Mathematisches Forschungsinstitut Oberwolfach

Report No. 35/2006

Four-dimensional Manifolds

Organised by
Denis Auroux (MIT Cambridge)
Ronald Stern (Irvine)
Peter Teichner (Berkeley)

August 6th – August 12th, 2006

Abstract. This workshop has provided a broad panorama of recent developments in the study of topological, smooth, and symplectic 4-manifolds. Related invariants of knots and 3-manifolds were also discussed.

Mathematics Subject Classification (2000): 14Jxx, 14Nxx, 53Cxx, 53Dxx, 57Mxx, 57Rxx, 14J26, 14J29, 14N35, 14P99, 32Q60, 53C15, 53D40, 57M25, 57M27, 57M50, 57R15, 57R17, 57R55, 57R57, 57R58.

Introduction by the Organisers

There are many active areas in the theory of topological, smooth, and symplectic 4-manifolds, and the flurry of recent activity has led to many spectacular advances. Each flavor of 4-manifold topology has its own distinctive features, as first evidenced in the early 1980s by Freedman and Donaldson's results which exposed a huge gap between the classification of topological 4-manifolds (up to homeomorphism) and that of smooth 4-manifolds (up to diffeomorphism). However, the various strands of 4-manifold topology are in constant interaction with each other, and they draw on each other for inspiration and methods, as evidenced by many of the talks that were given in the workshop.

The workshop aimed to provide an overview of recent developments in the field, and to bring together experts in various flavors of 4-manifolds. In order to stimulate communication between the different groups of researchers, the talks were held in an informal atmosphere, conducive to questions and discussions, and their number was limited to four per day.

The following list attempts to capture the main themes developed in the talks of the workshop, arranging them into a somewhat arbitrary classification.

Topological 4-manifolds, surgery, and concordance invariants. The main reason why the topology of manifolds is intrinsically more complicated in dimension 4 than in higher dimensions is the failure of the Whitney trick. However, Freedman's spectacular results from the early 1980s have shown that, in the category of topological 4-manifolds, subtle considerations allow one to overcome this obstacle and obtain classification results (e.g., in the simply connected case). The applicability of Freedman's program for more general fundamental groups is an ongoing theme of investigation; see the talk by Slava Krushkal. In a somewhat related direction, Rob Schneiderman's talk describes new concordance invariants for links, defined by considering the Whitney towers that they bound. Shelly Harvey's talk also focuses on concordance invariants and on new techniques for showing that certain classes of knots are not topologically slice (i.e., they do not bound a topological disk in the 4-ball).

Knot homologies and smooth concordance invariants. A lot of activity has recently been devoted to the construction and study of knot invariants that "categorify" classical invariants, such as Khovanov and Khovanov-Rozansky homologies on one hand, and the knot Floer homology of Ozsváth-Szabó and Rasmussen on the other hand. Among other remarkable properties (e.g., recent results show that knot Floer homology detects the genus of a knot and whether it is fibered), these homology theories provide new concordance invariants and can be used to estimate the smooth slice genus of a knot by purely combinatorial means. These invariants and the comparison between them are the focus of the talks given by Jake Rasmussen and Matt Hedden. In related directions, Kevin Walker's talk explores the possibility of defining a 4+1-dimensional TQFT out of Khovanov homology, while the talk by Tom Mark and Slaven Jabuka discusses torsion phenomena in Heegaard-Floer homology.

Exotic smooth 4-manifolds and rational blowdowns. Another area of research that is currently very active deals with the construction of new examples of exotic smooth 4-manifolds. One approach, pioneered by Fintushel and Stern, relies on Seiberg-Witten theory to distinguish manifolds obtained by surgery along tori; see Ron Fintushel's talk. Another approach is the rational blowdown construction, in which the tubular neighborhoods of certain configurations of spheres are replaced by rational homology 4-balls. This can be used to construct interesting examples of exotic smooth structures on blowups of $\mathbb{CP}^2$, as discussed in the talks of András Stipsicz and Jongil Park. A closely related question, discussed in Paolo Lisca's talk, is to determine which lens spaces bound rational homology balls. Finally, Dieter Kotschick's talk deals with exotic smooth structures on parallelizable 4-manifolds.

Symplectic 4-manifolds. Symplectic topology has undergone a rapid development in the past few years. For example, one important question is to determine which smooth 4-manifolds carry a symplectic structure. Stefan Friedl and Stefano Vidussi's talk explores the question of which 4-manifolds of the form $S^1 \times N^3$ are symplectic. In a different direction, Michael Usher's talk discusses the minimality of symplectic sums of two symplectic 4-manifolds. However, the most important aspect of modern symplectic topology is probably the theory of pseudoholomorphic curves, which is at the heart of e.g. enumerative geometry problems and Gromov-Witten invariants. In the presence of an antiholomorphic involution, one can also try to enumerate real curves by defining suitable invariants; this is the topic of Jean-Yves Welschinger's talk. Another important direction in modern symplectic topology is to understand the topology of contact 3-manifolds and their fillability; in her talk, Gordana Matić explores the relation between the monodromy of an open book decomposition and the fillability of the contact structure.

Near-symplectic structures and Lefschetz fibrations. Recently, a lot of attention has been devoted to "near-symplectic" structures on 4-manifolds, i.e. symplectic forms that degenerate on a union of circles. Such forms exist on every oriented 4-manifold with $b_+ \neq 0$, and suggest a method for extending symplectic methods to more general smooth 4-manifolds, as evidenced by Taubes' work that aims to extend to the near-symplectic setting his celebrated results relating Seiberg-Witten invariants to enumerative geometry. Two of the talks in this workshop focus on analogues of Lefschetz fibrations in the near-symplectic setting: David Gay's talk looks into a purely topological construction of such "broken fibrations", while Tim Perutz discusses an invariant (conjecturally equivalent to Seiberg-Witten) that counts pseudoholomorphic curves in such a setting.

Workshop: Four-dimensional Manifolds

Table of Contents

Abstracts

Exotic structures on rational surfaces and the rational blow–down process

András I. Stipsicz

ABSTRACT: After reviewing the current state of art of existence of exotic smooth structures on rational surfaces, we give a necessary combinatorial condition for a negative definite plumbing tree suitable for rational blow–down and show new examples for which the geometric construction works.

Let X, Y denote two closed, simply connected smooth 4–manifolds. We say that Y is *an exotic structure* on X if X and Y are homeomorphic but not diffeomorphic. Many 4–manifolds are known to admit infinitely many distinct (i.e., pairwise nondiffeomorphic) exotic smooth structures; in fact there is no known example of a smooth 4–manifold for which we could prove that it admits at most finitely many exotic structures. One of the most interesting questions in 4–dimensional topology is to determine whether the 4–sphere S^4 admits exotic smooth structures ('the smooth 4–dimensional Poincaré conjecture') and the same question for the complex projective plane $\mathbb{CP}^2$. By defining X_k as $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ we have

Theorem 1. *The rational surface X_k admits infinitely many exotic smooth structures for all $k \geq 5$.* □

This result is the combination of results of Donaldson, Friedman and Morgan, Okonek and Van de Ven, Kotschick, Fintushel and Stern, and more recently Park, Szabó and Stipsicz [2, P, 6, 7]. Recall that by celebrated results of Freedman and Donaldson it is known that the simply connected, closed 4–manifold Y is homeomorphic to X_k (for some k) if and only if $\chi(Y) = \chi(X_k)$ and $\sigma(Y) = \sigma(X_k)$, and the (small perturbation) Seiberg–Witten invariant $SW_Y : H^2(Y; \mathbb{Z}) \to \mathbb{Z}$ is a diffeomorphism invariant for smooth 4–manifolds homeomorhic to X_k with $k \leq 9$, moreover $SW_{X_k} \equiv 0$.

In the recent proofs of Theorem 1 for low values of k these facts were exploited for 4–manifolds constructed by the application of the rational blow–down process, which was first introduced by Fintushel and Stern in [1] and extended by J. Park in [4].

In a joint project with Z. Szabó and J. Wahl [8] we initiated a systematic study of plumbing trees which give rise to smooth 4–manifolds which can be blown down in a similar fashion. We have showed the following

Theorem 2. *The graphs given by Figures 1 – 3 can be rationally blown down. In addition, if the minimal plumbing tree Γ gives rise to a 4–manifold which can be rationally blown down in such a way that the canonical spin^c structure from Γ extends to the rational disk and gives a Seiberg–Witten moduli space with equal* *dimension, then* Γ *is either one of the graphs of Figures 1 – 3 (or certain degenerations of them for* $p = 0$ *and/or* $r = 0$*) or is constructed from one of the graphs of Figure 4 by blowing up either* (-1)*–vertices or edges emanating from* (-1)*–vertices and finally replacing the* (-1) *framing with* (-4) *in Case (a),* (-3) *in Case (b) and* (-2) *in Case (c).*

Examples of rational blow–downs of graphs from the family given by Figure 1 is given in [3].

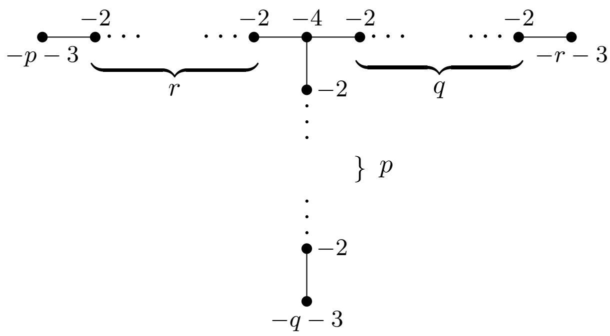


FIGURE 1. The graph $\Gamma_{p,q,r}$ for $p, q, r \geq 0$

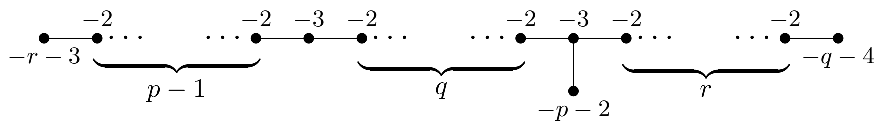


FIGURE 2. The graph $\Delta_{p,q,r}$ for $p \geq 1$ and $q, r \geq 0$

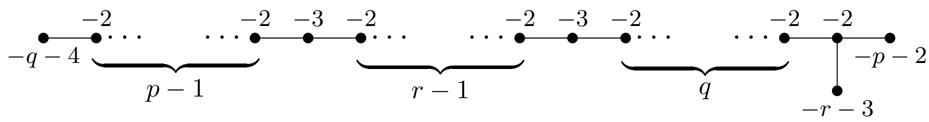


FIGURE 3. The graph $\Lambda_{p,q,r}$ for $p, r \geq 1$ and $q \geq 0$

REFERENCES

[1] R. Fintushel and R. Stern, *Rational blowdowns of smooth 4–manifolds*, J. Diff. Geom. **46** (1997), 181–235.

[2] R. Fintushel and R. Stern, *Double node neighborhoods and families of simply connected 4–manifolds with* $b_2^+ = 1$, J. Amer. Math. Soc. **19** (2006), 171–180.

[3] M. Michalogiorgaki, *Rational blow–down along Wahl type plumbing trees of spheres and exotic smooth structures on* $\mathbb{CP}^2\$k\overline{\mathbb{CP}^2}$, $k \in \{6, 7, 8, 9\}$, math.GT/0607608.

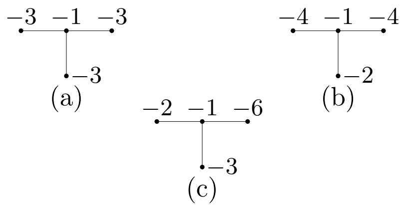


FIGURE 4. Plumbing trees giving rise to appropriate graphs of Theorem 1

[4] J. Park, *Seiberg–Witten invariants of generalized rational blow–downs*, Bull. Austral. Math. Soc. **56** (1997), 363–384.

[5] J. Park, *Simply connected symplectic 4–manifolds with* $b_2^+ = 1$ *and* $c_1^2 = 2$, Invent. Math. **159** (2005), 657–667.

[6] J. Park, A. Stipsicz and Z. Szabó, *Exotic smooth structures on* $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$, Math. Res. Lett. **12** (2005), 701–712.

[7] A. Stipsicz and Z. Szabó, *An exotic smooth structure on* $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$, Geom. Topol. **9** (2005), 813–832.

[8] A. Stipsicz, Z. Szabó and J. Wahl, *In preparation* (2006)

Lagrangian matching invariants and isotropic blow-up

TIM PERUTZ

Lagrangian matching invariants [3]

$$\mathcal{L}_{X,f}\colon \operatorname{Spin}^c(X)^{admiss} \to \mathbb{Z}[U] \otimes \Lambda^* H^1(X)$$

are invariants of a smooth, closed, oriented 4–manifold X together with a 'broken fibration' $f\colon X \to S^2$. Here $\operatorname{Spin}^c(X)^{admiss}$ denotes the 'admissible' Spin^c-structures; admissibility is a techical condition depending on f which one could hope to make redundant. The ring $\mathbb{Z}[U]\otimes_{\mathbb{Z}}\Lambda^* H^1(X)$ becomes graded upon declaring U to have degree 2 (and $H^1(X)$ degree 1), wherepon $\mathcal{L}_{X,f}(\mathfrak{s})$ becomes homogeneous of degree $[c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)]/4$.

The construction of $\mathcal{L}_{X,f}$ generalises Donaldson–Smith's 'standard surface count' for Lefschetz fibrations [2].

When X is homology-oriented, and $b_2^+(X) > 1$, the Seiberg-Witten invariants of X can also be formulated as a map

$$SW_X\colon \operatorname{Spin}^c(X) \to \mathbb{Z}[U] \otimes \Lambda^* H^1(X).$$

I conjecture that (for one homology orientation) equality holds whenever $\mathfrak{s}$ is admissible, hence that $\mathcal{L}_{X,f}$ is independent of f.

Definition 1. *A* **broken fibration** *is a proper map $f\colon X \to S$ from a 4-manifold to a 2-manifold, injective on its critical set* $\mathrm{crit}(f)$*, such that* $\mathrm{crit}(f)$ *is the union of isolated points and circles. Near an isolated critical point, the map must be equivalent to $\subset^2 \to \subset$, $(z_1, z_2) \mapsto z_1^2 + z_2^2$; near a critical point on a circle, to*

$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \quad (x, t) \mapsto (x_1^2 + x_2^2 - x_3^2, \pm t).$$

It is also required that there exists $w \in H^2(X)$ which evaluates positively on every component of every fibre of X.

It follows from the work of Auroux–Donaldson–Katzarkov [1] that, for any X with $b^+(X) > 0$, sufficiently high blow-ups $X \# k\overline{\mathbb{CP}}^2$ admit broken fibrations.

Example 1. *$f\colon X \to S^2$ is built as follows. Write $S^2 = D_+ \cup D_0 \cup D_-$, where $D_\pm$ are closed discs (the 'polar' regions) and D_0 a closed annulus (the 'equatorial' region), and adjacent regions meet along their common boundary circles. Let $X_+ := f^{-1}(D_+)$ be a product $\Sigma \times D_+$, and f a trivial Σ-bundle. Here Σ is a surface of genus $g > 0$. Let $X_- := f^{-1}(D_-)$ be $\bar{\Sigma} \times D_-$, where now $\bar{\Sigma}$ has genus $g - 1$. Let $X_0 := f^{-1}(D_0)$ be $S^1 \times W^3$, where W is the elementary cobordism from Σ to $\bar{\Sigma}$. The map $f\colon S^1 \times W \to S^1 \times I = D_0$ is $m \times id_{S^1}$, where $m\colon W \to I$ is a Morse function with only one critical point.*

The three pieces X_+, X_0 and X_- are glued together in the obvious way. One can show that their union X is then diffeomorphic to $(S^1 \times S^3) \# (\bar{\Sigma} \times S^2)$.

Example 2. *$\widehat{f}\colon \widehat{X} \to S^2$ is built as follows. As before, there is a circle δ of critical values in D_0. However, there is also an isolated critical value c in D_+. Fix an arc $\gamma \subset S^2$ connecting c to a point on δ. Over the complement of a neighbourhood of γ, $\widehat{f}$ is just f. There is a 2-sphere $E \subset \widehat{X}$ lying over γ, such that $\widehat{f}|_E\colon E \to \gamma$ is a Morse function attaining its extrema at c and a point on the circle of critical points. One checks that $[E]^2 = -1$. In fact, $\widehat{X} \cong X \# \overline{\mathbb{CP}}^2$.*

The passage from **1** to **2** is an instance of a general procedure for blowing up a broken fibration noticed by Auroux (see [1]). It is called 'isotropic blow-up', because E can be made isotropic with respect to a near-symplectic form compatible with the broken fibration.

In my talk, I claimed that the Lagrangian matching invariants are compatible with the isotropic blow-up procedure, in the sense that there is a blow-up formula as in Seiberg-Witten theory. Unfortunately, at present this formula is incomplete: it applies to those Spin^c-structures on the blow-up whose Chern class evaluates as ± 1 on the exceptional sphere.

The proof involves an explicit analysis of moduli spaces underlying Lagrangian matching invariants for the particular examples **1** and **2**. In these examples the moduli spaces (for a specific sequence of Spin^c-structures relevant to the proof) are in fact diffeomorphic to the corresponding moduli spaces of (perturbed) Seiberg-Witten monopoles, though they are defined in a quite different way.

Idea of the construction. I cannot here describe the invariants in any detail, but I will briefly indicate the idea. Suppose there is just one circle of critical points, as in (i) and (ii), and let $D_\pm$ be as in those examples. Replace the Lefschetz fibrations $X_\pm \to D_\pm$ by auxiliary fibrations:

- Over the 'high-genus' side D_+, we replace each fibre X_s by its nth symmetric product $\mathrm{Sym}^n(X_s)$, some $n > 0$. (When the fibre is singular, we instead use the punctual Hilbert scheme $\mathrm{Hilb}^{[n]}(X_s)$.)
- Over the 'low-genus' side D_-, we replace each fibre X_s by $\mathrm{Sym}^{n-1}(X_s)$.
- Call these auxiliary fibrations $E_+ \to D_+$ and $E_- \to D_-$. Give them symplectic forms $\Omega_\pm$ and compatible almost complex structures making the fibrations pseudo-holomorphic. Note that ∂E_+ and ∂E_- both fibre over S^1.
- We relate them by constructing a *Lagrangian matching condition*: a fibrewise middle-dimensional sub-fibre bundle $\mathcal{V} \subset \partial E_+ \times_{S^1} \partial E_-$ such that $(-\Omega_+) \oplus \Omega_-$ vanishes on $\mathcal{V}$.
- We then consider moduli spaces of pairs (u_+, u_-) of pseudo-holomorphic sections of $E_+ \to D_+$ and $E_- \to D_-$, such that $(u_+(z), u_-(z)) \in \mathcal{V}_z$ for each $z \in S^1$.

Why is this sensible? Of course, the answer depends on the construction of $\mathcal{V}$, which I cannot squeeze into this report. Note, though, that the mechanism is (a) symplectic; (b) *not* gauge-theoretic; (c) *not* 'tautological'.

1. It gives a general procedure for obtaining a moduli space of pseudo-holomorphic curves, with reasonable compactness properties, from a broken fibration. The auxiliary choices are from a path-connected space.

2. It fits heuristically with Seiberg-Witten theory. Think of $\mathrm{Sym}^n(X_s)$ as a moduli space of solutions to the dimensionally-reduced Seiberg-Witten equations on X_s. The parameter n is determined by a choice of Spin^c-structure $\mathfrak{s}$ as $2n+(2-2g) = \langle c_1(\mathfrak{s}), [X_s]\rangle$ (the RHS is constant over $s \in S^2$ but the genus is not, hence neither is n). It is a well-documented phenomenon (e.g. [5]) that, in an 'adiabatic limit' in which the metric on the base is expanded relative to those on the fibres, Seiberg-Witten monopoles approach pseudo-holomorphic sections of the bundle of symmetric products.

Why Lagrangian matching conditions? Consider the 3-dimensional Seiberg-Witten equations over a 3-cobordism W as in Example **1**. These determine a correspondence between the Seiberg-Witten moduli spaces associated with its two boundary components. This correspondence is Lagrangian with respect to canonical symplectic forms.

3. It fits, still more heuristically, with Taubes' programme in near-symplectic geometry [4]. Taubes suggests we should study pseudo-holomorphic curves C in $X \setminus Z$, where Z is the union of the critical circles. Moreover, C should have homological boundary $[\delta C] = [Z] \in H_1(Z;\mathbb{Z})$. Such a curve C has positive intersections with the fibres (themselves complex). By an adjuntion-formula argument C hits each fibre over D_+ n times, where n is determined by a Spin^c-structure as above. It hits each fibre over D_- $n-1$ times.

References

[1] D. Auroux, S. Donaldson and L. Katzarkov, *Singular Lefschetz pencils.* Geom. Topol. 9 (2005), 1043–1114.

[2] S. Donaldson and I. Smith, *Lefschetz pencils and the canonical class for symplectic four-manifolds.* Topology 42 (2003), no. 4, 743–785.

[3] T. Perutz, *Lagrangian matching invariants for fibred four-manifolds: I, II*, preprints, submitted. ArXiv: `math.SG/0606061, math.SG/0606062`.

[4] C. Taubes, *The geometry of the Seiberg-Witten invariants.* Proceedings of the ICM, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 493–504.

[5] C. Taubes, *Gr ⇒ SW: from pseudo-holomorphic curves to Seiberg-Witten solutions.* J. Differential Geom. 51 (1999), no. 2, 203–334.

Surgery on nullhomologous tori

Ronald Fintushel
(joint work with Ronald J. Stern)

The fact that surgery on nullhomologous tori can be an effective method for changing the smooth structure of a 4-manifold is demonstrated by the proof of the knot surgery theorem [FS2]. This proof relates the Seiberg-Witten invariant of a knot-surgered 4-manifold X_K to the Seiberg-Witten invariant of X and the Alexander polynomial of K by using the fact that a knot can be unknotted by crossing changes, which in turn can be achieved by ± 1-surgeries on loops λ_i which are nullhomologous in the complement of the knot K. The manifold X_K is the fiber sum of X and $S^1\times$ (0-surgery on K) and the nullhomologous tori we are alluding to are $S^1 \times \lambda_i$.

We are interested in applying this surgery technique to produce new simply connected 4-manifolds with $b^+ = 1$. In $E(1)$, consider the neighborhood $N_F \cong T^2 \times D^2$ of a smooth elliptic fiber F. View N_F as $S^1 \times (S^1 \times D^2)$ and consider the Whitehead double λ of the core circle $f = S^1 \times \{0\}$ as shown in Figure 1.

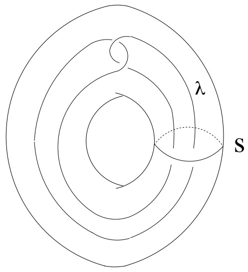


Figure 1

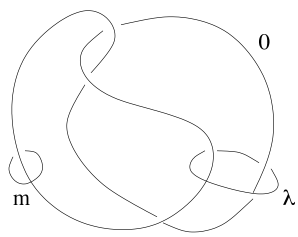


Figure 2

Then $F = S^1 \times f$, and s is the intersection of a section S of $E(1)$ with ∂N_F. The torus $\Lambda = S^1 \times \lambda$ is a nullhomologous torus in $E(1)$ — the Whitehead double of F. We wish to calculate the result of surgery on Λ. Since we don't wish to change the homeomorphism type of $E(1)$, we are interested in $1/n$-surgery.

This is $S^1\times$ ($1/n$-surgery on λ). Denote the result by $E(1)_{\Lambda,1/n}$. According to the Morgan-Mrowka-Szabo formula [MMS], the resulting Seiberg-Witten invariant depends only on the Seiberg-Witten invariants of $E(1)$ and of $E(1)_{\Lambda,0}$, the result of 0-surgery. (One needs to be careful here since the formula of [MMS] depends on some appropriate choices. See [FS4].) Since $E(1)$ admits a metric of positive scalar curvature, $\mathcal{SW}_{E(1)} = 0$.

The manifold $E(1)_{\Lambda,0}$ has $b_1 = 1$ and $H_2 = H_2(E(1)) \oplus \tilde{H}$ where $\tilde{H}$ is a hyperbolic pair generated by the core torus of the surgery and a dual torus which is the union of a punctured torus bounded by λ in Figure 2 and the surgery disk. Note that $b^+(E(1)_{\Lambda,0}) = 2$.
To obtain this figure, note that the Whitehead link is symmetric, and isotope one of the components slightly. The component labelled '0' is 's' of Figure 1. It is an unknot with meridian m. Perform knot surgery on $(E(1), F)$ by removing N_F and replacing it with $S^1 \times (S^3 \setminus s)$. The result is again $E(1)$, and it can be equally well described as the fiber sum of $E(1)$ with $S^1\times$(the 3-manifold of Figure 2) using $F = S^1 \times m$. We now see that $E(1)_{\Lambda,0} = E(1)\#_{F=S^1\times m}Y$ where Y is the result of 0-surgery on λ in Figure 2. Its Seiberg-Witten invariant was calculated in the proof of the knot surgery theorem [FS2] by applying a theorem of J. Hoste [H]: $\mathcal{SW}_{E(1)_{\Lambda,0}} = t^{-1} - t$, where t corresponds to the homology class of F. Then for $1/n$-surgery on Λ in $E(1)$, the Morgan-Mrowka-Szabo formula gives $\mathcal{SW}_{E(1)_{\Lambda,1/n}} = n(t^{-1} - t)$. Thus, surgery on the nullhomologous torus Λ provides an infinite family of distinct 4-manifolds homeomorphic to $E(1)$.

Since $E(1) \cong \mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$, we would like to blow down (perhaps even several times) and find Λ in $\mathbf{CP}^2\#k\overline{\mathbf{CP}}^2$ for $k < 9$. The dream is to obtain exotic structures by $1/n$-surgery on Λ without using the many blowups and rational blowdowns as in [P, SS, FS3, PSS]. The intersection number of Λ with an exceptional curve is 0, but intersection points of Λ and an exceptional curve cannot be removed. (For example, see Figure 1, and recall that a section of $E(1)$ is an exceptional curve.)

Instead, with a bit of work one can find 'Λ' in a manifold R_k homeomorphic to $\mathbf{CP}^2\#k\overline{\mathbf{CP}}^2$ for $k = 5, \ldots, 8$ and with $\mathcal{SW}_{R_k} = 0$. (One hopes that in fact $R_k \cong \mathbf{CP}^2\#k\overline{\mathbf{CP}}^2$.) This is done as follows in the $k = 8$ case, and the other cases are similar. Begin by applying the "double node trick" of [FS3] (to the manifold of Figure 2) to represent the class of a section in $E(1)$ by an immersed sphere Σ with one positive double point. It is easy to find a -2-sphere which is otherwise disjoint from this construction, and which intersects Σ in a single point. Now blow up the double point of Σ to obtain the configuration of spheres $\{-5, -2\}$ in $E(1)\#\overline{\mathbf{CP}}^2$. This configuration can be rationally blown down [FS1] to obtain R_8. It is an easy exercise to find an embedded sphere of self-intersection $+1$ in R_8; hence $\mathcal{SW}_{R_8} = 0$.

Since Λ is disjoint from all these constructions, it descends to R_8. As before, in order to calculate $\mathcal{SW}_{(R_8)_{\Lambda,1/n}}$, we need to find $\mathcal{SW}_{(R_8)_{\Lambda,0}}$. Now $(R_8)_{\Lambda,0}$ is obtained by:

1. Double node surgery with $K =$ the unknot, blowing up, then rationally blowing down.
2. 0-surgery on Λ.

Since Λ is disjoint from all the constructions in (1), the order in which (1) and (2) are performed is irrelevant. (As an aside, note that if we could exactly "see" Λ embedded in $\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$, step (1) would be unnecessary, and we could then use $\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$ rather than R_8.)

We already know that Step (2) gives $E(1)_{\Lambda,0}$ whose Seiberg-Witten invariant is $t^{-1} - t$. Now apply Step (1), the blowup formula and the rational blowdown theorem [FS1] give $\mathcal{SW}_{(R_8)_{\Lambda,0}} = \tau^{-1} - \tau$. Then another application of [MMS] tells us that $\mathcal{SW}_{(R_8)_{\Lambda,1/n}} = n(\tau^{-1} - \tau)$, and we see that we obtain an infinite family of distinct 4-manifolds, all homeomorphic to $\mathbf{CP}^2\#8\overline{\mathbf{CP}}^2$ (and also to R_8, of course) by surgeries on the nullhomologous torus $\Lambda \subset R_8$.

References

[FS1] R. Fintushel and R. Stern, *Rational blowdowns of smooth 4-manifolds*, Jour. Diff. Geom. **46** (1997), 181–235.

[FS2] R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363–400.

[FS3] R. Fintushel and R. Stern, *Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$*, J. Amer. Math. Soc. 19 (2006), 171–180.

[FS4] R. Fintushel and R. Stern, *Surgery on nullhomologous tori and simply connected 4-manifolds with* $\mathbf{b^+ = 1}$, in preparation.

[H] J. Hoste, *Sewn-up r-link exteriors*, Pacific J. Math. **112** (1984), 347–382.

[MMS] J. Morgan, T. Mrowka, Z. Szabo, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997), 915–929.

[P] J. Park, *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$*, Invent. Math. **159** (2005), 657–667.

[PSS] J. Park, A. Stipsicz and Z. Szabo, *Exotic smooth structures on* $\mathbf{CP}^2\#5\,\overline{\mathbf{CP}}^2$, Math. Res. Lett. **12** (2005), no. 5-6, 701–712.

[SS] A. Stipsicz and Z. Szabo, *An exotic smooth structure on* $\mathbf{CP}^2\#6\,\overline{\mathbf{CP}}^2$, Geom. Topol. **9** (2005), 813–832.

New Phenomena in Knot and Link Concordance

Shelly Harvey

(joint work with Tim Cochran, Constance Leidy)

We report the partial resolution of two long-standing questions about whether or not certain natural families of classical knots and links contain slice links.

A *link* $L = \{K_1, ..., K_m\}$ of m-components is an ordered collection of m oriented circles disjointly embedded in S^3. A *knot* is a link of one component. A *slice link* is a link whose components bound a disjoint union of m 2-disks smoothly embedded in B^4. The question of which links are slice links lies at the heart of the classification of 4-dimensional manifolds. The connected sum operation gives the set of all knots, modulo slice knots, the structure of an abelian group, called the *smooth knot concordance group*. Using (locally flat) topological embeddings, one gets the *topological knot concordance group* $\mathcal{C}$ which is a quotient of its smooth partner. For general links one must consider *string* links to get a well-defined group structure, and this operation is not commutative. This paper gives new information about all of these groups, using techniques of noncommutative algebra and analysis.

In the late 60's Levine [9] defined an epimorphism from $\mathcal{C}$ to the *algebraic concordance group*, which he showed was isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, given by the Arf invariant, certain discriminants and twisted signatures associated to the infinite cyclic cover of the knot complement. In the early 70's Casson and Gordon defined new invariants via dihedral covers; these were used to show that the kernel of Levine's map has infinite rank [1] [2]. Shortly after the work of Casson and Gordon the self-referencing sequences of knots shown in Figure 1 were considered by Casson, Gordon, Gilmer and others.

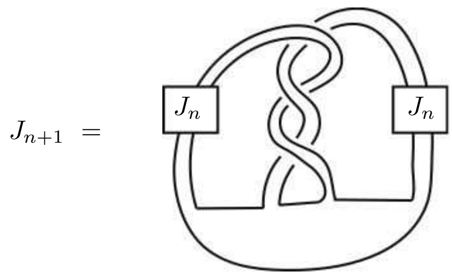


FIGURE 1. The knot $J_{n+1}, n \geq 0$

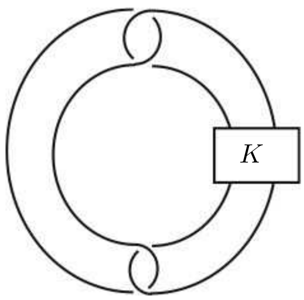
It was well known that if J_0 is a slice knot then each J_n is also slice. On the other hand, it was known that all of the knots J_n, $n \geq 1$, are algebraically slice, that is Levine's obstructions vanish for them. It was known to Casson-Gordon and Gilmer that if certain Levine signatures of J_0 are not zero, then Casson-Gordon invariants can be used to show that J_1 is not a slice knot, but that the invariants of both Levine and of Casson-Gordon-Gilmer vanish for J_n if $n \geq 2$. It was asked whether or not J_n is a slice knot assuming that some classical signature of J_0 is non-zero.

In [4], Cochran, Orr and Teichner used arbitrary solvable covers of the knot complement to find non-slice knots that could not be detected by the invariants of Levine or Casson-Gordon. However the status of the knots J_n above remained open. In fact the techniques of [4], [5] and [6] were inherently limited to knots of genus at least 2 (whereas the J_n all have genus 1). We prove:

Theorem 1. *For any $n \geq 0$ there is a constant C_n such that, if the absolute value of the integral of the Levine-Tristram signatures of K is greater than C_n, then no non-zero multiple of $J_n(K)$ is a slice knot.*

There are other results about members of this family wherein even higher-order signatures of J_0 obstruct J_n from being a slice knot.

Parallel to this family of knots, similar natural families of links have been considered. In particular, if K is any knot then the *Bing-Double of* K, $BD(K)$ is the 2-component link shown in Figure 2.



$= BD(K)$

Figure 2. Bing Double of K

Again, if K is slice then it is easy to see that $BD(K)$ is a slice link. It was asked whether or not the converse was true. It had previously been shown by Harvey that if the integral over the circle of the Levine signatures of K is non-zero then $BD(K)$ is not slice [7]. We have results that go beyond this and cover many cases where the classical signatures of K are zero.

Theorem 2. *Assume the classical Levine signatures of K are zero. There is a constant C such that if certain higher-order signatures of K are greater than C then $BD(K)$ is not slice. For example, there is a constant C such that, if the absolute value of the integral of the Levine-Tristram signatures of J_0 is greater than C, $BD(J_1)$ is not a slice link.*

Furthermore recall that [4] exhibited a new geometrically significant filtration of $\mathcal{C}$

$$\cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_{0.5} \subseteq \mathcal{F}_0 \subseteq \mathcal{C}.$$

It was shown that the filtration exhibits all of the previously known concordance invariants in its associated graded quotients of low degree. It was also shown that there is *new* information in the filtration. In particular, it was shown in [5] that $\mathcal{F}_2/\mathcal{F}_{2.5}$ contains an infinite rank summand of concordance classes of knots not detectable by previously known invariants.

Our second major result is a simplified proof of the following recent result of Cochran and Teichner.

Theorem (Cochran-Teichner [6])**.** *For any $n \in \mathbb{N}_0$, the quotient groups $\mathcal{F}_n/\mathcal{F}_{n.5}$ contain a subgroup isomorphic to $\mathbb{Z}$.*

In fact we show this using the family of knots J_n (for suitable chosen J_0 depending on n). This family is also simpler than the families of Cochran and Teichner. Moreover our examples are different enough that we can show

Theorem 3. *For any $n \in \mathbb{N}_0$, the quotient groups $\mathcal{F}_n/\mathcal{F}_{n.5}$ contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

We note that our construction of examples is all done in the smooth category so that we actually also prove the corresponding statements about the smooth knot concordance group.

Our new technique is to expand upon previous results of Constance Leidy about higher-order Blanchfield forms *without localizing the coefficient system* [8]. This is used to show that certain elements of π_1 of a slice knot (or link) exterior cannot lie in the kernel of the map into any slice disk(s) exterior. We also use recent results of Harvey on the *torsion-free derived series of groups* [3, 7].

References

[1] A. J. Casson and C. McA. Gordon, *On Slice Knots in Dimension Three*, Proc. Symp. in Pure Math. XXX, part 2, (1978), 39–53.

[2] A. J. Casson and C. McA. Gordon, *Cobordism of classical knots*, printed notes, Orsay, 1975, published in A la recherche de la topologie perdue 62, ed. Guilllou and Marin, Progress in Mathematics, 1986.

[3] T. Cochran and S. Harvey, *Homology and Derived Series of Groups*, Geom. Topol. **9** (2005), 2159–2191.

[4] T. D. Cochran, K. E. Orr, and P. Teichner. *Knot Concordance, Whitney Towers and L^2-Signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.

[5] T. Cochran, K. Orr and P. Teichner, *Structure in the Classical Knot Concordance Group*, Comment. Math. Helv. **79** (2004), no. 1, 105–123.

[6] T. Cochran and P. Teichner, *Knot Concordance and von Neumann ρ-invariants*, Duke Math. Journal (to appear), `math.GT/0411057`

[7] S. Harvey, *Homology Cobordism Invariants and the Cochran-Orr-Teichner Filtration of the Link Concordance Group*, preprint.

[8] C. Leidy, *Higher-Order Linking Forms for Knots*, Comment. Math. Helv. (to appear).

[9] J. P. Levine, *Knot Cobordism Groups in Codimension Two*, Comm. Math. Helv. **44** (1969), 229–244.

Differentials on Khovanov-Rozansky homology

Jacob Rasmussen

The HOMFLY homology of Khovanov and Rozansky [6] is an invariant of knots and links in S^3. Let us normalize the HOMFLY polynomial P so that it satisfies the skein relation

$$a^{1/2}P(L_-) - a^{-1/2}P(L_+) = (q^{1/2} - q^{-1/2})P(L_0) \tag{0.1}$$

and evaluates to 1 on the unknot. Then to a knot $K \subset S^3$, the HOMFLY homology assigns a triply-graded homology group $\overline{H}^{i,j,k}$ whose graded Euler characteristic is given by the HOMFLY polynomial:

$$P(K) = \sum_{i,j,k} (-1)^{k-j} q^i a^j \dim \overline{H}^{i,j,k}(K). \tag{0.2}$$

The aim of my talk was to describe certain spectral sequences relating $\overline{H}$ to another family of homology theories $\overline{H}_N$ (the $sl(N)$ homologies) also introduced by Khovanov and Rozansky [5].

As motivation for why this question might be relevant at a conference on four-manifolds, I should remark that KR-homologies share some remarkable formal similarities with the knot Floer homology [9], [12]. It is well known that the latter group is the E_1 term of a spectral sequence which converges to $\widehat{HF}(S^3) \cong \mathbb{Z}$, and the filtration grading of the surviving copy of $\mathbb{Z}$ in the sequence defines an invariant $\tau(K)$. Ozsváth and Szabó showed that this invariant provides a lower bound for the four-ball genus: $|\tau(K)| \leq g_*(K)$ [10]. Interestingly, work of Lee [7], Bar-Natan [1], and Turner [15] shows that the homology theory $\overline{H}_2$ (the original Khovanov homology) also admits a spectral sequence converging to $\mathbb{Z}$, and the filtration grading of the surviving term is again a lower bound for g_* [13].

In light of this, it seems worthwhile to understand the differential structure of the KR-homologies. The first progress in this field was made by Gornik [3], who defined a differential analogous to Lee's on the $sl(N)$ homology. Based on his work and some comparisons with the knot Floer homology, it was conjectured in [2] that the KR-homology can be equipped with a rich structure of differentials.

To explain what is known about this structure, we recall that the $sl(N)$ polynomial $P_N(K)$ is obtained by subsituting $a = q^N$ into $P(K)$. The $sl(N)$ homology $\overline{H}_N^{I,J}(K)$ is a doubly-graded group whose graded Euler characteristic satisfies

$$\sum_{I,J}(-1)^J q^I \dim \overline{H}^{I,J}(K) = P_N(K). \tag{0.3}$$

Theorem 1. [14] *For each $N > 0$, there's a spectral sequence $E_k(N)$ which starts at $\overline{H}(K)$ and converges to $\overline{H}_N(K)$. This sequence is an invariant of K.*

The differentials in the sequence $E_k(N)$ all preserve the quantity $I = i + Nj$, which corresponds to the power of q we get by substituting $a = q^N$ in the term $q^i a^j$. Moreover, the homological grading J on H_N corresponds to the quantity $k-j$ which appears in equation (2). Thus the theorem is a homological generalization of the relation $P_N(K) = P(K)|_{a=q^N}$.

The theorem has a number of applications to understanding the structure of KR-homology. Using it, it is not difficult to verify the first prediction of [2]:

Corollary 1. *For N sufficiently large, $\overline{H}(K) \cong \overline{H}_N(K)$.*

Combining this with previous work on the $sl(N)$ homology [11] gives

Corollary 2. *If K is a two-bridge knot, $\overline{H}(K)$ is determined by the HOMFLY polynomial and signature of K.*

The theorem also provides a natural context in which to view the results of Lee, Turner, and Gornik. In brief, their work shows that there is a spectral sequence $E_k(N, 1)$ starting at $\overline{H}_N(K)$ and converging to $\mathbb{Q}$. The same is true for $\overline{H}$. Indeed, $\overline{H}_1(K) \equiv \mathbb{Q}$ for any knot K, so $E_k(1)$ is such a sequence. Moreover, it is the universal sequence of this form, in the sense that the sequence $E_k(N,1)$ is induced by $E_k(1)$ after passing from $\overline{H}$ to $\overline{H}_N$ by $E_k(N)$.

Although these results represent some progress towards the conjectures of [2], much remains unknown. In particular, substituting $a = 1$ into the HOMFLY polynomial gives the Alexander polynomial, so in analogy with Theorem 1, we hope that there should be a spectral sequence $E_k(0)$ relating $\overline{H}$ to the knot Floer homology. More generally, the conjecture states that there should be spectral sequence $E_k(N)$ for each $N \in \mathbb{Z}$, and a symmetry $\phi : \overline{H}(K) \to \overline{H}(K)$ which exchanges $E_k(N)$ and $E_k(-N)$. ϕ generalizes the well-known symmetry of the HOMFLY polynomial: $P_K(a,q) = P_K(a,q^{-1})$.

At present, we do not know any viable candidate for the spectral sequences $E_k(N)$ for $N < 0$, with the exception of $N = -1$. Recall that $H_1(K) \equiv \mathbb{Q}$ for any knot K. By symmetry, we expect that the sequence $E_{-1}(K)$ should converge to $\mathbb{Q}$ as well. In [14] it is shown that

Theorem 2. *There's a spectral sequence $E_k(-1)$ which starts at $\overline{H}(K)$ and converges to $\mathbb{Q}$.*

References

[1] D. Bar-Natan. Khovanov's Homology for Tangles and Cobordisms. *Geom. Topol.*, 9:1443-1499, 2005. math.GT/0410495.

[2] N. Dunfield, S. Gukov, and J. Rasmussen. The superpolynomial for knot homologies. *Experiment. Math.*, 15:129-160, 2006. math.GT/0505662.

[3] B. Gornik. Note on Khovanov link cohomology. math.QA/0402266, 2004.

[4] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101:359–426, 2000.

[5] M. Khovanov and L. Rozansky. Matrix factorizations and link homology. math.QA/0401268, 2004.

[6] M. Khovanov and L. Rozansky. Matrix factorizations and link homology II. math.QA/0505056, 2005.

[7] E. S. Lee. An endomorphism of the Khovanov invariant. *Adv. Math.* 197:554-586, 2005. math.GT/0210213.

[8] P. Ozsváth and Z. Szabó. Holomorphic disks and link invariants. math.GT/0512286.

[9] P. Ozsváth and Z. Szabó. Heegaard Floer homology and alternating knots. *Adv. Math.*, 186:225–254, 2002. math.GT/0209056.

[10] P. Ozsváth and Z. Szabó. Knot Floer homology and the four-ball genus. *Geom. Topol.*, 7:615–639, 2003. math.GT/0301026.

[11] J. Rasmussen. Khovanov-Rozansky homology of two-bridge knots and links. math.GT/0508510.

[12] J. Rasmussen. Floer homology and knot complements. math.GT/0306378, 2003.

[13] J. Rasmussen. Khovanov homology and the slice genus. math.GT/04020131, 2004.

[14] J. Rasmussen. Some differentials on Khovanov-Rozansky homology. math.GT/0607544, 2006.

[15] P. Turner. Calculating Bar-Natan's characteristic two Khovanov homology. math.GT/0411225, 2004.

Simply connected surfaces of general type with $p_g = 0$ and $K^2 > 0$

Jongil Park
(joint work with Yongnam Lee)

One of the fundamental problems in the classification of complex surfaces is to find a new family of simply connected surfaces with $p_g = 0$ and $K^2 > 0$. Although a large number of non-simply connected complex surfaces of general type with $p_g = 0$ and $K^2 > 0$ have been known ([BHPV], Chapter VII), until now the only previously known simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 > 0$ was Barlow surface [B]. Barlow surface has $K^2 = 1$. The natural question arises if there is a simply connected surface of general type with $p_g = 0$ and $K^2 \geq 2$.

Recently, I constructed new simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $1 \leq K^2 \leq 2$ by using a rational blow-down surgery [P]. After this construction, it has been a very intriguing question whether such symplectic 4-manifolds admit a complex structure.

The aim of this talk is to confirm an affirmative answer for the question above. Precisely, we construct a new family of simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $1 \leq K^2 \leq 2$ by modifying Park's symplectic 4-manifolds in [P]. Our main techniques are a $\mathbb{Q}$-Gorenstein smoothing theory [KSB, M] and a rational blow-down surgery [FS].

In this talk, I would like to review some basic facts about a rational blow-down surgery and a $\mathbb{Q}$-Gorenstein smoothing theory. And then I'll sketch how to construct a new family of simply connected symplectic 4-manifolds using a rational blow-down surgery and how to show that such 4-manifolds admit a complex structure using a $\mathbb{Q}$-Gorenstein smoothing theory. Finally we show that such surfaces are in fact minimal surfaces of general type with $p_g = 0$ and $1 \leq K^2 \leq 2$.

References

[B] R. Barlow, *A simply connected surface of general type with $p_g = 0$*, Invent. Math. **79** (1984), 293–301.

[BHPV] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact complex surfaces*, 2nd ed. Springer-Verlag, Berlin, 2004.

[FS] R. Fintushel and R. Stern, *Rational blowdowns of smooth 4-manifolds*, Jour. Diff. Geom. **46** (1997), 181–235.

[KSB] J. Kollár and Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988) 299–338.

[M] M. Manetti, *Normal degenerations of the complex projective plane*, J. Reine Angew. Math. **419** (1991), 89–118.

[P] J. Park, *Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$*, Invent. Math. **159** (2005), 657–667.

Recent developments in the theory of smooth knot concordance invariants

Matthew Hedden

In the past three years, several new invariants of smooth knot concordance have been discovered [6, 9, 13]. This lecture focused on two of these invariants, denoted $\tau(K)$ and $s(K)$, respectively. Here K denotes a knot in the three-sphere. The former invariant was discovered by Ozsváth and Szabó in [9] and independently by Rasmussen [13], and is defined using the Floer homology theory for knots introduced by the aforementioned authors [11, 13]. $s(K)$ was introduced by Rasmussen [14] and is defined in the context of Khovanov knot homology [4]. The invariants share several formal properties and agree for many knots. In particular, each invariant is a homomorphism from the smooth knot concordance group to the integers, and each bounds the smooth four-genus, $g_4(K)$:

$$|\tau(K)| \leq g_4(K)$$
$$|s(K)| \leq 2g_4(K).$$

Moreover, the above inequalities are sharp for the (p, q) torus knot, hence providing new proofs of Milnor's famous conjecture on the four-genus and unknotting number of these knots. Indeed, it was conjectured by Rasmussen that 2τ and s agree for all knots. If confirmed, this conjecture would point to a surprising connection between the analytically defined Ozsváth-Szabó homology theory and the combinatorially defined Khovanov homology. Moreover, it would seem to indicate a relationship between the gauge theory of three and four-manifolds and the quantum framework underlying the Jones polynomial.

The primary purpose of the lecture was to explore Rasmussen's conjecture by discussing evidence for its validity and then presenting the first counterexamples, discovered by the lecturer and Ording [2]. The examples come from the Whitehead double construction:

Theorem 1 [2] *Let K denote the right-handed trefoil knot, and let $D(K, t)$ denote the t-twisted, positive-clasped Whitehead double of K (see Figure 1). Then $2\tau(D(K, 2)) \neq s(D(K, 2))$.*

A secondary aim was to discuss a relationship between $\tau(K)$ and algebraic curves in $\mathbb{C}^2$:

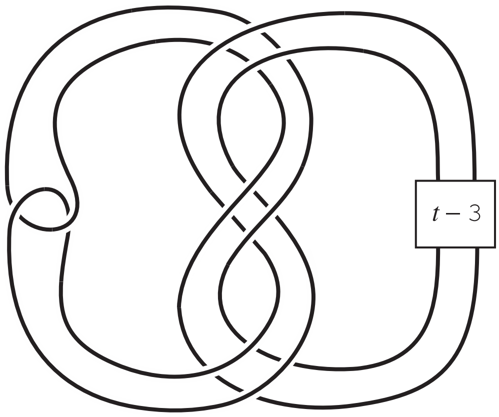
Theorem 2 [3] *Let K be a fibered knot in the three-sphere, and suppose that*

$$\tau(K) = g_4(K) = g(K),$$

where $g(K)$ denotes the Seifert genus of K. Then there exists a complex curve $V \subset \mathbb{C}^2$ which intersects the three-sphere in a knot K' isotopic to K i.e. $K \simeq K' = V \cap \{S^3 = \partial B^4 \subset \mathbb{C}^2\}$. Moreover, $g(V) = g(K)$.

This theorem should be viewed in the light cast by the various forms of the Thom conjecture which, loosely speaking, say that holomorphic curves minimize

FIGURE 1. The t-twisted positive Whitehead double of the right-handed trefoil. The box indicates the number of full right-handed twists to insert.



genus in their homology class. Since there are now several gauge and Ozsváth-Szabó theoretic proofs of the Thom conjecture, it was not surprising to see that an even stronger version of the converse of Theorem 2 holds - namely, that if K arises from the intersection of a complex curve V with the three-sphere, then $\tau(K) = g(V)$. Indeed, this result follows easily from work of Plamenevskaya [12], Boileau and Orevkov [1], and Rudolph [15] and is explicitly spelled out in [3] (see also [5] for the case where $g(V) = g(K)$). In contrast we found Theorem 2 to be quite unexpected.

Coupled with a theorem of Ozsváth and Szabó in [10] and recent work of Ni [7] we also have the following corollary:

Corollary 3 [3] *Let K be a knot in S^3 and suppose that positive Dehn surgery on K yields a lens space or, more generally, an L-space. Then there exists a complex curve V which intersects the three-sphere transversely in a knot isotopic to K. Furthermore, $g(V) = g(K)$.*

In fact, both Theorem 2 and Corollary 3 are slightly stronger - in both cases we showed that K is isotopic to the closure of a braid $\beta \in B_n$ consisting of very specific conjugates of the positive generators of B_n (here B_n is the n-stranded braid group). In the language of Rudolph [15], such braids are called *strongly quasipositive.* The theorem and corollary then follow from [15], which shows that the closure of a strongly quasipositive braid arises from a complex curve as stated.

The final part of the lecture dealt with an extension of some of the above ideas to other three-manifolds. We very roughly discussed how to use the knot Floer homology filtration together with an invariant $c(\xi) \in \widehat{HF}(-Y)/\{\pm 1\}$ of a contact structure, ξ, on Y to define an integer-valued invariant, denoted $\tau_\xi(K)$, of the triple

(Y, K, ξ). In the case of the standard contact structure on S^3, ξ_{std}, we recover the original invariant, i.e. $\tau_{\xi_{std}}(K) = \tau(K)$. We then briefly mentioned how $\tau_\xi(K)$ serves as an upper bound for the Thurston-Bennequin and rotation numbers of Legendrian realizations of (Y, K, ξ), generalizing [12]. In the case where (Y, ξ) admits a symplectic filling, we expect $\tau_\xi(K)$ to provide an obstruction for (Y, K) to arise as the intersection of Y with a J-holomorphic curve in the symplectic filling. We also speculated about an analogue of Theorem 2 for fibered knots in specific families of three-manifolds admitting symplectic fillings.

References

[1] M. Boileau and S.Y. Orevkov. Quasipositivite dune courbe analytique dans une boule pseudo-convexe. *C. R. Acad. Sci. Paris.* 332: 825-830. 2001.

[2] M. Hedden and P. Ording. The Ozsváth-Szabó and Rasmussen Concordance invariants are not equal. To appear, *Amer. J. Math.*

[3] M. Hedden. Notions of Positivity and the Ozsváth-Szabó concordance invariant. arXiv:math.GT/0509499 v1 21 Sep 2005.

[4] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.* 101(3):359-426, 2000.

[5] C. Livingston. Computations of the Ozsváth-Szabó concordance invariant. *Geom. Topol.* 8:735-742, 2004.

[6] C. Manolescu and B. Owens. A concordance invariant from the Floer homology of double branched covers. math.GT/0508065 2005.

[7] Y. Ni. Knot Floer homology detects fibered knots. arXiv:math.GT/0607156 v2 17 Jul 2006.

[8] P. S. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Annals of Math.* 159(3):1027-1158, 2004.

[9] P. S. Ozsváth and Z. Szabó. Knot Floer Homology and the four-ball genus. *Geom. Topol.*, 7:615-639, 2003.

[10] P. S. Ozsváth and Z. Szabó. On knot Floer homology and lens space surgeries. arxiv:math.GT/0303017.

[11] P. S. Ozsváth and Z. Szabó. Holomorphic disks and knot invariants. *Adv. in Math.*, 186(1):58-116, 2004.

[12] O. Plamenevskaya. Bounds for the Thurston-Bennequin number from Floer homology *Algebr. Geom. Topol.*4:399-406, 2004.

[13] J. Rasmussen. *Floer homology and knot complements.* PhD thesis, Harvard University, 2003.

[14] J. Rasmussen. Khovanov homology and the slice genus. arXiv:math.GT/0402131 v1 9, Feb 2004.

[15] L. Rudolph. Knot theory of complex plane curves. arxiv:math.GT/0411115 v2 7, Nov 2004.

Khovanov homology as a 4+1-dimensional TQFT

Kevin Walker
(joint work with Scott Morrison)

We start by repairing some sign issues in the usual definition of Khovanov homology. To do this, we will follow the approach of Bar-Natan [1], but we will replace unoriented (and/or oriented) links and surfaces with their "disoriented" counterparts. Recall that a disoriented 1- or 2-manifold is a piecewise oriented manifold in which each component of the codimension-1 submanifold separating oppositely oriented pieces is equipped with a preferred side (equivalently, equipped with an oriented normal vector). In a surface, changing the oriented pieces by a morse move changes the coefficient of the surface by $\pm i$, where the sign depends on the index of the Morse move and the direction of the preferred side. Plugging the above disoriented manifolds into the machinery of [1], we obtain sign-corrected versions of the results of Khovanov [3] and Jacobsson [2]:

Theorem 1. *There is a well-defined functor* Kh *from the category of disoriented links and (homotopy classes of) isotopies to the category of bigraded vector spaces and linear isomorphisms.*

Theorem 2. *More generally, there is a well-defined functor* Kh *from the category of disoriented links and isotopy classes of disoriented cobordisms in $S^3 \times I$ to the category of bigraded vector spaces and graded linear maps.*

If L is an oriented link, then our $\mathrm{Kh}(L)$ is canonically-up-to-sign isomorphic to the usual $\mathrm{Kh}(L)$.

The cobordisms in the theorem can be replaced by cobordisms modulo the neck cutting relation of [1] and modulo the above disorientation relations.

Recall from [1] that we get different versions of Kh depending on the value α of a closed genus 3 surface. We will let α be a free variable of q-degree -4, and our bigraded vector spaces will become bigraded $\mathbb{C}[\alpha]$-modules. Setting $\alpha = 0$ gives the original version of Khovanov homology. Setting α to a non-zero complex number gives Lee homology, but we lose the q-grading in this case.

Now let W be a 4-manifold and L a disoriented link in ∂W. We will define a graded $\mathbb{C}[\alpha]$-module $A_{\mathrm{Kh}}(W;L)$. $A_{\mathrm{Kh}}(W;L)$ will be defined as "pictures mod relations" (i.e. as a 4-dimensional skein module) and satisfies the the type of gluing law one expects for the Hilbert space of a 4+1-dimensional TQFT. (See [6] for more details on this approach to TQFTs.)

Let $\mathcal{P}_{\mathrm{Kh}}(W;L)$ be the set of all Kh-pictures on (W,L), where a Kh-picture consists of:

- a collection $\{B_i\}$ of disjoint 4-balls in the interior of W;
- a disoriented surface Σ properly embedded in $W \setminus \coprod B_i$, with $\partial\Sigma \cap \partial W = L$ and $\partial\Sigma \cap \partial B_i = L_i$, where L_i is a disoriented link in ∂B_i; and
- a label $x_i \in \mathrm{Kh}(L_i)$ for each B_i.

Define $A_{\mathrm{Kh}}(W;L)$ to be the set of all finite $\mathbb{C}[\alpha]$-linear combinations of elements of $\mathcal{P}_{\mathrm{Kh}}(W;L)$, modulo the following relations:

- If $P \in \mathcal{P}_{\mathrm{Kh}}(W;L)$ has $x_j = ay + z$, for some $a \in \mathbb{C}[\alpha]$ and $y, z \in \mathrm{Kh}(L_j)$, then $P \sim aP_y + P_z$, where P_y is the same as P except that the j^{th} label has been replaced by y, and P_z is defined similarly.
- If P' is obtained from $P \in \mathcal{P}_{\mathrm{Kh}}(W;L)$ by introducing an additional 4-ball, disjoint from Σ, labeled by $1 \in \mathrm{Kh}(\emptyset) \cong \mathbb{C}$, then $P' \sim P$.
- Let $P \in \mathcal{P}_{\mathrm{Kh}}(W;L)$ and $B'_j \supset B_j$ with B'_j disjoint from the other 4-balls of P. Let $\Sigma' = \Sigma \setminus B'_j$, $\Sigma'' = \Sigma \cap B'_j$, and $L'_j = \partial B'_j \cap \Sigma$. Then Σ'' is a cobordism from L_j to L'_j and by Theorem 2 there is an element

$x'_j = \mathrm{Kh}(\Sigma'')(x_j) \in \mathrm{Kh}(L'_j)$. Let P' be P with Σ, B_j and x_j replaced by Σ', B'_j and x'_j. Then $P' \sim P$.

- Let $P \in \mathcal{P}_{\mathrm{Kh}}(W; L)$, B_j and B_k be two 4-balls of P, and let $N \subset W$ be a neighborhood of an arc connecting B_j to B_k, disjointly from Σ. Let P' be obtained from P by replacing B_j and B_k with the single 4-ball $B_j \cup N \cup B_k$, labeled by $x_j \otimes x_k$. (Σ meets $B_j \cup N \cup B_k$ in the disjoint union of L_j and L_k, and there is a natural isomorphism $\mathrm{Kh}(L_j \coprod L_k) \cong \mathrm{Kh}(L_j) \otimes \mathrm{Kh}(L_k)$.) Then $P' \sim P$.

One can show that $A_{\mathrm{Kh}}(B^4; L) \cong \mathrm{Kh}(L)$, so A_{Kh} is a generalization of Khovanov homology to links in the boundary of arbitrary 4-manifolds. If one were to reduce all dimensions by one in the above procedure, replacing Σ with a ribbon tangle and replacing the x_i with appropriate morphisms from a ribbon category, then one could obtain the Witten-Chern-Simons invariants of a 3-manifold. Thus it is reasonable to think of A_{Kh} as a categorification of the Witten-Chern-Simons TQFT.

In order to state the gluing theorem for $A_{\mathrm{Kh}}(W^4; L)$ we need to introduce categories associated to 3-manifolds and representations of these categories associated to 4-manifolds with boundary.

Let M be a 3-manifold and let c be a collection of oriented framed points in ∂M, thought of as boundary conditions for disoriented tangles in M. Let $A_{\mathrm{Kh}}(M; c)$ be the category defined as follows. Objects are disoriented tangles in M with boundary c. Morphisms from tangle T to tangle T' are elements of $A_{\mathrm{Kh}}(M \times I; T \cup T')$. Composition is given by gluing the pictures representing elements of $A_{\mathrm{Kh}}(M \times I; T \cup T')$ and $A_{\mathrm{Kh}}(M \times I; T' \cup T'')$. Note that $A_{\mathrm{Kh}}(-M; c)$ is naturally isomorphic to $A_{\mathrm{Kh}}(M; c)^{\mathrm{op}}$. If M is closed, we omit c from the notation.

Let W be a 4-manifold with boundary. Then the collection of $\mathbb{C}[\alpha]$-modules $\{A_{\mathrm{Kh}}(W; L)\}$, indexed by links $L \in \partial W$, affords a right representation of $A_{\mathrm{Kh}}(\partial W)$ (or a left representation of $A_{\mathrm{Kh}}(-\partial W)$). The action is again given by gluing pictures. Denote this representation by $A_{\mathrm{Kh}}(W; \bullet)$.

We can now state the gluing theorem for the 4-dimensional invariants. Let $M = \partial W = -\partial W'$. Then there is a natural isomorphism

$$A_{\mathrm{Kh}}(W \cup_M W') \cong A_{\mathrm{Kh}}(W; \bullet) \otimes_{A_{\mathrm{Kh}}(M)} A_{\mathrm{Kh}}(W'; \bullet).$$

(There is a more general gluing (with corners) theorem for gluing a 4-manifold to itself along a 3-manifold with boundary, but it would be too cumbersome to state in this short abstract.)

It is too early to state with certainty that A_{Kh} is an interesting and/or computable invariant. One indication that it might be interesting is that Rasmussen's result [5] on genus bounds for surfaces in B^4 generalizes naturally to give genus bounds for surfaces representing a given homology class in W^4.

References

[1] D. Bar-Natan, *Khovanov's homology for tangles and cobordisms*, Geometry & Topology **9** (2005), 1443–1499

[2] M. Jacobsson, *An invariant of link cobordisms from Khovanov homology*, Algebr. Geom. Topol. **4** (2004) 1211–1251
[3] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000) 359–426
[4] S. Morrison and K. Walker *Disoriented and confused: fixing the functoriality of Khovanov homology*, in preparation
[5] J. A. Rasmussen, *Khovanov homology and the slice genus*, e-print (2004) arXiv:math.GT/0402131
[6] K. Walker *TQFTs*, preprint, http://canyon23.net/math/

Symplectic $\mathbf{S^1} \times \mathbf{N^3}$ and subgroup separability

Stefan Friedl, Stefano Vidussi

Let N be a closed 3–manifold. Thurston [Th76] showed that if N admits a fibration over S^1, then $S^1 \times N$ is symplectic, i.e. it can be endowed with a closed, non–degenerate 2–form ω.

It is natural to ask whether the converse of this statement holds true. We can state this problem in the following form:

Conjecture 1. *Let N be a closed 3–manifold. If $S^1 \times N$ is symplectic, then there exists $\phi \in H^1(N;\mathbb{Z})$ such that (N, ϕ) fibers over S^1.*

Here we say that (N, ϕ) fibers over S^1 if the homotopy class of maps $N \to S^1$ determined by $\phi \in H^1(N;\mathbb{Z}) = [N, S^1]$ contains a representative that is a fiber bundle over S^1.

Assuming the Geometrization Conjecture, it is possible to prove that the problem is reduced to the study of irreducible 3–manifolds, and we will henceforth make that assumption for N.

In [FV06a] we related this problem to the study of twisted Alexander polynomials of N, and in particular we proved the following, that is a weaker version of the main result of [FV06a]:

Theorem 1. *Let N be an irreducible 3–manifold such that $S^1 \times N$ admits a symplectic structure. Then there exists a primitive $\phi \in H^1(N;\mathbb{Z})$ such that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G the associated 1–variable twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is non–zero.*

Recall that the 1–variable twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi}$ associated to the pair (N, ϕ) is defined as the $\mathbb{Z}[t^{\pm 1}]$–order of the twisted Alexander module $H_1(N;\mathbb{Z}[G][t^{\pm 1}])$. Note that $\Delta^{\alpha}_{N,\phi} \neq 0$ if and only if $H_1(N;\mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion.

Given $\alpha : \pi_1(N) \to G$, denote the corresponding regular cover of N cover by N_G. Note that, if $S^1 \times N$ is symplectic, so is $S^1 \times N_G$. The ingredients of the proof of Theorem 1 are now the following: Taubes' results on the Seiberg-Witten invariants of symplectic 4–manifolds, the relation, proved by Meng and Taubes, between the Seiberg–Witten invariants of N_G and the ordinary ordinary Alexander polynomial Δ_{N_G}, and finally a relation obtained in [FV06a] between Δ_{N_G} and $\Delta^{\alpha}_{N,\phi}$.

Theorem 1 says in particular that the following conjecture implies Conjecture 1 for irreducible manifolds.

Conjecture 2. *Let $\phi \in H^1(N;\mathbb{Z})$ be a primitive class such that $\Delta^{\alpha}_{N,\phi} \neq 0$ for all $\alpha : \pi_1(N) \to G$, then (N,ϕ) fibers over S^1.*

To state our main theorem we need the following definition.

Definition 1. *A subgroup $A \subset \pi$ is separable if for all $g \in \pi \setminus A$, there exists an epimorphism $\alpha : \pi \to G$ to a finite group G such that $\alpha(g) \notin \alpha(A)$.*

We have the following result, proven in [FV06b]:

Theorem 2. *Let N be an irreducible 3–manifold. Let $\phi \in H^1(N;\mathbb{Z})$ be a primitive class such that $\Delta^{\alpha}_{N,\phi} \neq 0$ for all epimorphisms $\alpha : \pi_1(N) \to G$ to a finite group. Let $\Sigma \subset N$ be an embedded surface dual to ϕ having minimal genus. If $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, then (N,ϕ) fibers.*

The question of which subgroups of the fundamental group of a Haken manifold are separable has been studied extensively. In particular, the fact that abelian subgroups are separable (cf. [LN91] and [Ha01]) and that incompressible surfaces in Seifert fibered spaces are classified leads to the following corollary.

Corollary 1. *Conjecture 1 holds for irreducible manifolds with vanishing Thurston norm and for graph manifolds.*

This corollary in particular implies that if N_K is the 0–surgery on a knot K of genus $g(K) = 1$, and $S^1 \times N_K$ is symplectic, then K is a trefoil or the figure–8 knot. This answers a question of Kronheimer [Kr98].

Scott [Sc78] showed that any subgroup of a hyperbolic 2–manifolds is separable. It has been conjectured by Thurston [Th82] that all (surface) subgroups of hyperbolic 3–manifolds are separable. Clearly a positive solution to Thurston's conjecture would imply Conjecture 1 for hyperbolic manifolds. Furthermore suitable subgroup separability properties of the hyperbolic pieces in the geometric decomposition can be shown to imply Conjecture 1 for all irreducible manifolds.

We conclude with a short outline of the proof of Theorem 2. Let $M = N \setminus \nu\Sigma$. We have two embeddings $i_\pm : \Sigma \to \partial M$. By Stallings' theorem, (N,ϕ) fibers if the inclusion induced maps $i_\pm : \pi_1(\Sigma) \to \pi_1(M)$ are isomorphisms. Since Σ has minimal genus we know that the maps $i_\pm : \pi_1(\Sigma) \to \pi_1(M)$ are injective and that $\pi_1(M) \to \pi_1(N)$ is injective.

Assume, by contradiction, that one of the $i_\pm$ is not an isomorphism. We use the corresponding inclusion to view $\pi_1(\Sigma)$ and $\pi_1(M)$ as subgroups of $\pi_1(N)$.

Given an epimorphism $\alpha : \pi_1(N) \to G$ to any finite group G we have a long exact Mayer–Vietoris sequence

$$H_1(N;\mathbb{Z}[G][t^{\pm1}]) \to H_0(\Sigma;\mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm1}] \to \\ H_0(M;\mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm1}] \to H_0(N;\mathbb{Z}[G][t^{\pm1}]).$$

Now consider the ranks of the modules over $\mathbb{Z}[t^{\pm 1}]$: we have

$$\begin{aligned}
\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_1(N;\mathbb{Z}[G][t^{\pm 1}])) &= 0 \text{ since } \Delta^{\alpha}_{N,\phi} \neq 0,\\
\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N;\mathbb{Z}[G][t^{\pm 1}])) &= 0 \text{ since } \phi \neq 0,\\
\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma;\mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) &= \mathrm{rank}_{\mathbb{Z}}(H_0(\Sigma;\mathbb{Z}[G])),\\
\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(M;\mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) &= \mathrm{rank}_{\mathbb{Z}}(H_0(M;\mathbb{Z}[G])).
\end{aligned}$$

Therefore

$$\frac{|G|}{|\alpha(\pi_1(\Sigma))|} = \mathrm{rank}_{\mathbb{Z}}(H_0(\Sigma;\mathbb{Z}[G])) = \mathrm{rank}_{\mathbb{Z}}(H_0(M;\mathbb{Z}[G])) = \frac{|G|}{|\alpha(\pi_1(M))|}.$$

In particular we get that $\alpha(\pi_1(\Sigma)) = \alpha(\pi_1(M)) \subset G$. On the other hand it follows immediately from the assumption that $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, and from the assumption that $\pi_1(\Sigma) \to \pi_1(M)$ is not an epimorphism, that there exists an epimorphism $\alpha : \pi_1(N) \to G$ to a finite group with $\alpha(\pi_1(\Sigma)) \neq \alpha(\pi_1(M))$. This contradiction concludes the proof of Theorem 2.

References

[FV06a] S. Friedl, S. Vidussi, *Twisted Alexander polynomials and symplectic structures*, Preprint (2006)

[FV06b] S. Friedl, S. Vidussi, *Symplectic $S^1 \times N^3$, subgroup separability, and vanishing Thurston norm*, Preprint (2006)

[Ha01] E. Hamilton, *Abelian Subgroup Separability of Haken 3–manifolds and Closed Hyperbolic n–orbifolds*, Proc. London Math. Soc. 83 no. 3: 626–646 (2001)

[Kr98] P. Kronheimer, *Embedded surfaces and gauge theory in three and four dimensions*, Surveys in differential geometry, Vol. III (Cambridge, MA, 1996), 243–298, Int. Press, Boston, MA (1998)

[LN91] D. Long, G. Niblo, *Subgroup separability and 3-manifold groups*, Math. Z. 207, no. 2: 209–215 (1991)

[Sc78] P. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) 17, no. 3: 555–565 (1978)

[Th76] W. P. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. 55 (1976), no. 2, 467–468.

[Th82] W. P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982)

Torsion in Heegaard Floer homology

Stanislav Jabuka, Thomas Mark

ABSTRACT: We study the Heegaard Floer homology groups of a genus g surface times a circle. We exhibit that their HF^+ groups carry 2-torsion (for $g \geq 3$) and 3-torsion (for $g \geq 5$). These are the first known examples to contain any torsion in $HF^{\pm}$, $\widehat{HF}$ or HF^{∞}.

1. Introduction

Heegaard Floer homology, as introduced by P. Ozsváth and Z. Szabó in [1, 2], assigns to a spin^c 3-manifold $(Y, \mathfrak{s})$ a collection of Abelian groups: $HF^{\pm}(Y, \mathfrak{s})$, $\widehat{HF}(Y, \mathfrak{s})$, $HF^{\infty}(Y, \mathfrak{s})$, of which this article will focus exclusively on HF^+. Heegaard Floer homology groups carry additional structure such as an action of $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*(H_1(Y;\mathbb{Z})/Tor)$ and a $\mathbb{Q}$-grading when $c_1(\mathfrak{s})$ is torsion. In the latter case we write $HF_d^+(Y, \mathfrak{s})$ for the degree d portion of $HF^+(Y, \mathfrak{s})$.

Since their inception in 1999, the Heegaard Floer homology groups have been calculated for many classes of 3-manifolds. These include, among others, lens spaces, 3-manifolds obtained by negative definite plumbings, some mapping tori of surfaces, surgeries and double branched covers of knots, etc. Yet, among all these examples there is not one instance whose Heegaard Floer homology groups contain torsion elements. It is therefore natural to ask:

Question Are there any 3-manifolds for which there is torsion in any of the integer-coefficient Heegaard Floer groups?

The purpose of this note is to answer this question in the affirmative. Specifically we prove

Theorem 1.1. *Let Σ_g be a surface of genus g. Then $HF^+(\Sigma_g \times S^1, \mathfrak{s}_0)$ contains 2-torsion for all $g \geq 3$ and 3-torsion for all $g \geq 5$. Here $\mathfrak{s}_0 \in Spin^c(\Sigma_g \times S^1)$ is the unique spin^c-structure with $c_1(\mathfrak{s}_0) = 0$.*

2. The Heegaard Floer homology of a surface times a circle

Let us make the identification

$$Spin^c(\Sigma_g \times S^1) \cong H^2(\Sigma_g \times S^1; \mathbb{Z}) \cong H^2(\Sigma_g; \mathbb{Z}) \oplus H^1(\Sigma_g; \mathbb{Z})$$

With respect to this identification we shall denote by $\mathfrak{s}_k$ the unique spin^c-structure of the form $(\alpha, 0)$ with $\langle c_1(\alpha), [\Sigma_g] \rangle = 2k$. It is easy to see (using the adjunction inequality) that $HF^+(\Sigma_g \times S^1, \mathfrak{s}) = 0$ unless $\mathfrak{s}$ equals some $\mathfrak{s}_k$ with $|k| \leq g - 1$.

2.1. The case of $k \neq 0$.

The groups $HF^+(\Sigma_g \times S^1, s_k)$ with $0 < |k| \leq g - 1$ have been calculated by Ozsváth and Szabó [5] for all genera $g \geq 2$, and are given by

$$HF^+(\Sigma_g \times S^1, s_k) \cong H^*(Sym^m(\Sigma_g); \mathbb{Z}) \qquad \text{with} \qquad m = g - 1 - |k|$$

In particular they are all free Abelian groups.

2.2. The case of $k = 0$ and $g \leq 2$.

The groups in these cases are also known by work of Ozsváth and Szabó. As $\mathbb{Z}[U]$-modules they are given by $HF^+(\Sigma_g \times S^1, \mathfrak{s}_0) \cong \widehat{HF}(\Sigma_g \times S^1, \mathfrak{s}_0) \otimes \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$ where $\widehat{HF}(\Sigma_g \times S^1, \mathfrak{s}_0)$ are free Abelian groups of total rank 2, 6 and 20 for the genera 0, 1 and 2 respectively [1, 3, 4]. Once again, these groups are torsion free.

2.3. **The case of $k = 0$ and $g \geq 3$.** The groups in this collection were unknown prior to the work of the authors. Our main tool for calculating them is the surgery long exact sequence for Heegaard Floer homology [2]. To explain this, let K be a nullhomologous knot in a 3-manifold Y and denote the result of n-framed surgery along K by $Y_n = Y_n(K)$. Then for all sufficiently large n, and with suitable choices of spin^c-structures (which we suppress from our notation), there is a long exact sequence

$$... \to HF^+(Y_n) \xrightarrow{F} HF^+(Y) \to HF^+(Y_0) \to HF^+(Y_n) \to ... \tag{2.1}$$

Let $B(0,0)$ be the knot in $(S^1 \times S^2)\#(S^1 \times S^2)$ given by the third component of the Borromean rings after performing 0-framed surgery on the first two components. Taking $Y = \#^{2g}(S^1 \times S^2)$ and $K = \#^g B(0,0) \subset Y$, we have $Y_0 = \Sigma_g \times S^1$. Thus (2.1) can be used to find $HF^+(\Sigma_g \times S^1)$ provided one can get a handle on $HF^+(Y)$, $HF^+(Y_n)$ and the map F. All three of these can be computed quite explicitly from the knot Floer homology of K which has been determined by Ozsváth and Szabó, see [5].

2.3.1. *The $\mathbb{Z}_2$-coefficients case.* The form of the homomorphism F becomes particularly simple if one uses $\mathbb{Z}_2$-coefficients. Omitting details:

Theorem 2.1. *For any $g \geq 0$ and for all d sufficiently large one obtains*

$$dim_{\mathbb{Z}_2} HF_d^+(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{Z}_2) = 2^{2g-1} + 2^{g-1} \tag{2.2}$$

2.3.2. *The $\mathbb{C}$-coefficients case.* With $\mathbb{C}$-coefficients the map F from (2.1) is more intricate. Its kernel turns out to have a form familiar from Kähler geometry. Namely, consider $H^1(\Sigma_g; \mathbb{C})$ together with the cup product pairing as a symplectic vector space and let $e^1, e^2, ..., e^{2g-1}, e^{2g}$ be a symplectic basis. Write $\omega = e^1 \wedge e^2 + ... + e^{2g-1} \wedge e^{2g} \in \Lambda^2 H^1(\Sigma_g; \mathbb{C})$ for the symplectic form (here we identify $H^1(\Sigma_g, \mathbb{C})$ with its dual using the symplectic pairing). Define the *primitive forms* of degree j to be $\mathcal{P}^j = \Lambda^j H^1(\Sigma_g; \mathbb{C}) \cap \mathrm{Ker}(\iota_\omega)$ where ι_ω denotes contraction with ω. It is easily checked that $\dim_{\mathbb{C}} \mathcal{P}^j = \binom{2g}{j} - \binom{2g}{j-2}$, $j = 0, ..., g$.

With this notation in place, the kernel of F in sufficiently high degrees can be identified with the primitive forms $\mathcal{P}^0 \oplus \mathcal{P}^1 \oplus ... \oplus \mathcal{P}^g$ and the cokernel of F is isomorphic to $\mathrm{Ker}(F)$ by elementary linear algebra. Putting the calculations together one arrives at

Theorem 2.2. *For any $g \geq 0$ and all d sufficiently large one obtains*

$$dim_{\mathbb{C}} HF_d^+(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{C}) = \binom{2g+1}{g} \tag{2.3}$$

Since (2.3) is strictly smaller than (2.2) as soon as $g \geq 3$, we have that $HF_d^+(\Sigma_g \times S^1, \mathbb{Z})$ must contain 2-torsion for all such g. A comparison of (2.2) and (2.3) is summarized in the table below.

g	$\dim_{\mathbb{Z}_2} HF^+_d(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{Z}_2)$	$\dim_{\mathbb{C}} HF^+_d(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{C})$	Torsion?
0	1	1	No
1	3	3	No
2	10	10	No
3	36	35	Yes
4	136	126	Yes
5	528	462	Yes
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Using the explicit form of F in (2.1), one can calculate the groups $HF^+(\Sigma_g \times S^1, \mathfrak{s}_0, \mathbb{Z})$ "by hand" for several low values of g. Doing so for genus 5 one finds that

Lemma 2.3. *$HF^+_d(\Sigma_5 \times S^1, \mathfrak{s}_0; \mathbb{Z})$ has elements of order 3 for all sufficiently large values of d.*

On the other hand, an inductive argument on g yields

Lemma 2.4. *If $HF^+(\Sigma_g \times S^1, \mathfrak{s}_0; \mathbb{Z})$ contains p-torsion then so does $HF^+(\Sigma_{g+1} \times S^1, \mathfrak{s}_0; \mathbb{Z})$.*

The results from theorem 1.1 are a combination of the above observations and lemmas 2.3 and 2.4.

References

[1] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed 3-manifolds*, *Ann. of Math.* **159** (2004), no. 3, 1027–1158.
[2] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, *Ann. of Math.* **159** (2004), no. 3, 1159–1245.
[3] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261.
[4] Peter Ozsváth and Zoltán Szabó, *On the Floer homology of plumbed 3-manifolds* Geom. Topol. **7** (2003), 185–224
[5] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116

Invariant count of holomorphic disks in the presence of a real structure

Jean-Yves Welschinger

Let (X, ω) be a symplectic four-manifold and $L \subset X$ be a lagrangian surface. Following the spirit of Gromov-Witten invariants' theory, one can count the number of holomorphic disks sitting on L which realize a given relative homotopy class. This count however almost always depends on the various choices made to get a finite number (choice of an almost complex structure, incidence conditions...). When the lagrangian surface is in the fixed locus of some antisymplectic involution, it is however possible to count these disks with respect to some sign in order to make the result independent of any auxiliary choices and define a deformation invariant of the symplectic manifold together with its involution. This sign is actually determined by the parity of the number of transversal intersection points between the interior of the disk and the lagrangian surface. The integer valued invariant we thus obtain obviously bounds from below the number of pseudo-holomorphic disks sitting on L (similar phenomena appear in higher dimensions as well, see [2]). I did explain how techniques from symplectic field theory sometimes makes it possible to prove the sharpness of these lower bounds, as well as to get some computations and arithmetic properties of these invariants (divisibility by some power of two linear with respect to the degree).

References

[1] J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. **162** (2005), 195–234.
[2] J.-Y. Welschinger, *Enumerative invariants of strongly semipositive real symplectic manifolds*, Preprint math ArXiv AG/0509121 (2005).

A counterexample to the strong version of Freedman's conjecture

Slava Krushkal

A conjecture due to Michael Freedman asserts that the 4-dimensional topological surgery conjecture fails for non-abelian free groups, or equivalently that a family of canonical examples of links (the generalized Borromean rings) are not $A-B$ slice. A stronger version of the conjecture, that the Borromean rings are not even weakly $A-B$ slice, where one drops the equivariant aspect of the problem, has been the main focus in search for an obstruction to surgery. We show that the Borromean rings, and more generally all links with trivial linking numbers, are in fact weakly $A-B$ slice. This result shows the lack of a non-abelian extension of Alexander duality in dimension 4, and of an analogue of Milnor's theory of link homotopy for general decompositions of the 4-ball.

The $A-B$ slice problem is a reformulation of the 4−dimensional topological surgery conjecture, introduced in [2]. The surgery conjecture is known to hold for the class of *good* fundamental groups, including the groups of subexponential growth [4], [5], [8]. It holds for all fundamental groups if and only if a certain family of canonical examples of links (the generalized Borromean rings) are $A-B$ slice. Freedman's conjecture asserts that the Borromean rings are not $A-B$ slice, cf [1].

The $A-B$ slice problem may be roughly summarized as asking whether in dimension 4 there exists a non-abelian extension of Alexander duality. It concerns smooth codimension zero decompositions of the 4−ball. Here a *decomposition* of D^4, $D^4 = A \cup B$, is an extension of the standard genus one Heegaard decomposition of the 3−sphere $\partial D^4 = S^1 \times D^2 \cup D^2 \times S^1$. The attaching curves $\alpha \subset \partial A, \beta \subset \partial B$ (the cores of the two solid tori) form the Hopf link in S^3.

Algebraic and geometric properties of the two parts A, B of a decomposition are tightly correlated. Algebraically this is reflected, in particular, by Alexander duality. A more precise geometric information is given by handle structures: under a mild assumption on the handle decompositions which can be assumed without loss of generality, there is a one-to-one correspondence between $1-$handles of each side and $2-$handles of its complement. In general the interplay between the topologies of the two sides is very subtle.

A link $L = (l_1, \dots, l_n)$ in S^3 is *$A-B$ slice* if there exist decompositions $(A_1, B_1), \dots, (A_n, B_n)$ of D^4 and disjoint embeddings of all manifolds A_i, B_i into D^4 so that the attaching curves $\alpha_1, \dots, \alpha_n$ form the link L and the curves $\beta_1, \dots, \beta_n$ form an untwisted parallel copy of L. In addition, the complement of the embedding of each A_i must be homeomorphic to B_i and the complement of the embedding of each B_i is homeomorphic to A_i. The connection of the $A-B$ slice problem for the (generalized) Borromean rings to the surgery conjecture is provided by considering the universal cover of the hypothetical solutions to the canonical surgery problems, and the action of the free group by the covering transformations [2].

We present two approaches to the $A-B$ slice problem. The first one involves a new invariant of $4-$manifolds, *link groups*, introduced in [6]. These invariants are defined using maps of certain special handlebodies into $4-$manifolds, and we apply them to *model decompositions* of D^4, constructed in [3]:

Theorem 1. [7] *Link groups provide an obstruction in the $A-B$ slice problem, restricted to model decompositions of D^4, for any homotopically essential link.*

A stronger version of Freedman's conjecture was believed to be true, that the Borromean rings are not even *weakly $A-B$ slice* [3]. More precisely, a link L is weakly $A-B$ slice if in the definition above the manifolds $\{A_i, B_i\}$ are still embedded disjointly, but the complement of the embedding of A_i is not necessarily homeomorphic to B_i and similarly the complement of the embedding of each B_i is not necessarily homeomorphic to A_i. This stronger statement which disregards the equivariant aspect of the problem, has been the focus in search for an obstruction to surgery. In contrast, we prove

Theorem 2. *Let L be the Borromean rings, or more generally any link is S^3 with trivial linking numbers. Then L is weakly $A-B$ slice.*

This result shows the lack of a non-abelian extension of Alexander duality in dimension 4. In particular, if there is an obstruction in the $A-B$ slice program, it has to take into account the properties of not just the manifolds $\{A_i, B_i\}$, but also of their embeddings into D^4.

References

[1] M. Freedman, *The disk theorem for four-dimensional manifolds*, Proc. ICM Warsaw (1983), 647-663.

[2] M. Freedman, *Are the Borromean rings (A, B)-slice?*, Topology Appl., 24 (1986), 143-145.

[3] M. Freedman and X.S. Lin, *On the (A, B)-slice problem*, Topology Vol. 28 (1989), 91-110.

[4] M. Freedman and F. Quinn, *The topology of 4-manifolds*, Princeton Math. Series 39, Princeton, NJ, 1990.
[5] M. Freedman and P. Teichner, 4-*Manifold Topology I: Subexponential groups*, Invent. Math. 122 (1995), 509-529.
[6] V.Krushkal, *Link groups of* $4-$*manifolds*, arXiv:math.GT/0510507.
[7] V.Krushkal, *Link groups and Alexander duality*, arXiv:math.GT/0602105.
[8] V. Krushkal and F. Quinn, *Subexponential groups in* 4-*manifold topology*, Geom. Topol. 4 (2000), 407-430.

Research Report: Right veering diffeomorphisms and the contact class in Heegaard-Floer homology

Gordana Matić

ABSTRACT: We study the monoid of right-veeing diffeomorphisms on a compact surface with boundary and its connection with questions of tightness and fillability of the corresponding contact structure. In the case of the pseudo-Anosov diffeomorphism we relate symplectic fillability to the work of R. Roberts. We then present an alternate description of the Ozsváth-Szabó contact class in Heegaard Floer homology. Using our version of the contact class, we prove that if a contact structure (M, ξ) has an adapted open book decomposition whose page S is a once-punctured torus, then the monodromy is right-veering if and only if the contact structure is tight.

This is a short report presenting joint work with Ko Honda and Will Kazez done in [HKM1, HKM2, HKM3] .

Thurston and Winkelnkemper [TW] introduced open books into contact geometry and showed that every open book defines an "adapted" contact structure. Let $Aut(S, \partial S)$ denote the set of isotopy classes of diffeomorphisms of a surface with boundary S which restrict to the identity on the boundary. For $h \in Aut(S, \partial S)$ we will denote by (S, h) the open book with monodromy h and by $\xi_{(S,h)}$ the adapted contact structure on the mapping torus of h. By the work of Giroux [Gi], there is a 1-1 correspondence between isomorphism classes of open book decompositions modulo positive stabilization and isomorphism classes of contact structures in dimension three.

Relying on the work of Loi and Piergallini on Stein structures [LP], Giroux showed that the contact structure ξ is Stein fillable, i.e. the contact boundary of a Stein domain, if and only if there exists an adapted open book (S, h) for which the diffeomorphism h is a composition of right Dehn twists. Stein fillable contact structures are tight, and since not all tight structures are Stein fillable it is an interesting question to see if the difference can be described from the point of view of the compatible open books.

We will say that a diffeomorphism $h \in Aut(S, \partial S)$ is *right-veering* if for any properly embedded arc α the image $h(\alpha)$ is "to the right" of α, i.e. if the tangent vectors $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation on S at x (after α and $h(\alpha)$ have been isotoped to intersect transversally and "efficiently", i.e. with a minimal number of intersections). The main result of [HKM1] is:

Theorem 1. *A contact 3-manifold (M, ξ) is tight if and only if all of its adapted open book decompositions have right-veering monodromy.*

This is an is improvement of the "sobering arc" criterion for overtwistedness given by Goodman [Go]. It is important to note that, while existence of an open book with monodromy which is a composition of right Dehn twists implies Stein fillability, the right veering of the monodromy of a single open book does not imply tightness. In fact, any open book can be positively stabilized to be right veering. On the other hand, it is not known if every open book of a Stein fillable contact structure is a product of positive Dehn twists, but it is enough to have one such to conclude Stein fillability. If we denote by $Veer(S, \partial S)$ the monoid of right-veering diffeomorphisms and by $Dehn^+(S, \partial S)$ the monoid of diffeomorphisms that are products of right Dehn twists, it is easy to see that $Dehn^+(S, \partial S) \subset Veer(S, \partial S)$. In order to understand the difference between, on the one hand, right veering monodromies and tight open books and contact structures, and on the other hand, tight and Stein fillable contact structures, we need to understand the difference between $Veer(S, \partial S)$ and $Dehn^+(S, \partial S)$. In [HKM2] we study the monoid $Veer(S, \partial S)$ of right-veering diffeomorphisms, concentrating on the case when the surface S is a once punctured torus. We exhibit the difference between $Veer(S, \partial S)$ and $Dehn^+(S, \partial S)$ with the help of the Rademacher function.

In [HKM2] we also investigate the relationship of the right-veering property of a diffeomorphism with symplectic fillability, especially in the case of pseudo-Anosov monodromy. It turns out that it is important to understand the "amount" of twisting that a diffeomorphism has near the boundary. This amount can be described by the notion of a fractional Dehn twist. If the contact structure is supported by (S, h) where h is isotopic to pseudo-Anosov diffeomorphism ψ for which the stable foliation has n prongs at the boundary, the isotopy from h to ψ moves the boundary by a fractional Dehn twist of $c = k/n$. A pseudo-Anosov diffeomorphism is right veering if and only if c is positive.

Theorem 2. *Let S be a bordered surface with connected boundary and h be pseudo-Anosov with fractional Dehn twist coefficient c. If $c \geq 1$, then the contact structure $\xi_{(S,h)}$ supported by (S, h) is isotopic to a perturbation of a taut foliation. Hence (S, h) is (weakly) symplectically fillable and universally tight if $c \geq 1$.*

The taut foliation that the contact structure is isotopic to is in fact the one constructed on the mapping torus of the diffeomorphism by Rachel Roberts [Ro1, Ro2].

Hence, when a contact structure is supported by an open book with "sufficiently" right-veering monodromy, it is symplectically fillable and therefore tight as a consequence of a theorem of Eliashberg and Gromov [El]. Unfortunately, as we said, a right-veering diffeomorphism with a small amount of rotation does not always correspond to a tight contact structure. We might optimistically conjecture that a minimal (i.e., not destabilizable) right-veering open book defines a tight contact structure. If we specialize to the case of a once-punctured torus, we can show that this is in fact true. The proof relies on use of the Ozsváth-Szabó contact class in Heegard-Floer homology.

In the paper [OS], Ozsváth and Szabó associated an element in Heegaard-Floer homology to an open book decomposition and showed that its homology class is independent of the choice of the open book compatible with the given contact structure. They also showed that this invariant $c(\xi)$ is zero if the contact structure is overtwisted, and that it is nonzero if the contact structure is symplectically fillable. The *contact class* $c(\xi)$ has proven to be extremely powerful at (i) proving the tightness of various contact structures and (ii) distinguishing tight contact structures, especially in the hands of Lisca-Stipsicz [LS1, LS2] and Ghiggini [Gh].

In [HKM3] we introduce an alternate, more hands-on, description of the contact class in Heegaard-Floer homology. It is defined using a somewhat simpler Heegard decomposition related than the one used by Ozsváth and Szabó. This approach is more suitable for direct claculation of the contact class, without resorting to surgery exact sequences. For example, it makes it absolutely obvious from the definition that a contact structure with non-right veering open book has zero contact invariant. The simplification enables us to prove directly that the contact element of a right veering diffeomorphism on the once punctured torus is non-zero, thus showing:

Theorem 3. *Let (M, ξ) be a contact 3-manifold which is supported by an open book decomposition (S, h), where S is a once-punctured torus. Then ξ is tight if and only if h is right-veering.*

Very recently John Baldwin [Ba] also obtained results similar to Theorem 3. His method relies on standard exact sequence computations in Heegard-Floer homology.

References

[Ba] J. Baldwin, *Tight contact structures and genus one fibered knots*, preprint 2006. `ArXiv:math.SG/0604580.`

[El] Y. Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), 45–67, London Math. Soc. Lecture Note Ser. **151**, Cambridge Univ. Press, Cambridge, 1990.

[Gh] P. Ghiggini, *Strongly fillable contact 3-manifolds without Stein fillings*, Geom. Topol. **9** (2005), 1677–1687 (electronic).

[Gi] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

[Go] N. Goodman, *Overtwisted open books from sobering arcs*, Algebr. Geom. Topol. **5** (2005), 1173–1195 (electronic).

[H1] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. **4** (2000), 309–368 (electronic).

[HKM1] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary I*, preprint 2005. `ArXiv:math.GT/0510639.`

[HKM2] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary II*, preprint 2006. `ArXiv:math.GT/0603626.`

[HKM3] K. Honda, W. Kazez and G. Matić, *On the contact class in Heegard Floer Homology* , preprint 2006.

[LS1] P. Lisca and A. Stipsicz, *Ozsváth-Szabó invariants and tight contact three-manifolds. I*, Geom. Topol. **8** (2004), 925–945 (electronic).

[LS2] P. Lisca and A. Stipsicz, *Ozsváth-Szabó invariants and tight contact three-manifolds, II*, preprint 2004. `ArXiv:math.SG/0404136.`

[LP] A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of* B^4, Invent. Math. **143** (2001), 325–348.

[OS] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and contact structures*, Duke Math. J. **129** (2005), 39–61.

[Ro1] R. Roberts, *Taut foliations in punctured surface bundles, I*, Proc. London Math. Soc. (3) **82** (2001), 747–768.

[Ro2] R. Roberts, *Taut foliations in punctured surface bundles, II*, Proc. London Math. Soc. (3) **83** (2001), 443–471.

[TW] W. Thurston and H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.

Lens spaces, rational balls and the ribbon conjecture

PAOLO LISCA

ABSTRACT: We apply Donaldson's theorem on the intersection forms of definite 4–manifolds to characterize the lens spaces which smoothly bound rational homology 4–dimensional balls. Our result implies, in particular, that every smoothly slice 2–bridge knot is ribbon, proving the ribbon conjecture for 2–bridge knots.

It is a well–known fact that every ribbon knot is smoothly slice. The ribbon conjecture states that, conversely, a smoothly slice knot is ribbon. We prove that the ribbon conjecture holds for 2–bridge knots, deducing this result from a characterization of the 3–dimensional lens spaces which smoothly bound rational homology 4–dimensional balls (Theorem 1 below).

A link in S^3 is called 2*–bridge* if it can be isotoped to have exactly two local maxima with respect to a standard height function. Each pair of coprime integers $p > q > 0$ determines a standard 2–bridge link $K(p,q)$ in such a way that every 2–bridge link is isotopic to $K(p,q)$ for some p and q [1]. When p is even, $K(p,q)$ is a 2–component link, when p is odd $K(p,q)$ is a knot, and $K(p,p-q)$ is isotopic to the mirror image of $K(p,q)$. The 2–fold cover of S^3 branched along $K(p,q)$ is the lens space $L(p,q)$. To state our main result we shall need the following:

Definition 1. *Let $\mathbb{Q}_{>0}$ denote the set of positive rational numbers, and define maps $f,g\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ by setting, for $\frac{p}{q} \in \mathbb{Q}_{>0}$, $p > q > 0$, $(p,q) = 1$,*

$$f\left(\frac{p}{q}\right) = \frac{p}{p-q}, \qquad g\left(\frac{p}{q}\right) = \frac{p}{q'},$$

where $p > q' > 0$ and $qq' \equiv 1 \pmod{p}$. Define $\mathcal{R} \subset \mathbb{Q}_{>0}$ to be the smallest subset of $\mathbb{Q}_{>0}$ such that $f(\mathcal{R}) \subseteq \mathcal{R}$, $g(\mathcal{R}) \subseteq \mathcal{R}$ and $\mathcal{R}$ contains the set of rational numbers $\frac{p}{q}$ such that $p > q > 0$, $(p,q) = 1$, $p = m^2$ for some $m \in \mathbb{N}$ and q is of one of the following types:

(1) *$mk \pm 1$ with $m > k > 0$ and $(m,k) = 1$;*

(2) *$d(m \pm 1)$, where $d > 1$ divides $2m \mp 1$;*
(3) *$d(m \pm 1)$, where $d > 1$ is odd and divides $m \pm 1$.*

Casson, Gordon and Conway [3] showed that every knot of the form $K(p,q)$ with $\frac{p}{q} \in \mathcal{R}$ is ribbon. The interior of any ribbon disk can be radially pushed inside the 4–ball B^4 to obtain a smoothly embedded disk, and the 2–fold cover of B^4 branched along a slicing disk for $K(p,q)$ is a smooth rational homology ball with boundary the lens space $L(p,q)$. Therefore if $K(p,q)$ is a knot (i.e. if p is odd) we have the implications:

$$(*) \qquad \frac{p}{q} \in \mathcal{R} \;\Rightarrow\; K(p,q) \text{ ribbon } \;\Rightarrow K(p,q) \text{ smoothly slice } \;\Rightarrow\; L(p,q) = \partial W,$$

where W is a smooth 4–manifold with $H_*(W;\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$. Our main result is the following.

Theorem 1. *Let $p > q > 0$ be coprime integers. Then, the following statements are equivalent:*

(1) *$\frac{p}{q}$ belongs to $\mathcal{R}$.*
(2) *There exist:*
   (a) *A surface with boundary S, homeomorphic to a disk if p is odd and to the disjoint union of a disk and a Möbius band if p is even;*
   (b) *A ribbon immersion $i\colon S \looparrowright S^3$ with $i(\partial S) = K(p,q)$.*
(3) *The lens space $L(p,q)$ smoothly bounds a rational homology ball.*

Theorem 1 immediately implies the following result, which settles the ribbon conjecture for 2–bridge knots.

Corollary 1. *The implications in $(*)$ can all be reversed. In particular, the ribbon conjecture holds for 2–bridge knots.* □

Proof of Theorem 1 (outline): To show that (1) implies (2) we use explicit ribbon moves to prove the existence of ribbon surfaces in all the required cases.

If (2) holds, an easy topological argument shows that that the 2–fold cover of B^4 branched along a copy of Σ with interior pushed inside B^4 is a rational homology ball. Therefore (2) implies (3).

To prove that (3) implies (1) we use the following idea. If a lens space $L(p,q)$ smoothly bounds a rational homology ball $W(p,q)$, one can form a smooth negative definite 4–manifold $X(p,q)$ by taking the union of $-W(p,q)$ with a canonical 4–dimensional plumbing $P(p,q)$ bounding $L(p,q)$. Since $X(p,q)$ is negative definite, Donaldson's celebrated theorem [2] implies that the intersection form $Q_{X(p,q)}$ of $X(p,q)$ is isomorphic to the standard diagonal form $\mathbf{D}^n$, where $n = b_2(X(p,q))$. Therefore there is an embedding $Q_{P(p,q)} \hookrightarrow \mathbf{D}^n$, and since $-L(p,q) = L(p,p-q)$ smoothly bounds the rational homology ball $-W(p,q)$, there is an embedding $Q_{P(p,p-q)} \hookrightarrow \mathbf{D}^m$ as well. The existence of both embeddings (it is easy to see that a single embedding is not enough) gives constraints on the pair (p,q) which eventually allow us to prove that (1) follows from (3). In contrast to the simplicity of this idea, the algebro–combinatorial machinery we must set up to work out such constraints is fairly complex. Here is the gist of what we do. We can write

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}, \qquad \frac{p}{p-q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_m}}},$$

for some integers $a_i, b_j \geq 2$ for $i = 1, \dots, n$, $j = 1, \dots, m$. It turns out that $\sum_{i=1}^n (a_i - 3) + \sum_{j=1}^m (b_j - 3) = -2$, therefore up to replacing (p, q) with $(p, p - q)$ we may assume

$$I := \sum_{i=1}^{n} (a_i - 3) < 0. \tag{0.1}$$

Choosing a suitable set of generators of $H_2(P(p, q); \mathbb{Z})$, the embedding $Q_{P(p,q)} \hookrightarrow \mathbf{D}^n$ gives rise to a subset $S = \{v_1, \dots, v_n\} \subset \mathbf{D}^n$ with

$$v_i \cdot v_j = \begin{cases} -a_i & \text{if } |i - j| = 0, \\ 1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

We call such subsets *standard*. The bulk of our work consists of analyzing the standard subsets of $\mathbf{D}^n$, $n \geq 3$, satisfying Equation (0.1). For technical reasons, in order to understand standard subsets we need to understand certain more general subsets for which the intersection numbers $v_i \cdot v_j$ are allowed to vanish when $|i - j| = 1$. We prove that such subsets are obtained from similar subsets of $\mathbf{D}^3$ via a finite sequence of operations we call *expansions*. In its turn, this requires understanding the potential obstructions coming from the fact that during a sequence of expansions a subset might develop what we call *bad components*. At the end of the day we show that for each standard subset $S \subset \mathbf{D}^n$ with $I < 0$, we have $I \in \{-1, -2, -3\}$ and S is obtained from a standard subset of $\mathbf{D}^3$ by expansions. This allows us to describe explicitely the string of integers $(a_1, \dots, a_n)$ associated to standard subsets with $I < 0$, and gives the constraints mentioned above. □

Acknowledgments: I am grateful to Andrew Casson for generous help, and to Cameron Gordon for useful e–mail correspondence.

References

[1] **G Burde, H Zieschang**, *Knots*, de Gruyter Studies in Mathematics, **5**. Walter de Gruyter & Co., Berlin, 1985.

[2] **S K Donaldson**, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26** (1987), no. 3, 397–428.

[3] **L Siebenman**, *Exercises sur les noeuds rationnels*, mimeographed notes, Orsay, 1975.

Whitney towers and link concordance

Rob Schneiderman
(joint work with Peter Teichner)

Techniques of algebraic topology and surgery theory have provided a deep understanding of the structure of manifolds of dimension greater than or equal to 5, essentially in terms of abelian invariants. The *Whitney move* is a procedure for resolving singularities which plays a critical role in the surgery program by providing a key link between "algebraic cancellation" (the vanishing of homotopy-theoretic obstructions) and "geometric cancellation" (the removal of pairs of intersections among submanifolds by a homotopy). In dimension 4 the Whitney move fails in general, since Whitney disks (which guide Whitney moves) will generically contain intersection and self-intersection points, and this failure is intimately connected to the emergence of a plethora of low-dimensional phenomena. The theory of Whitney towers provides an obstruction theoretic approach to studying low-dimensional problems by "measuring" the failure of the Whitney move. In particular, a main goal of Whitney towers is to provide invariants of immersed spheres in 4-manifolds which are obstructions to homotoping the spheres to disjoint embeddings.

A *Whitney tower* is a 2-complex of iterated Whitney disks that pair intersections between immersed surfaces (and intersections among Whitney disks) in a 4–manifold. Whitney towers are parameterized by unitrivalent trees, and the basic measure of complexity of a Whitney towers is its *order*, which corresponds to the number of trivalent vertices in its associated tree. A Whitney tower of higher order is in some sense a "better approximation" to a homotopy eliminating singularities among the underlying immersed surfaces, and an order n Whitney tower $\mathcal{W}$ has an order n *intersection tree* $\tau_n(\mathcal{W})$ representing a group element whose vanishing is sufficient for the existence of an order $n+1$ Whitney tower on the underlying immersed surfaces. Thus, a certain degree of "algebraic cancellation implies geometric cancellation" is recovered from the higher dimensional theory. This obstruction element takes values in abelian groups $\mathcal{T}_n$ generated by decorated trivalent trees modulo antisymmetry and Jacobi relations – essentially the same groups that appear in the 3-dimensional finite-type theories. The univalent vertices in the trees correspond to the underlying immersed surfaces, the trivalent vertices correspond to Whitney disks, and the edges correspond to sheet-changing paths between adjacent Whitney disks.

In the setting of Whitney towers on properly immersed disks in the 4-ball, the intersection tree gives concordance invariants of the bounding link in S^3, including the Arf and Milnor invariants. For example, if a link $L \subset S^3$ bounds an order n Whitney tower $\mathcal{W}$ in the 4-ball, then $\tau_n(\mathcal{W}) = K_n(L) \in \mathcal{T}_n \otimes \mathbb{Q}$, where $K_n(L)$ is the leading term of the tree part of the Kontsevich invariant of L. With the same hypothesis, $\tau_n(\mathcal{W})$ is equal to the first non-vanishing Milnor invariant $\mu_n(L)$ also in $T_n \otimes \mathbb{Q}$. By restricting to *non-repeating* Whitney towers, whose trees have univalent vertices corresponding to distinct immersed disks, one recovers Milnor's classification of link-homotopically trivial links: The components of L bound disjoint immersed disks in the 4-ball if and only if all non-repeating intersection trees vanish in the corresponding (torsion-free) groups.

The (first order) intersection tree also detects the Arf invariants of the components of L. If a component l_i of L bounds D_i in B^4, with all double points of D_i paired by (framed) Whitney disks, then each intersection between an interior of a Whitney disk and D_i corresponds to a Y-tree, and the Arf invariant of l_i is the sum of such Y-trees modulo 2.

In fact, Whitney towers suggest a formulation of infinite families of "higher order Arf invariants" which we conjecture are well-defined concordance invariants. The first test case is the Bing-double of the Figure eight knot, which as far as we know has not been shown not to be slice; this two component link $L = l_1 \bigcup l_2 \subset S^3$ bounds a Whitney tower in B^4 consisting of two embedded disks D_i bounded by the l_i, with an immersed (order 1) Whitney disk $W_{(1,2)}$ pairing the intersections between D_1 and D_2, and a Whitney disk $W_{((1,2),(1,2))}$ pairing the self-intersections of $W_{(1,2)}$. This (order 3) Whitney disk $W_{((1,2),(1,2))}$ has a single interior intersection with $W_{(1,2)}$ which we conjecture represents a non-trivial concordance invariant of L. The situation is quite subtle, since for instance the unpaired (order 4) intersection can be "pushed up" to order 5, or $W_{((1,2),(1,2))}$ can be embedded at the cost of destroying its framing. It is also possible that this invariant corresponds to the mod 2 reduction of an isotopy invariant contained in the loop part of the Kontsevich integral of L.

References

[1] R. Schneiderman, *Whitney towers and Gropes in 4–manifolds*, Trans. Amer. Math. Soc. 358 (2006), 4251-4278.

[2] R. Schneiderman, *Simple Whitney towers, half-gropes and the Arf invariant of a knot*, Pacific Journal of Mathematics, Vol. 222, No. 1, Nov 2005.

[3] R. Schneiderman, P. Teichner, *Higher order intersection numbers of 2-spheres in 4–manifolds*, Alg. and Geom. Topology 1 (2001) 1–29.

[4] R. Schneiderman, P. Teichner, *Whitney towers and the Kontsevich integral*, Proceedings of a conference in honor of Andrew Casson, UT Austin 2003, Geometry and Topology Monograph Series, Vol. 7 (2004), 101-134.

[5] R. Schneiderman, P. Teichner, *Pulling apart 2-spheres in 4-manifolds*, In preparation.

[6] R. Schneiderman, P. Teichner, *Grope concordance of classical links*, In preparation.

The topology of symplectic sums of four-manifolds

Michael Usher

Let (X_i, ω_i) $(i = 1, 2)$ be symplectic 4-manifolds containing embedded symplectic surfaces F_i of equal area, equal positive genus, and opposite self-intersection. Given an orientation-reversing identification ψ of the normal bundles to the F_i, a construction of R. Gompf [2] yields a natural isotopy class of symplectic structures on the normal connect sum

$$Z = X_1 \#_{F_1=F_2} X_2 = (X_1 \setminus nd(F_1)) \cup_\psi (X_2 \setminus nd(F_2)) .$$

Our aim is to make general statements about the topology of manifolds Z obtained in this fashion under modest assumptions on the F_i. The main result is:

Theorem 1. *1) Provided that each F_i meets every embedded symplectic sphere of square -1 in X_i and neither of the X_i is a ruled surface having F_i as a section, the sum Z is* minimal.
2) If neither of the X_i is a ruled surface having F_i as a section, the minimal model of Z is neither rational nor ruled.

(Note that if, say, X_2 is a ruled surface with section F_2, then for at least one choice of framing ψ $X_1 \#_{F_1=F_2} X_2$ will be diffeomorphic to X_1, which explains the necessity of one of the conditions.)

A corollary of this theorem is a conjecture of Stipsicz asserting that fiber sums of nontrivial relatively minimal Lefschetz fibrations are minimal.

The theorem is readily implied by (indeed, is equivalent to) the following statement, which directly speaking is what we prove: Under the conditions of Theorem 1, the genus-zero Gromov–Witten invariants of Z all vanish.

To prove this latter statement, we rely on the analysis carried out by E. Ionel and T. Parker [3] of pseudoholomorphic curves in Z in terms of those in the X_i. Their results imply that a nonvanishing genus-zero Gromov–Witten invariant for Z (say in the homology class A) would give rise via "pinching the neck" to trees of (potentially reducible and nonreduced) J_i-holomorphic curves in the X_i, where the J_i are almost complex structures on X_i preserving the respective TF_i. Moreover the homology classes $A_i \in H_2(X_i;\mathbb{Z})$ satisfy the relation

$$\langle \kappa_Z, A\rangle = \langle \kappa_{X_1} + PD(F_1), A_1\rangle + \langle \kappa_{X_2} + PD(F_2), A_2\rangle.$$

But considerations of virtual dimension force the left hand side above to be negative, and so the theorem follows from the following:

Lemma 1. *Let F be an embedded symplectic surface of positive genus in a symplectic 4-manifold (X,ω) which meets all embedded symplectic (-1)-spheres in X and which, in case X is a ruled surface, is not a section. Then if J is an almost complex structure making F pseudoholomorphic and $A \in H_2(X;\mathbb{Z})$ is represented by a J-holomorphic sphere, we have*

$$\langle \kappa_X + PD(F), A\rangle \geq 0.$$

When the minimal model of X is neither rational nor ruled, the above inequality is proven by appealing to results of C. Taubes [5] and (in case $b^+(X) = 1$) A.K. Liu [4] which arise from Seiberg–Witten theory and imply that the canonical class of X enjoys certain positivity properties. Of course, when X is rational or ruled, the canonical class is no longer positive; however, our inequality amounts to the statement that a symplectic surface of positive genus which passes through all (-1)-spheres and is not a section of a ruled surface is necessarily "large" enough that the sum $\kappa_X + PD(F)$ does exhibit the desired positivity. In case X is an irrational ruled surface, this can be seem by fairly direct calculations using the adjunction formula. When X is rational, the story is more subtle due to the large number of classes A that are represented by holomorphic spheres. Using the analysis of the chamber structure in the cohomology of X that was carried out by R. Friedman and J. Morgan in their study [1] of the diffeomorphism types of algebraic surfaces, however, it can be shown that the hypotheses of the lemma ensure that $\kappa_X + PD(F)$ lies in the closure of the "forward time cone" of $H^2(X)$, from which it follows fairly easily that $\kappa_X + PD(F)$ evaluates nonnegatively on all holomorphic spheres. For full details of the proof of Lemma 1 see [6].

References

[1] R. Friedman and J. Morgan. *On the diffeomorphism types of certain algebraic surfaces. I.* J. Differential Geom. **27** (1988), no. 2, 297–369.

[2] R. Gompf. *A new construction of symplectic manifolds.* Ann. Math. **142** (1995), 527–595.

[3] E. Ionel and T. Parker. *The symplectic sum formula for Gromov–Witten invariants.* Ann. Math. **159** (2004), 935–1025.

[4] A. K. Liu. *Some new applications of the general wall crossing formula, Gompf's conjecture and its applications.* Math. Res. Lett. **3** (1996), 569–585.

[5] C. Taubes. *The Seiberg–Witten and Gromov invariants.* Math. Res. Lett. **2** (1995), 221–238.

[6] M. Usher. *Minimality and symplectic sums.* To appear in Int. Math. Res. Not.

Constructing broken Lefschetz fibrations and pencils

David T. Gay

(joint work with Robion Kirby)

Let X be a smooth, compact, oriented 4-manifold. We are interested in constructing smooth maps f from X to a surface Σ maintaining tight controls on the types of singularities allowed. If X has nonempty boundary, one should also impose some appropriate boundary conditions. If there are no singularities (and appropriate boundary conditions are imposed) π is an honest fibration. Consider the following types of singularities, described by local models (the local models should respect orientations):

Definition 1. *A critical point $p \in X$ of f is a* Lefschetz singularity *if f is locally modelled near p by the map $g : (w, z) \mapsto w^2 - z^2$ from $\mathbb{C}^2$ to $\mathbb{C}$. The point p is an* anti-Lefschetz singularity *if f is locally modelled near p by $(w, z) \mapsto \overline{w}^2 - z^2$.*

Definition 2. *An embedded circle $S \subset X$ of critical points of f is a* round 1-handle singularity *if f is locally modelled near S by the map $h : (\theta, x, y, z) \mapsto (\theta, -x^2 + y^2 + z^2)$ from $S^1 \times \mathbb{R}^3$ to $S^1 \times \mathbb{R}$. Note that the genus of a fiber on one side of $f(S)$ is one higher than the genus on the other side.*

Definition 3. *If the critical set of a map f is precisely a disjoint union of Lefschetz and anti-Lefschetz singularities and round 1-handle singularities, we say the map is a* broken achiral Lefschetz map. *We omit the adjective "broken" if there are no round 1-handle singularities and we omit the adjective "achiral" if there are no anti-Lefschetz singularities.*

Definition 4. *A (broken) (achiral) Lefschetz map from a closed 4-manifold X to S^2 is called a (broken) (achiral)* Lefschetz fibration *on X.*

Definition 5. *A (broken) (achiral)* Lefschetz pencil *on a closed* 4*-manifold* X *is a nonempty finite set* $B \subset X$ *(the base locus) and a (broken) (achiral) Lefschetz map* $f : X \setminus B \to S^2$ *which is locally modelled near each point* $b \in B$ *by the canonical map* $\pi_0 : \mathbb{C}^2 \setminus (0,0) \to \mathbb{C}P^1$.

Note that if we blow up each point in the base locus of a (broken) (achiral) Lefschetz pencil we get a (broken) (achiral) Lefschetz fibration. Algebraic geometers might like us also to prepend the adjective "topological" to these definitions, to distinguish from algebraic or holomorphic Lefschetz fibrations and pencils.

The relevant history of course goes back to Lefschetz and begins in the algebraic world. However, we begin our account with Donaldson, who proved [2] that every closed symplectic 4-manifold supports a Lefschetz pencil. Gompf and Stipsicz [6] observed that there are topological obstructions to the existence of achiral Lefschetz fibrations on a given closed 4-manifold, but Etnyre and Fuller [4] proved that every closed 4-manifold, after surgery along some circle, does support an achiral Lefschetz fibration. Using the fact that any closed 4-manifold X with $b_2^+(X) > 0$ has a near-symplectic form, Auroux, Donaldson and Katzarkov [1] extended Donaldson's techniques to show that any X with $b_2^+(X) > 0$ supports a broken Lefschetz pencil (and hence, after blowups, a broken Lefschetz fibration). Note that Etnyre and Fuller's result is essentially constructive, using handlebody decompositions and open books, and relying on Giroux's results relating contact structures to open books and Eliashberg's classification of overtwisted contact structures. The Auroux-Donaldson-Katzarkov result is quite non-constructive, being essentially an analytic result. We originally set out to reprove this result constructively, in the spirit of Etnyre and Fuller.

As motivation for this program, note that Donaldson and Smith [3] defined a way of counting sections and "multi-sections" of Lefschetz fibrations which Usher [8] showed was equivalent to Taubes' Gromov-Witten invariants, and hence to the Seiberg-Witten invariants. Perutz [7] has initiated a program to extend these results to the setting of broken Lefschetz fibrations. This program depends strongly on symplectic and pseudo-holomorphic geometry and is thus not particularly tolerant of anti-Lefschetz singularities, so if we use this program as a motivation for constructive results, we should try to avoid anti-Lefschetz singularities as much as possible.

We have proved the following result:

Theorem 1. *Given any closed* 4*-manifold* X *and any surface* $F \subset X$ *with* $F \cdot F = 0$*, there exists a broken achiral Lefschetz fibration* $f : X \to S^2$ *with* F *as a fiber.*

To give an idea of how this is proved, and to state some results that are interesting in their own right, we now look at compact 4-manifolds with boundary. Here we need to impose boundary conditions, which take the form of open book decompositions. For us, an open book decomposition is a smooth map f from a 3-manifold M to a disk D such that $f^{-1}(\partial D)$ is a compact 3-dimensional submanifold on which f is a fibration over S^1 and such that $f^{-1}(D \setminus \partial D)$ is a disjoint union of open solid tori on each of which f is the projection $S^1 \times D^2 \to D^2$. Recall

Giroux's results [5] that an open book decomposition supports a unique contact structure and that any two open books supporting the same contact structure are related by "positive stabilizations" (Murasugi sums with left-handed Hopf bands). We will also work with "negative stabilizations" (Murasugi sums with right-handed Hopf bands) of open books; these change the contact structures and in particular always produce overtwisted contact structures.

Definition 6. *A* convex *(broken) (achiral) Lefschetz fibration is a (broken) (achiral) Lefschetz map f from a compact* 4*-manifold X with nonempty boundary to D^2 which restricts to an open book on ∂X. A* concave *(broken) (achiral) Lefschetz fibration is a (broken) (achiral) Lefschetz map f from a compact* 4*-manifold X with nonempty boundary to S^2 such that $f(\partial X)$ is the southern hemisphere D_- and such that $f|\partial X$ is an open book. If we allow a base locus, then we have concave (broken) (achiral) Lefschetz pencils.*

It should then be clear that if we split a closed 4-manifold X into pieces X_- and X_+, and if we construct a convex (B)(A)LF on X_- and a concave (B)(A)LF on X_+ inducing the same open book on $\partial X_- = -\partial X_+$, then the two fibrations can be glued together to yield a (B)(A)LF on X.

We have the following result:

Theorem 2. *Given any compact* 4*-manifold X with a handlebody decomposition involving only* 0*-,* 1*- and* 2*-handles, and given any open book decomposition on ∂X supporting an overtwisted contact structure, there exists a convex broken Lefschetz fibration on X restricting to a positive stabilization of the given open book on ∂X.*

Note that here we have avoided anti-Lefschetz singularities, keeping the symplectic geometers happy so far. The proof of this theorem begins with a given handlebody decomposition and is completely explicit except when appealing to Eliashberg's classification of overtwisted contact structures and Giroux's results.

On the other side we have:

Proposition 1. *Given any surface F, there exists a concave broken Lefschetz fibration on $F \times D^2$.*

Note that this is considerably less trivial than the statement that $F \times D^2$ fibers over D^2; here we are constructing a map to S^2 with constrained boundary behavior. In fact this result generalizes part of a construction in [1] of a broken Lefschetz fibration on S^4.

Theorem 1 then follows from Theorem 2, the above proposition, and lemmas to the effect that (1) if we add a 1-handle to a concave BLF, we can extend the map to S^2 across the 1-handle to produce a new concave BLF, and (2) we can change the open book on the boundary of a concave BLF by positive or negative stabilizations by changing the BLF (and not changing the manifold), except that to achieve positive stabilizations we need to introduce anti-Lefschetz singularities. This last point is when we could not avoid "achirality".

References

[1] **Denis Auroux**, **Simon K Donaldson**, **Ludmil Katzarkov**, *Singular Lefschetz pencils*, Geom. Topol. 9 (2005) 1043–1114 (electronic)

[2] **S K Donaldson**, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. 53 (1999) 205–236

[3] **Simon Donaldson**, **Ivan Smith**, *Lefschetz pencils and the canonical class for symplectic four-manifolds*, Topology 42 (2003) 743–785

[4] **John B Etnyre**, **Terry Fuller**, *Realizing 4-manifolds as achiral Lefschetz fibrations*, Int. Math. Res. Not. (2006) Art. ID 70272, 21

[5] **Emmanuel Giroux**, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, from: "Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)", Higher Ed. Press, Beijing (2002) 405–414

[6] **Robert E Gompf**, **András I Stipsicz**, 4-*manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI (1999)

[7] **Timothy Perutz**, *Surface-fibrations, four-manifolds, and symplectic Floer homology*, Ph.D. thesis, University of London (2005)

[8] **Michael Usher**, *The Gromov invariant and the Donaldson-Smith standard surface count*, Geom. Topol. 8 (2004) 565–610 (electronic)

Parallelizable four-manifolds

D. Kotschick

In spite of our growing knowledge of smooth four-manifolds, we actually know very little about closed parallelizable four-manifolds. For example, we still do not know whether the four-torus admits an exotic smooth structure. In this talk I presented examples of compact parallelizable four-manifolds with distinct smooth structures, and gave some applications of their construction. I also discussed geometric structures on parallelizable four-manifolds.

1. Exotic smooth structures

Here are the first examples of exotic smooth structures on closed parallelizable four-manifolds:

Theorem 1. ([2]) *If k is odd and large enough, then there are infinitely many pairwise non-diffeomorphic smooth manifolds Y_k homeomorphic but not diffeomorphic to*

$$X_k = k(S^2 \times S^2)\#(1+k)(S^1 \times S^3)\ .$$

After a single stabilization with $S^2 \times S^2$ all the Y_k become standard.

The exotic Y_k arise from exotic, in fact, symplectic, manifolds Z_k homeomorphic to $k(S^2 \times S^2)$. These can be constructed in such a way as to dissolve after a single stabilization. The Y_k are then constructed from Z_k by summation with copies of $S^1 \times S^3$. After this summation the numerical Seiberg–Witten invariants vanish, but the homological invariants corresponding to moduli spaces of positive dimension still distinguish the Y_k from each other, and from X_k. The Y_k inherit the property of dissolving after a single stabilization.

Similar constructions can be made in other situations to produce exotic manifolds that are known to dissolve after a single stablization. There is still no example of an exotic four-manifold known to need more than one stabilization with $S^2 \times S^2$ to become standard. Yang–Mills theory offers some potential tools to attack this question, but these tools have no analog in Seiberg–Witten theory.

2. Geometry on parallelizable four-manifolds

2.1. **Minimal volume and circle actions.** The construction in the proof of Theorem 1 arose from considerations of minimal volumes and entropies in [2]. On the one hand, the standard smooth structure on X_k was shown to admit a smooth free circle action, implying that its minimal volume in the sense of Gromov vanishes. On the other hand, the existence of generic monopole classes in the sense of [1] on the Y_k shows that these exotic manifolds do not collapse with bounded scalar curvature, in particular their minimal volumes are strictly positive. Thus one obtains:

Corollary 1. ([2]) *The minimal volume is not invariant under homeomorphisms. Even the vanishing or nonvanishing of the minimal volume is not preserved by homeomorphisms.*

To have vanishing minimal volume, one has to consider manifolds with vanishing real characteristic numbers, so one is almost forced to look at parallelizable manifolds.

These examples also show that the existence of smooth fixed-point-free circle actions does depend on the smooth structure.

After the Oberwolfach meeting I realized that these considerations are very much related to Perelman's invariant of four-manifolds, cf. [3].

2.2. **Engel structures.** Most parallelizable four-manifolds do not have complex or symplectic structures. However, there are interesting geometric structures that do exist on parallelizable four-manifolds, and that might be useful in their investigation. These are the Engel structures. They are closely related to contact structures on three-manifolds. For their definition we first have to define even contact structures.

Definition 1. *An even contact structure on a $2n$-dimensional manifold M is a maximally non-integrable smooth hyperplane field $\mathcal{E}$.*

In dimension four, that is for $n = 2$, every closed manifold with zero Euler number admits an even contact structure by the h-principle (Gromov, McDuff). For this and other reasons it seems that even contact structures may not be geometrically interesting.

Definition 2. *An Engel structure on a 4-dimensional manifold M is a smooth rank 2 distribution $\mathcal{D}$ with the property that $[\mathcal{D}, \mathcal{D}]$ is an even contact structure $\mathcal{E}$.*

Every C^2 small perturbation on an Engel structure is again an Engel structure, moreover a generic rank 2 distribution on a four-manifold is Engel almost everywhere. In the classification of stable germs of distributions due to Montgomery, Engel structures occupy a special place, in that they are the only sporadic entry in the list, whose other entries are line fields, contact structures and even contact structures. This is one of several motivations for the study Engel structures.

It is not hard to show that an orientable four-manifold admitting an orientable Engel structure is parallelizable. Conversely, one has the following recent existence theorem:

Theorem 2 (Vogel [4]). *Every parallelizable four-manifold admits an orientable Engel structure.*

This can not be proved by applying convex integration or Gromov's h-principle, although the conclusion can of course be interpreted as an h-principle. Vogel's proof [4] is constructive, using round handle decompositions to build Engel structures. In this construction contact structures on the boundaries of round handles arise, and here Vogel uses a lot of the modern machinary of contact topology. Perhaps one of the reasons Engel structures have not been studied much before is that there was a lack of interesting examples on closed manifolds. Now that we have Vogel's existence theorem, it remains to be seen what Engel structures can tell us about four-dimensional topology. For example, we would like to know how the (non-empty!) classification of Engel structures on a parallelizable four-manifold depends on the underlying smooth structure.

References

[1] D. Kotschick, *Monopole classes and Einstein metrics*, Intern. Math. Res. Notices **2004** no. 12 (2004), 593–609.

[2] D. Kotschick, *Entropies, volumes, and Einstein metrics*, Preprint arXiv:math.DG/0410215 v1 8Oct2004.

[3] D. Kotschick, *Monopole classes and Perelman's invariant of four-manifolds*, Preprint arXiv:math.DG/0608504 v1 20Aug2006.

[4] T. Vogel, *Existence of Engel structures*, Preprint arXiv:math.GT/0411217, to appear in Ann. of Math. (2).

Reporter: Rosa Sena-Dias

Participants

Prof. Dr. Selman Akbulut
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
USA

Prof. Dr. Denis Auroux
MIT
Department of Mathematics
Room 2-169
77 Massachusetts Avenue
Cambridge MA 02139-4307
USA

Prof. Dr. Stefan Alois Bauer
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld

Michael Bohn
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Dr. David Cimasoni
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Prof. Dr. Tim D. Cochran
Dept. of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
USA

Andrew Cotton
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Prof. Dr. Sergey Finashin
Dept. of Mathematics
Middle East Technical University
06531 Ankara
TURKEY

Prof. Dr. Ronald A. Fintushel
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
USA

Dr. Stefan Klaus Friedl
Dept. of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
USA

Dr. Kim A. Froyshov
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Prof. Dr. Terry Fuller
Department of Mathematics
California State University at
Northridge
Northridge CA 91330-8313
USA

Prof. Dr. Mikio Furuta
Graduate School of
Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Dr. David T. Gay
Department of Mathematics and
Applied Mathematics
University of Cape Town
Private Bag X3
Rondebosch 7701
SOUTH AFRICA

Prof. Dr. Robert Gompf
Department of Mathematics
University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
USA

Julia Elisenda Grigsby
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Prof. Dr. Ian Hambleton
Department of Mathematics and
Statistics
McMaster University
1280 Main Street West
Hamilton, Ont. L8S 4K1
CANADA

Mark Hamilton
Mathematisches Institut
Universität München
Theresienstr. 39
80333 München

Prof. Dr. Shelly Harvey
Dept. of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
USA

Dr. Matthew Hedden
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
USA

Dr. Benjamin Himpel
Mathematisches Institut
Universität Bonn
Beringstr. 6
53115 Bonn

Dr. Slaven Jabuka
Department of Mathematics
University of Nevada
Reno, NV 89557
USA

Dr. Taehee Kim
Department of Mathematics
Konkuk University
1 Hwayang-dong,
Gwangjin-gu
Seoul
KOREA

Prof. Dr. Robion C. Kirby
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Prof. Dr. Paul A. Kirk
Dept. of Mathematics
Indiana University of Bloomington
Rawles Hall
Bloomington, IN 47405-5701
USA

Prof. Dr. Dieter Kotschick
Mathematisches Institut
Universität München
Theresienstr. 39
80333 München

Prof. Dr. Dr. h.c. Matthias Kreck
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Slava Krushkal
Dept. of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
USA

Prof. Dr. Paolo Lisca
Dip. di Matematica "L.Tonelli"
Universita di Pisa
Largo Bruno Pontecorvo,5
I-56127 Pisa

Prof. Dr. Thomas E. Mark
Dept. of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
USA

Prof. Dr. Gordana Matic
Department of Mathematics
University of Georgia
Athens, GA 30602
USA

Prof. Dr. Grigory Mikhalkin
Department of Mathematics
University of Toronto
40 St.George Street
Toronto, Ont. M5S 2E4
CANADA

Dr. Alexandra Mozgova
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Tomasz S. Mrowka
Department of Mathematics
MIT
77 Massachusetts Avenue
Cambridge, MA 02139-4307
USA

Prof. Dr. Kent Orr
Dept. of Mathematics
Indiana University at Bloomington
Swain Hall East
Bloomington, IN 47405
USA

Prof. Dr. Jongil Park
Department of Mathematical Sciences
Seoul National University
San 56-1, Sillim-dong
Gwanak-gu
Seoul 151-747
KOREA

Dr. Tim Perutz
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Frank S. Quinn
Department of Mathematics
Virginia Polytechnic Institute and
State University
Blacksburg, VA 24061-0123
USA

Dr. Jacob A. Rasmussen
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
USA

Prof. Dr. Daniel Ruberman
Department of Mathematics
Brandeis University
Waltham, MA 02454-9110
USA

Prof. Dr. Nikolai Saveliev
Dept. of Mathematics
University of Miami
P.O. Box 249085
Coral Gables, FL 33124-4250
USA

Prof. Dr. Rob Schneiderman
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395
USA

Konrad Schöbel
LATP, UMR-CNRS 6632
Centre de Mathematiques et Inform.
Universite de Provence, Aix-Mars.I
39, rue Joliot Curie
F-13453 Marseille Cedex 13

Rosa Sena-Dias
Dept. of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Prof. Dr. Bernd Siebert
Mathematisches Institut
Universität Freiburg
Eckerstr.1
79104 Freiburg

Dr. Ivan Smith
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Ronald J. Stern
Department of Mathematics
University of California at Irvine
Irvine, CA 92697-3875
USA

Prof. Dr. Andras Stipsicz
Alfred Renyi Institute of
Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Prof. Dr. Saso Strle
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
1000 Ljubljana
Slovenia

Prof. Dr. Peter Teichner
Department of Mathematics
University of California
Berkeley, CA 94720-3840
USA

Dr. Michael Usher
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
USA

Prof. Dr. Stefano Vidussi
Dept. of Mathematics
University of California
Riverside, CA 92521-0135
USA

Prof. Dr. Oleg J. Viro
Matematiska institutionen
Uppsala Universitet
Box 480
S-751 06 Uppsala

Prof. Dr. Kevin Walker
Microsoft Project Q
c/o Kavli Institute for Theoretical
Physics - UC Santa Barbara
Kohn Hall
Santa Barbara, CA 93106
USA

Prof. Dr. Jean-Yves Welschinger
Unite de Mathematiques Pures et
Appliquees
Ecole Normale Superieure de Lyon
46 Allee d'Italie
F-69364 Lyon Cedex 07

Raphael Zentner
Laboratoire d'analyse, de topologie
et de probabilites
Universite de Provence
F-13013 Marseille