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Komplexe Analysis

Organised by
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August 27th – September 2nd, 2006

Abstract. The main aim of this workshop was to discuss recent developments in several complex variables and complex geometry. The topics included: classification of higher dimensional varieties, mirror symmetry, hyperbolicity, Kähler geometry and classical geometric questions.

Mathematics Subject Classification (2000): 14xx, 32xx.

Introduction by the Organisers

The workshop *Komplexe Analysis*, organised by Jean-Pierre Demailly (Grenoble), Klaus Hulek (Hannover) and Thomas Peternell (Bayreuth) was held August 27th–September 2nd, 2006. This meeting was well attended with over 50 participants from Europe, US, Israel and Far East. The participants included several leaders of the field as well as many young (non-tenured) researchers, including four doctoral students.

The aim of the meeting was to present recent important results in several complex variables and complex geometry with a particular emphasis on topics linking different areas of the field, as well as to discuss new directions and open problems. Altogether there were eighteen talks of 60 minutes each, a programme which left sufficient time for informal discussions and joint work on research projects.

One of the topics under discussion was the classification theory of higher dimensional varieties. The key note lecture was given by J. M^{c}Kernan who announced a proof of the main conjecture in Mori theory, which says that all smooth projective varieties of general type have a minimal and a canonical model. This is equivalent to finite generation of the canonical ring. Outside the official programme McKernan also explained some of the technical details of the proof. In his talk

S. Kovács gave a proof of a conjecture of Viehweg on families of canonically polarized varieties over a surface. On the other side of the classification problem, the geometry of Fano manifolds and their classification also played an important role. P. Jahnke lectured on recent progress in the classification of Fano manifolds and A. V. Pukhlikov spoke on the birational geometry of these varieties. Finally, J.-M. Hwang discussed the problem which Fano manifolds can be rigid targets of holomorphic maps and M. Brion presented his work on the structure of log homogeneous pairs.

Mirror symmetry and related topics remain an important aspect of the field. This was addressed in three talks. B. Siebert spoke on tropical games and mirror symmetry. Tropical geometry was also the topic of A. Gathmann's talk who discussed recent progress in enumerative geometry in connection with tropical methods. On the other end of the temperature scale D. van Straten explained arctic monodromy calculations.

Hyperbolicity was the central theme of the talks of E. Rousseau, who proved hyperbolicity type results for generic hypersurfaces of sufficiently high degree in 4-dimensional projective space, and in the programmatic lecture by M. McQuillan. The talks by Ph. Eyssidieux on singular Kähler-Einstein metrics and by T. Ohsawa on Levi flat hypersurfaces in complex manifolds were predominantly analytic in nature.

More classically geometric topics were discussed by F. Catanese (slope of Kodaira fibrations), F. Bogomolov (symmetric tensors and the geometry of secant varieties) and P. M. H. Wilson (geometry of Kähler moduli). Finally, A. Sommese discussed techniques on how to decompose certain varieties into irreducible components. These questions were motivated by engineering problems and their solution has many practical applications.

Workshop: Komplexe Analysis

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Abstracts

Log homogeneous varieties

Michel Brion

Consider a complete nonsingular algebraic variety X over the field of complex numbers. If X is homogeneous, then its structure is well understood by a classical result of Borel and Remmert: X is the product of an abelian variety (the Albanese variety) and a flag variety (the basis of the "Tits fibration").

More generally, one would like to classify the almost homogeneous varieties X, i.e., those where a connected algebraic group G acts with an open orbit X_0. However, this central problem of equivariant geometry seems to be too general, and complete results are only known under assumptions of smallness (in various senses) for the boundary $X \setminus X_0$. We refer to [5] for an exposition of such classification results, in the setting of holomorphic actions on complex analytic spaces.

In another direction, there is a well-developed structure theory for certain classes of almost homogeneous varieties, where the acting group and the open orbit are prescribed: torus embeddings, spherical embeddings and, more generally, equivariant embeddings of a homogeneous space X_0 under an affine reductive group G (see the recent survey [6]). A subclass of special interest is that of G-regular varieties in the sense of [3]; in loose words, their orbit structure is that of a nonsingular toric variety. Another important subclass consists of quasi abelian varieties; in loose words again, they are toric bundles over abelian varieties. (Quasi abelian varieties are also called semiabelic varieties in [1], where their role in degenerations of abelian varieties is investigated.)

These restrictive assumptions on G and X_0 are convenient in the setting of algebraic transformation groups, but somewhat unnatural from a geometric viewpoint. This motivates the search for a class of almost homogeneous varieties containing all regular varieties under a reductive group and all quasi abelian varieties, having geometric significance and an accessible structure. Also, it is natural to impose that the boundary be a divisor with normal crossings.

In the preprint [4], we introduce the class of log homogeneous varieties and show that it fulfills the above requirements. Given a normal crossings divisor D in X, we say that X is log homogeneous with boundary D if the associated logarithmic tangent bundle $T_X(-\log D)$ is generated by its global sections. It follows readily that X is almost homogeneous under the connected automorphism group $G := \mathrm{Aut}^0(X, D)$, with boundary being D. More generally, the G-orbits in X are exactly the strata defined by D; in particular, their number is finite.

All G-regular varieties are log homogeneous, as shown in [2]. Quasi abelian varieties satisfy a stronger property, namely, the bundle $T_X(-\log D)$ is trivial; we then say that X is log parallelizable with boundary D. Conversely, log parallelizable varieties are quasi abelian by a theorem of Winkelmann [7] which has been the starting point for our investigations. The class of log homogeneous varieties turns out to be stable under several natural operations: induction, equivariant blow-ups, finite étale covers and taking invariant subvarieties or irreducible components of fibers of morphisms.

Any log homogeneous variety X comes with two natural morphisms which turn out to play opposite roles. The first one is the Albanese map $\alpha : X \to \mathcal{A}(X)$; in our algebraic setting, this is the universal map to an abelian variety. We show that α is a fibration with fibers being log homogeneous under the maximal connected affine subgroup G_{aff} of G, and spherical under any Levi subgroup of G_{aff}.

The second morphism is the Tits map τ to a Grassmannian, defined by the global sections of the logarithmic tangent bundle. Let $\sigma : X \to \mathcal{S}(X)$ be the Stein factorization of τ, so that σ is the morphism associated with the globally generated divisor $-K_X - D$ (the determinant of $T_X(-\log D)$). We show that $\mathcal{S}(X)$ is also a spherical variety under any Levi subgroup of G_{aff}, and the irreducible components of fibers of σ are quasi abelian varieties. It follows that the product morphism $\alpha \times \sigma$ is surjective with fibers being finite unions of toric varieties. This generalizes the Borel–Remmert theorem.

We refer to [4] for details, and also for the classification of log homogeneous surfaces. It shows that the variety $\mathcal{S}(X)$ is generally singular, and that only few two-dimensional homogeneous spaces admit a log homogeneous equivariant completion. The classification of such homogeneous spaces in an arbitrary dimension is an open question. Also, is it true that $\mathcal{S}(X)$ has only quotient singularities?

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Tropical Games and Mirror Symmetry

Bernd Siebert
(joint work with Mark Gross)

A classical construction, due to Mumford, produces a degeneration of toric varieties out of an integral polyhedral decomposition $\mathscr{P}$ of $\mathbb{R}^n$. This works by simply taking the fan Σ in $\mathbb{R}^{n+1}$ defined by the closures of the cones $\mathbb{R}_{\geq 0} \cdot (\Xi \times \{1\})$ over the cells $\Xi \in \mathscr{P}$; the projection to the last coordinate defines a map of this fan to the fan of $\mathbb{A}^1$, hence a toric morphism of toric varieties $\pi : X = X(\Sigma) \to \mathbb{A}^1$. General fibers of π are isomorphic to the toric variety given by the *asymptotic fan* $\{\lim_{t\to 0} t \cdot \Xi \,|\, \Xi \in \mathscr{P}\}$ of the polyhedral decomposition. The central fiber $X_0 = \pi^{-1}(0)$ can also readily be read off from $\mathscr{P}$ as follows. To a vertex v of the polyhedral decomposition is associated the complete, n-dimensional fan with cones $\mathbb{R}_{\geq 0} \cdot (\Xi - v)$ generated by the cells $\Xi \in \mathscr{P}$ containing v. The corresponding toric varieties are the irreducible components of X_0. They are glued pairwise by identifying toric prime divisors torically.

This construction was one of the motivations for my joint program with Mark Gross (UCSD) for a comprehensive explanation of the mirror phenomenon [3]. The central idea is to view mirror symmetry as a duality of degeneration data associated to *maximal, toric degenerations.* These exist for large classes of varieties, including Calabi-Yau varieties. The central fiber of a toric degeneration is a union of toric varieties glued along toric divisors just as in the Mumford construction above. $\mathbb{R}^n$ is now replaced by a topological manifold B, which is still obtained by gluing polyhedra $\Xi \in \mathscr{P}$ by integral affine maps. Moreover, the fans defining the toric components of X_0 endow a neighbourhood of each vertex with an integral affine structure that is compatible with the affine structures on the cells. In agreement with recent developments in "tropical algebraic geometry" this discrete part of our degeneration data is what one may call a "tropical Calabi-Yau variety", in case B is compact. This is the arena where according to our program mirror symmetry should be understood. The mirror transformation itself is a discrete version of the Legendre transformation. This requires a polarization on X_0, giving rise to a (multi-valued) convex, piecewise affine function φ on B. The triple $(B, \mathscr{P}, \varphi)$ is the complete set of discrete degeneration data. Results on cohomology and base change [4] relate the topology of B with the topology of the general fiber of the degeneration. For example, for a toric degeneration of Calabi-Yau varieties, B is a $\mathbb{Q}$-homology sphere, while for a degeneration of Hyperkähler manifolds, B is a $\mathbb{Q}$-homology complex projective space (these statements require a technical hypothesis on maximality of the degeneration).

While possibly not absolutely necessary for the investigation of mirror symmetry, it is natural to ask if one can revert the logic and construct a toric degeneration for any given $(B, \mathscr{P})$? The paper [3] already goes a long way toward this aim by establishing a one-to-one correspondence between certain cohomological data on B and spaces X_0 glued from complete toric varieties together with a *log-smooth* *structure*. Such log-structures allow to work with non-normal spaces in many respects as if they were smooth [5]. Log-smooth deformation theory (or Friedman's smoothability result for "d-semistable K3-surfaces" [1]) now implies that the answer to our question in dimension two is affirmative. In higher dimensions there is first a problem in applying known results off the shelves because the log structures itself have singularities and because many of the results are only available for the normal crossings case. These problems can be overcome [4]. However, in the non-normal crossings case there is still a problem with non-vanishing obstruction groups that does not seem to have a solution within this framework.

This talk was about a different idea that we first pursued in the first half of 2004. Deformation theory usually relies on patching thickenings of an open affine cover to produce a k-th order deformation X_k over $\operatorname{Spec}\mathbb{C}[t]/(t^{k+1})$. However, for reducible, reduced X_0 the (unique) primary decomposition for X_k exhibits the deformation by gluing k-th order thickenings of the irreducible components of X_0. This suggests a closed cover approach to deformation theory as opposed to the traditional open cover approach. These thickenings are not flat deformations itself, so much care has to be taken. Nevertheless, the Mumford construction above does have a neat interpretation in these terms; in this case the naive thickenings provided by the polyhedral decomposition are indeed consistent. In the more "non-linear" Calabi-Yau world this is not true anymore and corrections become necessary. This means that we have to change the gluings of the thickenings by (log-) automorphisms. Unfortunately, the automorphisms tend to affect also the other gluings, with the affected number of gluings increasing with k. The natural setup to study these effects is the dual picture, where $(B, \mathscr{P}, \varphi)$ is interpreted as (polarized) *intersection complex*. The rings to be glued can then be interpreted as monoid rings given locally by integral points in $\mathbb{R}^{n+1}$ lying above the graph of φ, in an integral affine chart.

At this point Kontsevich and Soibelman gave a rigid-analytic version of our two-dimensional reconstruction theorem mentioned above [6]. While technically a very different approach, which works on the affine manifold with singularities B alone and with a choice of Riemannian metric, it developed a picture of automorphisms propagating along (curvilinear) rays and producing new rays whenever two rays intersect. Possibly infinitely many new rays are inserted in such a way that the product of automorphisms along a small loop about the intersection point is the identity.

Translated to our language and to arbitrary dimensions the relevant automorphisms fix the subrings generated by monomials over a local affine hyperplane (a *wall*) in B along which it propagates. The affected order is measured by the change of slope of φ, hence is strictly increasing when passing into new cells. The rule to generate new walls follows from a unique decomposition lemma for a group of automorphisms of the relevant ring just as in [6]. The picture that one obtains is a refinement of the polyhedral decomposition of B by parts of such hyperplanes, which becomes ever more complicated when we increase the order. This is the "tropical game" alluded to in the title. It is completely determined by the starting data given by the polarized intersection complex $(B, \mathscr{P}, \varphi)$ and the log structure. It is then possible to construct a k-th order thickening of X_0 by taking a copy of the relevant ring for each cell of this refined decomposition and gluing them via the automorphisms of those walls that a connecting path crosses. General results of deformation theory then show that there also exists an actual deformation.

We thus obtain the following result.

Theorem 1. *Every polarized tropical manifold $(B, \mathscr{P}, \varphi)$ arises as the intersection complex of a polarized toric degeneration as defined in* [3]*, Section 4.2.*

For mirror symmetry it is important to note that our proof gives explicit access to the corrections of the complex structure that are necessary to glue, say, a Calabi-Yau manifold to finite order near a "large complex structure limit". This realizes Fukaya's dream of geometric access to the instanton corrections on the B-model (holomorphic) side of mirror symmetry [2]. It should now be possible to formulate the period computations on the B-model side of mirror symmetry in purely tropical terms. In fact, a preliminary check reproduced instanton numbers arising in a local Calabi-Yau example. It is also interesting to observe that our walls naturally support (families of) tropical curves; on the mirror side these relate to holomorphic curves and Gromov-Witten invariants [7],[8]. We believe that these observations will eventually lead to the geometric explanation of the relation between period computations and Gromov-Witten invariants in mirror symmetry. We also believe that an analogous connection involving tropical disks is the right framework for studying homological mirror symmetry.

Apart from applications to mirror symmetry our result also gives the exciting perspective of constructing new, interesting complex manifolds. For example, to obtain Hyperkähler manifolds in dimension 4 one needs to study affine structures on $\mathbb{CP}^2$ minus a codimension two locus.

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Symmetric tensors and the geometry of secant varieties in a projective space

Fedor Bogomolov
(joint work with Bruno de Oliveira)

In this talk we consider symmetric tensors on smooth projective varieties. Namely the space $H^0(X, S^m(\Omega^1 \times L))$ is the space of symmetric tensors on a smooth projective variety X with coefficients in a line bundle L. If $L = \mathcal{O}$ the corresponding space $H^0(X, S^m(\Omega^1))$ is a birational invariant of the projective variety X. This invariant is more subbtle then the canonical class. While the dimension of $H^0(X, nK)$ does not vary by Siu's theorem under smooth deformations of X the dimension of the spaces $H^0(X, S^m(\Omega^1))$ vary for $m > 1$. There are two types of examples of varieties where these dimensions vary. The first one is given in the article [2].

Another example is given by smooth hypersurfaces in $\mathbb{P}^3$. Bruckman [4] proved that $H^0(Y, S^m(\Omega^1)) = 0$ for any smooth hypersurface in $\mathbb{P}^n$, $n \geq 3$. However if X a hypersurface of degree d with only nodal singularities in $\mathbb{P}^3$ then there are symmetric tensor on the desingularization $\tilde{X}$ of X if the number of nodal singular points on X is sufficently big ($> 8/3(d^2 - 5/2d)$). There are surfaces with a suffiently big number of nodal singular points for $d \geq 6$. Since $\tilde{X}$ is deformationally equivalent to a smooth surface of the same degree d we obtain the second set of examples of the "jumping" phenomenon [3]. However in this case we get more.

In the case of smooth hypersurfaces $Y \subset \mathbb{P}^n$ we have a stronger vanishing theorem. We show that $H^0(Y, S^m(\Omega^1 \times \mathcal{O}(1))) = 0$ for any $m > 0$ if Y is not quadric. Note that $\mathcal{O}(1) = 1/d\mathcal{O}(K)$ for a hypersurface. Thus under a smooth deformation of a surface X of degree d in $\mathbb{P}^3$ the dimension of the space of tensors $H^0(\tilde{X}, S^{dm}(\Omega^1 \times \mathcal{O}(K)))$ jumps for sufficiently big m if $d \geq 6$. In order to show this we consider first the extension $\tilde{\Omega}^1$ of Ω^1 of the cotangent bundle induced from a standard extension Ω^1 on $\mathbb{P}^n$. Note that for $\mathbb{P}^n$ the bundle $(\tilde{\Omega}^1 \times \mathcal{O}(1)) = ((n+1)\mathcal{O})$ and the space $H^0(\mathbb{P}^n, S^m\tilde{\Omega}^1 \times \mathcal{O}(1))$ can be identified with $H^0(\mathbb{P}^n, \mathcal{O}(m))$. The imbedding $Y \subset \mathbb{P}^n$ induces a map of the completion $T(Y)$ of the tangent bundle to Y to $\mathbb{P}^n$. We show that $H^0(Y, S^m(\tilde{\Omega}^1 \times \mathcal{O}(1)))$ is equal to $H^0(\mathbb{P}^n, \mathcal{O}(m))$ if the tangent bundle $T(Y)$ surjects onto $\mathbb{P}^n$ and the preimage of a generic point in $\mathbb{P}^n$ is a connected subvariety of $T(Y)$. Using the results of [5] and Zak's theorem we show that both conditions hold for smooth Y with $\dim Y > (2/3)(n-1)$. Now $S^m(\Omega^1 \times \mathcal{O}(1)) \subset S^m(\tilde{\Omega}^1 \times \mathcal{O}(1))$ and the space of sections $H^0(Y, S^m(\Omega^1 \times \mathcal{O}(1)))$ depends on the properties of some special secant varieties for Y.

We define $C_Y X$ as a union of lines tangent to smooth points of X and intersecting Y. In particular an element of $H^0(Y, S^m(\tilde{S}^m(\Omega^1 \times \mathcal{O}(1))))$ belongs to $H^0(Y, S^m(\Omega^1 \times \mathcal{O}(1)))$ if the zero set of the corresponding form in $H^0(\mathbb{P}^n, \mathcal{O}(m))$ contains $C_X X$ and in fact any variety $(C_X^k X)$ obtained by the iteration of the procedure. Note that $(C_X^k X)$ is always contained in the set of quadrics containing X. The converse is also true if codim $X \leq 2$ and $\dim X \geq 3$. In order to show we prove more precise results on the structure of secant varieties. We show that $C_X X$ coincides with a component of a threesecant variety $Tr(X)$ if Y is a smooth subvarity in $\mathbb{P}^n$ and $\dim Y > 2/3(n-1)$ using a generalization of Zak's lemma. Then we apply results of [6] in order to complete the proof. We think that for any smooth $X \subset \mathbb{P}^n$, $\dim X > (n/2)+1$ the variety $(C^n_X X)$ coincides with the intersection of all quadrics containing X.

In particular for smooth subvarieties of codimension two in $\mathbb{P}^n$, $n \geq 5$ the space $H^0(\tilde{Y}, S^m(\Omega^1 \times \mathcal{O}(1)))$ depends on whether Y is contained in a quadric, or in the intersection of two quadrics. These results are contained in the article [1].

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Singular Kähler-Einstein metrics

Philippe Eyssidieux

(joint work with Vincent Guedj, Ahmed Zeriahi)

We report on our recent work in collaboration with V. Guedj and A. Zeriahi (math.AG/0603431).

Thirty years ago, in a celebrated article [9], S. T. Yau solved the Calabi conjecture by studying complex Monge-Ampère equations on a compact Kähler manifold.

Two major developments in the theory of complex Monge-Ampère equations occurred in the last decade. In the local theory, a deeper analysis of the image of the complex Monge-Ampère operator [2], [4] has followed the pioneering work of E. Bedford and A. Taylor [1].

In the global theory a new proof of the $\mathcal{C}^0$-estimate [4] and the study initiated in [3] has allowed one to treat complex Monge-Ampère equations with more degenerate R.H.S. We obtain:

Theorem A. *Let X be a compact Kähler manifold, ω a semi positive (1,1)-form such that $\int_X \omega^n > 0$ and $0 \leq f \in L^p(X, \omega^n)$, $p > 1$, a density such that $\int_X f\omega^n = \int_X \omega^n$. Then there is a unique continuous function ϕ on X such that $\omega + dd^c\phi \geq 0$ and*

$$(\omega + dd^c\phi)^n = f\omega^n \quad \textit{with} \quad \sup_X \phi = -1.$$

Furthermore $f \mapsto \phi$ *is a continuous map from* $L^p(X, \omega^n)$ *to* $\mathcal{C}^0(X)$.

When ω has algebraic singularities, then μ can be assumed to have L^p density with respect to the Lebesgue measure. An independant proof of this statement was given in [10], see also [7].

With this $\mathcal{C}^0$-estimate, it is possible to adapt classical ideas and prove:

Theorem B. *Let* X *be a projective algebraic complex manifold,* ω *a smooth semi Kähler form that is Kähler outside a complex subvariety* $S \subset X$. *Let* Ω *be a Kähler form on* X. *Assume furthermore that* $\omega^n = D\Omega^n$ *where* $D^{-\epsilon}$ *is in* $L^1(\Omega^n)$ *and that* $[\omega], [\Omega] \in NS_{\mathbb{R}}(X)$.

Let $s_1, ..., s_p$ *(resp.* $t_1, ..., t_q$*) be holomorphic sections of some line bundle. (resp. of some other line bundle). Assume* $k, l \in \mathbb{R}_{\geq 0}$ *and* $F \in \mathcal{C}^\infty(X, \mathbb{R})$ *are fixed so that*

$$\int_X \frac{1}{|t_1|^{2l} + \ldots + |t_q|^{2l}} \Omega^n < \infty \quad \text{and} \quad \int_X \frac{|s_1|^{2k} + \ldots + |s_p|^{2k}}{|t_1|^{2l} + \ldots + |t_q|^{2l}} e^F \Omega^n = \int_X \Omega^n.$$

Then the unique continuous function ϕ *such that* $\omega + dd^c\phi \geq 0$ *and*

$$(\omega + dd^c\phi)^n = \frac{|s_1|^{2k} + \ldots + |s_p|^{2k}}{|t_1|^{2l} + \ldots + |t_q|^{2l}} e^F \Omega^n, \quad \text{with} \ \sup_X \phi = -1,$$

is smooth outside $B = S \cup \cap_i \{s_i = 0\} \cup \cap_i \{t_i = 0\}$.

The assumption that X is projective and $[\omega], [\Omega] \in NS_{\mathbb{R}}(X)$ was removed by [5].

As a by-product of his results, S. T. Yau constructed Kähler-Einstein metrics on smooth canonically polarized manifolds and Ricci-flat metrics on what is now known as Calabi-Yau manifolds. It had been soon realized that this also yields Kähler-Einstein metrics on Kähler orbifolds hence on the canonical models of surfaces of general type since they have isolated quotient singularities.

In higher dimension, in spite of the development of the MMP during the 1980s, there was no fully satisfying analog of these Kähler-Einstein metrics.

For smooth minimal general type projective manifolds, H. Tsuji [8] proved that an appropriate Kähler-Ricci flow starting with an arbitrary Kähler datum exists in infinite time and stated it converges towards a current representing the canonical class which is smooth outside the exceptional locus of the map to the canonical model, and defines a Kähler-Einstein metric there. The conjecture made there that the current has continuous local potentials partly motivated our work. Tsuji's work was revisited, and fully clarified, in [7] using an annoucement of the results in [10].

We obtain a more general definition of singular Kähler-Einstein metrics as a consequence of the following result:

Theorem 1. *Let* (V, Δ) *be a projective klt pair such that* $K_V + \Delta$ *is an ample* $\mathbb{Q}$*-divisor. Then there is a unique semi-Kähler current in* $[K_V + \Delta]$ *with continuous potentials, which satisfies a global degenerate Monge-Ampère equation on* V *and defines a smooth Kähler-Einstein metric of negative curvature on* $(V - \Delta)^{reg}$.

Let (V, Δ) *be a projective klt pair such that* $K_V + \Delta \cong 0$ *(*$\mathbb{Q}$*-linear equivalence of* $\mathbb{Q}$*-Cartier divisors). Then in every ample class in* $NS_{\mathbb{R}}(V)$ *there is a unique* *semi-Kähler current with continuous potentials, which satisfies a global degenerate Monge-Ampère equation on V and defines a Ricci flat metric on $(V - \Delta)^{reg}$.*

Corollary 2. *Let X be a projective manifold of general type. Assume X has a (unique) model V with only canonical singularities and K_V ample. Then K_V contains a unique singular Kähler-Einstein metric ω_{KE} of negative curvature.*

The problem of constructing a singular Kähler-Einstein metric on a canonically polarized projective variety with canonical singularities X_{can} had already been considered in [6] with a Monge-Ampère approach. Theorem 5.6 there and its proof imply that given $\pi : X \to X_{can}$ a log resolution, there is a closed positive current T_{KE} in $\pi^* K_{X_{can}}$ with zero Lelong numbers such that T_{KE} is smooth on $\pi^{-1}(X_{can}^{reg})$ and defines a KE metric there. His constuction agrees with ours, and our contribution is that T_{KE} has a continuous potential coming from X_{can}.

Corollary 3. *Let X be a projective variety with only canonical singularities such that $K_V \equiv 0$. Then every ample class of X contains a unique singular Ricci flat metric.*

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The geometry of Kähler moduli

Pelham M.H. Wilson

If X is a compact Kähler manifold of dimension n, then cup-product $\cup$ determines a degree n form on $H^2(X, \mathbb{R})$. The *index cone* $W \subset H^{1,1}(X, \mathbb{R})$ is defined by

$$W := \{D \in H^{1,1}(X, \mathbb{R}) \ : \ D^n > 0 \text{ and the quadratic form } L \mapsto D^{n-2} \cup L^2 \text{ has signature } (1, h^{1,1} - 1)\}.$$

We let $W_1 := \{D \in W \ : \ D^n = 1\}$ denote the level set in W. The tangent space of W_1 at a point D is then $\{L \in H^{1,1} \ : \ D^{n-1} \cup L = 0\}$, and there is a Riemannian metric defined on W_1 via

$$(L_1, L_2) \mapsto -D^{n-2} \cup L_1 \cup L_2,$$

for L_1, L_2 elements of the tangent space at D. We consider therefore W_1 as a Riemannian manifold.

If for instance X is a surface ($n = 2$), the cup-product on $H^{1,1}(X, \mathbb{R})$ is just a Lorentzian real quadratic form, and a connected component of W_1 may be identified as real hyperbolic space of curvature -1.

The cone of Kähler classes $\mathcal{K}$ is a convex open subcone of W, by the Hodge Index theorem; we set $\mathcal{K}_1 = \mathcal{K} \cap W_1$ to be the corresponding level set. Note that $\mathcal{K}_1$ is an open submanifold of W_1; it represents part of the Kähler moduli space of X, and its geometry is studied in detail in [5]. If for instance X is a complex torus of dimension n, then one can identify $\mathcal{K}_1$ as the symmetric space $SL(n, \mathbb{C})/SU(n)$, and prove that the sectional curvatures of $\mathcal{K}_1$ are bounded between $-n(n-1)/2$ and 0.

In the case when the degree n form is diagonal, $x_0^n - x_1^n - \dots x_m^n$, one checks easily that $\mathcal{K}_1$ has constant sectional curvature $-(n/2)^2$.

We observe that $\mathcal{K}$ is isometric to the *curved Kähler cone* $\tilde{\mathcal{K}}$, as defined by Huybrechts in [1]; if we fix the complex structure and the volume form $\omega_0^n/n!$, then

$$\tilde{\mathcal{K}} := \{\text{Kähler } \textit{forms } \omega \ : \ \omega^n = c\,\omega_0^n \quad \text{for some } c > 0\}.$$

Here we need the Aubin–Calabi–Yau theorem. Moreover, $\mathcal{K}_1$ is then isometric to

$$\tilde{\mathcal{K}_1} := \{\text{Kähler forms } \omega \ : \ \omega^n = \omega_0^n\}.$$

Using this identification, one can show that any failure of the sectional curvatures to be bounded between $-n(n-1)/2$ and 0 is accounted for by the failure of squares of harmonic 2-forms to be harmonic. In particular, this explains why the bounds hold when X is any hermitian symmetric space of compact type.

In the case of smooth complex projective threefolds X with $b_3 = 3$ and $h^{2,0} = 0$, we may denote the ternary cubic form by $F(X_0, X_1, X_2)$ say; the metric on W_1 is then defined by the matrix $(-\frac{1}{6}\partial^2 F/\partial X_i\,\partial X_j)$. We let $h(X_0, X_1, X_2)$ denote the determinant of this matrix (also a cubic form), and let S be the basic degree 4 classical invariant of F. The curvature R at a given point of W_1 may then be seen [5] to be given by

$$R = -9/4 + S/(4h^2).$$

If F has the form $X_0^3 + G(X_1, X_2)$, then $S = 0$ and $R = -9/4$, in agreement with our calculation of $-(n/2)^2$ for the diagonal case in dimension n.

Continuing our study of threefolds of the above type, we suppose there is a point on the boundary of the closed cone $\bar{\mathcal{K}}$ where h vanishes. Then analysis of the possible F shows that $S \geq 0$. It is known that $\bar{\mathcal{K}}$ is locally finite polyhedral away from the cone $\{D \ : \ D^3 = 0\}$, and that the corresponding codimension one faces of $\bar{\mathcal{K}}$ determine birational contraction morphisms on X which lower the

Picard number by one. If there is a codimension one face corresponding to the contraction of a *surface* on X, then the comment above ensures that $S = 0$, and so the curvature R is constant. We might however have a (necessarily rational) point on the boundary of $\bar{\mathcal{K}}$ lying on the intersection of two codimension faces and corresponding to a morphism contracting a surface on X (and hence h vanishes at the point); this contraction lowers the Picard number by two and there is then no reason in general why we should have $S = 0$. If, under the above assumption, we have $S > 0$, then $R \to +\infty$ on $\mathcal{K}_1$, and in particular is positive on parts of $\mathcal{K}_1$. This latter case occurs for instance for Weierstrass fibrations over ruled surfaces $\mathbb{F}_e$ $(e = 0, 1, 2)$ — in these examples, we can check that $R > 0$ on all of $\mathcal{K}_1$ (although there are examples where this is not true). For these three examples, we deduce that the product of two harmonic forms is harmonic only if one of the forms is a multiple of the Kähler form. In the general case under consideration, the above arguments show however that we cannot have have $R \to -\infty$ on $\mathcal{K}_1$.

The above three examples, and other similar examples obtained from certain hypersurfaces in weighted projective 4-space, suggest that the Weil–Petersson metric on the complex moduli space of the mirror could in certain examples have some sectional curvatures being positive for points near large complex structure limit. This is against conventional expectations, indicating that perhaps the sectional curvatures of Weil–Petersson are the wrong curvatures to consider. It is incidentally a firm expectation of the physicists that the scalar curvature will be negative near large complex structure limit [3]

Computer calculations suggest that, for all smooth complex projective threefolds X with $b_3 = 3$ and $h^{2,0} = 0$, we should have $S \geq 0$ and hence $R \geq -9/4$ (cf. [6]). Motivated by Mirror Symmetry and results on the Weil–Petersson metric for the complex moduli space of the mirror [4], [2], and further computer calculations, one is led to make a general conjecture for Calabi–Yau n-folds, that the Ricci curvature is bounded below by $-(n/2)^2(h^{1,1} - 2)$ on $\mathcal{K}_1$. This is true for all known examples, even those (like complex tori) where the sectional curvatures can get down to $-n(n-1)/2$. This is consistent with results on the Ricci curvature of the Weil–Petersson metric. The conjecture takes a very nice form, if instead of the Riemannian metric on W_1, we consider the corresponding Lorentzian metric on the cone W. If the degree n form determined by cup-product is denoted by $F(X_0, X_1, \ldots, X_m)$ say, then this Lorentzian metric is just given by the matrix

$$\left(-\frac{1}{n(n-1)} \partial^2 F / \partial X_i \, \partial X_j \right).$$

This Lorentzian metric restricts to the usual Riemannian metric on W_1. The above conjecture is equivalent to the Lorentzian metric having *non-negative* Ricci curvature at all points of $\mathcal{K}$. This latter form of the conjecture is very convenient for computer calculations, and has also been checked to hold for various weighted hypersurface Calabi–Yau threefolds with $h^{1,1} > 3$. Such an example is $X_{13} \subset \mathbb{P}(1, 2, 3, 3, 4)$, whose smooth model is a Calabi–Yau threefold with $h^{1,1} = 5$.

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Existence of minimal models for varieties of log general type

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(joint work with Caucher Birkar, Paolo Cascini, Christopher Hacon)

Let us start with a motivating:

Conjecture 1. *Let X be a smooth projective variety.*
Then either
(1) $H^0(X, \mathcal{O}_X(mK_X)) \neq 0$ *for some* $m > 0$*, or*
(2) X *is covered by rational curves.*

In words, either X has a pluricanonical form or there is some good geometric reason why it cannot have any such form, namely it is covered by rational curves. One natural approach to the birational classification of projective varieties is to study the linear series $|mK_X|$, for all $m > 0$. (1) is the beginning of such a classification since it tells us when these linear series are all empty.

One approach to (1) is to find a nice birational representative Y of X:

Definition 2. Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal algebraic spaces, and let $\phi\colon X \dashrightarrow Y$ be a birational map over U, whose inverse does not contract any divisors. Let D be an $\mathbb{R}$-Cartier divisor on X.

We say that ϕ is D**-negative** if $D' = \phi_* D$ is $\mathbb{R}$-Cartier and for some resolution $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$, we have
$$p^* D = q^* D' + E,$$
where $E \geq 0$ and E is the full exceptional locus.

The key point behind this definition is that
$$H^0(X, \mathcal{O}_X(\llcorner mD \lrcorner)) \simeq H^0(Y, \mathcal{O}_Y(\llcorner mD' \lrcorner)).$$
Going back to the original problem of finding a good representative Y for X, note that this is exactly the property of Y that we want, namely that we have not altered the dimensions of the linear systems $|mK_X|$.

We recall the definition of log pairs and of log terminal singularities:

Definition 3. Let (X, Δ) be a log pair, so that X is normal and $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) is **kawamata log terminal** if for some (equivalently all) log resolution $f\colon Y \longrightarrow X$, if we write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where Γ and E are effective with no common components, $f_*\Gamma = \Delta$ and E is exceptional, then $\llcorner \Gamma \lrcorner = 0$.

In other words all the coefficients of Δ lie between zero and one, and this continues to hold even after we blow up.

Definition 4. Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal algebraic spaces. Let (X, Δ) be a kawamata log terminal pair.

A log terminal model for $K_X + \Delta$ over U is a $(K_X + \Delta)$-negative rational map $\phi\colon X \dashrightarrow Y$ over U, whose inverse does not contract any divisors, where $Y \longrightarrow U$ is projective and $K_Y + \Gamma$ is nef.

Now we are ready to state the main result:

Theorem 5 (Birkar, Cascini, Hacon, M$^{\mathrm{c}}$Kernan)**.** *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal algebraic spaces.*

If $K_X + \Delta$ is π-pseudo-effective and Δ is big over U, then $K_X + \Delta$ has a log terminal model over U.

(5) has some well-known consequences:

Corollary 6. *Let X be a smooth projective variety of general type.*

Then

(1) *X has a minimal model.*
(2) *X has a canonical model.*
(3) *the canonical ring*

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is finitely generated.

Proof. Since K_X is big, we may pick k such that $|kK_X|$ is non-empty. Pick $B \in |kK_X|$ and let $\Delta = \epsilon B$. Then $K_X + \Delta$ is kawamata log terminal for ϵ sufficiently small and we may apply (5). □

Corollary 7. *Let X be a smooth projective variety.*

If K_X is not pseudo-effective then there is a birational map $\phi\colon X \dashrightarrow Y$, whose inverse does not contract any divisors, which is K_X-negative and there is a morphism $f\colon Y \longrightarrow Z$ which is a Mori fibre space.

Proof. Pick H very ample smooth such that $K_X + H$ is ample. Let

$$c = \inf\{\, t \in [0,1] \mid K_X + tH \text{ is pseudo-effective} \,\}.$$

Then $0 < c < 1$ so that $K_X + \Delta$ is kawamata log terminal, where $\Delta = cH$ and $K_X + \Delta$ is not big. By (5) we may find a log terminal model $\phi\colon X \dashrightarrow Y$. Then ϕ

is K_X-negative as it is $K_X + \Delta$-negative and Δ is ample. By the base point free theorem, there is a morphism $f\colon Y \longrightarrow Z$, such that $K_Y + \Gamma = f^*A$, where A is ample. Note that $-K_Y$ is big over Z. Running an appropriate MMP, we reduce to the case when $-K_Y$ is ample over Z and f has relative Picard number one (the existence of this MMP will be a consequence of (5)). □

Perhaps the two most studied spaces in algebraic geometry are the moduli space of smooth curves with n marked points $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, which gives a geometrically meaningful compactification of $\mathcal{M}_{g,n}$, by adding stable curves. We have two consequences of (5) which are of interest. In the first we generalise the moduli space of curves to higher dimensions:

Corollary 8. *There is a geometrically meaningful compactification of the moduli space of varieties of general type.*

In fact this compactification, at least in the case of surfaces, was proposed some time ago by Kollár and Shepherd-Barron [3], using the space of stable log pairs. Since then Alexeev [1] proved that this works for surfaces and then Karu [2], using the work of Abramovich and Karu on semistable reductions, was able to reduce (8) to the existence of log canonical models.

Corollary 9. *Fix an ample divisor A on $X = \overline{\mathcal{M}}_{g,n}$. Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be the boundary divisors. If $K_X + A + \Delta$ is log canonical and big, where $\Delta = \sum a_i\Delta_i$, then there is a log terminal model for $K_X + A + \Delta$. Moreover, as we vary Δ, there are only finitely many such models.*

In fact (9) is part of a much bigger picture, cf. Theorem (C).

However before that, I would like to give one application of (5) to Moishezon spaces. Recall that a Moishezon space M is a compact integral analytic space, whose space of meromorphic functions has transcendence degree equal to the dimension. It is not hard to see that this is equivalent to requiring that M is birational to a projective variety. Note that it was proved by Artin that the category of algebraic integral spaces of finite type over $\mathbb{C}$ and the category of Moishezon spaces are equivalent.

Let us start with some motivation. Let X be a smooth projective threefold. Pick two smooth curves C and D which intersect in a node at $x \in X$. Let Y_1 be the blow up of X along C and then along D and let Y_2 be the blow up of X along D and then along C. Then the natural map is not an isomorphism; in fact it is a flop.

Now suppose that C and D intersect in two points x and y. In the category of algebraic spaces, we can arrange to blow up the two curves C and D in two different orders at x and y. The resulting space Y is not projective. Note though that in this example, Y contains a rational curve. Using arguments of Shokurov, cf. [5] we see that this is always the case:

Corollary 10. *Let $\pi\colon X \longrightarrow U$ be a proper morphism of normal algebraic spaces. Suppose that $K_X + \Delta$ is kawamata log terminal. If X does not contain any rational curves contracted by π, then π is a log terminal model.*

In particular

- *π is projective, and*
- *$K_X + \Delta$ is nef over U.*

We now turn to a sketch of the proof. The proof is by induction on n. We will prove (in fact the induction is a little more complicated than stated, but this will give the reader an idea of how we proceed):

Theorem A. *Let X be a projective variety of dimension n.*

If $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$, $K_X + \Delta$ is kawamata log terminal and Δ is big, then $K_X + \Delta$ has a log terminal model.

Theorem B. *Let X be a projective variety of dimension n.*

If $K_X + \Delta$ is kawamata log terminal, $K_X + \Delta$ is pseudo-effective and Δ is big, then $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

Theorem C. *Let X be a projective variety of dimension n.*

Fix an ample divisor A. Fix divisors $\Delta_1, \Delta_2, \dots, \Delta_k$. Then the set

$$\{\, Y \mid (Y, \Gamma) \text{ is the log terminal model of } K_X + A + \Delta,\ \Delta = \sum a_i \Delta_i \,\},$$

is finite.

The plan of the proof is as follows

(1) Theorem B_{n-1} and Theorem C_{n-1} imply Theorem A_n.
(2) Theorem A_n implies Theorem B_n.
(3) Theorem A_n and Theorem B_n imply Theorem C_n.

We now quickly indicate how to prove each step. We leave (1) to the last, since this is the most interesting step.

To prove (2) we use the ideas behind Shokurov's non-vanishing theorem. As in the proof of the non-vanishing theorem there are two cases, a numerically trivial case and the remaining cases. The first case follows from some results of Nakayama. As in the proof of the non-vanishing theorem, in the second case, we are able to create a component S of Δ of coefficient one. By induction, when we restrict to S we have a section. The key point is to lift this section, and for this we would like the first cohomology group of an appropriate sheaf to vanish. It is at this point we use Theorem A_n to conclude that we may find a log terminal model for an appropriate big divisor and then we may simply apply Kawamata-Viehweg vanishing as in the classical case of the non-vanishing theorem. Note that a hidden, but crucial, technical difficulty is that we need to deal with real divisors.

To prove (3) we use the ideas behind Shokurov's paper, 3-fold log models [4]. Suppose that $(X, \sum \Delta_i)$ is log smooth. Then we have

$$\{\, (a_1, a_2, \dots, a_k) \mid K_X + \Delta = \sum a_i \Delta_i \text{ is log canonical} \,\} = [0,1]^k.$$

The key point is that this set is compact (it at this point that it is crucial that we work with real coefficients). Thus we can check finiteness locally. In fact, using the ample divisor A, we can even perturb Δ so that we are always in the interior, and the argument follows by a straightforward induction.

We now turn to (1). To get an idea of how the proof proceeds, let us first show how to prove the much weaker result that Theorem C_n implies Theorem A_n. We start with a kawamata log terminal pair (X, Δ). Our aim is to find a log terminal model for $K_X + \Delta$. Pick an ample divisor H such that $K_X + \Delta + H$ is kawamata log terminal and nef. Let V be the vector space spanned by the components of Δ and H. Pick the smallest value of λ such that $K_X + \Delta + \lambda H$ is nef. If $\lambda = 0$ then we are done. Otherwise X is a log terminal model for $K_X + \Delta + \lambda H$ but it is not a log terminal model for any $t < \lambda$. By finiteness of log terminal models, there is another projective variety Y and a birational map $X \dashrightarrow Y$ such that Y is a log terminal model for $K_X + \Delta + tH$, for $t \in [\lambda - \epsilon, \lambda]$, for some $\epsilon > 0$. Continuing in this way, we reduce to the case when $\lambda = 0$ after finitely many steps, and we are done. In practice, the best way to proceed is to realise the steps X, Y, and so on, as a MMP for $K_X + \Delta$, with scaling of H.

Now suppose that we only know Theorem C_{n-1} holds. The idea is to work with the augmented divisor $K_X + S + \Delta$, where the components of S have coefficient one. As before we run a MMP, with scaling of H. We start with $K_X + S + \Delta + tH$. The key point is to choose S so that for every step of the MMP, the curves being contracted lie in S. We need this for two reasons. Firstly, we need to know that the relevant flips exist (technically these are called pl flips) and by the work of Hacon and M^{c}Kernan these flips exist by induction. Secondly, we don't know that there are only finitely many models on X, we only know that there are finitely many models on S. By standard arguments (cf. Shokurov's reduction to pl flips), we know that only finitely many flips intersect S. Since by our choice of S, the flips we use always intersect S, we are reduced to the case that $K_X + S + \Delta$ is nef. Again the key point is to choose S in such a way that knowing that $K_X + S + \Delta$ is nef implies that $K_X + \Delta$ is nef.

Finally observe that the major obstruction to extending these methods to prove the existence of log terminal models when Δ is not necessarily big seems to be proving (1).

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Rigid targets of surjective holomorphic maps

Jun-Muk Hwang

Given two compact complex manifolds X, Y and a surjective holomorphic map $f : Y \to X$, an obvious way to deform f as maps from Y to X is by the automorphisms of X. More precisely, let $\{g_t, t \in \mathbf{C}, |t| < 1\}$ be a 1-parameter family of automorphisms of X with $g_0 = \mathrm{Id}_X$. Then the family of holomorphic maps $\{g_t \circ f, |t| < 1\}$ defines a deformation of f. It turns out that in many cases, this is essentially the only way to deform f.

To simplify the discussion let us assume that X is a simply connected projective algebraic manifold. We will say that X is a *rigid target* if any projective subvariety Y in the tangent bundle $T(X)$ which is dominant over X is biregular over X, in other words, Y defines a section of the tangent bundle. It is not difficult to see that this definition is equivalent to saying that for any surjective holomorphic map $f : Y \to X$ from a compact complex manifold Y, $H^0(Y, f^*T(X)) = f^*H^0(X, T(X))$. Since $H^0(Y, f^*T(X))$ is the tangent space of the hom-scheme $Hom(Y, X)$ at $[f]$, this implies that all deformations of f come from automorphisms of X. In particular, when X is a rigid target with no non-zero holomorphic vector field, then any surjective holomorphic map $f : Y \to X$ is isolated in $Hom(Y, X)$. This appears to be a strong condition on X, but it turns out that a large class of projective manifolds are rigid targets. In fact, [3] says that a simply connected projective manifold which is not uniruled is a rigid target. Thus the interesting problem is to investigate which uniruled projective manifold X is a rigid target. Furthermore the preprint [5] shows that this question can be essentially reduced to the case when X is rationally connected. Most typical examples of rationally connected manifolds are Fano manifolds. Thus a natural question is: *which Fano manifold is a rigid target?*

In [2], this question is completely answered for Fano surfaces. For higher dimensional Fano manifolds, this seems to be a very hard question and there is not even a reasonable conjectural answer. However, for Fano manifolds with $b_2 = 1$, the following conjecture seems reasonable.

Conjecture 1. *Let X be a Fano manifold with second Betti number $b_2 = 1$, which is different from the projective space. Then X is a rigid target.*

The strongest evidence for Conjecture 1 is the following result, which is Theorem 3 of [4].

Theorem 2. *Let X be a Fano manifold with $b_2 = 1$, different from the projective space. If X contains no immersed $\mathbb{P}_k$ with trivial normal bundle for any $1 < k < \dim X$, then X is a rigid target.*

The technical assumption in Theorem 1 on the non-existence of $\mathbb{P}_k$ with trivial normal bundle holds for all examples of Fano manifolds with $b_2 = 1$ known to the author. In fact, the following stronger form of Conjecture 1 is expected.

Conjecture 3. *Let X be a Fano manifold with $b_2 = 1$, different from the projective space. Then it contains no immersed $\mathbb{P}_k$ with trivial normal bundle for any $1 < k < \dim X$.*

Since there seems to be no good approach to Conjecture 2 at the moment, it is worthwhile studying Conjecture 1 and trying to replace the technical assumption in Theorem 1 by less technical conditions. One result along this line is the following which is Theorem 2 of [1].

Theorem 4. *Let X be a Fano manifold with $b_2 = b_4 = 1$, different from the projective space. Then X is a rigid target.*

By Theorem 1, Theorem 2 is equivalent to the following.

Theorem 5. *Let X be a Fano manifold which contains an immersed $\mathbb{P}_k$ with trivial normal bundle for some $1 < k < \dim X$. If X has $b_2 = b_4 = 1$, then X is a rigid target.*

The key idea of the proof of Theorem 3 is the following. From the definition of a rigid target, we are dealing with some holomorphic map $f : Z \to W$ between projective varieties and some vector bundle V on W, and we need to show $H^0(Z, f^*V) = f^*H^0(W, V)$. Now there is one obvious case when this equality holds: this is so if V is a trivial vector bundle on W. In our case, we need to show for a surjective holomorphic map $f : Y \to X$, $H^0(Y, f^*T(X)) = f^*H^0(X, T(X))$. Although $T(X)$ itself is not a trivial bundle, we know that the triviality of the normal bundle of $\mathbb{P}_k$ in X implies a partial triviality of $T(X)$ along $\mathbb{P}_k$. By deformation, we have many $\mathbb{P}_k$ with trivial normal bundle covering X and exploiting the geometry of this family of $\mathbb{P}_k$, $k \geq 2$, we can prove Theorem 3. The condition $b_4 = 1$ is used in this process to get information about the fundamental class of $\mathbb{P}_k$.

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Tropical enumerative geometry

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(joint work with Hannah Markwig)

Tropical algebraic geometry is a recent field of mathematics that tries to attack complicated algebraic or geometric problems (e.g. in enumerative geometry) by combinatorial methods. Ideally, every construction in algebraic geometry should have a combinatorial counterpart in tropical geometry. If this tropical set-up is easier to understand one can then try to transfer the tropical results back to the original algebro-geometric setting.

Tropical (affine or projective) varieties may be defined as images of algebraic varieties over the field of Puiseux series under the valuation map. They are polyhedral complexes in $\mathbb{R}^n$ (resp. in the "tropical projective space" that is a certain compactification of $\mathbb{R}^n$). For hypersurfaces given as the zero locus of a polynomial f the corresponding tropical variety can easily be computed as the non-differentiability locus of the so-called tropicalization of f. For general varieties there is an explicit algorithm developed by Sturmfels et al. to compute their tropicalizations from the defining ideal.

Many results from classical geometry — e.g. Bézout's theorem, the degree-genus formula, or the Riemann-Roch formula — can be translated into tropical geometry. Some of the most important recent applications of tropical to algebraic geometry have however been in the area of enumerative geometry, starting with Mikhalkin's Correspondence Theorem stating that the numbers of complex plane algebraic curves of given degree and genus through a set of given points is the same as the corresponding number of tropical curves (counted with suitable multiplicities).

As an example we show how Kontsevich's formula to count rational plane curves of degree d through $3d-1$ general points can be proven in the tropical setting. The tropical proof follows the classical one very closely and thus also gives an idea how the concepts of moduli spaces of stable curves, stable maps, and morphisms between them can be translated into the tropical language.

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Birationally rigid Fano fiber spaces

Aleksandr V. Pukhlikov

1. We say that a surjective morphism $\pi\colon X \to S$ of projective varieties with connected fibers is a *rationally connected fiber space*, if a general fiber $\pi^{-1}(s)$, $s \in S$, is a rationally connected variety. A *structure of a rationally connected fiber space* (or, briefly, a *rationally connected structure*) on a rationally connected variety X is an arbitrary rational dominant map $\varphi\colon X \dashrightarrow S$, the fiber of general position of which $\varphi^{-1}(s)$, $s \in S$, is irreducible and rationally connected. If the base S is a point, then the structure is said to be trivial. Alternatively, a rationally connected structure on a variety X can be defined as a birational map $\chi\colon X \dashrightarrow X^\sharp$ onto a variety $X^\sharp$ equipped with a surjective morphism $\pi\colon X^\sharp \to S$ realizing $X^\sharp$ as a rationally connected fiber space. We identify the structures of a rationally connected fiber space $\varphi_1\colon X \dashrightarrow S_1$ and $\varphi_2\colon X \dashrightarrow S_2$, if there exists a birational map $\alpha\colon S_1 \dashrightarrow S_2$ such that the following diagram commutes:

$$
\begin{array}{ccccc} & X & \stackrel{\text{id}}{\leftrightarrow} & X & \\ \varphi_1 & \downarrow & & \downarrow & \varphi_2 \\ & S_1 & \stackrel{\alpha}{\dashrightarrow} & S_2, & \end{array} \tag{1}
$$

that is, $\varphi_2 = \alpha \circ \varphi_1$. In other words, φ_1 and φ_2 have the same fibers. The set of non-trivial structures of a rationally connected fiber space on the variety X (modulo the identification above) is denoted by $RC(X)$.

On the set $RC(X)$ there is a natural relation of partial order: for $\varphi_1, \varphi_2 \in RC(X)$ we have $\varphi_1 \leq \varphi_2$, if there is a rational dominant map $\alpha\colon S_1 \dashrightarrow S_2$ such that the diagram (1) commutes. In other words, the fibers of φ_1 are contained in the fibers of φ_2. For a general point $s \in S_2$ we have $\alpha^{-1}(s) = \varphi_1(\varphi_2^{-1}(s))$, therefore $\alpha \in RC(S_1)$ is a structure of a rationally connected fiber space on S_1. It is easy to see that the correspondence $\varphi_2 \mapsto \alpha$ determines a bijection of the sets $\{\psi \in RC(X) | \psi \geq \varphi_1\}$ and $RC(S_1)$. Therefore from the geometric viewpoint of primary interest are the *minimal* elements of the ordered set $RC(X)$. Denote the set of minimal elements by $RC_{\min}(X)$.

Also, on the set $RC(X)$ there is a natural relation of *fiber-wise birational equivalence*: $\varphi_1 \sim \varphi_2$ if there exists a birational transformation $\chi \in \operatorname{Bir} X$ and a birational map $\alpha\colon S_1 \dashrightarrow S_2$ such that $\varphi_2 \circ \chi = \alpha \circ \varphi_1$: in other words, the birational self-map χ transforms the fibers of φ_1 into the fibers of φ_2. It seems that the quotient set $RC_{\min}(X)/\sim$ is a crucial birational invariant of the variety X.

2. The following example shows how big the quotient $RC_{\min}(X)/\sim$ can be. Let $V \subset \mathbb{P}^M$ be a generic Fano hypersurface of degree $M - 1 \geq 4$.

Theorem 1. *Any two distinct generic linear projections $\varphi_1, \varphi_2\colon \mathbb{P}^M \dashrightarrow \mathbb{P}^1$, restricted onto V, determine distinct elements $\varphi_i|_V\colon V \dashrightarrow \mathbb{P}^1$ of the quotient $RC_{\min}(X)/\sim$.*

Proof. The fact that $\varphi_1|_V \not\equiv \varphi_2|_V$ can be derived from [1]. Minimality of $\varphi_i|_V$ follows from birational superrigidity of the fibers [2, 3]. $\square$

Recall that the *threshold of canonical adjunction* of a divisor D on a variety X is the number

$$c(D,X)=\sup\{\varepsilon\in\mathbb{Q}_+|D+\varepsilon K_X\in A^1_+X\},$$

where A^1_+X is the pseudo-effective cone of X. If Σ is a non-empty linear system on X, then we set $c(\Sigma,X)=c(D,X)$, where $D\in\Sigma$ is an arbitrary divisor. For a movable linear system Σ on the variety X define the *virtual threshold of canonical adjunction* by the formula

$$c_{\mathrm{virt}}(\Sigma)=\inf_{X^\sharp\to X}\{c(\Sigma^\sharp,X^\sharp)\},$$

where the infimum is taken over all birational morphisms $X^\sharp\to X$, $X^\sharp$ is a smooth projective model of $\mathbb{C}(X)$, $\Sigma^\sharp$ the strict transform of the system Σ on $X^\sharp$.

Definition 2. The variety V is said to be *birationally superrigid*, if for any movable linear system Σ on V the equality $c_{\mathrm{virt}}(\Sigma)=c(\Sigma,V)$ holds. The variety V (respectively, the Fano fiber space V/S) is said to be *birationally rigid*, if for any movable linear system Σ on V there exists a birational self-map $\chi\in\mathrm{Bir}\,V$ (respectively, a fiber-wise birational self-map $\chi\in\mathrm{Bir}(V/S)$), providing the equality $c_{\mathrm{virt}}(\Sigma)=c(\chi_*\Sigma,V)$.

3. The following example belongs to I. Sobolev [4]. Let $V\subset\mathbb{P}^3\times\mathbb{P}^1$ be a generic divisor of bidegree $(3,2)$. Obviously, $\pi\colon V\to\mathbb{P}^1$ realizes V as a pencil of cubic surfaces $|F|$. Let $\varepsilon\colon V\to\mathbb{P}^3$ be the projection, which is of generic degree 2 (but not a finite morphism), and $\tau\in\mathrm{Bir}\,V$ the corresponding Galois involution.

Theorem 3. (i) *Every pencil of rational surfaces on V is of the form $\chi(|F|)$ for some $\chi\in\mathrm{Bir}\,V$.* (ii) *There are no conic bundle structures on V.* (iii) $\mathrm{Bir}\,V=\mathrm{Bir}(V/\mathbb{P}^1)*<\tau>$*, where* $\mathrm{Bir}(V/\mathbb{P}^1)\subset\mathrm{Bir}\,V$ *is the subgroup of fiber-wise birational self-maps with respect to $\pi\colon V\to\mathbb{P}^1$.*

For the description of $\mathrm{Bir}(V/\mathbb{P}^1)$, see [5, 6]. The proof of theorem 2 is based on the method of [6].

4. A smooth projective variety F is a *primitive Fano variety*, if $\mathrm{Pic}\,F=\mathbb{Z}K_F$, the anticanonical class is ample and $\dim F\geq 3$. We say that a primitive Fano variety F is *divisorially canonical*, or satisfies the condition (C) (respectively, is *divisorially log canonical*, or satisfies the condition (L)), if for any effective divisor $D\in|-nK_F|$, $n\geq 1$, the pair $(F,\frac{1}{n}D)$ has canonical (respectively, log canonical) singularities. If this pair has canonical singularities for a general divisor $D\in\Sigma\subset|-nK_F|$ of any *movable* linear system Σ, then we say that F satisfies the condition of *movable canonicity*, or the condition (M).

Theorem 4. [7] *Assume that primitive Fano varieties $F_1,\dots,F_K$, $K\geq 2$, satisfy the conditions (L) and (M). Then their direct product $V=F_1\times\dots\times F_K$ is birationally superrigid. In particular, every structure of a rationally connected fiber space on the variety V is given by a projection onto a direct factor. The groups of birational and biregular self-maps of the variety V coincide:* $\mathrm{Bir}\,V=\mathrm{Aut}\,V$.

5. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus(M-1)} \bigoplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, $M \geq 4$, be a locally free sheaf on $\mathbb{P}^1$, $X = \mathbb{P}(\mathcal{E})$ the corresponding projective bundle in the sense of Grothendieck. Obviously,

$$\operatorname{Pic} X = \mathbb{Z}L_X \oplus \mathbb{Z}R, \quad K_X = -(M+1)L_X,$$

where L_X is the class of the tautological sheaf, R is the class of a fiber of the morphism $\pi\colon X \to \mathbb{P}^1$. Let $V \in |ML_X|$ be a generic divisor. Obviously, $\pi\colon V \to \mathbb{P}^1$ is a standard Fano fiber space, $\operatorname{Pic} V = \mathbb{Z}L \oplus \mathbb{Z}F$, where $L = L_X|_V$, $F = R|_V$.

Theorem 5. *The variety V is birationally superrigid. On the variety V there are exactly two non-trivial rationally connected structures: the morphism $\pi\colon V \to \mathbb{P}^1$ and the map $\varphi\colon V \dashrightarrow \mathbb{P}^1$, given by the movable linear system $|-K_V - F|$. The structures π and φ are not fiber-wise equivalent. There exists a unique, up to an isomorphism, variety $V^+ \in |ML_X|$ and a flop $\chi\colon V \dashrightarrow V^+$ such that the following diagram commutes:*

$$\begin{array}{cccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^+ & \\ \varphi & \downarrow & & \downarrow & \pi^+ \\ & \mathbb{P}^1 & = & \mathbb{P}^1. & \end{array}$$

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The slope of Kodaira Fibrations

Fabrizio Catanese and Sönke Rollenske

Throughout this abstract $\psi : S \to B$ will be a smooth fibration of a compact complex projective surface over a complete curve of genus b with fibre F a curve of genus g.

1. History

It is well known that the Euler characteristic e is multiplicative for fibre bundles and in 1957 Chern, Hirzebruch and Serre showed that the same holds true for the signature σ if $\pi_1(B)$ acts trivially on the cohomology of the fibre. In 1967 Kodaira

[3] constructed examples where multiplicativity of the signature does not hold true, and in his honour these surfaces are nowadays called Kodaira Fibrations.

Definition 1. A *Kodaira fibration* is a fibration $\psi : S \to B$ of a surface over a curve, which is a differentiable but not a holomorphic fibre bundle.

For surfaces we have the Arakelov inequality for the relative canonical bundle

$$\omega_{S/B}^2 = K_S^2 - 8(g-1)(b-1) \geq 0$$

with equality iff ψ is isotrivial, i.e. all fibres are biholomorphic to each other. Recalling the following formulae

$$c_1(S)^2 = K_S^2, \qquad c_2(S) = e(S) = 4(g-1)(b-1), \qquad \sigma(S) = \frac{c_1^2(S) - 2c_2(S)}{3}$$

we see that the signature of a Kodaira fibration is strictly positive.

Remark 2.
(1) The fibration ψ determines a non constant map $\mu : B \to \mathcal{M}_g$ to the moduli space of (smooth) curves. Since $\mathbb{P}^1$ is the only curve of genus one and $\mathcal{M}_1$ and $\mathcal{M}_2$ are affine. the genus g of the fibre is at least 3.
(2) The genus of the base B is at least 2 since $\mathbb{P}^1$ is simply connected and elliptic curves are uniformized by $\mathbb{C}$.
(3) The existence of the Satake compatification shows that one gets plenty of Kodaira fibrations by taking general complete intersections in the moduli space of curves.

2. The slope of Kodaira fibrations

Definition 3. The slope of a compact surface S is defined by

$$\mu(S) := \frac{c_1^2(S)}{c_2(S)} = \frac{K_S^2}{e(S)}$$

If S is a Kodaira fibration then $2 < \mu(S) \leq 3$ by the above and the Bogomolov-Miyaoka-Yau inequality, but in fact Kefeng Liu [2] showed that strict inequality holds also on the right hand side.

The classical examples by Kodaira, Atiyah and Hirzebruch had slope at most $2+\frac{1}{3}$ and one can calculate that the slope of a general complete intersection curve in $\mathcal{M}_g$ is around 2.18. The starting point of our research was the following question of Claude Le Brun: *Is there an effective upper bound for the slope of Kodaira fibrations strictly smaller than 3?* Our result in this direction is the following:

Theorem 4. *There exist a Kodaira Fibration S with slope* $\mu(S) = 2 + \frac{2}{3}$.

3. Construction

3.1. Tautological construction.

Definition 5. A pair (S', D') consisting of $\psi' : S' \to B'$ a smooth fibration with fibre F' and $D' \subset S'$ a divisor such that the map $D' \to B'$ is étale and the fibration of pointed curves $(F', F' \setminus D')$ is not isotrivial, is called a *log-Kodaira Fibration.*

Proposition 6. *Let (S', D') be a log-Kodaira Fibration as in the definition. Then for all surjections $\rho : \pi_1(F' \setminus D') \to G$ there exist an étale covering $f : B \to B'$ such that ρ is induced by a surjection $\pi_1(f^*(S' \setminus D')) \to G$. In other words, every Galois cover of the fiber, ramified exactly over $F' \setminus D'$ extends, after an étale pullback $f : B \to B'$ of the base, to a Galois cover of f^*S' ramified over f^*D'.*

Thus to construct Kodaira Fibrations it suffices to construct log-Kodaira Fibrations and in order to do so we restrict our attention to a special class of surfaces:

Definition 7. S is called a *double Kodaira fibred surface* if it admits a surjective holomorphic map $\psi : S \to B_1 \times B_2$ yielding two Kodaira fibrations $\psi_i : S \to B_i$. Let $D \subset B_1 \times B_2$ be the branch divisor of ψ. If both projections $pr_{B_j}|_D : D \to B_j$ are étale we call $\psi : S \to B_1 \times B_2$ a *double étale Kodaira Fibration.*

Remark 8. There is a topological characterisation for double Kodaira Fibrations similar to the one given by Kotschick in [4] for ordinary Kodaira fibrations.

We now remark that the slope of a surface does not change if we take an étale cover, that is we can try to simplify the branching divisor D by taking étale pullbacks.

Definition 9. A double étale Kodaira Fibration $S \to B_1 \times B_2$ is said to be *simple* if there exist étale maps $\phi_1, \ldots, \phi_m$ from B_1 to B_2 such that $D = \dot{\bigcup}_{k=1,\ldots,m} \Gamma_{\phi_k}$, i.e., each component of D is the graph of one of the ϕ_k's. We say that S is *very simple* if $B_1 = B_2$ and all ϕ_k's are automorphisms.

Lemma 10. *Every double étale Kodaira Fibration admits an étale pullback which is simple.*

Since the lemma does not hold if we replace simple by very simple we pose the following

Definition 11. A double étale Kodaira fibration is called standard, if it admits an étale pullback which is very simple.

3.2. **Packings of automorphism groups.** We will now focus on the construction of interesting log Kodaira fibrations leading to the proof of Theorem 4.

Definition 12. Let B be a curve of genus $b \geq 2$. A subset $\mathcal{S} \subset \mathrm{Aut}(B)$ is said to be a good set if the corresponding union of graphs $\bigcup_{\phi \in \mathcal{S}} \Gamma_\phi$ is a smooth divisor in $B \times B$. The packing number is then defined as $\alpha := |\mathcal{S}|/(b-1)$.

Proof of the theorem: We construct a triangle curve of genus 2 as a ramified cover over $\mathbb{P}^1$ with Galois group $\mathbf{Sl}(2, \mathbb{Z}/3)$ and branching indices (3,3,4), together with 3 automorphisms which yield a good set with $\alpha = 3$. Applying the tautological construction to the resulting log-Kodaira fibration we get a (standard) Kodaira fibration with $\mu(S) = 8/3$. □

As one would guess a large α would yield a surface with high slope, for instance examples with $\alpha = 4$ would yield a slope equal to $2,75$. In fact, the result of Liu and the tautological construction imply $\alpha < 8$.

Question 13. What is the maximal possible value for the packing number α?

4. The moduli space

We can describe the moduli space for double étale and standard Kodaira fibrations explicitly:

Theorem 14. *Double étale Kodaira Fibrations yield a union of connected components (of moduli spaces of surfaces).*

Theorem 15. *In the case of standard Kodaira Fibrations with fixed discrete data we get an irreducible component isomorphic to the moduli space of pairs (B,G), where B is a curve of genus at least 2 and G is a fixed group of automorphisms of B (G is generated by the Galois groups G_i of $B \to B_i$ and by the good set $\mathcal{S}$).*

Corollary 16. (1) *The surface in Theorem 4 is rigid.*
(2) *There exists a rigid curve in $\mathcal{M}_7$, the moduli space of curves of genus 7..*

(The talk was given by the first author.)

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Viehweg's conjecture for two-dimensional bases

Sándor Kovács
(joint work with Stefan Kebekus)

Let B° be a smooth quasi-projective complex curve and $q > 1$ a positive integer. Shafarevich conjectured [8] that the set of non-isotrivial families of smooth projective curves of genus q over B° is finite. Shafarevich further conjectured that if the logarithmic Kodaira dimension, for a definition see below, satisfies $\kappa(B^\circ) \leq 0$, then no such families exist. This conjecture, which later played an important role in Faltings' proof of the Mordell conjecture, was confirmed by Parshin [7] for B° projective and by Arakelov [1] in general. We refer the reader to the survey articles [9], [5], and [6] for a historical overview and references to related results.

It is a natural and important question whether similar statements hold for families of higher dimensional varieties over higher dimensional bases. Families over a curve have been studied by several authors in recent years and they are now fairly well understood—the strongest results known were obtained in [10, 11], and [4]. For higher dimensional bases, however, a complete picture is still missing and subvarieties of the corresponding moduli stacks are not well understood. As a first step toward a better understanding, Viehweg proposed the following:

Conjecture 1 ([9, 6.3])**.** *Let $f^\circ : X^\circ \to S^\circ$ be a smooth family of canonically polarized varieties. If f° is of maximal variation, then S° is of log general type.*

We briefly recall the relevant definitions, as they will also be important in the statement of our main result. The first is the variation, which measures the birational non-isotriviality of a family.

Definition 2. Let $f : X \to S$ be a projective family over an irreducible base S defined over an algebraically closed field k and let $\overline{k(S)}$ denote the algebraic closure of the function field of S. The variation of f, denoted by $\mathrm{Var} f$, is defined as the smallest integer ν for which there exists a subfield K of $\overline{k(S)}$, finitely generated of transcendence degree ν over k and a K-variety F such that $X \times_S \mathrm{Spec}\overline{k(S)}$ is birationally equivalent to $F \times_{\mathrm{Spec} K} \mathrm{Spec}\overline{k(S)}$.

Remark 3. In the setup of Definition 2, if the fibers are canonically polarized complex varieties, moduli schemes are known to exist, and the variation is the same as either the dimension of the image of S in moduli, or the rank of the Kodaira-Spencer map at the general point of S.

Definition 4. Let S° be a smooth quasi-projective variety and S a smooth projective compactification of S° such that $D := S \setminus S^\circ$ is a divisor with simple normal crossings. The logarithmic Kodaira dimension of S°, denoted by $\kappa(S^\circ)$, is defined to be the Kodaira-Iitaka dimension, $\kappa(S, D)$, of the line bundle $\mathcal{O}_S(K_S + D) \in \mathrm{Pic}(S)$. The variety S° is called of *log general type* if $\kappa(S^\circ) = \dim S^\circ$, i.e., the divisor $K_S + D$ is big.

Remark 5. It is a standard fact in logarithmic geometry that a compactification S with the described properties exists, and that the logarithmic Kodaira dimension $\kappa(S^\circ)$ does not depend on the choice of the compactification S.

Statement of the main result

Our main result describes families of canonically polarized varieties over quasi-projective surfaces. We relate the variation of the family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg's Conjecture 1 for families over surfaces.

Theorem 6 ([2, 1.4])**.** *Let S° be a smooth quasi-projective complex surface and $f^\circ : X^\circ \to S^\circ$ a smooth non-isotrivial family of canonically polarized complex varieties. Then the following holds.*

(1) *If $\kappa(S^\circ) = -\infty$, then* $\mathrm{Var}(f^\circ) \leq 1$.
(2) *If $\kappa(S^\circ) \geq 0$, then* $\mathrm{Var}(f^\circ) \leq \kappa(S^\circ)$.

In particular, Viehweg's Conjecture holds for families over surfaces,

For the special case of $\kappa(S^\circ) = 0$, this statement was proved by Kovács [3, 0.1] when S° is an abelian variety and more generally by Viehweg and Zuo [11, 5.2] when $T_S(-\log D)$ is weakly positive. A slightly weaker statement holds for families of minimal varieties [2, §8].

Remark 7. Notice that in the case of $\kappa(S^\circ) = -\infty$ one cannot expect a stronger statement. For an easy example take any non-isotrivial smooth family of canonically polarized varieties over a curve $g : Z \to C$, set $X := Z \times \mathbb{P}^1$, $S^\circ := C \times \mathbb{P}^1$, and let $f^\circ := g \times \mathrm{id}_{\mathbb{P}^1}$ be the obvious morphism. Then we clearly have $\kappa(S^\circ) = -\infty$ and $\mathrm{Var}(f) = 1$.

In view of Theorem 6, we propose the following generalization of Viehweg's conjecture.

Conjecture 8 ([2, 1.6])**.** *Let $f^\circ : X^\circ \to S^\circ$ be a smooth family of canonically polarized varieties. Then either $\kappa(S^\circ) = -\infty$ and $\mathrm{Var}(f^\circ) < \dim S^\circ$, or $\mathrm{Var}(f^\circ) \leq \kappa(S^\circ)$.*

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Recent developments on hyperbolicity of complex algebraic varieties

Erwan Rousseau

My works deal with the hyperbolicity of complex varieties of dimension 3 and the study of Demailly-Semple jets. In 1970, S. Kobayashi conjectured in [4] that a generic hypersurface X in $\mathbb{P}^n_{\mathbb{C}}$ is hyperbolic provided that $d = \deg(X) \geq 2n-1$, for $n \geq 3$. For $n = 3$, it was obtained by Demailly and El Goul in [2] that $d \geq 21$ implies the hyperbolicity of very generic hypersurfaces X in $\mathbb{P}^3_{\mathbb{C}}$.

The main result I have obtained is the following:

Theorem 1. *Let $X \subset \mathbb{P}^4_{\mathbb{C}}$ be a generic hypersurface such that $d = deg(X) \geq 593$. Then every entire curve $f : \mathbb{C} \to X$ is algebraically degenerated, i.e there exists a proper subvariety $Y \subset X$ such that $f(\mathbb{C}) \subset Y$.*

This result is a weaker version of the conjecture in dimension 3, because to obtain the full conjecture one needs to prove that the entire curve is constant.

Main notions : Let X be a complex manifold of dimension n and $f : (\mathbb{C}, 0) \to X$ a germ of holomorphic curve. Following [1], we introduce the vector bundle $E^{GG}_{k,m}T^*_X \to X$ whose fibers are complex valued polynomials $Q(f', f'', ..., f^{(k)})$ on the fibers of J_kX, the bundle of k-jets of germs of parametrized curves in X, of weighted degree m with respect to the $\mathbb{C}^*$ action:

$$Q(\lambda f', \lambda^2 f'', ..., \lambda^k f^{(k)}) = \lambda^m Q(f', f'', ..., f^{(k)}).$$

We define the subbundle $E_{k,m}T^*_X \subset E^{GG}_{k,m}T^*_X$, called the bundle of invariant jet differentials of order k and degree m, i.e :

$$Q((f \circ \phi)', (f \circ \phi)'', ..., (f \circ \phi)^{(k)}) = \phi'(0)^m Q(f', f'', ..., f^{(k)})$$

for every $\phi \in G_k$ the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$.

The interest of these bundles is underlined by the following theorem formulated by Green and Griffiths [3] and proved by Demailly, Siu :

Theorem 2. [3, 1] *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that*

$$H^0(X, E_{k,m}T^*_X \otimes L^{-1})$$

has a non zero section P. Then every entire curve $f : \mathbb{C} \to X$ must satisfy the algebraic differential equation $P(f) = 0$.

We define $A_k = \bigoplus_m (E_{k,m}T^*_X)_x$ the algebra of the germs of differential operators at a point $x \in X$. This algebra can be seen as a representation of the general linear group Gl_n. Demailly [1] has given a characterization of 2-jets of degree m :

$$Gr^\bullet E_{2,m}T^*_X = \bigoplus_{\lambda_1 + 2\lambda_2 = m} \Gamma^{(\lambda_1, \lambda_2, 0)} T^*_X,$$

where Γ is the Schur functor.

The algebraic study has led me to a characterization of 3-jets of degree m in dimension 3:

Theorem 3. [5] *If X is a complex variety of dimension 3, then:*

$$Gr^\bullet E_{3,m}T_X^* = \bigoplus_{0\leq\gamma\leq\frac{m}{5}} \Big(\bigoplus_{\{\lambda_1+2\lambda_2+3\lambda_3=m-\gamma;\lambda_i-\lambda_j\geq\gamma, i<j\}} \Gamma^{(\lambda_1,\lambda_2,\lambda_3)}T_X^* \Big)$$

where Γ is the Schur functor.

A Riemann-Roch computation has given:

Corollary 4. [5] *Let $X \subset \mathbb{P}^4$ be an hypersurface of degree d. For $d \geq 43$, $\chi(X, E_{3,m}T_X^*) \sim \alpha(d)m^9$ with $\alpha(d) > 0$.*

To obtain some results on the weak hyperbolicity of generic hypersurfaces in $\mathbb{P}^4$ of sufficently high degree, the strategy is based on Demailly's techniques [1] associated to some generalization of results of Clemens, Ein, Voisin [9] and Siu [8].

The first step to obtain global differential operators, vanishing on an ample divisor, reduces to a control of the dimension of the cohomology group $H^2(X, E_{3,m}T_X^*)$ where X is a smooth irreducible hypersurface in $\mathbb{P}^4$. Here, one may use the cohomology on flag varieties. Indeed, let $Fl(T_X^*)$ be the flag variety of T_X^* i.e the variety of sequences of vector spaces

$$D = \{0 = E_3 \subset E_2 \subset E_1 \subset E_0 = T_{X,x}^*\}.$$

Let $\pi : Fl(T_X^*) \to X$. This is a locally trivial fibration of relative dimension: $N = 1 + 2 = 3$.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition such that $\lambda_1 > \lambda_2 > \lambda_3$. Let $\mathcal{L}^\lambda$ be the line bundle on $Fl(T_X^*)$ whose fiber above the preceding flag is

$$\mathcal{L}_D^\lambda = \bigotimes_{i=1}^{3} \det(E_{i-1}/E_i)^{\otimes\lambda_i}.$$

According to Bott's theorem, if $m \geq 0$:

$$\begin{aligned} \pi_*(\mathcal{L}^\lambda)^{\otimes m} &= \Gamma^{m\lambda}T_X^*, \\ \mathcal{R}^q\pi_*(\mathcal{L}^\lambda)^{\otimes m} &= 0 \text{ if } q > 0. \end{aligned}$$

Therefore the bundles $\Gamma^{m\lambda}T_X^*$ and $(\mathcal{L}^\lambda)^{\otimes m}$ have the same cohomology.

Moreover, we have a tool to control the dimension of cohomology groups: Demailly's holomorphic Morse inequalities. Let $L = F - G$ be a line bundle on a compact kähler manifold X, with F and G nef line bundles. Then for $0 \leq q \leq n = \dim X$

$$h^q(X, kL) \leq \frac{k^n}{(n-q)!q!}F^{n-q}.G^q + o(k^n).$$

Therefore we can obtain a good control of the cohomology and thanks to the positivity of the Euler characteristic, we have the existence of global differential operators. This is done in [6].

The second step consists in proving the existence of enough independant differential operators.

Let $S = \mathbb{P}_N$ be the moduli space of all the hypersurfaces of degree d in $\mathbb{P}_n$, and $\chi \to S$ the universal family of hypersurfaces of bidegree $(d, 1)$ in $\mathbb{P}_n \times \mathbb{P}_N$ defined by $f = \sum_{\nu \in \mathbb{N}^{n+1}, |\nu|=d} \alpha_v z^\nu$ and smooth. Clemens has proved that $T_\chi \otimes \mathcal{O}_{\mathbb{P}_n}(1)$ is globally generated. Siu has introduced the space of vertical jets $J_{n-1}^{vert}(\chi)$ defined by $f = df = ... = d^{n-1}f = 0$ in $J_{n-1}(\chi)$ with the coefficients α_v considered as constants in the computation of $d^j f$. First, we generalize Clemens result and prove that there exists constants c_n and c'_n such that

$$T_{J_{n-1}^{vert}(\chi)} \otimes \mathcal{O}_{\mathbb{P}_n}(c_n) \otimes \mathcal{O}_{\mathbb{P}_N}(c'_n)$$

is globally generated, for $n = 4$.

Therefore, it becomes possible to produce enough independant global differential operators in the following way: the first step produces a global differential operator vanishing on an ample divisor, for a generic hypersurface X_{s_0}, $s_0 \in S$. This operator can be seen as a holomorphic function ω on $J_3(X_{s_0})$. We can extend it holomorphically to a neighbourhood U_{s_0} of s_0. Then we use meromorphic vector field $v_1, ..., v_p$ on $J_3^{vert}(\chi)$: we differentiate ω and obtain $v_1...v_p\omega$ which gives in restriction to X_{s_0} a differential operator. It is important to verify that this operator still vanishes on an ample divisor by a control of the poles order of the v_i and that the collection of operators obtained this way has no common zeros except the linear degeneracy locus. Thus, it can be proved that any entire curve $f : \mathbb{C} \to X_{s_0}$ must be contained in a proper subvariety $X'_{s_0} \subset X_{s_0}$. This is done in [7].

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On the Picard number of almost Fano threefolds

Priska Jahnke
(joint work with C. Casagrande, I. Radloff)

Let Y be a projective threefold with at most canonical Gorenstein singularities. If $-K_Y$ is big and nef, we call Y *almost Fano*. By the Base Point Free Theorem, $|-mK_Y|$ is base point free for $m \gg 0$ defining a birational morphism

$$f\colon Y \longrightarrow X,$$

called *anticanonical map*. Here X is a Fano threefold, again with at most canonical Gorenstein singularities.

Conversely, given any Fano threefold X with at most canonical Gorenstein singularities, there exists by Kawamata and Reid a *partial crepant resolution* $f\colon Y \longrightarrow X$ as above, such that Y has at most terminal and $\mathbb{Q}$–factorial singularities and $K_Y = f^*K_X$. Hence Y is almost Fano.

Smooth Fano threefolds are completely classified by Iskovskikh, Mori and Mukai; there are 106 families. Also in the singular case a classification is possible: Borisov proved boundedness for Gorenstein Fano threefolds with at most canonical singularities ([3]). But it may be more reasonable to classify the crepant resolution Y instead of X: following the minimal model program, a "good model" for X is supposed to have only terminal and $\mathbb{Q}$–factorial singularities, as is the case for Y. Moreover, members of the same deformation family of Fano threefolds X may admit a completely different fiber space structure, as the following example shows ([7], A.2.2 and A.4.8):

(1) Let $X \subset \mathbb{P}_4$ be a quartic with canonical singularities along a smooth curve of degree 8, which is the intersection of 3 quadrics. Then a partial crepant resolution Y of X is smooth with $\rho(Y) = 2$ and admits a del Pezzo fibration over $\mathbb{P}_1$.

(2) Let $X \subset \mathbb{P}_4$ be a quartic with canonical singularities along the rational normal curve. Then a partial crepant resolution Y is smooth with $\rho(Y) = 2$ as above, but now Y is the blowup along a curve of a smooth Fano threefold V of index 2, degree 4 and with $\rho(V) = 1$.

Besides a complete classification we ask for effective bounds of numerical data of (almost) Fano threefolds. Prokhorov proved in [9]

$$(-K_X)^3 = (-K_Y)^3 \leq 72$$

is effective, with equality for two weighted projective spaces only. In [5] we show $\rho(Y) \leq 10$ for any almost Fano threefold Y with $\iota_Y > 1$, and $\rho(Y) \leq 3$ in case $Y = X$ is Fano. Here ι_Y denotes the *pseudo-index* of Y, that is the minimal positive intersection number of $-K_Y$ with a rational curve. We moreover characterize the cases, where equality is achieved.

As a corollary we obtain the generalized Mukai conjecture in the singular case:

$$\rho(X)(\iota_X - 1) \leq \dim(X) \quad \text{with eq. iff} \quad X \simeq \mathbb{P}_{\iota_X - 1}^{\rho(X)}.$$

The conjecture says this equality should hold for smooth Fano varieties in any dimension and is known to be true up to dimension 5 ([1], [2]) and in the toric case ([4]). Originally, Mukai formulated his conjecture using the Fano index r_X instead of the pseudo-index.

Main tool for the proof of our bound in [5] is Cutkosky's (Mori's) classification of possible elementary extremal contractions for Gorenstein threefolds with at most terminal and $\mathbb{Q}$–factorial singularities ([6], [8]). Since K_Y is not nef, by the cone theorem there always exists a contraction

$$\phi\colon Y \longrightarrow Z,$$

which we now study in detail. The assumption $\iota_Y > 1$ guarantees that we do not leave our "category" in case ϕ is birational. Dropping the assumption on the pseudo-index, we would have to deal with non–Gorenstein almost Fano threefolds and with threefolds, where the anticanonical divisor is not nef anymore on some curves.

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Using fiber products to compute exceptional sets

Andrew J. Sommese
(joint work with Charles W. Wampler)

It is common in engineering problems to have polynomial systems

$$f(x;q) := \begin{bmatrix} f_1(x_1,\ldots,x_N;q_1,\ldots,q_M) \\ \vdots \\ f_n(x_1,\ldots,x_N;q_1,\ldots,q_M) \end{bmatrix} = 0,$$

with the values $(q_1,\ldots,q_M) \in \mathbb{C}^M$ regarded as parameters. For some problems, e.g., finding overconstrained mechanisms [1, 4], it is important to compute certain exceptional sets of the mapping of the solution set of the system $f(x;q) = 0$ to the parameter space $\mathbb{C}^M$.

To be precise, let $\pi : X \to Y$ be an algebraic map of irreducible and reduced quasiprojective algebraic sets. Assume that π is dominant, i.e., $\pi(X)$ is dense in Y and assume that Y is locally irreducible. Let $D_m(\pi)$ denote the closure of the set of points $x \in X$ such that $\dim_x \pi^{-1}(\pi(x)) = m$.

Problem. Compute the irreducible decomposition of $D_m(\pi)$ for all m.

One result of [3] is that each irreducible component Z of $D_m(\pi)$ corresponds to special irreducible components of the j-th fiber product of X with itself over Y for all $j \geq \text{cod } Z + 1$.

To understand this correspondence, let Z denote an irreducible subset of X and let

$$\Pi_Y^k X := \underbrace{X \times_Y \cdots \times_Y X}_{k \text{ factors}}$$

denote the k-th fiber product of X with itself over Y. We let $\pi_Y : \Pi_Y^k X \to Y$ denote the induced projection and we let $\pi_j : \Pi_Y^k X \to X$ denote the projection of $\Pi_Y^k X$ onto its j-th factor for any $j = 1,\ldots,k$.

There is a well-defined irreducible set $Z_k(\pi) \subset \Pi_Y^k X$ corresponding to Z, i.e., the image in $\Pi_Y^k X$ of the component of $\Pi_Y^k Z$ containing the set of points $(\underbrace{z,\ldots,z}_{k-\text{tuple}})$ for $z \in Z$. Note that $\dim Z_k(\pi) = k(\dim Z - \dim \pi(Z)) + \dim \pi(Z)$ and that $\pi_j(Z_k(\pi)) = Z$ for any $j = 1,\ldots,k$. We call $Z_k(\pi)$, the main component of Z in $\Pi_Y^k X$.

The main concern of the article [3] is to carry out the computation of the sets $D_m(\pi)$ in the numerical framework of [2]. In this framework, an easy algorithm is given to check to see if an irreducible component A of a fiber product $\Pi_Y^k X$ of X with itself over Y is a main component of an irreducible component of $D_m(\pi)$, and if it is, specify the component of $D_m(\pi)$. The algorithm is to check that A is invariant under the natural action of the symmetric group S_k; that $\dim A = k(\dim \pi_j(A) - \dim \pi_Y(A)) + \dim \pi_Y(A)$ for some j between 1 and k; and that given a general point $(z_1,\ldots,z_k) \in A$, the point $(z_1,\ldots,z_1)$ is in A. In this case, the $\pi_j(A)$ is an irreducible component of $D_{\dim \pi_j(A) - \dim \pi_Y(A)}(\pi)$.

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Levi flat hypersurfaces in complex manifolds

Takeo Ohsawa

In a complex manifold, real analytic hypersurfaces which have locally pluriharmonic defining functions are said to be Levi flat. They are naturally related to global existence questions in several complex variables and seem to deserve studying.

Grauert's counterexample to the Levi problem on complex manifolds shows that a Levi flat hypersurface may bound a domain without any nonconstant holomorphic functions (cf. [4]). It suggests that the methods for the domains in $\mathbb{C}^n$ might be useless to explore the world of Levi flat hypersurfaces.

Nevertheless, in virtue of the theory of harmonic integrals and L^2 estimates, a theorem of Hartogs for extending holomorphic functions inside the "holes" can be generalized to complex manifolds effectively enough (cf. [5, 8, 13, 2]) to prove the following.

Theorem 1. *(cf. [11]) Let M be a compact Kähler manifold of dimension ≥ 3 and let X be a real analytic compact Levi flat hypersurface in M. Then $M \setminus X$ does not admit any C^2 plurisubharmonic exhaustion function whose Levi form has at least 3 positive eigenvalues outside a compact subset of $M \setminus X$.*

Since every locally pseudoconvex proper domain in $\mathbb{P}^n$ is Stein (cf. [3]) we have

Corollary 2. *(cf. [6]) There exist no compact real analytic Levi flat hypersurfaces in $\mathbb{P}^n$ if $n \geq 3$.*

Notes and remarks. To put our result into a right context, it may be worthwhile to note that there exist compact Kähler surfaces and compact non-Kähler manifolds of dimension ≥ 3 which contain Levi flat hypersurfaces whose complements are Stein (cf. [9, 7, 10, 12]). $C^{k,\alpha}$-smooth real hypersurfaces are called Levi flat if they are foliated by complex submanifolds of maximal dimension. It is easy to see that the two definitions coincide for real analytic hypersurfaces. The real-analyticity condition in the corollary was relaxed to C^∞-smoothness by Siu [14] and recently to $C^{0,1}$-smoothness by Cao and Shaw [1]. However it remains open whether or not there exists a domain in $\mathbb{P}^2$ with real analytic Levi flat boundary, since M. Brunella recently pointed out that there remains some works to be done in the proof of Siu [15].

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Siu's Invariance of Plurigenera: a One-Tower proof

Mihai Păun

The main purpose of our talk was to present the following more general version of the invariance of plurigenera theorem originally proved by Y.–T. Siu in [2].

Theorem 1. [1] *Let $\pi : X \to \Delta$ be a projective family over the unit disk, and let (L, h) be a hermitian (eventually singular) line bundle, which is semipositively curved. Then any section of $(mK_{X_0} + L) \otimes I(h_{L|X_0})$ extends over X.*

Thus, we are able to replace the L^∞ hypothesis in the theorem of Siu by a L^2 condition.

In the argument presented in [2], there are two main technical tools, namely the global generation statement, and the Ohsawa-Takegoshi extension theorem. Here

we will present a proof which only makes use of the second technique. Although our approach is very similar to the "classical" one, we point out now very briefly the main differences.
Let $\sigma \in H^0(X_0, mK_{X_0})$ be a pluricanonical section defined on the central fiber (we assume that L is trivial, to simplify the discussion). There exists $A \to X$ an ample line bundle on X, such that $pK_X + A$ is generated by its global sections, say $(s_j^{(p)})$, for $0 \leq p \leq m-1$, $1 \leq j \leq N_p$, and such that any section of the bundle $mK_{X_0} + A$ extends over a neighbourhood of the origin. Consider now the family of sections $(\sigma \otimes s_j^{(0)})_{j=1,..,N_0}$ over the central fiber (so we are at the bottom of the tower). By the properties of A, all these sections extend to X, and denote by $h^{(0)}$ the metric induced on $mK_X + A$. Observe that the singularities of $h^{(0)}$ restricted to X_0 are precisely the zeroes of σ. We take now the adjoint bundle $K_X + mK_X + A$; over X_0 each of the sections $\sigma \otimes s_j^{(1)}$, $j = 1, ..., N_1$ are integrable with respect to $h^{(0)}$ and the Ohsawa-Takegoshi theorem (more precisely, the version given by Siu in [2]) will imply that we can extend this family of sections of the adjoint bundle over X. Then we iterate this procedure, and we obtain metrics for the $(km + p)K_X + A$, such that their restriction to X_0 is the metric given by the family of sections $(\sigma^{\otimes k} \otimes s_j^{(p)})_{j=1,...,N_p}$. An important point is that we can perform each step in a very effective manner; so by extracting roots and passing to the limit, we produce a metric for mK_X, with all the properties required by the extension theorem to end the proof.

The original argument of Siu goes as follows: he starts (on the top of the first tower) with the section $\sigma^{\otimes k} \otimes s_A$ (where $k \gg 0$, and s_A is a non-zero section of A), which he contracts with a local section of the anti-canonical bundle. The result is a local section of the bundle $(km-1)K_{X_0} + A$, but the (effective) global generation property will show that it is a combination of elements $\sigma_j^{(km-1)} \in H^0(X_0, (km-1)K_{X_0} + A)$, with holomorphic functions as coefficients, which are divisible by large powers of σ (remark that in our case, the corresponding sections are just $\sigma^{\otimes(k-1)} \otimes s_j^{(m-1)}$). At this point, the L^∞ hypothesis is needed, in order to obtain effective bounds for the coefficients. Next he applies this procedure inductively, and at the bottom of the tower the Ohsawa-Takegoshi extension theorem is used. Finally, a "second tower" is needed, to extend step by step the sections $\sigma_j^{(p)}$ on X. Again, at the end the metric on mK_X is produced by extracting roots.

As a by-product of our approach, we have the next extension result for the closed positive currents (which is part of a joint project with J.–P. Demailly).

Theorem 2. *Let $\pi : X \to \Delta$ be a projective family over the unit disk, and let*

$$T_0 := \Theta_\omega(K_{X_0}) + \sqrt{-1}\partial\overline{\partial}\varphi_0$$

be a closed positive current over the central fiber. Then there exist

$$\widetilde{T}_0 := \Theta_\omega(K_{X_0}) + \sqrt{-1}\partial\overline{\partial}\varphi \geq 0$$

over $X_{1-\eta}$, such that $\varphi_0 \leq \varphi$ on X_0.

Corollary 3. *Let $\pi : X \to \Delta$ be a projective family over the unit disk. Assume that K_{X_0} is pseudo-effective; then so is $K_{X_{1-\eta}}$.*

The complete proof of this results and related topics will appear elsewhere.

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Arctic Computation of Monodromy

Duco van Straten
(joint work with K. Jung)

We consider a Lefschetz fibration $f : \mathcal{X} \longrightarrow \mathbb{C}$ with critical set $\Sigma \subset \mathcal{X}$ and set of critical values $D := f(\Sigma) := \{p_1, p_2, \ldots, p_N\} \subset \mathbb{C}$. The fibre over each p_i has a single A_1-singularity. By choosing a base point $b \in \mathbb{C}$ and paths connecting b the critical values in the usual way we get a natural basis (up to sign) in the relative group $H_n(\mathcal{X}, X_b)$ consisting of *Lefschetz thimbles*, the traces of the cycles $\delta_i \in H_n(X)$ that vanish at p_i, see [6]. The Picard-Lefschetz formula implies that the monodromy representation $\rho : \pi_1(\mathbb{C} \setminus D, b) \longrightarrow \mathrm{Aut}(H_n(\mathcal{X}, X_b))$ is determined by the intersection numbers $(\delta_i \cdot \delta_j)$, which are conveniently summarized in the Stokes-Seifert matrix S, see for example [1].

We are interested in the case where $\mathcal{X} = (\mathbb{C}^*)^n$ and $f \in \mathbb{C}[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]$ a Laurent-polynomial with Newton-diagram Δ. It follows from the Bernstein-Koushnirenko formula that the number of critical points N is equal to $n!\,\mathrm{Vol}(\Delta)$.

The paper [7] provides a beautiful way of seeing the vanishing cycles and their intersection for the case $n = 2$. In this case $X := X_b$ is a Riemann surface of genus $g :=$ number of internal lattice points of Δ with $r :=$ number of lattice points on the boundary of Δ punctures. The exact sequence

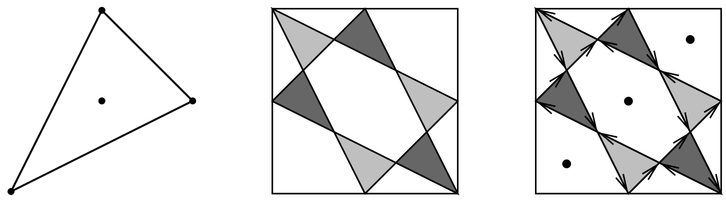
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_2(\mathcal{X}) & \longrightarrow & H_2(\mathcal{X}, X) & \longrightarrow & H_2(X) & \longrightarrow & H_1(\mathcal{X}) & \longrightarrow & 0 \\ & & =\downarrow & & =\downarrow & & =\downarrow & & =\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^N & \longrightarrow & \mathbb{Z}^{2g+r-1} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \end{array}$$

leads to the relation $N = 2g + r$, which is equivalent to Pick's formula.

The *amoeba* of a set $X \subset (\mathbb{C}^*)^2$ is defined as the image under the map $(\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2, (x, y) \mapsto (\log|x|, \log|y|)$ and is well studied, see e.g. [4], [8], [9]. In the *Harnack case* the amoeba has g holes and r *tentacles* that approach its spine, which is a tropical curve that lies in the tropical plane $\mathbb{R}^2$. The map from X to its amoeba is then two-to-one.

The *alga* of a set $X \subset (\mathbb{C}^*)^2$ is the image under the map $(\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2, (x, y) \mapsto (\arg(x), \arg(y)) \in \mathbb{T}$. Here $\mathbb{T} = (\mathbb{R}/2\pi\mathbb{Z})^2$ is the real 2-torus of arguments, which we call the *arctic or argtic plane*, which complements to the usual tropical plane.

As an example, let $f = x + y + 1/xy$. The alga of $X = f^{-1}(0)$ is shown below



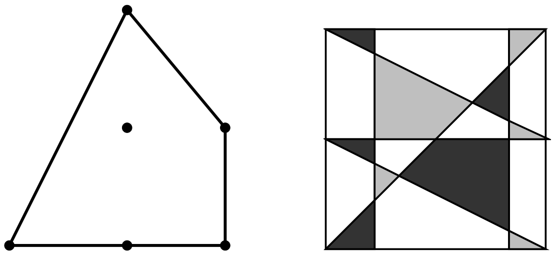
Newton diagram of $f = x + y + 1/xy$ and alga of $X = f^{-1}(0)$.

The bounding straight lines are circles on $\mathbb{T}$ and are the images of the asymptotic boundary circles that go 'around' the tentacles, and are orthogonal to the boundary edges of Δ. In the example, the map to its alga is generically one-to-one, and from the alga one can reconstruct a model of the Riemann surface X, very much like the construction of a Seifert surface of a link. The map f has $(1,1)$, (ω,ω),(ω^2,ω^2) ($\omega = \exp(2\pi i/3)$) as critical points, which are mapped by the alga map into the three 'free' regions complementary to the alga. Moreover, if we move the base point along a straight line towards one of the critical values, one can observe that the alga changes its shape and closes up the corresponding hole. From this we see that the cycles that vanish along these paths are exactly the cycles on the abstract Riemann surface that surround these free regions. From this one sees that in the example the vanishing cycles intersect pairwise in three points.

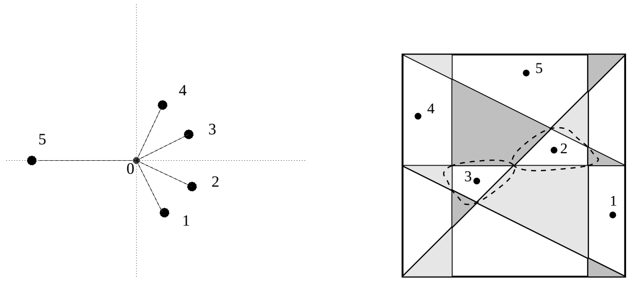
The beautiful observation of [7] is that this state of affairs holds in much greater generality, although the precise limits of applicability are not clear yet.

One starts by drawing arctic lines, orthogonal to the sides of Δ (as many as the integral length of that side). These lines come with an orientation in the direction of the outward normals to Δ. One avoids triple intersections of these lines using appropriate parallel shifts. We bi-color, according to orientation, the regions that are enclosed by oriented polygons. (The dual graph is a bipartite graph on the torus and leads to what is called a *brane tiling* in the physics literature.)

In this way one can obtain, almost without calculation, the monodromy for many Newton-diagrams, including the 16 reflexive polytopes in dimension two, [5]. We give one example.



The alga of $-x + y + 1/xy + 1/y + x/y = 0$ is close to the above picture, and the five critical points, ordered according to increasing argument, land under the alga projection in the five free regions.



The five critical values and $(\delta_2 \cdot \delta_3) = \pm 1$

From this one reads off all intersection numbers and obtains the following Stokes-Seifert matrix

$$S = \begin{pmatrix} 1 & -1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which leads to the monodromy $h := S^{-t} \cdot S$ around infinity that has all eigenvalues equal to 1, and a single Jordan-block of size two, as it should be.

Mirror symmetry relates these families of Laurent polynomials to certain 3-dimensional toric singularities Y, which geometrically are cones over toric surfaces in their anti-canonical embedding. The geometry of the vanishing cycles is then related to exceptional collections of line bundles on these surfaces, [3]. The first example $f = x + y + 1/xy$ corresponds to $Y = \mathbb{C}^3/(\mathbb{Z}/3)$, which is the cone over $\mathbb{P}^2$ in its anti-canonical embedding. For general $\mathbb{C}^3/A$ with A a finite abelian subgroup of $SL_3(\mathbb{C})$, corresponds to the case that Δ a triangle, see [10].

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Residues Hyperbolicity and Abundance

Michael M^{c}Quillan

The talk attempted to explain the relation between the relative hyperbolicity of foliations by curves, and the dynamics of the latter around their singularities. Indeed, supposing that the foliation were not hyperbolic outside of some algebraic, or even countable algebraic, subset one finds a transverse invariant measure $d\mu$ satisfying

$$K_f.d\mu \leq 0$$

where, necessarily, one further supposes that the singularities of $X \to [X/F]$ are canonical. Profiting from the minimal model theorem for foliations we may without loss of generality suppose K_f nef, so that if the cone generated by K_X and K_f contains a big divisor, the foliation is hyperbolic as soon as the residue class

$$\mathrm{Res}(d\mu) \in \mathrm{Ext}^n(K_f \otimes \mathcal{O}_{\mathrm{sing}(f)}, K_X)$$

is zero. A strategy to relate this to the winding number of $d\mu$ around the singularities and its consequences was explained.

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