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Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

Organised by
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Abstract. Geometric Group theory, Hyperbolic Dynamics and Symplectic Geometry are young and rapidly developing fields of mathematics. The scientific goal of the workshop was twofold: first, to present some recent breakthroughs, and, second, to discuss and digest a surprisingly fruitful interaction of these seemingly remote mathematical disciplines.

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Introduction by the Organisers

The workshop *Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry*, organized by Gerhard Knieper, Leonid Polterovich and Leonid Potyagailo brought together leading mathematicians working in these rapidly developing fields. The scientific goal of the workshop was twofold: first, to present some recent breakthroughs, and, second, to discuss and digest a surprisingly fruitful interaction of these seemingly remote mathematical disciplines. The topics of talks included rigidity of group actions (A. Katok, D. Witte Morris, M. Kanai), ergodic properties of group actions (U. Hamenstädt, V. Kaimanovich), dynamics on homogeneous spaces (S. Mozes), hyperbolic groups and hyperbolic manifolds (G. Courtois, F. Dahmani, M. Kapovich, I. Mineyev, N. Peyerimhof), hyperbolic dynamics (D. Kotschick, G. Paternain, M. Pollicott), geodesic flows (H. Koehler, E. Leschinsky), asymptotic geometry of finitely generated groups and Lie groups (E. Breuillard, G. Margulis).

Interestingly enough, quasi-morphisms – a group theoretical notion which was originated in the framework of bounded cohomology theory, arose in the lectures of several researchers in quite a different context. Quasi-morphisms were discussed by M. Burger and A. Iozzi from the viewpoint of representation theory and by D. Calegari and K. Fujiwara who studied commutator length on hyperbolic groups and mapping class groups. At the same time M. Entov, G. Ben Simon and P. Py discussed their appearance and applications in the framework of symplectic topology.

The workshop witnessed another interesting trend. It seems that geometric group theory is trying to conquer a new area: bi-invariant geometries on groups of diffeomorphisms were discussed by D. Burago and D. McDuff.

Some talks reflected a fruitful application of a powerful tool of modern symplectic topology, Floer homology, to "non-traditional" (from the symplectic viewpoint) areas such as the study of topological entropy of Reeb flows (F. Schlenk) and theory of the mapping class groups (A. Fel'shtyn). Some very recent developments in the foundation of Floer theory were presented by O. Cornea.

An important feature of the workshop was the presence of many excellent students from several countries who had an indispensable opportunity to profit from the intense interaction with the established mathematicians. In fact the students contributed a lot to the success of the workshop by active participation in the discussions. Some of the students gave high quality talks on their results during informal "late night" sessions.

One of the hightlights of the meeting was the open problem session on Wednesday night, skillfully moderated by Anatole Katok, where many participants formulated questions and research programs which, as we believe, will develop our fields.

Workshop: Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

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Measure rigidity beyond uniform hyperbolicity

Anatole Katok

This is a report on the recent and on-going work with Boris Kalinin and Federico Rodriguez Hertz in various combinations.

A geometric approach to measure rigidity was introduced in [5]. It is based on the study of conditional measures on various invariant foliations for the action. Over the last decade it has been successfully applied to the study of invariant measures of algebraic actions.

We make the first step in extending measure rigidity from algebraic actions to the general non-uniformly hyperbolic case, i.e. to positive entropy ergodic invariant measures for actions of higher rank abelian groups all of whose Lyapunov characteristic exponents do not vanish. Such measures are usually called *hyperbolic measures.*

The main thrust of our work is to show that under certain conditions of either homotopical or geometric/dynamical nature an invariant measure for a smooth action of a higher rank abelian group is forced to be absolutely continuous. No similar phenomena appear for classical cases of diffeomorphism and flows. Specifically we consider two situations when the rank of the action is the maximal possible compatible with non-uniform hyperbolicity:

Theorem 1. [2] *Every C^2 action α of $\mathbb{Z}^k$, $k \geq 2$, on the $(k+1)$-dimensional torus whose elements are homotopic to corresponding elements of an action α_0 by hyperbolic linear maps has exactly one invariant measure which is projected to Lebesgue measure by the semiconjugacy between α and α_0. This measure is absolutely continuous and the semiconjugacy provides a measure-theoretic isomorphism. The semiconjugacy has certain monotonicity properties and pre-images of all points are connected.*

Furthermore, there are many periodic points for α for which eigenvalues for α and α_0 coincide.

Theorem 2. [1, 4] *Let μ be an ergodic invariant measure for C^2 action α of $\mathbb{Z}^k$, $k \geq 2$, on a $(k+1)$-dimensional manifold. Assume that at least one element of the action has positive entropy and the all Lyapunov hyperplanes (kernels of the Lyapunov characteristic exponents) are different. Then μ is absolutely continuous.*

Possible extensions and generalizations of those results are discussed in my contribution to the Oberwolfach problem session [3]

Scheme of proof of Theorem 1

Part 1. Every leaf of the Lyapunov foliation $\tilde{\mathcal{W}}^{-}_{\mathcal{C}_i}$ has smooth α-invariant affine structure and if $\mu \in \mathcal{M}$ is ergodic then for μ almost every leaf the semiconjugacy maps the leaf affinely onto a line in $\mathbb{R}^{k+1}$. Furthermore, conditional measures induced by μ on leaves of $\tilde{\mathcal{W}}^-_{\mathcal{C}_i}$ are given by constant densities with respect to the affine structure.

This already implies that μ is absolutely continuous.

Step 1. For any element $\mathbf{m} \in \mathbb{Z}^k \setminus \{0\}$ the following inclusions hold

$$h(\tilde{\mathcal{W}}^-_{\alpha(\mathbf{m})}(x)) \subset \mathcal{W}^-_{\alpha_0(\mathbf{m})}(hx) \text{ and } h(\tilde{\mathcal{W}}^+_{\alpha(\mathbf{m})}(x)) \subset \mathcal{W}^+_{\alpha_0(\mathbf{m})}(hx),$$

on the set of full measure μ where $\tilde{\mathcal{W}}^-_{\alpha(\mathbf{m})}(x)$ and $\tilde{\mathcal{W}}^+_{\alpha(\mathbf{m})}(x)$ exist.

Step 2. The Lyapunov half-spaces and Weyl chambers for α with respect to the measure μ are the same as the Lyapunov half-spaces and Weyl chambers for α_0.

Hence the Lyapunov exponents for α can be numbered $\tilde{\chi}_i,\ i = 1, \dots, k+1$ so that $\tilde{\chi}_i = c_i\chi_i$ where c_i is a positive scalar.

Later on we will show that $c_i = 1$

For every Lyapunov exponent $\tilde{\chi}_i$ its Lyapunov distribution integrates μ a.e. to an invariant family of one-dimensional manifolds *the Lyapunov foliation corresponding to* $\tilde{\chi}_i$.

The semiconjugacy h maps these local (global) manifolds to the local (global) affine integral manifolds for the exponents χ_i.

Step 3. Existence and uniqueness of smooth invariant affine structures on the leaves of Lyapunov foliations

Step 4. Uniform growth estimates along the walls of the Weyl chambers

Step 5. Ergodicity along the walls of Weyl chambers (the π-partition trick).

Step 6. Invariance and absolute continuity of conditional measures on the Lyapunov foliations.

Step 7. Rigidity of entropy

Step 8 Ergodicity of singular elements and rigidity of the expansion coefficients.

Part 2

Step 1. Every leaf of the codimension one foliation $\tilde{\mathcal{W}}^+_{\mathcal{C}_i}$ has a smooth α invariant affine structure and for almost every leaf the semiconjugacy maps the leaf affinely onto a codimension one linear subspace in $\mathbb{R}^{k+1}$. Such a leaf is a smooth embedded codimension-one submanifold of $\mathbb{R}^{k+1}$. Call such leaves *good leaves* and any regular point on a good leaf a *good point.*

Step 2. The semiconjugacy preserves the closed half-spaces into which any good leaf divides $\mathbb{R}^{k+1}$.

Step 3. The order in the set of good leaves of $\tilde{\mathcal{W}}^+_{\mathcal{C}_i}$ is given by the order of their unique intersection points with almost any leaf of the complementary Lyapunov foliation $\tilde{\mathcal{W}}^-_{\mathcal{C}_i}$.

Step 4. Regular boxes are constructed as intersections of "slices" between two good leaves of $\tilde{\mathcal{W}}^+_{\mathcal{C}_i}$ for $i = 1, \dots, k+1$.

Step 5. Almost every point belongs to an arbitrary small regular box.

In fact, the pre-image of every point under the semi-conjugacy is a countable intersection of regular boxes and is hence connected.

In the proof arguments of three kinds are used:

- *Geometric:* Existence of invariant densities and affine structures; those are fairly general and are based on contraction estimates for various elements of the action and on commutativity.

  Properly modified versions of those arguments extend to considerably more general situations than actions with Cartan homotopy data.
- *Ergodic:* Smoothness of conditional measures and rigidity of affine structures almost everywhere. Those arguments essentially use recurrence. One-dimensionality of Lyapunov foliations is not essential but uniform C^0 bounds along certain directions are.

  These bounds rely on existence of semi-conjugacy with a linear action. Their absence in the more general case requires essentially new ideas in the proof of Theorem 2.
- *Topological:* Separation of the space into invariant blocks by codimension one manifolds. Here existence of at least one codimension one invariant foliation is essential.

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Positive topological entropy of Reeb flows

Felix Schlenk
(joint work with Leonardo Macarini)

A *contact manifold* is a pair (P, ξ) consisting of a smooth manifold P and a maximally non-integrable codimension-1-distribution of the tangent bundle TP. Assume that ξ is co-orientable in the sense that $\xi = \ker \alpha$ for a globally defined 1-form α on P. Such a 1-form α is called a *contact form.* The maximal non-integrability of ξ then translates to $\alpha \wedge (d\alpha)^n$ being a volume form on the $(2n+1)$-dimensional manifold P. To α one associates a dynamical system in the following way: The equations

$$d\alpha\,(\cdot, R) = 0, \quad \alpha(R) = 1$$

determine a vector field R on P, called the *Reeb field*. Its flow φ_R^t is the *Reeb flow*. A prominent example is the geodesic flow on the unit cotangent bundle over a Riemannian manifold. For a nowhere vanishing function $f\colon P \to \mathbb{R}$ we also have $\xi = \ker(f\alpha)$. The dynamics of the Reeb flows associated with α and $f\alpha$ can be very different, however.

With a smooth manifold M one associates its spherization (SM, ξ). This contact manifold can be thought of as the unit cosphere bundle over M (with respect to some Riemannian metric on M) together with the contact structure $\ker \sum_i p_i\, dq_i$. Choosing a contact form α on (SM, ξ) is the same thing as choosing a hypersurface $\Sigma \subset T^*M$ such that $\Sigma \cap T_q^*M$ is *starshaped* with respect to the origin 0_q for each $q \in M$.

Theorem. *Assume that M is a closed manifold which is energy-hyperbolic. Then $h_{top}(\varphi_R^t) > 0$ for any Reeb flow φ_R^t on (SM, ξ).*

Here, the *topological entropy* $h_{top}(\varphi_R^t)$ is a measure for the orbit complexity of the flow φ_R^t, see [4] for the definition and more information on topological entropy. "Energy-hyperbolic" means that the homology of the based loop space grows exponentially. To be more precise, fix a point $q_0 \in M$ and consider the based loop space

$$\Omega(M, q_0) \;=\; \left\{ q(t)\colon S^1 \to M,\ q(1) = q_0,\ q \text{ is piecewise smooth} \right\},$$

and for a fixed Riemannian metric g on M let

$$\mathcal{E}_g\colon \Omega(M, q_0) \to [0, \infty), \quad q(t) \mapsto \frac{1}{2}\int_0^1 |\dot q(t)|_g^2 dt$$

be the energy functional. For $a \geq 0$ set $\mathcal{E}_g^a = \{q \in \Omega(M, q_0) \mid \mathcal{E}_g(q) \leq a\}$ and $\iota^a\colon \mathcal{E}_g^a \hookrightarrow \Omega(M, q_0)$. Then M is said to be *energy-hyperbolic* if

$$\sup_{\mathbb{K}} \liminf_{m\to\infty} \frac{1}{m} \log \dim \iota_*^{\frac{m^2}{2}} H_*\left(\mathcal{E}_g^{\frac{m^2}{2}}; \mathbb{K}\right) \;>\; 0$$

where the supremum is taken over all fields $\mathbf{K}$. "Most" closed manifolds are energy-hyperbolic. In particular, rationally hyperbolic manifolds and manifolds whose fundamental group grows exponentially are energy-hyperbolic.

For the special case of geodesic flows or Finsler flows (corresponding to fiberwise *convex* hypersurfaces $\Sigma \subset T^*M$), the theorem was proved by Gromov [3], Paternain [6] and Paternain–Petean [7] using Morse theory, and later on in [2] using Lagrangian Floer homology. Our proof of the above theorem again uses Floer homology for Lagrangian intersections, which is used to translate the homological growth in the based loop space to the volume growth of certain Lagrangian submanifolds in T^*M. This transition relies on recent work of Abbondandolo and Schwarz and on various stability properties of filtered Floer homology. The positivity of $h_{top}(\varphi_R^t)$ then follows from Yomdin's theorem in [8].

Consider again a closed energy-hyperbolic manifold M, let $\Sigma \subset T^*M$ be a hypersurface enclosing the zero-section 0_M, and choose a smooth Hamiltonian function $H\colon T^*M \to [0, \infty)$ such that $H(0_M) = 0$ and $H^{-1}(1) = \Sigma$ with 1 a regular value. We denote by φ_H^t the Hamiltonian flow of H. If Σ is fiberwise starshaped, our theorem says that $h_{top}(\varphi_H^t|_\Sigma) > 0$. Indeed, the flow $\varphi_H^t|_\Sigma$ is a time-change of the Reeb flow on Σ, and positivity of topological entropy does not change under time-change. The hypersurface Σ is said to be of *restricted contact type* if there exists a 1-form α on T^*M such that $d\alpha = \sum_i dp_i \wedge dq_i$ and such that $\alpha|_\Sigma$ is a contact form. Fiberwise starshaped hypersurfaces are of this sort (take $\alpha = \sum_i p_i\, dq_i$). Contrary to the property "fiberwise starshaped", the property "restricted contact type" is invariant under symplectomorphisms of T^*M.

Open Problem. Does the above theorem extend to hypersurfaces of restricted contact type, i.e.: $h_{top}(\varphi_H^t|_\Sigma) > 0$ provided that Σ is of restricted contact type?

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Convex projective structures on Gromov–Thurston manifolds

Michael Kapovich

Gromov and Thurston in [3] constructed, for each $n \geq 4$, examples of compact n-manifolds which admit metrics of negative curvature, with arbitrarily small pinching constants, but do not admit metrics of constant curvature. The main goal of this paper is to put *convex projective structures* on Gromov–Thurston examples. Suppose that $\Omega \subset \mathbb{R}P^n$ is an open subset and $\Gamma \subset PGL(n+1,\mathbb{R})$ is a subgroup acting properly discontinuously on Ω. The quotient orbifold $Q = \Omega/\Gamma$ has natural projective structure c. The structure c is said to be *(strictly) convex* if Ω is a (strictly) convex proper subset of $\mathbb{R}P^n$. In this case we refer to Q as (strictly) convex projective orbifold. Our main result then is:

Theorem 1. *Gromov-Thurston examples admit strictly convex projective structures.*

Corollary 2. *For each $n \geq 4$ there exists a compact n–manifold M with a strictly convex projective structure, so that M is not homotopy–equivalent to a hyperbolic n–manifold.*

This theorem is proven in [4] via "bending" of the original hyperbolic structure on a certain hyperbolic n–manifold M' (used to as a building block for the construction of Gromov-Thurston examples).

There are two parts in this proof: (1) Producing a projective structure, (2) proving that the structure is convex. Then strict convexity of the structure follows from Benoist's theorem below (Theorem 3), since Gromov-Thurston examples have Gromov-hyperbolic fundamental groups.

Part (1) is dealt with by solving a certain *product of matrices* problem, which is a special case of a Lie-theoretic problem interesting on its own right:

Let $G_i, i = 1, ..., n$ be maximal tori in a simple Lie group G. Determine the image of the map

$$\prod_{i=1}^{n} G_i \to G, (g_1, ..., g_n) \mapsto g_1 \dots g_n.$$

The projective manifolds M' are then built by gluing convex subsets of the hyperbolic manifolds M. By passing to the universal cover we obtain a tessellation of $\tilde{M}'$ by convex polyhedra in $\mathbb{H}^n$, each of which has infinitely many facets.

Dealing with (2) is especially interesting, since, at present, there is only one general method for proving convexity of projective structures, namely via Vinberg–Tits fundamental domain theorem [5]. Unfortunately, this theorem applies only to reflection groups, which cannot be used in higher dimensions. Our approach to proving convexity is to adapt Vinberg's arguments in a more general context of manifolds obtained by gluing convex cones with *infinitely many faces.* In this setting, Vinberg's arguments (requiring polyhedrality of the cones) do not directly apply and we modify them by appealing to the *small cancelation theory.*

The main motivation for Theorem 1 comes from the following beautiful

Theorem 3. *(Y. Benoist,* [2]*) Suppose that a convex projective orbifold M is compact. Then M is strictly convex iff $\Gamma = \pi_1(M)$ is Gromov-hyperbolic.*

Examples of convex-projective structures on compact orbifolds are provided by the quotients of round balls in $\mathbb{R}P^n$ by discrete cocompact groups of automorphisms. The Hilbert metric on such examples is a Riemannian metric of constant negative sectional curvature. Thus such orbifolds are *hyperbolic.* By deforming the above examples in $\mathbb{R}P^n$ one obtains other examples of strictly convex projective manifolds/orbifolds.

We construct examples of compact strictly convex projective manifolds which are not obtained by deforming hyperbolic examples. Independently, such examples were constructed by Yves Benoist in dimension 4 using reflection groups, see [1]. The paper [1] also produces "exotic" strictly convex subsets Ω in $\mathbb{R}P^n$ for all $n \geq 3$: The metric space (Ω, d_H) is Gromov-hyperbolic but is not quasi-isometric to $\mathbb{H}^n$, where d_H is the Hilbert metric on Ω. However these examples do not appear to admit discrete cocompact groups of automorphisms.

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The cohomological equation for magnetic flows and thermostats

Gabriel P. Paternain
(joint work with Nurlan Dairbekov)

Let M be a closed manifold endowed with a Riemannian metric g. We consider a *generalized isokinetic thermostat*. This consists of a semibasic vector field $E(x,v)$ (the external field), that is, a smooth map $TM \ni (x,v) \mapsto E(x,v) \in TM$ such that $E(x,v) \in T_xM$ for all $(x,v) \in TM$. The equation

$$\frac{D\dot\gamma}{dt} = E(\gamma,\dot\gamma) - \frac{\langle E(\gamma,\dot\gamma),\dot\gamma\rangle}{|\dot\gamma|^2}\,\dot\gamma.$$

defines a flow ϕ on the unit sphere bundle SM.

Suppose now that M is a closed oriented surface. We can write

$$E(x,v) = \kappa(x,v)v + \lambda(x,v)iv$$

where i indicates rotation by $\pi/2$ according to the orientation of the surface and κ and λ are smooth functions. The evolution of the thermostat on SM can now be written as

$$\frac{D\dot\gamma}{dt} = \lambda(\gamma,\dot\gamma)\, i\dot\gamma. \tag{1}$$

If λ does not depend on v, then ϕ is the *magnetic flow* associated with the magnetic field $\Omega := \lambda\Omega_a$, where Ω_a is the area form of M. Of course, magnetic flows are Hamiltonian. If λ depends linearly on v, we obtain the *isokinetic Gaussian thermostat* which has served as a very useful model in non-equilibrium statistical mechanics.

Let $\pi : SM \to M$ be the canonical projection.

Main Theorem [2]**.** *Let M be a closed oriented surface and consider a generalized isokinetic thermostat (1). Suppose the flow ϕ is Anosov and let F be the vector field generating ϕ. Let $h \in C^\infty(M)$ and let θ be a smooth 1-form on M. Then the cohomological equation*

$$F(u) = h \circ \pi + \theta$$

has a solution $u \in C^\infty(SM)$ if and only if $h = 0$ and θ is exact.

The Theorem has two consequences.

The entropy production of the SRB measure ρ is given by

$$e_\phi(\rho) := -\int \mathrm{div} F\, d\rho = -\sum \text{Lyapunov exponents} \geq 0$$

where $\mathrm{div} F$ is the divergence of F with respect to any volume form in SM.

Theorem A [2]**.** *An Anosov Gaussian thermostat on a closed surface has zero entropy production if and only if the external field E has a global potential.*

A system with $e_\phi(\rho) > 0$ is referred to as *dissipative*. Dissipative Gaussian thermostats provide a large class of examples to which one can apply the Fluctuation Theorem of Gallavotti and Cohen.

Let E^s and E^u be the strong stable and unstable bundles of ϕ.

Theorem B [1]**.** *Let M be a closed oriented surface endowed with a Riemannian metric g and let Ω be an arbitrary 2-form. Suppose that the magnetic flow ϕ of the pair (g, Ω) is Anosov. We have:*

(1) *If Ω is exact, then $E^s \oplus E^u$ is C^1 if and only if Ω vanishes identically, i.e. ϕ is a geodesic flow;*
(2) *If Ω is non-exact, then $E^s \oplus E^u$ is C^1 if and only if the Gaussian curvature is constant and Ω is a constant multiple of the area form.*

The Main Theorem also holds for magnetic flows in arbitrary dimensions [3].

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Surface group representations with maximal Toledo invariant

Marc Burger

(joint work with Alessandra Iozzi, Anna Wienhard)

Let Σ_g be a compact oriented surface of genus $g \geq 2$, $\Gamma_g = \pi_1(\Sigma_g)$ its fundamental group and G a Lie group. This talk report on some aspects of the general theme which is the study of $\mathrm{Hom}(\Gamma_g, G)$. There are two objectives:

(1) Find components in $\mathrm{Hom}(\Gamma_g, G)$ analogous to Teichmüller space.
(2) Interpret the representations in these components in terms of geometric structures on Σ_g.

The above topics have been considered by various authors for various classes of (semisimple) Lie groups G, generalizing the case $G = \mathrm{PU}(1,1)$. Here we report on recent work done in the case where G is of hermitian type, that is G is semisimple connected with finite center and its associated symmetric space $\mathcal{X}$ carries a G-invariant complex structure. In this case one associates to every representation $\rho : \Gamma_g \to G$ its Toledo invariant $\mathrm{T}(\rho)$ constructed using the Kähler form on $\mathcal{X}$. We recall them the basic properties of $\mathrm{T}(\rho)$, the Milnor–Wood type inequality $|\mathrm{T}(\rho)| \leq |\chi(\Sigma_g)| \operatorname{rank}(\mathcal{X})$, and the fact that $\mathrm{T}(\rho)$ is locally constant on $\mathrm{Hom}(\Gamma_g, G)$. Then we introduce maximal representations, that is the ρ's for which $|\mathrm{T}(\rho)| = |\chi(\Sigma_g)| \operatorname{rank}(\mathcal{X})$, and state a result due to Burger–Iozzi–Wienhard [6] which says that maximal representations are injective with discrete image. Then we proceed with the question of how "large" $\rho(\Gamma_g)$ can be when ρ is maximal. This question is completely answered by a result of [6] giving complete information on the Zariski closure of $\rho(\Gamma_g)$; this result implies in particular that $\rho(\Gamma_g)$ always stabilizes a maximal tube-type subdomain in $\mathcal{X}$.

Finally, we report on recent work of Burger–Iozzi–Labourie–Wienhard in the case where $G = \mathrm{Sp}(V)$ is the real symplectic group of a symplectic vector space V. The main result is that the flat symplectic bundle over the unit tangent bundle T_1S – where S is a hyperbolization of Σ_g – associated to a maximal representation $\pi_1(S) \to \mathrm{Sp}(V)$, carries an Anosov structure in the sense of Labourie. This result has two noteworthy corollaries. First, any maximal representation $\rho : \Gamma_g \to G$ is a quasi isometric embedding for the word metric on Γ_g and any left-invariant metric on G. Second, if one considers the space of Lagrangians $\mathcal{L}(V)$ as the Shilov boundary of the bounded symmetric domain associated to $\mathrm{Sp}(V)$, then the limit set in $\mathcal{L}(V)$ of any maximal representation is a rectifiable circle. This is in sharp contrast with the behaviour of limit sets of quasi-Fuchsian groups in real hyperbolic 3-space.

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Metrics on Reductive Groups

Gregory Margulis

The talk is based on the joint paper [1] with Herbert Abels.

Let G be a connected Lie group, and L a bounded symmetric ($L = L^{-1}$) open subset of G. We assume that $1 \in L$. Then we can define the associated left invariant word metric ρ_L on G as follows: $\rho_L(g,h) \stackrel{def}{=} \rho_L(1, g^{-1}h)$ and $\rho_L(1,g) \stackrel{def}{=} \min\{\mathrm{i} : \mathrm{g} \in \mathrm{L}^{\mathrm{i}}\}$ where $L^0 = \{1\}, L^1 = L$ and $L^i = L^{i-1} \cdot L$. The group G is discrete with respect to ρ_L. Thus ρ_L is not related to the "small scale" geometry of G. On the other hand, one can easily see that ρ_L is quasi-isometric to the metric induced by any left invariant Riemannian metric on G.

We write $a =_C b$ if $|a - b| \leq C$. Two pseudometrics ρ_1 and ρ_2 on a space X are called *coarsely equal* if there exists $C > 0$ such that $\rho_1(x,y) =_C \rho_2(x,y)$ for an $x, y \in X$.

Let $G = \mathbf{G}(\mathbf{R})^0$ be the connected component of the identity of the group $\mathbf{G}(\mathbf{R})$ of $\mathbf{R}$-rational points of a reductive $\mathbf{R}$-group G. Let $\mathbf{A}$ be a maximal $\mathbf{R}$-split torus in G and $A = \mathbf{A}(\mathbf{R})^0$. Let $A^+ \subset A$ be a Weyl chamber for the Weyl group $W = \mathbf{N}_G(A)/Z_G(A)$. The group G admits a *Cartan decomposition*

$$G = KA^+K$$

where K is a suitable maximal compact subgroup of G.

Let $\|\cdot\|$ be a W-invariant norm on A. We can extend this norm to a function θ on G as follows:

$$\theta(k_1 a k_2) = \|a\|, k_1, k_2 \in K, a \in A^+.$$

It is clear that $\theta(KgK) = \theta(g), \theta(g) \geq 0$ and $\theta(g) = 0$ iff $g \in K$. It can be also shown that $\theta(g_1 g_2) \leq \theta(g_1) + \theta(g_2)$. Now we can define a left invariant pseudometric $\rho(g,h) \stackrel{def}{=} \theta(g^{-1}h)$ on G. Such pseudometrics will be called *normlike*. The following theorem is a special case of the main result of [1].

Theorem 1. *Let* $\mathbf{G}$ *be a reductive* $\mathbf{R}$*-group and* $G = \mathbf{G}(\mathbf{R})^0$*. Then for any bounded symmetric open subset* $1 \in L \subset G$*, the word metric* ρ_L *is coarsely equal to a normlike pseudometric on* G*.*

To put Theorem 1 in a more general context, we need to introduce some definitions.

Let $a \geq 0$. A parametrized curve $\{x(s) : s \in [0,a]\}$ in a pseudometric space (X, ρ) is called a *C-coarse geodesic* if $\rho(x(s), x(t)) =_C |s - t|$ for all $s, t \in [0, a]$. The space (X, ρ) is called *C-coarsely geodesic* if any two points $x, y \in X$ can be connected by a C-coarse geodesic. We will call the space (X, ρ) *coarsely geodesic* if it is C-coarsely geodesic for some $C > 0$.

We say that a pseudometric space (X, ρ) is *C-discretely geodesic* if for any $x, y \in X$ one can find a finite sequence $\{x_i : 0 \leq i \leq n\}$ such that $x_0 = x$, $x_n = y$, $\rho(x_i, x_{i+1}) \leq C$, and $\rho(x,y) = \sum_{i=0}^{n-1} \rho(x_i, x_{i+1})$. We will call the space (X, ρ) *discretely geodesic* if it is C-discretely geodesic for some $C > 0$. It is easy to see that if the space (X,ρ) is C-discretely geodesic, then it is C-coarsely geodesic. But, as simple examples show, the converse is not true.

The word metric construction can be generalized in the following well known way. Let $1 \in L \subset G$ be a symmetric generating set of a group G, and let f be a function on L such that $f(1)=0$ and $0 \le f(x) \le C$ for all $x \in L$. We define $\rho_f(g,h) \overset{def}{=} \rho_f(1, g^{-1}h)$ where $\rho_f(1,y), y \in G$, is defined as the infinum of $\sum_{i=1}^{n} f(g_i)$ over all possible representations $y = g_1 \cdot \ldots \cdot g_n$ of y as the product of elements $g_1, \ldots, g_n$ in L. It is easy to see that the metric ρ_f is C-discretely geodesic.

Let ρ be a left invariant pseudometric on G. We denote $\rho(1,g)$ by $\theta(g)$ and notice that $\theta(1)=0$, $\theta(g) \le 0$, $\theta(g^{-1}) = \theta(g)$ and $\theta(g_1 g_2) \le \theta(g_1) + \theta(g_2)$. We say that ρ is *proper* if the function θ is proper, i.e. the set $\{g \in G : \theta(g) < a\}$ is bounded in G for any $a > 0$. We say that ρ is *locally bounded* if the function θ is locally bounded. Now we can formulate a more general version of Theorem 1.

Theorem 2. *Let G be the same as in Theorem 1, and let ρ be a left invariant coarsely geodesic pseudometric on G. We assume that ρ is proper and locally bounded. Then ρ is normlike.*

We identify A with $\mathbf{R}^n$ and use additive notation for the product of elements in A. Since $\theta(x_1+x_2) \le \theta(x_1) + \theta(x_2)$ for any $x_1, x_2 \in A = \mathbf{R}^n$, the following limit

$$\|x\|_\rho \overset{def}{=} \lim_{m\to\infty} \frac{1}{m}\, \theta(mx)$$

exists for every $x \in \mathbf{R}^n$ and $\|\cdot\|_\rho$ is a norm. Also $\|\cdot\|$ is W-invariant because $Wx \subset KxK$. We show in [1] that ρ is coarsely equal to the pseudometric $\hat\rho$ on G corresponding to the norm $\|\cdot\|_\rho$. This statement can be easily deduced from the following

Proposition. $\sup_{x\in\mathbf{R}^n} |\theta(x) - \|x\|_\rho| < \infty$ or, equivalently, $\sup_{g\in G} |\rho(1,g) - \hat\rho(1,g)| < \infty$.

Remarks. (1) In [1] we prove analogs of Theorems 1 and 2 for a more general class of groups which we call "groups with weak Cartan decomposition". This class includes reductive groups over local fields and also includes the group $\mathbf{Z}^n$. In the case of the group $\mathbf{Z}^n$, the results of [1] are essentially due to Dmitry Burago [2] but our proof and the proof by Burago are quite different.

(2) Let $c_1 \ge 1$ and $c_2 \ge 0$. Two pseudometrics ρ_1 and ρ_2 on a space X are called (c_1,c_2)-*quasiisometric* if

$$\rho_1(x,y) \le c_1 \rho_2(x,y) + c_2$$

and

$$\rho_2(x,y) \le c_1 \rho_1(x,y) + c_2$$

for any $x, y \in X$. The pseudometrics ρ_1 and ρ_2 are coarsely equal iff they are $(1,c)$-quasiisometric for some $C > 0$. We say that two left invariant proper locally bounded pseudometrics ρ_1 and ρ_2 on a compactly generated group G are *asymptotically isometric* if the following two equivalent conditions are satisfied:

(a) $\lim_{g\to\infty} \frac{\rho_1(1,g)}{\rho_2(1,g)} = 1$;

(b) For every $\varepsilon > 0$, one can find $C_\varepsilon > 0$ such that ρ_1 and ρ_2 are $(1+\varepsilon, C_\varepsilon)$-quasiisometric.

It is quite easy to show that the pseudometrics ρ and $\hat{\rho}$ are asymptotically isometric.

(3) Part(a) of the following conjecture is a reformulation of a conjecture by D.Burago, and part (b) is an analog of (a) for non-discrete groups.

Conjecture. (a) Let G be a finitely generated group, and let ρ_1 and ρ_2 be left invariant proper asymptotically isometric pseudometrics on G. Then ρ_1 and ρ_2 are coarsely equal.

(b) Let G be a completely generated locally compact group, and let ρ_1 and ρ_2 be left invariant proper locally bounded asymptotically isometric pseudometrics on G. Then ρ_1 and ρ_2 are coarsely equal.

The results of [1] can be interpreted as the proof of this conjecture for groups with weak Cartan decompositions. Because of the large number of recent examples of "wildly behaved" groups, it is rather unlikely that the conjecture is true in general. Nevertheless, it seems that part (b) of the conjecture is true for connected Lie groups.

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Uniform growth of discrete groups of isometries in negative curvature

Gilles Courtois

(joint work with G. Besson and S. Gallot)

Let Γ be a finitely generated group. For a finite set of generators S, the word lengh $l_S(\gamma)$ of an element $\gamma \in \Gamma$ is defined as the smallest integer $n \geq 0$ such that there exist $s_1, ..., s_n \in S \cup S^{-1}$ with $\gamma = s_1 s_2 .... s_n$. This gives rise to a left invariant distance d_S on Γ defined by $d_S(\gamma_1, \gamma_2) = l_S(\gamma_2^{-1}\gamma_1)$. The entropy of Γ associated to the set of generators S, is defined as $ent_S\Gamma = \lim_{n\to\infty} n^{-1} log\#\{\gamma \in \Gamma, l_S \leq n\}$, and the algebraic entropy of Γ as $ent\Gamma = inf_S\{ent_S\Gamma\}$, where S runs the set of finite generating subset of Γ.

A finitely generated group Γ is said to have exponential growth if for some (and hence every) finite generating subset S, $ent_S\Gamma > 0$, and to have uniform exponential growth if $ent\Gamma > 0$.

In 1980, M. Gromov raised the following question: does exponential growth imply uniform exponential growth? cf. [8]. J.S. Wilson recently answered negatively this question, cf. [11]. Nevertheless, there are classes of groups for which the answer is yes, for example for hyperbolic groups, cf. [9], solvable groups, cf.[10], see [5] for a survey.

In 2005, A. Eskin, S. Mozes and H. Oh proved that the answer is yes for finitely generated linear groups over a field of non zero characteristic.

Theorem 1. [7] *Let Γ be a finitely generated subgroup of $GL(d,K)$, where K is a field of non zero characteristic. Then either Γ is virtually solvable or $ent\Gamma > 0$.*

Recently, E. Breuillard an T. Gelander has announced the following improvement of the above theorem showing that if Γ is a discrete finitely generated subgroup of $GL(d,K)$, where K is a field of any characteristic, then either Γ is virtually solvable or $ent\Gamma \geq C(d)$, where $C(d)$ is a positive number depending only on d and not on Γ. In particular, the entropy of a discrete finitely generated linear group is either 0 or is larger than a constant not depending on the group.

In this note we prove the

Theorem 2. *Let (X^n,g) be an n-dimensional Cartan Hadamard manifold with sectional curvature $-a^2 \leq K_g \leq -1$. There exist a positive number $C(n,a)$ depending on n and a such that if Γ is a discrete group of isometries acting on X^n, then either Γ is almost nilpotent or $ent\Gamma \geq C(n,a)$.*

In particular the theorem says that any such group Γ either has zero entropy or its entropy is bounded below by a uniform constant.

In a private communication M.Kapovich told us he has a personal and different proof of the above theorem under the assumption that Γ does not contain any elliptic elements.

In [1] R.C. Alperin and G.A. Noskov proved that for a non elementary geometrically finite group Γ acting on (X^n,g), then $ent\Gamma > C > 0$ but the constant C is not explicit and depends on Γ.

Sketch of proof. For a Cartan Hadamard manifold (X^n,g), let us denote ρ the distance associated to the riemannian metric g. For an isometry γ of (X^n,g), the displacement of γ is defined by $l(\gamma) = inf_{x\in X^n}\rho(x,\gamma x)$. Let us fix a non virtually nilpotent discrete group Γ and $S = \{\sigma_i\}_{i=1,..,k}$ a finite set of generators of Γ. By Margulis lemma, there is a constant $\mu(n,a) > 0$ such that $L =: inf_{x\in X} max_i \rho(x,\sigma_i x)$ satisfies $L \geq \mu(n,a)$. Moreover there exist $x_0 \in X$ such that $L = inf_{x\in X} max_i \rho(x,\sigma_i x) = max_i \rho(x_0,\sigma_i x_0)$ (if not, Γ would fix a point on the ideal boundary of X and would be virtually nilpotent).

There are basically two cases.

First case : there exist an element $\gamma \in \Gamma$ such that $l_S(\gamma) \leq 2$ whose minimal displacement $l(\gamma)$ is "large", in which case we conclude by a ping-pong argument.

Second case : all elements $\gamma \in \Gamma$ such that $l_S(\gamma) \leq 2$ have "small" minimal displacement. In that case, we construct a map $f : (\mathcal{G}(\Gamma,S), d_S) \to (X,g)$ from the Cayley graph $\mathcal{G}(\Gamma,S)$ of Γ endowed with the distance d_S into X satisfying the two following properties.

1) f is equivariant for the actions of Γ on $\mathcal{G}(\Gamma,S)$ and X.

2) $|f'| \leq L + ent_S\Gamma - \eta$

where $0 < \eta =: \eta(n,a) < L$ is a constant depending only on n and a.

We then observe the following: denoting by e the neutral element of Γ, we have

$$L \leq \rho(f(e), f(\sigma_i)) = \rho(f(e), \sigma_i f(e)) \leq L + ent_S\Gamma - \eta$$

where σ_i is such that $L = \rho(x_0, \sigma_i x_0) = max_j \rho(x_0, \sigma_j x_0)$.

The above inequality immediately gives $ent_S\Gamma \geq \eta(n, a)$.

Construction of the map f. Let us consider the following Poincaré series:

$$P_c(s, y) = \Sigma_\gamma e^{\rho(y,\gamma x_0)} e^{d_S(s,\gamma)},$$

where c is a positive number, $s \in \mathcal{G}(\Gamma, S)$ and $y \in X$.

We consider a c such that the series converges for one (and thus any) choice of $s \in \mathcal{G}(\Gamma, S)$ and $y \in X$. The Poincaré series satisfies the following two properties:

1) $P_c(s, y)$ is equivariant, i.e. $P_c(\gamma s, \gamma y) = P_c(s, y)$ for any $\gamma \in \Gamma$.

2) For any $s \in \mathcal{G}(\Gamma, S)$, the function $y \to P_c(s, y)$ is stricly convex and tends to $+\infty$ when $y \to \infty$.

This allows to define the map $f_c : (\mathcal{G}(\Gamma, S), d_S) \to (X, g)$ by $f_c(s) = y$ where y is the point at which the function $y \to P_c(s, y)$ achieves its minimum.

This map is equivariant and one can prove that $|f_c'| \leq c$.

The end of the proof then boils down in showing that the Poincaré series converges for any $c > L + ent_S\Gamma - \eta(n, a)$ where $0 < \eta(n, a) < L$ is a constant depending only on a and n. This is a consequence of the following geometric lemma which relies on the curvature assumption $K \leq -1$.

Lemma 3. *let $L > 0$ be a positive number. There exist two constant $\delta(L) > 0$ and $0 < \eta(L) < L$ such that for any pair $\{\sigma_1, \sigma_2\}$ of isometries of (X, g) with minimal displacement $l(\sigma_i) \leq \delta(L)$ and such that $\rho(x, \sigma_i x) \geq L - \eta(L)$ for some point $x \in X$, then $\rho(x, \sigma_1\sigma_2 x) \leq 2(L - \eta(L))$.*

The fact that the Poincaré series $P_c(s, y)$ then converges $c > L + ent_S\Gamma - \eta(L)$ then follows easily. We conclude by taking $L = \mu(n, a)$.

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Some arithmetic groups that cannot act on the line

Dave Witte Morris
(joint work with Lucy Lifschitz and Vladimir Chernousov)

A large, complicated group should not be able to act on a very small manifold. In particular, it is known that if Γ is a finite-index subgroup of $\mathrm{SL}(3,\mathbb{Z})$, then Γ has no nontrivial actions by orientation-preserving homeomorphisms of the real line $\mathbb{R}$. (More generally, the same is true if Γ is any finite-index subgroup of the integer points of any connected, almost-simple, algebraic group over $\mathbb{Q}$, with $\mathbb{Q}\text{-rank}\, G \geq 2$.) It has been conjectured that the same conclusion is true much more generally:

Definition 1. *A subgroup Γ of a Lie group G is an* irreducible lattice *in G if*

(1) *Γ is discrete,*
(2) *G/Γ has finite volume, and*
(3) *ΓN is dense in G, for every noncompact, closed, normal subgroup N of G.*

Conjecture 2. *Suppose*

- *G is a connected, semisimple Lie group with finite center,*
- *$\mathbb{R}\text{-rank}\, G \geq 2$, and*
- *Γ is any irreducible lattice in G.*

Then Γ has no nontrivial, orientation-preserving action on $\mathbb{R}$.

This conjecture can be restated in geometric terms:

Conjecture 3. *Suppose Γ is the fundamental group of an irreducible locally symmetric space M of finite volume, such that*

- *M is of noncompact type and has no Euclidean factors locally, and*
- *$\mathrm{rank}\, M \geq 2$,*

Then Γ has no nontrivial, orientation-preserving action on $\mathbb{R}$.

We prove this conjecture in the special case where the locally symmetric space M is *not* compact and its universal cover $\widetilde{M}$ is reducible:

Theorem 4. *Assume*

- *G and Γ are as in Conjecture 2,*
- *the adjoint group of G is* ***not*** *simple, and*
- *G/Γ is* ***not*** *compact.*

Then Γ has no nontrivial, orientation-preserving action on $\mathbb{R}$.

Example 5. *The theorem implies that no finite-index subgroup of* $\mathrm{SL}\big(2,\mathbb{Z}[\sqrt{3}]\big)$ *has a nontrivial, orientation-preserving action on* $\mathbb{R}$*. (Such subgroups are non-cocompact, irreducible lattices in* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$*). Furthermore,* $\sqrt{3}$ *can be replaced with any irrational algebraic integer* α*, such that either* α *is real or* α *is* not *a root of any quadratic polynomial with rational coefficients.*

The theorem is a consequence of the following two results:

Definition 6. *A subgroup* U *of* $\mathrm{SL}(\ell,\mathbb{C})$ *is* unipotent *if it is conjugate to a subgroup of*

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

Theorem 7. *Suppose*

- $\mathbb{F}$ *is an algebraic number field that is neither* $\mathbb{Q}$ *nor an imaginary quadratic extension of* $\mathbb{Q}$*,*
- $\mathcal{O}$ *is the ring of integers of* $\mathbb{F}$*,*
- *no proper subfield of* $\mathbb{F}$ *contains a finite-index subgroup of the group* $\mathcal{O}^\times$ *of units of* $\mathcal{O}$*,*
- Γ *is a finite-index subgroup of* $\mathrm{SL}(2,\mathcal{O})$*, and*
- U *is any unipotent subgroup of* Γ*.*

Then, for every orientation-preserving action of Γ *on* $\mathbb{R}$*, the* U*-orbit of each point in* $\mathbb{R}$ *is a bounded set.*

Theorem 8 (D. Carter, G. Keller, and E. Paige)**.** *If* $\Gamma \approx \mathrm{SL}(2,\mathcal{O})$ *is as described in Theorem 7, then some finite-index subgroup of* Γ *is a product of finitely many unipotent subgroups.*

Our methods will yield much more general results if one can generalize the Carter-Keller-Paige Theorem to lattices in $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{C})$:

Conjecture 9. *If* Γ *is any noncocompact lattice in either* $\mathrm{SL}(3,\mathbb{R})$ *or* $\mathrm{SL}(3,\mathbb{C})$*, then some finite-index subgroup of* Γ *is a product of finitely many unipotent subgroups.*

Theorem 10. *Assume*

- *Conjecture 9 is true,*
- G *is a connected, semisimple Lie group with finite center and no compact factors,*
- $\mathbb{R}\text{-rank}\, G \geq 2$*,*
- G *is **not** isogenous to* $\mathrm{SO}(6,2)$*,*
- G *is **not** an exceptional (almost) simple group of type* E_6*, and*
- Γ *is a noncocompact, irreducible lattice in* G*.*

Then Γ *has no nontrivial orientation-preserving action on* $\mathbb{R}$*.*

Combining this result with a beautiful fixed-point theorem of É. Ghys (also proved in most cases by M. Burger and N. Monod) yields the following conclusion:

Corollary 11. *Assume*

- Γ *and* G *are as in Theorem. 10,*
- *Conjecture 9 is true, and*
- *no simple factor of* G *is isogenous to* $\mathrm{SL}(2,\mathbb{R})$.

Then any action of Γ *on the circle* S^1 *factors through a finite quotient of* Γ.

Regrettably, our methods do not apply to cocompact lattices, because these do not have any unipotent subgroups.

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Towards quantization of the Morse complex

Octav Cornea

I. Recalls from Morse theory. Fix: L a compact, closed manifold; $f : L \to \mathbb{R}$ a Morse function, g a generic Riemannian metric on L. The Morse complex has the form

$$C(f) = (\mathbb{Z}/2 < \mathrm{Crit}(f) >, d) \ .$$

The main point is that $d^2 = 0$ which is due to the fact that broken trajectories between critical points of indexes different by 2 are in bijection with the ends of 1-dimensional moduli spaces of flow lines of $-\nabla_g(f)$. This complex computes the homology of L. One major advantage of the construction is that the algebraic structure is simple so that the notion extends easily leading to, for example, Floer theory. Simplicity becomes also a disadvantage as the algebra is *so* simple that many interesting phenomena can not be encoded in it. **Two such examples to be discussed here:**

i. Higher dimensional moduli spaces of trajectories.
ii. Bubbling.

II. Higher Dimensional Moduli Spaces. Let the negative gradient flow be $\gamma : \mathbf{R} \times L \to L$. The stable/unstable manifold(s) of $x \in Crit(f)$ are $W^s(x) = \{y \in L : \lim_{t\to\infty} \gamma_t(y) = x\}$, and $W^u(x) = \{y \in M : \lim_{t\to-\infty} \gamma_t(y) = x\}$. Generically, $M(P, Q) = W^u(P) \cap W^s(Q) \cap f^{-1}(a)$ is a manifold of dimension $ind(P) - ind(Q) - 1$. *Natural problem 1.* "Measure" the connecting manifolds $M(P, Q)$, $ind(x) - ind(y) > 1$. A few results: - Starting point due to Franks [5]: $M(P, Q)$ is a framed manifold. If P, Q consecutive in the flow, then $M(P, Q)$ is a closed manifold whose cobordism class is computable. - Loop representation of moduli spaces of flow lines (O.C., 1998): there is a natural map $l_{P,Q} : M(P, Q) \to \Omega L$ obtained as follows (ΩX is the space of based Moore loops). First, identify all critical points of f to a single base point (by contracting to a point a simple path that goes through all critical points). The resulting quotient space has the same homotopy type as L. Then associate to each flow line from P to Q the closed loop it defines in this quotient space. A framed bordism class is now defined $[M(P, Q)] \in \Omega^{fr}_*(\Omega L)$ and is again computable when $M(P, Q)$ is closed [2]. *Natural problem 2.* How to measure $M(P, Q)$ if P and Q are not consecutive? Due to broken orbits $M(P, Q)$ is, in general, non-compact but there is a *natural compactification* $\overline{M}(P, Q)$ which is a (compact) manifold so that:

$$\partial\overline{M}(P, Q) = \bigcup_R \overline{M}(P, R) \times \overline{M}(R, Q) \tag{1}$$

Moreover, (1) is compatible with the maps $l_{-,-}$ so that these maps provide a representation of the moduli spaces $M(-, -)$ inside ΩL. A key idea at this point (J.-F. Barraud, O.C. [1]) is to *enlarge the ring over which the Morse complex is defined.* Take $\mathcal{R} = S_*(\Omega M)$ where $S_*(-)$ are cubical chains and define the *extended* Morse complex:

$$\mathcal{C}(f) = (\mathcal{R} \otimes \mathbb{Z}/2 < Crit(f) > \ , \ \delta) \ , \ \delta x = \sum_y a_{xy} \otimes y$$

where, essentially, the a_{xy} represent the fundamental classes of $M(x, y)$ rel boundary so that if we put $A = (a_{xy})$, then (1) gives $dA = A^2$ and so $\delta^2 = 0$. There is a natural filtration

$$F^k\mathcal{C} = \mathcal{R} \otimes \mathbb{Z}/2 < Crit_{\leq k}(f) >$$

which leads to a spectral sequence E^r. The remarkable property of this is that E^r is invariant for $r \geq 2$ - these terms are in fact identified with the terms of the Serre spectral sequence of the path-loop fibration over L - and the differentials represent higher dimensional moduli spaces. Moreover, the construction is "robust" and carries over to Lagrangian Floer theory (J-strips instead of flow lines) *when* $\omega|_{\pi_2(M,L)} = 0$ which leads to symplectic applications.

III. Difficulties with relative Gromov-Witten invariants. Let now (M^{2n}, ω) be symplectic, $L^n \hookrightarrow M$ a *closed* Lagrangian in M. Fix $\lambda \in \pi_2(M, L)$. For $\forall$ J almost complex structure compatible with ω consider the moduli spaces of J-disks:

$$\mathcal{M}(\lambda, J) = \{u : (D^2, S^1) \to (M, L) : \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0; \ [u] = \lambda\}$$

Recall that J-holomorphic disks are quantified: $\exists$ smallest energy and there are only finitely homotopy classes containing J-disks below any fixed, positive energy value. *Natural problem 3.* Count J-disks through cycles ! Under good circumstances $\mathcal{M}(\lambda, J)$, is a manifold of dimension $n + \mu(\lambda) - 3$ where $\mu(\lambda) =$ Maslov index. However, $\mathcal{M}(\lambda, J)$ compactifies to a manifold *with boundary* and due to bubbling the count is not invariant. Instead, one may construct a homology theory - *cluster homology* (O.C., F. Lalonde, 2004 [3]). The main idea in this construction is to *use "quantized" flow lines which combine negative gradient flow lines and J-holomorphic disks (and spheres)* in an arrangement modeled on trees where the vertices are replaced by pseudo-holomorphic objects and the edges by flow lines of f. Such objects are assembled in *cluster moduli* spaces. *Remark.* a. "Linear" such objects have been considered by Oh [6] (following an idea of Fukaya) in the 1990's. These linear objects suffice in the monotone case when $\mu_{min} \geq 2$. b. An alternative way to deal with the bubbling of disks is due to Fukaya-Oh-Ono-Ohta (2000) [4] and also leads to a homology theory. c. The transversality required for the regularity of cluster moduli spaces is work in progress along two different methods, the first due to Hofer - Wysocki - Zehnder and, the second to Cieliebak - Mohnke. We then define a differential graded commutative rational algebra: $\mathcal{C}l(L, J, f) = (S(\mathbb{Q} < \mathrm{Crit(f)} >) \otimes \Lambda, d)$ (rational coefficients are necessary here !) where $S(V)$ is the free commutative DGA on the vector space V, Λ is an appropriate Novikov ring and d counts elements in 0-dimensional cluster moduli spaces. We use $\mathcal{R}' = \mathcal{C}l(L, J, f)$ as a "rich" ring which encodes bubbling and can then define Morse-Floer theory over it (for oriented, relative spin Lagrangians):

$$\mathbb{F}C_*(L, J, H, f) = (\mathcal{R}' \otimes \mathbb{Q} < P_0^H >, D')$$

where $H : M \times [0, 1] \to \mathbb{R}$ hamiltonian, P_0^H are time-1 contractible orbits of X_H with ends on L. It is likely that a theory dealing simultaneously with the phenomena described in II, and III is possible and will lead to interesting applications. A few applications of various parts of this machinery and related structures have been mentioned in the talk. I will list here only two examples which are true for monotone Lagrangians L with minimal Maslov class at least 2 (P. Biran, O.C. 2006): in other words, $\exists \rho > 0$ so that $\omega(\lambda) = \rho\mu(\lambda)$, $\forall \lambda \in \pi_2(M, L))$ and $\mu_{min} \geq 2$.

i. If $L = T^n$, then $HF_*(L) = 0$ or $HF_*(L) = H_*(L; \mathbb{Z}/2) \otimes \Lambda$. In the first case the Gromov radius, R, of L verifies $\pi R^2/2 \leq 2\rho$ (here $H(; \mathbb{Z}/2)$ is singular homology; $HF(-)$ is Floer homology, Λ is the appropriate Novikov ring).
ii. Assume $L \subset \mathbb{C}P^n$ verifies $HF_*(L) \neq 0$. If there is a symplectic embedding of a standard ball $B(r) \hookrightarrow \mathbb{C}P^n \backslash L$, then

$$\pi r^2 \leq \frac{n}{n+1} \ .$$

(Normalization: the maximal symplectic ball in $\mathbb{C}P^n$ is so that $\pi r^2 = 1$.)

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Bounded cohomology, maximal representations, rotation numbers and quasimorphisms

Alessandra Iozzi
(joint work with Marc Burger, Anna Wienhard)

Let Σ be a compact connected oriented surface of genus g with $n \geq 0$ boundary components. Let $\mathcal{X}$ be a Hermitian symmetric space and let $G = \mathrm{Iso}(\mathcal{X})^\circ$ be the connected component of the identity of its isometry group. If Σ has no boundary, one can associate to any representation $\rho : \pi_1(\Sigma) \to G$ a numerical invariant which is uniformaly bounded and varies continuously in $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$, and, since it takes integer values, is constant on connected components of $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ – see [6].

We describe in cohomological terms how to extend the definition of the above Toledo invariant to include surfaces with boundary: we define the invariant by evaluating the pullback of the bounded Kähler class $\rho^*(\kappa_{\mathcal{X}}^{\mathrm{b}})$, appropriately interpreted as a class in the singular relative bounded cohomology of Σ with respect to $\partial\Sigma$, against the relative fundamental class $[\Sigma, \partial\Sigma] \in \mathrm{H}_2(\Sigma, \partial\Sigma, \mathbb{R})$, thus obtaining a real number, [7]. While the continuity and uniform boundedness are preserved, now the invariant takes a whole interval of values and hence is not anymore deformation invariant. However, the Toledo invariant is additive with respect to the connected sum of surface, thus implying that if we realize our surface as connected sum of Σ_1 and Σ_2, the subset of maximal representations of $\pi_1(\Sigma)$ (that is of representation such that the Toledo invariant $\mathrm{T}(\Sigma, \rho)$ takes its maximal value $|\chi(\Sigma)|\,\mathrm{rank}(\mathcal{X})$) is the fibered product of the space of maximal representations of Σ_1 and of maximal representations of Σ_2.

The study of maximal representation of $\pi_1(\Sigma)$ then follows the same lines as for surfaces with empty boundary. Namely we can prove that a maximal representation is always discrete and faithful; the Zariski closure of the image is a reductive group and hence there is associated a symmetric space $\mathcal{Y}$ of noncompact type; the symmetric space $\mathcal{Y}$ is not only Hermitian but always of tube type and, lastly, the group $\pi_1(\Sigma)$ preserves, via ρ, a maximal tube type subdomain $\mathcal{T}$, with $\mathcal{Y} \subset \mathcal{T} \subset \mathcal{X}$.

Just like for surfaces without boundary – see [8] and [1] – the Toledo invariant of a representation of the fundamental group of a surface with boundary can be computed explicitly by using the standard presentation of $\pi_1(\Sigma)$. To this purpose, one can give a definition of rotation number of any element $g \in G$ with respect to a bounded integral class $\kappa \in \mathrm{H}^2_{\mathrm{b}}(G, \mathbb{Z})$, which, if $G = \mathrm{Homeo}_+(S^1)$ and κ is the bounded Euler class, generalizes the classical rotation number. The Toledo invariant can then be explicitly computed by means of the rotation numbers of the images in G under ρ of simply closed loops homotopic to the boundary components.

While this description of the Toledo invariant this is already of independent interest, one can use rotation numbers also to construct more refined invariant on $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$.

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Nielsen-Reidemeister theory and Floer homology

Alexander Fel'shtyn

(joint work with E. Troitsky)

1. Twisted conjugacy separable groups

Definition 1. *Let G be a countable discrete group and $\phi : G \to G$ an endomorphism. Two elements $x, x' \in G$ are said to be ϕ*-conjugate *or* twisted conjugate, *iff there exists $g \in G$ with $x' = gx\phi(g^{-1})$. We shall write $\{x\}_\phi$ for the ϕ*-conjugacy *or* twisted conjugacy *class of the element $x \in G$. The number of ϕ-conjugacy classes is called the* Reidemeister number *of an endomorphism ϕ and is denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.*

If G is a finite group, then the classical Burnside-Frobenius theorem says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of G. Let $\widehat{G}$ be the *unitary dual* of G, i.e. the set of equivalence classes of unitary irreducible representations of G.

In the present report we study the following property for a countable discrete group G and its automorphism ϕ: we say that the group is ϕ-*conjugacy separable* if its Reidemeister classes can be distinguished by homomorphisms onto finite groups, and we say that it is *twisted conjugacy separable* if it is ϕ-conjugacy separable for any automorphism ϕ with $R(\phi) < \infty$ (*strongly twisted conjugacy separable*, if we remove this finiteness restriction). This notion was used in [5] to prove the twisted Burnside-Frobenius theorem for polycyclic-by-finite groups with the finite-dimensional part of the unitary dual $\widehat{G}$ as an appropriate dual object.

The main results:

(1) **Classes of twisted conjugacy separable groups:** Polycyclic-by-finite groups are strongly twisted conjugacy separable groups[5]
(2) **Twisted conjugacy separability respects some extensions:** Suppose, there is an extension $H \to G \to G/H$, where the group H is a characteristic twisted conjugacy separable group; G/H is finitely generated FC-group (i.e., a group with finite conjugacy classes). Then G is a twisted conjugacy separable group[5].
(3) **Examples of groups, which are not twisted conjugacy separable:** HNN, Ivanov and Osin groups [5].
(4) **The affirmative answer to the twisted Dehn conjugacy problem for polycyclic-by-finite groups**[10].
(5) **Residually finite groups are twisted conjugacy separable,** in particular **twisted Burnside-Frobenius theorem is true for them** in the following formulation: Let G be a residually finite group and ϕ its automorphism with $R(\phi) < \infty$. Then $R(\phi) = S_f(\phi)$, where $S_f(\phi)$ is the number of fixed points of $\widehat{\phi} : \widehat{G}_f \to \widehat{G}_f$, $\widehat{\phi}(\rho) = \rho \circ \phi$, where $\widehat{G}_f$ is the part of the unitary dual $\widehat{G}$, which is formed by the finite-dimensional representations [10].

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [1]), in Selberg theory , and Algebraic Geometry .

2. R_∞ property

It is important to describe the class of groups G, such that $R(\phi) = \infty$ for any automorphism $\phi : G \to G$. We say that a group G has *property* R_∞ if any its automorphism ϕ has $R(\phi) = \infty$. First attempts to localize this class of groups go up to [3].

After that it was proved that the following groups belong to this class:

non-elementary Gromov hyperbolic groups [2, 13], Baumslag-Solitar groups $BS(m,n) = \langle a, b | ba^m b^{-1} = a^n \rangle$ except for $BS(1,1)$ [11], generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [12], the solvable generalization Γ of $BS(1,n)$

given by the short exact sequence

$$1 \to \mathbb{Z}[\frac{1}{n}] \to \Gamma \to \mathbb{Z}^k \to 1$$

as well as any group quasi-isometric to Γ [15], groups which are quasi-isometric to $BS(1,n)$ [14] (while this property is not a quasi-isometry invariant), the chameleon group of R. Thompson, a wide class of weakly branch groups including the Grigorchuk group and the Gupta-Sidki group[7].

Conjecture 2. *Any relatively hyperbolic group has* property R_∞. *In particular, any Kleinian group has* property R_∞.

3. Symplectic Floer homology and Nielsen fixed point theory [9]

A mapping class of a surface M is called *algebraically finite* if it does not have any *pseudo-Anosov* components in the sense of Thurston's theory of surface diffeomorphism. The term *algebraically finite* goes back to J. Nielsen. Diffeomorphism of *finite type* are special representatives of algebraically finite mapping classes adopted to the symplectic geometry.

Theorem 3. *[4] If ϕ is a diffeomorphism of finite type of a compact connected surface M of Euler characteristic $\chi(M) < 0$ and if ϕ has only isolated fixed points , then ϕ is monotone with respect to some ϕ-invariant area form and*

$$HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}, \ \dim HF_*(\phi) = N(\phi),$$

where $N(\phi)$ denotes the Nielsen number of ϕ.

Let $\Gamma = \pi_0(Diff^+(M))$ be the mapping class group of a closed connected oriented surface M of genus ≥ 2. Pick an everywhere positive two-form ω on M. A isotopy theorem of Moser says that each mapping class of $g \in \Gamma$, i.e. an isotopy class of $Diff^+(M)$, admits representatives which preserve ω. We can pick a monotone representative $\phi \in \mathrm{Symp}^m(M, \omega)$ of g. Then $HF_*(\phi)$ is an invariant of g, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω. Symplectomorphisms ϕ^n are also monotone for all $n > 0$.

Definition 4. *We define the* asymptotic invariant $F^\infty(g)$ *of mapping class $g \in \Gamma = \pi_0(Diff^+(M))$ to be the growth rate of the sequence $a_n = \dim HF_*(\phi^n)$ for a monotone representative $\phi \in \mathrm{Symp}^m(M, \omega)$ of g:* $F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n))$.

Conjecture 5. *For any mapping class $g \in \Gamma = \pi_0(Diff^+(M))$ there is a monotone representative $\phi \in \mathrm{Symp}^m(M, \omega)$ with respect to some ϕ-invariant area form ω such that*

$$HF_*(\phi) = H_*(M_{id}, \partial_{M_{id}}; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{N(\phi|M \setminus M_{id})},$$

where by M_{id} we denote the union of the components of $M \setminus int(U)$, where ϕ restricts to the identity. Suppose ψ is a standard(Thurston canonical form) representative of g and λ is the largest stretching factor of pseudo-Anosov pieses of ψ($\lambda := 1$ if there is no pseudo-Anosov pieces).Then asymptotic invariant $F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda = h(\psi) = \limsup_{n\to\infty} |N(\psi^n)|^{1/n}$

"Arnold-Nielsen" conjecture. If $\phi \in \mathrm{Symp}^m(M,\omega)$ has only non-degenerate fixed points, then from previous conjecture it follows that

$$\#Fix(\phi) \geq \dim HF_*(\phi) = \dim H_*(M_{id}, \partial M_{id}; \mathbb{Z}_2) + N(\phi | M \setminus M_{id}).$$

Due to P. Seidel $\dim HF_*(\phi)$ is a new symplectic invariant of a four-dimensional symplectic manifold with nonzero first Betti number. I hope that and *asymptotic invariant* also give rise to a new invariants of contact 3- manifolds and symplectic 4-manifolds.

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Symplectomorphism groups – an introduction

Dusa McDuff

This talk aimed to give an overview of basic results on the symplectomorphism group $\mathrm{Symp}(M,\omega)$ of a closed symplectic manifold (M,ω) and its subgroup of Hamiltonian symplectomorphisms $\mathrm{Ham}(M,\omega)$.

I explained why $\mathrm{Symp}(M,\omega)$ is C^0-closed in the diffeomorphism group, an application of Gromov's nonsqueezing theorem [4] due to Eliashberg and Ekeland–Hofer. (cf. [11, Ch 1.2])

I discussed the exact sequence

$$\mathrm{Ham}(M,\omega) \to \mathrm{Symp}_0(M,\omega) \stackrel{\mathrm{Flux}}{\to} H^1(M;\mathbb{R})/\Gamma,$$

(cf. [11, Ch 10]) and Ono's recent proof[12] of the Flux conjecture (that the Flux subgroup Γ is always discrete.)

Question *Is* $\mathrm{Ham}(M,\omega)$ *always* C^0 *closed in* $\mathrm{Diff}_0(M)$*? (cf.* [8]*)*

Most of the rest of the talk discussed properties of the Hofer norm (or metric) on $\mathrm{Ham}(M,\omega)$; cf. Hofer [5, 5, 13]. This is a biinvariant Finsler norm on $\mathrm{Ham}(M,\omega)$ in which the length of the path $\phi^H_t, t \in [0,1]$, generated by the Hamiltonian $H : M \times I \to \mathbb{R}$ is defined to be

$$\mathcal{L}(\{\phi^H_t\}) := \int_0^1 \big(\max_{x\in M} H(x,t) - \min_{x\in M} H(x,t)\big)\, dt.$$

The norm $\|\phi\|$ of an element $\phi \in \mathrm{Ham}(M,\omega)$ is the infimum of the lengths of all paths from the identity element to ϕ, and the distance function is given by $d(\phi,\psi) := \|\phi\psi^{-1}\|$. The geodesics on Ham with this metric are now well understood (see [1, 7]). It is shown in [9] that there are plenty of paths with the property that sufficiently short pieces of them minimize the Hofer distance between their endpoints. Therefore this is the natural definition of geodesic. However, it is not true that every element of Ham may be joined to the identity by such a path; see [7].

It is hard in general to find lower bounds for the Hofer norm, specially when $\pi_1(M) = 0$. Therefore the following basic question is still open.

Question *Does the metric space* $\mathrm{Ham}(M,\omega), \|\cdot\|)$ *always have infinite diameter?*

The most interesting results here are due to Polterovich ([13]) who used Lagrangian Floer homology to show that the diameter of $\mathrm{Ham}(S^2)$ is infinite, and Entov–Polterovich's work on quasimorphisms [3].

I ended by briefly mentioning Entov's [2] lovely application of Hofer geometry to the problem of understanding the Agnihotri–Belkale–Woodward inequalities for the eigenvalues of products of unitary transformations. An overview of his proof can be found in [10].

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Length and stable length

Danny Calegari
(joint work with Koji Fujiwara)

In a hyperbolic manifold, Margulis' lemma gives a universal scale on which one can say that a geodesic is *short*. In dimension 3, most short geodesics have a small stable commutator length; conversely, for every $\epsilon > 0$ and every dimension n, there is a universal constant $\delta(\epsilon, n) > 0$ such that if M is a hyperbolic n-manifold, and α is an element of $\pi_1(M)$ with stable commutator length less than δ, the length of the corresponding geodesic is less than ϵ.

This observation (proved in [1]) can be generalized to word-hyperbolic groups and groups acting on word-hyperbolic spaces as follows:

Theorem 1: Let G be a word-hyperbolic group which is δ-hyperbolic with respect to a generating set S. There is a positive constant $C(\delta, |S|)$ such that every element of G either has some power conjugate to its inverse, or it has stable commutator length at least C.

Stable commutator length defines a function from conjugacy classes to reals. It follows that in a torsion-free word-hyperbolic group G, the image of this map is discrete up to a certain point and has a first accumulation point $\delta_\infty(G)$.

Theorem 2: Let G be a nonelementary torsion-free word-hyperbolic group. The first accumulation point $\delta_\infty(G)$ for stable commutator length satisfies

$$\frac{1}{4} \le \delta_\infty(G) \le \frac{1}{2}$$

The lower bound in this theorem is sharp.

Finally,

Theorem 3: Let S be a surface with negative Euler characteristic and let $\mathrm{MCG}(S)$ denote the mapping class group of S. There is a positive constant $C(S)$ such that every pseudo-Anosov element $\phi \in \mathrm{MCG}(S)$ either has a power which is conjugate to its inverse, or it has stable commutator length at least C.

See [2] for details.

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Quasi-norms on groups of geometric origin

Dimitri Burago

(joint work with Sergei Ivanov and Leonid Polterovich)

We say that a group is *bounded* if it is bounded (as a metric space) with respect to any bi-invariant metric. We start with a couple of general remarks and examples illustrating this definition.

One can see that a group G is unbounded if it admits a *quasi-norm* (for brevity, q-norm), that is a function $q : G \to \mathbb{R}$ such that the following three properties hold:

(1) q is quasi-subadditive: there is a constant c such that

$$q(ab) \le q(a) + q(b) + c \;\; \forall a, b \in G \; ;$$

(2) q is quasi-conjugacy-invariant: there is a constant c such that

$$|q(b^{-1}ab) - q(b)| \le c \;\; \forall a, b \in G \; ;$$

(3) q is unbounded.

In particular, a group which admits a non-trivial homogeneous quasi-morphism (see e.g. [2]) is unbounded. There are however examples of unbounded finitely generated groups that do not admit non-trivial quasi-morphisms.

The group $SL(2,\mathbb{Z})$ carries an abundance of quasi-morphisms and hence is unbounded. In contrast, $SL(n,\mathbb{Z})$ is bounded for $n \geq 3$. Furthermore, $SL(n,R)$ is bounded for all n.

By our definition some seemingly "small" groups are unbounded, for instance the abelian group S^1. The corresponding q-norm can be constructed with the help of the infinite basis of $\mathbb{R}/\mathbb{Q}$ over $\mathbb{Q}$. In particular it is highly discontinuous with respect to the natural topology on S^1. In fact, one can show that an abelian group is bounded if and only if it is finite. On the other hand, if the commutator norm on $[G,G]$ is unbounded, then so is G.

However, most of examples we consider have the following property: there exists a finite subset of elements whose conjugates generate the whole group. We say that such groups are *finitely c-generated*. This requirement rules out "fuzzy" examples such as S^1.

Definition 1. *We say that a set $K \subset G$ c-generates G in N steps (where N is a positive integer or infinity) if every element $h \in G$ can be represented as a product $h = \tilde{h}_1\tilde{h}_2 \dots \tilde{h}_n$, where $n \leq N$ and each $\tilde{h}_i$ is conjugate to some element $h_i \in K$: $\tilde{h}_i = \alpha_i h_i \alpha_i^{-1}$, $\alpha_i \in G$. Hence a finitely c-generated group is c-generated by a finite subset (but not necessarily in finitely many steps).*

The following simple example is of key importance:

Example 2. *If a group is c-generated in infinitely many steps, one can define a norm q_K: $q_K(b)$ is the length of a shortest word representing b and such that each letter is a conjugate to an element from K. One can show that for any q-norm q there is a constant c such that $q \leq cq_K + c$; hence, in a sense, q_K is a maximal norm.*

Furthermore, if a group is c-generated by a finite subset K in finitely many steps, then G is bounded.

The main theme of our project is boundedness/unboundedness of various diffeomorphism groups (smooth, volume-preserving or symplectic). For instance, for all known symplectic manifolds, the group of compactly supported Hamiltonian diffeomorphisms is unbounded with respect to Hofer's norm (cf. [4]). On the other hand, the identity component of the group of C^∞-smooth compactly supported diffeomorphisms is bounded in all known to us examples! Some partial results in this direction are summarized in the following theorem (work in progress):

Theorem 3. (i) *The identity component $Diff_0^{comp}(\mathbb{R}^n)$ of the group of all compactly supported diffeomorphisms of $\mathbb{R}^n$, $Diff^{comp}(\mathbb{R}^n)$, is bounded.*

(ii) *For every connected manifold M, the identity component $Diff_0^{comp}(M \times \mathbb{R})$ of the group of compactly supported diffeomorphisms of $M \times \mathbb{R}$ is bounded. In particular, the group of compactly supported diffeomorphisms of the open annulus is bounded.*

(iii) *The identity component $Diff_0(S^n)$ of the group of all diffeomorphisms of the sphere S^n is bounded.*
(iv) *The identity component $Diff_0(M^3)$ of the group of all diffeomorphisms of an arbitrary closed 3-dimensional manifold M is bounded.*

Our proof of this theorem motivates the following definition.

Definition 4. *Let M be a smooth manifold. A subset $D \in M$ along with a diffeomorphism $\phi : B^n \to D \subset M$ is said to be a disc in M. We say that a diffeomorphism is elementary if its support lies in some disc in M.*

Let $f \in Dif_0^{comp}(M)$. We define the complexity *$L(f)$ of f to be the smallest k such that f can be represented as a product of k elementary diffeomorphisms. Note that $L(f)$ is finite due to the classical fragmentation lemma (see e.g.* [1]*).*

In a sense, L is a "universal candidate" for a q-norm: one can show that $Diff_0^{comp}(M)$ is unbounded iff L is unbounded. Furthermore, for every q-norm q on $Diff_0^{comp}(M)$, there is a positive constant c such that $q \leq cL$.

Open Problems: Does there exist a constant C such that every diffeomorphism of the two-dimensional torus isotopic to the identity can be represented as a product of no more than C elementary diffeomorphisms (diffeomorphisms supported in discs)? In other words, is L bounded on $Diff_0(T^2)$? What about $Diff_0^{comp}(\mathbb{R}^2 \# RP^2)$, the group of compactly supported diffeomorphisms of the (open) Möbius strip?

The case of groups of volume-preserving diffeomorphisms is even more intriguing. Denote by $Diff_0(M, vol)$ the identity component of the group of volume-preserving diffeomorphisms of a closed manifold M equipped with a volume form. In the case when $\pi_1(M)$ has the trivial center and admits non-trivial quasi-morphisms, the group $Diff_0(M, vol)$ is known to admit non-trivial quasi-morphisms as well (see [3, 5]), and hence is unbounded.

Open Problem: Is the group $Diff_0(S^n, vol)$ bounded or unbounded for $n \geq 3$? Note that for $n = 2$ this group coincides with the group of Hamiltonian diffeomorphisms of S^2 and hence is unbounded with respect to Hofer's norm.

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Infinitesimal rigidity of the Weyl chamber flow via the vanishing theorem of Weil

Masahiko Kanai

The aim of the present note is to give an account of the recent work by the author [**Ka2**], in which revealed is an unforeseen link between the classical vanishing theorems of Matsushima and Weil on the one hand, and rigidity of the Weyl chamber flow on the other. The old vanishing theorems, such as those of Weil or of Matsushima, have emerged in a series of attempts to grasp rigidity phenomena which were observed in group homomorphisms into *finite-dimensional* Lie groups: Indeed, the vanishing theorem of Matsushima says that there is no homomorphism of a lattice Γ of a higher-rank Lie group into the additive group $\mathbb{R}$ other than the trivial one, while Weil's vanishing is a key step in the proof of his local rigidity of the inclusion map of a lattice Γ into the ambient Lie group G. On the contrary, the Weyl chamber flow, which is a dynamical system arising from a higher-rank Lie group, is a smooth group action: It is a continuous homomorphisms into a diffeomorphism group, a typical example of *"infinite-dimensional"* Lie groups. The infinite-dimensionarity causes serious difficulties which seem inevitable to understand actions of noncompact groups (cf. [**Ka1**], [**KS2**]). There must lie a huge gap between finite- and infinite-dimensional realms. However, the gap is bridged over by our "extension theorems" with rather less effort, surprisingly.

Let $G = SL(n+1,\mathbb{R})$, and Γ a cocompact lattice of G (Although all the results in the present note hold for any semisimple Lie group G of $\mathbb{R}$-rank ≥ 2 with finite center and without simple factor locally isomorphic to a compact one, $SO(k,1)$ or $SU(k,1)$, we confine ourselves to the above spacial case in order to make description simpler). Then the following vanishing theorems hold, where $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$, and Γ acts on $\mathfrak{g}$ through the adjoint representation of G:

(**Matsushima**) $$H^1(\Gamma;\mathbb{R}) = 0;$$

(**Weil**) $$H^1(\Gamma;\mathfrak{g}) = 0.$$

These are the classical vanishing theorems with which we concern ourselves.

Meanwhile, a dynamical system that intrigues us is constructed in the following manner. Let $A \cong \mathbb{R}^n$ be the subgroup of $G = SL(n+1,\mathbb{R})$ consisting of those diagonal matrices with positive diagonal entries. The action of A on $V = \Gamma \setminus G$ from right is called the *the Weyl chamber flow*, and has been studied extensively in the past decade (cf. [**KS1**, **KS2**]). Since the action is locally free, the orbits form a nonsingular foliation of V, which is denoted by $\mathcal{F}$.

Let denote by $H^*(V,\mathcal{F};\mathbb{R})$ the *tangential* (or *leafwise*) *de Rham cohomology* of the foliated manifold $(V,\mathcal{F})$. It is the cohomology of the tangential de Rham complex $\{\Omega^*(V,\mathcal{F};\mathbb{R}), d_{\mathcal{F}}\}$ of the foliated manifold $(V,\mathcal{F})$: An element of $\Omega^p(V,\mathcal{F};\mathbb{R})$ is an $\mathbb{R}$-*valued tangential* p-*form*, which is by definition a C^∞ section of the vector bundle $\bigwedge^p T^*\mathcal{F}$ over V with $T^*\mathcal{F}$ being the cotangent bundle of $\mathcal{F}$, while the coboundary operator $d_{\mathcal{F}}$ is the *tangential exterior derivative* that is defined in the same manner as the ordinary exterior derivative except that the differential is performed only in the direction tangent to $\mathcal{F}$. One of the results established by Katok and Spatzier [**KS1**] is the following theorem, in which $\mathfrak{a} \cong \mathbb{R}^n$ denotes the Lie algebra of A:

(**Katok–Spatzier**) $\qquad H^1(V, \mathcal{F}; \mathbb{R}) \cong \mathfrak{a}^*.$

More precisely, their theorem claims the following: A tangential 1-form $\alpha \in \Omega^1(V; \mathbb{R})$ is tangentially exact whenever it is tangentially closed and satisfies $\int_V \alpha(H) = 0$ for all $H \in \mathfrak{a}$ which are thought of as vector fields tangent to $\mathcal{F}$ through the action of A; the integration is taken with respect to the standard volume from on V.

The theorem of Katok and Spatzier readily implies the **parameter rigidity** of the Weyl chamber flow ([**KS1**]): *Any smooth action of A on V that is smoothly orbit equivalent to the Weyl chamber flow has to be smoothly conjugate to the the Weyl chamber flow up to a (smooth) automorphism of A.*

A cochain homomorphism of the ordinary de Rham complex $\Omega^*(V; \mathbb{R})$ into the tangential one $\Omega^*(V, \mathcal{F}; \mathbb{R})$ is given by ignoring the direction transverse to the foliation $\mathcal{F}$. The following theorem provides a kind of the inverse procedure; namely, **transverse extension** of the tangential 1-form.

Theorem 1 ([**Ka2**]). *For any tangential 1-form $\alpha \in \Omega^1(V, \mathcal{F}; \mathbb{R})$ such that $d_{\mathcal{F}}\alpha = 0$ and that $\int_W \alpha(H) = 0$ for all $H \in \mathfrak{a}$, there exists a unique 1-form $\theta \in \Omega^1(V; \mathbb{R})$ that extends α.*

Theorem 1 yields

$$(\textbf{Matsushima}) \quad \Longleftrightarrow \quad (\textbf{Katok–Spatzier}) :$$

Matsushima's vanishing and the theorem of Katok–Spatzier are equivalent to each other. However, this could not be regarded as a new proof of the theorem of Katok–Spatzier, for *we do need the theorem of Katok–Spatzier in the proof of Theorem 1.*

The parameter rigidity is a rigidity in the direction tangent to the orbits. As to rigidity in the transverse direction, Katok and Spatzier [**KS2**] proved the **local rigidity** of the foliation $\mathcal{F}$: *A smooth foliation of V that is of the same dimension as $\mathcal{F}$ and has the tangent bundle close to that of $\mathcal{F}$ in an appropriate topology is smoothly conjugate to $\mathcal{F}$ (i.e., there is a diffeomorphism of V that sends the original foliation $\mathcal{F}$ to the perturbed one).*

Another transverse rigidity is the **infinitesimal rigidity** of the foliation $\mathcal{F}$, which could be interpreted as a linearized version of the local one, and is formulated in terms of a variant of the tangential de Rham cohomology; namely, the cohomology $H^*(V, \mathcal{F}; N\mathcal{F})$ of the cochain complex $\{\Omega^*(V, \mathcal{F}; N\mathcal{F}), d^D_{\mathcal{F}}\}$. The cochains $\omega \in \Omega^p(V, \mathcal{F}; N\mathcal{F})$ are the tangential p-forms *taking values in the normal bundle $N\mathcal{F}$* of the foliation $\mathcal{F}$, i.e., the C^∞ sections of $\bigwedge^p T^*\mathcal{F} \otimes N\mathcal{F}$. In the meantime, the tangential exterior derivative $d^D_{\mathcal{F}}$ is defined by means of the linear holonomy D of $\mathcal{F}$: The foliation $\mathcal{F}$ is said to be *infinitesimally rigid* if the following vanishing holds:

(**Infinitesimal Rigidity**) $\qquad H^1(V, \mathcal{F}; N\mathcal{F}) = 0.$

The infinitesimal rigidity of the foliation $\mathcal{F}$ is also tied to a classical vanishing theorem: Indeed, in [**Ka2**] it was proved that

$$(\textbf{Weil}) \quad \Longrightarrow \quad (\textbf{Infinitesimal Rigidity});$$

namely, that the infinitesimal rigidity of the foliation $\mathcal{F}$ follows from the vanishing theorem of Weil. In consequence, we are given

Theorem 2. *The orbit foliation $\mathcal{F}$ of the Weyl chamber flow is infinitesimally rigid.*

This is established again by means of a transverse extension theorem for 1-forms with coefficients in the normal bundle $N\mathcal{F}$ of $\mathcal{F}$.

Finally, we make a few comments on literatures. Our extension theorems are inspired by Matsumoto–Mitsumatsu [**MM**]. They proved an extension theorem for the tangential de Rham cohomology with trivial coefficients of the Anosov foliation of a closed surface of constant negative curvature. One can directly generalize their theorem to the Anosov foliations of the higher-dimensional rank-one locally symmetric spaces (see [**Ka2**]). It should also be mentioned that Kononenko had worked on quite similar problems. In particular, one can derive Theorem 1 from his theorem [**Ko1**, Theorem 6.1] and vice versa. Meanwhile, in [**Ko2**, Theorem 11.1], he proved a claim essentially equivalent to Theorem 2 under some extra assumptions, which include the assumptions that the $\mathbb{R}$-rank of G is at least three, and that G is split. During the conference at MFO, the author was informed by Anatoly Katok that he and S. Ferleger proved Theorem 1 in their unpublished paper which was prepared in 1997. The author expresses his thanks to A. Katok, who provided the author a copy of their unpublished paper.

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Engel structures and weakly hyperbolic flows on four-manifolds

Dieter Kotschick
(joint work with Thomas Vogel)

In this talk I gave an introduction to the geometry of Engel structures and of even contact structures on four-manifolds. The main focus was on the equivalence between even contact structures whose characteristic foliations satisfy a suitable weak hyperbolicity condition, and pairs of Engel structures subordinate to the given even contact structure and inducing opposite orientations on it. This equivalence is analogous to the equivalence discovered by Eliashberg–Thurston [1] and by Mitsumatsu [2] between projectively or conformally Anosov flows on three-manifolds and pairs of contact structures inducing opposite orientations. There are many other relations and parallels between contact structures on three-manifolds and Engel structures on four-manifolds that deserve further investigation.

1. Weakly hyperbolic flows

We study a weak notion of hyperbolicity for flows which are tangent to a fixed distribution, and which preserve this distribution.

Let M be a closed manifold, $\mathcal{E} \subset TM$ a smooth subbundle, and $\mathcal{W} \subset \mathcal{E}$ an orientable line field with $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. This ensures that $\mathcal{E}$ is preserved by any flow tangent to $\mathcal{W}$. Moreover, such a flow then acts on the quotient bundle $\mathcal{E}/\mathcal{W}$.

Definition 1. *The flow φ_t on M generated by a non-zero vector field W spanning $\mathcal{W}$ is said to be weakly hyperbolic if there is a continuous φ_t-invariant splitting*

$$\mathcal{E}/\mathcal{W} = \mathcal{E}_+ \oplus \mathcal{E}_-$$

with $\mathcal{E}_\pm$ of positive rank, and a constant $c > 0$ such that

$$\frac{||D\varphi_t(v_+)||}{||v_+||} \geq e^{ct} \, \frac{||D\varphi_t(v_-)||}{||v_-||} \tag{1}$$

for all non-zero vectors $v_\pm \in \mathcal{E}_\pm$, and all $t > 0$, with the norms taken with respect to a suitable continuous metric g on $\mathcal{E}/\mathcal{W}$.

This condition is independent of the spanning vector field W chosen for $\mathcal{W}$, as long as we fix an orientation for $\mathcal{W}$. If we change this orientation, by replacing W with $-W$, say, then weak hyperbolicity is preserved, but the roles of $\mathcal{E}_\pm$ are interchanged. The holonomy of $\mathcal{W}$ preserves $\mathcal{E}$ and acts naturally on the quotient $\mathcal{E}/\mathcal{W}$, and the condition in the definition is that the holonomy is much more expanding on $\mathcal{E}_+$ than on $\mathcal{E}_-$. This does not preclude the possibility that the holonomy could be expanding (or contracting) on both $\mathcal{E}_\pm$, as long as the expansion (or contraction) rates are such that (1) is satisfied. In the case that $\mathcal{E}$ is the tangent bundle of a three-manifold, Definition 1 reduces to the definition of conformally Anosov or projectively Anosov (pA) flows, see [1, 1].

If the distribution $\mathcal{E}$ is integrable, then it defines a foliation, and a flow tangent to $\mathcal{W} \subset \mathcal{E}$ restricts to every leaf of this foliation. The flow is weakly hyperbolic in the sense of Definition 1 if and only if its restriction to every leaf is conformally Anosov.

For the purposes of this talk we are interested in the case when $\mathcal{E}$ is an even contact structure on a four-manifold, and $\mathcal{W}$ is its characteristic foliation.

2. Even contact structures

Definition 2. *An even contact structure on a $2n$-dimensional manifold M is a maximally non-integrable smooth hyperplane field $\mathcal{E}$.*

Such a hyperplane field can be defined locally by a one-form α with the property that $\alpha \wedge (d\alpha)^{n-1}$ is nowhere zero. A global defining form exists if and only if $\mathcal{E}$ is coorientable. The two-form $d\alpha$ has maximal rank on $\mathcal{E}$. If one changes the defining form α, then the restriction of $d\alpha$ to $\mathcal{E}$ changes only by multiplication with a function, so its conformal class is intrinsically defined. The kernel of $d\alpha$ restricted to $\mathcal{E}$ coincides with the kernel of the $(2n-1)$-form $\alpha \wedge (d\alpha)^{n-1}$. This kernel is a line field $\mathcal{W} \subset \mathcal{E}$ giving rise to the characteristic foliation of $\mathcal{E}$, and the quotient bundle $\mathcal{E}/\mathcal{W}$ carries a conformal symplectic structure.

In dimension four, that is for $n = 2$, every closed manifold with zero Euler number admits an even contact structure by the h-principle (Gromov, McDuff).

3. Engel structures

Definition 3. *An Engel structure on a 4-dimensional manifold M is a smooth rank 2 distribution $\mathcal{D}$ with the property that $[\mathcal{D}, \mathcal{D}]$ is an even contact structure $\mathcal{E}$.*

If $\mathcal{E}$ is an even contact structure and $\mathcal{D}$ is an Engel structure whose derived distribution $[\mathcal{D}, \mathcal{D}]$ coincides with $\mathcal{E}$, we say that $\mathcal{E}$ is induced by $\mathcal{D}$, and that $\mathcal{D}$ is subordinate to $\mathcal{E}$.

Every C^2 small perturbation on an Engel structure is again an Engel structure, moreover a generic rank 2 distribution on a four-manifold is Engel almost everywhere. In the classification of stable germs of distributions due to Montgomery, Engel structures occupy a special place, in that they are the only sporadic entry in the list, whose other entries are line fields, contact structures and even contact structures. This is one of several motivations for the study Engel structures.

Lemma 4. *If $\mathcal{D}$ is subordinate to $\mathcal{E}$, then the characteristic foliation $\mathcal{W}$ of $\mathcal{E}$ is contained in $\mathcal{D}$.*

We now discuss orientations for the distributions involved in the definition of an Engel structure subordinate to a given even contact structure.

Lemma 5. 1. *Every Engel structure defines a canonical orientation on its induced even contact structure.*

2. *The following conditions on a* 4*-manifold* M *endowed with an Engel structure are equivalent:*

(a) M *is orientable,*
(b) $\mathcal{W}$ *is orientable,*
(c) $\mathcal{E}$ *is coorientable.*

These Lemmas quickly lead to the following:

Proposition 6. *A closed oriented four-manifold admitting an orientable Engel structure is parallelizable.*

Conversely, one has the following recent existence theorem:

Theorem 7 (Vogel [3])**.** *Every parallelizable four-manifold admits an orientable Engel structure.*

Vogel's proof [3] is constructive, using round handle decompositions to build Engel structures, and leads to even contact structures $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ whose characteristic foliation is Morse-Smale. Here we investigate a class of Engel structures for which $\mathcal{W}$ has very different dynamical properties.

4. Bi-Engel structures

The first part of Lemma 5 motivates the following:

Definition 8. *A bi-Engel structure on a* 4*-dimensional manifold* M *is a pair of Engel structures* $(\mathcal{D}_+, \mathcal{D}_-)$ *inducing the same even contact structure* $\mathcal{E}$*, defining opposite orientations for* $\mathcal{E}$*, and having one-dimensional intersection.*

By Lemma 4, the two Engel structures making up a bi-Engel structure must both contain the characteristic foliation $\mathcal{W}$ of the induced even contact structure $\mathcal{E}$. Thus their intersection is precisely $\mathcal{W}$, and their span is $\mathcal{E}$.

The following is our main theorem:

Theorem 9. *Let* $\mathcal{E}$ *be an even contact structure on a closed oriented four-manifold* M*, and* $\mathcal{W}$ *its characteristic foliation. Then* $\mathcal{W}$ *is weakly hyperbolic if and only if* $\mathcal{E}$ *is induced by a bi-Engel structure* $(\mathcal{D}_+, \mathcal{D}_-)$*.*

We would like to point out that the weakly hyperbolic flows corresponding to bi-Engel structures do not usually admit a strong Anosov splitting $\mathcal{E} = \mathcal{W} \oplus \mathcal{E}_+ \oplus \mathcal{E}_-$ lifting the splitting $\mathcal{E}/\mathcal{W} = \mathcal{E}_+ \oplus \mathcal{E}_-$. In fact, we can characterize the existence of the strong splitting through the geodesibility of $\mathcal{W}$ inside $\mathcal{E}$. The absence of the strong splitting in the general case is one of the difficulties encountered in the proof of Theorem 9. This same difficulty arises in the proof of the analogous three-dimensional result in [1], and our treatment covers that case as well.

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Quasi-morphisms and quasi-states in symplectic topology

Michael Entov
(joint work with Paul Biran, Leonid Polterovich, Frol Zapolsky)

The talk, based on joint works [5], [4], [6], [7] with L.Polterovich, P.Biran and F.Zapolsky, concerns the following interplay between symplectic topology, group theory and functional analysis.

For simplicity assume that M is one of the following symplectic manifolds (all of them simply connected): $\mathbb{C}P^n$, a complex Grassmannian or $(\mathbb{S}^2 \times \ldots \times \mathbb{S}^2, \omega \oplus \ldots \oplus \omega)$. Let $Symp_0(M)$ be the identity component of the group of symplectomorphisms of such an M and $\widetilde{Symp}_0(M)$ its universal cover. We use the Hamiltonian Floer theory in order to construct certain functionals $\mu : \widetilde{Symp}_0(M) \to \mathbb{R}$ and $\zeta : C^0(M) \to \mathbb{R}$ with remarkable algebraic properties. These functionals, which are of interest by themselves, also lead to important applications in symplectic topology.

I will briefly explain what are the properties of μ and ζ and what sort of applications can be obtained by means of these functionals.

First of all recall that there is a fundamental rigidity phenomenon in symplectic topology – for simply connected symplectic manifolds M it can be formulated as follows: certain subsets of M cannot be completely displaced from themselves by a symplectic isotopy of M while it is possible to do so by a smooth isotopy. Formally, we will say that X is *displaceable* if there exists $\phi \in Symp_0(M)$ such that $\phi(X) \cap \bar{X} = \emptyset$ and *non-displaceable* otherwise.

Now the functional μ is a *homogeneous quasi-morphism*, i.e. it satisfies the following properties:

(1) $\exists K = K(\mu) > 0$ such that $\forall x, y \in \widetilde{Symp}_0(M) \quad |\mu(xy) - \mu(x) - \mu(y)| \leq K$.

(2) $\mu(x^k) = k\mu(x)$ for all $x \in \widetilde{Symp}_0(M)$, $k \in \mathbb{Z}$.

Note that, according to Banyaga's theorem [2], for a closed simply connected M the group $\widetilde{Symp}_0(M)$ is perfect. Hence it does not admit any non-trivial real-valued homomorphism and thus finding a non-trivial homogeneous quasi-morphism is, in a sense, the best result one can hope for.

In fact, our homogeneous quasi-morphism μ has a number of additional properties. First, if an element $\phi \in \widetilde{Symp}_0(M)$ is represented by an identity-based path of symplectomorphisms supported in a displaceable open set then $\mu(\phi)$ is the classical Calabi invariant of ϕ. Secondly, $Symp_0(M)$ (for a simply connected M) is equipped with a remarkable bi-invariant *Hofer metric* (see e.g. [8] for a survey of Hofer geometry) which lifts to a (pseudo)metric on $\widetilde{Symp_0}(M)$ and the quasi-morphism μ is Lipschitz with respect to this (pseudo)metric.

The functional $\zeta : C^0(M) \to \mathbb{R}$ is a *non-linear* and satisfies the following properties:

(i) $\zeta(1) = 1$;
(ii) $F \le G \Rightarrow \zeta(F) \le \zeta(G)$;
(iii) $\zeta(aF + bG) = a\zeta(F) + b\zeta(G)$ for all $a, b \in \mathbb{R}$ and all functions $F, G \in C^\infty(M)$ whose Poisson bracket $\{F, G\}$ vanishes.

Such a ζ is called a *symplectic quasi-state* (see [6] for a discussion concerning the origin of the term "quasi-state", first introduced by J.Aarnes, and its relation to quantum mechanics). Moreover, ζ is invariant with respect to the natural $Symp_0(M)$-action on $C^0(M)$ and vanishes on functions with displaceable support.

The relation between μ and ζ is the following: consider $\widetilde{Symp_0}(M)$ as an infinite-dimensional Lie group whose Lie algebra is naturally identified with the space $C_0^\infty(M)$ of smooth functions with zero mean on M. With this language the restriction of ζ to $C_0^\infty(M)$ is simply the pullback of quasi-morphism $-\mu$ on the group to the Lie algebra via the exponential map.

Now the theorem of Aarnes [1], which generalizes the classical Riesz representation theorem, says that to any quasi-state (like ζ) one can associate a *quasi-measure*. A quasi-measure on M is a function $\tau : \mathcal{S} \to [0, 1]$ on the collection $\mathcal{S}$ of all subsets of M *which are either open or closed*, so that

1) $\tau(M) = 1$;

2) $X_1 \subset X_2 \Rightarrow \tau(X_1) \le \tau(X_2)$ for all $X_1, X_2 \in \mathcal{S}$;

3) $\tau(X_1 \sqcup \ldots \sqcup X_k) = \tau(X_1) + \ldots + \tau(X_k)$ for all $X_1, ..., X_k \in \mathcal{S}$ with $X_1 \sqcup \ldots \sqcup X_k \in \mathcal{S}$;

4) For every open subset X one has $\tau(X) = \sup \tau(A)$, where the supremum is taken over all closed subsets $A \subset X$.

A quasi-measure associated to ζ as above is $Symp_0(M)$-invariant and vanishes on displaceable subsets.

Below I list a few types of applications that can be obtained by means of μ, ζ and τ:

Applications concerning the algebraic structure of $\widetilde{Symp_0}(M)$. Here is an example of such an application: using general group theory (see e.g. [3]) one can deduce from the existence of μ that the commutator length of $\widetilde{Symp_0}(M)$ is infinite and the commutator length of an individual element $\phi \in \widetilde{Symp_0}(M)$ can be estimated from below by means of $\mu(\phi)$.

Existence and detection of non-displaceable subsets of M. Here is an example of an application of this sort that can be proved using the symplectic quasi-state ζ and its generalizations – not only for M mentioned above but also for a much wider class of symplectic manifolds: if $H_1, \ldots, H_k$ are smooth functions on M which pairwise commute with respect to the Poisson brackets then at least one common level set $H_1^{-1}(c_1) \cap \ldots \cap H_k^{-1}(c_k)$ is non-displaceable.

C^0-robustness of Poisson brackets on M. The definition of the Poisson bracket $\{F, G\}$ of two functions F, G involves first derivatives of the functions. Thus *a priori* there is no restriction on possible changes of $\{F, G\}$ when F and G are perturbed in the uniform norm. Nevertheless ζ can be used to produce effectively such a restriction. One can also use ζ to find a restriction on partitions of unity on M subordinate to coverings by displaceable sets – this can be also proved (using a generalization of ζ) for a much wider class of symplectic manifolds M than those mentioned above.

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Ergodic properties of isoperimetric domains in spheres of negatively curved manifolds

Norbert Peyerimhoff

(joint work with Gerhard Knieper)

In this talk we present an equidistribution result for averages over subsets of increasing spheres which satisfy a certain isoperimetric condition and discuss a few applications.

We first introduce the general situation: Let M be a compact manifold of nonpositive sectional curvature $K_M \leq 0$ and $\pi : \tilde{M} \to M$ be the Riemannian universal covering map. Then $M = \tilde{M}/\Gamma$, where Γ is a discrete subgroup of the isometry group of $\tilde{M}$. Let SM and $S\tilde{M}$ denote the unit tangent bundles of M and $\tilde{M}$.

Every continuous function $f \in C(SM)$ can be lifted to a Γ-periodic function $f \circ \pi \in C(\tilde{S})$. We always denote this lift by $\tilde{f}$. Note that all functions under consideration are defined on unit tangent bundles. Functions on the manifolds can be viewed as particular cases of functions on the unit tangent bundles which are constant in the fibres.

Let Φ^r denote the geodesic flow on SM or $S\tilde{M}$, $S_r(p) \subset \tilde{M}$ be the metric sphere of radius $r > 0$ about $p \in \tilde{M}$ and $S_r(p)^+ \subset S\tilde{M}$ be the set of outward unit normal vectors with base points in $S_r(p)$. A dynamical description of $S_r(p)^+$

is $\Phi^r(S_pM)$. The footpoint projection $S_r(p)^+ \to S_r(p)$ yields a diffeomorphism between $S_r(p)^+$ and $S_r(p)$. $S_r(p)^+$ inherits via this diffeomorphism a volume form from the induced Riemannian volume of the submanifold $S_r(p) \subset \tilde{M}$. We denote this volume form by $\lambda_{r,p}$. Sometimes we simplify notation by using only the symbol λ for this measure.

One can average a periodic lift $\tilde{f}$ over the increasing sets $S_r(p)^+$ and look at possible limits. In the case of negative curvature, the following result of G. Knieper is known:

Theorem(see [4]): *Let M be a compact Riemannian manifold with* strictly negative *curvature $K_M < 0$. Then there exists a probability measure μ_H* (horospherical measure) *such that we have, for all $p \in \tilde{M}$ and all $f \in C(SM)$:*

$$\lim_{r\to\infty} \frac{1}{\mathrm{vol}(S_r(p))} \int_{S_r(p)^+} \tilde{f}\, d\lambda_{r,p} = \int_{SM} f\, d\mu_H.$$

It may surprise that the horospherical measure μ_H is generally not invariant under the geodesic flow. But it was also proved in [4] that absolute continuity of μ_H to a ϕ^r-invariant measure implies an extremely restrictive situation: In this case, $\tilde{M}$ is already asymptotically harmonic (i.e., all horospheres have constant mean curvature) and therefore $\tilde{M}$ is a rank one symmetric space of noncompact type (the latter conclusion uses deep dynamical results of [3, 1, 2]).

Spherical means on nonpositively curved higher rank symmetric spaces (with possible Euclidean factors) have been considered in [6]. In this case we have convergence to a particular Φ^r-invariant probability measure μ_{max} with support in the barycentric directions of the Weyl chambers. The limit μ_{max} is a measure of maximal entropy.

For the rest of the talk our standing assumption is that M is compact with **strictly negative** sectional curvature $K_M < 0$. Our main result is the following generalization of the above theorem to certain (isoperimetric) subsets of increasing spheres:

Theorem(see [5]): *Let (p_j, r_j) be a sequence with $p_j \in \tilde{M}$ and $r_j \to \infty$. Let $U_j \subset S_{r_j}(p_j)^+ \subset S\tilde{M}$ be measurable sets satisfying the isoperimetric condition*

$$\lim_{j\to infty} \frac{\lambda_{r_j,p_j}(\partial_\epsilon U_j)}{\lambda_{r_j,p_j}(U_j)} = 0 \tag{1}$$

for some $\epsilon > 0$, where

$$\partial_\epsilon U_j := \{v \in S_{r_j}(p_j)^+ \mid d(v, \partial U_j) < \epsilon\}.$$

Then we have for all $f \in C(SM)$,

$$\lim_{j\to\infty} \frac{1}{\lambda(U_j)} \int_{U_j} \tilde{f}\, d\lambda = \int_{SM} f\, d\mu_H.$$

It is natural to ask whether spherical domains *evolving with the geodesic flow* are contained in this result as a particular case: i.e., let $U_r := \Phi^r(A) \subset S_r(p)^+$

for some $A \subset S_p\tilde{M}$ with, let's say, piecewise smooth boundary $\partial A \subset S_p\tilde{M}$. Do we have

$$\lim_{r\to\infty} \frac{1}{\lambda(U_r)} \int_{U_r} \tilde{f}\, d\lambda = \int_{SM} f\, d\mu_H?$$

So far, we can only answer this question in the affirmative if we have

$$\mu_p(\Phi^\infty(\partial A)) = 0, \tag{2}$$

where μ_p is the Patterson-Sullivan measure, centered at p, of the ideal boundary $\tilde{M}(\infty)$ and $\Phi^\infty : S_p\tilde{M} \to \tilde{M}(\infty)$ is the homeomorphism assigning to each vector $v \in S_p\tilde{M}$ the ideal end point of the corresponding geodesic ray, starting in direction v. (2) implies the isoperimetric property (1). Moreover, condition (2) can always be achieved by a perturbation of ∂A within $\partial_\delta A \subset S_p\tilde{M}$ for arbitrarily small fixed $\delta > 0$.

By easy approximation arguments one obtains the following result about spherical means *with a continous density function* ρ:

Corollary: *Let $\rho : S_p\tilde{M} \to (0,\infty)$ be a continuous function. Then we have, for all $f \in C(SM)$:*

$$\lim_{r\to\infty} \frac{\int_{S_r(p)^+} \tilde{f}\,(\rho\circ\Phi^{-r})\, d\lambda}{\int_{S_r(p)^+} \rho\circ\Phi^{-r}\, d\lambda} = \int_{SM} f\, d\mu_H.$$

In fact, this result answers a question posed to us by C. Vernicos [7] which was the starting point of our investigations.

Finally, we discuss a *mixing type property* which can be deduced from the above theorem. The horospherical measures come in a pair μ_{H^+}, μ_{H^-} and satisfy

$$\lim_{r\to\infty} \frac{1}{\mathrm{vol}(S_r(p))} \int_{S_r(p)^\pm} \tilde{f}\, d\lambda_{r,p} = \int_{SM} f\, d\mu_{H^\pm}, \tag{3}$$

where $S_r(p)^- = \Phi^{-r}(S_p\tilde{M})$ is the set of inward normal vectors with footpoints in $S_p(r) \subset \tilde{M}$. (3) implies that $\mu_{H^-} = F^*\mu_{H^+}$, where $F : SM \to SM$ is the flip map $F(v) = -v$. Let $a(p) = \lim_{r\to\infty} \frac{\mathrm{vol}(S_r(p))}{e^{hr}}$ be Margulis' asymptotic function with h equals the topological entropy of $\Phi_r : SM \to SM$. (Note that $a(p)$ is the total mass of the Patterson-Sullivan measure μ_p centered at p.) An additional integration over all of M yields:

Corollary: *We have, for all $f, g \in C(SM)$:*

$$\lim_{r\to\infty} \frac{\int_M a(p) \int_{S_pM} f\circ\Phi^r\, g\, d\mu_r\, d\mathrm{vol}(p)}{\int_M a(p)\, d\mathrm{vol}(p)} = \int_{SM} f\, d\mu_{H^+} \int_{SM} g\, d\mu_{H^-},$$

where μ_r denotes the normalized *pull back of the measure λ_r on the sphere $S_r(p) \subset M$ to the unit tangent space S_pM.*

Note that in the case of a rank one symmetric space this result reduces to ordinary mixing.

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The geodesic flow and symmetric join for hyperbolic groups.

Igor Mineyev

My area of interest is geometric group theory. In this abstract I will concentrate on the constructions that are related to hyperbolic dynamics in groups. This work is built up on ideas of many people, the reader is very much advised to see the numerous references and further discussion in [1], [2], and [1].

In [2] I describe the functor $\circledast$ which assigns to every set X its symmetric join $\circledast X$. As a set, $\circledast X$ is a the "obvious" formal union of intervals connecting each ordered pair of points in X, so that X is naturally embedded in $\circledast X$. When X is a topological space, $\circledast X$ is identified with a natural quotient of the usual (topological) join of X with itself, so it can be given the canonical quotient topology. If X is a metric space, a metric d_* is constructed so that the embedding of X into $\circledast X$ is an isometry. Furthemore, the $\mathrm{Isom}(X)$-action on X extends canonically to an $\mathrm{Isom}(X)$-action on $\circledast X$, and d_* is preserved by $\mathrm{Isom}(X)$. A (modification of) metric $\hat{d}$ constructed in [3] is used in the definition of d_*.

When, additionally, X is a hyperbolic complex X, for example a Cayley graph of a Gromov hyperbolic group, one can talk about the symmetric join $\circledast \bar{X}$ of $\bar{X}$, and d_* canonically extends to a generalized metric on $\circledast \bar{X}$ (with obvious infinite values allowed for points at infinity). Classical concepts known for $\mathbb{H}^n$ and negatively curved manifolds can now be defined in a precise way for hyperbolic complexes. We define a double difference, a cross-ratio and horofunctions in the compactification $\bar{X} = X \sqcup \partial X$. They are continuous, $\mathrm{Isom}(X)$-invariant, and satisfy sharp identities. We characterize the translation length of a hyperbolic isometry $g \in \mathrm{Isom}(X)$.

Parameterizing each line by $\bar{\mathbb{R}} := [-\infty, \infty]$ provides a flow on $\circledast \bar{X}$. If a line connects two points at infinity, then this flow on this line is isometric with respect to d_*. This flow space is defined for any hyperbolic complex X and has sharp properties.

The geodesic flow space $\mathcal{F}(X)$ now can be defined as the subspace of $\circledast \bar{X}$ consisting of lines connecting points at infinity. This is an analogue of the unit tangent bundle on a simply connected negatively curved manifold with the usual geodesic flow on it.

[2] also gives a construction of the asymmetric join $X \circledast Y$ of two metric spaces. These concepts are canonical, i.e. functorial in X, and involve no "quasi"-language.

Another application of the metric $\hat{d}$ provides a conformal (rather than quasi-conformal) structure on the ideal boundary of an arbitrary hyperbolic complex [1]. More precisely, this is a metric $\check{d}$ on ∂X with respect to which the $\mathrm{Isom}(X)$ action on ∂X is conformal in a very natural sense. This gives rise to a natural definition of *hyperbolic dimension* for a hyperbolic group, and one can ask questions in analogy with questions in geometry and geometric analysis (see [1]).

Both the symmetric join and the conformal metric on the boundary are of interest in particular because of their relation to the Cannon's conjecture: if the ideal boundary of a hyperbolic group Γ is homeomorphic to the 2-sphere, then Γ acts isometrically on the 3-dimensional hyperbolic space $\mathbb{H}^3$ properly discontinuously and cocompactly. The symmetric join construction provides a space on which Γ acts isometrically, though the space is not necessarily a 3-manifold. The metric $\check{d}$ provides a conformal structure on $\partial\Gamma$, though not necessarily the canonical conformal structure of the sphere. Both constructions can be viewed as steps toward the conjecture. Proving the full conjecture will require just a bit more work...

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The Isomorphy problem for some relatively hyperbolic groups

François Dahmani
(joint work with Daniel Groves)

The isomorphy (or isomorphism) problem asks for a general algorithm which will, given two finite group presentations, decide whether or not the presentations define isomorphic groups. For finitely presented groups in general, Adian [1] and Rabin [4] proved that there is no such algorithm. One can then ask whether there is a solution within a class $\mathcal{C}$ of groups. Namely, is there an algorithm which, given two finite group presentations and the knowledge that they define groups in $\mathcal{C}$, decides whether or not the presentations define isomorphic groups?

For interesting classes of groups, the expected answer to the above question is 'no'. In fact, there are very few large classes of groups for which the isomorphy problem is known to be solvable.

In recent years, geometric group theory has provided solutions to the isomorphy problem for some classes of groups. In particular Sela [5] solved the isomorphy problem for torsion-free hyperbolic groups which do not admit a small essential action on an $\mathbb{R}$-tree, and has an (unfortunately unpublished) proof for arbitrary torsion-free hyperbolic groups. A solution to the isomorphy problem for Sela's limit groups was provided by Bumagin, Kharlampovich and Miasnikov [2].

Relatively hyperbolic groups are defined as coarse analogues of fundamental groups of finite volume hyperbolic manifolds. Such a group acts on a Gromov-hyperbolic space, cocompactly on the complement of an invariant system of horoballs, and the stabilisers of the horoballs are called parabolic subgroups. We call such groups *toral* when they are torsion free, and when the parabolic subgroups are abelian. In [3], we prove:

Theorem 1. *The isomorphy problem is solvable for the class of toral relatively hyperbolic groups.*

As special cases of Theorem 1 we recover the above-mentioned results of Sela, and of Bumagin, Kharlampovich and Miasnikov.

Using Mostow rigidity, and yoga about finite index subgroups we also obtain:

Theorem 2. *The homeomorphy problem is solvable for finite volume hyperbolic n-manifolds, for $n \geq 3$.*

The presence of torsion seems to be a real problem. In particular the isomorphy problem for hyperbolic groups with torsion remains open.

It is worth remarking that in the case of torsion-free hyperbolic groups which admit no essential small action on an $\mathbb{R}$-tree (this is the case from [5]), our methods provide significant simplifications. The major innovation in our approach is the use of equations with inequations and *rational constraints* rather than equations alone, for chasing the morphisms from one group to the other. This greatly streamlines the solution to the isomorphy problem.

Let us outline the strategy. A first goal is to find explicit lists $\mathcal{C}_1$ and $\mathcal{C}_2$ where $\mathcal{C}_1$ is a finite collection of homomorphisms from H_1 to H_2 which contains a representative of each conjugacy class of monomorphism from H_1 to H_2, and $\mathcal{C}_2$ is an analogous list of homomorphisms from H_2 to H_1.

If we can find such lists, then we may not be able to decide which homomorphisms from our lists are monomorphisms. However, if there *is* an isomorphism between H_1 and H_2, then there will be some $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ so that $\psi \circ \phi$ and $\phi \circ \psi$ are inner automorphisms. We can check if a given map from H_1 to itself is an inner automorphism with a solution to the simultaneous conjugacy problem.

Thus, one of our key tools is the following theorem:

Theorem 3. *Suppose that H_1 and H_2 are toral relatively hyperbolic groups. There is an algorithm (explicit from the presentations of H_1 and H_2) which terminates if and only if there is some finite subset $\mathcal{B}$ of H_1 so that there are only finitely many conjugacy classes of homomorphisms from H_1 to H_2 which are injective on $\mathcal{B}$.*

In case the algorithm terminates, it provides a finite list of homomorphisms which contains a representative of every conjugacy class of monomorphism from H_1 to H_2.

Our approach to proving Theorem 3 is to use equations with inequations and rational constraints. Briefly, any morphism, sufficiently injective, and 'minimal' in its conjugacy class is a solution to a system of equations, and inequations in the target group. The minimality in the conjugacy class can be encoded with finite state automata in order to comply with some delicate definition of rational constrainst in this context. Thus a machine solving equations of this form will eventually add the morphism to the list, until there is nothing more to add.

If the algorithm of Theorem 3 does not terminate, then there are non-conjugate homomorphisms from H_1 to H_2 which are injective on larger and larger balls in H_1. Passing to a limit gives a faithful action of H_1 on an $\mathbb{R}$-tree. This is in fact the standart Bestvina-Paulin construction in the hyperbolic case. Using the Rips machine, this yields an abelian splitting of H_1.

Conversely, if H_1 has primary splittings for, it has infinitely many outer automorphisms, and thus there are infinitely many conjugacy classes of monomorphisms from H_1 to H_2, if any; in this case the algorithm from Theorem 3 will not terminate.

Then, the next key step in our proof of Theorem 1 is the algorithmic construction of the (primary) JSJ decomposition of a freely indecomposable toral relatively hyperbolic group, thus recognising (in parallel) the above situation:

Theorem 4. *There is an algorithm which takes a finite presentation for a freely indecomposable toral relatively hyperbolic group,* Γ *say, as input and outputs a graph of groups which is a primary (or essencial in the hyperbolic case) JSJ decomposition for* Γ.

Now consider two freely indecomposable groups H_1 and H_2. We can find their JSJ decompositions, and decide which vertex groups of H_1 are isomorphic to vertex groups of H_2. Using the canonical properties of the JSJ decomposition, one can then decide whether or not H_1 and H_2 are isomorphic.

For torsion-free hyperbolic groups, an algorithm finding the essential JSJ decomposition is due to Sela (unpublished). This result is of independent interest and should be useful for many other applications. For example, the automorphism group of a hyperbolic group can be calculated from the JSJ decomposition and a similar analysis applies to toral relatively hyperbolic groups. Also, the JSJ decomposition is one of the key tools in Sela's work on the Tarski problem. Thus to be able to effectively find the JSJ decomposition is an important first step for many algorithmic questions about the elementary theory of free (and possibly torsion-free hyperbolic or toral relatively hyperbolic) groups.

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Stable commutator length on mapping class groups

Koji Fujiwara
(joint work with Danny Calegari)

Let G be a group. Suppose g is an element in the commutator subgroup $[G, G]$. The commutator length, $cl(g)$, of g is the minimal number of commutators, c_i, g can be written as product $g = c_1 \cdots c_k$. If g is not in $[G, G]$, we define $\mathrm{cl}(g)$ to be infinite. The stable commutator length of g, $scl(g)$, is defined by

$$\mathrm{scl}(g) = \liminf_{n\to\infty} \mathrm{cl}(g^n)/n.$$

$\mathrm{scl}(g)$ may be infinite. Also $\mathrm{scl}(g)$ can be finite even if $\mathrm{cl}(g)$ is infinite.

Recently, Danny Calegari [Ca] showed the following "gap theorem" using pleated surfaces.

Theorem[Ca] Let M be a closed hyperbolic manifold, and G its fundamental group. Then, there is a constant $C(M) > 0$ such that $\mathrm{scl}(g) \geq C(M)$ for any non-trivial element $g \in G$.

An element g in a group G is called *essential* if for any $n > 0$ and any $h \in G$, $g^n \neq hg^{-n}h^{-1}$. In a joint work with Calegari, we show the following.

Theorem 1[CaF] Let G be a word-hyperbolic group. Then, there exists a constant $C(G) > 0$ such that for every element $g \in G$ which is essential, $\mathrm{scl}(g) \geq C$.

Theorem 2[CaF] Let S be a compact orientable surface, and $\mathrm{Mod}(S)$ its mapping class group. Then, there exists a constant $C(M) > 0$ such that for every pseudo-Anosov element g which is essential, $\mathrm{scl}(g) \geq C$.

Our tool is quasi-homomorphisms on G. A map $f : G \to \mathbb{R}$ is called a *quasi-homomorphism* if there exists a constant D such that for any elements $g, h \in G$,

$$|f(g) + f(h) - f(gh)| \leq D.$$

D is called a *defect* of f.

A quasi-homomorphism f is called *homogeneous* if for any $g \in G$ and any $n > 0$, $f(g^n) = nf(g)$. If f is a homogeneous quasi-homomorphism, then for any $g, h \in G$, $f(hgh^{-1}) = f(g)$, and $|f([g, h])| \leq D$, where D is a defect of f.

Given a quasi-homomorphism f with defect D, if one defines

$$\bar{f}(g) = \liminf_{n\to\infty} f(g^n)/n,$$

then $\bar{f}$ is a homogeneous quasi-homomorphism with defect at most $4D$.

If f is a homogeneous quasi-homomorphism with defect D on G with $f(g) = 1$ for some $g \in G$, then

$$\mathrm{scl}(g) \geq 1/2D.$$

To see this ([Ba]), suppose $\mathrm{cl}(g^n) = k$, namely, there are k commutators c_i such that

$$g^n = c_1 \cdots c_k.$$

Apply f, and obtain

$$n = f(g^n) \leq |f(c_1)| + \cdots + |f(c_k)| + (k-1)D \leq (2k-1)D.$$

Therefore,

$$\mathrm{scl}(g) = \liminf_{n \to \infty} k/n \geq 1/2D.$$

To give a (uniform) lower bound of $\mathrm{scl}(g), g \in G$, we try to construct a homogeneous quasi-homomorphism f on G such that $f(g) = 1$ with the defect $D(f)$ controlled from above. For our construction, we need hyperbolic geometry in the sense of Gromov.

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Dynamical properties of the mapping class group

Ursula Hamenstädt

Let S be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$. The *mapping class group* $\mathcal{M}(S)$ of all isotopy classes of orientation preserving diffeomorphisms of S acts smoothly and properly discontinuously on the *Teichmüller space* $\mathcal{T}(S)$ preserving the *Teichmüller metric* d, with the *moduli space* $\mathrm{Mod}(S)$ as the quotient orbifold.

Even though the Teichmüller metric is not non-positively curved in any reasonable sense, its *geodesic flow* Φ^t which is defined on the space $\mathcal{Q}(S)$ of area one holomorphic quadratic differentials over moduli space exhibits some hyperbolic behavior. For example, *periodic* orbits of Φ^t are in one-to-one correspondence with conjugacy classes of so-called *pseudo-Anosov* elements of the mapping class group. Moreover, Φ^t admits two invariant transverse continuous foliations W^{ss}, W^u which are called the *strong stable* and the *unstable* foliation. There is a natural Φ^t-invariant probability measure on $\mathcal{Q}(S)$ in the Lebesgue measure class which is absolutely continuous with respect to the strong unstable and the strong stable foliation. It admits a family of conditionals on strong stable manifolds which are uniformly contracted under Φ^t, with contraction rate $6g - 6 + 2m$ [4]. This measure is exponentially mixing [1] under the Teichmüller flow.

The space of leaves for the lift of the foliation W^{ss} to a foliation on the bundle $\mathcal{Q}^1(S)$ of area one quadratic differentials over Teichmüller space can be identified with the space $\mathcal{ML}$ of *measured geodesic laminations*. There is a natural $\mathcal{M}(S)$-invariant Radon measure on $\mathcal{ML}$ in the Lebesgue measure class. This measure gives full mass to the *recurrent* measured geodesic laminations which correspond to orbits of the Teichmüller flow in $\mathcal{Q}(S)$ which return to some fixed compact set for arbitrarily large times. We discuss the action of $\mathcal{M}(S)$ on $\mathcal{ML}$ which has properties similar to the linear action of a lattice in $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ [3].

Theorem A: *An $\mathcal{M}(S)$-invariant Radon measure on $\mathcal{ML}$ which gives full measure to the recurrent measured geodesic laminations coincides with the Lebesgue measure up to scale.*

Let $x \in \mathcal{T}(S)$ be a fixed point in Teichmüller space. The *Poincaré series* of $\mathcal{M}(S)$ with exponent α and basepoint x is the series

$$\sum_{g\in\mathcal{M}(S)} e^{-\alpha d(x,gx)}.$$

The *critical exponent* of $\mathcal{M}(S)$ is the infimum of all numbers $\alpha > 0$ such that the Poincaré series converges. We explain the calculation of this critical exponent [3].

Theorem B:

(1) *The critical exponent of $\mathcal{M}(S)$ equals $6g-6+2m$, and the Poincaré series diverges at the critical exponent.*
(2) *For a compact subset K of $\mathcal{Q}(S)$ and $r > 0$ let $n_K(r)$ be the number of periodic orbits for the Teichmüller flow on $\mathcal{Q}(S)$ of period at most r which intersect K; then*

$$\lim_{r\to\infty} \frac{1}{r} \log n_K(r) = 6g - 6 + 2m.$$

The second part of Theorem B does however not give a precise logarithmic asymptotic for the number of periodic orbits of the Teichmüller flow Φ^t. Namely, in [2] we constructed for a closed surface S of genus at least 4 and every compact subset K of $\mathcal{Q}(S)$ a periodic orbit for Φ^t which does *not* intersect K.

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Ergodic properties of boundary actions

Vadim A. Kaimanovich
(joint work with Rostislav Grigorchuk and Tatiana Nagnibeda)

The boundary theory occupies an important place in various mathematical fields, as, to name just a few, geometric group theory, rigidity theory, theory of Kleinian groups, potential analysis, etc. The free group is one of the central objects in the study of boundaries of groups, because its simple combinatorial structure makes of it a convenient test-case which contributes to the understanding of general concepts, both in the group-theoretic (as the free group is the universal object in the category of discrete groups) and geometric (as the homogeneous tree is a discrete analogue of the constant curvature hyperbolic space) frameworks.

There exist many different compactifications and associated boundaries of a group: the space of ends, the Martin boundary, the visual boundary, the Busemann boundary, the Floyd boundary, etc. In the case of the free group F freely generated by a finite set A, all these notions coincide, and the boundary ∂F can be realized as the space of infinite freely reduced words in the alphabet $A \cup A^{-1}$. The action of the group on itself extends by continuity to a continuous action on ∂F.

The choice of the generating set A determines a natural *uniform* probability measure m on ∂F which is quasi-invariant under the action of F. This measure can also be interpreted in a number of ways. Namely, as the measure of maximal entropy of the unilateral Markov shift in the space of infinite irreducible words, as a conformal density (Patterson measure), or as the hitting ($\equiv$ harmonic) measure of the simple random walk on the group. In the latter interpretation the measure space $(\partial F, m)$ is actually isomorphic to the Poisson boundary of the random walk, and it is this interpretation that plays the crucial role in our work.

Recall that an action of a countable group with respect to a quasi-invariant measure is called *ergodic* if it admits no non-trivial invariant set. Any action (on a Lebesgue space) admits a unique *ergodic decomposition* into its *ergodic components*. An action is called *conservative* if it admits no non-trivial *wandering set* (i.e, such that its translations are pairwise disjoint). If there is a wandering set of positive measure, then there also exists maximal such a set, and the union of its translations is called the *dissipative part* of the action. Any action admits the so-called *Hopf decomposition* into the conservative and dissipative parts. These parts can also be described as the unions of all purely non-atomic, and, respectively, of atomic ergodic components.

Properties of the boundary action of the subgroup H are closely connected with two geometric objects associated with H. One is the *Schreier graph* structure $\Gamma(X, A)$ on the quotient homogeneous space $X = H\backslash F$, which is a straightforward generalization of the notion of a Cayley graph. The other is the *limit set* $\Lambda_H \subset \partial F$ as well as its various subsets obtained from specifying the type of boundary convergence.

We use the combinatorial machinery of Nielsen and Schreier in order to describe explicitly the Hopf decomposition of the boundary $(\partial F, m)$ with respect to the action of an arbitrary subgroup H. We give a necessary and sufficient condition for the action to be conservative in terms of growth of the Schreier graph X with respect to the generators A. Our characterization of the conservative part allows us to construct examples of actions with both the conservative and the dissipative parts of positive measure. We then study the conservative part of the action and prove a necessary and sufficient condition for a conservative action to be ergodic.

Problem session

General audience

The uniform Kazhdan property for $SL_n(\mathbb{Z}), n \geq 3$

Goulnara N. Arzhantseva

Let Γ be a discrete group, and let S be a finite subset of Γ. For a unitary representation π of Γ in a separable Hilbert space $\mathcal{H}$ we define the number

$$K(\pi, \Gamma, S) = \inf_{0 \neq u \in \mathcal{H}} \max_{s \in S} \frac{\|\pi(s)u - u\|}{\|u\|}.$$

Then the *Kazhdan constant* of Γ with respect to S is defined as

$$K(\Gamma, S) = \inf_{\pi} K(\pi, \Gamma, S),$$

where the infimum is taken over unitary representations π having no invariant vectors. We also define the *uniform Kazhdan constant* of Γ as

$$K(\Gamma) = \inf_{S} K(\Gamma, S),$$

where the infimum is taken over all finite generating sets S of Γ.

A group Γ is said to have *Kazhdan property* (T) (or to be a Kazhdan group) if there exists a finite subset S of Γ with $K(\Gamma, S) > 0$. A group Γ is *uniform Kazhdan* if $K(\Gamma) > 0$.

Shortly after its introduction by David Kazhdan in the mid 60's, property (T) was used by Gregory Margulis to give a first explicit construction of infinite families of expander graphs of bounded degree. In particular, a major problem of practical application in the design of efficient communication networks was solved.

A classical example of a Kazhdan group is the group $SL_n(\mathbb{Z})$ for $n \geq 3$ (for more details and a general context of locally compact groups see a recent book [1]. Surprisingly, the following question is still open.

Question. *Is the group $SL_n(\mathbb{Z})$, for $n \geq 3$, uniform Kazhdan ?*

Infinite finitely generated uniform Kazhdan groups were discovered very recently [2], [3]. However, these groups are neither finitely presented nor residually finite. The latter construction provides an infinite uniform Kazhdan group that weakly (see [4]) contains an infinite family of expanders in its Cayley graph.

An affirmative answer to the above question would give, in particular, the first example of a residually finite (and, in addition, finitely presented) infinite uniform Kazhdan group. It is crucial for applications: infinite families of expanders could be constructed independently of the choice of the group generating set.

A negative answer would be interesting as well. In that case, this classical group would belong to the class of non-uniform Kazhdan groups. First examples of such groups were obtained using Lie groups [5]. Then, all word hyperbolic groups were also shown to have zero uniform Kazhdan constant [6].

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The entropy of a group endomorphism

Goulnara N. Arzhantseva

Let Γ be a group and let S be a finite generating set of Γ. We denote by $\ell_S(\gamma)$ the *word length* of an element $\gamma \in \Gamma$ *with respect to* S. The *growth function* of the pair (Γ, S) is given by

$$\beta(n;\Gamma,S) := \#\{\gamma \in \Gamma \mid \ell_S(\gamma) \leq n\}$$

and the *exponential growth rate* is the number

$$\omega(\Gamma,S) := \lim_{n\to\infty} \sqrt[n]{\beta(n;\Gamma,S)}.$$

Let us suppose that Γ is the fundamental group $\pi_1(M,*)$ of a compact Riemannian manifold (M,g). We denote by $L_g(\gamma)$ the length of a shortest geodesic from the base point $*$ to itself representing $\gamma \in \Gamma$. In particular, for a generator $s \in S$ we have a real number $L_g(s)$ instead of word length 1. The *geometric length* of $\gamma \in \Gamma$ *with respect to* S is the function $L_{g,S} : \Gamma \to \mathbb{R}$ defined by

$$L_{g,S}(\gamma) := \inf\left\{\sum_{i=1}^{n} L_g(s_i) \mid \gamma = s_1 \dots s_n,\ s_i \in S^{\pm 1}, n \in \mathbb{N} \cup \{0\}\right\}.$$

The *geometric growth function*

$$\beta(n;g,\Gamma,S)$$

and the *geometric exponential growth rate*

$$\omega(g, \Gamma, S)$$

are defined naturally. The following result is rather surprising from the combinatorial group theory viewpoint.

Theorem [1]**.** *Let $h(g)$ denote the volume entropy of (M, g). Then*

$$h(g) = \sup\{\ \omega(g, \Gamma, S) \mid S \text{ is a finite generating set of } \Gamma = \pi_1(M, *)\}.$$

Now let $f : M \to M$ be a continuous map and $f_* : \Gamma \to \Gamma$ be the induced endomorphism of the fundamental group $\Gamma = \pi_1(M, *)$. Let $h(f)$ denote the topological entropy of f. The *exponential growth rate of f_** is

$$\omega(f_*) := \max_{s \in S} \limsup_{n \to \infty} \left(\sqrt[n]{\ell_S(f_*^n(s))} \right)$$

By a result of Bowen,

$$h(f) \geq \log \omega(f_*).$$

We define the *geometric exponential growth rate* $\omega(g, f_*)$ of f_* in an obvious way by taking $L_{g,S}$ as the length function instead of the word length in the preceding definition. It is not hard to see that

$$h(f) \geq \log \omega(g, f_*).$$

The following question is a pure curiosity.

Question. *What is an analogy of Manning's theorem relating the entropy of a continuous map f and the (geometric) exponential growth rate of the corresponding endomorphism f_* ?*

Remark. The invariants $\omega(f_*)$ and $\omega(g, f_*)$ do not depend of the set of generators used.

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Mikhail Belolipetsky

Question. *Is it true that every maximal hyperbolic arithmetic reflection group is congruence?*

This question appeared in [1] and [2] where we prove the finiteness of the number of conjugacy classes of the maximal arithmetic reflection groups. If the answer to the question is positive then the argument in [2] can be significantly simplified and, moreover, the quantitative estimates will become effective. In a sence, the

whole subject of [2] is how to get around this question.
We refer to the cited papers for the precise definitions an related discussion.

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Misha Kapovich[1]

1. Informally speaking, the group of Hamiltonian diffeomorphisms Ham(M, ω) has both "positive and negative curvature features" (compare with the Teichmüller space and the mapping class group). An example of the "positive curvature feature" is existence of geodesics with respect to Hofer's metric (see [5]) which remain at bounded distance from the identity. An example of "hyperbolic-type behavior" is the fact that for certain (M, ω) the space of non-trivial homogeneous quasi-morphisms on Ham is infinite-dimensional (see [1, 4]). It would be interesting to detect some "negative curvature features" of Ham with respect to Hofer's metric. As a first step in this direction, consider the "annulus"

$$A_t = \{f \in \text{Ham} \ : \ t \leq \rho(\text{id}, f) \leq 2t\} \ ,$$

where ρ stands for the Hofer metric. Is it true that A_t is path connected for large t? If it is indeed so, the Hofer length structure induces a distance ρ_t on A_t. What happens with geometry of (A_t, ρ_t) as $t \to \infty$? For instance, fast growth of diam(A_t, ρ_t) can be interpreted as a "hyperbolic feature". At the moment even the following basic question seems to be open: is diam(A_t, ρ_t) finite or infinite for a given t?

2. It is an important problem to understand finitely generated subgroups of Ham(M, ω). An interesting source of examples is given by the right-angled Artin groups (see e.g. [2]). Given such a group, say Γ, it would be interesting to describe the class of symplectic manifolds (M, ω) such that Γ can be realized as a subgroup of Ham(M, ω). Note that right-angled Artin groups contain some interesting subgroups, like some lattices in $PO(n, 1)$. However, embedding of lattices in $PU(n, 1)$ into right-angled Artin groups are unknown for $n > 1$, compare [3], page 10.

Question 1. Let $G \subset PU(n, 1)$, $n \geq 2$ be a lattice. Does there exist a monomorphism of G to Ham(M, ω) for some symplectic manifold (M, ω)? If yes, can one build such a monomorphism with "interesting" dynamical properties?

An interesting particular case of this question is when (M, ω) is a closed oriented surface Σ_g of genus g equipped with an area form. Note that currently we do not

[1] Sections 1,2,5 are composed by L. Polterovich who carries full responsibility for all potential mistakes

have a single example of G which admits an embedding into $\mathrm{Ham}(M,\omega)$ for some M.

Question 2. Does there exist a finitely generated group which admits a monomorphism to $\mathrm{Ham}(\Sigma_g)$ but not to $\mathrm{Ham}(\Sigma_h)$ where $g > h$? In the opposite direction, is it true that every finitely generated subgroup of $\mathrm{Ham}(\Sigma_h)$ admits a monomorphism into $\mathrm{Ham}(\Sigma_g)$?

The positive answer would justify the intuitive feeling that $\mathrm{Ham}(\Sigma_g)$ "has more room" for larger g.

3. Suppose $\Gamma < PU(2, I)$ is a uniform arithmetic lattice generated by complex reflections of order 5 (one of Deligne-Mostow examples). Is there an equivariant totally-geodesic real embedding $(\mathbb{H}^2_{\mathbb{R}} \curvearrowleft \pi_1(\Sigma_g)) \longrightarrow (\mathbb{H}^2_{\mathbb{C}} \curvearrowleft \Gamma)$ whose image is disjoint from the fixed-point sets of reflections?
Motivation: Agol's approach to pseudo-Anosov surface subgroup problem in the mapping class group, Problem 5 below.

4. Is it true that all lattices in $Isom(\mathbb{H}^n), n \geq 4$, are non-coherent?
Remark: There is evidence that all arithmetic groups are not coherent.

5. The following question is open since the beginning of the 90-ies: Does there exist a hyperbolic group Γ which admits an exact sequence

$$1 \to \pi_1(\Sigma_g) \to \Gamma \to \pi_1(\Sigma_h) \to 1 \quad ?$$

If yes, is it true that there is a finite number of such groups for each g and h?

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Gerhard Knieper

Metrics on T^2.
The following question is motivated by the upcoming thesis of E. Leschinsky on metrics on T^2:
Question. *Is there a Riemannian metric on T^2 and a constant $n_0 \in \mathbb{N}$ such that in each homotopy class the number of closed geodesics is less than n_0?*

Parallel postulate problem.
Let g be a complete metric on $\mathbb{R}^2$ with the following property: For each geodesic c and each point p which is not located on c there exists a unique geodesic c' with $c'(0) = p$ which does not intersect c.
Question. *Is the metric g flat?*

Dmitri Burago

Question. *Let M be a complete length (or coarse length) space w.r.t. two metrics d_1, d_2. Let Γ act by isometries w.r.t. both metrics d_1 and d_2, cocompactly. Assume*

$$\lim_{d_1(x,y)\to\infty} \frac{d_1(x,y)}{d_2(x,y)} = 1.$$

Does this imply $|d_1 - d_2| \leq C$ for some constant C (true if Γ is e.g. abelian or hyperbolic)?

Danny Calegari

Define a complex as follows:
The n-simplices are configurations $\Delta := (\sigma_0, ..., \sigma_n)$ of $n+1$ properly embedded rays in the plane, up to (orientation-preserving) homeomorphism. By definition,

$$\partial\Delta = \sum (-1)^i (\sigma_0, ..., \hat{\sigma_i}, ..., \sigma_n) .$$

One can impose various conditions, for instance

(i) The rays are in general position (i.e. any two rays are transverse, and there are no triple points)

or

(ii) The rays are monotone (the order of intersections of any two rays is "increasing").

One can also restrict the equivalence relation somewhat; for instance, one can consider configurations of quasi-isometrically (qi) embedded proper rays up to the equivalence relation of a global quasi-isometry of R^2. (The complex in this case is a kind of qi boundary for R^2 , and might generalize to natural qi boundaries for CAT(0) spaces).

Problem: *Under each of the circumstances above, compute the homotopy type (and cohomology) of the complex.*

A brief summary and some rudimentary observations are contained in the reference

`http://www.its.caltech.edu/~dannyc/notes/8_20_2006.pdf`

Gregory Margulis [2]

Question 1. *What is the answer to the question posed by D. Burago when considering a connected Lie group?*
Question 2. *Let X be a symmetric space and Γ a uniform (nonuniform) lattice. For $T \in \mathbb{R}^+$ let $P(T)$ be the number of conjugacy classes in Γ of length less than T. What can be said about the growth rate of $P(T)$? In particular consider the special case: $X = \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$.*

Dusa McDuff[3]

1. Let (M, ω) be a closed symplectic manifold. Identify the Lie algebra of the group $G := \mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of (M, ω) with the space $\mathcal{F}$ of all smooth functions on M with zero mean. Consider any norm $|| \; ||$ on $\mathcal{F}$ which is invariant under the canonical action of G on $\mathcal{F}$. Such a norm gives rise in the standard way to a Finsler pseudo-distance on G. A priori this pseudo-distance can be degenerate. For instance, the pseudo-distance associated to the L_p-norm with $p \in [1; +\infty)$ vanishes [5]. On the other hand the norm

$$\mathrm{osc}(F) = \max F - \min F$$

on $\mathcal{F}$ generates a genuine distance ρ_H called the Hofer distance (see [11, 9]).

Question 1. *Suppose that a G-invariant norm $|| \; ||$ on $\mathcal{F}$ gives rise to a genuine distance on G. Is it true that this norm is equivalent to the norm* osc*?*

An indication that the answer may be "yes" is given by the recent work by Ostrover and Wagner [12] , who solved "one half" of the problem. They showed that if

$$||F|| \leq \mathrm{const} \cdot \mathrm{osc}(F) \quad \forall F \in \mathcal{F}$$

then necessarily

$$\mathrm{osc(F)} \leq \mathrm{const} \cdot ||F|| \quad \forall F \in \mathcal{F} .$$

2. We start with the following notion due to Hofer. Let ρ be any bi-invariant distance on G. For a subset $X \subset M$ define its *displacement energy* with respect to ρ as

$$e(\rho, X) = \inf \rho(\mathrm{id}, f) ,$$

where the infimum is taken over all Hamiltonian diffeomorphisms $f \in G$ which displace X:

$$f(X) \cap \mathrm{Closure(X)} = \emptyset .$$

One can show [5] that $e(\rho, X) > 0$ for every non-empty open subset X.

[2] Question 2 is based on a discussion of G. Margulis with G. Knieper. It has been composed by G. Knieper who takes full responsibility for potential mistakes

[3] Problems presented by Dusa McDuff composed by L. Polterovich who carries full responsibility for all potential mistakes

We say that $B \subset M$ is *a symplectic ball of capacity* a if B is the image of the standard symplectic ball

$$\{\pi(|p|^2 + |q|^2) < a\} \subset \mathbf{R}^{2n}$$

under a symplectic embedding (here $2n = \dim M$). For the Hofer distance one has the following energy-capacity inequality [9]: given any symplectic ball B of capacity a in M,

$$e(\rho_H, B) \geq \frac{1}{2}a$$

(the factor $\frac{1}{2}$ can be removed for a wide class of symplectic manifolds, see [6]). This inequality motivates the next question.

Question 2. *Let ρ be a bi-invariant distance on G. Does there exist a constant $C > 0$ such that*

$$e(\rho, B) \geq C \cdot a$$

for every symplectic ball $B \subset M$ of capacity a?

An affirmative answer to Question 1 would yield an affirmative answer to Question 2 in the case when ρ is generated by a G-invariant norm on $\mathcal{F}$.

3. It is an interesting problem to explore closed geodesics with respect to the Hofer metric. For instance, every Hamiltonian S^1-action corresponds to a closed geodesic.

Question 3. [4] *Is every element of $\pi_1(G)$ represented by a closed geodesic with respect to the Hofer metric?*

It is known (see e.g. [10, 8, 2]) that in general not every element of $\pi_1(G)$ is represented by an S^1-action.

4. [5] The group G is known to coincide with its commutator subgroup [1], and hence it carries a remarkable bi-invariant distance ρ_{com} called the commutator distance:

$$\rho_{com}(f, g) = cl(fg^{-1}) ,$$

where *cl* stands for the commutator length (the definition is given in Fujiwara's abstract [7], see also [3]).

It is an interesting problem to compare the commutator distance with other bi-invariant distances on G. For instance [5] if two Hamiltonian diffeomorphisms f and g are supported in a subset $U \subset M$ then

$$\rho(\mathrm{id}, [f, g]) \leq \frac{1}{4}e(\rho, U)$$

for every bi-invariant distance ρ on G. Here $[f, g] = fgf^{-1}g^{-1}$. Furthermore, the diameter of G with respect to ρ_{com} and ρ_H is known to be infinite for various symplectic manifolds, though no result is available in full generality (see McDuff's

[4] According to McDuff this question was motivated by a discussion with the participants after her talk.

[5] Joint with Polterovich.

abstract [11] for a discussion of the Hofer diameter). Interestingly enough, the Calabi quasi-morphism (quasi-morphism μ in Entov's abstract [4]), when defined, serves as a common tool for proving that

$$\mathrm{diam}(G, \rho_H) = \mathrm{diam}(G, \rho_{com}) = \infty \ .$$

The following problem would shed some light on the comparison between ρ_H and ρ_{com}. Given a closed symplectic manifold M, define

$$A(M) = \sup_{f,g \in G} \rho_H(\mathrm{id}, [f,g]) \ .$$

Question 4. *Is $A(M)$ finite or infinite?*

We have no answer for *any* symplectic manifold.

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Two problems in measure rigidity

Anatole Katok

1. Existence and arithmeticity of maximal rank hyperbolic measures

Theorem 1. [4] *Let $k \geq 2$, α be a $\mathcal{C}^{1+\epsilon}$, $\epsilon > 0$ action of $\mathbb{Z}^k$ on a $k+1$-dimensional manifold , μ an ergodic invariant measure of α with no proportional Lyapunov exponents and at least one element of α has positive entropy.*

Then μ is absolutely continuous.

The only known model for such action is the algebraic Cartan action on the torus $\mathbb{T}^{k+1}$, i.e. the action by hyperbolic maps with real eigenvalues. All known examples are differentiably conjugate to a Cartan action on an invariant open set. Notice however that there are many manifolds which can carry such actions even if one requires topological transitivity in addition. Those manifolds are constructed by blowing up periodic orbits of a Cartan action and either glueing in projective spaces (a σ-process) or identifying boundary spheres of different holes.

Problem 1. *What compact manifolds carry actions satisfying assumptions of Theorem 1?*

The answer may be different for real-analytic actions where certain restrictions are plausible and smooth (C^∞) actions which are likely to exist on any compact manifold. The key case is that of the ball $\mathbb{D}^{k+1}$. In fact existence of an action on the ball which is sufficiently "flat" at the boundary would imply existence on any compact manifold as in [1].

The most interesting problem concerns certain arithmetic structure present in such actions. It is motivated by the following result for the torus.

Theorem 2. [3, 5] *Let α be a $\mathcal{C}^{1+\epsilon}, \epsilon > 0$, $\mathbb{Z}^k$ action on $\mathbb{T}^{k+1}$ Cartan homotopy data i.e. each element is homotopic to the corresponding element of a linear Cartan action α_0. Then*

- *The set $\mathcal{M}$ consists of a single measure μ.*
- *The measure μ is absolutely continuous.*
- *The semi-conjugacy h is bijective on a set of full measure and thus effects a measurable isomorphism between (α, μ) and (α_0, λ).*
- *The semi-conjugacy is differentiable along almost every leaf of each Lyapunov foliation.*

Problem 2. *What are possible values of entropy for elements of an action α satisfying assumptions of Theorem 1?*

The following conjecture represents a cautiously optimistic view of the situation.

Conjecture. *The entropy values are algebraic integers of degree at most $k+1$.*

2. Higher rank global symplectic rigidity?

Theorem 2 represents the first case of global measure rigidity on the torus. Its proof is based on ideas developed in [6].

Proofs of measure rigidity of linear actions other than linear Cartan or more general *totally non-symplectic* requires different methods; see [2].

As a representative example for a possible global rigidity result consider a linear action α_0 of a maximal abelian subgroup of $SP(4,\mathbb{Z})$ on $\mathbb{T}^4$ diagonalizable over $\mathbb{R}$. Let α be a $\mathbb{Z}^2$ action whose elements are homotopic to the corresponding elements of α_0 and let μ be an α-invariant Borel probability measure such that $h_*\mu = \lambda$ where h is the semiconjugacy between α and α_0 as before.

Problem 3. *Show that μ is absolutely continuous and that $\mu = \omega \wedge \omega$ where ω is a Lebesgue measurable exterior 2-form closed in a properly defined sense.*

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