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Classical Algebraic Geometry

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Introduction by the Organisers

The Workshop on Classical Algebraic Geometry was notable for a relaxed atmosphere (18 talks) and an abundance of young people. A wide variety of themes related to classical topics were discussed with a very modern point of view. Although it is tempting to summarize each of the talks, we limit ourselves to four highlights:

- There has been a great deal of interest in the question: are there structural characterizations of rational varieties in higher dimensions? Rationality itself is elusive: the notion of "rationally connected variety" (a variety where any two points can be connected by a rational curve) seems much more tractable. Brendan Hassett described work of his with Yuri Tschinkel showing that these varieties exhibit an analogue of a famous arithmetic property of quadrics: if a family of varieties has smooth rationally connected fibers, then given a collection of share at least some properties of quadrics in low dimensions. In many cases, a collection of "local sections" can be connected by a global section.
- A central theme of algebraic geometry is that the set of algebraic varieties of a particular kind often itself is naturallly an algebraic variety. It was classically assumed that such families would be generically nice in some sense. This has turned out not to be the case: to understand them, one must accept non-reduced components. The first examples of this phenomenon were given by David Mumford in a very famous paper. A second

highlight of our conference was a re-interpretation and generalization of Mumford's example by Shigeru Mukai.

- A number of combinatorial and computational applications in algebraic geometry have recently come from what the statisticians have for a long time called the "max-plus" algebra—in algebraic geometry it now goes under the name "tropical". Sean Keel showed off a new application of these ideas, found jointly with Paul Hacking and Eugene Tevelev: the "tropical fan" associated with certain toric varieties provides an extremely nice and natural compactification of these varieties. Among the remarkable classical examples that Keel gave is that of the moduli space of smooth cubic surfaces.
- Among the algebraic varieties of algebraic varieties, the moduli space of curves of genus g and some of its variants is by far the most important, with applications ranging from string theory in physics (Witten) to new versions of resolution of singularities (de Jong). Constructions of Severi and others from the early part of the 20th century showed that for low genus (≤ 10) the moduli space is rational, and Severi believed that he had proved rationality for all genera. The error in his argument was soon found, and it has been an important problem to decide which moduli spaces were actually rational.

  The importance of this work comes as much from the technique involved — studying the divisor class group of the moduli space, which really means describing conditions on an algebraic curve that are locally given by just one equation — as from the results. At our conference Farkas spoke on a very far-reaching generalization of what was known systematically using syzygies to describe conditions on curves that lead to new divisors.

Workshop: Classical Algebraic Geometry

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Abstracts

Rational connectedness over small fields

Yuri Tschinkel
(joint work with Fedor Bogomolov)

Let k be a field, $\bar{k}$ its separable closure, X an algebraic variety over k and $X(k)$ its set of rational points. We are interested in rational, resp. algebraic, points on X and in rational curves on X, defined over k or $\bar{k}$. For $k = \bar{k}$ of chacteristic zero we have (at least) two notions of "connectivity" via rational curves:

(1) For all $x_1, x_2 \in X(k)$ there exists a chain of rational curves $C_1 \cup \ldots \cup C_r \subset X$ connecting x_1 and x_2;
(2) For all $x_1, x_2 \in X(k)$ there exists a *free* rational curve $C \subset X$ connecting x_1, x_2.

For smooth projective X these two properties are equivalent. The situation is less clear for quasi-projective X. For example, we don't know whether (2) holds for the smooth locus of a singular Del Pezzo surface, or its partial desingularization.

There are versions involving arbitrary (or general) finite sets of points, prescribed local behavior at finitely many points, etc. (see [4]).

In arithmetic situations, when $k \neq \bar{k}$, there are more logical possibilities: one could ask for irreducible curves defined over the groundfield k, or for curves over k connecting points over $\bar{k}$. Of particular interest are *small* ground fields, such as finite fields $\mathbb{F}_q$ or the rationals $\mathbb{Q}$. A prototype result is the following theorem of Kollár and Szabó:

Theorem 1 ([5]). *Let X be a smooth projective separably rationally connected variety over $k = \mathbb{F}_q$. There is a function $\phi = \phi(\deg(X), \dim(X), n)$ such that for $q > \phi$ and for every set of n points $x_1, \ldots, x_n \in X(k)$ there exists a geometrically irreducible rational curve C, defined over k with $x_1, \ldots, x_n \in C(k)$.*

The theorem applies, e.g., to hypersurfaces $X \subset \mathbb{P}^N$ of low degree $d \leq N$. It turns out that rational connectivity holds sometimes even for $d = N + 1$:

Theorem 2 ([1]). *Let $X = \widetilde{A/G}$ be a Kummer surface over $k = \mathbb{F}_q$ (with q suffiently large), and $X^\circ \subset X$ the complement to exceptional curves. Then for every set $x_1, \ldots, x_n \in X^\circ(\bar{k})$ there exists a geometrically irreducible rational curve $C \subset X$, defined over k, such that $x_1, \ldots x_n \in C(\bar{k})$.*

This theorem applies, for example, to quartic Kummer surfaces. Choosing nonsupersingular A gives examples of nonuniruled K3 surfaces, which are "rationally connected". Using this we provide examples of nonuniruled surfaces of general type over finite fields with the same property.

The proof of Theorem 2 relies on a fact of independent interest.

Theorem 3 ([2]). *Let C be a smooth projective curve of genus $\mathsf{g} > 2$ over $k = \mathbb{F}_q$ (with q sufficiently large) and J its Jacobian. Fix a point $c_0 \in C(k)$ and the embedding $C \hookrightarrow J$, via $c \mapsto c - c_0$. Then*

$$J(\bar{k}) = \cup_{m=1}^{\infty} m \cdot C(\bar{k}).$$

In fact, one can let m run through arithmetic progressions.

These results over finite fields have surprizing applications over number fields. Namely, let $X = \widetilde{J/G}$ be Kummer surface over a (sufficiently large) number field K, with J the Jacobian of a curve of genus 2. Choose models for X, J over the integers $\mathscr{O}_K$ and a finite set of nonarchimedian places of (sufficiently) good reduction S. Finally, for $v \in S$, choose $\bar{x}_v \in X(\mathbf{k}_v)$ - points in the reduction modulo v.

Theorem 4 ([3]). *There exists a K-rational point $x \in X(K)$ such that $x_v = \bar{x}_v$ modulo v, for all $v \in S$.*

Such a version of weak approximation, for first order jets, is unknown even for cubic surfaces over number fields.

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Computing certain invariants of topological spaces of dimension three

Barbara Fantechi

(joint work with Kai Behrend)

This is a report on the paper [4]. Let X be a scheme (we will write scheme for scheme or DM algebraic stack throughout) of finite type over the complex numbers. An obstruction theory (as in [3] is a morphism $\alpha : E \to \tau_{\geq -1} L_X$ in $D^{\leq 0}_{coh}(X)$ such that $h^0\alpha$ is an isomorphism and $h^{-1}\alpha$ is surjective. This implies that at every point $x \in X$, $H^0(E^\vee(x))$ is isomorphic to $T_x X$ and $H^1(E^\vee(x))$ is an obstruction space $T^2_x X$ for X at x. If the obstruction theory is 1-perfect, i.e. isomorphic to a complex $[E^{-1} \to E^0]$ of locally free sheaves, then X carries an induced virtual fundamental class $[X]^{vir} \in A_d(X)$ which has good properties and can be used to define enumerative invariants when X is proper: here $d = rk\, E^0 - rk\, E^{-1} = \dim T^1_x X - \dim T^2_x X$ is the expected dimension of X.

In case where X is a moduli space of simple sheaves on a smooth projective manifold V with fixed determinant, then it carries an obstruction theory with $T^i_F X = Ext^i_0(F,F)$ (where the subscript zero means traceless: see e.g. [5] chapter 10). This is 1-perfect when V is a surface (this can be used for an algebraic definition of Donaldson invariants) and also when V is a threefold with $(-K_V)$ effective, since $Ext^3_0(F,F)$ is dual to $Hom_0(F, F(-K_V)) \subset Hom_0(F,F) = 0$; in this way R. Thomas defines holomorphic Casson invariants, also called Donalson-Thomas invariants [9].

If the threefold V is Calabi-Yau, in the sense that K_V is trivial, then the obstruction theory described above is *symmetric*, i.e. there exixts an isomorphism $\phi : E \to E^\vee[1]$ such that $\phi^\vee[1] = \phi$. A symmetric obstruction theory necessarily has expected dimension zero, and if X is prope one can consider its virtual degree, i.e., the degree of the virtual fundamental class.

It is easy to see that any obstruction theory arising by $X = Z(\omega)$ where ω is a closed one-form on a smooth variety is naturally symmetric. We prove a partial converse to this, namely every symmetric obstruction theory can be locally described as induced by $X = Z(\omega)$ where ω is an almost closed one-form on a smooth variety, i.e. $d\omega|_X = 0$.

Using this, K. Behrend proved in [1] that if X has a symmetric obstruction theory, then there exists a constructible integer valued function on X whose integral is the virtual degree, if X is proper. Here by constructible function we mean a finite linear combination of characteristic functions of closed subvarieties; the integral is defined by $\int 1_V = e(V)$ the Euler characteristic, and extended by linearity. Moreover the constructible function is local in the étale topology and multiplicative in products; if X is smooth, its value is $(1)^{\dim X}$.

If the scheme X admits a torus action, we define naturally the notion of equivariant symmetric obstruction theory; we prove that if $x \in X$ is a scheme-theoretic isolated fixed point of the torus action, then the value of the constructible function in x is $(-1)^{\dim T_x X}$. As an application, we prove a conjecture of Maulik, Nekrasov, Okounkov, Pandharipande [8] asserting that for every n

$$\deg[Hilb^n\, V]^{vir} = (-1)^n e(Hilb^n\, V).$$

To complete the proof we need to know the parity of dimension of the tangent space to the Hilbert scheme of points on affine three-space at points corresponding to monomial ideals; this is already determined in [8].

A more general version of the conjecture, determining $\deg[Hilb^n\, V]^{vir}$ for an arbitrary smooth projective threefold V has been proven independently by Jun Li in [7] and by M. Levine and R. Pandharipande in [6] using a different approach, namely by proving that the virtual degree above is (in a suitable sense) cobordism invariant and reducing to the case of V toric.

The method outlined here should prove useful for computing other Donaldson-Thomas invariants, see e.g. [2]. Stronger results should be possible if one could prove that every space with symmetric obstruction theory is locally the zero locus of a closed one-form; it is possible that however this is not true, and one must instead strengthen the assumption, but still in such a way that it still applies to Donaldson-Thomas moduli spaces. One possibility would be to rephrase the symmetry as a condition on an appropriate dg-moduli scheme.

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Some new local properties of multiplier ideals

Robert Lazarsfeld

This report is a summary of – and adapted from – the paper [3].

Let X be a smooth complex algebraic variety of dimension d, and let $\mathfrak{b} \subseteq \mathscr{O}_X$ be an ideal sheaf. Given a rational or real number $c > 0$ one can construct the *multiplier ideal*

$$\mathscr{J}(\mathfrak{b}^c) = \mathscr{J}(X, \mathfrak{b}^c) \subseteq \mathscr{O}_X$$

of $\mathfrak{b}$ with weighting coefficient c. This is a new ideal on X that measures in a somewhat subtle manner the singularities of functions $f \in \mathfrak{b}$. Multiplier ideals appear naturally in the Kawamata–Viehweg–Nadel vanishing theorem, and in recent years thay have found many applications in local and global algebraic geometry.

There has been considerable interest — especially from the algebraic side of the field — in trying to understand how general or special multiplier ideals may be among all ideal sheaves. Multiplier ideals are always integrally closed, but up to now they have not been known to satisfy any other local properties. In fact, Favre–Jonsson [1] and Lipman–Watanabe [5] proved that in dimension $d = 2$, every integrally closed ideal can locally be realized as a multiplier ideal.

The corresponding statement in dimensions ≥ 3 was open for several years until Kungyong Lee [4] recently succeeded in proving that the ideal of a suitable number of general lines through the origin in $\mathbf{C}^3$ couldn't arise as a multiplier ideal. However his argument didn't pinpoint any general features of multiplier ideals that might be violated: rather the idea was to follow a potential resolution of singularities of the data with enough care that one could eventually get a contradiction.

In joint work with Lee, we have put these matters in a new light. Specifically, the main result of [3] asserts that multiplier ideals satisfy some unexpected properties of an algebraic nature. In the following, we work in the local ring $(\mathscr{O}, \mathfrak{m})$ of X at a point $x \in X$, and as above $d = \dim X$.

Theorem 1. *Let* $\mathscr{J} = \mathscr{J}(\mathfrak{b}^c)_x \subseteq \mathscr{O}$ *be (the germ at* x *of) any multiplier ideal. If* $p \geq 1$*, then no minimal* p^{th} *syzygy of* $\mathscr{J}$ *vanishes modulo* $\mathfrak{m}^{d+1-p}$.

Let me explain the statement more precisely in the case $p = 1$. Fix minimal generators $f_1, \ldots, f_b \in \mathscr{J}$, and let $g_1, \ldots, g_b \in \mathfrak{m}$ be functions giving a minimal syzygy

$$\sum g_i f_i \ = \ 0$$

among the f_i. Then the claim is that

$$\mathrm{ord}_x(g_i) \leq d - 1$$

for at least one index i. When $p \geq 2$ the meaning is similar. Note however that there aren't any restrictions on the order of vanishing of *generators* of a multiplier ideal, since for instance all powers of $\mathfrak{m}$ occur as multiplier ideals.

The theorem implies that if $d \geq 3$, then many integrally closed ideals cannot arise as multiplier ideals. For example consider $2 \leq m \leq d - 1$ functions

$$f_1, \ldots, f_m \ \in \ \mathscr{O}$$

vanishing to order $\geq d$ at x. If the f_i are chosen generally, then the complete intersection ideal $\mathscr{I} = (f_1, \ldots, f_m)$ that they generate will be radical, hence integrally closed. On the other hand, the Koszul syzygies among the f_i violate the condition in Theorem 1, and hence $\mathscr{I}$ is not a multiplier ideal. If $d \geq 3$ a modification of this construction yields $\mathfrak{m}$-primary integrally closed ideals having a syzygy vanishing to high order.

Theorem 1 follows from a more technical statement involving the vanishing of a map on Tor's:

Theorem 2. *The natural maps*

$$\mathrm{Tor}_p\big(\mathfrak{m}^{d-p}\mathscr{J}, \mathbf{C}\big) \longrightarrow \mathrm{Tor}_p\big(\mathscr{J}, \mathbf{C}\big)$$

vanish for all $0 \leq p \leq d$.

This in turn is proved by noting that an exact "Skoda complex" [2, Section 9.6.C] sits inbetween the two Koszul complexes computing the groups in question.

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Invariants of singularities in positive characteristic

Mircea Mustaţă
(joint work with Manuel Blickle, Karen E. Smith)

We study invariants of singularities in positive characteristic that are analogues of the jumping numbers of multiplier ideals in characteristic zero. For simplicity, we deal here only with the case of principal ideals, though the reults below generalize to arbitrary ideals.

Suppose that k is a perfect field of positive characteristic p, and let f be a nonzero polynomial in $R = k[x_1, \ldots, x_n]$. On R we have the Frobenius morphism $F\colon R \to R$, $F(x) = x^p$ that is flat and finite. If I is an ideal in R and if e is a positive integer, then we put

$$I^{[p^e]} := (u^{p^e} \mid u \in I).$$

We now recall the definition of the F-thresholds of f from [3]. Suppose that J is a fixed ideal in $k[x_1, \ldots, x_n]$ such that $f \in \mathrm{rad}(J)$. For every positive integer e, let $\nu_f^J(p^e)$ be the largest r such that $f^r \not\in J^{[p^e]}$ (if there is no such r, we put $\nu_f^J(p^e) = 0$). Since the Frobenius morphism is flat, it follows that

$$\nu_f^J(p^{e+1}) \geq p \cdot \nu_f^J(p^e),$$

and therefore we may define the *F-threshold* of f with respect to J by

$$c^J(f) := \sup_{e \geq 1} \frac{\nu_f^J(p^e)}{p^e} = \lim_{e \to \infty} \frac{\nu_f^J(p^e)}{p^e}.$$

It is easy to see that this is a finite number.

Example 1. Let $f = x^2 + y^3$. Note that in characteristic zero, the log canonical threshold $\mathrm{lc}(f)$ of f is equal to $\frac{1}{2}+\frac{1}{3} = \frac{5}{6}$. Suppose now that we are in characteristic $p > 3$ and let us compute $c^{\mathfrak{m}}(f)$, where $\mathfrak{m} = (x, y)$. One can show that if $p \equiv 1 \pmod 3$, then $c^{\mathfrak{m}}(f) = \frac{5}{6}$, while if $p \equiv 2 \pmod 3$, then $c^{\mathfrak{m}}(f) = \frac{5}{6} - \frac{1}{6p}$.

In fact, there are the following results of Hara, Takagi and Watanabe on the connection between the log canonical threshold of a pair in characteristic zero and the F-thresholds of its reductions mod p. Suppose that $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and for every prime p we denote by f_p the class of f in $\mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]$. We denote by $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n)$ in $\mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]$.

Theorem 1. *With the above notation, for all $p \gg 0$ we have*

$$\mathrm{lc}_0(f) \geq c^{\mathfrak{m}}(f_p),$$

where $\mathrm{lc}_0(f)$ denotes the log canonical threshold of f in a neighborhood of 0.

Theorem 2. *Moreover, we have*

$$\lim_{p\to\infty} c^{\mathfrak{m}}(f_p) = \mathrm{lc}_0(f).$$

Probably the most interesting question in this area is the following

Conjecture 1. *There are infinitely many p such that $\mathrm{lc}_0(f) = c^{\mathfrak{m}}(f_p)$.*

Note that the log canonical threshold can be computed in characteristic zero via resolution of singularities. We stress that in characteristic p this is not the case, even in cases when we have such a resolution: in Example 1, we always have a resolution, and this is independent of the characteristic.

The F-thresholds are positive characteristic analogues of the jumping numbers of the multiplier ideals in characteristic zero (see [2] p. 168). In fact, they can be realized as jumping numbers for some generalized test ideals introduced by Hara and Yoshida in [1]. We give now a definition that is equivalent to the one in [1].

For every $e \geq 1$ and $r \geq 0$, let $I_{r,e}(f)$ be the smallest ideal (with respect to inclusion) such that

$$f^r \in I_{r,e}(f)^{[p^e]}.$$

It is easy to see that if $c \in \mathbb{R}_+$, then for every e we have $I_{\lceil cp^e \rceil,e}(f) \subseteq I_{\lceil cp^{e+1} \rceil,e+1}(f)$, where we denote by $\lceil \alpha \rceil$ the smallest integer $\geq \alpha$. Since R is Noetherian, it follows that there is an ideal denoted $\tau(f^c)$ such that for $e \gg 0$ we have $\tau(f^c) = I_{\lceil cp^e \rceil,e}(f)$.

It is clear from definition that if $c_1 < c_2$, then $\tau(f^{c_2}) \subseteq \tau(f^{c_1})$. One can also show that for every c there is $\epsilon > 0$ such that $\tau(f^c) = \tau(f^{c'})$ for every $c' \in [c, c+\epsilon)$. We call c a *jumping number* of f if $\tau(f^c) \neq \tau(f^{c'})$ for every $c' < c$.

We show that the set of jumping numbers of f is equal to the set of F-thresholds of f (when we let the ideal J vary). In characteristic zero, the jumping numbers for the multiplier ideals are determined by a log resolution of f. This implies, for example, that the numbers are rational and discrete. In characteristic p, as we have seen, the F-thresholds are not determined by a resolution, even when such a resolution exists.

Our main results are

Theorem 3. *The set of F-thresholds of f is discrete, i.e. we have finitely many such thresholds in every bounded interval.*

Theorem 4. *All F-thresholds of f are rational numbers.*

We sketch the proofs of the above theorems. Note that the first result easily implies the second. Indeed, if α is an F-threshold, then so is $p\alpha$. Moreover, if $\alpha > 1$, then also $\alpha - 1$ is an F-threshold. This implies that for every F-threshold α, all fractional parts $\{p^e\alpha\}$ are F-thresholds (we make the convention that 0 is an F-threshold). Theorem 5 implies that there only finitely many such numbers, hence there are $e_1 \neq e_2$ such that $(p^{e_1} - p^{e_2})\alpha$ is an integer.

In order to prove Theorem 5, it is enough to show that if $\deg(f) = d$, then for every $c \in \mathbb{R}_+$, $\tau(f^c)$ is generated by polynomials of degree $\leq cd$ (this shows that the set $\{\tau(f^\alpha) \mid \alpha \leq c\}$ is finite). This in turn can be deduced from the following description of the ideals $I_{r,e}(f)$. Consider the basis $\{x^u \mid u = (u_i)_i \in \mathbb{N}^n, 0 \leq u_i \leq p^e - 1\}$ of R over R^{p^e}. If we write f^r in this basis

$$f^r = \sum_u a_u^{p^e} x^u,$$

then $I_{r,e}(f) = (a_u \mid u)$. In particular, $I_{r,e}(f)$ is generated by polynomials of degree $\leq \frac{dr}{p^e}$.

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Approximation results for rationally connected varieties

Brendan Hassett

(joint work with Yuri Tschinkel)

We work over an algebraically closed field k of characteristic zero.

Let B be a smooth connected curve over k and $\pi : \mathscr{X} \to B$ a *rationally connected fibration*, i.e., a flat proper morphism whose generic fiber X is smooth and rationally connected. We shall often assume that the total space $\mathscr{X}$ is nonsingular. For each $b \in B$, we write $\mathscr{X}_b = \pi^{-1}(b)$ and $\mathscr{X}_b^{sm}$ for the smooth locus of $\mathscr{X}_b$. A rationally connected fibration $\mathscr{Y} \to B$ with generic fiber X is called a *model* of X.

Our point of departure is the following pair of results: Each rationally connected fibration $\mathscr{X} \to B$ admits a section $s : B \to \mathscr{X}$. [2] Furthermore, given $b_1, \ldots, b_r \in B$ so that each $\mathscr{X}_{b_i}$ is smooth and $x_i \in \mathscr{X}_{b_i}$, then there exists a section with $s(b_i) = x_i$ for each i. [6]

A smooth projective variety X over $k(B)$ satisfies *geometric weak approximation* at $b_1, \ldots, b_r \in B$ if either of the following two equivalent conditions holds:

- For *any* nonsingular model $\mathscr{Y} \to B$ and points $y_i \in \mathscr{Y}_{b_i}^{sm}$, there exists a section $t : B \to \mathscr{Y}$ with $t(b_i) = y_i$ for each i.
- There exists *one* model $\mathscr{X} \to B$ so that, for any $N \geq 0$ and any collection of formal sections

  $$\hat{s}_i : \widehat{B}_{b_i} \to \mathscr{X} \times_B \widehat{B}_{b_i}, \quad \widehat{B}_{b_i} = \operatorname{Spec} \widehat{\mathscr{O}}_{B,b_i},$$

  there is a section $s : B \to \mathscr{X}$ so that

  $$s \equiv \hat{s}_i \pmod{\mathfrak{m}_{B,b_i}^{N+1}}, \quad i = 1, \ldots, r.$$

X satisfies *geometric weak approximation* if these hold for any finite collection of points in B.

In general, geometric weak approximation holds for:

- **rational varieties:** X is rational over $k(B)$.
- **Del Pezzo surfaces of degree ≥ 4:** This reduces to the case of quartic Del Pezzo surfaces addressed in [1].
- **hypersurfaces of very small degree:** $X \subset \mathbf{P}^n$ a hypersurface of degree d provided $n \geq \phi(d)$ where $\phi(d)$ satisfies the recursion

$$\phi(d) = \binom{\phi(d-1)+d-1}{d-1}, \quad \phi(1) = 1,$$

i.e., $n \geq 2$ for $d = 2$, $n \geq 6$ for $d = 3$, $n \geq 84$ for $d = 4$, etc. [5]

The case of Del Pezzo surfaces of degree ≤ 3 remains open.

Assuming that X is rationally connected, geometric weak approximation holds at $b_1, \ldots, b_r \in B$ for:

- **places of good reduction:** By definition, $b_i \in B$ is of *good reduction* if there exists a smooth local model near b_i, i.e., a smooth proper morphism $\widehat{\mathscr{Y}} \to \widehat{B}_{b_i}$ with generic fiber equal to X. [3]
- **places with strongly rationally connected smooth locus:** X admits a local model $\widehat{\mathscr{Y}} \to \widehat{B}_{b_i}$ such that, for each $y \in \mathscr{Y}^{sm}_{b_i}$, there exists a rational curve $\mathbf{P}^1 \to \mathscr{Y}^{sm}_{b_i}$ joining y and the generic point. [4]
- **nice cubic surfaces:** X is a cubic surface so that each b_i of bad reduction satisfies one of the following:
  i. $\mathscr{X}_{b_i}$ is a cubic surface with at worst rational double points and $\mathscr{X}$ is nonsingular along $\mathscr{X}_{b_i}$, e.g., cubic surfaces over $k(B)$ with square-free discriminant; [4]
  ii. $\mathscr{X}_{b_i}$ is a cubic surface with at worst ordinary double points, not isomorphic to the *Cayley cubic surface*

$$wxy + xyz + yzw + zwx = 0;$$

  here $\mathscr{X}$ need not be smooth. [5]

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Termination of (many) 4-dimensional log flips

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(joint work with Valery Alexeev, Christopher Hacon)

This is a report on the paper [1]. Due to a recent result by Hacon and McKernan [2] which used the extension theorem of log canonical forms and Shokurov's saturation method, the only remaining unknown part of the Minimal Model Program is the termination conjecture of flips for klt pairs. The conjecture claims that there does not exist an infinite sequence of flips

$$(X, B) = (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \dots$$

projective over a fixed variety S starting from a klt pair (X, B) where B is an $\mathbb{R}$-divisor. It was already proved to be true if $\dim X = 3$, and if $\dim X = 4$ and (X, B) is terminal.

The difficulty of the termination conjecture is to give a correct definition of the *difficulty*. In the case of the 3-dimensional terminal vatrieties, the proof of the termination by Shokurov based on the first definition of the difficulty was surpringly simple. But if one applies the same definition to the log terminal case, then the difficulty becomes infinity. The correct definition of the difficulty should satisfy the following requirements: (1) it is well-defined, (2) it decreses after a flip, and (3) it satisfies DCC.

Let (X, B) be a klt pair of dimension 4. Assume that the termination conjecture is true in the case where (X, B) is terminal. Then there exists a $\mathbb{Q}$-factorial terminalization $(X', B') \to (X, B)$. The termination for the klt case is more difficult than the terminal case because the coefficients of B' decreases even if those of B are fixed. Especially, the case where the coefficients go to 0 in the limit is the most difficult open case.

We define the difficulty of the klt pair (X, B) as that of the terminalization (X', B'). So assume that the pair (X, B) is terminal. Roughly speaking, the difficulty $d(X, B)$ counts the number of DVR's whose discrepancies are less than 1. Let B_i be an irreducible component of B with coefficient b_i. Repeated blowings up along a codimension 2 subvariety on B_i yield *echos* of the subvariety whose discrepancies are $k(1 - b_i)$ for positive integers k. Since there are infinitely many such varieties, the naive difficulty is infinity. In order to compensate the echos, we add a renormalization term to the defining formula of the difficulty which comes from the Picard number of the boundary divisor. We also use a weighted counting method in order to obtain a simpler formula.

The same proof for the termination works for both the terminal pair and the klt pair. Therefore, we run the proof twice to obtain the result as in the case of latex program.

The main theorem is the following:

Theorem 1. *Let (X, B) be a 4-dimensional klt pair. Assume that the boundary B is decomposed into two effective $\mathbb{R}$-divisors $B = B' + B''$ such that the strict transforms of B' are ample for all flips in the sequence. Then the sequence terminates.*

Corollary 1. *Let $B = \sum b_i B_i$. Assume that there exist $c_i \in \mathbb{R}$ such that $c_0 K_X + \sum c_i B_i$ is big over the base S. Then there exists* some *finite sequence of MMP for the pair (X, B) which produces the final result, i.e., a Mori fiber space or a minimal model.*

By using [2], we obtain:

Corollary 2. *There exists a flip for any small contraction in dimension* 5.

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$\mathrm{SL}_2(\mathbb{R})$-invariant loci in the one-form bundle over M_g

Martin Möller

A special case of a theorem by Ratner ([7]) states the following: Let $\Gamma \subset \mathrm{SO}(1, \mathrm{n})$ be a lattice. The quotient $\mathrm{SO}(1, \mathrm{n})/\Gamma =: FM$ is the frame bundle over the homogenous manifold $M := \mathbb{H}^n/\Gamma$. Consider a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SO}(1, \mathrm{n})$. Then for each point in $x \in FM$ the closure of the orbit $\mathrm{SL}_2(\mathbb{R}) \cdot x$ is again the orbit of a subgroup in $\mathrm{SO}(1, \mathrm{n})$.

The total space ΩM_g of the one-form bundle over the moduli space M_g of curves of genus g has a natural action by $\mathrm{SL}_2(\mathbb{R})$ as follows: One integrates locally on the Riemann surface the one-form. On the image in $\mathbb{C} \cong \mathbb{R}^2$ one can apply matrices in $\mathrm{SL}_2(\mathbb{R})$. The composition map defines a new complex structure on the surface when identifying $\mathbb{R}^2$ by $\mathbb{C}$. Although ΩM_g is not a homogeneous manifold the $\mathrm{SL}_2(\mathbb{R})$-action behaves similarly. McMullen has shown ([3] and subsequent work) that in genus $g = 2$ all orbit closures are indeed algebraic manifolds. They are now completely classified.

In higher genera this problem can be decomposed into two steps: First one wants show that orbit closures are complex, hopefully even algebraic manifolds. Second one wants to characterize and classify such manifolds. The latter is a purely algbro-geometric problem for the following reason. There is a natural stratification of ΩM_g according to the number and multiplicities of zeros of the one-form. The strata posses a structure of a linear manifold. In fact choosing locally a basis of relative periods and integrating them against the one-form defines a local diffeomorphism between the strata and $\mathbb{C}^N$. The next proposition, well-known to the experts, provides the translation between group action and algebraic geometry.

Proposition 1. *A closed analytic subspace in* $\mathbb{C}^N$ *is* $\mathrm{GL}_2(\mathbb{R})$*-invariant if and only if it is linear and can be defined by* $\mathbb{R}$*-linear equations.*

For the lowest dimensional orbit closures there is the following solution to the characterization problem. Orbits that are already closed project to Teichmüller curves C in M_g. These are algebraic curves that are geodesic for the Teichmüller metric. See e.g. [2] for more background on Teichmüller curves and relation to billiards. Given $C \to M_g$ consider the pullback $f : X \to C_1$ of the universal family over the moduli space of curves $M_g^{[n]}$ with some level-n-structure to the corresponding unramified cover $C_1 \to C$.

Theorem 1. *([4]) A family* $f : X \to C_1$ *of curves arises as above from a Teichmüller curve if and only if the local system* $\mathbb{V} = R^1 f_* \mathbb{C}$ *contains a rank two subsystem* $\mathbb{L}$ *whose Kodaira-Spencer mapping*

$$\mathbb{L}^{(1,0)} \to \mathbb{L}^{(0,1)} \otimes \Omega^1_{C_1}(\log(\overline{C_1} \setminus C_1))$$

is an isomorphism.

This characterization of Teichmüller curves can be used to construct Teichmüller curves and also billiard tables with special dynamical properties. See [5] or [1] for more details on this joint work with I.Bouw. Nevertheless the classification of Teichmüller curves is still wide open for $g \geq 3$.

Towards the classification of higher dimensional linear manifolds in $g \geq 3$ we have the following partial result:

Theorem 2. *([6]) A linear manifold in a stratum of* ΩM_3 *is either a whole stratum, the intersection of a stratum with the hyperelliptic locus or it parameterizes curves whose Jacobian has non-trivial endomorphisms.*

A completion of the picture in $g = 3$ seems to depend, among other things, on deciding if the preimages in M_3 of Hilbert modular threefolds always intersect the hyperelliptic locus.

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Singularities of divisors of low degrees on abelian varieties

Olivier Debarre
(joint work with Christopher Hacon)

We work over the complex numbers.

Let D be an effective $\mathbb{Q}$-divisor on a smooth projective variety A, let $\mu : A' \to A$ be a *log resolution* of the pair (A, D), and write

$$\mu^*(K_A + D) = K_{A'} + \sum a_i D_i$$

where the D_i are distinct prime divisors on A'. The pair (A, D) is

- *log canonical* if $a_i \leq 1$ for all i;
- *(Kawamata) log terminal* if $a_i < 1$ for all i;
- (D prime) *canonical* if $a_i \leq 0$ for all i such that D_i is μ-exceptional.

These properties have consequences for the singularities of D: for any positive integers m and k,

- if $(A, \frac{1}{m}D)$ is log canonical, $\operatorname{codim}_A(\operatorname{Sing}_{mk} D) \geq k$;
- if $(A, \frac{1}{m}D)$ is log terminal, $\operatorname{codim}_A(\operatorname{Sing}_{mk} D) > k$;
- (D prime) if (A, D) is canonical, D is normal with rational singularities and $\operatorname{codim}_A(\operatorname{Sing}_k D) > k$ for $k \geq 2$.

Let now A be an abelian variety of dimension g. The degree of an ample divisor D on A is defined by

$$\deg(D) = \frac{1}{g!} D^g = h^0(A, D).$$

An ample divisor of degree 1 is called a theta divisor.

Kollár was the first to use vanishing theorems to prove results on the singularities of divisors in abelian varieties. Here is a sample result.

Theorem 1 (Kollár for $m = 1$; Ein–Lazarsfeld)**.** *Let (A, Θ) be a complex principally polarized abelian variety and let $D \equiv m\Theta$, where m is a positive integer. The pair $(A, \frac{1}{m}D)$ is log canonical.*

The proof is very simple: to any effective $\mathbb{Q}$-divisor on A one can associate a *multiplier ideal* and log canonicity is equivalent to $\mathscr{I}(A, \frac{t}{m}D) = \mathscr{O}_A$ for all rationals $t \in (0, 1)$. Let Z be the subscheme of A defined by one of these ideals. The Nadel vanishing theorem yields

$$H^i(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0 \qquad \text{for all } i > 0 \text{ and all } a \in A.$$

If Z is nonempty, we have $Z \not\subset \Theta_a$ for a general, hence $H^0(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0$. It follows that

$$\chi(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = H^0(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0$$

for $a \in A$ general hence for all a because the Euler characteristic is a numerical invariant. We conclude that

$$H^i(A, \mathscr{O}_A(\Theta_a) \otimes \mathscr{I}_Z) = 0 \qquad \text{for all } i \text{ and all } a \in A.$$

By the Fourier–Mukai theory, this implies $\mathscr{O}_A(\Theta) \otimes \mathscr{I}_Z = 0$, which is absurd.

Ein and Lazarsfeld also prove that if Θ is irreducible, (A, Θ) is canonical. In particular, Θ is normal and has rational singularities. This was known before only for Jacobians of curves, in which case the result holds in any characteristic (Kempf). It is unknown whether this is still true for any principally polarized abelian variety in positive characteristic.

As explained above, this implies

$$\operatorname{codim}_A(\operatorname{Sing}_k \Theta) > k$$

for $k \geq 2$. For Jacobians, and more generally for generalized Prym varieties (hence for any indecomposable principally polarized abelian variety of dimension ≤ 5), one actually has

$$\operatorname{codim}_A(\operatorname{Sing}_k \Theta) \geq 2k - 1$$

for $k \geq 2$ by work of Casalaina–Martin. Is it true that this holds in any dimension?

We prove the following extension of the above theorem to ample divisors whose degree is small with respect to the dimension.

Theorem 2. *Let (A, L) be a* simple *polarized abelian variety of degree d and dimension $g > (d+1)^2/4$ and let $D \equiv mL$.*

- *if $m = 1$, the divisor D is prime and the pair (A, D) is canonical;*
- *if $m \geq 2$, the pair $(A, \frac{1}{m}D)$ is log terminal unless $D = mE$.*

If A is not simple, $|L|$ may very well contain reducible elements. There are also examples, in any dimension $g \geq 2$, and for any $d \geq 3$ and $m \geq d - 1$, of pairs $(A, \frac{1}{m}D)$ that are not log canonical.

The proof goes as follows (when $m \geq 2$, we will only prove log canonicity; log terminality is harder). The point is to show that

- if $m = 1$, the *adjoint ideal* (which I won't define here) $\mathscr{J}(A, D)$ is trivial;
- if $m \geq 2$ and $t \in \mathbb{Q} \cap (0, 1)$, the multiplier ideal $\mathscr{I}(A, tD)$ is trivial.

In each case, let Z be the subscheme of D defined by the ideal and set

$$h = h^0(A, L_a \otimes \mathscr{I}_Z) \in [0, d]$$

for a general in A.

If $h = d$, all sections of L contain all translates of Z, which must be empty.

Assume $h = 0$. In the case $m \geq 2$, we conclude as in the Ein–Lazarsfeld proof $\mathscr{O}_A(\Theta) \otimes \mathscr{I}_Z = 0$, which is absurd. In the case $m = 1$, work of Ein and Lazarsfeld implies that D is fibered by nonzero abelian varieties, which is also absurd.

So we assume $0 < h < d$ (and Z nonempty). Set

$$J = \{(s, a) \in \mathbf{P}H^0(A, L) \times A \mid s|_{Z+a} \equiv 0\}.$$

The fiber of a point a of A for the second projection $q : J \to A$ is $\mathbf{P}H^0(A, L \otimes \mathscr{I}_{Z+a}) \simeq \mathbf{P}^{h-1})$ and a unique irreducible component I of J dominates A. It has dimension $g + h - 1$.

Let $p : I \to \mathbf{P}H^0(A, L)$ be the first projection. Nonempty fibers $F_s = q\big(p^{-1}(s)\big)$ have dimension $\geq g + h - d$. For a general in A, the subvariety $p(q^{-1}(a)) = \mathbf{P}H^0(A, L \otimes \mathscr{I}_{Z+a})$ of $\mathbf{P}H^0(A, L)$ is a linear subspace of dimension $h - 1$. It must vary with a, because a nonzero s does not vanish on all translates of Z. It follows that the linear span of $p(I)$ has dimension at least h. For $s_1, \ldots, s_{h+1}$ general elements in $p(I)$, one has, since A is simple,

$$\dim(F_{s_1} \cap \cdots \cap F_{s_{h+1}}) \geq g - (h+1)(d-h) \geq g - (d+1)^2/4.$$

For $a \in F_{s_1} \cap \cdots \cap F_{s_{h+1}}$, the sections $s_1, \ldots, s_{h+1}$ all vanish on $Z + a$, hence $h^0(A, L_a \otimes \mathscr{I}_Z) \geq h+1$. Since the Euler characteristic $\chi(A, L_a \otimes \mathscr{I}_Z)$ is independent of a, this proves that

$$V_{>0} = \{a \in A \mid H^i(A, L_a \otimes \mathscr{I}_Z) \neq 0 \text{ for some } i > 0\}$$

has dimension $\geq g - (d+1)^2/4 > 0$.

When $m = 1$, it follows from the Green–Lazarsfeld theory that every irreducible component of the set

$$V_i = \{a \in A \mid H^i(A, L_a \otimes \mathscr{I}_Z) \neq 0\}$$

is a translated abelian subvariety of A of codimension $\geq i$; since A is simple, it is finite for $i > 0$. When $m \geq 2$, we have $V_{>0} = \emptyset$ by Nadel vanishing. In each case, we get a contradiction.

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A generalization of Mumford's example

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(joint work with H. Nasu)

Let $\mathrm{Hilb}^{sc}\, V$ be the Hilbert scheme parametrizing smooth curves in a smooth projective variety V. In [3], Mumford showed that $\mathrm{Hilb}^{sc}\, \mathbb{P}^3$ has a generically non-reduced component. More precisely the following is proved:

Example 1. Let S be a smooth cubic surface in $\mathbb{P}^3$, E a (-1)-$\mathbb{P}^1$ in S and $C \subset S$ a smooth member of the linear system $|4h + 2E| \simeq \mathbb{P}^{37}$ on S. (C is of degree 14 and genus 24.) Such space curves C are parametrized by $W^{56} \subset \mathrm{Hilb}^{sc}\, \mathbb{P}^3$, an open subset of a $\mathbb{P}^{37}$-bundle over $|3H| \simeq \mathbb{P}^{19}$. Here H is a plane in $\mathbb{P}^3$ and h is its restriction to S. Then W^{56} is an irreducible component of $(\mathrm{Hilb}^{sc}\, \mathbb{P}^3)_{red}$ and $\mathrm{Hilb}^{sc}\, \mathbb{P}^3$ is nowhere reduced along W^{56}.

It is well known that every infinitesimal (embedded) deformation of $C \subset V$ is unobstructed if $H^1(N_{C/V}) = 0$. Conversely we find a sufficient condition for a first order infinitesimal deformation of a curve C in a 3-fold V to be obstructed, abstracting an essence from the arguments in [1] and [4]. As application we construct generically non-reduced components of the Hilbert schemes of uniruled 3-folds V including Examples 1 and 2 as special cases:

Example 2 ([2])**.** Let V_3 be a smooth cubic 3-fold in $\mathbb{P}^4$, S its general hyperplane section, E a (-1)-$\mathbb{P}^1$ in S and $C \subset S$ a smooth member of $|2h+2E| \simeq \mathbb{P}^{12}$. ($C$ is of degree 8 and genus 5.) Such curves C in V_3 are parametrized by $W^{16} \subset \mathrm{Hilb}^{sc}\, V$, an open subset of $\mathbb{P}^{12}$-bundle over the dual projective space $\mathbb{P}^{4,\vee}$. Then W^{16} is an irreducible component of $(\mathrm{Hilb}^{sc}\, V_3)_{red}$ and $\mathrm{Hilb}^{sc}\, V_3$ is nowhere reduced along W^{16}.

The curves C of genus 24 in Example 1 are not (moduli-theoretically) general but the curves C of genus 5 in Example 2 are general. Hence, with the help of Sylvester's pentahedral theorem ([5]), Example 2 gives a counterexample to the following problem:

Problem 1. Is every component of the Hom scheme $\mathrm{Hom}(X, V')$ generically smooth for a smooth curve X with general modulus and for a general member V' in the Kuranishi family of V?

Let $\mathrm{Hom}_8(X_5, V_3)$ be the Hom scheme of morphisms of degree 8 from a curve X_5 of genus 5 with general modulus to a smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$.

Theorem 1 ([2])**.** *If V_3 is also moduli-theoretically general, then* $\mathrm{Hom}_8(X_5, V_3)$ *has a generically non-reduced component of expected dimension ($= 4$).*

The following seems still open:

Problem 2. Let G/P be a projective homogeneous space, *e.g.*, a Grassmann variety and X a curve with general modulus. Is every component of $\mathrm{Hom}(X, G/P)$ generically smooth?

The answer is affirmative for the projective space $\mathbb{P}^n$ by virtue of Gieseker's theorem (= Petri's conjecture).

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A Step in Castelnuovo theory via Gröbner bases

Ivan Petrakiev

I report on a recent result in Castelnuovo theory concerning the extrinsic properties of projective curves of high genus ([9]).

Let C be a reduced, irreducible and nondegenerate curve of degree d and arithmetic genus g in $\mathbf{P}^n$. A celebrated theorem of Castelnuovo (1889) gives an explicit upper bound $\pi_0(d,n)$ on g in terms of d and n. Moreover, curves that attain the maximal genus, the so called *Castelnuovo curves*, have rather special extrinsic properties and are well understood. In particular, as long as $d \geq 2n+1$, such curves always lie on surfaces of minimal degree $n-1$.

Castelnuovo's theorem has been reconsidered and extended further by several classical geometers, including G. Halphén ([6]), G. Fano ([5]) and, later, by Eisenbud-Harris ([3]). The main philosophy of the modern Castelnuovo theory is that curves of sufficiently high genus should lie on surfaces of some small degree.

Extending Castelnuovo's bound, Eisenbud-Harris ([3]) defined a decreasing string of numbers

$$\pi_\alpha(d,n) \approx \frac{d^2}{2(n-1+\alpha)} + O(d),$$

where $\alpha = 0,1,\ldots,n-1$, and made a conjecture:

Conjecture 1. *If C is a curve of genus $g > \pi_\alpha(d,n)$ and $d \geq 2n+2\alpha-1$, then C must lie on a surface of degree at most $n+\alpha-2$.*

In [3], a proof is given for the case $\alpha = 1$, although a similar result has been already known to Fano. The Eisenbud-Harris conjecture is also known to be true any α, as long as $d >> 0$ (the explicit bound on d is exponential in n).

In [9] we settle the next case $\alpha = 2$ ($n \geq 8$) of the Eisenbud-Harris conjecture. The only previous work in this direction known to us is the paper of C. Ciliberto ([2]) , where some partial results were obtained by different methods.

Recall the main circle of ideas involved in Castelnuovo theory. Let $\Gamma = C \cap \mathbf{P}^{n-1}$ be a general hyperplane section of C. We will say, that Γ is in *symmetric position*, which generalizes the notion of *uniform position*, first introduced by Harris in [7].

As Castelnuovo observed, if C is to have high genus, then Γ must have a "small" Hilbert function $h_\Gamma(l)$ and, in particular, Γ must fail to impose many conditions on quadrics in $\mathbf{P}^{n-1}$. Assume $d \geq 2n+1$. Then, according to the well-known Castelnuovo's lemma, $h_\Gamma(2)$ takes its minimal value $2n-1$ precisely when Γ is a set of points lying on a *rational normal curve* in $\mathbf{P}^{n-1}$. This allowed Castelnuovo to determine his bound $\pi_0(d,n)$ on the genus of C and describe the curves that achieve it.

By generalizing Castelnuovo's lemma, one is naturally lead to conjecture the following (see also [4] for a generalization).

Conjecture 2. *If $\Gamma \subset \mathbf{P}^{n-1}$ is a set of $d \geq 2n+2m-1$ points in symmetric position, with $h_\Gamma(2) \geq 2n+m-2$ (where $1 \leq m \leq n-3$), then Γ must lie on a curve of degree at most $n+m-2$.*

This conjecture also appears in [8] in a different context.

In [9] we establish the first previously unknown cases $m = 3$ ($n \geq 5$) and $m = 4$ ($n \geq 7$) of Conjecture 2. This in turn implies the corresponding result on Conjecture 1.

The starting point in our work is the fact, that under the assumptions of Conjecture 2, Γ lies on an m-fold rational normal scroll (this was already observed by Fano and rediscovered by Eisenbud-Harris and Reid). We use this, together with the symmetry of Γ, to write the beginning of a Gröbner basis for the homogeneous ideal of Γ in degree 2, in a suitable coordinate system and monomial order. It turns out, that there are only few quadrics missing in our Gröbner basis, precisely $\binom{m-1}{2}$. We make a conjecture about the "missing" $\binom{m-1}{2}$ quadrics and support it with some evidence, that comes from an elementary geometric observation. In the cases $m = 3, 4$, we are actually able to complete the whole Gröbner basis of Γ in degree 2.

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Preconceptions and misconceptions on relative stable maps in the normal crossings case

Dan Abramovich
(joint work with Barbara Fantechi)

I report on joint work in progress with Barbara Fantechi; a closely related work in symplectic geometry is being developed by Joshua Davis of Duke University.

The theory of relative stable maps was first introduced by Ziv Ran in his paper on the degree of the Severi variety under a different name; the subject was developed within Gromov-Witten theory by a number of people, including A.M. Li–Y. Ruan, E. Ionel–T. Parker, and J. Li. Working in algebraic geometry, we must follow the work of Jun Li. Related work appeared through the years, including Harris–Mumford, Alexander-Hirschowitz, Gathmann, Caporaso–Harris, Vakil.

Stable maps were introduced by Kotsevich as a tool in Gromov–Witten theory, which from the point of view of this workshop, serves as a tool in enumerative geometry. The main goal is to count the number of curves of given genus g and homology class β on a variety X meeting given cycles $\gamma_1, \dots \gamma_n$.

Tools in Gromov–Witten theory include the famous WDVV equation, but much more powerful are the methods of localization and degeneration. I concentrate on the degeneration method.

Previous work concentrated on the case of a family of varieties parametrized by a curve, with smooth total space and special fiber consisting of two smooth components meeting transversally along a divisor Σ. The issue is that, although a space of stable map fibered over the base exists, the formalism of virtual fundamental classes fails in genus > 0 in case there are components mapping to the singular locus Σ of the fiber. The solution involves expanding the fiber by sticking a chain of $\mathbb{P}^1$ bundles over Σ between the two original components of the fiber.

One then defines degenerate stable maps to the fibers, and similarly relative stable maps to each component, and one proves a gluing formula for degenerate stable maps in terms of relative stable maps to each of the components.

The problem we set out to solve is:

(1) define relative stable maps to (Y, D), where D is a normal crossings divisor on a smooth variety Y,
(2) define degenerate stable maps to a variety obtained by gluing such relative (Y_i, D_i) appropriately along the divisors, in such a way that Gromov–Witten invariants are defined and are deformation invariant, and
(3) prove a gluing formula comparing the two.

The aim of the talk was to describe a good number of places where one finds welcome and unwelcome surprises in the project. Due to ill planning, most of the talk ended up in explaining the earlier work.

A welcome surprise mentioned in the talk is the following: much grief was brought on previous writers in analyzing the so called "predeformability condition", a closed condition on relative and degenerate stable maps which is unpleasant to work out. Techniques of stacks and twisted stable maps of Olsson and Abramovich–Vistoli enable one to transform this into an open condition on a modified space, thus avoiding much of the grief.

An unwelcome surprise mentioned in the talk is in describing the gluing formula: whereas in the previously studied case the Gromov–Witten numbers of a two-component variety was described in terms of its decomposition to exactly two components, in our case, where the degenerate variety has at least three intersecting components, our formalism requires summing over further decompositions, where each component of the original variety is further "expanded".

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The log canonical and stable pair compactifications of moduli of del Pezzo surfaces

Sean Keel
(joint work with Paul Hacking, Eugene Tevelev)

Let Y^n be the moduli space of isomorphism classes of smooth marked del Pezzo surfaces of degree $9-n$, where a marking of S means an expression of S a blowup of $\mathbb{P}^2$ in n-marked points (and two marked surfaces are iso if the expressions differ by an element of PGL_3). Let $B \subset S$ be the union of its (-1)-curves. Let $Y^n_\times \subset Y^n$ be the open subset corresponding to surfaces such that B has normal crossings. There are two natural Mori theoretic compactifications of Y^n, each of which turns out to be very nice:

Theorem 1. *Y_n is log minimal. For $n \leq 7$, its log canonical compactification $\overline{Y}^n_{\mathrm{lc}}$ exists and is smooth and projective, and the boundary $\overline{Y}^n_{\mathrm{lc}} \setminus Y^n$ is a union of smooth divisors with simple normal crossings.*

Theorem 2. *$Y^n_\times$ embeds in the Kollár–Shepherd-Barron–Alexeev moduli stack of stable surfaces. Now suppose that $n \leq 6$.*

This embedding extends to the embedding of Y^n. Let $Y^n \subset \overline{Y}^n_{\mathrm{ss}}$ be the closure. $\overline{Y}^n_{\mathrm{ss}}$ is a smooth and projective scheme. The boundary $\overline{Y}^n_{\mathrm{ss}} \setminus Y^n$ is a union of smooth divisors with simple normal crossings. Let $\pi : (\mathscr{S}, \mathscr{B}) \to \overline{Y}^n_{\mathrm{ss}}$ be the restriction of the universal family. Its fibres (S, B) have at worst stable toric singularities.

The functorial morphism $Y^{n+1} \to Y^n$ extends to the morphism $\overline{Y}^{n+1}_{\mathrm{lc}} \to \overline{Y}^n_{\mathrm{lc}}$ and we have commutative diagrams

$$\begin{array}{ccc} \mathscr{S} & = & \overline{Y}^6_{\mathrm{lc}} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \overline{Y}^5_{\mathrm{ss}} & = & \overline{Y}^5_{lc} \end{array} \qquad \begin{array}{ccc} \mathscr{S} & \longrightarrow & \overline{Y}^7_{\mathrm{lc}} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \overline{Y}^6_{\mathrm{ss}} & \longrightarrow & \overline{Y}^6_{lc}, \end{array}$$

where the horisontal arrows in the second diagram are log crepant.

It turns out that $\overline{Y}^6_{lc}$ is isomorphic to the Naruki space of cubic surfaces and $\overline{Y}^7_{lc}$ is isomorphic to Sekiguchi's space of degree 2 del Pezzo surfaces. Sekiguchi gave a root system interpretation of Naruki's construction, and then a unified way of constructing $\overline{M}_{0,n}$, $\overline{Y}^6$, $\overline{Y}^7$ from D_n, E_6, E_7, using certain cross ratio maps associated to sub D_4 root systems (he attributes this construction of $\overline{M}_{0,n}$ to Terada). We observe that Sekiguchi's D_4 cross ratios have the following geometric expression: Given a Del Pezzo surface S and an arrangement L, L_1, L_2, L_3, L_4 consisting of 4 pairwise disjoint (-1)-curves L_i each incident to a (-1)-curve L, intersecting gives 4 points

$$\{L_1 \cap L,\ L_2 \cap L,\ L_3 \cap L,\ L_4 \cap L\} \subset L \tag{1}$$

This induces a regular map $Y^n \to M_{0,4} \subset \mathbb{P}^1$. Sekiguchi observes that taking all possible arrangments gives an immersion (for some large integer N):

$$Y^n \subset (\mathbb{P}^1)^N. \tag{2}$$

Theorem 3 (Naruki). *For $n = 6$, the closure $\overline{Y}^6$ is smooth and the boundary $\overline{Y}^6 \setminus Y^6$ has smooth components and simple normal crossings.*

We will show that in fact $\overline{Y}^6 = \overline{Y}^6_{lc}$. Sekiguchi conjectured that Theorem 3 also holds for $n = 7$. We prove that this is indeed the case:

Theorem 4 (Sekiguchi's conjecture). *For $n = 7$, the closure in* (2) *is equal to $\overline{Y}^7_{\mathrm{lc}}$, in particular smooth, with simple normal crossing boundary.*

The above condition that the L_i be pairwise disjoint is rather unnatural – we can (on $Y^n_\times$) define such a map for any four (-1) curves all incident to a fifth curve L Doing so yields $\overline{Y}^n_{ss}$:

Theorem 5. *For $n \leq 6$, the product of all cross ratio maps gives a closed embedding $\overline{Y}^n_{ss} \subset (\mathbb{P}^1)^m$, where m is the number of arrangements of four (-1)-curves incident to a fifth curve.*

We obtain $\overline{Y}^n_{lc}$, and $\overline{Y}^n_{ss}$ together with its universal family (and thus its functorial meaning) canonically from the interior Y^n by applying elementary ideas from Mori theory and tropical algebraic geometry. The same construction applied to $M_{0,n}$ yields $\overline{M}_{0,n}$. Next we explain the procedure:

We begin with a brief review of tropical algebraic geometry: Let k be an algebraically closed field. For any variety Y defined over k, $M_Y := \mathscr{O}^*(Y)/k^*$ is a lattice. Let T_Y (the intrinsic torus of Y) be the corresponding algebraic torus, $T_Y := \mathrm{Hom}(M_Y, \mathbb{G}_m)$. There is a canonical evaluation map $Y \to T_Y$ (unique up to translation by an element of T_Y). One checks that Y is very affine (i.e. is a closed subvariety of some algebraic torus) iff $Y \to T_Y$ is a closed embedding.

Let K be the field of Puiseux series over k. Let $\deg : K^* \to \mathbb{Q}$ be the non-archimedean valuation. Let $Y(K)$ (resp. $T_Y(K)$) be the set of K-points of Y (resp. T_Y). The valuation gives rise to the map

$$\deg : T_Y(K) \to U_Y \otimes \mathbb{Q}.$$

The Tropical Variety $\mathrm{trop}(Y)$ of Y is defined as the image of the composition

$$Y(K) \to T_Y(K) \overset{\deg}{\to} U_Y \otimes \mathbb{Q}.$$

We make use of the following definition, due to Tevelev:

Definition 1. Let $\mathscr{F}$ be a fan in $U_Y \otimes \mathbb{Q}$, $T_Y \subset X(\mathscr{F})$ the corresponding toric variety and $\overline{Y}(\mathscr{F}) \subset X(\mathscr{F})$ the closure of $Y \subset X(\mathscr{F})$. We call $\mathscr{F}$ (and $\overline{Y}(\mathscr{F})$) tropical if $\overline{Y}(\mathscr{F})$ is complete, and the structure map $\overline{Y}(\mathscr{F}) \times T_Y \to X(\mathscr{F})$ is flat and surjective.

Tevelev proves the following:

Theorem 6. *Tropical fans exist, and if $\mathscr{F}$ is tropical then $|\mathscr{F}| = \mathrm{trop}(Y)$. Any refinement of a tropical fan is tropical.*

It is natural to wonder if every fan structure on trop(Y) is tropical, and more importantly, if there is a canonical choice –so the closure in the corresponding toric variety gives a canonical way of compactifying Y. We observe that under some natural conditions, this is indeed the case:

Tevelev proved that if the structure map for one tropical compactification is smooth, then this is true for any tropical compactification (thus this is a property of Y). In this instance we say that Y is Schön. Note in this case any tropical compactification (together with its boundary) has toric singularities.

Definition 2. We say that a Schön very affine variety Y is Hübsch if it is log minimal (i.e. some log pluricanonical map is an immersion) and the log canonical compactification is tropical. We call the corresponding fan structure on trop(Y) the log canonical fan.

We prove the following:

Theorem 7. *If Y is Hübsch, then any fan supported on* trop(Y) *is tropical, and is a refinement of the log canonical fan.*

Thus in the Hübsch case the log canonical fan – and thus the log canonical compactification – can be recovered from the set trop(Y).

There turn out to be many nice examples:

Theorem 8. *Complements of connected hyperplane arrangments, Y_n ($n \leq 7$), and $M_{0,n}$ are all Hübsch. The corresponding tropical (= log canonical) compactifications are Kapranov's visible contour compactification, the compactification of Theorem 1, and the Grothendieck-Knutsen compactification $M_{0,n} \subset \overline{M}_{0,n}$.*

Next we describe the tropical fans for Y_n and $M_{0,n}$. Building on ideas of Sekeguchi these turn out to be canonically associated with the root systems E_n and D_n. For that reason we also denote $Y(E_n) = Y_n$, $Y(D_n) = M_{0,n}$.

Here we set $E_4 = A_4$ and $E_5 = D_5$. This notation is consistent because $M_{0,5} = Y_5$. Combinatorics of the fibers of various universal families is encoded in two towers of embeddings of root systems

$$\begin{array}{ccccccccc} & & & & E_4 & & & & \\ & & & & \cap & & & & \\ D_3 & \subset & D_4 & \subset & D_5/E_5 & \subset & D_6 & \subset & \dots \\ & & & & \cap & & & & \\ & & & & E_6 & & & & \\ & & & & \cap & & & & \\ & & & & E_7 & & & & \\ & & & & \cap & & & & \\ & & & & E_8 & & & & \end{array} \tag{3}$$

Theorem 9. *Let Δ be D_n or E_n and let $Y = Y(\Delta)$. Let W be the Weyl group. Let $\mathbb{Z}^{\Delta_+}$ be the lattice with basis vectors $[\alpha]$ for positive roots α. Let*

$$U(\Delta)^\vee = \left\{\sum n_\alpha[\alpha] \,\middle|\, \sum n_\alpha \alpha^2 = 0\right\} \subset \mathbb{Z}^{\Delta_+}.$$

Then $U(\Delta)^\vee$ is an irreducible W-module of rank equal to the number of roots in Δ_+ with three-legged support. We have an isomorhism of W-modules

$$\mathscr{O}^*(Y)/k^* = U(\Delta)^\vee.$$

Let $\psi : \mathbb{Z}^{\Delta_+} \to U(\Delta)$ be the dual map. For any root subsystem $\Theta \in \Delta$, let $\psi(\Theta) := \psi(\sum[\alpha])$, the sum over $\alpha \in \Theta \cap \Delta_+$. Let $\mathscr{F}(\Delta) \subset U(\Delta) \otimes \mathbb{Q}$ be the fan defined as follows. Its rays are spanned by $\psi(\Theta)$ for all subsystems of type:

- $D_2, \ldots, D_n$ for $\Delta = D_{2n+1}$;
- $D_2, \ldots, D_{n-1}, D_n \times D_n$ for $\Delta = D_{2n}$;
- $A_1, A_2 \times A_2 \times A_2$ for $\Delta = E_6$;
- $A_1, A_2, A_3 \times A_3, A_7$ for $\Delta = E_7$.

Rays that correspond to subsystems $\Theta_1, \ldots, \Theta_k$ span a cone if and only if any two subsystems are either orthogonal or nested with one exception: in the case of E_7, we exclude cones that correspond to seven-tuples of orthogonal A_1's.

Theorem 10. *$\mathscr{F}(\Delta)$ is the log canonical fan of $Y(\Delta)$, and is strictly simplicial, i.e. the corresponding toric variety $X(\mathscr{F}(\Delta))$ is smooth. ($n \leq 7$ for E_n).*

Tevelev proved that a dominant map of very affine varieties $Y' \to Y$ induces a surjective map of intrinsic tori $T_{Y'} \to T_Y$ and a surjective map of tropical varieties $\text{trop}(Y') \to \text{trop}(Y)$. Let $T_{Y'/Y}$ be the kernel of $T_{Y'} \to T_Y$.

Theorem 11. *Let $\mathscr{F}_n := \mathscr{F}(E_n)$. Let $n \leq 6$. The natural morphism $Y^{n+1} \to Y^n$ induces a surjective map of fans $\mathscr{F}_{n+1} \to \mathscr{F}_n$. This induces a commutative diagram:*

$$\begin{array}{ccc} \overline{Y}^{n+1}_{\text{lc}} & \longrightarrow & X(\mathscr{F}_{n+1}) \\ \downarrow & & \downarrow \pi \\ \overline{Y}^{n}_{\text{lc}} & \longrightarrow & X(\mathscr{F}_{n}) \end{array}$$

π has reduced fibres. There are canonical minimal refinements $\mathscr{F}'_n$ of $\mathscr{F}_n$, $\tilde{\mathscr{F}}_{n+1}$ of $\mathscr{F}_{n+1}$ inducing a commutative diagram with flat π':

$$\begin{array}{ccccccc} \tilde{Y}^{n+1} & \longrightarrow & Z^n & \longrightarrow & X(\tilde{\mathscr{F}}_{n+1}) & \longrightarrow & X(\mathscr{F}_{n+1}) \\ \downarrow & & \downarrow & & \downarrow \pi' & & \downarrow \pi \\ \overline{Y}^{n}_{ss} & = & \overline{Y}^{n}_{ss} & \longrightarrow & X(\mathscr{F}'_n) & \longrightarrow & X(\mathscr{F}_n) \end{array}$$

Here $\overline{Y}^n_{ss}$ is the closure of Y^n in $X(\mathscr{F}'_n)$, $\tilde{Y}^{n+1}$ is the closure of Y^{n+1} in $X(\tilde{\mathscr{F}}_{n+1})$, and the diagram in the middle is Cartesian. $\overline{Y}^n_{ss}$ is isomorphic to the closure of $Y^n_\times$ in the Kollar-Shepherd-Barron–Alexeev space of stable surfaces. Consider the multiplication map

$$\Psi : \tilde{Y}^{n+1} \times T_{Y^{n+1}/Y^n} \to Z^n.$$

For $n \leq 5$, $\pi = \pi'$ is flat, $\overline{Y}^n_{ss} = \overline{Y}^n_{lc}$, ¡$\tilde{Y}^{n+1} = \overline{Y}^{n+1}_{lc}$, Ψ is smooth, and $\tilde{Y}^{n+1} \to \overline{Y}^n_{ss}$ is the universal family $\mathscr{S} \to \overline{Y}^n_{ss}$.

Let $n = 6$. Then Ψ is smooth outside of Eckhart points, i.e. smooth points of the fibre where three boundary divisors meet transversally. Blowing up the union of Eckhart points (which is a smooth subvariety contained in the smooth locus of $\tilde{Y}_{n+1}$) yields the universal family of stable cubic surfaces $\mathscr{S} \to \overline{Y}^6_{ss}$.

Let us stress one point of the construction which strikes us as remarkable: We begin with the canonical universal family $p : Y^{n+1} \to Y^n$ (whose fibre over a surface is the complement in the surface to the union of its -1 curves) and obtain from it the functorial compactification, together with its universal family (and thus the functor itself) by a canonical, combinatorial procedure, uniquely determined by p and the map (of sets!) $\mathrm{trop}(Y^{n+1}) \to \mathrm{trop}(Y^n)$. We note that the family of stable pairs we obtain is highly non-trivial. For example, in the simplest possible degeneration, corresponding to a generic point of a boundary divisor of $\overline{Y}^5_{ss} = \overline{M}_{0,5}$, the limit surface has 6 (smooth) components.

A natural smooth compactification of the space of elliptic curves in projective space via blowing up the space of stable maps

Ravi Vakil

(joint work with Aleksey Zinger)

The moduli space of stable maps $\overline{\mathscr{M}}_{g,k}(X, \beta)$ to a complex projective manifold X (where g is the genus, k is the number of marked points, and $\beta \in H_2(X, \mathbb{Z})$ is the image homology class) is the central tool and object of study in Gromov-Witten theory. The open subset corresponding to maps from smooth curves is denoted $\mathscr{M}_{g,k}(X, \beta)$.

The protean example is $\overline{\mathscr{M}}_{0,k}(\mathbb{P}^n, d)$. This space is wonderful in essentially all ways: it is irreducible, smooth, and contains $\mathscr{M}_{0,k}(\mathbb{P}^n, d)$ as a dense open subset. The boundary

$$\Delta := \overline{\mathscr{M}}_{0,k}(\mathbb{P}^n, d) \setminus \mathscr{M}_{0,k}(\mathbb{P}^n, d)$$

is normal crossings. The divisor theory is fully understood, and combinatorially tractable [4]. In some sense, this should be seen as the natural generalization of the space of complete conics compactifying the space of smooth conics.

It is natural to wonder if such a beautiful structure exists in higher genus. In arbitrary genus, however, there is no reasonable hope: $\mathscr{M}_g(\mathbb{P}^n, d)$ is badly behaved. (We emphasize that even the *interior* of the moduli space of stable maps is badly-behaved.) More precisely, $\mathscr{M}_g(\mathbb{P}^n, d)$ (as g, n, and d vary) is arbitrarily singular in a well-defined sense — it can have essentially any singularity, and can have components of various dimension meeting in various ways with various nonreduced structures [6]. In short, there is no reasonable hope of describing a desingularization, as this would in essence involve describing a resolution of singularities.

In genus one, however, the situation remains remarkably beautiful. Although $\overline{\mathscr{M}}_{1,k}(\mathbb{P}^n, d)$ in general has many components, it is straightforward to show that $\mathscr{M}_{1,k}(\mathbb{P}^n, d)$ is irreducible and smooth. Let $\overline{\mathscr{M}}^0_{1,k}(\mathbb{P}^n, d)$ be the closure of this open subset (the "main component" of the moduli space).

We will describe a natural desingularization of this main component

$$\widetilde{\mathscr{M}}_{1,k}(\mathbb{P}^n, d) \to \overline{\mathscr{M}}^0_{1,k}(\mathbb{P}^n, d).$$

(Details appear in [7]. In particular, it is proved there that this construction actually gives a desingularization.) This desingularization has several desirable properties.

- It leaves the interior $\mathscr{M}_{1,k}(\mathbb{P}^n, d)$ unchanged.
- The boundary $\widetilde{\mathscr{M}}_{1,k}(\mathbb{P}^n, d) \setminus \mathscr{M}_{1,k}(\mathbb{P}^n, d)$ is simple normal crossings, with an explicitly described normal bundle.
- The points of the boundary have explicit geometric interpretations.
- The desingularization can be interpreted as blowing up "the most singular locus", then "the next most singular locus", and so on, but with an unusual twist.
- The divisor theory is explicitly describable, and the intersection theory is tractable. (For example, one can compute the top intersection of divisors using [9].)
- The compactification is natural in the following senses.
  - (i) The desingularization is equivariant: it behaves well with respect to the symmetries of $\mathbb{P}^n$. Hence we can apply Atiyah-Bott localization to this space — not just in theory, but in practice.
  - (ii) It behaves well with respect to the inclusion $\mathbb{P}^m \hookrightarrow \mathbb{P}^n$.
  - (iii) It behaves well with respect to the marked points (forgetful maps, ψ-classes, etc.).
  - (iv) Consider the universal map $\pi : \mathscr{C} \to \mathbb{P}^n$ over $\overline{\mathscr{M}}_{g,k}(\mathbb{P}^n, d)$, where $\rho : \mathscr{C} \to \overline{\mathscr{M}}_{g,k}(\mathbb{P}^n, d)$ is the structure morphism. An important sheaf in Gromov-Witten theory is $\rho_* \pi^* \mathscr{O}_{\mathbb{P}^n}(a)$. When $g > 0$, this is not a vector bundle, which causes difficulty in theory and computation. However, in genus 1, "resolving $\overline{\mathscr{M}}^0_{1,k}(\mathbb{P}^n, d)$ also resolves this sheaf": when the sheaf is pulled back to the desingularization, it "becomes" a vector bundle. More precisely, it contains a natural vector bundle, and is isomorphic to it on the interior. This vector bundle is explicitly describable.

We find it interesting that such a natural naive approach as we will describe actually works, and yields a desingularization with these nice properties. For example, if $n > 2$, this desingularization can be interpreted as a natural compactification of the Hilbert scheme of smooth degree d curves in projective space, and thus could be seen as the genus 1 version of the complete conics.

This construction also has a number of applications:

- enumerative geometry of genus 1 curves via localization.

- Gromov-Witten invariants in terms of enumerative invariants [8].
- the Lefschetz hyperplane property: effective computation of Gromov-Witten invariants of complete intersections [3] (see also [2] for the special case of the quintic threefold).
- algebraic version of "reduced" Gromov-Witten invariants in symplectic geometry [8].
- an approach to hopefully prove physicists' predictions [1] about genus 1 Gromov-Witten invariants (work of Zinger, in progress).

We finally describe the construction explicitly. (In the lecture, the geography of $\overline{\mathscr{M}}_{1,k}(\mathbb{P}^n, d)$ was sketched as motivation.) It is straightforward to show that $\overline{\mathscr{M}}_{1,k}(\mathbb{P}^n, d)$ is nonsingular on the locus where there is no contracted genus 1 (possibly nodal) curve (for example, the proof of [5, Prop. 4.21] applies). We say a stable map is in the *m-tail locus* if there is an arithmetic genus 1 contracted curve, with precisely m points of the contracted curve that are either marked, or meet the rest of the curve. The algorithm is then as follows: blow up the one-tail component (which actually does nothing to $\overline{\mathscr{M}}^0_{1,k}(\mathbb{P}^n, d)$), then the proper transform of all two-tail components, then the proper transform of all three-tail components, etc.

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Attempt to construct the general simply connected Godeaux surface

Miles Reid

A surface S with $p_g = 0$, $K^2 = 1$ is a Godeaux surface. Its bicanonical pencil $|2K_S|$ consists of curves of genus 4. I assume that $|2K|$ has 4 distinct base points P_i, and every $C \in |2K|$ is 2-connected. This already implies that there is no torsion in $\operatorname{Pic} S$ (and I conjecture moreover that the surface is simply connected).

The divisor class $A = K_{S|C}$ has $3A = K_C$ by the adjunction formula, $H^0(C, A) = 0$ and $2A = P_1 + \cdots + P_4$. The graded ring $R(C, A)$ has Hilbert series

$$\begin{aligned} 1 + t^2 + 4t^3 + 5t^4 + 7t^5 + 9t^6 + 11t^7 + 13t^8 + \cdots \\ &= \frac{1 - 2t + 2t^2 + 2t^3 - 2t^4 + t^5}{(1-t)^2} \\ &= \frac{1 + 3t^3 + 4t^4 + 3t^5 + t^8}{(1-t^2)(1-t^3)} \quad \text{etc.} \end{aligned}$$

Thus for all the generators one must embed $C \hookrightarrow \mathbf{P}(2, 3^4, 4^4, 5^3)$ (codimension 10), where it needs at least 46 defining equations. The following procedure gives a much more efficient presentation:

Write $e_i \colon \mathscr{O}_C \hookrightarrow \mathscr{O}_C(P_i)$ for the trivial inclusion and

$$x = e_1e_2e_3e_4 \colon \mathscr{O}_C \hookrightarrow \mathscr{O}_C(2A)$$

for the basis section of $H^0(2A)$ vanishing at $P_1 + \cdots + P_4$. Next, for each i write $f_i \in H^0(K_C - P_j - P_k - P_l)$ for a basis element; then $y_i = f_ie_je_ke_l \in K_C$ forms a basis of $H^0(K_C)$ with $y_i(P_j) = \delta_{ij}$ (so mapping the P_i to the coordinate points of $\mathbf{P}^3$).

The point of the construction is that these e_i, f_i also give rise to sections of $4A$, $5A$ and $6A$ with many "tautological" monomial relations between them; for "general" C, these embed C into a toric variety $w(\mathbf{P}^1)^4$, where it only needs 4 equations tied by a single syzygy.

More specifically, these generators are

$$\begin{aligned} z_{ij} &= f_if_je_ke_l = y_iy_j/x \in H^0(4A) \\ t_l &= f_if_jf_ke_l = y_iy_jy_k/x^2 \in H^0(5A) \\ u &= f_1f_2f_3f_3 = y_1y_2y_3y_4/x^3 \in H^0(6A) \end{aligned}$$

Then the 16 generators $x, y_i, z_{ij}t_l, u$ are the vertexes of a 4-cube, and the relations holding between them are the 2×2 minors of this array, corresponding to the 55 equations defining the Segre embedding of $(\mathbf{P}^1)^4 \subset \mathbf{P}^{15}$ (in straight projective space).

The remaining equations defining $C \subset w(\mathbf{P}^1)^4$ are in degrees $4, 4, 5, 6$, tied by one syzygy in degree 8. The parameter space for these equations is a concretely given rational variety.

The above analysis of $R(C, A)$ for a single curve C extends to the relative case: write $\widetilde{S} \to S$ for the blowup of the 4 base points P_i; of the bicanonical pencil $|2K_S|$; then $\widetilde{S} \to \mathbf{P}^1$ is a fibre space of curves marked with a relative 1/3-canonical divisor. It is easy to write the relative algebra over $\mathbf{P}^1$ in each degree as explicit sums of line bundles. This gives a strategy to write out the general S. Essentially the same equations were written out (starting out from different principles) by Frank Schreyer, and his probabilistic calculations (although not rigorous) make clear that the moduli space is 8-dimensional and irreducible.

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Koszul divisors on moduli spaces of curves

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The aim of this work is to describe a general method of constructing special effective divisors on moduli spaces using the syzygies of the parametrized objects. We present a unified framework for doing divisor class calculations on $\overline{\mathcal{M}}_g$ and try to show that Koszul divisors are the most intrinsic and, from the point of view of birational geometry, most useful divisors on a moduli space.

The idea of using geometric divisors to study the birational geometry of $\overline{\mathcal{M}}_g$ goes back to Harris and Mumford (cf. [HM]) who proved that $\overline{\mathcal{M}}_g$ is of general type for large g by studying the Hurwitz divisor $\overline{\mathcal{M}}^1_{2k-1,k} := \{[C] \in \mathcal{M}_{2k-1} : \exists C \overset{k:1}{\to} \mathbf{P}^1\}$. In [FP] and [F1], for each genus $g = 6i + 10$, we constructed a divisor defined on a Hurwitz scheme over $\overline{\mathcal{M}}_g$ and showed that its pushforward to $\overline{\mathcal{M}}_g$ always violates the Slope Conjecture on effective divisors on $\overline{\mathcal{M}}_g$ (see [HMo] and [FP] for background on the Harris-Morrison Conjecture).

We fix integers $i \geq 0$ and $s \geq 1$ and set $r := 2s + si + i, g := rs + s$ and $d := rs + r$. We denote by $\mathfrak{G}^r_d$ the stack parametrizing pairs $[C, L]$ with $[C] \in \mathcal{M}_g$ and $L \in W^r_d(C)$ and denote by $\sigma : \mathfrak{G}^r_d \to \mathcal{M}_g$ the natural projection. We denote by $K_{i,j}(C, L)$ the (i, j)-th Koszul cohomology group of the pair (C, L) and we define a stratification of $\mathfrak{G}^r_d$ with strata $\mathcal{U}_{g,i} := \{(C, L) \in \mathfrak{G}^r_d : K_{i,2}(C, L) \neq 0\}$ and set $\mathcal{Z}_{g,i} := \sigma_*(\mathcal{U}_{g,i})$.

Theorem 1. *If $\sigma : \widetilde{\mathfrak{G}}^r_d \to \overline{\mathcal{M}}_g$ is the compactification of $\mathfrak{G}^r_d$ given by limit linear series, then there exists a natural morphism between torsion free sheaves of the same rank $\phi : \mathcal{A} \to \mathcal{B}$ over $\widetilde{\mathfrak{G}}^r_d$ such that $\overline{\mathcal{Z}}_{g,i}$ is the image of the degeneracy locus of ϕ. The class of the pushforward to $\overline{\mathcal{M}}_g$ of the virtual degeneracy locus of ϕ is given by*

$$\sigma_*(c_1(\mathcal{B} - \mathcal{A})) \equiv a\lambda - b_0\ \delta_0 - b_1\ \delta_1 - \cdots - b_{[g/2]}\ \delta_{[g/2]},$$

where $a, b_0, \ldots, b_{[g/2]}$ are explicitly given coefficients such that $b_1 = 12b_0 - a$ and

$$s\big(\sigma_*(c_1(\mathcal{B} - \mathcal{A}))\big) = \frac{a}{b_0} = 6\frac{f(s,i)}{(i+2)\ sg(s,i)},\ \text{with}$$

$$f(s,i) = (i^4+24i^2+8i^3+32i+16)s^7+(i^4+4i^3-16i-16)s^6-(i^4+7i^3+13i^2-12)s^5-(i^4+2i^3+i^2+14i+24)s^4+(2i^3+2i^2-6i-4)s^3+(i^3+17i^2+50i+41)s^2+(7i^2+18i+9)s+2i+2$$

and

$$g(s,i) = (i^3+6i^2+12i+8)s^6 + (i^3+2i^2-4i-8)s^5 - (i^3+7i^2+11i+2)s^4 - \\ -(i^3-5i)s^3 + (4i^2+5i+1)s^2 + (i^2+7i+11)s + 4i + 2.$$

Furthermore, we have that $6 < \frac{a}{b_0} < 6 + \frac{12}{g+1}$ *whenever* $s \geq 2$. *If the morphism* ϕ *is generically non-degenerate, then* $\overline{\mathscr{Z}}_{g,i}$ *is a divisor on* $\overline{\mathscr{M}}_g$ *which gives a counterexample to the Slope Conjecture for* $g = s(2s+si+i+1)$.

Despite its complicated appearance, the slope computed in Theorem 1 encodes a surprising amount of information about $\overline{\mathscr{M}}_g$, in particular, for suitable choices of s and i it specializes to the divisor class calculations carried out in [HM], [Kh], [FP] and [F1] which were originally obtained using a variety of ad hoc techniques.

Via Green's Conjecture for syzygies of canonical curves (cf. [V]), Theorem 1 provides a new way of calculating the class of the compactification of the Brill-Noether divisor first computed by Harris and Mumford (cf. [HM]):

Corollary 1. *The slope of the Harris-Mumford divisor* $\overline{\mathscr{M}}^1_{2i+3,i+2}$ *on* $\overline{\mathscr{M}}_{2i+3}$ *consisting of curves which cover* $\mathbf{P}^1$ *with degree* $\leq i+2$ *is given by the formula*

$$s(\overline{\mathscr{M}}^1_{2i+3,i+2}) = \frac{6(i+3)}{i+2} = 6 + \frac{12}{g+1}.$$

For $s=2$ and $g = 6i+10$ (that is, in the case $h^1(L)=2$ when $\mathfrak{G}^r_d$ is isomorphic to a Hurwitz stack parametrizing covers of $\mathbf{P}^1$), we recover the main result from [F1]:

Corollary 2. *The slope of the divisor* $\overline{\mathscr{Z}}_{6i+10,i}$ *on* $\overline{\mathscr{M}}_{6i+10}$ *consisting of curves possessing a pencil* $\mathfrak{g}^1_{3i+6}$ *such that if* $L = K_C(-\mathfrak{g}^1_{3i+6}) \in W^{3i+4}_{9i+12}(C)$ *denotes the residual linear system, then* $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^{3i+4}$ *fails to satisfy the Green-Lazarsfeld property* (N_i), *is given by the formula:*

$$s(\overline{\mathscr{Z}}_{6i+10,i}) = \frac{3(4i+7)(6i^2+19i+12)}{(12i^2+31i+18)(i+2)}.$$

In the case $i=0$ we have complete results in the sense that (1) we show that $\overline{\mathscr{Z}}_{g,0}$ is an actual divisor on $\overline{\mathscr{M}}_g$ and (2) we can compute the entire class $[\overline{\mathscr{Z}}_{g,0}]$ rather than the λ, δ_0 and δ_1 coefficients

Theorem 2. *For an integer* $s \geq 2$ *we set* $r := 2s, d := 2s(s+1)$ *and* $g := s(2s+1)$. *Then the degeneracy locus*

$$\mathscr{Z}_{g,0} := \{[C] \in \mathscr{M}_g : \exists L \in W^r_d(C) \text{ such that } C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^r \text{ is not projectively normal}\}$$

is a divisor on $\overline{\mathscr{M}}_g$ *of slope*

$$s(\overline{\mathscr{Z}}_{g,0}) = \frac{3(16s^7 - 16s^6 + 12s^5 - 24s^4 - 4s^3 + 41s^2 + 9s + 2)}{s(8s^6 - 8s^5 - 2s^4 + s^2 + 11s + 2)}$$

contradicting the Slope Conjecture.

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Examples of Rigid Varieties and the Action of the Absolute Galois Group

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(joint work with F. Catanese, F. Grunewald)

The key slogan of the following is: *the absolute Galois group acts on the set of components of moduli spaces*, e.g., let $\mathfrak{M}_{x,y}$ be the moduli space of isomorphism classes of minimal complex surfaces S of general type with $K_S^2 = x$. It is wellknown that $\mathfrak{M}_{x,y}$ is defined over the integers and therefore the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of irreducible (or connected) components of $\mathfrak{M}_{x,y}$. In particular, $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the 0-dimensional components of $\mathfrak{M}_{x,y}$, the *rigid* surfaces.

Example 1. Assume, you have a class $\mathfrak{M}$ of complex algebraic varieties such that the following condition is satisfied: if X, X' have the same characteristic numbers and $\pi_1(X) \cong \pi_1(X')$, then X and X' or X and $\bar{X}'$ are deformation equivalent.

Then $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts (factoring through $Gal(\bar{\mathbb{Q}}/\mathbb{Q})/ << \sigma >>$, where σ denotes complex conjugation) on the set of fundamental groups $\{\pi_1(X) | [X] \in \mathfrak{M}\}$.

A class of varieties satisfying the above condition are the so-calles *varieties isogenous to a higher product*, introduced and studied in [4].

Definition 1. A complex algebraic variety X of dimension n is called *isogenous to a higher product* if and only if there is a finite étale cover $C_1 \times \cdots \times C_n \to X$, where $C_1, \ldots, C_n$ are compact Riemann surfaces of respective genera $g_i := g(C_i) \geq 2$.

In fact, X is isogenous to a higher product if and only if there is a finite étale Galois cover of X isomorphic to a product of curves of genera at least two, ie., $X \cong (C_1 \times \cdots \times C_n)/G$, where G is a finite group acting freely on $C_1 \times \cdots \times C_n$.

For simplicity we will assume in the following $X = S$ to be a surface.

We have the following

Theorem 1. *(Catanese, [4]). Let* $S = (C_1 \times C_2)/G$ *be a surface isogenous to a product. Then any surface* X *with the same topological Euler number and the* *same fundamental group as S is diffeomorphic to S. The corresponding subset of the moduli space $\mathfrak{M}_S^{top} = \mathfrak{M}_S^{diff}$, corresponding to surfaces homeomorhphic, resp., diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

In particular, if X is orientedly diffeomorphic to S, then X is deformation equivalent to S or to $\bar{S}$.

Remark 1. (1) The class of varieties isogenous to a higher product provide a wide class of examples where one can test or disprove several conjectures and questions (cf. e.g., [5], [1], [2]).

(2) Notice, that given a surface $S = (C_1 \times C_2)/G$ isogenous to a product, we obtain always three more, exchanging C_1 with its conjugate curve $\bar{C}_1$, or C_2 with $\bar{C}_2$, but only if we conjugate both C_1 and C_2, we obtain an orientedly diffeomorphic surface. However, these four surfaces could be all biholomorphic to each other.

Definition 2. A surface S isogenous to a higher product is called a *Beauville surface* if and only if S is rigid.

The absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the points in the moduli space corresponding to Beauville surfaces and we have the following quite natural, but rather ambiguous questions.

Question 1. Let S be a Beauville surface. Then S is defined over a number field.

(1) What is a field of definition of S?
(2) What is the $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ - orbit of the point $[S]$ in the moduli space?

Beauville surfaces were extensively studied in [2] and the action of the complex conjugation $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on Beauville surfaces is completely understood. The key fact is, that the datum of a Beauville surface can be described group theoretically, since it is equivalent to the datum of two *triangle curves* (i.e., a curve C together with a finite group G acting on C, s.th. $C \to C/G \cong \mathbb{P}^1$ has exactly three branch points) with isomorphic groups with some additional condition assuring that the diagonal action on the product of the two curves is free.

Among others, we prove in [2]:

Theorem 2. *There are Beauville surfaces S not biholomorphic to $\bar{S}$ (i.e., σ acts non trivially on S) with group*

(1) *the symmetric group $\mathfrak{S}_n$ for $n \geq 7$,*
(2) *the alternating group $\mathfrak{A}_n$ for $n \geq 16$ and $n \equiv 0$ mod 4, $n \equiv 1$ mod 3, $n \not\equiv 3, 4$ mod 7.*

The following theorem gives examples of surfaces which are not real, but biholomorphic to their complex conjugates, or in other words, they give real points in their moduli space which do not correspond to real surfaces.

Theorem 3. *Let $p > 5$ be a prime with $p \equiv 1$ mod 4, $p \not\equiv 2, 4$ mod 5, $p \not\equiv 5$ mod 13 and $p \not\equiv 4$ mod 11. Set $n := 3p + 1$. Then there is a Beauville surface S with* *group $\mathfrak{A}_n$ which is biholomorphic to the complex conjugate surface $\bar{S}$, but is not real.*

To understand the action of the absolute Galois group on arbitrary Beauville surfaces seems impossible, but if we restrict ourselves to the case where one of the triangle curves is given by the Galois closure of a polynomial with exactly two critical values, the situation becomes much easier.

Let $P \in \mathbb{C}[z]$ be a polynomial with critical values $\{0, 1\}$. Then it follows that P has coefficients in $\overline{\mathbb{Q}}$, and in fact then in some number field K.

In order not to have infinitely many polynomials with the same branch set, we consider *normalized polynomials* $P(z) := z^n + a_{n-2}z^{n-2} + \ldots a_0$ with only critical values $\{0, 1\}$. Once we choose the types of the respective cycle decompositions $(m_1, \ldots, m_r)$ and $(n_1, \ldots, n_s)$ of the respective monodromies over 0 and 1, we can write our polynomial P in two ways, namely $P(z) = \prod_{i=1}^{r}(z - \beta_i)^{m_i}$, and $P(z) - 1 = \prod_{k=1}^{s}(z - \gamma_k)^{n_k}$.

We have the equations $F_1 = \sum m_i\beta_i = 0$ and $F_2 = \sum n_k\gamma_k = 0$ (P is normalized). Moreover, $m_1 + \ldots m_r = n_1 + \ldots n_s = n = degP$ and therefore, since $\sum_j(m_j - 1) + \sum_i(n_i - 1) = n - 1$, we get $r + s = n + 1$.

Since we have $\prod_{i=1}^{r}(z - \beta_i)^{m_i} = 1 + \prod_{k=1}^{s}(z - \gamma_k)^{n_k}$, comparing coefficients we obtain further $n - 1$ polynomial equations with integer coefficients in the variables β_i, γ_k which we denote by $F_3 = 0, \ldots, F_{n+1} = 0$. Let

$$\mathbb{V}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$$

be the algebraic set in affine $(n + 1)$-space corresponding to the equations $F_1 = 0, \ldots, F_{n+1} = 0$. Mapping a point of this algebraic set to the vector $(a_0, \ldots, a_{n-1})$ of coefficients of the corresponding polynomial P we obtain an algebraic set $\mathbb{W}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))$ (by elimination of variables) in affine $(n - 1)$ space. Both these algebraic sets are defined over $\mathbb{Q}$ and by Riemann's existence theorem follows that they are either empty or have dimension 0.

Example 2. We calculate that $\mathbb{W}(7; (2, 2, 1, 1, 1); (3, 2, 2))$ is irreducible over $\mathbb{Q}$, which implies that $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on $\mathbb{W}$. Looking at the possible monodromies, we see that there are two (non complex conjugate) non equivalent polynomials. The two permutations of types $(2, 2)$ and $(3, 2, 2)$ generate $\mathfrak{A}_7$ and the Galois closure of the two polynomial maps give two triangle curves (not diffeomorphic) triangle curves C_1, C_2.

We remark that $\mathfrak{A}_7$ has generators a_1, a_2 of order 5 such that their product has order five, yielding a triangle curve C. Obviously, $\mathfrak{A}_7$ acts freely on $C_1 \times C$ as well as on $C_2 \times C$ and we obtain two Galois conjugate Beauville surfaces S_1, S_2, which are not diffeomorphic and therefore have different fundamental groups by theorem 1.

This phenomenon was already observed by Serre, cf. [6]. The fact that we only obtain two surfaces, which are Galois conjugate, but not homeomorphic, comes from the low degree of our polynomial. Raising the degree, we will probably get arbitrary many Galois conjugate, not homeomorphic, surfaces.

Remark 2. The (topological) fundamental groups of S_1 and S_2 are not isomorphic, whereas their profinite completions $\pi_1^{alg}(S_i)$ (i.e., the algebraic fundamental groups) are isomorphic.

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Families of canonically polarized varieties over surfaces

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(joint work with Sándor Kovács)

Abstract

The Shafarevich hyperbolicity conjecture asserts that a family of curves over a quasi-projective 1-dimensional base is isotrivial unless the logarithmic Kodaira dimension of the base is positive. More generally it has been conjectured by Viehweg that the base of a smooth family of canonically polarized varieties is of log general type if the family is of maximal variation. Here, we relate the variation of a family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg's conjecture for families parametrized by surfaces.

1. Introduction

Let B° be a smooth quasi-projective complex curve and $q > 1$ a positive integer. Shafarevich conjectured [9] that the set of non-isotrivial families of smooth projective curves of genus q over B° is finite. Shafarevich further conjectured that if the logarithmic Kodaira dimension satisfies $\kappa(B^\circ) \leq 0$, then no such families exist —the definition of the logarithmic Kodaira dimension is recalled below. This conjecture, which later played an important role in Faltings' proof of the Mordell conjecture, was confirmed by Parshin [8] for B° projective and by Arakelov [1] in general. We refer the reader to the survey articles [10] and [7] for a historical overview and references to related results.

It is a natural and important question whether similar statements hold for families of higher dimensional varieties over higher dimensional bases. Families over a curve have been studied by several authors in recent years and they are now fairly well understood—the strongest results known were obtained in [11, 12], and [6]. For higher dimensional bases, however, a complete picture is still missing and subvarieties of the corresponding moduli stacks are not well understood. As a first step toward a better understanding, Viehweg proposed the following:

Conjecture 1 ([10, 6.3])**.** *Let $f^\circ : X^\circ \to S^\circ$ be a smooth family of canonically polarized varieties. If f° is of maximal variation, then S° is of log general type.*

We briefly recall the relevant definitions, as they will also be important in the statement of our main result. The first is the variation, which measures the birational non-isotriviality of a family.

Definition 1. Let $f : X \to S$ be a projective family over an irreducible base S defined over an algebraically closed field k and let $\overline{k(S)}$ denote the algebraic closure of the function field of S. The variation of f, denoted by $\operatorname{Var} f$, is defined as the smallest integer ν for which there exists a subfield K of $\overline{k(S)}$, finitely generated of transcendence degree ν over k and a K-variety F such that $X \times_S \operatorname{Spec} \overline{k(S)}$ is birationally equivalent to $F \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{k(S)}$.

In the setup of Definition 1, if the fibers are canonically polarized complex varieties, moduli schemes are known to exist, and the variation is the same as either the dimension of the image of S in moduli, or the rank of the Kodaira-Spencer map at the general point of S.

Definition 2. Let S° be a smooth quasi-projective variety and S a smooth projective compactification of S° such that $D := S \setminus S^\circ$ is a divisor with simple normal crossings. The logarithmic Kodaira dimension of S°, denoted by $\kappa(S^\circ)$, is defined to be the Kodaira-Iitaka dimension, $\kappa(S, D)$, of the line bundle $\mathscr{O}_S(K_S + D) \in \operatorname{Pic}(S)$. The variety S° is called of *log general type* if $\kappa(S^\circ) = \dim S^\circ$, i.e., the divisor $K_S + D$ is big.

It is a standard fact in logarithmic geometry that a compactification S with the described properties exists, and that the logarithmic Kodaira dimension $\kappa(S^\circ)$ does not depend on the choice of the compactification S.

2. Statement of the main result

Our main result describes families of canonically polarized varieties over quasi-projective surfaces. We relate the variation of the family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg's Conjecture 1 for families over surfaces.

Theorem 1 ([3, Thm. 1.4])**.** *Let S° be a smooth quasi-projective complex surface and $f^\circ : X^\circ \to S^\circ$ a smooth non-isotrivial family of canonically polarized complex varieties. Then the following holds.*

(1) *If* $\kappa(S^\circ) = -\infty$, *then* $\operatorname{Var}(f^\circ) \le 1$.
(2) *If* $\kappa(S^\circ) \ge 0$, *then* $\operatorname{Var}(f^\circ) \le \kappa(S^\circ)$.

In particular, Viehweg's Conjecture holds for families over surfaces. □

Examples show that Theorem 1 is sharp. In [3, Sect. 8], a slightly weaker statement is shown for families of minimal varieties. In view of Theorem 1, the following generalization of Viehweg's conjecture was proposed.

Conjecture 2 ([3, Conj. 1.6]). *Let $f^\circ : X^\circ \to S^\circ$ be a smooth family of canonically polarized varieties. Then either $\kappa(S^\circ) = -\infty$ and $\operatorname{Var}(f^\circ) < \dim S^\circ$, or $\operatorname{Var}(f^\circ) \leq \kappa(S^\circ)$.*

3. Brief outline of the proof

The proof of Theorem 1 relies on the following main ingredients.

(1) Viehweg-Zuo's existence results for pluri-log-forms on the base of the family, [12].
(2) Methods developed by Keel-McKernan in their solution [5] to the Miyanishi-Conjecture for surfaces. In particular, we employ a criterion to guarantee that in certain situations, the open part of a uniruled log-surface is covered by rational curves that meet the boundary set-theoretically in a small number of points.
(3) Miyaoka's theory of deformations along a foliation and his characterization of uniruledness.

In a nutshell, the line of argumentation in [3] goes as follows. We use the results of Viehweg-Zuo to prove instability of the tangent bundle on a logarithmic minimal model of the base. This implies the existence of a foliation on the base which, by Miyaoka's result, has rational curves as leaves. In relevant cases, Keel-McKernan's result applies to give the existence of $\mathbb{C}^*$'s in the open part of the base which —by the classical Shafarevich hyperbolicity result— are known to not exist.

4. Families over higher dimensional bases

Theorem 1 discusses families over surfaces. In order to discuss families over higher dimensional bases, we feel that a better understanding of foliations on the base is required. As a first step in this direction, Luis Solá, Matei Toma and the author have shown the following result, which can perhaps be seen as an effective version of the classical Miyaoka criterion.

Theorem 2 ([4], see also [2, thm. 0.1]). *Let X be a normal complex projective variety, $C \subset X$ a complete curve which is entirely contained in the smooth locus X_{reg}, and $\mathscr{F} \subset T_X$ a (possibly singular) foliation which is regular along C. Assume that the restriction $\mathscr{F}|_C$ is an ample vector bundle on C. If $x \in C$ is a general point, then the leaf through x is algebraic and rationally connected.* □

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Moduli of K3 Surfaces

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(joint work with V. Gritsenko, G.K. Sankaran)

This talk was a report on recent work about the Kodaira dimension of moduli spaces of K3 surfaces.

1. The main result

The main result is the following

Theorem 1. *The moduli space $\mathscr{F}_{2d}$ of polarized* K3 *surfaces of degree* $2d$ *is of general type for* $d > 61$ *and for* $d = 46, 50, 54, 58, 60$.

Moreover if $d > 39$*, but different from* $41, 44, 45, 47$ *then the Kodaira dimension of* $\mathscr{F}_{2d}$ *is non-negative.*

Mukai [4], [5], [6] has proved that the spaces $\mathscr{F}_{2d}$ are unirational for $d \leq 10, 17, 19$. In the other direction Kondo [2] has proved that $\mathscr{F}_{2p^2}$ is of general type for primes $p \gg 0$. His result is not effective. Finally, Gritsenko [1] has shown that the covers $\mathscr{F}_{2d}(n)$ given by a level-n structure are of general type for $n \geq 3$.

2. The general set-up

Let L be a lattice of signature $(2, n)$. This defines a period domain

$$\Omega_L = \{[\mathbf{w}] \in \mathbf{P}(L \otimes \mathbb{C});\ (\mathbf{w}, \mathbf{w})_L = 0,\ (\mathbf{w}, \bar{\mathbf{w}})_L > 0\} = \mathscr{D}_L \cup \mathscr{D}'_L$$

with two connected components $\mathscr{D}_L$ and $\mathscr{D}'_L$. We denote by $\mathrm{O}(L)$ the group of orthogonal transformations of the lattice L and by $\mathrm{O}^+(L)$ the subgroup which fixes the connected components. If $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$ is the dual lattice then the group $A_L = L^\vee/L$ carries a canonical quadratic form. Let $\widetilde{\mathrm{O}}(L) = \ker(\mathrm{O}(L) \to \mathrm{O}(A_L))$ and $\widetilde{\mathrm{O}}^+(L) = \widetilde{\mathrm{O}}(L) \cap \mathrm{O}^+(L)$. In this situation we have the general

Theorem 2. *Assume that $n \geq 9$ and consider an arithmetic subgroup $\Gamma \subset \mathrm{O}^+(L)$. Let $\mathscr{F}_L(\Gamma) = \Gamma \backslash \mathscr{D}_L$. Then there exists a toroidal compactification $\overline{\mathscr{F}}_L(\Gamma)$ of $\mathscr{F}_L(\Gamma)$ which has only canonical singularities. Moreover, all branch divisors of the quotient map $\mathscr{D}_L \to \mathscr{F}_L(\Gamma)$ come from reflections in Γ and there are no branch divisors in the boundary of $\overline{\mathscr{F}}_L(\Gamma)$.*

The special case of K3 surfaces arises as follows. Let $d > 0$ and consider a vector

$$h \in L_{\mathrm{K3}} = 3U \oplus 2E_8(-1), \quad h^2 = 2d$$

where U is the hyperbolic plane and $E_8(-1)$ is the unique negative definite, even unimodular lattice of rank 8. Then

$$h^\perp_{L_{\mathrm{K3}}} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle =: L_{2d}$$

and

$$\mathscr{F}_{2d} = \widetilde{\mathrm{O}}^+(L_{2d}) \backslash \mathscr{D}_{L_{2d}}$$

is the moduli space of polarised K3 surfaces of degree $2d$.

In order to prove that quotients of symmetric domains by arithmetic groups are of general type one can use modular forms. More precisely, let F_{nk} be a modular form of weight nk and with character $\det^k$ with respect to an arithmetic group Γ. Then for a suitable n-form dZ on $\mathscr{D}_L$ the form $F_{nk} dZ^k$ is a Γ-invariant k-fold pluricanonical form on $\mathscr{D}_L$ and hence descends to a pluricanonical on $\mathscr{F}_L(\Gamma)$, at least away from the fixed point set of Γ. There are several onbstructions to extending such a form to a toroidal compactification $\overline{\mathscr{F}}_L(\Gamma)$. The first type are *cuspidal* obstructions: dZ^k picks up poles of order k along the boundary and this requires F_{nk} to vanish of order k along the boundary. The second type of obstructions are called *elliptic* obstructions: these come from the possible existence of non-canonical singularities on $\overline{\mathscr{F}}_L(\Gamma)$. Due to Theorem 2 these do not occur in dimension $n \geq 9$. The last type of obstructions come from quasi-reflections in Γ. These quasi-reflections do not lead to singularities in $\overline{\mathscr{F}}_L(\Gamma)$, however, the form $F_{nk} dZ^k$ still picks up poles of along the corresponding branch divisors and hence F_{nk} has to vanish along these divisors to suitable order. Again, in view of Theorem 2, the only elements in Γ which can occur for $n \geq 9$ are reflections and we refer to the corresponding obstructions as *reflective* obstructions.

One can then prove the following result

Proposition 1. *Let $\Gamma \subset \mathrm{O}^+(L)$ be an arithmetic subgroup and $n \geq 9$. Assume that there exists a character χ of Γ and a cusp form $F_a \in S_k(\Gamma, \chi)$ of weight $a < n$ which vanishes along the branch divisor of the quotient map $\mathscr{D}_L \to \mathscr{F}_L(\Gamma)$. Then $\mathscr{F}_L(\Gamma)$ is of general type.*

This result has a number of applications, of which we mention only one here. For this we consider the space

$$\mathscr{S}\mathscr{F}_{2d} = \widetilde{\mathrm{SO}}^+(L_{2d})\backslash \mathscr{D}_{L_{2d}}$$

which can be interpreted as the moduli space of polarised K3 surfaces S of degree $2d$ with a *spin structure*, i.e. an orientation of $H^2(S, \mathbb{Z})$. Then one can show

Theorem 3. *The spaces $\mathscr{S}\mathscr{F}_{2d}$ are of general type for $d \geq 3$.*

This is a straightforward application of Proposition 1. The branch divisor of $\mathscr{D}_{L_{2d}} \to \mathscr{S}\mathscr{F}_{2d}$ is given by reflection along vectors l with $l^2 = -2d$. Additive lifting of Jacobi forms produces weight 17 forms vanishing along these divisors. For $d = 2$ the space $\mathscr{S}\mathscr{F}_4$, which is a double cover of the space of quartic surfaces, has non-negative Kodaira dimension, whereas the space $\mathscr{S}\mathscr{F}_2 = \mathscr{F}_2$ is rational.

3. The Borcherds Form

The map $\mathscr{D}_{L_{2d}} \to \mathscr{F}_{2d}$ is branched along divisors obtained by reflections along (-2) and $(-2d)$-vectors. In order to obtain a suitable low weight cusp form we make use of Borcherds' modular form. Let

$$L_{2,26} = 2U \oplus 3E_8(-1).$$

Borcherds has constructed a modular form Φ_{12} on the domain $\mathscr{D}_{L_{2,26}}$ which has an infinite product expansion. Its divisor is the set of all hyperplanes given by (-2)-vectors. Given any vector $l \in E_8(-1)$ with $l^2 = -2d$ we can define an embedding of L_{2d} into $L_{2,26}$ and hence also of the corresponding homogeneous domains. Restricting Φ_{12} to such an embedded domain will normally give the zero-function. However we can proceed as follows: let $R_l = \{r \in E_8(-1) \mid r^2 = -2,\ r \cdot l = 0\}$, and $N_l = \# R_l$. Then

$$F_l = \left. \frac{\Phi_{12}(Z)}{\prod_{\{\pm r\} \in R_l} (Z, r)} \right|_{\mathscr{D}_{L_{2d}}} \in M_{12 + \frac{N_l}{2}}(\widetilde{\mathrm{O}}^+(L_{2d}), \det) \tag{1}$$

is a non-zero form of weight $12 + N_l/2$ on $\mathscr{D}_{L_{2d}}$ with character det. This form was also used by Kondo, as well as Borcherds, Katzarkov, Pantev and Shepherd-Barron in connection with K3 surfaces. It vanishes on all (-2)-divisors in $\mathscr{D}_{L_{2d}}$ and if $N_l > 0$ one can also show that it is cusp form. Finally, if the weight of F_l is less than 68, i.e., if $N_l < 112$, then F_l can be shown to vanish along all $(-2d)$-divisors in $\mathscr{D}_{L_{2d}}$. Using this result we see that it suffices to find vectors $l \in E_8(-1)$ with $l^2 = -2d$ and $0 < N_l < 14$. Indeed, this can be done for the values given in Theorem 1. The proof of this involves a subtle study of the geometry of the root lattice of E_8 and divides into a general part and a special part. The first part follows from

Proposition 2. *There exists a vector $l \in E_8$ with $l^2 = 2d$ and $0 < N_l < 14$ if*

$$4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d) \tag{2}$$

or

$$5N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d) + 378N_{D_5}(2d) \tag{3}$$

where $N_L(2d)$ denotes the number of representaions of the integer $2d$ in the lattice L.

Using explicit estimates of the function $N_L(2d)$ involved gives the required vectors for $d > 143$. The remaining values of d can be produced by exhibiting explicit vectors.

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