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Mini-Workshop: Hypercyclicity and Linear Chaos

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Abstract. The mini-workshop was devoted to the study of hypercyclic and chaotic operators within the wider framework of linear dynamical systems. Topics discussed included common hypercyclic vectors; hypercyclic and supercyclic subspaces; extensions of hypercyclicity like Cesàro-, Faber-, and disjoint hypercyclicity; hypercyclic N-tuples and hypercyclic direct sums; hypercyclic C_0-semigroups and hypercyclic polynomials; hypercyclicity in non-metrizable spaces; weak supercyclicity; hypercyclic composition operators; and the influence of the norms $\|T^n\|$ on the dynamical behaviour of T. A list of open problems is included in the report.

Mathematics Subject Classification (2000): 47A16.

Introduction by the Organisers

Chaos has long been thought of as being intrinsically linked to non-linearity. The investigations into hypercyclicity in the last two decades have thoroughly refuted this assumption. Many, even quite natural, linear dynamical systems exhibit chaos; this effect, however, only becomes visible when one studies infinite-dimensional state spaces. Starting from the seemingly innocent definition of a hypercyclic operator, that is, an operator with a dense orbit, the theory has developped into a very active research area, the Theory of Linear Dynamical Systems.

In recent years, several open problems have been solved. For example, Ansari and Bernal have given a positive answer to Rolewicz' question if every separable infinite-dimensional Banach space supports a hypercyclic operator, and Costakis and Peris have given a positive answer to Herrero's question if every multi-hypercyclic operator is hypercyclic. On the other hand, Herrero's problem if for every hypercyclic operator T also $T \oplus T$ is hypercyclic has turned out to be a major challenge. This Great Open Problem in hypercyclicity has, in fact, motivated a fair number of recent investigations.

The solution of another major problem in hypercyclicity was announced by A. Peris during the workshop. In his talk he showed that if $\{T_t : t \geq 0\}$ is a hypercyclic C_0-semigroup then every operator $T_t, t > 0$, in the semigroup must itself be hypercyclic. This problem had previously evaded the efforts of several researchers.

In an attempt to better understand the Great Open Problem, H. Petersson studied the set of hypercyclic vectors for direct sums of operators. He found conditions under which there are dense subspaces $U_i, 1 \leq i \leq N$, such that every vector $(u_i)_{1 \leq i \leq N} \in U_1 \times \ldots \times U_N$ with $u_i \neq 0$ for all i is hypercyclic for $T_1 \oplus \ldots \oplus T_N$.

Hypercyclic vectors for $T_1 \oplus \ldots \oplus T_N$ were also studied by J. Bès. However, motivated by an investigation of Furstenberg, he only considered vectors of the form $(x, \ldots, x)$, and called these vectors d-hypercyclic. He obtained examples of such vectors based on a d-Hypercyclicity Criterion.

In a different direction, N. Feldman called an N-tuple $(T_1, \ldots, T_N)$ of commuting operators hypercyclic if the orbit $\{T_1^{k_1} \cdots T_N^{k_N} x : k_i \geq 0\}$ is dense in X for some vector x. He discussed many examples and presented extensions of results in hypercyclicity to the new setting.

Spectral results in hypercyclicity often involve the unit circle. The talk of S. Grivaux explained why this is so. Replacing the polynomials z^n by the Faber-polynomials F_n^Ω of a non-empty simply connected domain Ω in $\mathbb{C}$ with compact closure and rectifiable boundary, she defined a vector x to be Ω-hypercyclic for an operator T if $\{F_n^\Omega(T)x : n \geq 0\}$ is dense in X. She presented analogs of results in hypercyclicity in this context, where now the rôle of the unit disk is taken over by Ω. In a similar vein, a vector x is called Cesàro-hypercyclic if the set $\{\frac{1}{n} \sum_{k=0}^{n} T^k x : n \geq 0\}$ is dense in X. For this setting, G. Costakis obtained a version of the somewhere dense orbit theorem of Bourdon and Feldman.

Another topic of the workshop were common hypercyclic vectors. F. Bayart obtained a new condition on a parametrized family $(T_\lambda)_{\lambda \in \Lambda}$ to have a dense G_δ-set of common hypercyclic vectors. The proof used both the Baire category theorem and a nontrivial result from probability theory. A different sufficient condition for the same problem was given by K. Chan. He employed his condition to show that any two hypercyclic unilateral weighted backward shifts can be connected by a path of such operators having a dense G_δ-set of common hypercyclic vectors.

H. N. Salas studied the existence of supercyclic subspaces, that is, closed infinite-dimensional subspaces of supercyclic vectors. He presented two sufficient conditions for their existence, and studied their necessity in the special case of backward shift operators. K.-G. Grosse-Erdmann extended the known sufficient condition for the existence of hypercyclic subspaces to the setting of F-spaces, and he obtained an analogous result on frequently hypercyclic subspaces.

The topic of supercyclicity was also taken up by É. Matheron. He first showed that unitary operators can be weakly supercyclic; his example was the multiplication operator M_z on a space $L^2(\mu)$, where μ is a probability measure on $\mathbb{T}$ of small support. He then presented recent work of S. Shkarin saying that the measure μ may even be such that its Fourier coefficients tend to 0.

J. Bonet studied hypercyclicity and chaos of the differentiation operator on some natural spaces of holomorphic functions. The special interest of these results was that he considered spaces that are non-metrizable, so that the Baire category theorem is not available. He also presented a characterization of topological transitivity and chaos of the backward shift on non-metrizable sequence spaces.

F. Martínez characterized hypercyclicity and chaos of certain polynomials on Köthe echelon spaces. He also discussed a result that links the infinite-dimensional dynamics of polynomials to the Julia set of a corresponding complex polynomial.

Two of the talks studied the behaviour of the sequence of norms $\|T^n\|, n \geq 1$, of operators T on Banach spaces. C. Badea characterized those sequences (n_k) such that, for any operator T on any separable Banach space X, $\sup_{k\geq 0} \|T^{n_k}\| < \infty$ implies that the unimodular point spectrum $\sigma_p(T) \cap \mathbb{T}$ is at most countable. As an application he obtained an example of a chaotic and frequently hypercyclic operator on Hilbert space that is not topologically mixing, thereby answering a question of Peris. V. Müller considered an opposite of hypercyclicity, vectors x for which $\|T^n x\|$ is big in some sense. He showed that, for any sequence (T_n) of operators and any positive sequence (a_n) with $\sum_n a_n < \infty$ there is a vector x so that $\|T^n x\| \geq a_n \|T_n\|$ for all $n \geq 0$. He also obtained analogues for weak orbits.

R. Mortini investigated composition operators $f \mapsto f \circ \phi_n, n \geq 1$, on the space $\mathcal{B} = \{f \in H(\Omega) : \sup_{z\in\Omega} |f(z)| \leq 1\}$, where Ω is a domain in $\mathbb{C}^N$ and the ϕ_n are automorphisms or, more generally, holomorphic self-maps of Ω. He characterized when there exists an $f \in \mathcal{B}$ so that $\{f \circ \phi_n : n \geq 1\}$ is locally uniformly dense in $\mathcal{B}$, and also presented an analogous result when $\mathcal{B}$ is replaced by the space $H(\Omega)$ of all holomorphic functions on Ω.

It is a distinctive feature of hypercyclicity that it unites researchers from various backgrounds, be it topological dynamics, operator theory, semigroup theory, the theory of locally convex spaces or complex analysis, which was also reflected in the composition of the participants. A very lively exchange of ideas characterized the mini-workshop, which was particularly apparent in the two Special Sessions that both ran overtime. Some of the problems discussed in the Problem Session on Wednesday morning and some of the contributions presented in the Informal Session on Friday morning are collected at the end of this report, while additional problems can be found in the abstracts of J. Bès, N. Feldman, F. Martínez, H. Petersson and H. N. Salas.

The mini-workshop was organized by Teresa Bermúdez (La Laguna), Gilles Godefroy (Paris), Karl-G. Grosse-Erdmann (Hagen), and Alfredo Peris (Valencia). Unfortunately, Teresa Bermúdez was unable to participate. The participants greatly appreciated the hospitality and the stimulating atmosphere of the Forschungsinstitut Oberwolfach.

Mini-Workshop: Hypercyclicity and Linear Chaos

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Abstracts

Common Hypercyclic Vectors for Paths of Shift Operators

Kit C. Chan
(joint work with Rebecca Sanders)

Let X be a separable, infinite dimensional Banach space, and $B(X)$ be the algebra of all bounded linear operators $T : X \to X$. A sequence of bounded linear operators $\{T_n \in B(X) : n \geq 1\}$ is said to be *universal* if there exists a vector x in X such that the set $\{T_n x : n \geq 1\}$ is dense in X. Such a vector x is said to be a *universal vector* for the sequence $\{T_n\}$. In the case that there is a single bounded linear operator T in $B(X)$ whose sequence of positive powers $\{T^n\}$ is universal, then we say that the operator T is *hypercyclic*, and the corresponding vector x is said to be a *hypercyclic vector.*

In 1982, Kitai [7] showed that whenever T is hypercyclic, the set of all hypercyclic vectors for T is a dense G_δ subset of X. It follows from the Baire Category Theorem that if each operator T_n in a countable family $\{T_n : n \geq 1\}$ is hypercyclic, then the set of vectors that are hypercyclic for each T_n is also a dense G_δ subset of X. Such a vector is said to be a *common hypercyclic vector* for the family $\{T_n\}$. Since the Baire Category Theorem does not go beyond countability, it is interesting to investigate the possibility of having common hypercyclic vectors when the family of hypercyclic operators is uncountable. For that, we introduce the concept of *a path of operators*, which is a continuous function F from a real interval I into the operator algebra $B(X)$. A path of hypercyclic operators is said to *have a dense G_δ set of common hypercyclic vectors,* if the set of all common hypercyclic vectors contains a dense G_δ subset, but not necessarily that the set itself is dense G_δ.

The first examples of hypercyclic operators on a Banach space were offered in 1969 by Rolewicz [9]. To explain his result, we first need a definition. An operator $T : \ell^p \to \ell^p$ on the Banach space ℓ^p, with $1 \leq p < \infty$, is said to be a *unilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_n > 0 : n \geq 1\}$ such that

$$T(a_0, a_1, a_2, \ldots) = (w_1 a_1, w_2 a_2, w_3 a_3, \ldots).$$

When all $w_n = 1$, the operator is simply said to be the *unilateral backward shift,* and is denoted by B. What Rolewicz [9] showed was that each unilateral weighted backward shift tB, where $t \in (1, \infty)$, is a hypercyclic operator. In 1999, Salas [11] raised the question whether all those operators tB have a common hypercyclic vector. Then in 2003, Abakumov and Gordon [1] answered Salas's question in the positive, and a year later Costakis and Sambarino [3] proved that the path tB, with $t \in (1, \infty)$, indeed has a dense G_δ set of common hypercyclic vectors. Meanwhile they also exhibited another path of unilateral weighted backward shifts that has a dense G_δ set of common hypercyclic vectors. About the same time, for general operators, León-Saavedra and Müller [8] proved that whenever T is a hypercyclic operator, every operator in the path of rotations $\{e^{it}T : t \in [0, 2\pi]\}$ has the same hypercyclic vectors as T itself. The above interesting examples motivate us to investigate whether every path of hypercyclic operators has a common hypercyclic vector. As it turns out, the answer is negative.

Theorem 1. *There is a path of hypercyclic unilateral weighted backward shifts with no common hypercyclic vector.*

Since it is easy to show that the set of all hypercyclic unilateral weighted backward shifts is path connected, Theorem 1 implies that between any two hypercyclic unilateral weighted backward shifts there is a path of such operators having no common hypercyclic vector. This may sound a little disappointing, but it raises the natural question whether we can actually find one path that has common hypercyclic vectors. Along this line, we are able to prove the following.

Theorem 2. *Between any two hypercyclic unilateral weighted backward shifts, there exists a path of such operators having a dense G_δ set of common hypercyclic vectors.*

Our proof for the above existence result is rather complicated, because the weights of a hypercyclic unilateral weighted backward shift may be very small; see Salas [10]. Nevertheless, it inspires us to find a necessary and sufficient condition for a general family of operators to have a dense G_δ set of common universal vectors.

Theorem 3. *Let $\{(F_{t,n}) \subset B(X) : n \geq 1, t \in [a, b]\}$ be a family of operators. Suppose the map $t \mapsto F_{t,n}$ is continuous on $[a, b]$ for each $n \geq 1$. Then $(F_{t,n})$ has a dense G_δ set of common universal vectors if and only if for any pair of nonempty open subsets U_1 and U_2 of X, there is a partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$, and positive integers $n_1, n_2, \ldots, n_k$, and a nonempty open set $V \subset U_1$, such that*

$$F_{t,n_i}(V) \subset U_2, \quad \textit{whenever} \;\; t \in [t_{i-1}, t_i] \textit{ and } 1 \leq i \leq k.$$

Using Theorem 3, one derives the above result of Costakis and Sambarino [3] as a corollary.

Corollary 4. *The path $\{tB : \ell^p \to \ell^p | t \in (1, \infty)\}$ has a dense G_δ set of common hypercyclic vectors.*

The necessary and sufficient condition in Theorem 3 enables us to put down a more concrete sufficient condition that can be applied to other paths of operators. Such a sufficient condition was first found by Costakis and Sambarino [3], but our condition is different in the sense that we do not require a comparison of certain growth rates with a convergent series.

Theorem 5. *Let $\{(F_{t,n}) \subset B(X) : n \geq 1, t \in [a, b]\}$ be a family of operators. Suppose the map $t \mapsto F_{t,n}$ is continuous on $[a, b]$ for each $n \geq 1$. Then $(F_{t,n})$ has a dense G_δ set of common universal vectors if there are dense sets D_1 and D_2 in X, an increasing sequence of integers (m_k), and mappings $S_{t,k} : D_1 \to X$ such that for each $h \in D_1$ and for each $\epsilon > 0$, there exists a $\delta > 0$ satisfying:*

(1) *for all* $t \in [a,b]$*, we have* $\|S_{t,k}h\| \to 0$*, as* $k \to \infty$*;*
(2) *for all* $f \in D_2$*, we have* $\|F_{t,m_k}f\| \to 0$ *uniformly for all* $t \in [a,b]$*;*
(3) *for all* $t \in [a,b]$ *and all* $N \geq 1$*, there exists an integer* $k \geq N$*, such that*

$$\|F_{t',m_k}S_{t,k}h - h\| < \epsilon, \quad \textit{whenever } |t'-t| < \delta.$$

When $a = b$, the interval $[a,b]$ in Theorems 3 and 5 becomes a singleton set and in that case we have exactly one sequence of operators $\{T_n\}$. Then Theorem 3 reduces to the classical results of Große-Erdmann [6] and Godefroy and Shapiro [5], and Theorem 5 reduces to the well-known hypercyclicity criterion. The criterion was first obtained by Kitai [7] in 1982, and it was rediscovered in a much greater generality by Gethner and Shapiro [4] with a different proof. The version to which Theorem 5 reduces was provided by Bès and Peris [2]. The hypercyclicity criterion is a sufficient condition for hypercyclicity and universality. It has been used widely to prove that certain operators are hypercyclic.

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Disjointness in Hypercyclicity

Juan Bès
(joint work with Alfredo Peris)

Given $N \geq 2$ operators $T_1, T_2, \dots, T_N$ on a Fréchet space X, it has been natural to study the cyclic properties that their direct sum $T_1 \oplus \cdots \oplus T_N$ may inherit from those of T_1, T_2,…,T_N. A significant example of this is Herrero's [5] central question on hypercyclicity and weak mixing:

Must $T \oplus T$ *be hypercyclic whenever* T *is?*

Another example is Salas's characterization of those weighted shifts T_1,…,T_N on $X = c_0$ or ℓ_p $(1 \leq p < \infty)$ for which $T_1 \oplus \cdots \oplus T_N$ is hypercyclic or supercyclic [7], [8]. But while the dynamical systems $(X, T_1), \dots, (X, T_N)$ may be independent of each other (i.e., for $i \neq j$ the action of T_i may not depend on T_j and vice-versa), they may still have a certain correlation (or lack of it) manifested in the behaviour of the co-orbits

$$Co - Orb(z; T_1, \dots, T_N) = \{(T_1^n z, T_2^n z, \dots, T_N^n z) : \quad n = 0, 1, \dots\} \qquad (z \in X).$$

In this talk we propose to consider the situation in which dense co-orbits exist, a weaker notion than Furstenberg's disjointness of fluid flows [4].

Definition 1. We say that $N \geq 2$ hypercyclic operators $T_1, \dots, T_N$ acting on a Fréchet space X are *disjointly hypercyclic* (in short, d-hypercyclic) provided there is some vector z in X such that $Co - Orb(z; T_1, \dots, T_N)$ is dense in X^N. We call such z a d-hypercyclic vector for the operators T_1, T_2,…,T_N.

Definition 2. We say that $N \geq 2$ sequences of operators $(T_{1,j})_{j=1}^\infty$, …,$(T_{N,j})_{j=1}^\infty$ in $B(X)$ are *d-topologically transitive* (respectively, *d-mixing*) provided for every non-empty open subsets V_0,…,V_N of X there exists $m \in \mathbb{N}$ so that $\emptyset \neq V_0 \cap T_{1,m}^{-1}(V_1) \cap \cdots \cap T_{N,m}^{-1}(V_N)$ (respectively, so that $\emptyset \neq V_0 \cap T_{1,j}^{-1}(V_1) \cap \cdots \cap T_{N,j}^{-1}(V_N)$ for each $j \geq m$). Also, we say that $N \geq 2$ operators T_1,…,T_N in $B(X)$ are d-topologically transitive (respectively, d-mixing) provided $(T_1^j)_{j=1}^\infty$,…, $(T_N^j)_{j=1}^\infty$ are d-topologically transitive sequences (respectively, d-mixing sequences).

The results presented here are taken from [2].

A. The d-Hypercyclicity Criterion.

Definition 3. We say that $T_1, \dots, T_N \in B(X)$ satisfy the *d-Hypercyclicity Criterion* provided there exists a strictly increasing sequence (n_k) of positive integers, dense subsets X_0, X_1,…,X_N of X, and mappings $S_{l,k} : X_l \to X$ $(1 \leq l \leq N,\ k \in \mathbb{N})$ satisfying

$$\begin{aligned} T_l^{n_k} &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_0, \\ S_{l,k} &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_l, \text{ and} \\ (T_l^{n_k} S_{i,k} - \delta_{i,l}\, Id_{X_l}) &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_l \ (1 \leq i \leq N). \end{aligned} \tag{1}$$

Theorem 4. *Let $T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion. Then for some sequence (n_k) of positive integers the sequences of operators $(T_1^{n_k})_k, \dots, (T_N^{n_k})_k$ are d-topologically mixing. In particular, $T_1, \dots, T_N$ have a dense manifold of d-hypercyclic vectors.*

As happens for the case of $N = 1$ operator [3], the converse of Theorem 4 also holds and may be expressed in terms of hereditary dense d-hypercyclicity [2, Theorem 2.7].

B. Disjoint differentiation operators.

Let $H(\mathbb{C})$ be the space of entire functions on the complex plane $\mathbb{C}$, and endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. For each $a \in \mathbb{C}$, we let T_a denote the translation operator on $H(\mathbb{C})$ given by $T_a(f)(z) = f(z+a)$ ($z \in \mathbb{C}, f \in H(\mathbb{C})$), and by D the derivative operator.

Theorem 5. *Let $a_1, \ldots, a_N \in \mathbb{C} \setminus \{0\}$ $(N \geq 2)$ be given. Then*

(a) *The translation operators $T_{a_1}, \ldots, T_{a_N}$ are d-hypercyclic on $H(\mathbb{C})$ if and only if the scalars $a_1, \ldots, a_N$ are pairwise distinct.*

(b) *The differentiation operators $a_1 D^{r_1}, \ldots, a_N D^{r_N}$ are d-hypercyclic on $H(\mathbb{C})$ for any selection of integers $1 \leq r_1 < \cdots < r_N$.*

Using different methods, Theorem 5(a) and generalizations were obtained independently by L. Bernal [1].

C. Disjoint powers of weighted shifts.

For each bounded sequence $w = (w_n)$ of scalars, let B_w be the unilateral backward shift on $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leq p < \infty)$ given by $B_w(x_n) = (w_{n+1}x_{n+1})$. Also, let B be the backward shift with constant weight sequence $w = (1)$. In [2] we characterize the d-hypercyclicity of different powers of (unilateral/bilateral) backward weighted shifts in terms of the different weight sequences. As a consequence, we have:

Theorem 6. *Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ $(1 \leq p < \infty)$, and let integers $0 = r_0 < 1 \leq r_1 < \cdots < r_N$ be given $(N \geq 2)$. Then for any non-zero scalars $a_1, \ldots, a_N$, the operators $a_1 B^{r_1}, \ldots, a_N B^{r_N}$ are d-hypercyclic if and only if $1 < |a_1| < \cdots < |a_N|$. Moreover, for any backward shift B_w the following are equivalent.*

(i) *$B_w^{r_1}, B_w^{r_2}, \ldots, B_w^{r_N}$ are d-hypercyclic on X.*

(ii) $\sup_{m \in \mathbb{N}} \left(\min \left\{ \prod_{i=1}^{m(r_l - r_s)} w_i : \quad 0 \leq s < l \leq N \right\} \right) = \infty.$

(iii) *$B_w^{r_1} \oplus \cdots \oplus B_w^{r_N} \oplus \oplus_{1 \leq s < l \leq N} B_w^{(r_l - r_s)}$ is hypercyclic on $X^{\frac{N(N+1)}{2}}$.*

In particular, $B_w, B_w^2, \ldots, B_w^N$ are d-hypercyclic if and only if the operator $B_w \oplus B_w^2 \oplus \cdots \oplus B_w^N$ is hypercyclic on X^N.

D. Disjoint hypercyclicity and Hilbert adjoints.

In [6], Salas constructed a hypercyclic operator T on a Hilbert space so that its Hilbert adjoint T^* is also hypercyclic. A simple argument shows that it is not possible to have an operator that is disjointly hypercyclic with its Hilbert adjoint. However, we have the following extension of Salas' result.

Theorem 7. *For each $N \geq 2$, there exist d-hypercyclic operators $T_1, \ldots, T_N$ on $\ell_2(\mathbb{Z})$ so that their Hilbert adjoints $T_1^*, \ldots, T_N^*$ are also d-hypercyclic.*

E. Final comments.

In joint work (in progress) with A. Peris we investigate spectral conditions for the existence of d-hypercyclic subspaces, that is, of closed, infinite dimensional subspaces of a Banach space X consisting entirely (except for the origin) of d-hypercyclic vectors for given operators $T_1, \dots, T_N$ on X.

We conclude with three problems:

Problem 8. *Let $T_1, \dots, T_N$ be d-hypercyclic and invertible. Must $T_1^{-1}, \dots, T_N^{-1}$ be d-hypercyclic?*

Problem 9. *Let $T_1, \dots, T_N$ be d-hypercyclic, and let $\lambda_1, \dots, \lambda_N$ be scalars of modulus 1, where $N \geq 2$. Must $\lambda_1 T_1, \dots, \lambda_N T_N$ be d-hypercyclic? Must $T_1, \dots, T_N$ and $\lambda_1 T_1, \dots, \lambda_N T_N$ have the same set of d-hypercyclic vectors?*

Problem 10. *Does every separable, infinite dimensional Banach space support a pair (or more) of d-hypercyclic operators?*

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Hypercyclic and chaotic polynomials on Fréchet spaces

Félix Martínez-Giménez
(joint work with Alfredo Peris)

N. Bernardes [1] proved that there are no hypercyclic homogeneous polynomials of degree $d > 1$ on any Banach space. Contrary to the Banach space case, A. Peris showed that the Fréchet space $\omega := \mathbb{C}^{\mathbb{N}}$ admits chaotic homogeneous polynomials of arbitrary degree [4]. For Banach spaces he showed that l_p also admits a (non-homogeneous) chaotic polynomial of degree d, for each $d > 1$ [5]. These examples motivated us to study them when they are defined on more general sequence spaces, namely Köthe echelon sequence spaces.

We recall that a continuous map $f : X \to X$ on a separable Fréchet space is said to be chaotic in the sense of Auslander and Yorke (AY-chaotic) if (1) it is topologically transitive, i.e. hypercyclic in our setting, and (2) it has sensitive dependence on initial conditions; f is said to be chaotic in the sense of Devaney (D-chaotic) if (1) it is topologically transitive, and (2) it has a dense set of periodic points.

A map $P : X \to X$ is a d-homogeneous polynomial if there exists a continuous and d-multilinear map $A : X \times \cdots \times X \to X$ such that $P(x) = A(x, \ldots, x)$. A continuous polynomial on X is a linear combination of homogeneous polynomials.

A Köthe matrix $A = (a_{j,k})_{j,k\in\mathbb{N}}$ is a scalar matrix such that for each $j \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with $a_{j,k} > 0$, and such that $0 \le a_{j,k} \le a_{j,k+1}$ for all $j, k \in \mathbb{N}$. We consider the Köthe echelon sequence space

$$\lambda_p(A) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \ : \ \|x\|_k := \left(\sum_{j=1}^{\infty} |x_j \ a_{i,k}|^p \right)^{1/p} < \infty, \ \forall \ k \in \mathbb{N} \right\},$$

and for $p = 0$

$$\lambda_0(A) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \ : \ \lim_{j\to\infty} x_j \ a_{j,k} = 0, \ \|x\|_k := \sup_{j\in\mathbb{N}} |x_j| a_{j,k}, \ \forall \ k \in \mathbb{N} \right\}.$$

Theorem 1. *Consider the d-homogeneous polynomial ($d > 1$)*

$$\begin{array}{rccc} P: & \lambda_p(A) & \longrightarrow & \lambda_p(A) \\ & (x_i)_{i\ge1} & \longmapsto & (x_{i+1}^d)_{i\ge1} \end{array}$$

The following are equivalent:

(1) *P has a non-zero periodic point.*
(2) *For each $k \in \mathbb{N}$, $\{a_{j,k}\}_{j\ge1} \in l_p$.*
(3) *$l_\infty \subset \lambda_p(A)$ canonically.*
(4) *P is D-chaotic.*

Examples 1. (1) If $A = (a_{j,k})_{j,k\in\mathbb{N}}$ satisfies that for all $k \in \mathbb{N}$ there exists $j(k)$ with $a_{j,k} = 0$ for each $j \ge j(k)$, then we have $\omega = \lambda_1(A)$ and P is chaotic on ω.

(2) For $A = (a_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{N}} = (e^{-j/k})_{j\in\mathbb{N}_0,k\in\mathbb{N}}$ we have $\lambda_p(A) \cong \mathcal{H}(\mathbb{D})$ endowed with the compact open topology and with the isomorphism

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j \mapsto \left(\frac{f^{(j)}(0)}{j!} \right)_{j\ge0}.$$

We thus obtain that the polynomial $P\left(f(z)\right) = \sum_{j=0}^{\infty} \left(\frac{f^{(j+1)}(0)}{(j+1)!} \right)^d z^j$ is chaotic on $\mathcal{H}(\mathbb{D})$.

(3) For $A = (a_{j,k})_{j\in\mathbb{N}_0,k\in\mathbb{N}} = (e^{jk})_{j\in\mathbb{N}_0,k\in\mathbb{N}}$ we have $\lambda_p(A) \cong \mathcal{H}(\mathbb{C})$ endowed with the compact open topology and with the same isomorphism as before. In that case P is not chaotic on $\mathcal{H}(\mathbb{C})$.

Theorem 2. *Consider the non-homogeneous polynomial ($d > 1$)*

$$\begin{array}{rccc} Q: & \lambda_p(A) & \longrightarrow & \lambda_p(A) \\ & (x_i)_{i\ge1} & \longmapsto & ((x_{i+1}+1)^d - 1)_{i\ge1} \end{array}$$

(1) *If $\liminf_j \frac{a_{j,k}}{d^j} = 0$ for each $k \in \mathbb{N}$, then Q is hypercyclic (even AY-chaotic).*
(2) *If $\left\{ \frac{a_{j,k}}{d^j} \right\}_{j=1}^{\infty} \in l_p$ for each $k \in \mathbb{N}$, then Q is D-chaotic.*

Examples 2. (1) For $A = (1)_{j,k\in\mathbb{N}}$ we have $\lambda_p(A) = l_p$ and Q is D-chaotic on l_p.

(2) For $A = (j^k)_{j,k\in\mathbb{N}}$ we have that $\lambda_p(A)$ is a nuclear space. This implies $\lambda_p(A) = \lambda_q(A) = \lambda_0(A)$ for any $1 \le p, q \le \infty$, and therefore the space of rapidly decreasing sequences $s := \left\{(x_j)_{j\ge 1} \in \mathbb{C}^{\mathbb{N}} \ : \ \lim_{j\to\infty} |x_j| j^k = 0, \ \forall\, k \in \mathbb{N}\right\} = \lambda_0(A)$. The polynomial Q is chaotic on s.

(3) The space of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \|f\|_k := \sup\{|x^\alpha f^{(\beta)}(x)| : x \in \mathbb{R},\ \alpha+\beta \le k\} < \infty,\ \forall k \in \mathbb{N} \right\}$$

is isomorphic to s with the isomorphism $f(x) = \sum_{n=0}^{\infty} a_n H_n(x) \mapsto (a_n)_{n\ge 0}$ where $a_n := \langle f, H_n \rangle = \int_{-\infty}^{\infty} f(x) H_n(x) dx$, and $H_n(x)$ is the n-th Hermite polynomial. We have $Q(f(x)) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{d} \binom{d}{k} a_{n+1}^k \right) H_n(x)$ is D-chaotic on $\mathcal{S}(\mathbb{R})$.

Concerning the proofs of Theorem 1 and 2 we will mention here that in both cases we make a "linearization" of the polynomial in order to construct a commutative diagram and then we use characterizations for hypercyclic and chaotic weighted backward shifts [3, 2] and the following lemma.

Lemma 1 ([3]). *Suppose $f_i : E_i \to E_i$ is a continuous map on a separable Fréchet space E_i, $i = 1, 2$, and $\phi : E_1 \to E_2$ is a continuous map with dense range such that $\phi \circ f_1 = f_2 \circ \phi$. If f_1 is hypercyclic (AY-chaotic, D-chaotic) then so is f_2.*

It should be pointed out that in all previous examples we get polynomials which are AY-chaotic or D-chaotic, in particular hypercyclic polynomials with sensitive dependence on initial conditions. The next result, due to A. Peris [5], shows that hypercyclicity is equivalent to sensitive dependence on initial conditions at least for a class of polynomials on l_p. First observe that both types of polynomials we have studied are included in the following class

$$\begin{array}{ccc} \lambda_p(A) & \xrightarrow{P} & \lambda_p(A) \\ (x_1, x_2, \dots) & \longmapsto & (q(x_2), q(x_3), \dots) \end{array}$$

where $q(x)$ is a fixed complex polynomial of degree $d > 1$ with $q(0) = 0$ (this is a necessary condition to have the polynomial well defined).

Theorem 3 ([5]). *For the particular case $\lambda_p(A) = l_p$ the following are equivalent*

(i) *P is AY-chaotic.*
(ii) *P is hypercyclic.*
(iii) *P has sensitive dependence on initial conditions.*
(iv) $0 \in \mathcal{J}(q)$ *(the Julia set of q).*

Remark 1. Observe that the chaotic behavior of P is also related with the complex dynamics of q. In other words, there is a "link" between the infinite dimensional dynamics of P and the finite dimensional complex dynamics of q.

Going one step further, chaos in the sense of Devaney seems to be a stronger condition than AY-chaos. By a result of Fatou and Julia, $\mathcal{J}(q)$ is the closure of the repelling periodic points of q. It is known that if 0 is a fixed repelling point of

q then P is D-chaotic [5], but we do not know if $0 \in \mathcal{J}(q)$ is a sufficient condition for P being D-chaotic.

Conjecture. $P : l_p \to l_p$ is D-chaotic if and only if 0 is a repelling fixed point of $q(x)$.

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Hypercyclic operators in C_0-semigroups

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(joint work with José A. Conejero, Vladimir Müller)

The investigation of hypercyclic C_0-semigroups was initiated by Desch, Schappacher and Webb in [10], although the first example of a hypercyclic semigroup was given by G.D. Birkhoff [3] who showed that, given any $a \in \mathbb{C} \setminus \{0\}$, the translation semigroup $\{T_t := T_{ta} : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})\ ;\ t \geq 0\}$, (where $(T_b f)(z) := f(b+z)$ for each $f \in \mathcal{H}(\mathbb{C})$) defined on the Fréchet space $\mathcal{H}(\mathbb{C})$ of entire functions on the complex plane endowed with the compact-open topology, is hypercyclic.

So far, several specific examples of hypercyclic semigroups have been studied, see for example [10, 8, 11, 15]. In [1] Bermúdez, Bonilla and Martinón proved that every separable infinite dimensional Banach space admits a hypercyclic semigroup. This result was extended to Fréchet spaces in [4].

We recall that a one parameter family $\mathcal{T} = \{T_t\}_{t\geq 0}$ of operators in the space $L(X)$ of (continuous and linear) operators on a separable F-space X is a *strongly continuous semigroup* (or C_0-semigroup) of operators in $L(X)$ if $T_0 = I$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$, and $\lim_{t\to s} T_t x = T_s x$ for all $s \geq 0$, $x \in X$. A C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is said to be hypercyclic if $Orb(\mathcal{T}, x) := \{T_t x\ :\ t \geq 0\}$ is dense in X for some $x \in X$.

Given $T \in L(X)$, let us denote by $HC(T)$ the set of all hypercyclic vectors of T, and analogously, denote by $HC(\mathcal{T})$ the set of hypercyclic vectors of a C_0-semigroup $\mathcal{T}$. It is easy to see that if $\mathcal{T} = \{T_t\}_{t\geq 0}$ is a C_0-semigroup and some operator T_t in the semigroup is hypercyclic, then the semigroup $\mathcal{T}$ itself is hypercyclic.

For the converse situation (from the continuous to the discrete case), as a consequence of an old result of Oxtoby and Ulam [14] it is possible to establish that, if $x \in HC(\mathcal{T})$ is hypercyclic, then there exists a residual set $G \subset \mathbb{R}_+$, such that

$x \in HC(T_t)$ for all $t \in G$ (see, e.g., [5]). The point here is whether $G = \mathbb{R}_+$. That is, if $\mathcal{T} = \{T_t\}_{t\geq 0}$ is a hypercyclic C_0-semigroup, is every operator T_t, $t > 0$, hypercyclic? This problem was explicitly stated in [1].

Our main result is the solution to this problem in the affirmative. To do this we adapt an argument due to León and Müller [13] on rotations of hypercyclic operators. This approach is not new: Several authors have tried to use similar arguments to the ones in [13] for the C_0-semigroups context without success (e.g., [6], [12] and [9]). The key point in the proof, proceeding by contradiction, is to construct a pair of continuous maps $f : HC(\mathcal{T}) \to \mathbb{T}$ and $g : \mathbb{D} \to HC(\mathcal{T})$ such that $f \circ g|_{\mathbb{T}}$ is homotopically nontrivial. Such a point has resisted previous attempts (notice that the homotopy in [9] does not yield any contradiction, which results in a serious gap).

We first need a technical result, which is a consequence of an adaptation to F-spaces of a result of Costakis and Peris [7], using ideas of Wengenroth [16].

Lemma 1. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a hypercyclic semigroup in $L(X)$. If $t > 0$, $(\lambda_1, \lambda_2) \neq (0, 0)$ and $x \in HC(\mathcal{T})$, then $\lambda_1 x + \lambda_2 T_t x \in HC(\mathcal{T})$.*

Theorem 2. *Let $\mathcal{T} = \{T_t\}_{t\geq 0}$ be a hypercyclic semigroup on $L(X)$, and let $x \in HC(\mathcal{T})$. Then $x \in HC(T_{t_0})$ for every $t_0 > 0$.*

Sketch of Proof. Without loss of generality, we may assume that $t_0 = 1$.

Let $\mathbb{T} := \{z \in \mathbb{C} \ : \ |z| = 1\}$ denote the unit circle, $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$ the closed unit disc, and let $\mathbb{R}_+ := \{t \in \mathbb{R} \ : \ t \geq 0\}$.

We define the map $\rho : \mathbb{R}_+ \to \mathbb{T}$ by $\rho(t) := e^{2\pi i t}$. For every pair $u, v \in X$ let

$$F_{u,v} := \left\{\lambda \in \mathbb{T} : \exists (t_n)_n \subset \mathbb{R} \text{ with } \lim_n t_n = \infty, \ \lim_n T_{t_n} u = v, \text{ and } \lim_n \rho(t_n) = \lambda\right\}.$$

The proof is divided into several steps. The first three steps follow in a straightforward way from the compactness of $\mathbb{T}$ and the definition of $F_{u,v}$.

Step 1 *If $u \in HC(\mathcal{T})$, then $F_{u,v} \neq \emptyset$ for all $v \in X$.*

Step 2 *If $\lim_k v_k = v$, $\lambda_k \in F_{u,v_k}$, and $\lim_k \lambda_k = \lambda$, then $\lambda \in F_{u,v}$.* (In particular, $F_{u,v}$ is a closed set for each $u, v \in X$.)

Step 3 *If $u, v, w \in X$, $\lambda \in F_{u,v}$, and $\mu \in F_{v,w}$, then $\lambda\mu \in F_{u,w}$.*

Fix now $x \in HC(\mathcal{T})$. By Steps 1, 2 and 3, $F_{x,x}$ is a nonempty closed subsemigroup of $\mathbb{T}$ with multiplication. Firstly, suppose that $F_{x,x} = \mathbb{T}$. Then, given $y \in X$, by Steps 1 and 3 we get $F_{x,y} = \mathbb{T}$. In particular $1 \in F_{x,y}$, which yields that (with some technical arguments) $T_1 y \in \overline{Orb(T_1, x)}$. Since T_1 has dense range and $y \in X$ is arbitrary, then x is hypercyclic for T_1.

For the rest of the proof we assume that $F_{x,x} \neq \mathbb{T}$, and we will show that it leads to a contradiction. The next step is a consequence of the obvious description of the closed subsemigroups of $\mathbb{T}$.

Step 4 *There exists some $k \in \mathbb{N}$ such that, for each $y \in HC(\mathcal{T})$, there is $\lambda \in \mathbb{T}$ satisfying $F_{x,y} = \{\lambda z \ : \ z^k = 1\}$.*

Step 5 *There is a continuous function $h : \mathbb{D} \to \mathbb{T}$, whose restriction to the unit circle is homotopically nontrivial. A contradiction.* Consider the function $f : HC(\mathcal{T}) \to \mathbb{T}$ as $f(y) := \lambda^k$, where $\lambda \in F_{x,y}$. Clearly, by Steps 2 and 4, f is well defined and continuous. It easily follows that $f(T_t x) = e^{2\pi i t k}$ for every $t \geq 0$.

We will find $g : \mathbb{D} \to HC(\mathcal{T})$ such that $h := f \circ g$ is the desired function which will give the contradiction. We first define $g : \mathbb{T} \to HC(\mathcal{T})$, and then extend it to $\mathbb{D}$. To do this, we fix $t_0 > 1$ so that $T_{t_0} x$ is "close enough" to x, and define $g : \mathbb{T} \to HC(\mathcal{T})$ by

$$g(e^{2\pi i t}) := \begin{cases} T_{2tt_0} x & \text{if } 0 \leq t < 1/2, \\ (2t-1)x + (2-2t)T_{t_0} x & \text{if } 1/2 \leq t < 1. \end{cases}$$

Clearly, g is well defined and continuous. The selection of t_0 gives that the index of $f \circ g$ at 0 is greater than or equal to $[t_0]k$.

We extend the function g to $\mathbb{D}$ by defining $g(z) := (1-|z|)x + |z| g(z/|z|)$ for each $z \neq 0$, and $g(0) = x$.

To sum up, we have a continuous function $h := f \circ g : \mathbb{D} \to \mathbb{T}$, such that its restriction to the unit circle is homotopically nontrivial, a contradiction.

□

Motivated by Birkhoff's ergodic theorem, Bayart and Grivaux introduced the notion of frequent hypercyclicity [2] by quantifying the frequency with which an orbit meets open sets. To be precise, let us define the *lower density* of a set $A \subset \mathbb{N}$ by $\underline{\mathrm{dens}}(\mathrm{A}) := \liminf_{\mathrm{N}\to\infty} \#\{\mathrm{n} \leq \mathrm{N} : \mathrm{n} \in \mathrm{A}\}/\mathrm{N}$. An operator $T \in L(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that, for every non-empty open subset $U \subset X$, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. Each such vector x is called a *frequently hypercyclic vector* for T.

Analogously, if we define the lower density of a measurable set $M \subset \mathbb{R}_+$ by $\underline{\mathrm{Dens}}(M) := \liminf_{N\to\infty} \mu(M \cap [0, N])/N$, where μ is the Lebesgue measure on $\mathbb{R}_+$, then a C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ in $L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set $\{t \in \mathbb{R}_+ : T_t x \in U\}$ has positive lower density. In both cases, frequent hypercyclicity is stronger than hypercyclicity.

We prove that, if a C_0-semigroup $\mathcal{T} = \{T_t\}_{t\geq 0}$ is frequently hypercyclic, then every single operator $T_t \neq I$ is frequently hypercyclic, and shares with the semigroup the same frequently hypercyclic vectors.

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Why the unit circle?

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(joint work with Catalin Badea)

1. Introduction

Let X be a complex infinite dimensional separable Banach space, and $T \in \mathcal{B}(X)$ a bounded linear operator on X. The operator T is said to be *hypercyclic* if there exists a vector $x \in X$ such that $\{T^n x \,;\, n \geq 0\}$ is dense in X. The motivation of our study here comes from the following observation: there are several results in the spectral theory of hypercyclic operators involving the unit disk $\mathbb{D}$ or the unit circle $\mathbb{T}$. The simplest of these spectral properties appears in the early work of Kitai: if $T \in \mathcal{B}(X)$ is hypercyclic, every connected component of the spectrum $\sigma(T)$ of T meets the unit circle $\mathbb{T}$. Or, according to a result of Herrero, if H is a Hilbert space, the norm-closure in $\mathcal{B}(H)$ of the set $HC(H)$ of hypercyclic operators on H can be completely described in terms of spectral properties of the operator, some of them involving the unit circle. Herrero's proof of this result uses in a crucial way a criterion for hypercyclicity due to Godefroy and Shapiro ([6]), which brought under the light the interplay between the behaviour of eigenvectors associated with eigenvalues inside or outside $\mathbb{D}$ and hypercyclicity properties: for any bounded operator T on X, consider the two spaces

$$H_+(T) = \text{sp}\,[\ker(T-\lambda I) \,;\, |\lambda| > 1] \text{ and } H_-(T) = \text{sp}\,[\ker(T-\lambda I) \,;\, |\lambda| < 1].$$

If $H_+(T)$ and $H_-(T)$ are dense in X, then T is hypercyclic. More recently, the connection between properties of the eigenvectors associated to eigenvalues of modulus 1 (such eigenvectors will be called $\mathbb{T}$-*eigenvectors* in the sequel) and properties of the dynamical system (X, T) was studied in [5], [3] and [2].

A possible explanation for this frequent occurrence of the unit circle in hypercyclicity results is given by the following remark. The unit disk is hidden in the definition of a hypercyclic operator in the sense that the iterates T^n coincide with $F_n^{\mathbb{D}}(T)$, where $F_n^{\mathbb{D}}(z) = z^n$ represent the basic Taylor polynomials associated to $\mathbb{D}$. Let now Ω be a non empty and simply connected open subset of $\mathbb{C}$ with compact closure $\overline{\Omega}$ and rectifiable boundary $\partial\Omega$. The *Faber polynomials* F_n^{Ω} associated to the domain Ω are a natural generalization of the Taylor polynomials of the disk.

Definition 1.1. A bounded operator T on X is said to be Ω-*hypercyclic* if there exists a vector x of X such that $\{F_n^{\Omega}(T)x \; ; \; n \geq 0\}$ is dense in X. Such a vector is an Ω-hypercyclic vector for T, and the set of such vectors is denoted by $HC_{\Omega}(T)$.

Then $\mathbb{D}$-hypercyclicity is exactly the same notion as hypercyclicity. It turns out that most of the results involving the unit disk/circle for hypercyclic operators have natural analogs for Faber-hypercyclic operators which involve the corresponding open domain Ω or its boundary. For another instance of such a relationship between the geometry of the domain and the behaviour of the Faber polynomials of an operator (in relation to the boundary point spectrum $\sigma_p(T) \cap \partial\Omega$), see [1].

2. Some basic properties of Ω-hypercyclic operators

As was already mentioned in the introduction, hypercyclicity implies several spectral restrictions on the operator. We give here some examples of the corresponding spectral restrictions entailed by Ω-hypercyclicity. For instance, here is the analog of Kitai's necessary spectral condition:

Proposition 2.1. *Let Ω be a simply connected bounded domain whose boundary is a rectifiable Jordan curve, and let $T \in \mathcal{B}(X)$ be a Ω-hypercyclic operator. Then every connected component of the spectrum $\sigma(T)$ of T meets the boundary of Ω.*

The following modified Godefroy-Shapiro Criterion is quite natural:

Theorem 2.2. *Let Ω be a simply connected bounded domain whose boundary is a rectifiable Jordan curve, and $T \in \mathcal{B}(X)$ a bounded operator on X such that the following two vector spaces*

$$H_+^{\Omega}(T) = \mathrm{sp}\,[\ker(T - zI) \; ; \; z \in \overline{\Omega}^c\,] \text{ and } H_-^{\Omega}(T) = \mathrm{sp}\,[\ker(T - zI) \; ; \; z \in \Omega\,]$$

are dense in X. Then T is Ω-hypercyclic.

This criterion immediately yields a variety of examples of Ω-hypercyclic operators, with corresponding applications. In all the forthcoming examples, we suppose that Ω is a bounded domain whose boundary is a rectifiable Jordan curve.

Example 2.3. Let $\Phi \in H^{\infty}(\mathbb{D})$ be a non constant bounded analytic function on $\mathbb{D}$, and M_{Φ} the associated multiplier on $H^2(\mathbb{D})$. Then M_{Φ}^* is Ω-hypercyclic if and only if $\overline{\Phi(\mathbb{D})} \cap \partial\Omega \neq \emptyset$, where $\overline{\Phi(\mathbb{D})}$ denotes the conjugate of the set $\Phi(\mathbb{D})$.

Example 2.4. Let B be the backward shift on ℓ_p, $1 \leq p < +\infty$, or c_0: $Be_0 = 0$ and $Be_n = e_{n-1}$ for $n \geq 1$, where $(e_n)_{n\geq 0}$ denotes the canonical basis of the space. For every complex number ω with $|\omega| > d(0, \partial\Omega)$, ωB is Ω-hypercyclic.

Example 2.5. Let H be a complex separable infinite dimensional Hilbert space. Every bounded operator on H can be written as the sum of two Ω-hypercyclic operators.

Example 2.6. The norm closure $\overline{HC_\Omega}(H)$ in $\mathcal{B}(H)$ of the class $HC_\Omega(H)$ of Ω-hypercyclic operators on a complex separable infinite dimensional Hilbert space H consists exactly of those operators which satisfy the following three conditions:

(1) $\sigma_W(T) \cap \partial\Omega$ is connected;
(2) $\sigma_0(T) = \emptyset$;
(3) $ind(z - T) \geq 0$ for every $z \in \rho_{SF}(T)$.

3. Faber-hypercyclicity and peripheral point spectrum

We are now concerned with the study of the influence of eigenvectors associated to eigenvalues belonging to the boundary of Ω on the Ω-hypercyclicity of the operator. We call such eigenvectors *$\partial\Omega$-eigenvectors*.

Definition 3.1. Let Ω be a (simply connected) bounded domain of $\mathbb{C}$ whose boundary is a rectifiable Jordan curve, and let $T \in \mathcal{B}(X)$. We say that T has *perfectly spanning $\partial\Omega$-eigenvectors* if there exists a continuous probability measure σ on $\partial\Omega$ such that for every $A \subseteq \partial\Omega$ with $\sigma(A) = 1$, we have

$$\overline{\text{sp}}[\ker(T - z) \ ; \ z \in A] = X.$$

In view of the results of [2], we would like to show that if $T \in \mathcal{B}(X)$ has perfectly spanning $\partial\Omega$-eigenvectors, then T is Ω-hypercyclic. But the proof of [2] seems to use the unit circle in a crucial way, since everything relies on the behavior of Fourier coefficients of measures. Surprisingly enough, it turns out that this Ω-version is still true under some mild smoothness assumptions on $\partial\Omega$:

Theorem 3.2. *Suppose that the boundary of Ω is a curve of regularity $\mathcal{C}^{1+\alpha}$ for some $\alpha \in]0, 1[$. If X is any separable infinite dimensional Banach space, and $T \in \mathcal{B}(X)$ has perfectly spanning $\partial\Omega$-eigenvectors, then T is Ω-hypercyclic.*

4. Frequent Ω-hypercyclicity

Definition 4.1. A bounded operator T on X is said to be *frequently Ω-hypercyclic* if there exists an $x \in X$ such that for every non empty open subset U of X,

$$\underline{\text{dens}}\{n \geq 0 \ ; \ F_n(T)x \in U\} := \liminf_{N\to+\infty} \frac{1}{N} \#\{n \leq N \ ; \ F_n(T)x \in U\} > 0.$$

The main idea of Theorem 4.2 comes from a frequent hypercyclicity result of [4].

Theorem 4.2. *Let Ω be a simply connected bounded domain of $\mathbb{C}$ whose boundary is a curve of class $C^{3+\alpha}$ for some $\alpha \in]0, 1[$, and T a continuous linear operator on a separable infinite dimensional F-space X. Suppose that there exists a countable* *family of functions $(E_i)_{i\geq 1}$ defined on $\partial\Omega$ with values in X, of class C^2, such that for every $z \in \partial\Omega$ we have* $\ker(T-z) = \overline{\mathrm{sp}}\,[E_i(z)\ ;\ i \geq 1]$. *We also suppose that the $\partial\Omega$-eigenvectors are spanning, i.e. $X = \overline{\mathrm{sp}}\,[\ker(T-z)\ ;\ z \in \partial\Omega]$. Then T is frequently Ω-hypercyclic.*

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Norms of iterates, unimodular eigenvalues and linear dynamics

Catalin Badea

(joint work with Sophie Grivaux)

The behaviour of the sequence of the norms of the iterates of a bounded linear Banach space operator $T \in \mathcal{B}(X)$ is closely related to the size of its unimodular point spectrum $\sigma_p(T)\cap\mathbb{T}$. Here $\mathbb{T}$ will denote the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ while $\sigma_p(T) = \{\lambda \in \mathbb{C} : \mathrm{Ker}(T-\lambda) \neq \{0\}\}$ is the point spectrum (the set of eigenvalues) of T. One of the fundamental results in this direction is due to Jamison ([3]): if X is a separable Banach space and T is power-bounded on X, i.e. $\sup_{n\geq 0} \|T^n\| < +\infty$, then the unimodular point spectrum $\sigma_p(T) \cap \mathbb{T}$ is at most countable. The influence of partial power-boundedness of T on the size of the unimodular point spectrum will be our main interest here:

Definition 1. Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers, and T a bounded linear operator on the Banach space X. We say that T is *partially power-bounded with respect to* (n_k) if $\sup_{k\geq 0} \|T^{n_k}\| < +\infty$.

It is clear that the spectrum $\sigma(T)$ of a partially power-bounded operator T is contained in the closed unit disk. Partially power-bounded operators have been studied in this setting by Ransford ([4]) and Ransford and Roginskaya ([5]). They have proved in particular that partial power-boundedness of an operator does not necessarily imply countability of the unimodular point spectrum. Whether this phenomenon can happen or not depends of course on the sequence (n_k). The following definition was introduced in [1]:

Definition 2. Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers. We say that $(n_k)_{k\geq 1}$ is a *Jamison sequence* if for any separable Banach space X and any bounded linear operator T on X, $\sigma_p(T) \cap \mathbb{T}$ is at most countable as soon as T is partially power-bounded with respect to (n_k).

Using this terminology, it was proved in [5] that any sequence (n_k) with the property that $\sup_{k\geq 0}(\frac{n_{k+1}}{n_k}) < +\infty$ is a Jamison sequence, but that the sequence $n_k = 2^{2^k}$, for instance, is not. Jamison sequences were further studied in [1], where it was proved that no sequence (n_k) with $\lim \frac{n_{k+1}}{n_k} = +\infty$ is ever a Jamison sequence. Some equidistribution criteria were also proved in [1], providing examples of Jamison sequences verifying $\liminf \frac{n_{k+1}}{n_k} = 1$ and $\limsup \frac{n_{k+1}}{n_k} = +\infty$.

A unified approach to all these results can be given using the following ([2]) characterization of Jamison sequences. It is important to remark that we can assume without loss of generality that n_0 is equal to 1.

Theorem 3. *Let $(n_k)_{k\geq 0}$ be an increasing sequence of integers with $n_0 = 1$. The following assertions are equivalent:*

(1) *$(n_k)_{k\geq 0}$ is a Jamison sequence;*
(2) *there exists a positive real number ε such that for every $\lambda \in \mathbb{T} \setminus \{1\}$, we have*
$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon;$$
(3) *there exists a positive real number ε and a countable set $E \subset \mathbb{T}$ such that for every $\lambda \in \mathbb{T} \setminus E$, we have*
$$\sup_{k\geq 0} |\lambda^{n_k} - 1| \geq \varepsilon.$$

We discuss then several old and new examples of Jamison sequences. We also discuss partially power-bounded operators acting on Banach or Hilbert spaces having peripheral point spectra with large Hausdorff dimension.

These results have applications in linear dynamics ([1]).

Theorem 4. *There exists a Hilbert space operator which is frequently hypercyclic, chaotic (in Devaney's sense), but not topologically mixing.*

This answers a question raised by Alfredo Peris.

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Weakly supercyclic unitary operators

Étienne Matheron

It was shown by C. Kitai ([3]) that hyponormal operators cannot be hypercyclic. This was generalized in two different ways by P. Bourdon ([2]) and R. Sanders ([4]). Bourdon showed that hyponormal operators cannot be *supercyclic*, and Sanders proved that they cannot be *weakly* hypercyclic either. It is therefore reasonable to consider the case of weak supercyclicity. In a joint work with F. Bayart, we were able to get the following result.

Theorem 1 ([1]). *Every weakly supercyclic hyponormal operator is a multiple of a unitary operator.*

This theorem is proved in two steps. First, one uses the Berger–Shaw Theorem to show that every weakly supercyclic hyponormal operator is in fact normal. Then, by some simple spectral considerations, one proves that if a multiplication operator M_φ acting on some $L^2(\mu)$-space is weakly supercyclic, then the function φ has constant modulus, which gives the desired result. □

From Theorem 1, it is of course natural to ask whether a unitary operator can be weakly supercyclic. Therefore, one has to look at the multiplication operator M_z acting on $L^2(\mu)$, where μ is a Borel probability measure on $\mathbb{T}$.

Example 1 ([1]). Assume μ has the following property: for each measurable set $Z \subset \mathbb{T}$ with $\mu(Z) = 1$, one can find a positive integer q and two distinct points $a, b \in Z$ such that $a^q = b^q$. Then M_z is not weakly supercyclic on $L^2(\mu)$.

This follows from the (weakly) supercyclic version of Ansari's theorem together with the following simple observation: *if $\varphi \in L^\infty(\mu)$ and if the multiplication operator M_φ is cyclic on $L^2(\mu)$, then one can find $Z \subset \mathbb{T}$ with $\mu(Z) = 1$ such that the function φ is 1-1 on Z.* Indeed, if M_z were weakly supercyclic, then all powers of M_z would be cyclic, so one could find $Z \subset \mathbb{T}$ with $\mu(Z) = 1$ such that all functions z^q, $q \in \mathbb{N}$ are 1-1 on Z. □

In particular, M_z is not weakly supercyclic if the measure μ is not singular with respect to Lebesgue measure. One can also find singular measures satisfying the above property; for example, this holds for the canonical "Lebesgue" measure on the usual Cantor ternary set.

On the other hand, it turns out that if the support of μ is very small, then M_z is indeed weakly supercyclic on $L^2(\mu)$. Recall that a compact set $K \subset \mathbb{T}$ is said to be a *Kronecker set* if, for every continuous function $f : K \to \mathbb{T}$, one can find a sequence of positive integers (k_i) tending to ∞ such that $z^{k_i} \to f(z)$ uniformly on K.

Example 2 ([1]). Assume the measure μ is continuous; that is, $\mu(\{a\}) = 0$ for each point $a \in \mathbb{T}$. If the support of μ is a Kronecker set, then M_z is weakly supercyclic on $L^2(\mu)$.

The proof relies on the following fact: *If the measure μ is continuous, then the measurable functions with constant modulus are weakly dense in $L^2(\mu)$.* By definition of a Kronecker set, this gives at once the above result. Notice that the continuity assumption on μ is necessary: it is not hard to check that M_z cannot be weakly supercyclic if the discrete part of μ is nonzero, unless μ is a point mass. □

From the point of view of Harmonic Analysis, Kronecker sets are "very small" sets. On the other hand, a compact set $K \subset \mathbb{T}$ is usually considered as "large" if it carries a probability measure μ whose Fourier coefficients vanish at infinity. Thus, it is natural to ask whether M_z can be weakly supercyclic on $L^2(\mu)$ if $\hat{\mu} \in c_0(\mathbb{Z})$. This problem was raised in [1], and recently solved by S. Shkarin ([5]).

Theorem 2 ([5]). *There exists a probability measure μ on $\mathbb{T}$ such that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, and yet M_z is weakly supercyclic on $L^2(\mu)$.*

The main part of my talk was devoted to the exposition of Shkarin's proof. I will now outline the main steps, following rather closely Shkarin's paper [5].

Lemma 1. *Let $(g_i)_{i\in I}$ be a family of vectors in H (separable Hilbert space), and let $(\alpha_i)_{i\in I}$ be a family of nonzero complex numbers, where I is a countable set. Assume the following properties hold.*

(1) $\sum_{i\in I}\sum_{j\neq i} |\langle g_j, g_i\rangle|^2 < \infty$;

(2) $\sum_{i\in I} |\alpha_i|^2 = \infty$.

Then 0 belongs to the weak closure of the set $\{\frac{1}{\alpha_i}\, g_i;\ i \in I\}$.

The proof of this lemma reads as follows. Condition (1) and Cauchy-Schwarz's inequality yield an estimate of the form $\|\sum a_i g_i\| \leq C\,\|(a_i)\|_2$, for all finite linear combinations of the g_i's. In other words, there is a bounded linear operator $L : l^2(I) \to H$ such that $L(e_i) = g_i$, $i \in I$, where (e_i) is the canonical basis of $l^2(I)$. Now, it is not hard to deduce from (2) that 0 is in the weak closure of the set $\{\frac{1}{\alpha_i}\, e_i;\ i \in I\}$, and this proves the lemma. □

As a very simple consequence of Lemma 1, one can now state a useful criterion for weak supercyclicity of the operator M_z on $L^2(\mu)$ (a more general result is proved in [5], but the version below is enough for our purpose). Let us fix a bijection $(p,q) \mapsto \langle p,q\rangle$ from $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$, and for $i \in \mathbb{N}$, write $i = \langle p_i, q_i\rangle$.

Criterion for weak supercyclicity. *Let $\mathcal{H} = \{h_p;\ p \in \mathbb{N}\} \subset L^2(\mu)$ be a countable set such that $\mathbb{C}\,\mathcal{H}$ is weakly dense in $L^2(\mu)$. Let also $(\alpha_i)_{i\in\mathbb{N}}$ be a sequence of nonzero complex numbers such that $\sum_{q=1}^{\infty} |\alpha_{\langle p,q\rangle}|^2 = \infty$ for each $p \in \mathbb{N}$, and put $f_i = \alpha_i h_{p_i}$. Assume one can find a sequence of positive integers $(k_i)_{i\in\mathbb{N}}$ such that $\sum_i \sum_{j\neq i} |\langle g_i, g_j\rangle|^2 < \infty$, where $g_i(z) = z^{k_i} - f_i(z)$. Then M_z is weakly supercyclic on $L^2(\mu)$, and $\mathbf{1}$ is a weakly supercyclic vector for M_z.*

This is indeed an immediate consequence of Lemma 1: setting $I_p := \{i \in \mathbb{N};\ p_i = p\}$, we see at once that each function h_p is in the weak closure of the set $\{\frac{1}{\alpha_i} z^{k_i};\ i \in I_p\} \subset \mathbb{C}\, Orb(M_z, \mathbf{1})$.

From this criterion, it is clear that Theorem 2 follows from the next lemma: just put $\delta_i = 2^{-i}$ and $f_i = \frac{2^{-p_i}}{\sqrt{q_i}}\, h_{p_i}$.

Lemma 2. *Let $(f_i)_{i\in\mathbb{N}} \subset \mathcal{C}(\mathbb{T})$ with $\|f_i\|_\infty \le 1$ and $\|f_i\|_\infty \to 0$. Let also (δ_i) be a sequence of positive numbers. Then one can find a probability measure μ on $\mathbb{T}$ with $\hat{\mu}(n) \to 0$, and a sequence of positive integers (k_i) such that $|\langle g_i, g_j\rangle_{L^2(\mu)}| \le \delta_i$ whenever $j < i$, where $g_i(z) = z^{k_i} - f_i(z)$.*

For the proof, one more lemma is needed. In the sequel, I will denote by $\mathcal{M}(\mathbb{T})$ the space of all complex measures on $\mathbb{T}$. Since $\mathcal{M}(\mathbb{T})$ is the dual space of $\mathcal{C}(\mathbb{T})$, one can speak of the w^*- topology on $\mathcal{M}(\mathbb{T})$. Let $\mathcal{R}$ be the family of all complex measures $\mu \in \mathcal{M}(\mathbb{T})$ such that $\hat{\mu}(n) \to 0$; I use the symbol $\mathcal{R}$ because these measures are usually called *Rajchman measures*. One important fact is that $\mathcal{R}$ is hereditary for absolute continuity: if $\mu \in \mathcal{R}$ and $\nu << \mu$, then $\nu \in \mathcal{R}$.

Lemma 3. *Let μ be a probability measure in $\mathcal{R}$. Let also $\varphi \in \mathcal{C}(\mathbb{T})$ with $\|\varphi\|_\infty \le 1$, and let $N \in \mathbb{N}$. Then one can find a probability measure $\nu \in \mathcal{R}$ and an integer $k \ge N$ such that*

(1) *ν is w^*- close to μ;*
(2) *$\|\hat{\nu} - \hat{\mu}\|_\infty \le 2\|\varphi\|_\infty$;*
(3) *$z^k\,\nu$ is w^*- close to $\varphi\,\nu$.*

Here is a short outline of proof. First, one chooses a probability measure μ_1 which is w^* - close to μ, and whose support is a Kronecker set contained in $\text{supp}(\mu)$. By definition of a Kronecker set, one can find an integer $k \ge N$ such that $z^k|\varphi|\,\mu_1$ is close (in norm) to $\varphi\,\mu_1$, and since $(1-|\varphi|)\,\mu \in \mathcal{R}$, one may assume that $z^k(1-|\varphi|)\,\mu$ is w^*- close to 0. Next, one picks a probability measure $\mu_2 \in \mathcal{R}$ which is w^*- very close to μ_1 (specifically, close enough to ensure that $z^k|\varphi|\,\mu_2$ is close to $z^k|\varphi|\,\mu_1$); this is possible because $\mu \in \mathcal{R}$ and $\text{supp}(\mu_1) \subset \text{supp}(\mu)$. Finally, one checks that $\nu := (1-|\varphi|)\,\mu + |\varphi|\,\mu_2$ works. □

Proof of Lemma 2. Let d be a compatible metric for the w^*- topology on the unit ball of $\mathcal{M}(\mathbb{T})$. Let also (ε_i) be a sequence of positive numbers to be chosen later. One constructs by induction two increasing sequences of integers (n_i) and (k_i), and a sequence (μ_i) of probability measures in $\mathcal{R}$, such that the following conditions are fulfilled, with $g_i(z) = z^{k_i} - f_i(z)$.

(a) $d(\mu_i, \mu_{i+1}) < \varepsilon_i$;
(b) $\|\hat{\mu}_{i+1} - \hat{\mu}_i\|_\infty \le 2\|f_{i+1}\|_\infty$;
(c) $|\hat{\mu}_{i+1}(n) - \hat{\mu}_i(n)| < \varepsilon_i$ if $|n| \le n_i$;
(d) $|\hat{\mu}_i(n)| < \varepsilon_i$ if $|n| > n_i$;
(e) $|\langle \overline{g_j} g_{j'}, \mu_{i+1} - \mu_i\rangle| < \varepsilon_i$ for all $j, j' \le i$;
(f) $|\int \overline{g_j} g_i\, d\mu_i| < \varepsilon_i$ for all $j < i$.

One starts with any probability measure $\mu_1 \in \mathcal{R}$, some positive integer n_1 such that (d) holds, and $k_1 := 1$. The measure μ_{i+1} and the integer k_{i+1} are given by Lemma 3 applied to $\mu := \mu_i$ and $\varphi := f_{i+1}$; more explicitly, conditions (a), (c), (e) follow from (1) in Lemma 3, condition (b) follows from (2), and (f) follows from (3). Then one can choose n_{i+1} according to (d) since $\mu_{i+1} \in \mathcal{R}$.
By (a), the sequence (μ_i) is w^*- convergent to some probability measure μ, provided $\sum_i \varepsilon_i < \infty$. If the ε_i's are small enough, one checks using (b), (c), (d) that $\mu \in \mathcal{R}$, while (e), (f) ensure $|\langle g_j, g_i\rangle| < \delta_i$ for all $j < i$. This concludes the proof of Lemma 2, and hence the full proof of Shkarin's theorem. □

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How to get common universal vectors

Frédéric Bayart
(joint work with Etienne Matheron)

Let $(T_\lambda)_{\lambda\in\Lambda}$ be a parametrized sequence of operators on some separable Fréchet space X. Suppose that each T_λ is hypercyclic. The question that we study is the existence of a common hypercyclic vector for the whole family. If Λ is countable, this follows from Baire's theorem. If Λ is uncountable, the first positive example was given by Abakumov and Gordon in [1] whereas Costakis and Sambarino in [2] have given a criterion for the existence of a common hypercyclic vector.

Our aim in this work is to improve the result of Costakis and Sambarino in order to take into account the geometry of the Fréchet space. We assume that the parameter space Λ is an interval, and that $T_\lambda(x)$ depends continuously on the pair (λ, x). Finally, we also assume that there exists a dense set $\mathcal{D} \subset X$ such that each operator T_λ has a right inverse $S_\lambda : \mathcal{D} \to X$.

Theorem 1. *Assume the Banach space X has type $p \in [1;2]$, and that for each $f \in \mathcal{D}$ and any compact set $K \subset \Lambda$, there exists a sequence of positive numbers $(c_k)_{k\in\mathbb{N}}$ such that the following conditions are satisfied:*

- (a) *(c_k) is nonincreasing, and $\sum_{k=0}^{\infty} c_k^p < \infty$;*
- (b$_1$) *$\|T_\lambda^{n+k} S_\alpha^n(f)\| \le c_k$, for any $n, k \in \mathbb{N}$ and $\lambda, \alpha \in K$, $\lambda \ge \alpha$;*
- (b$_2$) *$\|T_\lambda^n S_\alpha^{n+k}(f)\| \le c_k$ for any $n, k \in \mathbb{N}$ and $\lambda, \alpha \in K$, $\lambda \le \alpha$;*
- (c$_1$) *$\|(T_\lambda^{n+k} - T_\mu^{n+k})(S_\alpha^n f)\| \le (n+k)|\lambda - \mu|\, c_k$, for $n, k \in \mathbb{N}$ and $\lambda, \mu \ge \alpha \in K$;*

(c$_2$) $\|(T_\lambda^n - T_\mu^n)(S_\alpha^{n+k}(f))\| \leq n|\lambda - \mu|\, c_k$, *for* $n, k \in \mathbb{N}$ *and* $\lambda, \mu \leq \alpha \in K$.

Then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ *is a dense* G_δ *subset of* X.

An interesting feature of the proof of the previous theorem is that it mixes Baire Category arguments with a nontrivial result from Probability Theory, namely Dudley's majorization theorem for subgaussian stochastic processes.

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On large orbits

V. Müller

Let X be a complex Banach space and $T \in B(X)$. Vectors $x \in X$ such that the orbit $\{T^n x : n = 0, 1, \dots\}$ is "large" (in some sense) form the opposite extreme to the hypercyclic vectors. The talk was a survey of results concerning large orbits.

All the results here are formulated for complex Banach spaces. However, most of them are also true for real Banach spaces (sometimes with minor modifications).

Theorem 1 ([M], [B])**.** *Let* $T \in B(X)$, $a_n > 0$, $a_n \to 0$. *Then there exists* $x \in X$ *such that* $\|T^n x\| \geq a_n r(T^n)$ *for all* n.

Moreover, it is possible to find such an x *with* $\|x\| < \max_n a_n + \varepsilon$, *where* ε *is any given positive number.*

Theorem 2 ([MV])**.** *Let* X, Y *be Banach spaces,* $T_n \in B(X, Y)$, $a_n > 0$, $\sum a_n < \infty$. *Then there exists* $x \in X$ *such that* $\|T_n x\| \geq a_n \|T_n\|$ *for all* n.

Moreover, x can be chosen in any ball of radius greater than $\sum a_n$.

Corollary 3. *Let* $T \in B(X)$, $\sum \|T^n\|^{-1} < \infty$. *Then there exists a dense subset of vectors* $x \in X$ *such that* $\|T^n x\| \to \infty$.

In complex Hilbert spaces it is sufficient to assume that $\sum \|T^n\|^{-2} < \infty$.

Example 4. There exists an operator $T \in B(X)$ such that $\|T^n\| = n + 1$ for all n, but there is no $x \in X$ with $\|T^n x\| \to \infty$.

Similar results are true for weak orbits $\{\langle T^n x, x^*\rangle : n = 0, 1, \dots\}$, where $x \in X$ and $x^* \in X^*$ are fixed vectors.

Theorem 5 ([MV])**.** *Let* X, Y *be Banach spaces,* $T_n \in B(X, Y)$, $a_n > 0$, $\sum a_n^{1/2} < 1$. *Then there exist* $x \in X$, $y^* \in Y^*$ *such that* $\|x\| = 1 = \|y^*\|$ *and* $|\langle T_n x, y^*\rangle| \geq a_n \|T_n\|$ *for all* n.

Corollary 6. *Let* $T \in B(X)$, $\sum \|T^n\|^{-1/2} < \infty$. *Then there are vectors* $x \in X$ *and* $x^* \in X^*$ *such that* $|\langle T^n x, x^*\rangle| \to \infty$.

In complex Hilbert spaces it is sufficient to assume that $\sum \|T^n\|^{-1} < \infty$.

Example 7. There exists an operator T on a Hilbert space H such that $\|T^n\| = n+1$ for all n, but there are no $x, y \in X$ with $|\langle T^n x, y\rangle| \to \infty$.

The analogue of Theorem 1 for weak orbits is true only in some partial cases.

Theorem 8 ([BM]). *Let T be a completely non-unitary contraction on a Hilbert space H. Suppose that $r(T) = 1$. Let $a_n > 0$, $a_n \to 0$. Then there exists $x \in H$ such that $|\langle T^n x, x\rangle| \geq a_n$ for all n.*

Moreover, it is possible to choose x such that $\|x\| < \max a_n + \varepsilon$.

For unitary operators the questions concerning weak orbits reduce to the questions about the behaviour of the Fourier coefficients of L^1 functions.

Corollary 9. *Let μ be a probability measure with $\operatorname{supp}\mu \subset \mathbb{T}$. Suppose that μ is Rajchman (i.e., $\hat{\mu}(n) \to 0$ if $|n| \to \infty$). Let $a_n > 0$, $a_n \to 0$. Then there exists $f \in L^1(\mu)$ such that $h \geq 0$ a.e. and $|\hat{f}(n)| \geq a_{|n|}$ for all integers n.*

In particular, this is true if μ is absolutely continuous with respect to the Lebesgue measure.

Example 10. There exists a probability measure μ with $\operatorname{supp}\mu \subset \mathbb{T}$ and a sequence (a_n) of positive numbers converging to 0 such that there is no $f \in L^1(\mu)$ with $|\hat{f}(n)| \geq a_n$ for all $n \geq 0$.

Consequently, there exists a unitary operator $U \in B(H)$ and a sequence (a_n), $a_n > 0$, $a_n \to 0$ such that there are no $x, y \in H$ with $|\langle U^n x, y\rangle| \geq a_n$ for all $n \geq 0$.

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Hypercyclic Tuples of Operators

Nathan S. Feldman

In what follows X will denote a separable Banach space and $T = (T_1, T_2, \ldots, T_n)$ an n-tuple of *commuting* bounded linear operators on X. The semigroup of operators generated by T is $\langle T\rangle = \{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} : k_i \geq 0\}$. Notice that $\langle T\rangle$ is a finitely generated (discrete) Abelian semigroup. If $x \in X$, then the orbit of x under the n-tuple T (or equivalently under the semigroup $\langle T\rangle$) is $Orb(T, x) = Orb(\langle T\rangle, x) = \{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} x : k_i \geq 0\}$. We say that the n-tuple T is hypercyclic if there is a vector that has dense orbit under T.

Examples:

(1) $T = (B, 2I)$, where B is the (unilateral) backward shift on ℓ^2 and I denotes the identity operator is a hypercyclic pair, even though neither operator is hypercyclic.
(2) If A and B are hypercyclic operators, then the pair $T = (A \oplus I, I \oplus B)$ is hypercyclic.
(3) T is hypercyclic whenever $\langle T \rangle$ contains a hypercyclic operator.
(4) If A is a supercyclic operator on X and θ is an irrational multiple of π, then $T = (2I, \frac{1}{3}I, e^{i\theta}I, A)$ is a hypercyclic tuple on X. Thus the study of hypercyclic tuples of operators includes the study of supercyclic operators.

Theorems:

(1) If $T = (M^*_{f_1}, \dots, M^*_{f_n})$ is an n-tuple of coanalytic Toeplitz operators on $H^2(\mathbb{D})$, then T is hypercyclic if and only if the semigroup $\langle T \rangle$ contains a hypercyclic operator.
(2) If G is any bounded open set in $\mathbb{C}$ with infinitely many components, then there exists two bounded analytic functions, f, g, on G such that the pair of adjoint multiplication operators $T = (M^*_f, M^*_g)$ is hypercyclic on the Bergman space $L^2_a(G)$, but the semigroup generated by T, $\langle T \rangle$, does not contain any hypercyclic operators.
(3) On $\mathbb{C}^n$ there exists a hypercyclic $(n+1)$-tuple of diagonal matrices.
(4) There does not exist a hypercyclic n-tuple of diagonalizable matrices on $\mathbb{C}^n$.
(5) There does not exist a hypercyclic $n-$tuple of normal operators on an infinite dimensional Hilbert space.

In [3] Bourdon and Feldman proved that if a single operator on a real or complex locally convex space has a somewhere dense orbit, then that orbit must actually be dense. Next we present a version of this for tuples of operators.

A Somewhere Dense Theorem: Suppose that $T = (T_1, T_2, \dots, T_n)$ is an n-tuple of commuting operators on a separable Banach space X over $\mathbb{R}$ or $\mathbb{C}$. If $x \in X$ and $Orb(T, x)$ is somewhere dense in X, then $Orb(T, x)$ is dense in X provided one of the following holds:
(a) $\sigma_p(A^*) = \emptyset$ for each $A \in \langle T \rangle$, or
(b) there is a cyclic operator B that commutes with T and satisfies $\sigma_p(B^*) = \emptyset$.

An n-tuple T is said to be $k-$*hypercyclic* if there are k vectors in X such that the union of their orbits under T is dense in X. A tuple T is *multi-hypercyclic* if T is $k-$hypercyclic for some $k \geq 1$.

It was proven independently by Costakis [2] and Peris [4] that a multi-hypercyclic operator is hypercyclic. Also Ansari [1] proved that if T is a hypercyclic operator, then T^n is also hypercyclic. Next we state versions of these for tuples of operators.

Corollary Suppose that $T = (T_1, T_2, \ldots, T_n)$ is an n-tuple of commuting operators on a separable Banach space X and either condition (a) or (b) above is satisfied. Then the following hold:

(1) If T is multi-hypercyclic, then T is hypercyclic.
(2) If T is hypercyclic, then $T^k = (T_1^{k_1}, T_2^{k_2}, \ldots, T_n^{k_n})$ is also hypercyclic for any multi-index $k = (k_1, k_2, \ldots, k_n)$ of positive integers.

Examples:

(1) $\left\{ \frac{2^n}{3^k} : n, k \geq 0 \right\}$ is dense in $\mathbb{R}^+$.
(2) If $T = \left(\begin{bmatrix} 2 & \\ & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \begin{bmatrix} 1 & \\ & \frac{1}{3} \end{bmatrix} \right)$ and $x = (1, 1)$, then the orbit of x under T is dense in the first quadrant of $\mathbb{R}^2$. Hence T has a somewhere dense orbit that is not dense. In fact, if $x = (a, b)$ and $a, b \neq 0$, then the orbit of x under T is dense in that quadrant that contains x. Thus, T is $4-$hypercyclic on $\mathbb{R}^2$, but not $3-$hypercyclic on $\mathbb{R}^2$.
(3) If $T = (2I_{\mathbb{R}} \oplus I_{\ell^2_{\mathbb{R}}}, \frac{1}{3} I_{\mathbb{R}} \oplus 2B)$, then T is a $2-$hypercyclic pair on $\mathcal{H} = \mathbb{R} \oplus \ell^2_{\mathbb{R}}$ that is not hypercyclic. Here B denotes the unilateral backward shift on the (real) space $\ell^2_{\mathbb{R}}$.

Questions:

(1) If T is a hypercyclic tuple of operators, then must there be a cyclic operator B that commutes with T? Can B be chosen to also satisfy $\sigma_p(B^*) = \emptyset$?
(2) Is there an $n-$tuple of operators on a complex Banach space that has a somewhere dense orbit that is not dense?
(3) Does there exist a hypercyclic $n-$tuple of matrices on $\mathbb{C}^n$?
(4) Does there exist a hypercyclic $n-$tuple of hyponormal operators on an infinite dimensional Hilbert space?

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Questions on Supercyclic and Hypercyclic Subspaces

Héctor N. Salas

Let E be an infinite dimensional separable complex Banach space. Recall that a *supercyclic subspace* for the operator T on E is an infinite dimensional closed subspace such that any nonzero vector is supercyclic for T. In [5], it was proved the following:

Theorem 1. *Let $T \in \mathcal{L}(E)$. Suppose that T satisfies:*

(1) *The Supercyclicity Criterion for a sequence $(\lambda_{n_k}) \subset \mathbb{C} \setminus \{0\}$.*
(2) *There exists an infinite dimensional closed subspace B_0 of E such that $\lambda_{n_k} T^{n_k}(x) \to 0$ for every $x \in B_0$.*

Then T has a supercyclic subspace.

The second condition is called the B_0 condition. It was discovered in the context of hypercyclicity (actually universality) by Montes-Rodríguez, [4]. The converse is true in the class of backward weighted shifts, unilateral and bilateral. Moreover, for these operators the existence of supercyclic subspaces was also identified in terms of their weights, see Theorem 5.1 and Theorem 6.1 of [5]. In general, however, it is not known whether the converse of this theorem is true, see [5] and [6].

Question 1. *Let $T \in \mathcal{L}(E)$. Suppose that T satisfies the Supercyclicity Criterion for (λ_{n_k}) and has a supercyclic subspace. Must it also satisfy the B_0 condition for (λ_{n_k})?*

If supercyclicity is replaced by hypercyclicity, then the converse is true and is also equivalent to the fact that $\sigma_e(T)$, the essential spectrum of T, meets the unit disc, see sections 3 and 4 of [3]. What if we allow T to be unbounded? The concept of spectrum is not very useful in this situation but we can ask:

Question 2. *Let T be an unbounded operator defined on the whole space E such that T has a hypercyclic subspace. Can T be such that for any infinite dimensional closed subspace E_0 and any sequence (n_k), then there exists $x_0 \in E_0$ with $T^{n_k}(x_0) \nrightarrow 0$?*

In [8], it was proved that for every infinite dimensional separable Banach space there is an unbounded operator on it such that every nonzero vector is hypercyclic. (In view of the Read operator, [7], the most interesting case is when the underlying space is Hilbert.) So we may restate Question 2 with the additional requirement that every nonzero vector is hypercyclic. In [5], it was also proved the following. (The second condition is called the spectral condition.)

Theorem 2. *Let $T \in \mathcal{L}(E)$. Suppose that T satisfies:*

(1) *The Supercyclicity Criterion for (λ_{n_k}).*
(2) *$sup_k\{|\lambda^{n_k}\lambda_{n_k}|\} < \infty$ for some $\lambda \in \sigma_e(T)$.*

Then T has a supercyclic subspace.

The converse is not true. The counterexample, Corollary 7.5 of [5], is an operator of the form $T^{(\infty)} = T \oplus T \oplus ...$ It is somehow surprising that even for weighted shifts the spectral condition is not necessary for having supercyclic subspaces.

Proposition 3. *Let a_k be a positive sequence increasing to 1 and let ϵ_k be a positive sequence decreasing to 0. Let n_k be a rapidly increasing sequence such that $a_k^{n_k - n_{k-1}} < \epsilon_k$. Let T be the unilateral backward shift on $\ell^2(\mathbb{N})$ with weights*

$$\overbrace{1, 1, \cdots, 1}^{n_1}, \overbrace{a_1, a_1, \cdots, a_1}^{n_1}, \overbrace{a_1^{-1}, a_1^{-1}, \cdots, a_1^{-1}}^{n_1}, \overbrace{1, 1, \cdots, 1}^{n_2}, \overbrace{a_2, a_2, \cdots, a_2}^{n_2 - n_1},$$

$$\overbrace{a_1, a_1, \cdots, a_1}^{n_1}, \overbrace{a_1^{-1}, a_1^{-1}, \cdots, a_1^{-1}}^{n_1}, \overbrace{a_2^{-1}, a_2^{-1}, \cdots, a_2^{-1}}^{n_2 - n_1}, \overbrace{1, 1, \cdots, 1}^{n_3},$$

and so forth. Then T has a supercyclic subspace but doesn't satisfy the spectral condition.

Note that $\sigma_e(T)$ is the unit circle, but any sequence (λ_{n_k}) which satisfies the Supercyclicity Criterion for T is unbounded.

Proposition 4. *Let T be the bilateral backward shift on $\ell^2(\mathbb{Z})$ with weights*

$$\cdots, 1/2, 3/4, 2/3, 1/2, 2/3, 1/2, 1/2, \overbrace{1}^{0}, 1, 1, 1, 1, 1, 1, 1, \cdots$$

Then T has a supercyclic subspace but doesn't satisfy the spectral condition.

Note that $\sigma(T) = \sigma_e(T) = \{z : |z| = 1\}$ and no multiple of T may be hypercyclic. In both propositions the underlying space may be ℓ^p with $1 \leq p < \infty$.

Question 3. *Are there nice classes of operators for which the spectral condition is necessary for having supercyclic subspaces?*

To avoid the "easy cases" of $T^{(\infty)}$, we might restrict our attention to operators T such that whenever $E = E_1 \oplus E_2$ and E_1 and E_2 are invariant under T, then T has a supercyclic subspace if and only if $T|_{E_1}$ and $T|_{E_2}$ have supercyclic subspaces. Observe that even for operators which are multiples of hypercyclic operators, Question 3 is interesting since the nature of the hypercyclic subspaces and supercyclic subspaces is quite different. The following result can be seen as a special case of Theorem 7.1 of [5]. Recall that the weights are always positive.

Theorem 5. *Let S and T be bilateral backward shifts defined on $\ell^2(\mathbb{Z})$, and let (s_n) and (t_n) be their respective weight sequences. Suppose that $s_n = t_n$ for positive integers and $t_n \leq s_n$ otherwise. Then*

(1) *Every hypercyclic vector for S is also a hypercyclic vector for T.*
(2) *If S has a hypercyclic subspace, so does T.*

Recently there have been several papers studying families of operators having common hypercyclic vectors and common hypercyclic subspaces, see for instance [2] and [1]. The family $\mathcal{S}$ below is uncountable.

Corollary 6. *Let S be a hypercyclic bilateral backward shift with weight sequence (s_n) and let $\mathcal{S}$ be the family of bilateral backward shifts such that $T \in \mathcal{S}$ if and only if its weight sequence (t_n) satisfies that $s_n = t_n$ for positive integers and $t_n \leq s_n$ otherwise. Then*

(1) *$\mathcal{S}$ has a common hypercyclic vector.*
(2) *$\mathcal{S}$ has a common hypercyclic subspace whenever S has a hypercyclic subspace.*

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Hypercyclic Structures in Product Spaces

Henrik Petersson

The symbol X denotes a separable Fréchet space, if nothing else is specified, and $L(X)$ the algebra of continuous linear operators on X. "The big" open problem in Hypercyclicity is whether $T \oplus T$ (and hence any $\oplus_1^n T : X^n \to X^n$, $n \geq 2$) is hypercyclic whenever $T \in L(X)$ is hypercyclic. For some recent progress on this problem we refer to [2]. In fact, we know that the following are equivalent for $T \in L(X)$ (in particular, see [1]):

(i) T satisfies the Hypercyclicity Criterion (HC);
(ii) T is hereditarily hypercyclic (that is, T is hereditarily hypercyclic for some sequence (n_k), in the sense that $(T^{m_k})_{k\geq 0}$ is hypercyclic for any subsequence (m_k) of (n_k));
(iii) $T \oplus T$ is hypercyclic;
(iv) T is syndetically hypercyclic (that is, $(T^{n_k})_{k\geq 0}$ is hypercyclic for any strictly increasing sequence (n_k) with $\sup(n_{k+1} - n_k) < \infty$).

Further, if X admits a continuous norm, then (i) through (iv) are equivalent to any of the following:

(v) $L_T : S \mapsto TS$ is hypercyclic on $L(X)$ for the Strong Operator Topology;
(vi) There exists an $S \in L(X)$ such that $(Sx_1, Sx_2, ..., Sx_n)$ is hypercyclic for $\oplus_1^n T$ for any finite linearly independent set $\{x_1, ..., x_n\} \subset X$.

In fact, one can show that an operator $S \in L(X)$ is hypercyclic for L_T if and only if it satisfies the condition in (vi) (this holds even if X does not admit any continuous norm).

All this motivates us to study structures of hypercyclic vectors in X^n for operators $\oplus_1^n T$. In particular we know, from the above, that if $T \in L(X)$ is hereditarily hypercyclic and X admits a continuous norm, then there exists an infinite dimensional vector subspace $V \subseteq X$ such that $(v_i) \in X^n$ is hypercyclic for $\oplus_1^n T$ for any linearly independent set $\{v_i\}_1^n \subseteq V$. Indeed, just take $V := \mathrm{Im}S$ where S is any hypercyclic vector for L_T. A counter-example shows, however, that not every linearly independent set $\{x_i\}_1^n \subseteq X$ of hypercyclic vectors forms a hypercyclic vector (x_i) for $\oplus_1^n T$. (Is it true that the set HC(T) of hypercyclic vectors for any (hereditarily) hypercyclic operator $T \in L(X)$ contains the range ImS of a one-to-one operator $S \in L(X)$, and/or, if T is such a mapping, is every linearly independent n-tuple $(x_i) \subseteq \mathrm{Im}S$ hypercyclic for $\oplus_1^n T$?)

Next, we know that any hypercyclic operator admits a (dense) infinite dimensional subspace of, except for zero, hypercyclic vectors (a so-called *hypercyclic vector subspace*). The simplest type of subspaces of X^n are those of the form $U_1 \times ... \times U_n$, where $U_i \subseteq X$ are subspaces, but such a subspace cannot be a hypercyclic vector subspace for $\oplus_1^n T$, because $(0, x_2, ..., x_n)$ is never a hypercyclic vector. However, a simple observation, due to Vladimír Müller, is that if we fix a hypercyclic vector (x_i) for $\oplus_1^n T$ and put $U_i := \{p(T)x_i : p \text{ polynomial}\}$, then every vector in $U_1 \times ... \times U_n$ with nonzero coordinates is hypercyclic for $\oplus_1^n T$. Note also that every vector subspace U_i is dense, so $U_1 \times ... \times U_n$ is a dense "almost" hypercyclic vector subspace. On the other hand, this construction requires that we already know a hypercyclic vector (x_i) for $\oplus_1^n T$, which suggests to try to obtain such subspaces in a more constructive way.

We give a constructive proof of how we can obtain dense "almost" hypercyclic vector subspaces $U_1 \times ... \times U_n$ for $\oplus_1^n T$ where T is hereditarily hypercyclic. In fact, in contrast to the construction found by Müller, ours works also for sequences $(T_n)_{n\geq 0} \subset L(X)$ and so we have the following more general:

Theorem 1. *Assume $(T_m^{(i)})_{m\geq 0} \subset L(X_i)$, $i = 1, ..., n$ are hereditarily densely hypercyclic for a common sequence (m_j), where X_i are Fréchet spaces. Then there exist dense hypercyclic vector subspaces $U_i \subseteq X_i$ for $(T_m^{(i)})_{m\geq 0}$, respectively, such that every vector (u_i) of $U_1 \times ... \times U_n$ with $u_i \neq 0$ for all i, is hypercyclic for $\oplus_{i=1}^n (T_m^{(i)})$.*

From the proof (see especially the last part of Lemma 1) it follows that:

Corollary 1. *Assume (n_k) is a hereditarily hypercyclic sequence for T (and S, resp.). Then for any hypercyclic vector x for (T^{n_k}), there exists a dense vector subspace $U \subseteq X$ such that (x, y) is hypercyclic for $T \oplus T$ ($T \oplus S$, resp.) for any nonzero $y \in U$.*

Thus if T is mixing [2], i.e. hereditarily hypercyclic for the full sequence, we can find, for every hypercyclic vector x for T, a dense hypercyclic vector subspace U for T such that (x, y) is hypercyclic for $T \oplus T$ whenever $y \in U \setminus \{0\}$. Since every (infinite dimensional separable) Banach space supports a mixing operator [2], we also have:

Corollary 2. *Let Y be a complemented infinite codimensional subspace of a separable Banach space X, and assume $T \in L(Y)$ is hereditarily hypercyclic. Then T admits a hypercyclic extension $\hat{T} \in L(X)$. Further, $\hat{T}$ can be chosen so that for* *any hypercyclic vector $y \in Y$ for (T^{n_k}), where (n_k) is a hereditarily hypercyclic sequence for T, there exists an $x \in X$ such that $x + y$ is hypercyclic for $\hat{T}$.*

The proof of Theorem 1 rests on the following, which is of independent interest:

Lemma 1. *Let $\{x_k\}_{k\geq 1}$ be any countable dense set in X, and assume $(T_n^{(m)})_{n\geq 0} \subset L(X)$, $m = 1, 2, \ldots$ are hereditarily densely hypercyclic sequences for, respectively, the sequence $(n_j(m))_j$. Then there exist a linearly independent sequence $(z_k)_{k\geq 1} \subset X$ and for each m subsequences $(n_{i,j}^{k,l}(m))_j$, $i \geq 1$, $l \geq 2$, $k \leq l$, of $(n_j(m))_j$ such that*

a) $\lim_j T^{(m)}_{n_{i,j}^{k,l}(m)} z_k = x_i$ *for all $k \leq l$ and i, m;*
b) $\lim_j T^{(m)}_{n_{i,j}^{\nu,l}(m)} z_k = 0$ *for all $\nu \neq k$ and i, m.*

In particular, $V := \operatorname{span}\{z_k\}_{k\geq 1}$ forms a common hypercyclic vector subspace for the sequences $(T_n^{(m)})_{n\geq 0}$, $m \geq 1$. Further, $\{z_k\}_{k\geq 1}$, and thus V, can be chosen to be dense, and z_1 as any common hypercyclic vector for $(T^{(m)}_{n_j(m)})_{j\geq 0}$, $m \geq 1$ (which exists).

Recall that a sequence $(T_n)_{n\geq 0} \subset L(X)$ is called *hereditarily densely hypercyclic* for a sequence (n_k), if $(T_{m_k})_{k\geq 0}$ has a dense set of hypercyclic vectors for every subsequence (m_k) of (n_k). Since any hereditarily hypercyclic operator is hereditarily densely hypercyclic, we conclude from Lemma 1 that *any countable family of hereditarily hypercyclic operators on a Fréchet space has a common dense hypercyclic vector subspace.* This result has also been obtained by L. Bernal-González and M. C. Calderón-Moreno, and a much stronger result was obtained by Sophie Grivaux in the Banach space setting, namely – the same conclusion holds for any countable family of hypercyclic operators (on a Banach space).

It is a natural question to ask to what degree we can replace the hypercyclic vector subspaces U_i in Theorem 1 by hypercyclic subspaces, i.e., closed hypercyclic vector subspaces for T_i. We have some partial results on this.

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Bounded universal functions in one and several complex variables

Raymond Mortini

(joint work with P. Gorkin & F. Leon-Saavedra *and* F. Bayart, P. Gorkin & S. Grivaux)

Let Ω be a domain in $\mathbb{C}^N$ and let X denote either the Fréchet space $H(\Omega)$ of all holomorphic functions on Ω or the unit ball $\mathcal{B} = \{f \in H(\Omega) : \sup_{z\in\Omega} |f(z)| \leq 1\}$. A function $f \in X$ is called X-universal for the sequence (ϕ_n) of holomorphic selfmaps of Ω if the "orbit" $\mathcal{O}(f) = \{f \circ \phi_n : n \in \mathbb{N}\}$ is locally uniformly dense in X. Bernal-Gonzalez, Grosse-Erdmann and Montes-Rodriguez showed (see [3, 6, 2]), that if $\Omega \subseteq \mathbb{C}$ is not isomorphic to $\mathbb{C}\setminus\{0\}$, then the sequence (ϕ_n) of automorphisms of Ω admits a $H(\Omega)$-universal function if and only if (ϕ_n) is a runaway sequence; this means that for every compact set $K \subseteq \Omega$ there is $n \in \mathbb{N}$ such that $\phi_n(K)\cap K = \emptyset$.

During the Mini-workshop I reported on results developed with P. Gorkin, F. Leon-Saavedra ([4]) as well as with F. Bayart, P. Gorkin and S. Grivaux ([1]) and concerning the existence of X-universal functions in various contexts. One of the results tells us the following:

Theorem 1. [4] *The sequence (ϕ_n) of automorphisms on $\Omega \subseteq \mathbb{C}^N$ admits a $\mathcal{B}$-universal function H if and only if for some (for all) $z_0 \in \Omega$ the sequence $(\phi_n(z_0))$ contains an asymptotic interpolating sequence of type one; or equivalently if there exists $f \in \mathcal{B}$, f not constant, such that*

$$\limsup |f(\phi_n(z_0))| = 1. \tag{1}$$

These asymptotic interpolating sequences for general domains have been introduced and studied extensively in [5]. We also point out that the $\mathcal{B}$-universal functions above were obtained either with the use of a newly developed hypercyclicity criteria for $\mathcal{B}$-universality or explicitly:

$$H = \sum (p_j \circ \phi_{n_j}^{-1})(f_j g_1 \cdots g_{j-1}),$$

where $\{p_j : j \in \mathbb{N}\}$ is a countable, locally uniformly dense subset of $\mathcal{B}$, and (ϕ_{n_j}) is some suitably chosen subsequence of (ϕ_n). Here the functions f_j and g_j satisfy $|f_j| + |g_j| < 1$, in order that the series above converges.

Relations between runaway sequences and asymptotic interpolating sequences of type one are given, too. As a corollary we obtain

Corollary 2. [4] *Let $\Omega \subseteq \mathbb{C}^N$ be a domain for which $H^\infty(\Omega)$ is locally uniformly dense in $H(\Omega)$. Then condition (1) implies that the sequence (ϕ_n) of automorphisms of Ω admits a $H(\Omega)$-universal function.*

In the second part of the talk we focus on $\mathcal{B}$-universal functions for an arbitrary sequence (ϕ_n) of selfmaps on the unit disk $\mathbb{D}$ (see [1]). Suppose that $|\phi_n(0)| \to 1$

and that there exists a $\mathcal{B}$-universal function H for (ϕ_n). Then, for some subsequence (n_j), $H \circ \phi_{n_j} \to z$. Hence

$$(1 - |\phi_{n_j}(0)|^2)|H' \circ \phi_{n_j}(0)| \, \frac{|\phi'_{n_j}(0)|}{(1 - |\phi_{n_j}(0)|^2)} \to 1.$$

By the Schwarz-Pick lemma

$$\Delta_j := \frac{|\phi'_{n_j}(0)|}{1 - |\phi_{n_j}(0)|^2} \to 1 \tag{2}$$

The following result, proved in [1], shows that the converse holds, too:

Theorem 3. [1] *Suppose that for a sequence (ϕ_n) of selfmaps of $\mathbb{D}$ we have $|\phi_n(0)| \to 1$. Then the condition (2) is necessary* and *sufficient for the existence of a $\mathcal{B}$-universal function for the sequence (ϕ_n).*

It is also shown that whenever there exists a $\mathcal{B}$-universal function for (ϕ_n), then there is a Blaschke product whose orbit $\mathcal{O}(B)$ is locally uniformly dense in $\mathcal{B}$. As a corollary we obtain the following:

Corollary 4. [1] *If ϕ_n is the n-th iterate $\overbrace{\phi \circ \phi \cdots \circ \phi}^{n \text{ times}}$ of some self-map ϕ of $\mathbb{D}$, then (ϕ_n) admits a $\mathcal{B}$-universal function if and only if ϕ is a parabolic or hyperbolic automorphism.*

Results on the existence of infinite dimensional subspaces of $H^\infty(\mathbb{D})$ for which each element is universal for (ϕ_n), are discussed also. For example it is shown in [1] that whenever there is a $\mathcal{B}$-universal function for (ϕ_n), then there actually exists a uniformly closed subspace V of $H^\infty(\mathbb{D})$, linear isometric to ℓ^1 and locally uniformly dense in $H(\mathbb{D})$ such that the orbit $\mathcal{O}(f)$ is dense in $\{g \in H^\infty(\mathbb{D}) : ||g||_\infty \leq ||f||_\infty\}$ for every $f \in V$. On the other hand, such a "universal" space can never be chosen to be *locally* uniformly closed since every infinite dimensional, locally uniformly closed subspace of $H(\mathbb{D})$ necessarily contains an unbounded function.

We also give a necessary and sufficient condition on a sequence (ϕ_n) of selfmaps of $\mathbb{D}$ in order for this sequence to admit an $H(\mathbb{D})$-universal function:

Theorem 5. [1] *Let (ϕ_n) be a sequence of holomorphic self-maps of $\mathbb{D}$ such that $\phi_n(0) \to 1$. Then (ϕ_n) admits a $H(\mathbb{D})$-universal function if and only if there exists some subsequence n_j such that for all compact sets $K \subseteq \mathbb{D}$ there exists j_0 such that ϕ_{n_j} is injective on K for all $j \geq j_0$.*

In particular, the sequence $\phi_n(z) = \alpha_n z + 1 - \alpha_n$ where $\alpha_n \to 0$, admits a $H(\mathbb{D})$-universal function, but not a $\mathcal{B}$-universal function. Examples of sequences giving rise to $\mathcal{B}$-universal functions include the class of n-th roots of the atomic inner function $S(z) = \exp[-(1+z)(1-z)]$. Whereas the product of two $\mathcal{B}$-universal functions is not necessarily universal again (for example B^2, where B is a $\mathcal{B}$-universal Blaschke product), the composition of two $\mathcal{B}$-universal functions, is. Also, $B \times (B \circ B)$ is $\mathcal{B}$-universal.

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Hypercyclicity and chaos of operators on non-metrizable spaces

José Bonet

Let G denote the open unit disc $\mathbb{D}$ or the whole complex plane $\mathbb{C}$. In this lecture we study the dynamical behaviour of the differentiation operator D on several spaces of holomorphic functions $f \in H(G)$ on the open set G. A weight v on G is a continuous strictly positive function on G. We consider only weight functions v on G which are radial, i.e. $v(z) = v(|z|)$ for each $z \in G$, and non-increasing on the positive real numbers in G. Moreover, if $G = \mathbb{D}$ we assume that $v(r)$ tends to 0 as r tends to 1, and if $G = \mathbb{C}$, we suppose that $r^n v(r)$ tends to 0 for each n as r tends to ∞. The following weighted Banach spaces of holomorphic functions were investigated by Shields and Williams. Their isomorphic classification was clarified completely by Lusky [5].

$$Hv(G) := \{f \in H(G) \mid ||f||_v := \sup_{z\in G} v(z)|f(z)| < \infty\}.$$

Its closed subspace $Hv_0(G)$ consists of all the holomorphic functions $f \in H(G)$ such that $v|f|$ vanishes at infinity on G. The space $Hv_0(G)$ contains the polynomials and the polynomials are dense. It is known that the differentiation operator is bounded on $Hv(G)$ if and only if it is bounded on $Hv_0(G)$, and it is never a bounded operator from $Hv(\mathbb{D})$ into itself. However, Lusky (see also [4]) proved that, if the following condition introduced by Shields and Williams is satisfied:

$$\sup_n \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty,$$

then the differentiation operator D is continuous from $Hv_0(\mathbb{D})$ into $Hw_0(\mathbb{D})$ for $w(r) := (1-r)v(r), r \in [0,1[$. In particular, D is continuous from $H(v_n)_0(\mathbb{D})$ into $H(v_{n+1})_0(\mathbb{D})$ for $v_n(z) := (1-|z|)^n,\ z \in \mathbb{D}$. Therefore D is continuous (and surjective) from the Korenblum algebra $A^{-\infty} := \mathrm{ind}_n H(v_n)_0(\mathbb{D})$ into itself. This space is a complete separable countable inductive limit of Banach spaces, which is not metrizable, not Baire, nuclear, and is isomorphic to the strong dual s' of the space s of rapidly decreasing sequences. As a consequence of the comparison principle and a result of Godefroy and Shapiro, we have

Theorem 1 ([1])**.** *The differentiation operator* $D : A^{-\infty} \to A^{-\infty}$ *is hypercyclic and has a dense set of periodic points.*

A. Harutyunyan, W. Lusky [4] showed that if $Hv(\mathbb{C})$ is isomorphic to ℓ_∞, then the differentiation operator $D : Hv_0(\mathbb{C}) \to Hv_0(\mathbb{C})$ is bounded if and only if there are $\beta > 0$ and $r(0) > 0$ such that $v(r)e^{\beta r}$ is increasing in the interval $[r(0), \infty[$.

Theorem 2. *The differentiation operator* $D : Hv_0(\mathbb{C}) \to Hv_0(\mathbb{C})$ *is hypercyclic if and only if* $\underline{\lim}_{n\to\infty} \frac{||z^n||_v}{n!} = 0$.

As a consequence, $D : Hv_0(\mathbb{C}) \to Hv_0(\mathbb{C})$ for $v(r) = e^{-\beta r}$ is not hypercyclic if $0 < \beta < 1$, and it is hypercyclic if $\beta \geq 1$. Countable inductive limits of weighted Banach spaces of this type are called radial Hörmander algebras; they played an important role in interpolation theory and were investigated by C. Berenstein and B.A. Taylor. The most important example is $A_s(\mathbb{C}) := \text{ind}_n H(v_n)_0(\mathbb{C}), v_n(z) = \exp(-n|z|^s), z \in \mathbb{C}, s > 0$. In case $s = 1$ we obtain the space of entire functions of exponential type.

Theorem 3 ([1])**.** *The differentiation operator* $D : A_s(\mathbb{C}) \to A_s(\mathbb{C})$ *is not hypercyclic if* $0 < s < 1$ *and it is hypercyclic and has a dense set of periodic points if* $s \geq 1$.

Meise proved that the monomials $(z^n)_n$ constitute an absolute basis of radial Hörmander algebras $A_p(\mathbb{C})$ as soon as the subharmonic radial function $p(z)$ satisfies certain conditions. In this case, the space $A_p(\mathbb{C})$ is isomorphic to a Köthe co-echelon space, and the differentiation operator is a weighted backward shift on this sequence space. In joint work with Grosse-Erdmann and Peris [3], we investigated the dynamical behaviour of weighted backward shifts on complete non-metrizable sequence spaces. Recall that every hypercyclic operator is topologically transitive. The converse holds if the operator is defined on a separable, metrizable Baire space. Transitivity of linear operators on non-metrizable spaces was investigated in [2]. The following result permits us to obtain consequences about weighted backward shifts on non-metrizable sequence spaces.

Theorem 4 ([3])**.** *Let* E *be a complete, barrelled locally convex space with an unconditional Schauder basis* $(e_n)_n$. *Assume that the operator* $B(\sum a_n e_n) := \sum a_{n+1} e_n$ *is well defined and continuous. We have*

(1) B *is topologically transitive if and only if for each 0-neighbourhood* U *in* E *there is* n *such that* $e_n \in U$.

(2) The following conditions are equivalent: (a) B *is chaotic, (b)* B *has a non-trivial periodic point, and (c) the series* $\sum e_n$ *converges in* E.

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Extensions of a theorem of Bourdon and Feldman on somewhere dense orbits

George Costakis

Let $T : X \to X$ be a linear continuous operator acting on a complex Banach space X. The symbol $Orb(T, x)$ stands for the orbit of a vector x under the operator T, i.e.

$$Orb(T, x) = \{x, Tx, T^2x, \cdots\}.$$

The operator T is called hypercyclic (supercyclic) provided there exists $x \in X$ so that the set $Orb(T, x)$ ($\mathbb{C}Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C},\, n = 0, 1, 2, \cdots\}$) is dense in X. Our purpose is to provide certain extensions and versions of the following somewhere dense orbit theorem due to Bourdon and Feldman [1].

Theorem 1. *If the set $Orb(T, x)$ ($\mathbb{C}Orb(T, x)$) is somewhere dense then it is everywhere dense.*

In order to present our first result, [3], we introduce the following definition.

Definition 2. Let $T : X \to X$ be a linear continuous operator acting on a complex Banach space X. For every $x \in X$ the set

$$J(x) = \{y \in X : \text{ there exist a strictly increasing sequence of positive integers } \{k_n\} \text{ and a sequence } \{x_n\} \subset X \text{ such that } x_n \to x \text{ and } T^{k_n} x_n \to y\}$$

denotes the extended limit set of x under T.

Observe that $J(x)$ is always a closed set. We establish the following theorem, see [3].

Theorem 3. *Let $T : X \to X$ be a linear continuous operator acting on a complex Banach space X. If x is cyclic for T and the set $J(x)$ has non-empty interior then $J(x) = X$.*

It is not difficult to show that the above theorem implies the somewhere dense orbit theorem of Bourdon and Feldman. We would also like to point out that the above theorem also implies that T is hypercyclic and although $J(x) = X$ it is not necessary that the vector x is hypercyclic, see [3].

Our second version of the somewhere dense orbit theorem concerns the notion of Cesaro hypercyclicity. An operator T is called Cesaro hypercyclic provided there exists a vector $x \in X$ so that its Cesaro orbit

$$\{\frac{I + T + T^2 + \cdots + T^{n-1}}{n} x : n = 1, 2, \cdots\}$$

under T is dense in X. F. León-Saavedra showed that T is Cesaro hypercyclic if and only if there exists a vector $y \in X$ such that the sequence

$$\{\frac{T^n}{n} y : n = 1, 2, \cdots\}$$

is dense in X, see [4]. In [2] we establish the following.

Theorem 4. *Let $T : X \to X$ be a linear continuous operator acting on a complex Banach space X and let $x \in X$.*

i) Suppose T is hypercyclic. If the sequence $\{\frac{T^n}{n} x : n = 1, 2, \cdots\}$ is somewhere dense in X then it is everywhere dense and hence T is Cesaro hypercyclic.

ii) If the sequence $\{\frac{T^n}{n} x : n = 1, 2, \cdots\}$ is somewhere dense in X then for every $\epsilon > 0$ the set

$$\{\lambda \frac{T^n}{n} x : \lambda \in (0, \epsilon),\ n = 1, 2, \cdots\}$$

is everywhere dense in X.

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Hypercyclic and frequently hypercyclic subspaces

Karl.-G. Grosse-Erdmann

(joint work with Antonio Bonilla)

A *hypercyclic subspace* of an operator T is defined as an infinite-dimensional closed subspace of the underlying space X such that every non-zero vector in the subspace is hypercyclic for T. We study the corresponding notion for the new concept of a frequently hypercyclic operator. We begin, however, with two remarks on the well-studied classical notion.

1. The theorem of Montes

Montes [12] obtained the following sufficient condition on an operator to have a hypercyclic subspace.

Theorem (Montes)**.** *Let X be a separable Banach space and $T : X \to X$ a continuous linear operator. Suppose that there exists a sequence (n_k) such that*

(i) *T satisfies the Hypercyclicity Criterion for (n_k);*
(ii) *there exists an infinite-dimensional closed subspace Z_0 of X such that $T^{n_k}x \to 0$ for all $x \in Z_0$.*

Then T has a hypercyclic subspace.

By now, there exist three proofs of this result. Montes' own proof was based on a technique of perturbation of basic sequences, together with a rather lengthy construction. Chan [6], see also [7] and [13], found a short operator theoretic proof. A proof using tensor product techniques was given by Martínez and Peris [11].

Independently, Petersson [14], using basic sequences, and Bonet, Martínez and Peris [4], using tensor products, succeeded in generalizing Montes' result to separable Fréchet spaces, provided that the space possesses a continuous norm. Without this extra assumption, the result becomes false, see [4, Remark 3.6].

Further generalizations are due to León and Müller [10] (sequences of operators on Banach spaces), whose proof also considerably shortened Montes' original proof, and to Aron, Bès, León and Peris [1] (common hypercyclic subspaces for countably many sequences of operators on Banach spaces).

We present here the generalization of Montes' theorem to separable F-spaces that have a continuous norm. We remark that the setting of F-spaces, that is, complete metrizable topological vector spaces, seems to be the largest common ground for many results in hypercyclicity theory, see [9].

Theorem 1. *Let X be a separable F-space with a continuous norm and $T : X \to X$ a continuous linear operator. Suppose that there exists a sequence (n_k) such that*

(i) *T satisfies the Hypercyclicity Criterion for (n_k);*
(ii) *there exists an infinite-dimensional closed subspace Z_0 of X such that $T^{n_k}x \to 0$ for all $x \in Z_0$.*

Then T has a hypercyclic subspace.

Our proof follows Montes' original approach via basic sequences. The central new idea consists in the observation that since X admits a continuous norm it can be embedded continuously into some Banach space $\widehat{X}$. One may then work with basic sequences in that space. Using the arguments of León and Müller, we arrive at a very short proof of our result. We remark that, with the same proof, our result extends to sequences of operators and to countable families of sequences of operators, thereby including all known results.

2. Kit Chan's approach

Chan's proof of Montes' theorem is based on his observation that there is an intimate link between the hypercyclicity of the operator T and that of the *left* *multiplication operator* $L_T : B(X) \to B(X), L_T V = TV$, where X is a separable Banach space and $B(X)$ is the space of continuous linear operators on X. Since $B(X)$ is not separable in the operator norm topology, hypercyclicity of L_T has to be taken in a weaker topology. Chan showed that if one considers the strong operator topology (SOT), then the following implications hold:

- If T satisfies the Hypercyclicity Criterion, then L_T satisfies a so-called SOT-Hypercyclicity Criterion;
- if L_T satisfies the SOT-Hypercyclicity Criterion, then L_T is SOT-hypercyclic;
- if L_T is SOT-hypercyclic and T satisfies condition (ii) in Montes' theorem, then T has a hypercyclic subspace.

The second of these implications says, more precisely, the following.

Theorem (Chan)**.** *Let X be a separable Banach space and $L : B(X) \to B(X)$ a linear operator that is continuous with respect to the operator norm. Suppose there exists a countable, SOT-dense set $D \subset B(X)$ and a mapping $A : D \to D$ such that, for all $V \in D$,*

(i) $LAV = V$,
(ii) $\|L^n V\| \to 0$ *and* $\|A^n V\| \to 0$.

Then L is hypercyclic for the strong operator topology on $B(X)$.

So far, all proofs of this and similar results in the literature have been constructive, while the usual proof of the Hypercyclicity Criterion is based on the Baire category theorem. It was previously believed that no such proof is available for Chan's result because the strong operator topology is not metrizable. We show that a Baire category proof is, nonetheless, possible. It is based on the following notion of restricted universality that was introduced in [8]. A sequence (L_n) of continuous mappings $L_n : X \to Y$ between metric spaces X and Y is called *universal for a subset M of Y* if there exists some $x \in X$ such that $M \subset \overline{\{L_n x : n \geq 1\}}$.

3. Frequently hypercyclic subspaces

Bayart and Grivaux [2], [3] have recently introduced the concept of a frequently hypercyclic operator. They call an operator T *frequently hypercyclic* if there exists a vector $x \in X$ such that, for every non-empty open subset U of X, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. In that case, x is called a *frequently hypercyclic vector.* They also provide a sufficient condition for an operator to be frequently hypercyclic, which we recently improved to the following, see [5].

Theorem (Frequent Hypercyclicity Criterion)**.** *Let X be a separable F-space and $T : X \to X$ a continuous linear operator. Suppose that there are a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that, for all $x \in X_0$,*

(i) $\sum_{n=1}^{\infty} T^n x$ *converges unconditionally,*
(ii) $\sum_{n=1}^{\infty} S^n x$ *converges unconditionally,*
(iii) $TSx = x$.

Then T is frequently hypercyclic.

A variant of the proof of Theorem 1 provides the following result.

Theorem 2. *Let X be a separable F-space with a continuous norm and $T : X \to X$ a continuous linear operator. Suppose that*

(i) *T satisfies the Frequent Hypercyclicity Criterion;*
(ii) *there exists an infinite-dimensional closed subspace Z_0 of X such that $T^n x \to 0$ for all $x \in Z_0$.*

Then T has a frequently hypercyclic subspace.

As an application we show that the translation operator $T : H(\mathbb{C}) \to H(\mathbb{C})$, $Tf(z) = f(z+1)$ has a frequently hypercyclic subspace.

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List of open problems

The following list contains some of the problems that were posed and discussed during the Problem Session of the workshop.

1. J. Bonet: Two questions on hypercyclic operators on Fréchet spaces

1. The following problem was proposed by F. Martínez-Giménez, A. Peris and me in our joint article "A Banach space which admits no chaotic operator", Bull. London Math. Soc. 33 (2001) 196-198. It seems to be still unsolved.

Does every infinite dimensional nuclear Fréchet space admit a chaotic operator?

The problem seems to be open even for nuclear Fréchet spaces with basis, which are in fact isomorphic to a nuclear Köthe echelon space. It is known that the space of holomorphic functions on the disc, the space of entire functions and the space of rapidly decreasing sequences support chaotic operators.

2. *Does every nuclear Fréchet space E support a hypercyclic operator T such that λT is hypercyclic for all $\lambda \neq 0$?*

This is clearly a question on non-normable Fréchet spaces, since every operator T on a Banach space with norm smaller than 1 cannot be hypercyclic. The work of G. Godefroy and J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991), 229-269, contains several examples of nuclear Fréchet spaces (of entire functions) which admit a hypercyclic operator T such that λT is hypercyclic for all $\lambda \neq 0$.

2. Félix Martínez-Giménez: Existence of chaotic polynomials on Fréchet Spaces

In 1969 S. Rolewicz [5] asked about the existence of hypercyclic continuous linear operators on any separable infinite dimensional Banach space. The question was solved in the positive independently by Ansari [1] and Bernal [2]. Later, Bonet and Peris [3] were able to prove it for Fréchet spaces. The corresponding question for chaos in the sense of Devaney (hypercyclicity plus density of periodic points) has a negative answer [4].

Concerning polynomials, it is natural to ask: Does every separable infinite dimensional Fréchet space admit a chaotic polynomial of degree $d > 1$?

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3. V. Müller: Three problems

Problem. *Let T be an operator on a Hilbert space H. Suppose that for each $x \in H$, $\inf_n \|T^n x\| = 0$. Does it imply that $\inf_n(\|T^n u\| + \|T^n v\|) = 0$ for all $u, v \in H$?*

The problem is motivated by the notion of orbit-reflexive operators; it arose in a discussion with D. Hadwin.

Problem (Prajitura)**.** *Let $T \in B(X)$. Suppose that both T and $3T$ are hypercyclic. Is then $2T$ also hypercyclic?*

More generally, is the set $\{t > 0 : tT \text{ is hypercyclic}\}$ convex?

Problem. *Let $T \in B(X)$, $x \in X$. Suppose that for each $y \in X, y \neq 0$, there exists n such that $\|T^n x - y\| < \varepsilon \|y\|$, where ε is a fixed positive number (maybe $\varepsilon < 1/3$?). Does it imply that T is hypercyclic?*

The problem is inspired by the result of N. Feldman that for the hypercyclicity of T it is sufficient to have $x \in X$ with d-dense orbit.

4. Alfredo Peris: Disjointness of ergodic operators

Let X be a separable Banach (or Hilbert) space. Find operators $T, S \in L(X)$ that respectively admit invariant (Borel) probability measures μ, μ' whose topological support is X, and such that (X, T, μ), (X, S, μ') are ergodic and, for each $A, B \subset X$ Borel sets, there are $X_1, X_2 \subset X$ of full measure with

$$\lim_N \frac{1}{N+1} \sum_{k=0}^{N} \chi_A(T^n x_1)\chi_B(S^n x_2) = \mu(A)\mu'(B) \quad \forall x_i \in X_i,\ i = 1, 2.$$

Observe that, in particular, there would exist elements $x \in X$ so that, for each pair of non-empty open sets $U, V \subset X$,

$$\underline{\text{dens}}\{n \in \mathbb{N}\ ;\ (T^n x, S^n x) \in U \times V\} > 0.$$

5. Alfredo Peris and José A. Conejero: Chaotic operators in C_0-semigroups

Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a C_0-semigroup defined on a separable complex Banach space X. Assume that $\mathcal{T}$ is chaotic in the sense of Devaney (that is, $\mathcal{T}$ is hypercyclic and the set of periodic points $\mathcal{P} := \{x \in X\ ;\ \exists t > 0 \text{ with } T_t x = x\}$ is dense in X). Is every single operator T_t $(t > 0)$ chaotic? Does there exist at least a $t_0 > 0$ such that T_{t_0} is chaotic?

Contributions from the Informal Session

We include two of the contributions that were presented during the Informal Session of the workshop.

1. Karl-G. Grosse-Erdmann: On ω, every hypercyclic operator satisfies the Hypercyclicity Criterion

The Great Open Problem in hypercyclicity asks if every hypercyclic operator satisfies the Hypercyclicity Criterion, or, equivalently, if for every hypercyclic operator T also the direct sum $T \oplus T$ is hypercyclic, see [1]. It is conceivable that the solution to this problem depends on the underlying space X. For example, the answer may well be positive in Hilbert space but negative in some Banach spaces.

We want to point out that there is a space in which the problem can be solved, namely the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences. Endowed with the product topology, this space becomes a separable Fréchet space.

In 1993, Herzog and Lemmert [5] characterized the hypercyclic operators on ω. We denote by T' the dual operator to T.

Theorem (Herzog, Lemmert). *A continuous linear operator T on ω is hypercyclic if and only if it is torsion free, that is, if for any continuous linear functional λ on ω and every complex polynomial P, $P(T')\lambda = 0$ implies that $P = 0$ or $\lambda = 0$.*

As observed by Conejero [4], it is easy to see that T is torsion free if and only if T' has no eigenvector. Now, since $\omega \times \omega$ is isomorphic to ω, and since $(T \oplus T)'$ has an eigenvector if and only if T' does, we arrive at the following consequence of the Theorem of Herzog and Lemmert.

Theorem. *If T is a hypercyclic operator on ω, then $T \oplus T$ is hypercyclic on $\omega \times \omega$.*

We emphasize, however, that ω has to be considered as a pathological space in hypercyclicity, as is clearly indicated by several known results, see, for example, [3], [2]. Thus, the positive solution of the great problem in ω tells us nothing about its fate in general.

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2. V. Müller: Hypercyclic direct sums

Theorem 1. *Let M be any subset of the set of all positive rational numbers. Then there are operators A, B acting on a Banach space X such that, for $j, k \geq 1$, the operator $A^j \oplus B^k$ is hypercyclic if and only if $j/k \in M$.*

A variation of this is:

Theorem 2. *Let M be any subset of the set of all rational numbers r, $0 < r \leq 1$. Then there exists a Banach space operator A such that, for $1 \leq j \leq k$, the operator $A^j \oplus B^k$ is hypercyclic if and only if $j/k \in M$.*

Operators in Theorems 1, 2 are weighted bilateral shifts with properly chosen weights.

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