Mathematisches Forschungsinstitut Oberwolfach

Report No. 31/2006

Calculus of Variations

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July 9th – July 15th, 2006

Abstract. Research in the Calculus of Variations has always been motivated by questions generated within the field itself as well as by problems arising in other areas of Mathematics, and applied sciences. The purpose of this meeting was to illustrate the diversity of points of view and motivations in current research, with particular attention to applications.

Mathematics Subject Classification (2000): 26B25, 35B65, 35J60, 35J20, 35K55, 49J45, 49L25, 49Q10, 49Q20, 49R50, 52A40, 53C42, 58E20, 83C30, 94A17.

Introduction by the Organisers

The workshop "Calculus of Variations" took place from July 9 to 15, 2006, and was attended by almost fifty participants, mostly from European and North American universities and research institutes. There were 24 lectures on recent research topics, plus a review lecture on the Lieb-Thirring inequalities by Michael Loss (Georgia Tech, Atlanta). As the workshop had no specific focus, talks covered a wide range of topics, with the aim of featuring different research trends, bringing new problems to the fore, and stimulating interaction between mathematicians from different backgrounds.

Five lectures were focused on problems related to Continuum Mechanics and Materials Science. Gero Friesecke (Munich and Warwick) presented some results on a simplified model for molecules, where the aim is to give a rigorous explanation of the screening effect (i.e., the fact that the interaction of electrically balanced molecules due to electrostatic forces is short ranged); this problem is still open and presumably quite challenging in case of 'realistic' models. László Székelyhidy (ETH Zürich) presented new result about the structure of quasiconvex hulls for sets of 2×2 matrices, perhaps the most interesting development on this topic in recent years. Sergio Conti (Duisburg) considered the asymptotic behaviour of an energy functional that appears in the modeling of different physical problems, such as blistering in elastic films, magnetic thin films, etc.; the main result presented in his lecture adds one more piece to the work of many authors towards the proof of a conjecture by Aviles and Giga on the variational limit of such functional. The lecture by Felix Otto (Bonn) was focused on the rigorous analysis of pattern formation in micromagnetics: this type of pattern formation is particularly interesting because of the complexity of the observed behaviours – not yet fully explained in rigorous terms – and of the relative simplicity of the underlying continuum model. Related to this topic was also the lecture of Hans Knüpfer (Bonn).

Four lectures dealt with regularity problems of different sorts. G. Rosario Mingione (Parma) reviewed some recent developments on the regularity of solutions of nonlinear parabolic systems. Michael Struwe (ETH Zürich) presented a new approach to regularity for harmonic maps valued in a hypersurfaces, yielding new results when the domain dimension is larger than 2. The regularity of harmonic maps valued in Riemannian manifolds was also considered by Ernst Kuwert (Freiburg); these results stemmed from other results on the conformal structure of surfaces with suitable bounds on the Willmore energy. Mariel Saez (MPI for Gravitationl Physics, Potsdam) presented a Lipschitz regularity result for the pseudo-infinity Laplacian.

A certain number of lectures were related to shape optimization and optimal transport problems. Almut Burchard (Toronto) presented some partial results about the shape of closed curves in the three-dimensional space that minimize the first eigenvalue of the associated one-dimensional Schrödinger operator; it is conjectured that these curves are circles (among other things, the conjecture is related to the optimal constant in a particular Lieb-Thirring inequality). Jochen Denzler (Knoxville) and Giuseppe Buttazzo (Pisa) considered other optimization problems related to the first eigenvalue of (variants of) the Laplace operator on a given domain. Alexander Plakhov (Aveiro) studied bodies of minimal resistance moving through a rarefied particle gas. Francesco Maggi (Duisburg) and Aldo Pratelli (Pavia) presented some recent quantitative versions with optimal exponents of the classical isoperimetric inequality in the n-dimensional Euclidean space. Qinglan Xia (UC Davis) proposed a model for the shape formation in tree leaves which postulates a step-by-step optimized growth for the associated transport system (the venation of the leaf), where "optimized" refers to a given transport cost. Numerical simulations based on this simple model show that varying the two built-in parameters generates a wide variety of leaf shapes. Vladimir Oliker (Emory University, Atlanta) described a variational approach to the Aleksandrov problem about the existence of closed convex hypersurfaces with prescribed integral Gauss curvature. A similar approach is also used to design reflecting surfaces with prescribed irradiance properties; the functional underlying this variational principle is related to Monge-Kantorovich optimal transport theory.

Yann Brenier (Nice) considered the problem of foliating the three-dimensional Euclidean space and the four-dimensional Minkowski space by extremal surfaces (which in Minkowski space can be interpreted as classical relativistic strings). One way of obtaining such foliations is finding minimizers or critical points of suitable energy functionals, subject to certain nonlinear constraints; due to these constraints, standard methods do not apply in this case, and the existence of such minimizers is open. Pierre Cardaliaguet (Brest) studied a non-local geometric evolution problem for sets in the n-dimensional Euclidean space, which can be formally viewed as the gradient flow of a linear combination of volume and capacity. Since this flow preserves inclusion, it allows for a notion of weak solutions in the sense of viscosity; it is shown that such solutions agree with the limits of the the minimizing movements obtained by time discretization.

Diogo Gomes (Instituto Superior Tecnico, Lisbon) reviewed some recent results on the viscosity solution of Hamilton-Jacobi equations and the relations with the associated Hamiltonian dynamics, and Aubrey-Mather theory. Olvier Druet (ENS Lyon) presented new results on the bubbling phenomenon for the solutions (and also the Palais-Smale sequences) of sequences of variational elliptic equations in dimension two with critical nonlinearities. Robert Jerrard (Toronto) described a version of the Γ-convergence method designed for saddle points instead of minima, and used this abstract tool to obtain non-trivial solutions to the Ginzburg-Landau system in dimension three. Reiner Schätzle (Tübingen) gave a proof of (a modified version of) a conjecture by De Giorgi on the approximation of the Willmore functional for hypersurfaces in dimension three; the conjecture is still open in higher dimensions.

Keith Ball (University College London) presented the proof of a long-standing conjecture (due to Lieb) on the entropy gap between the normalized sum of N independent copies of a given random variable X and its limit as $N \to \infty$, i.e., the Gaussian distribution. A key role in the proof is played by a new variational characterization of Fisher information.

The lecture by Gerhard Huisken (MPI for Gravitationl Physics, Potsdam) was focused on the problem of defining mass in general relativity; in particular, he presented a new definition based on the isoperimetric inequality (more precisely, on the asymptotic behaviour of the isoperimetric profile), and some results on the properties of this mass. One of the advantages of this definition, compared to others based on the notion of curvature, is the relatively simple calculus that is required for handling it. Furthermore, it can be adapted so as to obtain a notion of localized mass.

Workshop: Calculus of Variations

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Abstracts

Minimum energy configurations of classical charges in the potential of an array of atomic nuclei: Large N asymptotics

Gero Friesecke

This work (joint with my graduate student Stéphane Capet) is motivated by the desire to understand mathematically basic "screening effects" in atoms, molecules and solids. By "screening effects" physicists mean the remarkable cancellation effects between the strong, long-range Coulomb forces among the constitutent elementary particles (electrons and atomic nuclei) which cause atoms to behave as if their mutual interaction was weak and short-range.

To shed mathematical light on screening, we introduce and analyze the following variational problem in which electrons are modelled as classical point charges: Minimize

$$V_{N,Z}(x_1,..,x_N) := \sum_{i=1}^{N} v(x_i) + \sum_{1\le i<j\le N} \frac{1}{|x_i - x_j|}, \tag{1}$$

where

$$v(x) = \sum_{\alpha=1}^{M} \frac{-Z_\alpha}{|x-R_\alpha|} \qquad (Z=(Z_1,..,Z_M),\ Z_\alpha>0,\ N\in\mathbb{N},\ R_\alpha\in\mathbb{R}^3), \tag{2}$$

over the set

$$\mathcal{A}_N := \{(x_1,..,x_N)\in\mathbb{R}^{3N} \,\Big|\, |x_i - R_\alpha| \ge d \text{ for all } i,\ \alpha\} \quad (d>0). \tag{3}$$

This model is a natural classical analogue of the fundamental time-independent electronic Schrödinger equation. It retains the notorious many-body Coulomb interactions of electrons and atomic nuclei exactly, whereas the hard core assumption (3) may be viewed as a crude "uncertainty principle" which prevents electrons from falling into the nucleus, with the hard-core radius d playing the role of $\hbar$. More precisely, the model (1), (2), (3) arises from the full quantum mechanical Hamiltonian of the electrons in a molecule,

$$H_{N,Z} = -\frac{1}{2}\Delta + V_{N,Z} = \sum_{i=1}^{N}\Big(-\frac{1}{2}\Delta_{x_i} + v(x_i)\Big) + \sum_{1\le i<j\le N}\frac{1}{|x_i-x_j|},$$

by retaining electron interaction exactly, but – crudely – replacing the one-body operator $-\frac{1}{2}\Delta_{x_i} + v(x_i)$ by the effective potential $v_{eff}(x_i) := v(x_i)$ when $|x_i - R_\alpha| \ge d$ for all i and all α, $+\infty$ otherwise.

Our main result is the following. For simplicity we state it here only in the case of atoms ($M=1$), in which case the nucleus may be assumed to be placed at the origin, d may be assumed to be equal to 1, and (2), (3) reduce to

$$v(x) = -\frac{Z}{|x|} \quad (Z>0), \qquad \mathcal{A}_N = \{(x_1,..,x_N)\in\mathbb{R}^{3N} \,\Big|\, |x_i|\ge 1 \text{ for all } i\}.$$

Theorem 1. *(Equidistribution and escape to infinity)*
For any sequence $\{(x_1^{(N,Z)}, .., x_N^{(N,Z)})\}$ *of approximate minimizers of* $V_{N,Z}$*, in the limit* $N \to \infty$*,* $Z \to \infty$*,* $N/Z \to \lambda \in (0, \infty)$*, the associated measures*

$$\mu^{(N,Z)} := \frac{1}{Z} \sum_{i=1}^{N} \delta_{x_i^{(N,Z)}}$$

satisfy

$$\mu^{(N,Z)} \rightharpoonup^* \min\{\lambda, 1\} \frac{H^2|_{S^2}}{4\pi} =: \mu_\lambda.$$

Moreover the ground state energy satisfies

$$\frac{\inf V_{N,Z}}{Z^2} \longrightarrow -\min\{\lambda, 1\} + \frac{1}{2}(\min\{\lambda, 1\})^2.$$

Note that for negative ions ($\lambda > 1$) the limit measure has less mass than the approximating measures,

$$\int d\mu_\lambda = 1 < \lambda = \lim \frac{N}{Z} = \lim \int d\mu^{(N,Z)}.$$

Physically this means that only $Z + o(Z)$ particles stay bound and $N - (Z + o(Z))$ particles move off to infinity. We note that in the more difficult case of quantum mechanical atoms, an anologous result was proved (via a different approach) by Lieb, Sigal, B.Simon, and Thirring.

In case of a molecule, in which we know of no previous results, neither classical nor quantum-mechanical, one investigates the limit

$$(4) \qquad N \to \infty, \quad |Z| := \sum_{\alpha=1}^{M} Z_\alpha \to \infty, \quad \frac{Z_\alpha}{|Z|} \to z_\alpha \in [0, 1], \quad \frac{N}{|Z|} \to \lambda \in (0, \infty);$$

one finds, e.g. in case of neutrality ($\lambda = 1$) and non-overlapping hard cores ($|R_\alpha - R_\beta| > 2d$ for all $\alpha \neq \beta$), that the limit measure is

$$\sum_{\alpha=1}^{M} z_\alpha \frac{H^2|_{S_\alpha^2}}{4\pi d^2},$$

where S_α^2 denotes the sphere of radius d centred at R_α. This is a *screening result*: asymptotically the total amount of electronic charge around each nucleus is exactly equal to the nuclear charge.

As one interesting consequence, one can read off that in the above asymptotic limit the interatomic potential for the classical model, i.e. the ground state energy minus the ground state energy of the individual atoms, is exactly zero when $|R_\alpha - R_\beta| > 2d$ for all $\alpha \neq \beta$. In particular the interatomic potential is short range, despite the fact that the interaction between the constituting charges is not.

To prove the theorem, one
– rewrites the classical problem as a variational problem for the associated measures
– passes to a Gamma-limit

– minimizes explicitly the Gamma-limit
– infers the asserted convergence from standard arguments in Gamma-convergence.

More precisely, we define

$$I^{(N,Z)}(\mu) := \\ -\int_{\mathbb{R}^3\setminus\Omega} \sum_{\alpha=1}^{M} \frac{Z_\alpha}{|Z|} \frac{1}{|x-R_\alpha|} d\mu(x) + \frac{1}{2} \int\int_{(\mathbb{R}^3\setminus\Omega)^2\setminus\mathrm{diag}} \frac{1}{|x-y|} d\mu(x)\, d\mu(y)$$

if $\mu = \frac{1}{|Z|}\sum_{i=1}^{N} \delta_{x_i}$ for some distinct $x_1,..,x_N \in \mathbb{R}^3\setminus B_1$, and set $I^{(N,Z)}(\mu) := +\infty$ otherwise. Here diag denotes the diagonal $\{(x,x)\,|\,x\in\mathbb{R}^3\setminus B_1\}$. Then for μ as in the first alternative, we have the identity

$$I^{(N,Z)}(\mu) = \frac{1}{|Z|^2} V_{N,Z}(x_1,...,x_N).$$

We then show:

Theorem 2. *In the limit (4), the sequence of functionals $I^{(N,Z)}$ Gamma-converges (with respect to weak* convergence of Radon measures) to the following functional on nonnegative Radon measures:*

$$I(\mu) := \begin{cases} -\int_{\mathbb{R}^3\setminus\Omega} \sum_{\alpha=1}^{M} \frac{z_\alpha}{|x-R_\alpha|} d\mu(x) + \frac{1}{2}\int\int_{(\mathbb{R}^3\setminus\Omega)^2} \frac{1}{|x-y|} d\mu(x)\, d\mu(y) & \\ & \text{if } \int d\mu \le \lambda, \\ +\infty & \text{otherwise.} \end{cases}$$

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Tartar's conjecture and localization of quasiconvex hulls

László Székelyhidi Jr.
(joint work with Daniel Faraco)

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Our interest lies in compactness properties of sequences of approximate solutions to inclusions of the type

$$(1) \qquad Du(x) \in K \text{ for almost every } x \in \Omega$$

for functions $u : \Omega \to \mathbb{R}^2$, where $K \subset \mathbb{R}^{2\times 2}$ is a given compact set.

It is well known that for such problems the main obstruction to compactness is due to the possible presence of rapid oscillations in the sequence of gradients Du_j. Indeed, if $A, B \in \mathbb{R}^{2\times 2}$ are any two matrices such that $\mathrm{rank}(A-B) = 1$, then one can construct a sequence of uniformly Lipschitz functions u_j whose gradients oscillate between A and B, and no subsequence of $\{Du_j\}$ converges strongly in $L^1(\Omega)$. If A and B are such that $\mathrm{rank}(A-B) = 1$, we say that A and B are *rank-one connected* and in general speak of rank-one connections. Thus a necessary condition for compactness in (1) is that K contains no rank-one connections. In [12] L. Tartar conjectured that in fact this condition should also be sufficient, although subsequently (see for example [13]) he produced an example of a set K consisting of four matrices where there are no rank-one connections but compactness fails. Such examples are nowadays called T_4 configurations. On the other hand the conjecture was verified by V. Šverák in [10] for *connected* sets $K \subset \mathbb{R}^{2\times 2}$.

One of our main results is that the additional condition that K contains no T_4 configurations is indeed sufficient for compactness. We remark that there is a very quick algorithm for testing for T_4 configurations, see [11].

Theorem 1 (D. Faraco - L.Sz. '06)**.** *Suppose $K \subset \mathbb{R}^{2\times 2}$ is a compact set without rank-one connections and K contains no T_4 configurations. Then for any uniformly Lipschitz sequence $u_j : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathrm{dist}\,(Du_j, K) \to 0$ in $L^1(\Omega)$, the sequence $\{Du_j\}$ is (pre)compact in $L^1(\Omega)$.*

In problems with lack of compactness one is led to consider the relaxed problem, and it is well known that for problem (1) this is characterized by the quasiconvex hull K^{qc}. A basic technique is to get a lower estimate from the rank-one convex hull K^{rc} and an upper estimate from the polyconvex hull K^{pc}, since $K^{rc} \subset K^{qc} \subset K^{pc}$. In estimating the rank-one convex hull a very useful fact is that the rank-one convex hull is *localizable.* This means that if we know *apriori* that K^{rc} is disconnected (for example by an estimate on the polyconvex hull), then K^{rc} can be calculated by considering just subsets of K contained in each connected component of K^{rc}, see for example [3]. This result, known as the "structure theorem" for rank-one convex hulls, is valid in any dimension, and the proofs rely heavily on the locality of rank-one convexity. Our second main result is that in the space of 2×2 matrices the structure theorem also holds for the quasiconvex hull.

Theorem 2 (D. Faraco - L.Sz. '06)**.** *If $K \subset \mathbb{R}^{2\times 2}$ is a compact set and $K^{qc} \subset \bigcup_{i=1}^n U_i$ for pairwise disjoint open sets U_i, then $K^{qc} \cap U_i = (K \cap U_i)^{qc}$.*

There is a close relationship between Theorem 2 and Morrey's conjecture regarding quasiconvexity and rank-one convexity. We recall that a variational integral of the form $\int_\Omega f(Du(x))dx$ is weak* lower-semicontinuous in the space $W^{1,\infty}(\Omega, \mathbb{R}^m)$ if and only if $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex. It is well known that every quasiconvex function is rank-one convex, and Ch.B. Morrey Jr. in [7] posed the problem of whether rank-one convexity implies quasiconvexity. In the higher dimensional case, where $m \geq 3$, V. Šverák in [9] constructed an ingenious counterexample, showing that quasiconvexity is not the same as rank-one convexity. However, the case $n = m = 2$ remains an outstanding open problem. Subsequently J. Kristensen used Šverák's counterexample in [5] to show that for $m \geq 3$ quasiconvexity is not a local condition. Also, the type of localization as in Theorem 2 is not possible. Theorem 2 suggests that if there is a difference between rank-one convexity and quasiconvexity in $\mathbb{R}^{2\times 2}$, it has to be of a much more subtle nature.

Our approach is based on the notion of *incompatible sets*. Following [1] we call two disjoint compact sets $K_1, K_2 \subset \mathbb{R}^{2\times 2}$ *homogeneously incompatible* if whenever $u_j : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a sequence of uniformly Lipschitz mappings which are affine on the boundary and such that $\operatorname{dist}(Du_j, K_1 \cup K_2) \to 0$ in L^1, then either $\operatorname{dist}(Du_j, K_1) \to 0$ or $\operatorname{dist}(Du_j, K_2) \to 0$. Our method to proving Theorems 1 and 2 is to find a decomposition of K into homogeneously incompatible sets. We build on the ideas developed in [11] to arrive at a sufficiently large class of sets which give rise to pairs of homogeneously incompatible sets. Such sets will be given as the *quasiconformal envelope* $\mathcal{E}_\Gamma$ of closed curves $\Gamma \subset \mathbb{R}^{2\times 2}$. The key point is to realize that the set $\mathcal{E}_\Gamma$ corresponds on the one hand to elliptic equations and on the other hand to families of quasiconformal mappings. More precisely, if $u \in W^{1,2}(\Omega, \mathbb{C})$ satisfies $Du(z) \in \overline{\mathcal{E}_\Gamma}$ for almost every $z \in \Omega$, then u solves a corresponding nonlinear Beltrami equation of the form

$$\partial_{\overline{z}} u = H(z, \partial_z u),$$

whereas when coupled with appropriate boundary conditions u gives rise to a family of quasiconformal mappings parametrized by the curve Γ as

$$u^t(z) = u(z) - \Gamma(t) z.$$

The former allows us to use the approach in [2] to construct certain nonlinear operators which act as projectors onto the set $\mathcal{E}_\Gamma$, whereas the latter, an idea which appeared in [4], leads to the required incompatibility result for solutions of the inclusion $Du(z) \in \overline{\mathcal{E}_\Gamma}$. Indeed, our proof of this incompatibility relies heavily on adapting the methods in Section 7 of [4] - where Γ is a straight line in the conformal plane - to our nonlinear setting.

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Calderón-Zygmund Theory for Parabolic Systems

Giuseppe Mingione

Calderón-Zygmund estimates are an ultra-classical tool in modern PDE theory. They allow, an elliptic equation/system being given, to infer the precise L^q-type regularity of solutions starting from that of the "data". For instance, considering the linear elliptic equation of the form

$$\text{div }(a(x)Du) = f \qquad \text{in} \qquad \Omega \subset \mathbb{R}^n \ , \tag{1}$$

where the coefficients $a : \Omega \to \mathbb{R}^n$ are assumed to be elliptic, and even VMO in the sense of Sarason [8], then we have that $f \in L^q(\Omega) \implies Du \in L^q(\Omega)$, for every $q > 1$; here f is the given datum. For these type of statements see for instance [5], Chapter 10. This kind of result has been first obtained in the simplest case $\triangle u = f$ by mean of Singular Integrals Theory, and eventually, using other Harmonic Analysis tools, such as nonlinear commutators, in order to treat the case of VMO coefficients. Such approaches heavily rely on the linearity of the problem, and, more dramatically, on the possibility of using *explicit representation formulas* for solutions, by mean of singular integrals. Eventually, different approaches via interpolation and regularity estimates have been offered by Campanato and Stampacchia, but they again rely on the linearity of the problem. It is then clear that such purely linear approaches cannot be used in the case of elliptic and parabolic equations of the type

$$\text{div } a(Du) = \text{ div } f \ , \qquad\qquad u_t - \text{ div } a(Du) = \text{ div } f \ . \tag{2}$$

The purpose of this note is to shortly report on recent developments concerning these last, non-linear cases. The first, extremely important result for non-linear equations was achieved by Tadeusz Iwaniec [6]. He considered the model case

$$\triangle_p u = \text{ div } (|Du|^{p-2}Du) = \text{ div } (|F|^{p-2}F) \ , \tag{3}$$

involving the p-Laplacean operator $\triangle_p$, essentially proving that $F \in L^q_{\text{loc}} \implies Du \in L^q_{\text{loc}}$, for any $q \geq p$. Extending the result to the full range $q \geq p-1$ still remains an open problem. We can say that Iwaniec's result opened the way to a *non-linear Calderón-Zygmund theory.* Eventually, developing Iwaniec's methods, DiBenedetto & Manfredi [4] extended this result to the case of systems, while

Kinnunen & Zhou [7] considered the case of a degenerate operator with VMO coefficients $a(x)$, of the type

$$\text{div } (a(x)|Du|^{p-2}Du) = \text{ div } (|F|^{p-2}F) \ . \tag{4}$$

Again, in [1] we obtained the first result for non-uniformly elliptic operator, considering the so called $p(x)$-Laplacean system

$$\text{div}(|Du|^{p(x)-2}Du) = \text{div}(|F|^{p(x)-2}F) \ , \tag{5}$$

arising in many physical models, as the one for electrorhelogical fluids elaborated by Růžička. For problems of the type "left hand side"= div f, we have $f \in L^{qp/(p-1)}_{\text{loc}} \implies Du \in L^q_{\text{loc}}$, again for any $q \geq p$. The techniques used in [6, 4, 7] avoid Singular Integrals by means of a suitable, combined use of refined regularity estimates for solutions to homogeneous problems, that is, when $f \equiv 0$, and Harmonic Analysis tools, such as Fefferman-Stein theorem on the so called Sharp Maximal Function, a tool anyway avoided in [1].

Very unfortunately, these techniques completely break down in the case of parabolic problems of the type of the non-homogeneous, parabolic p-Laplacean system

$$u_t - \text{div}(|Du|^{p-2}Du) = \text{div}(|F|^{p-2}F) \ . \tag{6}$$

The reason is the following: parabolic systems as (6) exhibit a different scaling with respect to space and type. This reflects in the fact that there is no natural, a priori determined family of cylinders on which one can find good estimates; on the contrary, cylinders good for obtaining useful integral estimates are *intrisically defined* by the solution itself. When $p \geq 2$, they are of the type

$$K := (x_0, t_0) + B_R(0) \times (-\lambda^{2-p}R^2, 0] \ . \tag{7}$$

where, roughly speaking

$$\lambda \approx \left(\frac{1}{|Q_K|} \int_{Q_K} |Du|^p \, dx \, dt \right)^{\frac{1}{p}} \ .$$

Note that the critical fact here is that λ appears on both the sides of the previous equivalence via the definition of K, and it is part of the proofs to show that such intrinsic cylinders can be actually constructed: this is the core of DiBenedetto's approach to parabolic regularity [3]. As a consequence of this phenomenon, for instance, one cannot use maximal function operators, that need to have an a priori defined underlying family of balls/cylinders. Using maximal operators is only possible in the non-degenerate case $p = 2$, therefore Calderón-Zygmund theory for systems as in (6) remained untouched. Though one cannot use Harmonic Analysis tools, one can use Harmonic Analysis ideas, and plugging them in, directly at a PDE estimates level; the outcome is the following result, taken from [2]:

Theorem 1. *Let* $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ *be a weak solution to* (6)*, where*

$$p > \frac{2n}{n+2} \ . \tag{8}$$

Then

$$F \in L^q_{\rm loc}(C, \mathbb{R}^{nN}) \implies Du \in L^q_{\rm loc}(C, \mathbb{R}^{nN}) \qquad \forall \ \ q \geq p \, .$$

Moreover there exists a constant $c \equiv c(n, N, p, \nu, L, q, \omega(\cdot)) > 1$ *such that if* $Q_{2R} \mathcal{C} C$ *then*

$$\left(\frac{1}{|Q_R|} \int_{Q_R} |Du|^{pq} \, dz \right)^{\frac{1}{q}} \leq c \left[\frac{1}{|Q_{2R}|} \int_{Q_{2R}} |Du|^p \, dz + \left(\frac{1}{|Q_{2R}|} \int_{Q_{2R}} |F|^{pq} \, dz + 1 \right)^{\frac{1}{q}} \right]^d \, , \tag{9}$$

where

$$1 \leq d := \begin{cases} \dfrac{p}{2} & \textit{if } p \geq 2 \\ \dfrac{2p}{p(n+2) - 2n} & \textit{if } p < 2 \, . \end{cases} \tag{10}$$

Here Q_R is a standard parabolic cylinder of the type $(x_0, t_0) + B_R(0) \times (-R^2, 0]$, and B_R is the standard Euclidean ball of radius R, centered at x_0. The problem is of course considered in the cylindrical domain $C := \Omega \times (0, T)$. Comments are in order. The lower bound in (8) is unavoidable in order to obtain this type of regularity, as shown by counterexamples; this is a peculiarity of the parabolic situation. Inequality (9) fails to be homogeneous but when $p = 2$, because of the presence of the exponent d, that is the "the scaling deficit" of the system (6) when $p \geq 2$. This is unavoidable too: multiplying a solution of (6) times a constant does not yield a solution of a similar system, therefore a priori estimates for solutions cannot be homogeneous on standard parabolic cylinders. When $p < 2$ the value of d perfectly reflects the fact that the result is not valid below the bound in (8): d approaches infinity when p approaches the lower bound in (8). Our techniques are flexible enough to apply in much more general situations. For instance, in [2] we obtain similar results in the case of systems with suitable VMO coefficients of the type

$$u_t - \operatorname{div}(a(x,t)|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F) \, .$$

Moreover, we can treat general non-linear systems with the so called "Uhlenbeck structure" i.e.: the non-linear dependence upon Du is via $|Du|$:

$$u_t - \operatorname{div}[g(|Du|)Du] = \operatorname{div}(|F|^{p-2}F) \, ,$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$, is in $C^1(\mathbb{R} \setminus \{0\})$, and

$$\text{if } p \geq 2 \, , \qquad g'(s) \geq 0 \qquad \forall \quad s > 0 \tag{11}$$

$$\nu s^{p-2} \leq g(s) \leq L s^{p-2} \qquad \forall \quad s > 0 \tag{12}$$

$$\frac{|g'(s)|s}{g(s)} \leq \begin{cases} L & \text{if } p \geq 2 \\ \theta(< 1) & \text{if } p < 2 \end{cases} \qquad \forall \quad s > 0 \tag{13}$$

$$\langle g(|w_2|)w_2 - g(|w_1|)w_1, w_2 - w_1 \rangle \geq \nu(\mu^2 + |w_1|^2 + |w_2|^2)^{\frac{p-2}{2}} |w_2 - w_1|^2 \, , \tag{14}$$

for all $w_1, w_2 \in \mathbb{R}^{nN}$. Finally, when dealing with equations, all those involving possibly degenerate/singular operators of Leray-Lions type can be considered, i.e.

$$u_t - \operatorname{div}[a(z)A(Du)] = \operatorname{div}(|F|^{p-2}F) \ ,$$

where the vector field $A : \mathbb{R}^N \to \mathbb{R}^N$ is $C^1(\mathbb{R} \setminus \{0\})$, and satisfies the following growth and ellipticity assumptions:

$$|A(w)| + |DA(w)|(\mu^2 + |w|^2)^{\frac{1}{2}} \leq L(\mu^2 + |w|^2)^{\frac{p-1}{2}} \ , \tag{15}$$

$$DA(w)\lambda \otimes \lambda \geq \nu(\mu^2 + |w|^2)^{\frac{p-2}{2}}|\lambda|^2 \ , \tag{16}$$

for every $w, \lambda \in \mathbb{R}^n$, where, as usual, $0 < \nu \leq L$, and $\mu \in [0, 1]$.

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A sharp-interface limit for the singularly perturbed Eikonal functional

Sergio Conti

(joint work with Camillo De Lellis)

We consider the functional

$$F_\varepsilon[u, \Omega] \ = \ \int_\Omega \left\{ \frac{(1 - |\nabla u(x)|^2)^2}{\varepsilon} + \varepsilon |D^2 u(x)|^2 \right\} dx \, . \tag{1}$$

Here ε is a positive number, Ω is a bounded open set of $\mathbb{R}^2$, $u \in W^{1,2}(\Omega; \mathbb{R})$, and $|D^2u|^2 = \sum_{ij}(D^2_{ij}u)^2$. These functionals have been proposed as models for different physical problems (liquid crystals [2], blistering in thin films [13], convection patterns [11], magnetism in thin films [9]). In most cases one seeks minimizers of F_ε among the u's such that

$$u|_{\partial\Omega} \ = \ 0 \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} \ = \ -1 \, . \tag{2}$$

In [2] (see also [3]) the following conjectures were made. Firstly, if $\limsup_{\varepsilon\downarrow 0} F_\varepsilon(u^\varepsilon) < \infty$, then u^ε converges, up to subsequences, to a Lipschitz function u solving the eikonal equation $|\nabla u| = 1$. Secondly, if $\{u^\varepsilon\}$ is a family of minimizers, then the limits u must minimize

$$F_0[v,\Omega] \;=\; \frac{1}{3}\int_{\Omega\cap J_{\nabla v}} |[\nabla v]|^3 \, d\mathcal{H}^1 \,, \tag{3}$$

among all v solving the eikonal equation. Here $J_{\nabla v}$ is the set where "∇v jumps", and $[\nabla v]$ is the "jump".

A technique to control the functional from below was devised by Jin and Kohn [12], which introduced a class of "entropies" whose divergence is controlled, in an appropriate sense, by F_ε. Building upon this work, compactness with respect to the strong $W^{1,3}$ topology was then proven independently in [1, 7] and [10]. A lower bound was also obtained; which for the case that the limit is BV can be written as a simple line integral [4]. Precisely:

Theorem 1 (From [1, 7])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $\varepsilon_i \to 0$ and u_i be such that $F_{\varepsilon_i}[u_i,\Omega] < C < \infty$. Then there is a subsequence converging strongly in $W^{1,3}$ to a function u_0 with $|\nabla u_0| = 1$. If additionally $\nabla u_0 \in BV$, then*

$$\liminf_{k\to\infty} F_{\varepsilon_{i_k}}[u_{i_k},\Omega] \ge \frac{1}{3}\int_{J_{\nabla u_0}} |\nabla^+ u_0 - \nabla^- u_0|^3 d\mathcal{H}^1 \,.$$

Notice that the compactness result does not give $\nabla u_0 \in BV$. Indeed, whereas the functional (3) is lower semicontinuous on BV, it is not coercive in the same space, as was shown in [1, 7] by constructing a sequence which is bounded in energy but converges to a limit outside BV.

An upper bound matching this lower bound had already been obtained in [4] for the case that the limit is a single straight interface, and in [7] for the case that finitely many straight interfaces are present. The presence of the gradient structure made it however difficult to conclude by density. We present here the derivation of an upper bound under the assumption that the limit is in BV, which was discussed in detail in [6]. An independent proof of the same result was obtained by A. Poliakovsky [14, 15].

Theorem 2 (From [6])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 set and $u_0 \in W^{1,\infty}(\Omega,\mathbb{R})$ with $\nabla u_0 \in BV(\Omega, S^1)$, and $\varepsilon_i \to 0$. Then there is a sequence $u_i \in C^\infty(\Omega)$ such that $u_i \to u_0$ in $W^{1,p}(\Omega)$ for every $p < \infty$ and*

$$\limsup_{i\to\infty} F_{\varepsilon_i}[u_i,\Omega] \le \frac{1}{3}\int_{J_{\nabla u_0}\cap\Omega} |\nabla^+ u_0 - \nabla^- u_0|^3 d\mathcal{H}^1 \,.$$

If u_0 obeys the boundary condition (2), then the sequence can be chosen so that the same holds for each i.

The full determination of the Γ-limit, i.e., removing the assumption $\nabla u_0 \in BV$ in both Theorem 1 and Theorem 2, is open. Notice that if ∇u_0 is not in BV but is the limit of a sequence bounded in energy, one can still give a meaning to the expression on the right hand side (see [8]): a natural conjecture is that this quantity coincides with the supremum among all lower bounds obtained with all entropies and that the optimal sequence exists also in this case.

The proof of Theorem 2 is based on taking a mollification of u (after an appropriate continuation outside Ω) and improving it locally, a technique already used in [5] to obtain Γ-convergence for a vectorial problem motivated by the theory of elasticity. Precisely, we fix a family of mollifiers ϕ_ε, and define

$$u_i = \phi_{\varepsilon_i} * u_0 . \tag{4}$$

Since $\nabla u_0 \in BV$, the sequence u_i is automatically bounded in energy, as can be shown using the following local estimate.

Lemma 1 (From [6])**.** *There exists a universal constant C such that the following holds for every $k \geq 1$. If $u_0 \in W^{1,\infty}(B_{2k\varepsilon})$ with $\nabla u_0 \in BV(B_{2k\varepsilon}, S^1)$, then*

$$F_\varepsilon[u_\varepsilon, B_{k\varepsilon}] \ \leq \ C\|D^2 u_0\|(B_{2k\varepsilon}) .$$

This estimate is in the end used only on the "bad" set, where the structure of u_0 on a scale $k\varepsilon$ does not permit any better estimate. Since $\nabla u_0 \in BV$, the jump measure $\|D^2 u_0\|$ does not concentrate on the bad set. On the part of the domain where ∇u_0 has a better behavior, sharper estimates can be obtained.

In order to show that the limiting energy concentrates on the jump one uses a quadratic estimate:

Lemma 2 (From [6])**.** *There exists a universal constant C such that the following holds for all $k \geq 1$. If $u_0 \in W^{1,\infty}(B_{2k\varepsilon})$ and $\nabla u_0 \in BV(B_{2k\varepsilon}, S^1)$, then*

$$F_\varepsilon[u_\varepsilon, B_{k\varepsilon}] \ \leq \ C\frac{1}{\varepsilon}\left[\|D^2 u_0\|(B_{2k\varepsilon})\right]^2 . \tag{5}$$

This proves that only the jump part of $D^2 u_0$ contributes to the limit.

Finally, on "good" points of the jump, i.e., on points of $J_{\nabla u_0}$ where the convergence of the blow-ups at scale $k\varepsilon$ is already "good enough", we replace u_i by the single-interface profile. The convergence of the blow-ups permits to show that the energy of the boundary layer is a small fraction of the total energy, and to conclude the proof.

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Foliations by 2-surfaces and Born-Infeld equations

Yann Brenier
(joint work with Yann Brenier)

We consider foliations of R^{1+3} by extremal surfaces, either for the euclidean metric or the Minkowski metric. In PDE words, this amounts to look for two fields E, B, valued in R^3, depending on $(t, x) \in R^{1+3}$, subject to the following differential and algebraic constraints

$$\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0, \quad E \cdot B = 0,$$

that are critical points of the functional

$$\int \sqrt{B^2 + \eta E^2}\, dxdt,$$

under appropriate boundary conditions, with $\eta = 1$ in the Euclidean case and $\eta = -1$ in the Lorentzian case.

In the Euclidean case, it is natural to look for minimizers. Because of the differential constraints, the algebraic expression $E \cdot B = 0$ can be written in divergence form and is weakly continuous in (E, B) with respect to the weak topology of L^2. Unfortunately, we expect minimizing sequences to converge only as vector valued measures. So, it is unclear what is left from the algebraic constraint under such a limiting process. A delicate interplay between concentration and oscillation effects may play a crucial role. At this level, our point is just to attract the attention of the Calculus of Variation community to what seems to us an interesting "supercritical compensated compactness" problem. Let us just observe that the algebraic constraint cannot be entirely relaxed. Indeed, an easy calculation shows that the convex hull of

$$\int \sqrt{B^2 + E^2}\, dxdt,$$

with constraint $E \cdot B = 0$, is

$$\int \sqrt{B^2 + E^2 + 2|E \cdot B|}\, dxdt.$$

Also notice that various simplified versions of the problem can be addressed. If the t dependence is given up, for instance, we may write (at least locally) E as a gradient $E = \nabla\phi$, and we can renormalize the algebraic constraint as $\nabla \cdot (\psi B) = 0$, where

$$\psi = \frac{\phi}{\sqrt{1 + \phi^2}}.$$

Then the problem becomes "critical", with the pairing of a vector valued measure (B) and a bounded Borel function (ψ). If, in addition, one space variable is dropped, the problem becomes explicitely integrable (and somewhat similar to a simple "disocclusion problem" in image processing).

Let us now move to the Lorentzian case, in which extremal surfaces can be physically interpreted as classical relativistic strings. We are, therefore, interested in foliating the classical 4D Minkowski space by a continuum of strings. We first observe (see [1]) that the problem can be related to the Born-Infeld functional:

$$\int \sqrt{r^2 + B^2 - E^2 - r^{-2}(E \cdot B)^2}\, dxdt,$$

for which r is a fixed parameter and one looks for critical points E, B subject to the linear differential constraint:

$$\partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0.$$

This model was introduced in 1934 by Born and Infeld as a nonlinear cutoff theory to substitute for Maxwell's electromagnetism (which allows infinite electrostatic field for point charged particles). Here r stands for an absolute bound for any electrostatic field (just as the speed of light is an absolute bound for any velocity in special relativity). Consistently, Maxwell's linear theory is recovered in the "low energy" limit $r \to \infty$. In the opposite "high energy" limit, as $r \to 0$, r becomess a penalty parameter for the algebraic constraint $E \cdot B = 0$. The high energy asymptotic analysis of the Born-Infeld model has been recently handled in collaboration with W.A. Yong in [4]. More generally, the study of the Born-Infeld model leads to many side issues. In particular, because of concentration and oscillation phenomena, the concept of extremal surface (i.e. relativistic strings) can be relaxed (subrelativistic strings and/or sticky strings) in different ways, as discussed in [2, 3].

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An isoperimetric concept for the Mass in General Relativity

Gerhard Huisken

The total mass of an isolated gravitating system in General Relativity, like a star or a black hole, should be mathematically defined as a geometric invariant of 3-dimensional asymptotically Euclidean Riemannian manifolds (M^3, g) that occur in the context of Einstein's field equations as spacelike hypersurfaces of a Lorentzian manifold modelling the isolated system. Traditionally such a geometric invariant has been defined by Arnowitt, Deser and Misner as a geometrically invariant limit of flux integrals over large 2-spheres in the asymptotically Euclidean region, involving first derivatives of the metric tensor (ADM-mass):

$$m_{ADM}(M^3, g) := \frac{1}{16\pi} \lim_{R\to\infty} \int_{M^3} (\partial_i g_{ij} - \partial_j g_{ii})\, d\nu^j.$$

The simplest versihon of the famous positive mass theorem of Schoen and Yau states that for asymptotically flat 3-manifolds of non-negative scalar curvature $R \geq 0$ the mass is non-negative with equality if and only if (M^3, g) is isometric to flat Euclidean 3-space.

The lecture expalins how the concept of mass can be reinterpreted from the point of view of the isoperimetric inequality: Note that for any region Ω of Euclidean 3-space we have the isoperimetric inequality

$$|\partial\Omega|^{3/2} \geq 6\pi^{1/2}\,\mathrm{Vol}^(\Omega),$$

which is encoded in the isoperimetric profile $\phi_0(s) = \frac{1}{6\sqrt{\pi}} s^{3/2}$ assigning to each area $s \geq 0$ the maximal volume that can be enclosed with that area. When studying the most important nontrivial examples of asymptotically flat Euclidean Riemannian 3-manifolds, the spatial Schwarzschild manifolds $\frac{(M^3, g_m) = (R^3\setminus\{0\}, g_m = \delta(1+m}{2r)^4)}$ of mass $m_{ADM} = m > 0$ we see that the corresponding isoperimetric profile ϕ_m satisfies the expansion

$$\phi_m(s) = \phi_0(s) + \frac{1}{2} m s + \text{lowerorder(s)}.$$

With this observation we define for any mildly asymptotically Euclidean Riemannian 3-manifolds the *isoperimetric mass* by

$$m_{ISO}(M^3, g) := \limsup_{|\partial\Omega|\to\infty} \frac{2}{|\partial\Omega|}\left(\mathrm{Vol}(\Omega) - \frac{1}{6\sqrt{\pi}}|\partial\Omega|^{3/2}\right)$$

The talk justifies this definition by indicating the proof of a positive mass theorem for this isoperimetric definition of mass and explaining its relation to the $ADM - mass$ and to the Hawking-mass of 2-spheres $\Sigma^2 = \partial\Omega \subset M^3$ given by

$$m_{Haw}(\Sigma^2) := \frac{|\Sigma^2|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int_{\Sigma^2} H^2\, d\mu\right),$$

where H is the mean curvature of the 2-sphere. In non-technical terms one of the results established is the following:

Theorem *If (M^3, g) is an asymptotically flat Riemannian 3-manifold of non-negative scalar curvature with well-defined ADM-mass, then we have the inequality*

$$\sup m_{Haw}(\Sigma^2) \le m_{ISO}(M^3, g) \le m_{ADM}(M^3, g),$$

where the supremum on the LHS is taken over all surfaces Σ^2 that are outward area-minimizing. In particular, $m_{ISO} \ge 0$ with equality only if (M^3, g) is isometric to Euclidean space.

The idea behind the proofs is to use the evolution of outward area-minimising 2-surfaces in (M^3, g) along both mean curvature flow and inverse mean curvature flow in their level-set formulations and to exploit the properties of these flows established by B. White [4] in the case of mean curvature flow and H.-Ilmanen in the case of inverse mean curvature flow [3]. In particular, in view of the work of White inverse mean curvature flow and the Geroch monotonicity can be used to control the Hawking mass on the level-sets of solutions to mean curvature flow. Notice that a relation between the mass and the isoperimetric inequality was first observed by Christodoulou and Yau [2]. We also note the work of Bray and Neves [1] concerning inverse mean curvature flow and the Yamabe problem and mention that a relation between mean curvature type flows and isoperimetric inequalities has also been observed in work of B. Andrews, P. Topping and F. Schulze.

We also remark that the method yields a version of the Penrose inequality for the isoperimetric mass and can be used to give a consistent definition of an isperimetric quasi-local mass that will be explored elsewhere.

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Conformal surfaces with bounds on their Willmore energy

Ernst Kuwert
(joint work with Reiner Schätzle)

For an immersed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined as the integral

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\vec{H}|^2 \, \mathrm{d}\mu_g,$$

where $\vec{H}$ is the mean curvature vector, $g = f^* g_{euc}$ is the pull-back metric and μ_g is the induced area measure on Σ. In this talk we present a bi-Lipschitz type estimate for surfaces of genus $p \geq 1$ whose Willmore energy is strictly less than a certain critical value. As the Willmore functional is invariant under the Möbius group of $\mathbb{R}^n$, i.e. under dilations and inversions, such an estimate can possibly hold only up to the action of the Möbius group, that is after application of a suitable Möbius transformation. Let us denote by β_p^n the infimum of the Willmore functional among all closed, genus p immersions $f : \Sigma \to \mathbb{R}^n$.

Theorem 1. *Let $f : \Sigma \to \mathbb{R}^n$, $n = 3, 4$, be an immersion of a closed, oriented surface of genus $p \geq 1$, satisfying the following conditions for some $\delta > 0$:*

$$(1) \qquad \mathcal{W}(f) \leq 8\pi - \delta,$$

$$(2) \qquad \mathcal{W}(f) - 4\pi \leq \sum_{i=1}^{k} (\beta_{p_i}^n - 4\pi) - \delta, \text{ whenever } p = \sum_{i=1}^{k} p_i \text{ for } 1 \leq p_i < p,$$

$$(3) \qquad \mathcal{W}(f) \leq \beta_p^4 + \frac{8\pi}{3} - \delta, \text{ if } n = 4.$$

Then there is a Möbius transformation ϕ, such that the metric g induced by $\phi \circ f$ is uniformly equivalent to a conformal metric g_0 of constant curvature:

$$g = e^{2u} g_0 \quad \text{where} \quad \max_\Sigma |u| \leq C(p, \delta) < \infty.$$

Immersions with $\mathcal{W}(f) < 8\pi$ are actually embeddings, as was first observed by Li and Yau [1]. It is also well-known that $\beta_p^n \geq 4\pi$ and that round spheres are the only closed surfaces with energy equal to 4π. Willmore conjectured that $\mathcal{W}(f) \geq 2\pi^2$ for any torus in $\mathbb{R}^3$, where equality is achieved for a specific torus of revolution. A proof of this conjecture has been submitted by M. Schmidt [3].

The existence of a smooth, minimizing torus in $\mathbb{R}^n$ was settled by L. Simon in [4]. As an important tool, we use his approximate graphical decomposition lemma for surfaces whose curvature is small in L^2 in a ball. The second main ingredient is an estimate by S. Müller & V. Šverak for conformal parametrizations of complete surfaces of the type of a plane [2]. In the case of tori, i.e. $p = 1$, we obtain as a corollary that the conformal type is estimated depending only on the bounds (1) and (3).

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Partial regularity for harmonic maps, revisited

Michael Struwe
(joint work with Tristan Rivière)

In [7], Tristan Rivière presented a new approach to the regularity result of Hélein [6] for weakly harmonic maps in dimension $m = 2$, where he succeeded in writing the harmonic map system in the form of a conservation law whose constituents satisfied elliptic equations with a Jacobian structure to which Wente's [10] regularity results could be applied.

In [8] we observe that the regularity result already follows from the novel interpretation given in [7] of the harmonic map system in the sense of gauge theory, and we succeed in obtaining analogous partial regularity results for stationary harmonic maps and related problems also in dimension $m \geq 3$.

Recall that the equation for a harmonic map $u = (u^1, \ldots, u^n) \in H^1(B; \mathbb{R}^n)$ from a ball $B^m = B \subset \mathbb{R}^m$ to a hypersurface $N \subset \mathbb{R}^n$ with normal ν may be written in the form

$$-\Delta u^i = w^i \nabla w^j \cdot \nabla u^j = (w^i \nabla w^j - w^j \nabla w^i) \cdot \nabla u^j,\ 1 \leq i \leq n, \tag{1}$$

where $w = \nu \circ u$; and similarly in higher codimension. The key idea then is to identify the anti-symmetry of the 1-form

$$\Omega^{ij} = (w^i dw^j - w^j dw^i),\ 1 \leq i, j \leq n, \tag{2}$$

as the essential structure of equation (1).

Interpreting $\Omega \in L^2(B; so(n) \otimes \wedge^1 \mathbb{R}^n)$ as a connection in the $SO(n)$-bundle u^*TN and following Uhlenbeck's approach the existence of Coulomb gauges [9], one succeeds in finding $P \in H^1(B; SO(n))$ and $\xi \in H^1(B)$ such that

$$P^{-1}dP + P^{-1}\Omega P = *d\xi, \tag{3}$$

where $*$ is the Hodge dual. Applying the gauge transformation P^{-1} to ∇u and observing the identity $dP^{-1} = -P^{-1}dPP^{-1}$, from (1) we obtain the equation

$$-div(P^{-1}\nabla u) = (P^{-1}\nabla P + P^{-1}\Omega P) \cdot P^{-1}\nabla u = *d\xi \cdot P^{-1}du, \tag{4}$$

where the right hand side already has the structure of a Jacobian – up to the harmless (bounded) factor P^{-1}. Also observe that ∇u may be recovered from the term $P^{-1}\nabla u$ without any difficulty.

For general elliptic systems of the form

$$-\Delta u = \Omega \cdot \nabla u \text{ in } B \tag{5}$$

we then obtain the following result.

Theorem 1. *For every $m \in \mathbb{N}$ there exists $\varepsilon(m) > 0$ such that for every $\Omega \in L^2(B^m, so(n) \otimes \wedge^1\mathbb{R}^m)$ and for every weak solution $u \in H^1(B^m, \mathbb{R}^n)$ of equation* (5)*, satisfying the Morrey growth assumption*

$$\sup_{x\in B,\, r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x)\cap B} (|\nabla u|^2 + |\Omega|^2)\, dx \right) < \varepsilon(m) \quad , \tag{6}$$

we have that u is locally Hölder continuous in B with exponent $0 < \alpha = \alpha(m) < 1$.

The preceding result readily yields the partial regularity of stationary harmonic maps. For a smooth, compact, oriented k-dimensional submanifold $N \subset \mathbb{R}^n$ and a ball $B \subset \mathbb{R}^m$ let

$$H^1(B; N) = \{u \in H^1(B; \mathbb{R}^n);\ u(x) \in N \text{ for almost every } x \in B\}. \tag{7}$$

Recall that a map $u \in H^1(B; N)$ is *stationary* if u is critical for the energy

$$E(u) = \int_B |\nabla u|^2\, dx$$

both with respect to variations of the map u and with respect to variations in the domain. It follows that u is weakly harmonic; that is, u satisfies the equation

$$-\Delta u = A(u)(\nabla u, \nabla u) = \sum_{l=1}^{n-k} \sum_{\alpha=1}^{m} \nu_l \langle d\nu_l \partial_\alpha u, \partial_\alpha u \rangle) = \sum_{l=1}^{n-k} w_l \langle \nabla w_l, \nabla u \rangle \tag{8}$$

in the sense of distributions, where A is the second fundamental form of N, defined locally via an orthonormal frame field ν_l, $1 \le l \le n-k$, for the normal bundle to N, and where we denote as $w_l = \nu_l \circ u$ the corresponding unit normal vector field along the map u, as in (1). Also denote as $\langle \cdot, \cdot \rangle$ the Euclidean inner product.

Moreover, as a consequence of the stationarity condition with respect to variations in the domain we have the monotonicity estimate

$$r^{2-m} \int_{B_r(x_0)} |\nabla u|^2\, dx \le R^{2-m} \int_{B_R(x_0)} |\nabla u|^2\, dx \tag{9}$$

for all balls $B_R(x_0) \subset B$ and all $r \le R$.

Thus, Theorem 1 may be applied and we obtain the following generalization of the partial regularity result of Evans [4] and Bethuel [1] for target manifolds of class C^2 (with second fundamental form of class C^0). Note that their approach in general requires the target manifold N^k to be of class C^5; see [6], Theorem 4.3.1 and Remark 4.3.2.

Theorem 2. *Let $N^k \subset \mathbb{R}^n$ be a closed submanifold of class C^2. Let $m \geq 3$ and suppose $u \in H^1(B^m; N)$ is a stationary harmonic map. There exists a constant $\varepsilon_0 > 0$ depending only on N with the following property. Whenever on some ball $B_R(x_0) \subset B$ there holds*

$$R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, dx < \varepsilon_0, \tag{10}$$

then u is Hölder continuous (and hence as smooth as permitted by the smoothness of N) on $B_{R/2}(x_0)$.

Proof. As in (1), equation (8) equivalently may be written in the form

$$-\Delta u^i = \Omega^{ij} \cdot \nabla u^j, \tag{11}$$

where $\Omega \in L^2(B; so(n) \times \wedge^1 \mathbb{R}^n)$ in view of our assumption on N, with components

$$\Omega^{ij} = \Omega^{ij}_\alpha dx^\alpha = \sum_{l=1}^{n-k} (w^i_l dw^j_l - w^j_l dw^i_l), \; 1 \leq i, j \leq n, \tag{12}$$

in the above local representation of A. Note that (9) and (10) imply that ∇u and Ω belong to the Morrey space $L^{2,m-2}(B)$ with

$$\sup_{x_0 \in B} r^{2-m} \int_{B_r(x_0) \cap B} |\Omega|^2 \, dx \leq C \sup_{x_0 \in B} r^{2-m} \int_{B_r(x_0) \cap B} |\nabla u|^2 \, dx \leq C\varepsilon_0. \tag{13}$$

The result now is an immediate consequence of Theorem 1. □

The proof of Theorem 1 only uses very classical tools in elliptic regularity theory, as described for instance in the lecture notes of Giaquinta [5].

It seems that the key technical improvement over previous approaches to the regularity problem consists in letting the gauge transformations act directly on the gradient of the maps (rather than on an associated moving frame). This idea therefore may be applicable in a variety of other contexts where properly gauged moving frames have been used, including equations of fourth or higher order in conformal geometry, or even hyperbolic equations such as the wave map system.

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Quantification of blow-up levels for some 2-d elliptic PDE's with critical exponential nonlinearity

Olivier Druet

Consider Ω a smooth bounded domain of $\mathbf{R}^2$ and let us consider a sequence of solutions (u_ε) of the equation

$$\Delta u_\varepsilon = f_\varepsilon(x, u_\varepsilon) \text{ in } \Omega\ ,\quad u_\varepsilon = 0 \text{ on } \partial\Omega\ . \qquad (E_\varepsilon)$$

where f_ε is a sequence of functions of critical growth. In order to simplify, let us assume that

$$f_\varepsilon(x,t) = h_\varepsilon(x) e^{\varphi_\varepsilon(t)} e^{4\pi t^2}$$

where $h_\varepsilon \to h_0$ in $C^2(\Omega)$ as $\varepsilon \to 0$ with $h_0 > 0$ and $\varphi_\varepsilon \to \varphi_0$ in $C^2_{loc}(\mathbf{R})$ as $\varepsilon \to 0$ with, moreover $\frac{\varphi'_\varepsilon(t)}{t} \to 0$ as $t \to +\infty$ uniformly in ε. We let J_ε be the energy associated to the equation (E_ε), namely

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F_\varepsilon(x,u)\, dx$$

where F_ε is a primitive in t of f_ε. Note that positive critical points in $H^1_0(\Omega)$ of J_ε are solutions of equation (E_ε). We then have the following result :

Theorem 1 (Druet, [1]**)** - *If (u_ε) is a sequence of solutions of equation (E_ε) with $J_\varepsilon(u_\varepsilon)$ bounded, then there exists u_0 solution of the limiting equation (maybe the zero solution) and $N \in \mathbf{N}$ such that*

$$J_\varepsilon(u_\varepsilon) \to J_0(u_0) + \frac{N}{2}$$

as $\varepsilon \to 0$.

This result is the analog of the following well-known result due to Struwe :

Theorem 2 (Struwe, [3]**)** - *Let Ω be a smmoth bounded domain in $\mathbf{R}^n$, $n \geq 3$, and let (u_ε) be a sequence of solutions, bounded in $H^1_0(\Omega)$, of*

$$\Delta u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{\frac{n+2}{n-2}} \text{ in } \Omega\ ,\quad u_\varepsilon = 0 \text{ on } \partial\Omega\ .$$

Assume that $h_\varepsilon \to h_0$ in $C^2(\Omega)$. Then there exists a solution u_0 of the limiting equation (maybe the zero solution) and $N \in \mathbf{N}$ such that

$$I_\varepsilon(u_\varepsilon) \to I_0(u_0) + N I_{min}$$

as $\varepsilon \to 0$ *where* I_{min} *is some universal dimensional constant and*

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + h_\varepsilon u^2\right) dx - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}}\, dx\ .$$

Let me compare these two theorems. First, they are interesting because of the lack of compactness hidden behind the nonlinearity. In theorem 2, it is due to the lack of compactness of the embedding of $H_0^1(\Omega)$ in $L^{\frac{2n}{n-2}}(\Omega)$. The analog in Theorem 1 is the critical inequality due to Trudinger [4] and Moser [2] : there exists $C(\Omega)$ such that

$$\int_M \exp^{4\pi \frac{u^2}{\|\nabla u\|_2^2}}\, dx \le C(\Omega)$$

for all $u \in H_0^1(\Omega)$. In this inequality, both 4π and the square power are optimal. In both situations, we are thus able to quantify the level at which a lack of compactness can appear for a sequence of solutions of the critical equation. The fact behind these results is that the lack of compactness is standard and is due to bubbles which have a standard form.

The main difference between these two results can be illustrated by the following fact : in theorem 2, one could replace "sequences of solutions" by "sequences of Palais-Smale sequences". In other words, theorem 2 continues to hold if one assumes that (u_ε) is a sequence of positive functions bounded in $H_0^1(\Omega)$ satisfying that

$$\int_\Omega (\nabla u_\varepsilon, \nabla\varphi_\varepsilon)\, dx + \int_M h_\varepsilon u_\varepsilon \varphi_\varepsilon\, dx - \int_M u_\varepsilon^{\frac{n+2}{n-2}}\varphi_\varepsilon\, dx = o\left(\|\nabla\varphi_\varepsilon\|_2\right)$$

for all sequences (φ_ε) in $H_0^1(\Omega)$. This is not anymore true for theorem 1 and one can produce counter-examples. The standard proof of theorem 2 is by substracting the weak limit u_0 to u_ε, to detect a bubble through the Levy concentration function, to substract this bubble and to go on with this process. Each time one substracts a bubble, one looses I_{min} in the energy. Thus the process has to stop and one can conclude. The crucial point here is that $u_\varepsilon - u_0 -$ bubbles is still a Palais-Smale sequence for a nice functional. In the case of the nonlinearity in $e^{4\pi u^2}$ of theorem 1, one can imagine the nightmare it becomes. Thus we have to use a pointwise method to detect bubbles. The proof of theorem 1 is of course much more technical and involved than the proof of theorem 2.

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On an isoperimetric conjecture for a Schrödinger operator depending on a curve

Almut Burchard

Let γ be a smooth closed curve in $\mathbb{R}^3$, and let $\kappa(s)$ be its curvature as a function of the arclength parameter. The curve determines a one-dimensional Schrödinger operator by

$$H_\gamma = -\frac{d^2}{ds^2} + k^2 .$$

We consider the principal eigenvalue $e_0(\gamma)$ of H_γ with periodic boundary conditions. It has been conjectured that $e_0(\gamma)$ is minimized among all curves of given length by circles [1, 2]. In recent joint work with L. E. Thomas, we show that circles minimize $e_0(\gamma)$ at least locally, subject to a length constraint [3].

The conjecture is related to the optimal constant in a particular Lieb-Thirring inequality [1], and to the equation for the tension of an inextensible, elastic loop moving in three-dimensional space [2]. Similar Schrödinger operators turn up in connection with interfaces between in reaction-diffusion equations [4, 5], quantum particles constrained to narrow channels [6], and Dirac operators on the sphere [7].

To describe our result more precisely, we fix the length of the curve to be 2π. In the case of the unit circle, the curvature satisfies $\kappa^2 \equiv 1$, the ground state eigenfunction is constant, and the principal eigenvalue of H_γ is $e_0(\gamma) = 1$. The eigenvalue $e_0(\gamma)$ assumes the same value as for a circle for planar loops whose tangent vector is the unit vector in the direction of $\big(\cos(s), \beta\sin(s), 0\big)$ for $s \in [0, 2\pi]$, where $\beta \neq 0$ is a constant. In recent joint work with Lawrence E. Thomas, we show that small deformations about any of these loops cause e_0 to strictly increase, provided the loop is not simply deformed to a translation or rotation of another loop in the same family. The conjecture itself remains open. The best lower bound known is that $e_0(\gamma) \geq .6085$ for curves of length 2π [8].

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Geometry of optimal windows

Jochen Denzler
(joint work with Almut Burchard)

We describe some recent results, joint with Almut Burchard, about optimal windows. Here, a window is defined to be a measurable subset D of the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. We consider the Laplace operator in this domain, with Dirichlet boundary conditions $u = 0$ on D and Neumann boundary conditions $\partial_\nu u = 0$ on $N := \partial\Omega \setminus D$; these latter can be understood as natural boundary conditions for the variational problem. The principal eigenvalue λ_1 is viewed as a function of D, subject to the constraint that the area ($n-1$ dimensional Hausdorff measure) of D is fixed. A minimizer for the principal eigenvalue is called an optimal window, and it was shown in [2] that optimal windows exist. The geometry of optimal windows is only known in the case where Ω is a ball. In this case, the optimal window is unique up to symmetry operations and null sets, as was shown in [2]. In the sequel, 'eigenvalue' always refers to the principal eigenvalue λ_1.

It may be noted that there is a fine and a coarse definition of the window eigenvalue problem, depending on whether the boundary conditions on D are required to be satisfied up to sets of zero capacity or up to sets of zero measure, and these definitions of the eigenvalue are not equivalent. For *optimal* windows however, the two formulations 'essentially' coincide, due to the coarse area constraint.

The results given here are published with full proofs in [1].

Limited symmetry still allows to get information about optimal windows in squares and rectangles. Firstly, if a segment D slides along one side of a rectangle, the corresponding eigenvalue decreases as D moves from the center position to a position adjacent to the corner. More generally, any window on one side of the rectangle has eigenvalue larger than a connected window of the same area shifted to be adjacent to a corner. These results rely on increasing rearrangements and Dirichlet–Neumann bracketing.

A variation formula gives the change of the eigenvalue as the window changes under the flow of a vector field. In 2D, according to results taken from [3], solutions to elliptic problems near a simple interface of Dirichlet and Neumann boundary (segment of D meeting segment of N) lie, locally near the interface point, in the Sobolev space $W^{2,2} \oplus S$ with a 1-dimensional singular space S. On the side of a rectangle, the space S consists of $c \operatorname{Im} \sqrt{z}$ ($c \in \mathbb{R}$), written in complex notation (an assuming the interface point to be at $z = 0$). It turns out that the size of the singular coefficient c determines the derivative of λ_1 as D is extended near the interface point.

If the interface point between N and D is in the corner of a rectangle, the singular coefficient vanishes. This is why a window segment adjacent to a corner can still be improved upon by shifting the window a bit around the corner.

In a square (unit square with no loss of generality), the techniques mentioned can be used to improve any window that is confined to two sides. The optimum among such windows is always an L shaped segment around a corner. Numerical evidence suggests that the optimal window in a square should be a segment centered either at a corner or in the middle of a side, depending on the length. Approximately in line with heuristics that optimal windows 'prefer' corners (low diffusive accessibility), the corner-centered segment is better than the side-centered, when the length of D is below 1.02 or between 2.04 and 3.15 sidelengths. In the other cases, the side-centered segment is better than the corner-centered one.

An interesting phenomenon of spontaneous symmetry breaking occurs however: Optimal windows in a square never have the full $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of a rectangle, nor even the 180° rotational symmetry. This can be shown purely analytically for windows up to one sidelength. In the general case, the result is subject to minor numerical ingredients, namely the calculation of eigenvalues for only a 1-parameter family of windows (all other reductions are still analytic in nature).

We finally give an example of a star-shaped domain in $\mathbb{R}^2$ in which an optimal window cannot be connected. This example is also interesting in view of the conjecture that optimal windows in convex domains should be connected.

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Optimal regularity for the pseudo-infinity Laplacian

Mariel Saez

(joint work with J.D. Rossi)

The main goal is to study the optimal regularity of viscosity solutions to the pseudo-infinity Laplacian. We find that the solutions are Lipschitz but not necessarily C^1. For the sake of completeness, we also prove existence and uniqueness for this operator.

The pseudo-infinity Laplacian is the second order nonlinear operator given by

$$\tilde{\Delta}_\infty u = \sum_{i \in I(\nabla u)} u_{x_i x_i} |u_{x_i}|^2, \tag{1}$$

where the sum is taken over the indices in $I(\nabla u) = \{i : |u_{x_i}| = \max_j |u_{x_j}|\}$. This operator appears naturally as a limit of the following $p-$Laplace type problems:

$$\tilde{\Delta}_p u = \sum_{i=1}^{N} (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0. \tag{2}$$

In the context of viscosity solutions we show

Theorem 1. *Let $u : \Omega \to \mathbb{R}$ be a viscosity solution to*

$$\tilde{\Delta}_\infty u = 0, \tag{3}$$

where $\Omega \subset \mathbb{R}^N$. Then u is locally Lipschitz.

Moreover, this result is optimal for $N \geq 2$, since

$$u(x, y) = x + \frac{1}{2}|y|, \tag{4}$$

is viscosity solution to (3) *that has no further regularity than Lipschitz.*

This result contrast with the one showed by Savin in [11]. In this paper he showed that a solution to the standard infinity Laplacian in two space dimensions is C^1.

As mentioned before, we also include an existence and uniqueness result. Namely,

Theorem 2. *Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$, for any Lipschitz boundary data $g(x)$, there exists a unique viscosity solution to*

$$\begin{aligned} \tilde{\Delta}_\infty u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

The existence result is proved using arguments from [4]. The strategy is to take limits (along subsequences) of variational solutions to (2) with right hand side equal to 0 as $p \to \infty$. Uniqueness follows by adapting results in [3].

On the other hand, the main ingredient of the proof of Theorem 1 is that solutions to (3) verify a comparison with l^1-cones property. This property is analogous to the one satisfied by solutions to the usual infinity Laplacian with l^2-cones, see [2] and [5].

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Pattern formation in micromagnetics

Felix Otto

(joint work with Ruben Cantero-Alvarez and Jutta Steiner)

From the point of view of mathematics, micromagnetics is an ideal playground for a pattern forming system in materials science: There are abundant experiments on a wealth of visually attractive phenomena and there is a well–accepted continuum model.

In this talk, I will focus on a specific experimental pattern for thin film ferromagnetic elements, the concertina pattern. Experiments indicate that this pattern arises out of a bifurcation at the onset of switching under an external magnetic field. Starting point for our analysis is the micromagnetic model which has three length scales (the width ℓ and the thickness t of the sample and a material length scale d) and thus many parameter regimes. For the concertina pattern, we identify the appropriate paramater regime and rigorously derive a reduced model via Γ–convergence. The Γ–limit combines a limit in parameter space with a blow-up in function space (which zooms in on the bifurcation point). It thus identifies the dominant nonlinear term and just has a single non-dimensional parameter (the appropriately renormalized external field $\hat{h}_{ext}$). We numerically simulate the reduced model and compare it to experimental data and find good agreement.

We further argue that the Γ-limit interpolates between the subcritical bifurcation at the critical value of $\hat{h}_{ext}$ and heuristic domain theory, which replaces the smooth transition layers (Néel walls) by sharp discontinuities of a given line energy. We show that the Γ-limit displays the same scaling as domain theory in the limit $\hat{h}_{ext} \uparrow \infty$. The lower bounds rely on an appropriate nonlinear interpolation inequality.

A variational approach to entropy and information

Keith Ball

One of the most natural ways to track convergence in the central limit theorem is by means of the gap between the entropy of the normalised sums

$$\frac{1}{\sqrt{n}}\sum_{1}^{n} X_i$$

of independent copies of a random variable, and that of the Gaussian limit.

If X is a real random variable with density $f : \mathbb{R} \to [0, \infty)$, the entropy is

$$\mathrm{Ent}(X) = -\int_{\mathbb{R}} f \log f.$$

Among random variables with a given variance, the Gaussian has the largest entropy. According to the Shannon-Stam inequality if X and Y are IID, then the normalised sum $(X + Y)/\sqrt{2}$ has entropy at least as large as that of X and Y. It has long been believed and was formally conjectured by Lieb in 1978 that the entropy of the normalised sums should increase with n: that there should be an analogue of the second law of thermodynamics for the central limit process.

My aim is to describe a variational approach to entropy and information developed by my collaborators S. Artstein, F. Barthe, A. Naor and I in [2] and [1]. The approach yields a proof of Lieb's conjecture and the first quantitative measures of entropy growth along the central limit process for a large class of random variables.

The method relies upon the connection between entropy and Fisher information provided by the Ornstein-Uhlenbeck semigroup. An analysis of a local version of the transportation proof of the Brunn-Minkowski inequality leads to a variational characterisation of the Fisher information. This enables us to estimate the information of normalised sums of independent random variables with a density f in terms of the spectral properties of the f-Laplacian:

$$L : s \mapsto -(fs')'/f.$$

The calculus of variations and spectral theory needed for the results above is elementary. My hope is that participants at this meeting will be motivated to study the variational problem more deeply.

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A gradient flow for Bernoulli free boundary problem

Pierre Cardaliaguet

In two joint works with O. Ley (U. Tours) [3] and [4] we investigate the geometric evolution equation

$$V_{t,x} = -1 + \lambda \bar{h}(x, \Omega(t)) \qquad \text{for all } t \geq 0,\ x \in \partial\Omega(t)\ . \tag{1}$$

In the above equation, $t \to \Omega(t)$ is an (unknown) evolving familly of bounded open subsets of $\mathbb{R}^N$, $V_{t,x}$ is the normal velocity of the set $\Omega(t)$ at time t and at the point x, λ is some positive parameter and $\bar{h} = \bar{h}(x, \Omega)$ is a non local term of Hele-Shaw type given, for any set Ω with smooth boundary, by

$$\bar{h}(x, \Omega) = |\nabla u(x)|^2\ ,$$

where $u : \Omega \to \mathbb{R}$ is the capacity potential of Ω with respect to S, i.e., the solution of the following p.d.e.

$$-\Delta u = 0 \text{ in } \Omega \backslash S,\ u = 1 \text{ on } \partial S,\ u = 0 \text{ on } \partial\Omega\ .$$

The set S is a fixed source and we always assume above that $S \subset\subset \Omega(t)$.

This geometric equation appears formally as a gradient flow for the energy

$$\mathcal{E}_\lambda(\Omega) = Vol(\Omega) + \lambda \text{cap}_S(\Omega)$$

where

$$\text{cap}_S(\Omega) = \inf \left\{ \int_{\Omega \backslash S} |\nabla u|^2\ ;\ u \in H^1(\Omega \backslash S),\ u = 0 \text{ on } \partial\Omega,\ u = 1 \text{ on } \partial S \right\}$$

The problem of minimizing $\mathcal{E}_\lambda$ is known as Bernoulli free boundary problem. It is not difficult to check that minimizers of $\mathcal{E}_\lambda$ are stationary solution of (1). It is thus natural to investigate large time behavior of solutions of (1) to find critical values of $\mathcal{E}_\lambda$.

The idea of using discrete gradient flows of the form (1) for minimizing functionals defined on sets has been successifully used by several authors for numerical reasons: see for instance [1]. Our aim is to understand better the underlying continuous flow. We have chosen for this (1) as a toy model.

The first remark about the velocity driving our geometric flow is that its preserves—at least formally—the inclusion. Indeed, if two bounded open subsets Ω_1 and Ω_2 of $\mathbb{R}^N$ satisfy $S \subset\subset \Omega_1 \subset \Omega_2$ and if $x \in \partial\Omega_1 \cap \partial\Omega_2$, then

$$-1 + \lambda \bar{h}(x, \Omega_1) \leq -1 + \lambda \bar{h}(x, \Omega_2)$$

It is known that such a property should entail an inclusion preserving property for the flow. Using this we define in [3] a geometric notion of "viscosity" solution for (1), i.e., a notion of (non smooth) solution where the equation is tested pointwisely by smooth evolving sets. We prove that these generalized solutions preserve the inclusion. This property naturally entails existence, generic uniqueness and stability of the flow.

The next step towards the interpretation of (1) as a gradient flow for the energy $\mathcal{E}_\lambda$ amounts to show that this energy is non increasing along the generalized flow.

This is what we investigate in [4]. This question is not obvious since the pointwise definition of viscosity solution seems very far from an energy estimate. To overcome this difficulty, we built minimizing movements for $\mathcal{E}_\lambda$ in the flavour of what is done in [2] for the perimeter, and we prove that such minimizing movements coincide with the solutions of (1).

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Variational Methods in Hamiltonian Systems

Diogo Aguiar Gomes

In recent years it has become increasingly clear that viscosity solutions of Hamilton-Jacobi equations provide an appropriate setting for the study of Hamiltonian dynamics and, conversely, the understanding of this dynamic gives rise to new estimates and properties of viscosity solutions. The objective of this report is to describe recent research and to present some contributions of the author to this area.

In this report, we assume that we are given a Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n$, $H = H(p, x)$, which is smooth, for each $p \in \mathbb{R}^n$, the mapping $x \mapsto H(p, x)$, is $\mathbb{Z}^n$-periodic, and there exist constants $\Gamma, \gamma > 0$ such that $\gamma|\xi|^2 \leq \sum_{i,j=1}^n \frac{\partial^2 H}{\partial p_i \partial p_j}\xi_i\xi_j \leq \Gamma|\xi|^2$, for each $p, x, \xi \in \mathbb{R}^n$. The associated Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^n$, $L = L(x, v)$, is the Legendre transform of H.

In the context of homogenization of Hamilton-Jacobi equations, Lions, Papanicolaou and Varadhan [LPV88], proved the following result:

Theorem 3 (Lions, Papanicolao, Varadhan)**.** *For each $P \in \mathbb{R}^n$ there exists a unique number $\overline{H}(P)$ and a function $u(x, P)$, $\mathbb{Z}^n$ periodic in x, which is a viscosity solution to*

$$H(P + D_x u, x) = \overline{H}(P). \tag{1}$$

The constant $\overline{H}(P)$ can be characterized through a variational formula

$$\overline{H}(P) = \inf_{\varphi \in C^1(\mathbb{T}^n)} \sup_x H(P + D_x\varphi, x), \tag{2}$$

originally proven in [CIPP98], which has interpretation in terms of a dual problem.

The theorem does not assert uniqueness of the viscosity solution u. However, as it was shown in [Gom03b], under certain hypothesis it is possible to prove uniqueness and continuity of the viscosity solution u with respect to parameters. These hypothesis can be formulated in terms of ergodic properties of certain measures, the Mather measures. These are measures supported on $\mathbb{T}^n \times \mathbb{R}^n$ which are invariant under the Hamiltonian dynamics and minimize the average action. The connection between classical mechanics, viscosity solutions and Mather measures is well known, and was explored by several authors. see, for instance, [Eva04] and the references therein.

1. Generalized Aubry-Mather problem

In stochastic optimal control it is important to understand the asymptotic behavior of controlled diffusions $\mathbf{x}(t)$ that satisfy certain stochastic differential equations such as $d\mathbf{x} = \mathbf{v}dt + \sigma dW_t$, in which W_t is a Brownian motion, and $\mathbf{v}$, the control, is a progressively measurable process, with respect to W_t, chosen to minimize a functional of the form $E\int_0^T L(\mathbf{x}, \mathbf{v})ds$. The stochastic Mather problem [Gom02] consists in minimizing $\int_{\mathbb{T}^n\times\mathbb{R}^n} Ld\mu$ over all probability measures in $\mathbb{T}^n \times \mathbb{R}^n$ which satisfy

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \left[vD_x\phi + \frac{\sigma^2}{2}\Delta\phi \right] d\mu = 0, \tag{3}$$

for all $\phi \in C^2(\mathbb{T}^n)$. In the next theorem we give a characterization of these minimizing measures:

Theorem 4. *There exists a unique probability measure μ^*, the stochastic Mather measure, which minimizes $\int_{\mathbb{T}^n} Ld\mu$ under the constraint (3). We have*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} [L(x, v) + P \cdot v]\, d\mu^* = -\overline{H}(P),$$

where $\overline{H}(P)$ is the unique number for which

$$-\frac{\sigma^2}{2}\Delta u + H(P + D_x u, x) = \overline{H}(P) \tag{4}$$

has a periodic viscosity solution.

The stochastic Mather measures can be used to study properties of viscosity solutions of the second-order Hamilton-Jacobi equation (4):

Theorem 5. *Suppose u solves (4) . Then for any $y \in \mathbb{R}^n$*

$$\int |D_x u(x + y) - D_x u(x)|^2 \, d\mu^* \le C|y|^2,$$

for some constant that does not depend on σ.

2. Stability of Viscosity Solutions

In order to understand the support of Mather measures, it is important to study the dependence on ϵ of solutions of

$$H_\epsilon(P + D_x u^\epsilon, x) = \overline{H}_\epsilon(P). \tag{5}$$

We describe briefly in this section some results in this direction from [Gom03a].

Theorem 6. *Let $M > 0$ and $P_0 \in \mathbb{R}^n$ such that $\omega_0 = D_P H_0(P_0)$ is Diophantine. It is possible to construct functions $\tilde{u}_N^\epsilon$, $\tilde{H}_\epsilon^N$ through finite expansions:*

$$\begin{aligned}\tilde{u}_N^\epsilon =& \epsilon v_1(x, P_0) + \epsilon(P - P_0)D_P v_1(x, P_0) + \epsilon^2 v_2(x, P_0) + \\ &+ \frac{1}{2}\epsilon(P - P_0)^2 D_{PP}^2 v_1(x, P_0) + \epsilon^2(P - P_0)_P v_2(x, P_0) + \\ &+ \epsilon^3 v_3(x, P_0) + \cdots,\end{aligned}$$

and

$$\begin{aligned}\tilde{H}_\epsilon^N(P) =& \overline{H}_0(P_0) + \epsilon\overline{H}_1(P_0) + (P - P_0)D_P\overline{H}_0(P_0) \\ &+ \epsilon^2\overline{H}_2(P_0) + \cdots\end{aligned}$$

such that

$$H_\epsilon(P + D_x\tilde{u}_N^\epsilon, x) = \tilde{H}_\epsilon^N(P) + O(\epsilon^N + |P - P_0|^N),$$

in a neighborhood of P_0 and $\epsilon = 0$, for some constant $N(M)$. Furthermore, for each ϵ sufficiently small, there exists P^ϵ such that $D_P\tilde{H}_\epsilon^N(P_\epsilon) = \omega_0$. Then, for any viscosity solution u^ϵ of $H_\epsilon(P_\epsilon + D_x u^\epsilon, x) = \overline{H}_\epsilon(P_\epsilon)$, we have

$$\operatorname*{esssup}_x |D_x u^\epsilon - D_x\tilde{u}_N^\epsilon|^2 \leq C\epsilon^M,$$

as $\epsilon \to 0$.

3. Numerical methods and converse KAM

The computation of the effective Hamiltonian, $\overline{H}$, is extremely importan both for the study of Hamiltonian systems and in the homogenization theory for Hamilton-Jacobi equations, [LS03], and references therein. In [GO04] we have used the the minimax formula (2) to compute numerically $\overline{H}$. In [GO05], in preparation, we have developed a series of necessary conditions for the existence of invariant tori which can be checked numerically and are based upon our computations of $\overline{H}$:

Theorem 7. *Let $(x, p) \in \mathbb{T}^n \times \mathbb{R}^n$ and $(\mathbf{x}, \mathbf{p})$ denote its trajectory through the Hamiltonian flow. Let*

$$S_P(t) = \int_0^t L(\mathbf{x}, \dot{\mathbf{x}}) + P \cdot \dot{\mathbf{x}} + \overline{H}(P)ds.$$

If (x, p) belongs to the Mather set, there exist constants such that

$$\inf_{P\in\mathbb{R}^n} \sup_{t\geq 0} \sup_{k\in\mathbb{Z}^n} \frac{|S_P(t)|}{|\mathbf{x}(t) - \mathbf{x}(0) + k|} \leq C.$$

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A unified variational approach to some problems in convexity theory and optics

Vladimir Oliker

In his book on convex polyhedra [1], section 7.2.4, A.D. Aleksandrov raised a general question of finding variational formulations and solutions to problems of existence and uniqueness of convex hypersurfaces (in particular, of convex polyhedra) in Euclidean space with prescribed geometric data. Particular cases of such data are integral Gauss-Kronecker curvature, surface area functional, areas of projections, etc. An example of such problem and its variational solution is the celebrated Minkowski problem.

It turns out that indeed there is a variational principle that can be applied in an almost canonical way to many such problems to prove existence and uniqueness. In particular, the Aleksandrov problem of existence of compact and complete noncompact convex hypersurfaces with prescribed integral Gauss-Kronecker curvature can be solved by applying this principle [5]. In addition, several classes of interesting and practically important problems of design of reflecting and refracting surface(s) transforming the radiation of a source into an output front with prescribed in advance irradiance properties can also be stated and often solved in the same framework [3], [4], [2]. The construction of the required functional is motivated by the Monge-Kantorovich optimal mass transport theory. The cost functions naturally arising in these problems vary and include in particular the quadratic as well as some of the less familiar logarithmic and concave costs.

In this talk I present such a variational approach to the Aleksandrov problem of existence of a closed convex hypersurface with prescribed integral Gauss curvature [6] and to the problem of designing a reflector with prescibed near-field scattering data[7].

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Problems of optimal resistance and Monge-Kantorovich mass transport

Alexander Plakhov

A body moves in a homogeneous medium consisting of point particles. The medium is very rare, so that mutual interaction of particles can be neglected. Interaction of particles with the body is absolutely elastic: each particle performs several (maybe zero) collisions and moves freely afterwards. It is required to find a body, from a given class of bodies, such that resistance of the medium to the body's motion is minimal.

This is general setting of the minimal resistance problem. In order to specify it, one needs to determine the class of bodies, the kind of motion (for example, translational motion or a combination of translational and rotational motion), the state of medium (the particles may stay at rest as well as perform thermal motion). The problem can be stated not only in $\mathbb{R}^3$, but also in spaces of other dimensions; besides, the problems of *maximal* resistance can be considered as well.

A particular case, where the class consists of convex axially symmetric bodies of fixed length and width and the body performs translational motion in a medium of motionless particles, is called Newton's minimal resistance (or aerodynamic) problem [1]. Since the beginning of 90th, there were obtained interesting results concerning the problem in various classes of non-symmetric and/or non-convex bodies, under the so-called *single impact assumption*, that is, each particle may collide with the body at most once [2]-[8]. It was also shown that typically, in classes of non-convex bodies where multiple collisions are admitted, infimum of resistance equals zero, that is, there exist almost perfectly streamlined bodies [9, 10]. This result is essentially three-dimensional: for the two-dimensional analogue of the problem, infimum is positive [10].

The problems involving rotational motion of bodies and/or thermal motion of particles seem to be more relevant to real life. Some of them are relatively easy, as applied to classes of *convex* bodies; see, e.g., [11] for the case of slowly uniformly rotating bodies of fixed volume and [12], for the case of of translational motion of convex axially symmetric bodies in media of positive temperature. Here we shall discuss these problems in classes of *non-convex* bodies. The discussion is based on the study of billiards in unbounded regions. We restrict our study to the two-dimensional case.

Our approach is as follows. Each body (compact connected subset of $\mathbb{R}^2$ with piecewise smooth boundary) is limited by a curve consisting of a convex part and a number of "cavities". Each particle, interacting with the body, either reflects only once from the convex part of curve, or, otherwise, gets into a cavity, performs there a series of reflections, and eventually gets out of the cavity and leaves the body forever. Define the angles of "getting in" and "getting out" by φ and φ^+; to any cavity one assigns a measure describing the joint distribution of φ and φ^+. Resistance of the body depends on the measures generated by the cavities, therefore characterization of these measures is a key question for a wide range of problems of minimal and maximal resistance.

This characterization is the main result of the present work; it is based on a detailed analysis of billiard dynamics in cavities; as a result, we determine the closure of the set of measures generated by cavities in weak topology. We then apply this result to the problems of minimal and maximal mean resistance for classes of bodies containing or being contained in a given convex bounded set K with nonempty interior. The bodies are subject to slow uniform motion. By mean resistance, we mean the time averaged value of resistance. The problems reduce to some special Monge-Kantorovich problems of mass transfer; solving them, one concludes that infimum and supremum of mean resistance are equal to 0.9878... and to 1.5, respectively, where resistance of K is taken to be 1.

Note that in [11], the value 0.9878... was proved to be a lower bound of mean resistance, but without having the characterization of measures result, it was impossible to prove that this value is the *upper* lower bound.

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On the formation of tree leaves

Qinglan Xia

Tree leaves have diverse and elaborate shapes and venation patterns. Why tree leaves grow in such an amazing way? What kinds of mathematics are behind the beauty of tree leaves? In this talk, I will describe a geometric variational model for it.

To understand the formation of a tree leaf, it is necessary to understand the main tasks of the tree leaf. A tree leaf will transport resources like water and solutes from its root to its tissues via xylem, absorb solar energy at its cells through photosynthesis, and then transport the chemical products (carbohydrates) synthesized in the leaf back to its root by phloem. Thus, a leaf tends to increase the surface area as large as possible to maximize metabolic capacity, because metabolism produces the energy and materials required to sustain and reproduce life. On the other hand, more importantly, the leaf tends to maximize internal efficiency by developing an efficient transporting system. The meaning of "efficient transport system" may vary as the type of the tree leaf varies. Here, we will demonstrate that tree leaves have different shapes and venation patterns mainly because they have adopted different efficient transport systems.

The efficient transport system of a tree leaf built here is a modified version of the optimal transport path, which was introduced by the author in [2][3][4] to study the phenomenon of ramifying structures in mass transportation. In the case of tree leaves, the cost functional on transport systems is controlled by two meaningful parameters. The first parameter describes the economy of scale which comes with transporting large quantities together, while the second parameter discourages the direction of outgoing veins at each node from differing much from the direction of the incoming vein. In general, the values of the parameters usually depend on the species of the tree leaf.

With a given cost function on transport systems, the formation of a leaf is mainly governed by a selection principle. As we know, the growing of a tree leaf is a dynamic process of generating new cells. It originates from a bud with a given initial growing direction. At every stage, the leaf will develop an optimal transport system to transport water between the root and the existing cells with respect to the given transport cost function. Also, as the environment changes, the leaf may generate some new cells (i.e. expanding the area) nearby its boundary. The selection of those new cells is not random. Under the same environmental conditions, each potential new cell outside the existing leaf produces about the same amount of revenue such as the absorbed solar energy. However, the expense corresponding to each potential new cell varies with respect to the position of the cell. Here, the expense is mainly the transport cost of water and nutrients between the cell and the root. A selection principle says that a new cell is generated only if the expense is less than the revenue it produces. This simple rule determines the selection of new cells during the generation process. When the environmental conditions change, the corresponding revenue that a cell can produce also changes. When the corresponding revenue of each cell increased to a certain degree, it becomes benefit to produce some new cells, and thus the leaf will grow. Due to limited resources, the revenue that a cell can possibly produce is bounded above. This fact forces the leaf to stop growing after some time. As a result, the final shape and venation pattern of a leaf are mainly determined by the cost function defined on the collection of all possible transport systems, as well as the actual environment.

Based on this model, we also provide some computer visualization of tree leaves, which resemble many known leaves including the maple and mulberry leaf. Under the same initial condition, efficient transport systems modeled by different parameters will provide tree leaves with different shapes and different venation patterns. It demonstrates that optimal transportation (i.e. internal efficiency) plays a key role in the formation of tree leaves.

In the end, we discuss a limiting process for our model by letting the size of the grid approach zero. The limiting leaf will correspond to a Radon measure with connected compact support, and its transport system becomes a vector measure whose divergence is the difference of the leaf (viewed as a measure) and the Dirac measure located at the root in the sense of distribution.

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A quantitative version of the isoperimetric inequality

Francesco Maggi & Aldo Pratelli
(joint work with Nicola Fusco)

The isoperimetric inequality states that, given a Borel set E of $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure $|E|$, its (distributional) perimeter $P(E)$ is greater or equal than the perimeter of a ball having the same volume as E. That is, if ω_n is the measure of the unit ball B of $\mathbb{R}^n$, we have

$$P(E) \geq n\omega_n^{1/n}|E|^{(n-1)/n}\,, \tag{1}$$

with equality if and only if $E = x + r_E B$ for some $x \in \mathbb{R}^n$ and $r_E := (|E|/\omega_n)^{1/n}$.

In a quantitative version of inequality (1) the isoperimetric deficit $D(E)$,

$$D(E) := \frac{P(E)}{n\omega_n^{1/n}|E|^{(n-1)/n}} - 1\,, \quad |E| > 0\,,$$

controls the distance of E from the set of balls $\{x + r_E B : x \in \mathbb{R}^n\}$. If we restrict our attention to the class of convex sets E it is natural to work with the Hausdorff distance, and the corresponding quantitative inequalities have been studied in depth, among others, by Bernstein [1], Bonnesen [2] (when $n = 2$) and Fuglede [4] (for $n \geq 2$). In the general case, instead, it is natural to adopt the Vitali distance $d(E, F) := |E\Delta F|$, defined as the Lebesgue measure of the symmetric difference between E and F, and introduce the notion of asymmetry of E as

$$A(E) := \inf\left\{\frac{d(E, x + r_E B)}{|E|} : x \in \mathbb{R}^n\right\}\,.$$

In this setting, a quantitative isoperimetric inequality was shown by Hall, Hayman and Weitsman [8] and Hall [7]. They prove that

$$A(E) \leq C(n)D(E)^{1/4}\,, \quad \text{i.e. } P(E) \geq n\omega_n^{1/n}|E|^{(n-1)/n}\left\{1 + \left(\frac{A(E)}{C(n)}\right)^4\right\}\,, \tag{2}$$

(here and in the following, $C(n)$ is a constant depending only on the dimension n and possibly changing its value from line to line). A stronger result, in terms of decay rate of A with respect to D, is in fact contained in Hall's paper [7], where it is shown that

$$A(E) \leq C(n)D(E)^{1/2}\,, \quad \text{whenever } E \text{ is axially symmetric.} \tag{3}$$

The decay rate here is sharp, as one can check considering the ellipses $E(r) := \{x \in \mathbb{R}^n : (rx_1)^2 + \sum_{i=2}^n x_i^2 = 1\}$ in the limit $r \to 1$. Hall conjectures the validity of (3) on arbitrary sets, i.e. that

$$A(E) \leq C(n)D(E)^{1/2}\,, \quad \text{for every Borel set } E. \tag{4}$$

In [5] we prove (4), in the way explained below.

Without loss of generality it is assumed that $|E| = \omega_n$. Furthermore, as $A(E) \leq 2$, up to taking $C(n) \geq 2/\sqrt{\delta(n)}$, one can assume that $D(E) \leq \delta(n)$ for some fixed $\delta(n)$. One can prove that $A(E) \to 0$ when $D(E) \to 0$, and this implies that E is somehow close to a ball, in a soft qualitative way, provided we choose $\delta(n)$ small enough. We try to replace E with a "more symmetric" set E', in such a way that the validity of (4) on E can be deduced from the validity of (4) on E'. This amounts in proving that

$$A(E) \leq C(n)A(E')\,, \qquad\qquad D(E') \leq C(n)D(E)\,. \tag{5}$$

If the set E' is obtained from E by a symmetrization procedure, one usually gets the second inequality for free (possibly with constant 1, if the given symmetrization decreases the perimeter and leaves the Lebesgue measure unchanged); however, on symmetrizing, we expect to lower the asymmetry too, so that the two inequalities are somehow in competition.

This kind of approach is adopted in [8]. They prove that, given E, a direction ν can be found so that E^*, the Schwarz symmetrization of E with respect to ν, satisfies

$$A(E) \leq C(n)A(E^*)^{1/2}\,. \tag{6}$$

Recall that E^* is the set which intersection E^*_t with $\{x \cdot \nu = t\}$ is a $(n-1)$-dimensional ball centered at $t\nu$ and $\mathcal{H}^{n-1}$-measure equal to $\mathcal{H}^{n-1}(E_t)$, where $E_t := E \cap \{x \cdot \nu = t\}$. The set E^* is axially symmetric and satisfies $P(E^*) \leq P(E)$ (thus $D(E^*) \leq D(E)$). The existence of ν such that (6) holds is clearly a non trivial fact, as one can easily produce a set E such that $A(E) > 0$ but $E^* = B$ with respect to a given ν.

By applying (3) to E^* one finds $A(E^*) \leq C(n)D(E^*)^{1/2} \leq C(n)D(E)^{1/2}$, so deriving (2) from (6). However, being the exponent $1/2$ in (6) optimal, Hall's conjecture cannot be proved this way.

The key notion of our approach is that of n-symmetric set. We say that a set E is n-symmetric if it is invariant by reflection with respect to the n coordinate hyperplanes. The crucial consequence of this definition is that the minimization problem defining $A(E)$ can be somehow trivialized. Indeed, if E is n-symmetric then a simple symmetry argument shows that

$$A(E) = \inf_{x \in \mathbb{R}^n} \frac{d(E, x + B)}{\omega_n} \leq \frac{d(E,B)}{\omega_n} \leq 3A(E)\,. \tag{7}$$

This property allows to prove (4) by induction on the class of n-symmetric sets. Indeed if E is n-symmetric and E^* is its Schwarz symmetrization with respect to, say, the x_1-axis, then

$$\omega_n A(E) \leq d(E,B) \leq d(E,E^*) + d(E^*,B)\,.$$

Since E is n-symmetric, E^* is n-symmetric too. Therefore by applying (7) and (3) to E^* we find

$$d(E^*,B) \leq 3\omega_n A(E^*) \leq C(n)D(E^*)^{1/2} \leq C(n)D(E)^{1/2}\,.$$

On the other hand, $E_t = E \cap \{x_1 = t\}$ is a $(n-1)$-symmetric set in $\{x_1 = t\}$, while E^*_t is an $(n-1)$-dimensional ball centered at the center of symmetry of E_t, and with the same $\mathcal{H}^{n-1}$-measure. Thus, again by (7),

$$d(E,E^*) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(E_t \Delta E_t^*)dt \le 3 \int_{\mathbb{R}} \mathcal{H}^{n-1}(E_t) A_{\mathbb{R}^{n-1}}(E_t)dt\,,$$

where $A_{\mathbb{R}^{n-1}}$ denotes the asymmetry in $\{x_1 = t\}$. If $D_{\mathbb{R}^{n-1}}$ is the corresponding notion of isoperimetric deficit, by induction one finds

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(E_t) A_{\mathbb{R}^{n-1}}(E_t)dt \le C(n) \int_{\mathbb{R}} \mathcal{H}^{n-1}(E_t) \sqrt{D_{\mathbb{R}^{n-1}}(E_t)}dt\,. \tag{8}$$

In turn this last quantity is controlled by $D(E)^{1/2}$. This can be heuristically justified by recalling that, if we define $v(t) = \mathcal{H}^{n-1}(E_t)$, $p(t) = \mathcal{H}^{n-2}(\partial E_t)$ and

$$q(t) = \mathcal{H}^{n-2}(\partial E_t^*) = (n-1)\omega_{n-1}^{1/(n-1)} v(t)^{(n-2)/(n-1)}\,,$$

then, by the Coarea Formula,

$$P(E) \ge \int_{\mathbb{R}} \sqrt{v'(t)^2 + p(t)^2}dt\,, \qquad P(E^*) = \int_{\mathbb{R}} \sqrt{v'(t)^2 + q(t)^2}dt\,.$$

As $P(E^*) \ge P(B)$ by the isoperimetric inequality, we have

$$\begin{aligned} P(B)D(E) &= P(E) - P(B) \ge P(E) - P(E^*) \\ &\ge \int_{\mathbb{R}} \sqrt{(v')^2 + q^2(1 + D_{\mathbb{R}^{n-1}}(E_t))^2} - \sqrt{(v')^2 + q^2}\, dt \\ &\gtrapprox \int_{\mathbb{R}} q(t)^2 D_{\mathbb{R}^{n-1}}(E_t)dt\,, \end{aligned} \tag{9}$$

so that, loosely speaking, one passes from the last term in (9) to the one in (8) by Hölder inequality. To make these arguments completely rigorous a crucial role is played by the aforementioned assumption $D(E) \le \delta(n)$, but this is too technical to be further discussed in here.

Summarizing, n-symmetric sets have some special properties that allow to deduce from (3) that

$$A(E) \le C(n)D(E)^{1/2} \quad \text{if } E \text{ is } n\text{-symmetric.} \tag{10}$$

In turn we can deduce (4) from (10) once we show that, given a set E, then a n-symmetric set E' can be found so that (5) holds true. We now pass to discuss this last step. We start by considering a simpler task, i.e. we just ask E' to be symmetric with respect to one hyperplane, say $\{x_1 = 0\}$. Up to translating E in the x_1-direction we achieve $|E \cap \{x_1 > 0\}| = |E \cap \{x_1 < 0\}|$. If we denote by E_1^+ the set obtained by reflecting $E \cap \{x_1 > 0\}$ w.r.t. $\{x_1 = 0\}$, and similarly define E_1^-, then $E_1^\pm$ are both symmetric with respect to $\{x_1 = 0\}$, have the same measure as E and satisfy $P(E_1^+) + P(E_1^-) \le 2P(E)$. Therefore $D(E_1^\pm) \le 2D(E)$, and the second inequality in (5) is certainly achieved. On the other hand it could be as well that $A(E) > 0$ but $A(E_1^\pm) = 0$, if for example

$$E = \big[B \cap \{x_1 > 0\}\big] \cup \big[(B + e_2) \cap \{x_1 < 0\}\big]\,. \tag{11}$$

Note that this set E exhibit the bad behavior with respect to symmetrization by reflection *only* in the x_1-direction. Luckily enough, this is a general fact, and one can prove that given two coordinate directions, say x_1 and x_2, and considered the four sets $E_1^{\pm}, E_2^{\pm}$, then there exists at least one set E' among them such that $A(E) \leq C(n)A(E')$. Being certainly $D(E') \leq 2D(E)$, we have found E' satisfying (5), and having an hyperplane of symmetry.

This procedure can be applied $(n-1)$-times so to find (up to a possible final rotation) a set E' symmetric with respect to the first $(n-1)$ coordinate hyperplanes and such that (5) holds. At this stage we are forced to symmetrize E' with respect to the x_n-direction, and clearly the above selection argument cannot be repeated further without possibly stepping into a loop. However, it comes out that one among $(E')_n^+$ and $(E')_n^-$ (defined in the obvious way after translating E' so that $|E' \cap \{x_n > 0\}| = |E' \cap \{x_n > 0\}|$) shall satisfy (5). This is basically due to the fact that being E' already symmetric with respect to $x_1, \ldots, x_{n-1}$, it is then impossible to meet in the x_n-direction the situation exemplified by (11).

Apart from being useful in proving inequality (4), these kind of arguments, and especially the notion of n-symmetry, can be effectively used in the study of quantitative versions of the Sobolev inequalities

$$S(n,p)\left(\int_{\mathbb{R}^n} |f|^{np/(n-p)}\right)^{(n-p)/np} \leq \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{1/p},$$

for $1 \leq p < n$. The cases $p = 1$ and $1 < p < n$ are of course quite different, and are considered, respectively, in [6] and [3].

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Γ-convergence and saddle points

Robert L. Jerrard
(joint work with Peter Sternberg)

We consider a family $\{E_X^\varepsilon\}_{\varepsilon\in(0,1]}$ of C^1 functionals on a Banach space X, such that E_X^ε Γ-converges as $\varepsilon \to 0$ to a limiting functional E_Y on a Banach space Y. We define a notion of saddle point for E_Y (which is typically only lower semicontinuous) and we prove a general result giving some conditions sufficient to guarantee that if E_Y has a saddle point y_* with corresponding critical value $c_* = E_Y(y_*)$, then E_X^ε has a critical point for every sufficiently small ε, and the associated critical values converge to c_* as $\varepsilon \to 0$. We then apply this general theorem to construct new solutions of Ginzburg-Landau and related equations.

All the new results discussed in this note are proved in [3].

We use the notation $Y_0 := \{y \in Y : E_Y(y) < \infty\}$.

We adopt the following definition of Γ-convergence: we say that E_X^ε Γ-converges to E_Y as $\varepsilon \to 0$ if there exists a continuous map $P_{YX} : X \to Y$ and, for every $\varepsilon \in (0.1]$, a map $Q_{XY}^\varepsilon : Y_0 \to X$ (not necessarily continuous) such that

lower bound: If $y_0 \in Y_0$ and $\{x_\varepsilon\} \subset X$ are such that $\|P_{YX}(x_\varepsilon) - y_0\|_Y \to 0$ as $\varepsilon \to 0$, then

$$\liminf E_X^\varepsilon(x_\varepsilon) \geq E_Y(y_0). \tag{1}$$

upper bound: For every $y \in Y_0$,

$$E_X^\varepsilon(Q_{XY}^\varepsilon(y)) \to E_Y(y) \text{ and } \|P_{YX}Q_{XY}^\varepsilon(y) - y\|_Y \to 0 \text{ as } \varepsilon \to 0. \tag{2}$$

(For fixed $y \in Y_0$, the sequence $\{Q_{XY}^\varepsilon(y)\}_{\varepsilon\in(0,1]} \subset X$ is what is sometimes called a recovery sequence for y.) We always require the following compactness condition:

$$\text{If } E_X^\varepsilon(x_\varepsilon) \leq K \text{ for } 0 < \varepsilon \leq 1 \text{ then } \{P_{YX}(x_\varepsilon)\}_{\varepsilon\in(0,1]} \text{ is precompact in } Y. \tag{3}$$

Next, we say that E_Y has a **saddle point** at $y_* \in Y_0$ if there exists a nonnegative integer ℓ, a number $\delta_0 > 0$, and a continuous map $P_{ZY} : Y \to \mathbb{R}^\ell$ such that

$$E_Y(y_*) < E_Y(y) \quad \text{for } y \in \{y \in Y : 0 < \|y - y_*\|_Y \leq \delta_0, P_{ZY}(y) = 0\} \tag{4}$$

and in addition, if we write $z_* := P_{ZY}(y_*)$, then there exists a neighborhood $Z \subset \mathbb{R}^\ell$ of z_* and a continuous map $Q_{YZ} : Z \to Y_0$ such that

$$\begin{cases} P_{ZY} \circ Q_{YZ}(z) = z \quad \text{for all } z \in Z, \\ \|Q_{YZ}(z) - y_*\|_Y \leq \delta_0 \text{ for all } z \in Z \\ Q_{YZ}(z_*) = y_*, \\ \text{and for every } r > 0, \sup_{\{z\in Z, |z-z_*|\geq r\}} E_Y(Q_{YZ}(z)) < E_Y(y_*). \end{cases} \tag{5}$$

Note that when $\ell = 0$, the above assumptions imply that E_Y has a local minimum at y_*. This easier situation is addressed by classical work of Kohn and Sternberg [4]. The following theorem, our main abstract result, can be seen as generalizing the Kohn-Sternberg theorem about local minimizers and Γ-limits.

Theorem 1. *Suppose that* X, Y *are Banach spaces and that* $\{E_X^\varepsilon\}_{\varepsilon\in(0,1]}$ *is a family of* C^1 *functionals* $X \to \mathbb{R}$ *that* Γ*-converge to a limiting functional* $E_Y : Y_0 \to \mathbb{R}$ *via maps* $P_{YX} : X \to Y$ *and* $Q^\varepsilon_{XY} : Y_0 \to X$*. Assume also that* (3) *holds and that* E_X^ε *satisfies the Palais-Smale condition for every* $\varepsilon \in (0, 1]$*.*

Let $y_* \in Y$ *be a saddle point in the above sense. Assume also (using notation from the definition of a saddle point) that*

$$Q^\varepsilon_{XZ} = Q^\varepsilon_{XY} \circ Q_{YZ} : Z \to X \text{ is continuous for all } \varepsilon, \tag{6}$$

$$P_{ZY} \text{ is uniformly continuous in } \{y \in Y : \|y - y_*\|_Y \le 2\delta_0\}, \tag{7}$$

$$\|P_{YX} \circ Q^\varepsilon_{XZ}(z) - Q_{YZ}(z)\|_Y \to 0 \text{ uniformly in } Z \text{ as } \varepsilon \to 0, \tag{8}$$

$$E_X^\varepsilon(Q^\varepsilon_{XZ}(z)) \to E_Y(Q_{YZ}(y)) \text{ uniformly in } Z \text{ as } \varepsilon \to 0. \tag{9}$$

Then there exists $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon \le \varepsilon_0$*, there exists a critical point* x^ε *of* E_X^ε *such that* $\lim_{\varepsilon\to 0} E_X^\varepsilon(x^\varepsilon) = E_Y(y_*)$*.*

On this level of generality, it need not be true that the critical points of E_X^ε converge in any sense to the limiting point y_* as $\varepsilon \to 0$. This can be shown by elementary examples in which $X = Y = \mathbb{R}^2$, $P_{YX}(x) = x$, and $E_X^\varepsilon \to E_Y$ in $C^{0,\alpha}$ for every $\alpha \in [0, 1)$, as $\varepsilon \to 0$.

To apply the above result, we next fix a bounded, open set $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 1$ with smooth boundary, and we consider the functional

$$E_Y(T) = \begin{cases} \mathbf{M}(T) & \text{if } T \in Y_0 \\ +\infty & \text{if not} \end{cases}$$

where[1] $Y_0 = \{$rectifiable, integer 1-currents $T : T = \partial S$ for some $S\}$ on the Banach space

$$Y := \{\partial S : S \text{ is a 2-current with finite mass in } \Omega\}$$

of 1-dimensional boundaries in Ω, endowed with norm

$$\|T\|_Y = \inf\{\mathbf{M}(S) \ : \ S \text{ is a 2-current}, \partial S = T\}.$$

The functional E_Y arises as the Γ-limit of sequences of suitably scaled Ginzburg-Landau functionals when $n = 2$; and of appropriate generalized Ginzburg-Landau functionals for arbitrary $n \ge 3$. The Γ-limit of the Allen-Cahn functional when $n = 1$ can also be expressed in terms of E_Y. Thus, the existence of suitable saddle points of E_Y can be used with Theorem 1 to produce solutions of the Euler-Lagrange equations of all of these functionals.

It is easy to find critical points of the arclength functional in topologies stronger than the Y-topology. The next theorem asserts that these are still critical points in the weaker sense required for the above result about Γ-limits.

[1] An element T of Y_0 can be identified with a sum of Lipschitz curves γ_i with no boundary in Ω (and satisfying certain topological conditions if $\partial\Omega$ is not connected), and $\mathbf{M}(T) = E_Y(T)$ is then the sum of the arclengths.

Theorem 2. *Assume that* (x_0, y_0) *is a critical point of the function* $d : \partial\Omega \times \partial\Omega \to \mathbb{R}$ *given by* $d(x, y) = |x - y|$, *and that the Hessian of* d *is nonsingular at* (x_0, y_0). *Assume also that the line segment joining* x_0 *to* y_0 *is contained in* Ω *and that* x_0 *and* y_0 *belong to the same component of* $\partial\Omega$. *Let* T_* *be the current corresponding to this segment (oriented from* x_0 *to* y_0 *say).*

Then T_* *is a saddle point of* $E_Y : Y \to \mathbb{R}$ *in the sense of Theorem 1.*

To prove the theorem we must build maps P_{ZY} and Q_{YZ} satisfying (4), (5). The hard part of the proof is the construction of P_{ZY} and verification of (4).

Finally, by combining the above and Γ-limit results of [1] for example, we prove the existence of solutions of certain nonlinear PDEs, such as

Theorem 3. *Suppose that* $n \geq 2$ *and let* T^* *be a saddle point of* E_Y, *as found in Theorem 2. Then there exists* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$ *there is a solution* $u_\varepsilon \in W^{1,n}(\Omega; \mathbb{R}^n)$ *of the equation*

$$-\nabla \cdot (|\nabla u_\varepsilon|^{n-2} \nabla u_\varepsilon) + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \qquad \text{in } \Omega \tag{10}$$

with natural boundary conditions $\nu \cdot \nabla u_\varepsilon = 0$ *on* $\partial\Omega$; *and such that*

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \mathbf{M}(T_*), \qquad E_\varepsilon(u) = \frac{1}{c(n)|\ln \varepsilon|} \int_\Omega \frac{|\nabla u|^n}{n} + \frac{(|u|^2 - 1)^2}{4\varepsilon^2}\, dx.$$

Essentially the same result can be proved when $n = 2$ for the Ginzburg-Landau system with magnetic field, and for the Allen-Cahn equation when $n = 1$. For the solutions found in Theorem 3, the energy presumably concentrates around the support of T_* as $\varepsilon \to 0$, but this cannot be proved from Theorem 1 alone.

For the Allen-Cahn equation, results along the lines of Theorem 3 (but considerably stronger) have been proved using techniques based on Liapuonv-Schmidt reduction, see [6], [5]. These techniques rely on good control over the spectrum of a suitable linearized operator and are harder to apply for the Ginzburg-Landau equation and various generalizations mentioned above. Other results related to Theorem 3 describe energy concentration around codimension 2 minimal surfaces for sequences of solutions of the Ginzburg-Landau equation ((10) with $n = 2$); see [2] for results valid in arbitrary dimensions.

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2–d Stability of the Néel wall

Hans Knüpfer
(joint work with Antonio DeSimone, Felix Otto)

The non–dimensionalized magnetization $m : \Omega \to S^2$ of a soft ferromagnet $\Omega \subset \mathbb{R}^3$ is characterized by the minimization of the Landau–Lifshitz energy functional [4]:

$$E_{3d}(m) = d^2 \int_\Omega |\nabla m|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx. \tag{1}$$

The first term, the exchange energy, favors alignment of neighboring spins; the second, the stray field energy, describes the energy of the magnetic stray field $\nabla u : \mathbb{R}^3 \to \mathbb{R}^3$. It is determined by

$$\Delta u = \nabla \cdot m,$$

where m is extended by $m = 0$ outside Ω. Therefore the stray field is generated by both divergence of m inside the magnetic sample ("volume charges") and a non–zero normal component of m on the surface of the magnetic sample ("surface charges"). The material constant d is called exchange length.

The competition of these two terms in the energy leads to the development of bulk regions with almost constant magnetization ("magnetic domains"), separated by narrow transition layers ("domain walls"). In the macroscopic view these walls are discontinuity lines of m, microscopically they have a rich inner structure.

We are interested in thin films: Two important wall types in thin films are the Néel wall and the cross–tie wall.

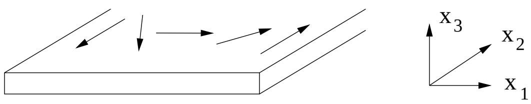

Figure 1. Néel wall

The Néel wall (see Fig. 1, Fig. 2, left hand side) is the dominating wall type in very thin films (thickness $t \ll d$) [6]. It is characterized by in–plane and 1–d rotation of the magnetization:

$$m_3 = 0 \qquad \text{and} \qquad m = m(x_1). \tag{2}$$

The cross–tie wall appears in somewhat thicker films (see Figure 2, right hand side). It is a complicated composite wall where a 180°–Néel wall is replaced by a pattern of Néel wall segments with transition angle of 90° or less [1]. There is a central Néel wall segment along the x_1–axis, where perpendicular Néel wall segments branch off periodically (lines in Figure 2), the so called cross–ties. Topologically this structure enforces the existence of point singularities of the magnetization in the macroscopic view, the so called Bloch lines. There are circular Bloch lines (white circle) and cross Bloch lines (black circles).

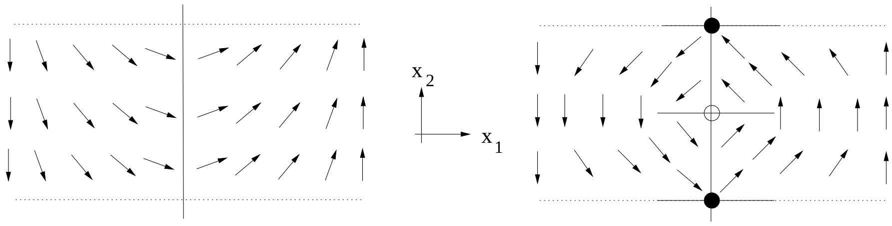

FIGURE 2. Néel wall and cross–tie wall

The reason for this microstructure formation is well understood: The angle dependency for the Néel wall energy is very pronounced. For small transition angles $\theta \ll 1$ one can compute

$$E_{\text{Néel}}(\theta) \sim \theta^4.$$

This explains why the energy of the cross–tie wall may be smaller than for a single 180°–Néel wall, although the total wall length in Figure 2 is larger. The conjecture why cross–tie walls are not observed in thin films is the following: The Bloch lines, which are part of the cross–tie wall structure, require relatively an increasingly high energy in the limit of thin films. We prove this conjecture by showing that the Néel wall is energetically optimal, if Bloch lines are suppressed. This amounts to prove stability of the Néel wall while keeping the first condition in (2) and removing the second one.

We confine our setting: Due to the exchange part of (1), it seems to be safe to assume that the magnetization be constant in the thickness direction x_3, i.e. $m = m(x_1, x_2)$. We furthermore enforce a macroscopic 180°–transition in x_1–direction by Dirichlet boundary conditions. In x_2–direction we assume ℓ–periodicity of m in order to assign an energy density. This leads to the following 2–d setting for $m : \mathbb{R}^2 \to \mathbb{R}^2$:

$$|m|^2 = 1, \tag{3}$$

$$m = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \qquad \text{for } \pm x_1 \geq 1, \tag{4}$$

$$m(x_1, x_2 + \ell) = m(x_1, x_2) \qquad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \tag{5}$$

We do not use the full setting as in (1), but rather a thin film approximation of the energy by assuming that all magnetic charges lie on the $(x_3 = 0)$–plane:

$$\Delta u_{\text{approx}} = \nabla \cdot m \ \mathcal{H}^2 \lfloor_{\{x_3=0\}}.$$

This leads to the following thin film energy (see [2], [3])

$$\ell \ E_\varepsilon(m) = \varepsilon \int_{\mathbb{R}\times[0,\ell)} |\nabla m|^2 \ dx + \frac{1}{2} \int_{\mathbb{R}\times[0,\ell)} |\nabla^{-1/2} \nabla \cdot m|^2 \ dx,$$

where $\varepsilon = d^2/t$. We will use the notation: $A \approx B$ for $\varepsilon \ll 1 \Leftrightarrow \lim_{\varepsilon\to 0} A/B = 1$.

Our main result [2] is:

Theorem 1. *For $\varepsilon \ll 1$, any ℓ and any smooth m we have*

$$\min_{m \text{ satisfies } (3),(4),(5)} E_\varepsilon(m) \approx \min_{m \text{ satisfies } (3),(4),(5), m=m(x_1)} E_\varepsilon(m) \approx \frac{\pi}{2\ln\frac{1}{\varepsilon}}.$$

It shows that asymptotically, the minimal energy is assumed by a 1–d transition layer. More precisely, variations of the optimal 1–d transition layer in x_2–direction cannot decrease the leading order coefficient of the energy.

The main part consists of proving the lower bound, i.e.

Proposition 1. *For $\varepsilon \ll 1$, any ℓ and smooth m with (3), (4) & (5) we have*

$$E_\varepsilon(m) \gtrapprox \frac{\pi}{2\ln\frac{1}{\varepsilon}}.$$

The proof of Proposition 1 relies on a dynamical systems argument which is based on the flow of the rotated magnetization $m^\perp$. Note that on both boundaries, $x_1 = \pm 1$, the rotated flow is directed inward. By the Poincaré–Bendixon theorem [7, Theorem 4.5] this implies the existence of a closed periodic orbit onto which m is normal. We identify this curve as a (not necessarily unique) "center line" of the wall and define an appropriate test function ϕ, which resembles the characteristic function of the left part of the wall. Then by the divergence theorem, ϕ can be controlled by the energy. A duality argument yields a lower bound for the energy.

In the way of the proof we need to derive the exact failure of a critically failing Sobolev inequality, which corresponds to $BV(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \not\subseteq H^{\frac{1}{2}}(\mathbb{R}^2)$. With exact constant there holds:

Lemma 3. *For $\varepsilon \ll w$ and any ϕ which is ℓ–periodic in x_2 we have*

$$\int_{\mathbb{R}\times\frac{2\pi}{\ell}\mathbb{Z}} \min\,\{\frac{1}{\varepsilon}, |k|, w|k|^2\}\ |\mathcal{F}\phi|^2\ dk \lessapprox \frac{2}{\pi}\ (\ln\frac{w}{\varepsilon}) \sup_{\mathbb{R}\times[0,\ell)} |\phi| \int_{\mathbb{R}\times[0,\ell)} |\nabla\phi|\ dx.$$

Finally, we would like to point out a result about compactness of energy minimizing sequences for the Néel wall [5], which is based on a localized version of the above dynamical systems argument.

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A mini review of Lieb-Thirring inequalities

Michael Loss

Consider the operator

$$-\Delta - V(x) \tag{1}$$

on $\mathbb{R}^n$, $n \geq 1$, where V is a nonnegative function vanishing at infinity. Denote by

$$-E_1(V) < -E_2(V) \leq -E_3(V) \cdots \tag{2}$$

its negative eigenvalues. A Lieb-Thirring inequality is an estimate of the type

$$\sum E_j(V)^\gamma \leq L(n,\gamma) \int V(x)^{\gamma+n/2} dx \ . \tag{3}$$

Here, γ is a non-negative number and in the case where $\gamma = 0$ the left side denotes the number of non-positive eigenvalues. The constant $L(n,\gamma)$ denotes the sharp constant in the inequality.

In [9] , with $n = 3$ and $\gamma = 1$, this inequality was proved for the first time, yielding a spectacularly simple proof of Stability of Matter. The cases $n = 1, \gamma > 1/2$ and $n \geq 2, \gamma > 0$ were settled in [10] . In all cases explicit upper bounds on the sharp constants were given. The case $n = 1, \gamma = 1/2$ was proved in [12] . The most difficult cases are $n \geq 3, \gamma = 0$ and these were settled in [3, 8, 11]. In all other cases the inequality cannot hold for a finite constant $L(n,\gamma)$.

The semi-classical constant $L_{\mathrm{sc}}(n,\gamma)$ is defined by the relation

$$(2\pi)^{-n} \int [p^2 - V(x)]_-^\gamma \, dpdx = L_{\mathrm{sc}}(n,\gamma) \int V(x)^{\gamma+n/2} dx \tag{4}$$

where $[f(x)]_- = -\min(0, f(x))$ is the negative part of the function $f(x)$. By Weyl's law it is known that

$$\lim_{\mu\to\infty} \mu^{-\gamma-n/2} \sum E_j(\mu V)^\gamma = L_{\mathrm{sc}}(n,\gamma) \int V(x)^{\gamma+n/2} dx \ , \tag{5}$$

and hence $L(n,\gamma) \geq L_{\mathrm{sc}}(n,\gamma)$.

It was shown in [1] that

$$L(n,\gamma) = L_{\mathrm{sc}}(n,\gamma) \Rightarrow L(n,\gamma') = L_{\mathrm{sc}}(n,\gamma') \text{ all } \gamma' \geq \gamma \ . \tag{6}$$

Further, in [10] it was proved that $L(1,3/2) = L_{\mathrm{sc}}(1,3/2)$ and hence $L(1,\gamma) = L_{\mathrm{sc}}(1,\gamma)$ for all $\gamma \geq 3/2$.

This result was extended to arbitrary dimensions by Laptev and Weidl ([7]). Consider the Schrödinger operators with a matrix valued potential,

$$-\frac{d^2}{dx^2} \otimes I_M - V(x) \ . \tag{7}$$

Here $x \to V(x)$ takes values in the positive semidefinite hermitean $M \times M$ matrices. Again, denote the negative eigenvalues by $-E_1(V)\ le - E_2(V) \leq \cdots$.

It was shown in [7] that

$$\sum E_j(V)^\gamma \leq L_{\mathrm{sc}}(1,\gamma) \int \mathrm{Tr}[V(x)^{\gamma+1/2}] dx \ , \tag{8}$$

for all $\gamma \geq 3/2$. It follows from trace identities in the case $\gamma = 3/2$, together with the fact that (6) carries over to matrix valued potentials as well (see [7]). For a direct proof of (8) in the case $\gamma = 3/2$ using commutation metods, see [2].

Writing the Schrödinger operator in $\mathbb{R}^n$ as

$$-\frac{\partial^2}{\partial x_1^2} - \Delta' - V(x_1, x') \geq -\frac{\partial^2}{\partial x_1^2} - [-\Delta' - V(x_1, x')]_- \tag{9}$$

Laptev and Weidl 'peal off' the variables using the result for matrix valued potentials in an inductive fashion to arrive at their main result

$$\sum E_j(V)^\gamma \leq L_{\rm sc}(n,\gamma) \int V(x)^{\gamma+n/2} dx \ , \tag{10}$$

for all $\gamma \geq 3/2$.

The only other instance where the sharp constant is known is the case $n = 1, \gamma = 1/2$. In the scalar case it was shown in [5] that $L(1, 1/2) = 1/2$ which is uniquely optimized by the δ-function potential. Note that the semiclassical constant is $1/4$ in this case. In [6] this result is extended to matrix valued potentials.

Otherwise, nothing is known about the sharp constants. In the scalar case, $n = 1$ and $1/2 \leq \gamma \leq 3/2$ Lieb and Thirring [10] conjectured that the potential that optimizes the Lieb-Thirring inequality should carry only one bound state, i.e., only one negative egenvalue. Assuming this, one can easily work out the optimizing potential to be $(\gamma^2 - 1/4)\cosh(x)^{-2}$. For details, see [10] .

Further, it is conjectured in [10] that $L(n, 1) = L_{\rm sc}(n, 1)$, $n \geq 3$. Besides being an interesting result in itself, it would imply that Thomas-Fermi theory, a statistical theory of quantum mechanics, is an exact lower bound to the Schrödinger functional (see [9]). All these conjectures are open. For improved constants, however, see [6].

One might suspect that the semiclassical constant is sharp for all γ provided that the dimension n is large enough. This is proved to be incorrect in [4] where it is shown that $L(n,\gamma) > Ł_{\rm sc}(n,\gamma)$ for all $\gamma < 1$.

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On a modified conjecture of De Giorgi

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(joint work with Matthias Röger)

We prove a conjecture of De Giorgi from 1991 in a modified form in three dimensions that the sum of the area and the Willmore functional is the $\Gamma-$limit of a diffuse Landau-Ginzburg approximation.

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Reinforcement Problems For Elastic Structures

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Reinforcing an elastic structure under a given load is a problem which arises in several applications. The literature on the topic is very wide, from the mathematical point of view as well as from the one of engineering. Here we limit ourselves to consider the simple case of an elastic membrane occupying a domain Ω and subjected to a given exterior load $f \in L^2(\Omega)$. The shape u of the membrane in the equilibrium configuration is then characterized as the solution of the partial differential equation

$$-\Delta u = f \quad \text{in } \Omega$$

together with the corresponding boundary conditions of Neumann or Dirichlet type on $\partial\Omega$ (or on a part of it).

The reinforcement of the membrane is usually performed at its boundary, by the addition of suitable stiffeners, whose total amount is prescribed. We are interested in finding the reinforcement of the membrane, in terms of the distribution of stiffeners, which minimizas the total compliance functional.

Mathematically, the reinforcement is described by a nonnegative coefficient $a(x)$ which acts on the Neumann part Γ of the boundary $\partial\Omega$ and modifies the boundary conditions into the new ones:

$$\frac{\partial u}{\partial \nu} + a(x)u = 0 \quad \text{on } \Gamma.$$

The rigorous deduction of the boundary condition above from an elliptic problem with a thin boundary layer where a high ellipticity coefficient is imposed was obtained by Brezis, Caffarelli and Friedman (1980) and by Acerbi and Buttazzo (1986).

The problem of finding an optimal reinforcement for the membrane then consists in the determination of a coefficient $a(x)$ which minimizes the so called elastic compliance, in order to provide a membrane as stiff as possible. More precisely, for every coefficient $a \in L^1(\partial\Omega)$ we consider the corresponding energy

$$\mathcal{E}(a) = \inf\Big\{ \int_\Omega |Du|^2\,dx + \int_{\partial\Omega} a(x)u^2\,d\mathcal{H}^{N-1} - 2\int_\Omega f(x)u\,dx \ :\ u \in H^1(\Omega)\Big\}$$

and the compliance $\mathcal{C}(a) = -\mathcal{E}(a)$. Then the optimal reinforcement problem can be written as

$$\min\Big\{\mathcal{C}(a)\ :\ a \in L^1(\partial\Omega),\ \int_{\partial\Omega} a\,d\mathcal{H}^{N-1} \le m\Big\}.$$

We show that the optimization problem above admits a solution a_{opt} and we study some properties of this optimal solution. Some numerical examples are also presented.

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