ϖ_A) and (B, D_B, ϖ_B) is in fact a pair (F, σ), where F is indeed a triangulated functor from A to B and σ is an isomorphism of functors from $F \circ D_A$ to $D_B \circ F$ satisfying certain properties with respect to ϖ_A and ϖ_B (see [5, Definition 2.6], for example). To prove the above theorem, we choose of course $F = Rf_*$ and $\sigma = \alpha$. Showing that α satisfies those properties is a long formal computation involving all kinds of adjunctions and compabilities between $(\ )^*, (\ )_*, (\ )^!$, the internal tensor product, the internal Hom restricted to suitable subcategories of $D(X)$, the triangulated structure and signs.

Let $f : X \to Y$ be a proper morphism of relative dimension d between schemes which are smooth equidimensional over a connected base scheme S, and let L be a line bundle on Y. Then there is a natural isomorphism $\beta : f^! L \xrightarrow{\simeq} f^* L \otimes \omega_X \otimes f^* \omega_Y^{-1}[d]$. Thus we also have transfer morphisms $f_* : W^{*+d}(X, f^* L \otimes_{\mathcal{O}_X} \omega_X) \to W^*(Y, L \otimes_{O_Y} \omega_Y)$. Notice that this forces us to consider twisted dualities when studying transfer morphisms. The transfer map respects the composition: $g_* \circ f_* = (g \circ f)_*$. We now state the base change and projection formulas and the dévissage theorem.

Theorem 2. *(base change) Assume that we have a cartesian square of schemes*

$$\begin{array}{ccc} V & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

where g and g' are flat and f and f' are proper. Let N be a line bundle on Z. Then we have a commutative square of Witt groups

$$\begin{array}{ccc} W^*(V, g'^* f^! N) & \xleftarrow{g'^*} & W^*(X, f^! N) \\ {\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ W^*(Y, g^* N) & \xleftarrow[g^*]{} & W^*(Z, N) \end{array}$$

We now use the (left) product $\mathrm{W}^i(X, L) \times \mathrm{W}^j(X, L') \to \mathrm{W}^{i+j}(X, L \otimes L')$ on the graded Witt ring as defined in [6]. Then for $f : X \to Y$ proper and flat, the usual projection formula $f_*(a.f^*(b)) = f_*(a).b$ holds for any a and b in $W^*(X)$ resp. $W^*(Y)$ with suitably twisted dualities (see [4, Theorem 2.16] for the precise statement).

2. Dévissage and further applications

Using the results of the previous section, we can establish the following.

Theorem 3. *(dévissage) Assume that $f : Z \to X$ is a closed embedding of codimension d between equidimensional varieties and L a line bundle on X. Then we have a dévissage isomorphism $f_* : W^*(Z, f^!L) \xrightarrow{\simeq} W^*_Z(X, L)$ and a localization long exact sequence*

$$... \to W^n(Z, f^!L) \xrightarrow{f_*} W^n(X, L) \xrightarrow{j^*} W^n(X - Z, j^*L) \xrightarrow{\partial} W^{n+1}(Z, f^!L) \to ...$$

Using transfers, base change and dévissage, it is possible to obtain certain computations for Witt groups of cellular varieties along the lines of the work of [8] for orientable cohomology theories.

Having transfers, base change and the projection formula, we may define the category of Witt correspondences over a fixed connected base scheme S. Its objects are couples (X, L) with X smooth and proper over S, and L a dualizing complex on X (which can be shown to be isomorphic to a shifted line bundle). The morphisms are defined by

$$Hom((X, L), (Y, M)) := \mathrm{W}^{d_X}(X \times Y, (\omega_X \otimes L^{-1}) \boxtimes M).$$

Taking the pseudo-abelian completion one obtains the category of (effective pure) Witt motives. Now consider (twisted) flag varieties, that is varieties of the form $X = G/P$ where P is a parabolic subgroup of a split reductive group G. Introducing the category of G-equivariant K-motives, Panin [9] establishes decompositions of X in the category of G-equivariant K-motives which follow from the knowledge of K_0 of the representation category of G and the equivalence of abelian categories with duality $Res : \mathrm{Vect}^G(G/P) \to \mathrm{Rep}(P)$ where $\mathrm{Rep}(P)$ denotes finite dimensional k-representations. For G a simply connected semisimple group, he shows that the K-groups of a projective homogeneous variety X under G decomposes as $K_*X \simeq K_*A$ where A is a finite dimensional semisimple F-algebra. His results are explicit, and in particular he recovers from them the computation of algebraic K-theory of quadrics and Brauer-Severi varieties. Several problems arise when trying to translate Panin's approach to Witt motives. The formula for Witt groups for representation categories are more complicated, and (unlike for K-theory) some decompositions in the category of Witt motives can be shown not to lift equivariantly. What we already did [3] is to compute the Witt groups of certain categories of finite-dimensional representations of an algebraic group G over k. For instance we prove the following result.

Theorem 4. *Let G be a split reductive group and let $(X^+)^o$ denote the set of dominant characters fixed under the duality. Then there is a morphism of $\mathrm{W}^0(F)$-algebras $\mathrm{W}^0(F) \otimes_{\mathbf{Z}} \mathbf{Z}[(X^+)^0] \to \mathrm{W}^*(\mathrm{Rep}(G))$ which is an isomorphism if G is semisimple and simply connected.*

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Gersten-Witt complexes for hermitian Witt groups of Azumaya algebras

Stefan Gille

Let X be a regular noetherian scheme and $\mathcal{A}$ an Azumaya algebra over X. Denote $\mathcal{M}_c(\mathcal{A})$ the abelian category of coherent $\mathcal{A}$-modules. The subcategory $M^p_{\mathcal{A}} \subseteq \mathcal{M}_c(\mathcal{A})$ consisting of $\mathcal{A}$-modules whose support has codimension $\geq p$ in X is a Serre subcategory of $\mathcal{M}_c(\mathcal{A})$ for all $p \geq 0$. These define a filtration $\mathcal{M}_c(\mathcal{A}) = M^0_{\mathcal{A}} \supseteq M^1_{\mathcal{A}} \supseteq M^2_{\mathcal{A}} \supseteq \ldots$ on $\mathcal{M}_c(\mathcal{A})$ and therefore by Quillen's [6] localization theorem give rise to a spectral sequence: $E_1^{p,q}(X) := K_{-p-q}(M^p_{\mathcal{A}}/M^{p+1}_{\mathcal{A}})$ which converges to the (Quillen-) K-theory of $\mathcal{A}$ if X has finite Krull dimension.

Using well known arguments one shows that the $(n+p+q)$-th line of this spectral sequence is isomorphic to the following complex which is called Gersten-complex of $\mathcal{A}$:

$$\bigoplus_{x\in X^{(0)}} K_n(\mathcal{A}(x)) \xrightarrow{d^0} \bigoplus_{x\in X^{(1)}} K_{n-1}(\mathcal{A}(x)) \xrightarrow{d^1} \bigoplus_{x\in X^{(2)}} K_{n-2}(\mathcal{A}(x)) \xrightarrow{d^2} \ldots$$

(here $X^{(i)} \subseteq X$ denotes the set of points of codimension i in X, and we have set $\mathcal{A}(x) := \mathcal{A} \otimes k(x)$, where $k(x)$ is the residue field of $x \in X$). This complex is exact and the kernel of the differential d^0 is naturally isomorphic to $K_n(\mathcal{A})$ if X is the spectrum of a local ring of a smooth variety over a field as shown by Colliot-Thélène and Ojanguren [2] (constant case, *i.e.* $\mathcal{A}$ is extended from the base field) and Panin and Suslin [7] (general case).

In [4] we have constructed such a complex for hermitian Witt groups of Azumaya algebras with involution of the first kind. Since our construction uses Balmer's [1] localization sequence we have to assume that $\frac{1}{2}$ is in the global sections of X.

We denote below $W^{\pm}(\mathcal{A},\tau)$ the (skew-)hermitian Witt group of the Azumaya algebra $\mathcal{A}$ with involution τ.

With this notation our main result reads as follows

Theorem. *Let X be a noetherian regular scheme of finite Krull dimension and $\mathcal{A}$ an Azumaya algebra with involution of the first kind τ over X. Then there is a spectral sequence $E_1^{p,q}(\mathcal{A},\tau)$ which converges to the derived hermitian Witt theory of $(\mathcal{A},\tau)$. The odd lines of the spectral sequence are zero, and the even lines $2i$ are isomorphic to the following complexes: The hermitian Gersten-Witt complex of $(\mathcal{A},\tau)$*

$$\bigoplus_{x\in X^{(0)}} W^+(\mathcal{A}(x),\tau(x)) \xrightarrow{d^0} \bigoplus_{x\in X^{(1)}} W^+(\mathcal{A}(x),\tau(x)) \xrightarrow{d^1} \ldots$$

$$\ldots \longrightarrow \bigoplus_{x\in X^{(i)}} W^+(\mathcal{A}(x),\tau(x)) \xrightarrow{d^i} \ldots$$

if i is even, and the skew-hermitian Gersten-Witt complex of $(\mathcal{A},\tau)$

$$\bigoplus_{x\in X^{(0)}} W^-(\mathcal{A}(x),\tau(x)) \xrightarrow{d^0} \bigoplus_{x\in X^{(1)}} W^-(\mathcal{A}(x),\tau(x)) \xrightarrow{d^1} \ldots$$

$$\ldots \longrightarrow \bigoplus_{x\in X^{(i)}} W^-(\mathcal{A}(x),\tau(x)) \xrightarrow{d^i} \ldots$$

is i is odd (here we have set $\tau(x) := \tau\otimes \mathrm{id}_{k(x)}$). These complexes are exact if X is the spectrum of a local ring of a smooth variety and the kernel of d^0 is naturally isomorphic to $W^+(\mathcal{A},\tau)$ in the hermitian case, i.e. for i even, and to $W^-(\mathcal{A},\tau)$ in the skew-hermitian case, i.e. for i odd.

We construct such hermitian Gersten-Witt complexes for any coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ with involution of the first kind τ if the scheme X has a dualizing complex $\mathcal{I}_\bullet$. This construction uses coherent Witt theory, which we introduce in [4], too. Let $D_c^b(\mathcal{M}(\mathcal{A}))$ be the bounded derived category of coherent $\mathcal{A}$-modules with coherent homology sheaves, and denote $\overline{\mathcal{F}}$ for a right $\mathcal{A}$-module $\mathcal{F}$ the same $\mathcal{O}_X$-module with left $\mathcal{A}$-structure $a\cdot x := x\tau(a)$. If τ is of the first kind

$$\mathcal{F}_\bullet \longmapsto \overline{\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_\bullet,\mathcal{I}_\bullet)}$$

is then a duality on $D_c^b(\mathcal{M}(\mathcal{A}))$ making this category a triangulated category with duality. The associated triangular Witt groups $\tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet)$ are called coherent hermitian Witt groups. Note that if $\mathcal{A}$ is not an Azumaya algebra these groups are even for X regular and $\mathcal{I}_\bullet$ an injective resolution of a line bundle of X not isomorphic to the derived Witt groups of $(\mathcal{A},\tau)$. However if $\mathcal{A}$ is an Azumaya algebra and X regular of finite Krull dimension there is always a dualizing complex $\mathcal{I}_\bullet$ of X, such that $\tilde{W}^i(\mathcal{A},\tau,\mathcal{I}_\bullet)$ is isomorphic to the i-th derived Witt group of $(\mathcal{A},\tau)$, and so in particular $\tilde{W}^0(\mathcal{A},\tau,\mathcal{I}_\bullet)\simeq W^+(\mathcal{A},\tau)$ and $\tilde{W}^2(\mathcal{A},\tau,\mathcal{I}_\bullet)\simeq W^-(\mathcal{A},\tau)$.

The codimension function of $\mathcal{I}_\bullet$ induces as for symmetric coherent Witt groups, see [3], a filtration on $D_c^b(\mathcal{M}(\mathcal{A}))$ from which we get as in K-theory using the localization sequence in triangular Witt theory a spectral sequence which converges to the coherent Witt groups of $(\mathcal{A}, \tau)$.

In a work in progress [5] we study the case that the involution τ of the Azumaya algebra $\mathcal{A}$ is of the second kind. This means that there is an automorphism σ of order two of X, and τ is a morphism of $\mathcal{O}_X$-algebras $\mathcal{A} \longrightarrow \sigma_*(\mathcal{A}^{op})$, where $\mathcal{A}^{op}$ denotes the opposite algebra of $\mathcal{A}$, such that $\sigma_*(\tau) \cdot \tau = \mathrm{id}_{\mathcal{A}}$. In this situation we get the same theorem as above except that we take direct sums only over the sets $X_\sigma^{(i)} := \{ x \in X^{(i)} \mid \sigma(x) = x \}$. The main difference in the construction of the spectral sequence is that we have in this case no coherent Witt theory available, and hence we do not get such complexes for Azumaya algebras over singular schemes as it is the case if τ is of the first kind.

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Tori and essential dimension

Mathieu Florence

(joint work with G. Favi)

In this talk, we are mainly concerned with upper bounds concerning the essential dimension of some algebraic groups (see [Re] for the definition of this notion; we will also briefly define it below). The best known upper bounds for many algebraic groups have been performed by considering group actions on certain *lattices*. This can be seen in the work of Ledet [JLY], Lemire [Lem], and the joint work by Lorenz, Reichstein, Rowen and Saltman [LRRS].

In this talk, we will develop a more geometrical and unified approach to these results using the language of tori.

In the following k will denote an arbitrary ground field.

We shall need the following general result (which is a particular case of Gabriel's theorem, see [DeGr] Exposé V, Théorème 8.1):

Theorem 1. *Let G/k be a linear algebraic group, acting* freely *on a k-variety X. Then there exists a G-invariant dense open subvariety U of X satisfying the following properties:*

i) There exists a quotient map $\pi : U \longrightarrow U/G$ in the category of schemes.

ii) π is onto, open and U/G is of finite type over k.

iii) $\pi : U \longrightarrow U/G$ is a G-torsor.

Definition 2. *Let G act on X. An open subscheme U which satisfies the conclusion of the above theorem will be called a* friendly *open subset of X.*

Let us now briefly recall the definition of a versal torsor.

Definition 3. *Let k be a field, G be a linear algebraic group over k and Y a k-scheme of finite type. A G-torsor $f : X \longrightarrow Y$ over Y is called* generic *for G (or* versal*, or* classifying*) if, for every extension k'/k, with k' infinite, and for every G-torsor $P' \longrightarrow \mathrm{Spec}(k')$, the set of points $y \in Y(k')$ such that $P' \simeq f^{-1}(y)$ is dense in Y.*

It is not hard to see that such a versal torsor always exists.

Definition 4. *Let G be a linear algebraic group over k. The smallest dimension $\dim(Y)$ of a generic G-torsor $X \longrightarrow Y$ is called the* essential dimension of G.

We are now able to state a proposition which gives a sufficient condition for a torsor to be versal.

Proposition 5. *Let k be a field and G be a linear algebraic group over k. Assume we are given a quasi-projective k-variety X, together with a generically free action of G on X. Suppose further that, for every extension k' of k with k' infinite, and for every G-torsor P over k', the twist of $X \times_k k'$ by P has a dense subset of k'-rational points. Let U be a friendly open subset of X for the action of G. Then the G-torsor $U \longrightarrow U/G$ is versal.*

We will now give a purely cohomological description of a versal torsor for $\mathbf{PGL}_n$, n odd. As a corollary, we recover a result due to Lorenz, Rowen, Reichstein and Saltman (see [LRRS]). To begin with, let us introduce some notations.

Let X be a finite set of cardinality n. Denote by $\mathbf{PGL}_X$ the group $\mathbf{PGL}(k^X)$. Let T_X be the diagonal maximal torus of $\mathbf{PGL}_X$ (with cocharacter module canonically isomorphic to $\mathbb{Z}^X/\mathbb{Z}$); its normalizer is the group $N_X = T_X \rtimes \mathfrak{S}_X$, where $\mathfrak{S}_X$ is the symmetric group of X. It is well-known that the map

$$H^1(K, N_X) \longrightarrow H^1(K, \mathbf{PGL}_X)$$

is surjective for any K. Thus, for finding a versal torsor for $\mathbf{PGL}_X$, it is enough to find one for N_X. Recall that we have the canonical Koszul complex (more precisely, its dual)

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^X \longrightarrow \bigwedge^2 \mathbb{Z}^X \longrightarrow \cdots \longrightarrow \bigwedge^n \mathbb{Z}^X \longrightarrow 0,$$

where the maps are just given by wedging (say, on the right) by $\sum_{x\in X} x$. In particular, for any action of a group G on X, this complex is G-equivariant. Let us cut the first part of this complex in two short exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^X \longrightarrow (T_X)_* \longrightarrow 0$$

and

$$0 \longrightarrow (T_X)_* \longrightarrow \bigwedge^2 \mathbb{Z}^X \longrightarrow Q_X \longrightarrow 0.$$

Let R_X be the k-torus with cocharacter module Q_X and let S_X be the k-torus with cocharacter module $\bigwedge^2 \mathbb{Z}^X$; i.e. $R_X = \mathrm{Spec}(k[Q_X])$ and $S_X = \mathrm{Spec}(k[\bigwedge^2 \mathbb{Z}^X])$. The last exact sequence gives a canonical sequence of k-tori

$$1 \longrightarrow T_X \longrightarrow S_X \longrightarrow R_X \longrightarrow 1.$$

Theorem 6. *Assume $n \geq 5$ is odd. Then, the natural action of N_X on S_X is generically free, and gives rise to a versal torsor for N_X.*

Corollary 7 (see [LRRS], Theorem 1.1)**.** *Assume $n \geq 5$ is odd. Then,*

$$\mathrm{ed}_k(\mathbf{PGL}_n) \leq \frac{(n-1)(n-2)}{2}.$$

Using a similar procedure, we can also give a geometric proof of a result originally due to Ledet (see [Led]). Note that our proof works also for finite fields. The proof of the case $r = 1$ of the theorem was communicated to us by Serre.

Theorem 8. *Let k be a field, $p > 2$ a prime number and r a positive integer. Assume p is not the characteristic of k. Let l/k be the field generated by p^r-th roots of unity, and G its Galois group, of order $t = p^d q$, where q divides $p-1$. We then have*

$$\mathrm{ed}_k(\mathbb{Z}/p^r\mathbb{Z} \rtimes G) \leq \varphi(q)p^d.$$

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Das elementare Hindernis und homogene Räume

Jean-Louis Colliot-Thélène

Es wurde über die jüngste Arbeit [3] (mit M. Borovoi und A. N. Skorobogatov) berichtet.

Sei k ein Körper der Charakteristik Null, $\overline{k}$ ein algebraischer Abschluß von k und $\mathfrak{g}$ die Galoisgruppe von $\overline{k}$ über k. Sei X eine glatte, absolut-irreduzible algebraische Varietät über dem Körper k, sei $k(X)$ der Funktionenkörper von X und $\overline{k}(X)$ der Funktionenkörper von $\overline{X} = X \times_k \overline{k}$.

Die exakte Folge von diskreten stetigen $\mathfrak{g}$-Moduln

$$1 \to \overline{k}^{\times} \to \overline{k}(X)^{\times} \to \overline{k}(X)^{\times}/\overline{k}^{\times} \to 1$$

definiert ein Element $ob(X) \in \mathrm{Ext}^1_{\mathfrak{g}}(\overline{k}(X)^{\times}/\overline{k}^{\times}, \overline{k}^{\times})$.

Lemma 1. [5] *Wenn X einen k-rationalen Punkt besitzt, dann ist $ob(X) = 0$.*

Die Klasse $ob(X)$ heißt das elementare Hindernis (zum Bestehen eines k-rationalen Punktes auf X).

Satz 2. *Sei k ein $\mathfrak{p}$-adischer Körper. Es ist $ob(X) = 0$ dann und nur dann, wenn die Abbildung von Brauergruppen $\mathrm{Br}k \to \mathrm{Br}k(X)$ injektiv ist.*

Satz 3. *Sei k ein Zahlkörper. Wenn X rationale Punkte in allen Komplettierungen k_v von k besitzt, und $ob(X) = 0$, dann liefert die Untergruppe der Elemente von $\mathrm{Br}X$, die überall lokal konstant sind, kein Brauer–Maninsche Hindernis zum Hasseschen Prinzip für X.*

Hier heißt ein Element $\alpha \in \mathrm{Br}X$ überall lokal konstant, wenn für jede Stelle v von k die Beschränkung von α in $\mathrm{Br}(X \times_k k_v)$ liegt im Bild von $\mathrm{Br}k_v$.

Von jetzt an, sei G eine zusammenhängende algebraische Gruppe über k und *X/k ein homogener Raum von G.* Die Gruppe G ist eine Erweiterung einer abelschen Varietät A durch eine lineare Gruppe L. Es sei ferner angenommen, daß *die geometrischen Isotropiegruppen von G* (bei ihrer Aktion über X) *zusammenhängend sind.* Was die folgenden Sätze betrifft, dies ist keine leichtsinnige Annahme, wie ein Beispiel von Florence [6] zeigt.

Satz 4. *Sei k ein $\mathfrak{p}$-adischer Körper. Ist $ob(X) = 0$, d.h. (Satz 2) ist $\mathrm{Br}k \to \mathrm{Br}k(X)$ injektiv, dann besitzt der homogene Raum X einen k-rationalen Punkt.*

Satz 5. *Sei k ein Funktionenkörper in zwei Variablen über* **C**. *Sei $G = L$ linear und ohne E_8-Komponente. Ist $ob(X) = 0$, dann besitzt der homogene Raum X einen k-rationalen Punkt.*

Beim Beweis wird ein Satz von de Jong [8] benutzt.

Satz 6. *Sei k ein Zahlkörper. Sei $G = L$ linear. Wenn der homogene Raum X rationale Punkte in allen reellen Komplettierungen von k besitzt, und $ob(X) = 0$, dann besitzt X einen k-rationalen Punkt.*

Beim Beweis wird ein Satz von Harari und Szamuely [7] benutzt.

Satz 7. *Sei k ein nichtformallreeller Zahlkörper. Sei G eine Erweiterung einer abelschen Varietät A durch eine lineare Gruppe L. Nehmen wir an, daß dieTate-Shafarevich Gruppe von A endlich ist. Ist $ob(X) = 0$, dann besitzt der homogene Raum X einen k-rationalen Punkt.*

Das folgende Beispiel sollte man den Sätzen 3, 6 und 7 entgegensetzen. Die Gruppe G, die hier vorkommt, ist eine Erweiterung einer elliptischen Kurve durch die spezielle Gruppe $SL(D)$, wobei D die Quaternionenalgebra von Hamilton bezeichnet.

Beispiel 8. *Es gibt eine Gruppe G über $\mathbf{Q}$ und einen prinzipalhomogenen Raum X von G mit den Eigenschaften : die Varietät X besitzt rationale Punkte in allen Komplettierungen $\mathbf{Q}_v$ von $\mathbf{Q}$, es ist $ob(X) = 0$, trotzdem besitzt X keinen $\mathbf{Q}$-rationalen Punkt.*

Letztere Tatsache wird durch ein Brauer-Maninsche Hindernis erklärt.

Diese Sätze stellen eine Verallgemeinerung von wohlbekannten Sätzen in der Theorie der abelschen Varietäten (Cassels, Tate, Lichtenbaum, Manin [10, 9, 7]) und in der Theorie der linearen algebraischen Gruppen (Kneser, Sansuc, Borovoi) dar. Von diesen klassischen Sätzen und von den Methoden von [1, 2, 4] wird in den Beweisen Gebrauch gemacht.

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Multiples of trace forms of G–Galois algebras

Eva Bayer–Fluckiger

Let k be a field of characteristic $\neq 2$, and let G be a finite group. Let L be a G–Galois algebra over k, and let

$$q_L : L \times L \to k, \quad q_L(x, y) = \mathrm{Tr}_{L/k}(xy),$$

be its trace form. Note that q_L is a G*–quadratic form*: we have $q_L(gx, gy) = q_L(x, y)$. The following question was studied in [1], [2], [3], [5], [7], [10]:

Question. *Let L and L' be two G–Galois algebras. When are the G–forms q_L and $q_{L'}$ isomorphic?*

This question is settled in some cases, for instance when G has odd order [3] or when the 2–Sylow subgroups of G are elementary abelian [7], but it is open in general. In [1], the following weaker problem was raised:

Question. *Let L and L' be two G–Galois algebras, and let ϕ be a quadratic form over k. When are the G–forms $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ isomorphic?*

Let us denote by $W(k)$ the Witt ring of k, and let I be its fundamental ideal. Let us suppose that $\mathrm{cd}_2(k) \leq d$. Then we have

Theorem. (Chabloz, [9]) *Let L and L' be two G–Galois algebras, and let $\phi \in I^d$. Then the G–forms $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ are isomorphic.*

Let k_s be a separable closure of k, and set $\Gamma_k = \mathrm{Gal}(k_s/k)$. Let us set $H^n(k) = H^n(\Gamma_k, \mathbf{Z}/2\mathbf{Z})$. Let

$$e_n : I^n/I^{n+1} \to H^n(k)$$

be the Milnor isomorphisms.

Let L be a G–Galois algebra, and let $f_L : \Gamma_k \to G$ be a continuous homomorphism corresponding to L. Then f_L induces a homomorphism $f_L^* : H^1(G, \mathbf{Z}/2\mathbf{Z}) \to H^1(k)$. For any $x \in H^1(G, \mathbf{Z}/2\mathbf{Z})$, set $x_L = f_L^*(x)$. The following is proved in [7]:

Lemma. *Let L and L' be two G–Galois algebras. Suppose that the G–forms q_L and $q_{L'}$ are isomorphic. Then $x_L = x_{L'}$.*

The aim of the talk was to present a sketch of the proof of the following result:

Theorem. *Let L and L' be two G–Galois algebras, and let $\phi \in I^{d-1}$. Then the G–forms $\phi \otimes q_L$ and $\phi \otimes q_{L'}$ are isomorphic if and only if $e_{d-1}(\phi) \cup x_L = e_{d-1}(\phi) \cup x_{L'}$.*

Special cases of this result were proved in [4], [5], [6], and [9]. The proof uses some results about algebras with involution as well as theorems of Parimala, Sridharan and Suresh [12] and of Berhuy [8].

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Is the Luna stratification intrinsic?

Zinovy Reichstein

(joint work with J. Kuttler)

Let k be an algebraically closed field of characteristic zero, $G \to \mathrm{GL}(V)$ be a representation of a reductive linear algebraic group G on a finite-dimensional vector space V, defined over k, and $\pi\colon V \to X = V /\!/ G$ be the categorical quotient map for the G-action on V. For the definition and a discussion of the properties of the categorical quotient in this setting, see, e.g., [MF], [PV] or [K].

There is a natural stratification on X, due to D. Luna; I shall refer to it as the Luna stratification. Recall that for every $p \in X$ the fiber $\pi^{-1}(p)$ has a unique closed orbit. Choose a point v_p in this orbit. Then the stabilizer subgroup $\mathrm{Stab}(v_p)$, is reductive, and its conjugacy class in G is independent of the choice of v_p. This subgroup determines the stratum of p. More precisely, the Luna stratum associated to the conjugacy class (H) of a reductive subgroup $H \subseteq G$ is defined as

$$X^{(H)} = \{p \in X \mid \mathrm{Stab}(v_p) \in (H)\}\,.$$

There are only finitely many Luna strata, and each stratum is a locally closed non-singular subvariety of X. Moreover, if we set

$$V^{\langle H\rangle} = \{v \in V \mid \ Gv \text{ is closed and } \mathrm{Stab}(v) = H\}$$

then the natural map $V^{\langle H \rangle} \to X^{(H)}$ is a principal $N_G(H)/H$-bundle. For proofs of these assertions, see [PV, Section 6.9] or [S, Section I.5].

The Luna stratification provides a systematic approach to the problem of describing the G-orbits in V; it also plays an important role in the study of the geometry and topology of the categorical quotient $X = V /\!/ G$. It is thus natural to ask the following questions.

(i) Is the Luna stratification of X intrinsic? In other words, is it true that for every automorphism $\sigma \colon X \to X$ and every reductive subgroup $H \subset G$ there is a reductive subgroup $H' \subset G$ such that $\sigma(X^{(H)}) = X^{(H')}$?

(ii) Are the Luna strata in X intrinsic? Here we say that $X^{(H)}$ is intrinsic if $\sigma(X^{(H)}) = X^{(H)}$ for every automorphism $\sigma \colon X \to X$.

In general, the Luna stratification is not intrinsic. Indeed, there are many examples, where $V /\!/ G$ is an affine space (cf. e.g., [PV, Section 8]) and the automorphism group of an affine space is highly transitive (cf. e.g., [Re$_1$, Theorem 3.1]), so that two distinct points in the same stratum can be taken, by an automorphism, to points in a different strata. Moreover, even in those cases where the Luna stratification is intrinsic, the individual strata may not be intrinsic.

In this talk, based on joint work with J. Kuttler, I will present three results that show that one can nevertheless give positive answers to (i) and (ii) for large classes of interesting representations.

The first result concerns finite groups. Recall that an element $g \in \mathrm{GL}(V)$ is called a pseudo-reflection if g has finite order and fixes (pointwise) a hyperplane in V. In other words, g is a pseudo-reflection, if its eigenvalues are $1, 1, \dots, 1$ ($\dim(V) - 1$ times) and ζ, where ζ is a root of unity. If G is generated by pseudo-reflections then, by a theorem of Chevalley and Shaphard-Todd, $V /\!/ G$ is an affine space. As I remarked above, in this case the Luna stratification cannot be intrinsic. The following theorem may be viewed as a partial converse.

Theorem 1. *Let $G \to \mathrm{GL}(V)$ be a linear representation of a finite group G. Suppose no non-identity element of G acts on V as a pseudo-reflection. Then for every automorphism σ of $X = V /\!/ G$ there is an automorphism τ of G such that $\sigma(X^{(H)}) = X^{(H^\tau)}$ for every reductive subgroup H of G. In particular, the Luna stratification in X is intrinsic.*

Note that under the assumptions of Theorem 1 the individual Luna strata are intrinsic in many cases (but not always). We also note that Theorem 1 may be viewed as an algebraic analogue of a result of Prill [Pri]. In the course of the proof we show that every automorphism of X can be lifted to an automorphism of V.

Next we turn to representations V of (possibly infinite) reductive groups G. Our main result addressing question (i) is the following theorem.

Theorem 2. *Let $G \to \mathrm{GL}(W)$ be a finite-dimensional linear representation of a reductive algebraic group G defined over k. Then the Luna stratification in $W^r /\!/ G$ is intrinsic if*

(a) $r \geq 2\dim(W)$,

(b) G preserves a nondegenerate quadratic form on W and $r \geq \dim(W) + 1$,

(c) $W = \mathfrak{g}$ is the adjoint representation of G and $r \geq 3$.

In this case we do not know under what circumstances an automorphism $\sigma \colon V /\!/ H \to V /\!/ G$ can be lifted to V, so our proof is indirect. Along the way we show that if $V = W^r$ is as in Theorem 2 and S is a Luna stratum in $V /\!/ G$ then $\overline{S}$ is singular at every point $x \in \overline{S} \setminus S$.

Our final result concerns the natural action of $G = \mathrm{GL}_n$ on $V = \mathrm{M}_n^r$ by simultaneous conjugation. The variety $X = \mathrm{M}_n^r /\!/ \mathrm{GL}_n$ has been extensively studied in the context of both invariant and PI theories; an overview of this research area can be found in [F], [Pro] or [DF]. In [Re$_1$, Re$_2$] I constructed a large family of automorphisms of $X = V^r /\!/ \mathrm{PGL}_n$, preserving the Luna strata. It is thus natural to ask whether or not every automorphism of X has this property.

Theorem 3. *Let $r \geq 3$. Then every Luna stratum in $\mathrm{M}_n^r /\!/ \mathrm{GL}_n$ is intrinsic.*

The fact that the principal stratum $X^{(\{e\})}$ is intrinsic is an immediate consequence of a theorem of Le Bruyn and Procesi [lBP, Theorem II.3.4], which says that $X^{(\{e\})}$ is precisely the smooth locus of X. This result served both as a motivation and as a starting point for our proof of Theorem 3.

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Graded Hermitian Forms and Springer's Theorem

Adrian R. Wadsworth
(joint work with J.-F. Renard and J.-P. Tignol)

We provide a new approach to the generalization of Springer's Theorem to Hermitian forms over a valued division algebra finite dimensional over its center with involution. To every such division algebra D with valuation w there is a value group Γ_D and a residue division algebra $\overline{D}$, and an associated graded division ring $gr(D)$ in which all homogeneous elements are invertible. If σ is an involution on D which is compatible with w (i.e., $w \circ \sigma = w$) there is an induced grade-preserving involution σ' on $gr(D)$. We give a theory of even graded Hermitian forms on graded vector spaces over $gr(D)$, which is analogous to the theory of even Hermitian forms over D, and an associated Witt group W_g of these graded Hermitian forms. This W_g has a canonical direct sum decomposition determined by the grade group $\Gamma_{gr(D)}$ of $gr(D)$. (Note that $\Gamma_{gr(D)} = \Gamma_D$.) The summands correspond noncanonically to Witt groups of $\overline{D}$ with respect to various involutions. By considering value functions on vector spaces over D that are compatible with Hermitian forms on the vector spaces, we obtain a canonical homomorphism from the Witt group W of even Hermitian forms over D to W_g. When the valuation on the center of D is Henselian, we show that this map $W \to W_g$ is an isomorphism. These results are valid whenever $char(\overline{D}) \neq 2$ (in which case all Hermitian forms are even), but also in many cases when $char(\overline{D}) = 2$, though only for forms where the isometry group is of symplectic or unitary type, and often with a tameness assumption on the valuation w.

A preprint of this work is available at the Linear Algebraic Groups and Related Structures preprint server: `http://www.mathematik.uni-bielefeld.de/LAG/`

Canonical dimension of spinor groups

Nikita A. Karpenko

Let G be a split simple algebraic group over a field F and let T be a G-torsor. A splitting field of T is called generic, if it has an F-place to any other splitting field of T. Canonical dimension $\operatorname{cd} T$ of T is the minimum of transcendence degree of all generic splitting fields of T. Canonical dimension $\operatorname{cd} G$, as introduced in [1], is the maximum of $\operatorname{cd} T$ for all G_K-torsors T, where K runs over all fields containing F.

There are not so many G for which the value of the invariant $\operatorname{cd} G$ is determined. However, fixing a prime p and neglecting the finite field extensions of degree coprime with p, one comes to the canonical p-dimension $\operatorname{cd}_p G$ which is now computed for any G (and any p), [4, 6].

Assume that the group G has only one torsion prime p. One knows that $\operatorname{cd} G = \operatorname{cd}_p G$ for any such G with the exception of the spinor and half-spinor groups, [2]. One hopes that $\operatorname{cd} G = \operatorname{cd}_p G$ also for the excepted G (it is not clear what should be the hope in the case of multiple torsion primes and there is not a single computation of $\operatorname{cd} G$ in that case yet).

For any even n, canonical dimensions $\operatorname{cd}\operatorname{Spin}_n$ and $\operatorname{cd}\operatorname{Spin}_{n-1}$ coincide and can be interpreted as the minimal transcendence degree of a generic splitting field of a generic n-dimensional quadratic form with trivial discriminant and Clifford invariant. The equality

$$\operatorname{cd}\operatorname{Spin}_n = \operatorname{cd}_2\operatorname{Spin}_n$$

is proved for all n up to 16, [3], and also for all n such that n or $n+1$ is a power of 2, [2]. The case $n = 10$ follows from Pfister's theorem on 10-dimensional quadratic forms in I^3, [5]. The key for $n = 12$ and $n = 14$ is the following inequality which we prove for any even n, [3]:

$$\operatorname{cd}\operatorname{Spin}_n < \operatorname{cd}\operatorname{Spin}_{n-2} + \frac{n}{2} \ .$$

For a half-spinor group $\operatorname{Spin}_n^{\sim}$ (where n is divisible by 4), a similar method produces the inequality

$$\operatorname{cd}\operatorname{Spin}_n^{\sim} < \operatorname{cd}\operatorname{Spin}_{n-2} + \frac{n}{2} + 2^{k-1} - 1$$

(k is the dyadic order of n) which shows that

$$\operatorname{cd}\operatorname{Spin}_n^{\sim} = \operatorname{cd}_2\operatorname{Spin}_n^{\sim} \quad \text{as far as} \quad \operatorname{cd}\operatorname{Spin}_{n-2} = \operatorname{cd}_2\operatorname{Spin}_{n-2} \ .$$

Example: $\operatorname{cd}\operatorname{Spin}_{16} = \operatorname{cd}_2\operatorname{Spin}_{16} = \operatorname{cd}_2\operatorname{Spin}_{17} = 21$, and the only information on $\operatorname{cd}\operatorname{Spin}_{17}$ we dispose is

$$21 \leq \operatorname{cd}\operatorname{Spin}_{17} \leq 28 \ .$$

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Relative Brauer groups and index reduction for genus 1 curves

Daniel Krashen
(joint work with M. Ciperiani and M. Lieblich)

The main question that this project aims to answer is the following: Let k be a field, and let C be a genus 1 curve over k. If α is an element of the Brauer group $Br(k)$, how do we compute the index of $\alpha_{k(C)}$? In particular, how do we tell when $\alpha_{k(C)}$ is split? To set up language for the second question, define the relative Brauer group $Br(C/k) = ker(Br(k) \to Br(k(C))$. The second question asks us how we compute this group.

Although there are good answers to both of these questions for various homegenous and toric varieties (see [MPW96, MPW98, Sal93, MP97]), there are no general answers known for varieties which are not geometrically rational. One may therefore think of the genus 1 case as the first nontrivial place to look for a new kind of example of this type of computation. This has been carried out in certain specific examples. The relative Brauer group is computed for certain hyperelliptic curves by Ilseop Han [Han], with explicit generators described for certain hyperelliptic curves defined over the rational numbers. Also, index reduction results have been obtained for certain hyperelliptic curves defined over a p-adic field by V. I. Yanchevskiĭ [Yan97].

In the first part of this project, joint with Mirela Ciperiani, we give an explicit presentation for the relative Brauer group of a genus 1 curve C in terms of the coordinates of the generators of the k-points on an elliptic curve $\mathcal{E}$ (the Jacobian of C). This is done be considering an exact sequence:

$$0 \to Pic_0(C) \to \mathcal{E}(k) \to Br(C/k) \xrightarrow{\mathfrak{a}} \frac{per(C)\mathbb{Z}}{ind(C)\mathbb{Z}} \to 0,$$

giving an explicit formula for the morphism $\mathfrak{a}$ and of the cokernel. In the case that the original curve has a nice property, which we call "cyclicity," this formula may be significantly simplified so that one may interpret the elements of the relative Brauer group as all arising as cup products in a suitable sense.

In the second part of this project, which is work in progress with Max Lieblich, we expect to reduce the computation of the index to computations of relative Brauer groups after finite extensions of the ground field. This could be interpreted also in the following way: Given a central simple algebra A/k, naively we may express the index of $A_{k(C)}$ as the minimal degree of an extension $E/k(C)$ which splits the algebra A i.e.

$$ind(A_{k(C)}) = min\{[E : k(C)] \mid A_E \text{ is split}\}.$$

Instead, we expect that the following stronger formula will hold:

$$ind(A_{k(C)}) = min\{[L : k] \mid A_{L(C)} \text{ is split}\},$$

or in other words, for index computations it suffices to only consider those field extensions which come from the ground field. In another perspective, this says that the computation of the index may be reduced to computation of when the class A_L is split by the curve C_L when L/k is a finite extension. In the case that the relative Brauer group is easy to compute, this gives very precise formulas for the index. In particular, the work of Roquette and Lichtenbaum tells us precisely how to calculate the relative Brauer group in the case that k is p-adic, and in this case we have in exact expression for an index reduction formula based only on certain invariants of the curve C. In the general case, this formula is a bit complicated, but we reproduce here an example of the formula when the curve has index p a prime:

Example of a consequence of the formula above: Suppose k is a local field, C/k is a genus one curve of index p, and A/k is a central simple algebra with $ind(A) = p^n$. Define

$$cap(C) = max\{m | \exists L/k, [L : k] = dp^m, p \not| d \text{ and } C(L) = \emptyset\}.$$

Then we have

$$ind(A_{k(C)}) = \begin{cases} 2^n & \text{if } m < n-1 \\ 2^{n-1} & \text{if } m \geq n-1 \end{cases}$$

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Purity for simple groups of the type G_2

Ivan Panin

(joint work with V. Chernousov)

Let $\mathcal{F}$ be a covariant functor from the category of commutative rings to the category of sets. We say that $\mathcal{F}$ *satisfies purity for R* if

$$\bigcap_{\mathrm{ht}\mathfrak{p}=1} Im[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)] = Im[\mathcal{F}(R) \to \mathcal{F}(K)].$$

Elements $\xi \in \mathcal{F}(K)$ from the left hand side of the relation are called *R-unramified.* If $\mathcal{F}'$ is one more covariant functor from the category of commutative rings to the category of sets and $f : \mathcal{F} \to \mathcal{F}'$ is a functor transformation then f takes R-unramified elements to R-unramified ones.

Let k be a characteristic zero field. By a linear algebraic k-group we mean in the text a reduced affine k-group scheme. In particular, a linear algebraic k-group is always k-smooth. The main result of this paper is the following purity theorem:

Theorem 1. *Let G be a simple split algebraic group of the type G_2. The functor $R \mapsto H^1_{\acute{e}t}(R, G)$ satisfies purity for regular local rings containing k.*

Remark 2. *This theorem gives the positive answer in the case of simple group of the type G_2 on a question raised by Colliot-Thélène and Sansuc in* [C-T/S] *(see Question 6.4 p. 124).*

A split simple algebraic group of the type G_2 is the automorphism group of the octonion algebra. So the theorem above can be easily reduced to the following result.

Theorem 3. *Let R be a regular local ring of the form $\mathcal{O}_{X,x}$ where X is a k-smooth affine variety. Let K be the quotient field of R. Let $\phi = \langle\langle a, b, c\rangle\rangle$ be a 3-fold Pfister form over K which is R-unramified as just a quadratic form. Then there are units $a', b', c' \in R^\times$ such that*

$$\langle\langle a', b', c'\rangle\rangle \otimes_R K \cong \phi.$$

The proof of the last Theorem uses results of [Oj] and [P].

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On some invariants of fields of characteristic $p > 0$

Ricardo Baeza

(joint work with J. Kr. Arason and R. Aravire)

Let K be a field of characteristic $p > 0$. We denote by Ω_K^n the K-vector space of absolute (i.e., over K^p) n-differential forms defined over K (see [C], [Gr]). Let $d : \Omega_K^n \longrightarrow \Omega_K^{n+1}$ be the exterior differential operator which extends the derivation $d : K \longrightarrow \Omega_K^1$. Let $\wedge : \Omega_K^m \times \Omega_K^n \longrightarrow \Omega_K^{m+n}$ be the exterior product. The usual Artin-Schreier operator $\wp : K \longrightarrow K$, $\wp(a) = a^p - a$, can be extended (see [C], [Ka 1], [Ka 2]) to the general Artin-Schreier map

$$\wp : \Omega_K^n \longrightarrow \Omega_K^n / \mathrm{d}\,\Omega_K^{n-1} \tag{1.1}$$

which is defined on generators by

$$\wp\left(x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right) = (x^p - x)\,\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \mod d\Omega_K^{n-1}.$$

We denote by $\nu_K(n)$ the kernel of $\wp$ and by $H_p^{n+1}(K)$ the cokernel of $\wp$. In [I] the groups $H_p^{n+1}(K)$ have been interpreted as the p-part of the Galois cohomology of K. A well known result ([Ka 2], [B-Ka]) asserts that $\nu_K(n)$ is additively generated by the logarithmic differential forms $\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$, $a_i \in K^*$. Obviously $H_p^{n+1}(K)$ is additively generated by the elements $\overline{a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}}$, $a, a_1, \ldots, a_n \in K^*$. Let

$\mathcal{B} = \{a_i, i \in I\}$ be a p-basis of K, i.e., the elements $\prod_{i\in I} a_i^{\varepsilon_i}$, $\varepsilon_i \in \{0, 1, \ldots, p-1\}$ and $\varepsilon_i = 0$ for almost all $i \in I$, are a basis of K over K^p (see [J], [C], [Gr]). Let us fix some ordering on I and define the set

$$\sum_n = \{\sigma : \{1, \ldots, n\} \longrightarrow I \,|\sigma(i) < \sigma(j) \text{ whenever } i < j\}$$

equipped with the lexicographic ordering. Then the set $\{\frac{da_\sigma}{a_\sigma}, \sigma \in \sum_n\}$ is a K-basis of Ω_K^n, where we set $\frac{da_\sigma}{a_\sigma} = \frac{da_{\sigma(1)}}{a_{\sigma(1)}} \wedge \cdots \wedge \frac{da_{\sigma(n)}}{a_{\sigma(n)}}$ for any $\sigma \in \sum_n$. For any $\alpha \in \sum_n$ we denote by $\Omega_{K,\alpha}^n$ the K-subspace of Ω_K^n generated by the elements $\frac{da_\beta}{a_\beta}$ with $\beta \leq \alpha$ and by $\Omega_{K,<\alpha}^n$ the K-subspace generated by the elements $\frac{da_\beta}{a_\beta}$ with $\beta < \alpha$. Thus we obtain a filtration $\{\Omega_{K,\alpha}^n, \Omega_{K,<\alpha}^n, \alpha \in \sum_n\}$ of Ω_K^n (see [Ka 1]). Fixing a p-basis $\mathcal{B}$ we can define a $\mathcal{B}$-dependent Artin-Schreier homomorphism

$$\wp : \Omega_K^n \longrightarrow \Omega_K^n \tag{1.2}$$

by

$$\wp\left(\sum_{\sigma\in\Sigma_n} c_\sigma \frac{da_\sigma}{a_\sigma}\right) = \sum_{\sigma\in\Sigma_n} (c_\sigma^p - c_\sigma) \frac{da_\sigma}{a_\sigma}$$

This homomorphism is only modulo $d\Omega_K^{n-1}$ independent of the choice of the p-basis $\mathcal{B}$, so that it makes sense to introduce the subgroup $\wp\Omega_K^n + d\Omega_K^{n-1}$ of Ω_K^n, and we get

$$H_p^{n+1}(K) = \Omega_K^n / \left(\wp\Omega_K^n + d\Omega_K^{n-1}\right).$$

The main goal of this work is to study the behavior under finite field extensions of the following invariant of a field K of characteristic p.

Definition. The ν_p-invariant of a field K of characteristic $p > 0$ is

$$\nu_p(K) = \min\{n \mid H_p^{n+1}(K) = 0\}.$$

If this minimum does not exist, we set $\nu_p(K) = +\infty$.

$\nu_p(K)$ is a sort of p-cohomological dimension for K, according to the cohomological interpretation of the groups $H_p^{n+1}(K)$, cf. [I]. If $p = 2$, this invariant was studied in [A-Ba 1], where it is shown that for any finite extension L/K of a field K with $2 = 0$, it holds

$$\nu_2(K) \leq \nu_2(L) \leq \nu_2(K) + 1.$$

In fact $\nu_2(K)$ coincides with the usual ν-invariant of a field which is defined in terms of quadratic forms, i.e.

$$\nu_2(K) = \nu(K) := \min\{n \mid I^n W_q(K) = 0\},$$

where $W_q(K)$ denotes the Witt group of quadratic forms over K and I is the maximal ideal of even dimensional non singular symmetric bilinear forms of the Witt ring $W(K)$ (see loc. cit.). The equality $\nu_2(K) = \nu(K)$ follows from Kato's isomorphism

$$H_2^{n+1}(K) \cong I^n W_q(K)/I^{n+1} W_q(K)$$

(see [Ka 1]) and the analogue of the Arason-Pfister main theorem for fields of characteristic 2 (see [Ba]). The proof of $\nu(K) \leq \nu(L) \leq \nu(K)+1$ uses quadratic forms techniques. This result has been the motivation (for us) to introduce the invariant $\nu_p(K)$ of a field K of characteristic $p>0$. The main result in this work is now

Theorem. *Let K be a field of characteristic $p>0$. Let L/K be a finite extension. Then*

$$\nu_p(K) \leq \nu_p(L) \leq \nu_p(K)+1.$$

Moreover, if L/K is purely inseparable, then $\nu_p(K)=\nu_p(L)$.

Since any finite extension L/K contains a subfield E, $K \subset E \subset L$, such that E/K is separable and L/E is purely inseparable, it is clear that to prove the theorem we may assume separately that L/K is separable and purely inseparable.To this end, we use the notion of trace maps $s_*:\Omega_L^n \longrightarrow \Omega_K^n$ associated to any trace map $s: L \longrightarrow K$, (see [C-T], [Al-K]) and show that for the usual trace map $\mathrm{Tr}: L \longrightarrow K$,

$$\mathrm{Tr}_*\left(\wp\Omega_L^n + d\Omega_L^{n-1}\right) \subset \wp\Omega_K^n + d\Omega_K^{n-1},$$

so that Tr_* induces a homomorphism $\mathrm{Tr}_*: H_p^{n+1}(L) \longrightarrow H_p^{n+1}(K)$. The proof of the theorem for a separable extension L/K relies on the existence of this map. Let us now consider the case of a purely inseparable extension. In this case the main tool used in the proof of our result is a characterization of the highest coefficient of a differential form (with respect to a given ordering of a p-basis of K and the corresponding ordered K-basis of Ω_K^n) contained in the group $\wp\Omega_K^n + d\Omega_K^{n-1}$ (see [A-Ba 2] for the case $p=2$). This technical result is based on a weak version of Kato's lemma 2 in [Ka 2]. Kato's original version of this lemma assumes that the field K has the property $K=K^{p-1}$, whereas our weak version holds for any field K of characteristic $p>0$.

Finally let us remark that one can also consider the groups $H_{p^m}^{n+1}(K)$, $m \geq 1$ an integer, introduced in [Ka 2] (see also [I]), to define the invariants

$$\nu_{p,m}(K) = \min\{n \mid H_{p^m}^{n+1}(K)=0\}.$$

We show that these invariants are all equal to $\nu_p(K)$.

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On the splitting of bilinear forms in characteristic two

Ahmed Laghribi

Let F be a field of characteristic 2. Our aim is to describe some recent results on the splitting theory of bilinear forms in characteristic 2 [6]. The expression "bilinear form" means "regular symmetric bilinear form of finite dimension". To a bilinear form B with underlying vector space V, we associate a quadratic form $(V, \widetilde{B})$ given by: $\widetilde{B}(v) = B(v, v)$ for $v \in V$. This quadratic form is unique up to isometry. The dimension of a bilinear (or quadratic) form B is denoted by $\dim B$. A metabolic plane is a 2-dimensional bilinear form given by the symmetric matrix $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ for some $a \in F$. We denote this form by $M(a)$. A bilinear form is called isotropic (*resp.* metabolic) if it contains a metabolic plane as a subform (*resp.* it is isometric to an orthogonal sum of metabolic planes). A bilinear form is called anisotropic if it is not isotropic. For $a_1, \cdots, a_n \in F^*$, let $\langle a_1, \cdots, a_n \rangle$ denote the diagonal bilinear form $\sum_{i=1}^n a_i x_i y_i$. The function field of a bilinear form B, denoted by $F(B)$, is by definition the function field of the quadratic form $\widetilde{B}$.

1. Witt decomposition. Before we discuss the standard splitting of bilinear forms, we give a refined version of the Witt decomposition for bilinear forms [6]: For any nonzero bilinear form B, there exist nonnegative integers r, s, t, and $a_1, \cdots, a_r, b_1, \cdots, b_s \in F^*$ such that:

$$\begin{cases} B \simeq \langle a_1, \cdots, a_r \rangle \perp M(b_1) \perp \cdots \perp M(b_s) \perp \underbrace{M(0) \perp \cdots \perp M(0)}_{t \text{ copies}}, \\ \langle a_1, \cdots, a_r, b_1, \cdots, b_s \rangle \text{ is anisotropic.} \end{cases}$$

The bilinear form $\langle a_1, \cdots, a_r \rangle$ is unique. We call it the anisotropic part of B, and we denote it by B_{an}. The bilinear form $C := \langle a_1, \cdots, a_r, b_1, \cdots, b_s \rangle$ is generally not determined uniquely up to isometry, but the quadratic form $\widetilde{C}$ is always determined uniquely up to isometry. The integers s, t and $s+t$ are called the metabolicity index, the hyperbolicity index and the Witt index of B, and they are denoted by $i_m(B)$, $i_h(B)$ and $i_W(B)$, respectively. By using the uniqueness of $(\widetilde{B})_{\text{an}}$ the anisotropic part of $\widetilde{B}$, it is clear that $\widetilde{C} \simeq (\widetilde{B})_{\text{an}}$, and then we get the following relations:

$$(\star) \qquad \begin{cases} \dim(\widetilde{B})_{\text{an}} = \dim B_{\text{an}} + i_m(B), \\ i_W(\widetilde{B}) = i_m(B) + 2i_h(B), \end{cases}$$

where $i_W(\widetilde{B})$ is the Witt index of the quadratic form $\widetilde{B}$ (in [2], [3] we denote $i_W(\widetilde{B})$ by $i_d(\widetilde{B})$ and we call it the defect of $\widetilde{B}$).

2. Standard splitting tower, height 1 and the degree invariant. The standard splitting tower of a nonzero bilinear form B is defined to be the following sequence of field extensions of F, and bilinear forms:

$$\begin{cases} F_0 = F \quad \text{and} \quad B_0 = B_{\text{an}}, \\ \text{For } n \geq 1: \quad F_n = F_{n-1}(B_{n-1}) \quad \text{and} \quad B_n = ((B_{n-1})_{F_n})_{\text{an}}. \end{cases}$$

The height of B, denoted by $\mathrm{h}(B)$, is the smallest integer h such that $\dim B_h \leq 1$. By using [5, Prop. 1.1, Cor. 5.5], we get a complete classification of anisotropic bilinear forms of height 1:

Proposition 1. *An anisotropic bilinear form B is of height 1 if and only if B is similar to a subform of a bilinear Pfister form π and* $\dim \pi - \dim B \leq 1$.

Let B be a nonzero bilinear form and $(F_i, B_i)_{0 \leq i \leq h}$ its standard splitting tower with $h = \mathrm{h}(B)$. Suppose $\dim B_{\text{an}} \geq 2$. Then, $h \geq 1$ and there exists a unique bilinear Pfister form π over F_{h-1} such that B_{h-1} is similar to a subform of π and $\dim \pi - \dim B_{h-1} \leq 1$ (Proposition 1). The form π is called the leading form of B, and we say that B is good if π is defined over F, i.e., $\pi \simeq \tau_{F_{h-1}}$ for some bilinear form τ over F (in this case τ is isometric to a unique bilinear Pfister form). If $\dim B$ is even (*resp.* odd), then B is called of degree d where $\dim \pi = 2^d$ (*resp.* degree 0). We denote by $\deg(B)$ the degree of B. We put $\deg(B) = \infty$ for the zero form B. Our main result on the degree invariant is the following theorem:

Theorem 2. *For any integer $d \geq 0$, the set of bilinear forms of degree $\geq d$ coincides with I^dF the d-th power of the fundamental ideal IF of the Witt ring $W(F)$ (we take $I^0F = W(F)$).*

The analogue of this theorem for quadratic forms is true in any characteristic. It is a consequence of some results of Orlov, Vishik and Voevodsky in characteristic $\neq 2$, and it is due to Aravire and Baeza in characteristic 2 [1]. To get Theorem 2, we proved the following result:

Proposition 3. *For any integer $n \geq 1$, and any anisotropic bilinear form C of dimension $> 2^n$, the natural homomorphisms $I^nF/I^{n+1}F \longrightarrow I^nF(C)/I^{n+1}F(C)$ and $I_q^{n+1}F/I_q^{n+2}F \longrightarrow I_q^{n+1}F(C)/I_q^{n+2}F(C)$ have trivial kernels.*

Here $I_q^{n+1}F = I^nF \otimes W_q(F)$, where $\otimes$ is the module action of $W(F)$ on the Witt group $W_q(F)$ of regular quadratic forms over F.

3. On bilinear forms of height 2. Concerning bilinear forms of height 2, we start with a general result: Let B be an anisotropic bilinear form of arbitrary height, and set $\dim B = 2^n + l$ with $0 < l \leq 2^n$. By using [3, Th. 1.1] and the norm field of $\widetilde{B}$ [2, Section 8], we construct a field extension K/F such that B_K is anisotropic and $i_W(\widetilde{B}_{K(B)}) = l$. It follows from the relations in $(\star)$ that:

$$\dim B = 2^{n+1} - \dim(B_{K(B)})_{\text{an}} + 2i_h(B_{K(B)}).$$

As a consequence we obtain:

Proposition 4. *Let B be an anisotropic bilinear form (good or not) of height 2 whose leading form is of dimension 2^d. Then, $\dim B \in \{2^{n+1} - 2^d + 2s + \epsilon \mid 0 \leq s \leq 2^{d-1} - \epsilon\}$, where $\dim B \in]2^n, 2^{n+1}]$, and $\epsilon = 0$ or 1 according as $\dim B$ is even or odd.*

Moreover, we obtain a complete classification of good bilinear forms of height 2 as follows:

Theorem 5. *Let B be an anisotropic good bilinear form of height 2, and standard splitting tower $(F_i, B_i)_{0 \leq i \leq 2}$. Let τ be a bilinear form over F such that τ_{F_1} is the leading form of B. Let $\dim \tau = 2^d$. We distinguish between two cases:*
(1) *$\dim B$ is odd: In this case, B is Witt-equivalent to $\rho_1 \perp \rho_2$, where ρ_1 (resp. ρ_2) is similar to the pure part of a d-fold bilinear Pfister form (resp. is similar to an n-fold bilinear Pfister form for some $n > d$), $\dim B > 2^{n-1}$ and $\widetilde{B}$ is similar to a subform of $\widetilde{\rho}_2$.*
(2) *$\dim B$ is even: We have three possibilities: $\dim B \in \{2^{d+1}\} \cup \{2^n - 2^d \mid n \geq d+2\} \cup \{2^m \mid m > d+1\}$.*
(2.1) *In the first case, B is Witt-equivalent to $x\tau \perp \nu$, where $x \in F^*$ and ν is similar to a $(d+1)$-fold bilinear Pfister form.*
(2.2) *In the second case, $B \simeq \rho \otimes \tau$ such that $\dim \rho$ is odd and $B \perp \langle \det \rho \rangle \otimes \tau$ is similar to an n-fold bilinear Pfister form.*
(2.3) *In the third case, we have no conclusion.*
Conversely, the conditions given in **(1)**, **(2.1)** *or* **(2.2)** *suffice to conclude that B is good of height 2 with leading form of dimension 2^d.*

Our proof of this theorem uses results in [2], [3], [5], and the following generalization of a result of Karpenko [4] to bilinear forms in characteristic 2:

Proposition 6. *If $B \in I^nF$ is anisotropic of dimension $< 2^{n+1}$ ($n \geq 1$), then* $\dim B \in \{2^{n+1} - 2^i \mid 1 \leq i \leq n+1\}$.

All the results presented here are related to the standard splitting of bilinear forms, and then a natural question that we can ask is the following:

Question. *Let B be an anisotropic bilinear form of dimension ≥ 2, and $(F_i, B_i)_{0 \leq i \leq h}$ its standard splitting tower with $h = \mathrm{h}(B)$. Let L/F be a field extension such that B_L becomes isotropic. Is it true that $i_W(B_L) = i_W(B_{F_k})$ for some $k \in \{1, \cdots, h\}$?*

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On the central part of the Rost invariant

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(joint work with S. Garibaldi)

In the 1990's, M. Rost proved that the group of degree 3 normalized invariants of an absolutely simple simply connected algebraic group G over k — that is, the group of natural transformations of the Galois cohomology functors

$$H^1(\star, G) \to H^3(\star, \mathbb{Q}/\mathbb{Z}(2))$$

— is a cyclic group with a canonical generator (see [Mer03]). This canonical generator is known as the *Rost invariant*. Roughly speaking, it is the "first" nonzero invariant, in that there are no non-zero normalized invariants

$$H^1(\star, G) \to H^d(\star, \mathbb{Q}/\mathbb{Z}(d-1))$$

for $d < 3$ [KMRT98, §31].

Example 1. If $G = \mathrm{Spin}(q_0)$ for some n-dimensional non degenerate quadratic form q_0 over k, with $n \geq 5$, then the exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathrm{Spin}(q_0) \xrightarrow{\pi} \mathcal{O}^+(q_0) \longrightarrow 1$$

induces a map

$$H^1(k, \mathrm{Spin}(q_0)) \xrightarrow{\pi^\star} H^1(k, \mathcal{O}^+(q_0)),$$

which maps $H^1(k, \mathrm{Spin}(q_0))$ to isomorphism classes of n dimensional quadratic forms q over k such that $[q] - [q_0]$ belongs to the third power $I^3(k)$ of the fundamental ideal $I(k) \subset W(k)$. The Rost invariant is given in that case by

$$R_{\mathrm{Spin}(q_0)}(u) = e^3(\pi^\star(u) - [q_0]),$$

where e^3 denotes the Arason invariant $e^3 : I^3(k) \to H^3(k, \mu_2) \subset H^3(k, \mathbb{Q}/\mathbb{Z}(2))$.

Example 2. If $G = SL(A)$ for some central simple algebra A of degree n over k, the exact sequence

$$1 \longrightarrow SL(A) \longrightarrow GL(A) \xrightarrow{\mathrm{Nrd}_A} \mathbb{G}_m \longrightarrow 1$$

induces an isomorphism $H^1(k, SL(A)) \simeq k^\times/\mathrm{Nrd}_A(A^\times)$. Denote by $[A]$ the Brauer class of A and $(x)_n$ the image of any $x \in k^\times$ in $H^1(k, \mu_n) \simeq k^\times/k^{\times n}$. The invariant R defined by

$$R(x.\mathrm{Nrd}_A(A^\times)) = (x)_n.[A] \in H^3(k, \boldsymbol{\mu}_n^{\otimes 2}) \subset H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

generates the group of degree 3 invariants of G. It is not known, however, whether R coincides with the Rost invariant.

Let Z be the center of G, and denote by i the inclusion $Z \hookrightarrow G$. It induces a natural transformation $i^\star : H^1(\star, Z) \to H^1(\star, G)$, so that any degree 3 invariant of $H^1(\star, G)$ restricts to an invariant

$$H^1(\star, Z) \to H^3(\star, \mathbb{Q}/\mathbb{Z}(2)),$$

which we call its central part. Note that, even though it is not obvious from its definition, it is a group invariant, i.e. the map $H^1(K, Z) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is a group homomorphism for any K/k, as was shown in [Gar01, 7.1] (see also [MPT03, Cor. 1.8]). We will denote by $\mathrm{Inv}^3(H^1(\star, Z))$ the group of degree 3 group invariants of $H^1(\star, Z)$.

Example 3. From the explicit description recalled in examples 1 and 2, one may easily check that the central part of $R_{\mathrm{Spin}(q_0)}$ is trivial, since $R_{\mathrm{Spin}(q_0)}(u)$ only depends on $\pi^\star(u)$. On the other hand, if $G = SL(A)$, then $Z = \mu_n$, and the central part of R is given by the cup product with $[A]$, now viewed as a map $H^1(k, \boldsymbol{\mu}_n) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$. This Brauer class $[A]$ appears to be the so-called Tits class of the algebraic group G (see [KMRT98, §31] for a definition).

For an arbitrary absolutely simple simply connected algebraic group G over k, we do not have an explicit description of the Rost invariant as in examples 1 and 2. Nevertheless, its central part does admit a nice description as in example 3. This was proven by Merkurjev, Parimala and Tignol in [MPT03] for classical groups; we complete their results and prove the same holds for exceptional groups (see [GQM06]). We may summarize the combined results in the following theorem[1]:

Theorem 4 ([MPT03] and [GQM06]). *Assume the characteristic of the base field k does not divide the exponent of the center Z of G. If G is of type A, C_n (n odd), D, E_6 or E_7, then the central part of the Rost invariant and the cup product with the Tits class of G generate the same subgroup of* $\mathrm{Inv}^3(H^1(\star, Z))$. *Otherwise the central part of the Rost invariant is zero.*

The cup product appearing in the theorem is induced by a bilinear form

$$Z(k_{sep}) \times Z(k_{sep}) \to \mathbb{Q}/\mathbb{Z}(2)$$

which has to be specified, especially for groups of type D_{2m} (see [MPT03, §1.3] for a definition of $\mathbb{Q}/\mathbb{Z}(2)$). If m is odd, then Z is a twisted group $\mu_{2[E]}$ for some quadratic étale algebra E/k, so that $Z(k_{sep}) = \mu_2(k_{sep}) \times \mu_2(k_{sep})$ and we take

$$((x, y), (x', y')) \mapsto xx' + yy' \in \mathbb{Q}/\mathbb{Z}(2).$$

If m is even, then Z is the kernel $R^1_{E/k}(\boldsymbol{\mu}_2)$ of the norm map

$$N_{E/k} : R_{E/k}(\boldsymbol{\mu}_2) \to \boldsymbol{\mu}_2,$$

where E is cubic étale over k, and E is a field if the group is of trialitarian D_4 type. In that case, we have $Z(k_{sep}) = \{(x, y, z) \in \boldsymbol{\mu}_2(k_{\mathrm{sep}})^3,\ xyz = 1\}$ and we take

$$((x, y, z), (x', y', z')) \mapsto xx' + yy' + zz'.$$

The new cases in the theorem are the groups of type E_6, E_7, and trialitarian D_4, but our method also applies to classical groups which are not of type A, for which we recover the results of [MPT03]. (Note that the exceptional groups of type E_8, F_4, and G_2 all have trivial center, so the central part of the Rost invariant is automatically zero for those groups.)

Part of our proof is actually borrowed from [MPT03]. Precisely, their corollaries 1.2 to 1.6, which are stated for a general cycle module, actually show in our context that any group invariant $H^1(\star, Z) \to H^3(\star, \mathbb{Q}/\mathbb{Z}(2))$ is given by some cup-product with a cohomology class $t \in H^2(k, Z)$. Hence, it only remains to prove that either $t = 0$ or t and t_G generate the same subgroup of $H^2(k, Z)$, for a well-chosen definition of the cup-product. This is done in [MPT03] using concrete interpretations of the classical groups. Instead, we first reduce to groups whose Tits index satisfies a certain condition, using some injectivity result which follows from [MT95, Th. B]. For those particular groups, the remarkable fact which enables us to conclude is that the center of G actually is contained in a semi-simple simply connected subgroup, which appears to be (in all cases) a product of $SL(A)$, for

[1]There is a typo in Th. 1.13 of [MPT03], which addresses the C_n case: the words "odd" and "even" should be interchanged. For keeping the C_{odd} and C_{even} cases straight, we find it helpful to recall that $B_2 = C_2$.

which the Rost invariant, and its central part, are known. Our proof includes, for each type of group, an explicit description of this subgroup G', and of the embedding $Z \hookrightarrow G'$.

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Zero cycles of degree one and rational points on principal homogeneous spaces

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Let k be a field and G a connected linear algebraic group defined over k. The following question (cf. [Se], p. 166) is wide open.

[Q] Let X be a principal homogeneous space for G defined over k which admits a zero cycle of degree one. Is $X(k)$ nonempty?

In other words, if there exist finite extensions L_i/k, $1 \leq i \leq r$ with $[L_i : k] = n_i$, $\gcd_i(n_i) = 1$ and $X(L_i)$ nonempty, one asks whether $X(k)$ is nonempty. Phrased in terms of Galois cohomology, one asks whether the map

$$H^1(k, G) \to \prod_i H^1(L_i, G)$$

has trivial kernel.

An affirmative answer to [Q] if k is a number field is due to Sansuc ([Sa], §4, Cor. 4.8). The main ingredients in the proof of Sansuc are the following results for a number field k.

I. (Hasse Principle) Let G be a semisimple simply connected linear algebraic group defined over k. The map

$$H^1(k, G) \to \prod_{v \text{ real}} H^1(k_v, G)$$

is injective.

II. For a finite abelian algebraic group μ defined over k, the map

$$H^1(k,\mu) \to \prod_{v \text{ real}} H^1(k_v,\mu)$$

is surjective.

The Hasse principle is due to Kneser, Harder and Chernousov (cf. [BP1]).
For a proof of II, we refer to ([Sa] Lemma 1.6).

Let k be a field of virtual cohomological dimension 2. Let Ω_k denote the space of orderings of k and for each $v \in \Omega_k$, let k_v denote the real closure of k at v. There is a conjecture due to Colliot-Thélène asserting the Hasse principle in this setting:

Conjecture HP. *For a semisimple simply connected linear algebraic group G defined over a perfect field k of virtual cohomological dimension 2, the map*

$$H^1(k,G) \to \prod_{v\in\Omega_k} H^1(k_v,G)$$

is injective.

If the cohomological dimension of k is 2, this conjecture coincides with Conjecture II of Serre. Both Conjecture II and Conjecture HP are settled in the affirmative for all groups of classical type and for groups of type G_2 or of type F_4 ([MS], [BP1], [BP2]). There is however no analogue of II in this general setting. For instance, if $\mu = Z/2Z$, the image

$$H^1(k,Z/2Z) \to \prod_{v\in\Omega_k} H^1(k_v,Z/2Z)$$

lands in the space of continuous maps $\mathcal{C}(\Omega_k, Z/2Z)$ and the surjectivity of $H^1(k,Z/2Z) \to \mathcal{C}(\Omega_k,Z/2Z)$ imposes SAP condition on the field k. One can avoid the use of the property II in Sansuc's proof, which leads to the following more general result.

For a connected linear algebraic group G defined over a field k, let G^u denote the unipotent radical of G, $G^{red} = G/G^u$, G^{ss} the derived group of G^{red} and G^{sc} the simply connected cover of G^{ss}.

Theorem 1. *Let k be a perfect field of virtual cohomological dimension at most 2. Let G be a connected linear algebraic group defined over k with G^{sc} satisfying the Hasse principle conjecture. Then every principal homogeneous space for G over k admitting a zero cycle of degree one admits a rational point.*

For a general field k, we have the following theorem of Bayer and Lenstra ([BL]) concerning unitary groups.

Theorem 2. *Let k be a field of characteristic not 2. Let D be a central division algebra over K with an involution τ of either kind over k. Let h be a hermitian form over (D,τ) and $U(h)$ the unitary group of h. Let L be a finite extension of k of odd degree. Then the map*

$$H^1(k, U(h)) \to H^1(L, U(h))$$

has trivial kernel.

In particular, every principal homogeneous space under $U(h)$ admitting a zero cycle of degree one admits a k-rational point. Starting from this result, using certain norm principles of Gille and Merkurjev ([Gi], [M]), one can prove the following result.

Theorem 3. *Let k be a field of characteristic not 2. Let G be a semisimple linear algebraic group defined over k which is either simply connected or adjoint, of classical type. Then every principal homogeneous space under G, which admits a zero cycle of degree one, has a k-rational point.*

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Outer automorphisms of Lie algebras of type D_4

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Let (V, q) be a nonsingular finite dimensional quadratic space of dimension 8 over a field F of characteristic different from 2 and 3. Let b_q be the polar form of q and let σ_q be the involution of the endomorphism algebra E of V associated with b_q. The Lie algebra $\mathfrak{O}(q)$ of skew-symmetric elements of E with respect to σ_q is a simple Lie algebra of type D_4. Automorphisms of $\mathfrak{O}(q)$ induced by proper similitudes of q are called inner. They form a normal subgroup of the group of automorphisms of $\mathfrak{O}(q)$ and it is well known that over an algebraically closed field the quotient is isomorphic to the group of automorphisms of the Dynkin diagram of type D_4. This group is isomorphic to S_3 and there is a unique cyclic subgroup of order 3 of outer automorphisms, modulo inner automorphisms. Moreover, over an algebraically closed field there are two conjugation classes of such cyclic subgroups. One class has as fixed point algebra a Lie algebra of type G_2 and the other class a Lie algebra of type A_2.

In this report we give a classification of conjugation classes of outer automorphisms over arbitrary fields of characteristic different from 2 and 3. For $\mathfrak{O}(q)$ to admit an outer automorphism of order 3, q has to be a 3-fold Pfister form. It is well known that 3-fold Pfister forms are norm forms of Cayley algebras. One conjugation class is given by the isomorphism class of the Cayley algebra associated with q. The fixed point algebra is the Lie algebra of derivations of the Cayley algebra and hence is of type G_2. The other classes correspond to central simple algebras of degree 3 with involutions of the second kind. More precisely, let K be the quadratic étale algebra of degree 2 over F obtained by formally adjoining a primitive cubic root of unity to F and let B be a central simple of degree 3 over K. Let τ be an involution of B which restricts to the nontrivial automorphism of K. The Lie algebra $\mathfrak{L}$ of trace zero skew-symmetric elements of (B, τ) is of type A_2. Conjugation classes of outer automorphisms of order 3 of orthogonal Lie algebras $\mathfrak{O}(q)$ having as fixed point algebras Lie algebras of type A_2 are classified by isomorphism classes of such central simple algebras. The form q is the quadratic trace form restricted to $\mathfrak{L}$ and is a 3-fold Pfister form if K is as described above. The proof uses Clifford algebras and the theory of symmetric compositions.

Coordonnées de Kac

Jean-Pierre Serre

1. Notations

Soit G un groupe algébrique quasi-simple, simplement connexe, sur un corps algébriquement clos k. On note $Z(G)$ le centre de G, et l'on pose $G^{ad} = G/Z(G)$. On choisit un sous-groupe de Borel B de G et un tore maximal T de B, d'où un système de racines R, muni d'une base $(\alpha_i)_{i \in I}$; les α_i appartiennent au groupe P des caractères de T, et forment une base de $P \otimes_{\mathbf{Z}} \mathbf{Q}$.

La plus grande racine $\tilde{\alpha}$ de R s'écrit

(1) $$\tilde{\alpha} = \sum n_i \alpha_i, \quad \text{avec} \quad n_i \in \mathbf{Z}, \quad n_i \geq 1.$$

On pose $\alpha_0 = -\tilde{\alpha}, n_0 = 1$ et $I_0 = I \cup \{0\}$, de sorte que (1) se récrit :

(2) $$\sum_{i \in I_0} n_i \alpha_i = 0.$$

L'ensemble I (resp. I_0) est l'ensemble des sommets du graphe de Dynkin (resp. du graphe de Dynkin complété) de R, cf. [1], Chap. VI.

2. Le cas de caractéristique 0

Supposons que k soit de caractéristique 0, et choisissons une paramétrisation des racines de l'unité de k, autrement dit un homomorphisme $\mathbf{e} : \mathbf{Q} \to K^*$ de noyau $\mathbf{Z}$. (Lorsque $K = \mathbf{C}$, un choix naturel est $\mathbf{e}(x) = e^{2\pi i x}$.)

Soit $x = (x_i)_{i \in I_0}$ une famille d'éléments de $\mathbf{Q}$ indexée par I_0. On associe à x l'élément t_x de $T(k)$ caractérisé par la propriété suivante : pour tout $\omega \in P$, on a

$$\omega(t_x) = \mathbf{e}(\sum_{i \in I} c_i(\omega) x_i), \tag{3}$$

où les $c_i(\omega)$ sont les coordonnées de ω par rapport à la base (α_i) de $P \otimes \mathbf{Q}$.

Les t_x sont des éléments d'ordre fini, et l'on a

$$\alpha_i(t_x) = \mathbf{e}(x_i) \ \text{ pour tout } \ i \in I, \tag{4}$$

ce qui suffit à caractériser t_x lorsque $Z(G) = 1$.

Notons C l'ensemble des $x = (x_i)$ satisfaisant aux deux conditions suivantes :

$$x_i \geq 0 \ \text{ pour tout } i \in I_0, \tag{5}$$

$$\sum n_i x_i = 1, \tag{6}$$

les n_i ($i \in I_0$) étant les entiers définis dans (2) ci-dessus.

Théorème 1 (Kac [3], [4], [5]). *Tout élément d'ordre fini de $G(k)$ est conjugué d'un t_x ($x \in C$), et d'un seul.*

L'ensemble C fournit ainsi une paramétrisation des classes de conjugaison de G d'ordre fini.

3. Démonstration du théorème 1

Lorsque $k = \mathbf{C}$, le théorème 1 peut se traduire en termes de groupes de Lie compacts ; on constate qu'il devient alors un corollaire de la description (due à Elie Cartan, cf. [2]) des classes de conjugaison au moyen du *simplexe fondamental* de l'algèbre de Lie de T. Les $(n_i x_i)$ s'interprètent comme les coordonnées barycentriques de ce simplexe.

Le cas général se traite de façon analogue. Les ingrédients principaux sont :

– le fait que tout élément semi-simple de $G(k)$ est conjugué d'un élément de $T(k)$, et que ce dernier est bien déterminé modulo l'action du groupe de Weyl sur $T(k)$.

– la description d'un domaine fondamental du groupe de Weyl affine donnée dans [1], VI, §2 et dans [2].

4. Quelques propriétés des coordonnées (x_i)

Les plus importantes sont les suivantes (on en trouvera d'autres dans Kac, *loc.cit.*) :

(i) Soit $x = (x_i)$ un élément de C, et soit m le plus petit commun dénominateur des x_i, de sorte que l'on a $x_i = s_i/m$, avec $s_i \in \mathbf{N}, m \geq 1$, pgcd $(s_i) = 1$ et :

$$\sum_{i \in I_0} n_i s_i = m. \tag{7}$$

Alors m est l'ordre de l'image de t_x dans G^{ad} : c'est *l'ordre adjoint* de t_x. (D'habitude, ce sont les s_i qui sont appelées les "coordonnées de Kac" de la classe de conjugaison considérée.)

(ii) Soit $x \in C$, et soit G_x le centralisateur de t_x. Alors G_x est un sous-groupe connexe de G, réductif, contenant T, et dont le graphe de Dynkin a pour sommets les $i \in I_0$ tels que $x_i = 0$.

En particulier, t_x est régulier si et seulement si $x_i > 0$ pour tout i ; l'ordre adjoint d'un tel élément est au moins égal au nombre de Coxeter $h = \sum n_i$.

Ces propriétés sont très commodes pour énumérer les classes de conjugaison d'ordre fixé. Ainsi, si G est de type G_2, et si l'on s'intéresse aux classes d'ordre 5, les entiers (n_i) sont : (1, 2, 3) et l'équation (7) s'écrit

$$s_0 + 2s_1 + 3s_2 = 5 \ \ , \ \ \text{avec} \ \ s_i \in \mathbf{N} \text{ et } \mathrm{pgcd}(s_0, s_1, s_2) = 1 \ .$$

Il y a quatre solutions : (3, 1, 0), (2, 0, 1), (1, 2, 0), (0, 1, 1). Dans chaque cas le centralisateur est de dimension 4 : son rang semi-simple est égal à 1.

5. Reformulation et extension à la caractéristique $p > 0$

La présentation donnée ci-dessus a deux inconvénients : elle dépend du choix de la paramétrisation $\mathbf{e} : \mathbf{Q} \to k^*$, et, si caract$(k) = p$, elle ne s'applique qu'aux éléments d'ordre premier à p.

On peut remédier à ces deux défauts en introduisant les "μ-éléments" de G. Alors qu'un élément d'ordre n peut être vu comme un plongement du schéma en groupes (étale) $\mathbf{Z}/n\mathbf{Z}$ dans G, un μ-élément d'ordre n est (par définition) un plongement $\mu_n \to G$, où μ_n est le schéma en groupes des racines n-ièmes de l'unité (autrement dit le groupe dual du groupe $\mathbf{Z}/n\mathbf{Z}$).

Si l'on note $M_n(G)$ l'ensemble des μ-éléments d'ordre n de G, la réunion $M(G)$ des $M_n(G)$ est un substitut de l'ensemble des éléments d'ordre fini de G ; on a $M(G) = \mathrm{Hom}\,(\varprojlim \mu_n, G)$. On transpose alors sans difficulté ce qui a été fait au §2 : tout $x = (x_i)$ définit un μ-élément θ_x de $M(T)$, et le th. 1 est remplacé par :

Théorème 1′. *Tout μ-élément de G est conjugué d'un θ_x $(x \in C)$ et d'un seul.*

La démonstration est la même.

Exemple. Supposons que k soit de caractéristique $p > 0$. Un élément de $M_p(G)$ s'identifie à un élément non nul X de la p-algèbre de Lie de G tel que $X^p = X$. On obtient ainsi la classification de ces éléments : leurs classes de conjugaison sont décrites par les mêmes coordonnées de Kac que celles des éléments d'ordre p en caractéristique 0, et leurs centralisateurs sont donnés par la même recette.

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