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Spatial Random Processes and Statistical Mechanics

Organised by
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Abstract. The workshop focused on the broad area of spatial random processes and their connection to statistical mechanics. The subjects of interest included random walk in random environment, interacting random walks, polymer models, random fields and spin systems, dynamical problems, metastability as well as problems involving two-dimensional conformal geometry. The workshop brought together many leading researchers in these fields who reported to each other on their recent achievements and exchanged ideas for new problems and potential solutions.

Mathematics Subject Classification (2000): 82xx, 60xx.

Introduction by the Organisers

The workshop *Spatial Random Processes and Statistical Mechanics* was organized by Kenneth Alexander (USC), Marek Biskup (UCLA), Remco van der Hofstad (Eindhoven) and Vladas Sidoravicius (IMPA Rio de Janeiro). Some fifty participants from four continents attended which included senior researchers as well as mid-carreer and junior scientists, and a few graduate students.

The workshop brought together leading experts in the following fields:

- Interacting random walks and polymers
- Scaling limits in high D and expansion techniques
- Statistical mechanics of interfaces
- Percolation and other systems in 2D
- Random walk in random environment

which represent topics in the wider area of statistical mechanics and interacting random processes where currently substantial progress is being made.

In total, there were 22 talks split between the five days so that each day was focused on one specific topic from the list above. Each day had a long lunch break between 12:30 and 16:30, which made it possible for researchers to discuss the latest developments, exchange new ideas or just enjoy the working conditions of the Institute.

On Wednesday evening, a Special Evening Lecture was delivered by Wendelin Werner on the subject of Conformal Loop Ensembles which was followed by a celebration of the Fields Medal that was awarded to Werner at the ICM Madrid two weeks prior to the meeting. This was considered an important event, as Werner's is the first Fields Medal awarded to probability theory as a field, and to statistical mechanics in particular.

We proceed by a brief overview of the 5 subjects above and a summary of the principal results of those areas that were reported on in the workshop:

Interacting random walks and polymers: The subject of interacting random walks and, in particular, polymer models seems more active than ever. Concerning polymer problems, the present research focuses on detailed understanding of the problem of heteropolymer absorption at an interface and, particularly, the critical line separating the localized and delocalized regimes.

Several polymer talks have been presented at the workshop. Francesco Caravenna talked about pinning of polymers whose monomers interact via a Laplacian interaction. Francis Comets outlined the approach to directed polymer models via multiplicative cascades. Fabio Toninelli presented results on the "rounding" of localization-delocalization transition due to the presence of quenched disorder. Stu Whittington discussed three models of a polymer subject to a force.

Another talk in this category concerned interacting random walk in relation to Bose-Einstein condensation. Here Wolfgang König presented results linking the Gross-Pitaievski approach to Bose-Einstein condensation to certain explicit problems for interacting Brownian motions. These arise as modifications of the standard, Feynman-Kac representation of the interacting Bose gas.

Scaling limits in high D and expansion techniques: High-dimensional statistical mechanics has enjoyed a lot of attention due to the success of the lace-expansion method. Many critical models were analyzed using this technique and mean-field exponents were established above the corresponding upper critical dimension. Several results in this category were presented at the workshop.

Akira Sakai explained his recent breakthrough in applying the lace expansion to the Ising model. The principal idea is to use the random current representation which permits the use of BK-like inequalities. Frank den Hollander's talk focused on the scaling limit of invasion-percolation cluster on a regular tree. Here the surprising fact is that, despite its close relation to the ordinary (critical) percolation, the scaling limits of this object is rather different. Martin Barlow reported on the study of simple random walk on the Incipient Infinite Cluster of directed (spread-out) percolation above 6 dimensions. The main result is that the spectral dimension of this walk is 4/3, in accord with the so-called Alexander-Orbach conjecture. Mark Holmes discussed expansion techniques for certain self-interacting random walks.

Statistical mechanics of interfaces: Droplets and interfaces have been a very active area of research in probability theory in the last 30 years. At present, the most intensely studied questions are those of detailed geometrical properties of interfaces, including description of their fluctuations and, potentially, scaling limits, and problems of phase separation.

Three talks on these subjects were presented at the workshop: Senya Shlosman outlined a proof of existence of non-translation invariant, Dobrushin-interface states in certain systems with continuum spins. The main novel idea was the use of chessboard estimates, rather than contour formalism which is not quite available in these cases. Ostap Hryniv reported on ongoing work dealing with droplet formation in 2D Ising model. Here the objective is to derive very precise, local-limiting type asymptotic for all quantities. Finally, Dima Ioffe discussed methods for sharp control of random paths—which arise as, e.g., interface lines—possessing a natural regeneration structure. One of the applications of this is the proof of the Ornstein-Zernike behavior which is instrumental for the conclusions mentioned in Ostap Hryniv's talk.

Percolation and other systems in 2D: Two-dimensional statistical mechanics enjoyed tremendous boost in the last 5 years due to Schramm's invention of the Stochastic Loewner Evolution and the ensuing proofs that this process describes interfaces in various models at criticality. One of the fundamental results is that of Smirnov who proved that, for critical percolation on the triangular lattice, the crossing probabilities converge, in the scaling limit, to Cardy's formulas.

The problem with Smirnov's result is that it is extremely special to the (regular) triangular lattice. Recently, a number of people attempted generalizations beyond this setting. At our workshop, Vincent Beffara presented a summary of some of his attempts in this direction and his analysis of various error bounds by comparison with predictions via SLE. On the other hand, Lincoln Chayes described a model of interacting percolation where Cardy's formulas can, after a non-trivial amount of work, be rigorously justified. In his Special Evening Lecture, Wendelin Werner sketched the definitions and properties of the Conformal Loop Ensembles which are the scaling limits of outer boundaries of a class of critical 2D systems.

Another 2D percolation problem was presented in the talk of Rob van den Berg, who explained the relation of the forest-fire models to the problem of self-destructive percolation. One of the main issues is the existence of the forest-fire model beyond a time when an infinite forest first appears; in self-destructive percolation this boils down to an intriguing conjecture on the continuity of the infinite cluster density.

Random walk in random environment: The subject of random walk in random environment is a place of unabating activity, particularly, in the area of quenched limit laws. The reversible situations are pretty much well understood so the focus is on irreversible problems or complicated geometries.

Four major results of this sort were presented at the workshop: Jean-Dominique Deuschel discussed the irreversible random walks that admit a finite cycle decomposition. While generally non-reversible, this decomposition allows the use of the theory developed for reversible situations. Noam Berger talked about the zero-one law for the speed in sufficiently high dimensions which extends the famous (and old) results of Kalikow. Ofer Zeitouni presented a new proof of diffusivity for RWRE in $d \geq 3$ in the perturbative regime; this result was published 20 years ago by Bricmont and Kupiainen but their proof is, to the present day, deemed quite impenetrable. Finally, as already mentioned, Martin Barlow presented a proof of the Alexander-Orbach conjecture for the simple random walk on the Incipient Infinite Cluster for directed percolation above 6 dimensions.

The atmosphere of the workshop was very friendly, and there was ample discussion, both during and after the lectures. The organizers wish to thank the "Mathematisches Forschungsinstitut Oberwolfach" for their help in the practical organization of the workshop, and in particular for providing a superb environment for this meeting. As the organizing team, we have discussed the possibility of organizing a similar workshop again in about three years. This idea was encouraged by many participants.

Kenneth Alexander
Marek Biskup
Remco van der Hofstad
Vladas Sidoravicius

Workshop: Spatial Random Processes and Statistical Mechanics

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Abstracts

Pinning models with Laplacian interaction in (1+1)–dimension

Francesco Caravenna
(joint work with Jean–Dominique Deuschel)

We study the path behavior of a class of random fields $\varphi : \{0, \ldots, N\} \to \mathbb{R}$ with Laplacian interactions and with in addition a delta–pinning reward for the field to touch the x–axis, that plays the role of a defect line.

1. The model

For $N \in \mathbb{N}$ and $\varepsilon \geq 0$, we consider the probability measure $\mathbb{P}_{\varepsilon,N}$ on $\mathbb{R}^{N-1}$ defined by

$$\mathbb{P}_{\varepsilon,N}\big(\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{N-1}\big) \;=\; \frac{\exp\big(-\mathcal{H}_N(\varphi)\big)}{\mathcal{Z}_{\varepsilon,N}} \prod_{i=1}^{N-1} \big(\varepsilon\, \delta_0(\mathrm{d}\varphi_i) + \mathrm{d}\varphi_i\big)\,, \tag{1}$$

where $\mathrm{d}\varphi_i$ is the Lebesgue measure on $\mathbb{R}$, $\delta_0(\cdot)$ is the Dirac mass at zero, $\mathcal{Z}_{\varepsilon,N}$ is the normalizing constant (*partition function*) and the Hamiltonian $\mathcal{H}_N(\varphi)$ is defined by

$$\mathcal{H}_N(\varphi) := \sum_{n=0}^{N} V\big(\Delta\varphi_n\big)\,, \qquad \text{where} \quad \Delta\varphi_n \;:=\; \varphi_{n+1} + \varphi_{n-1} - 2\varphi_n\,, \tag{2}$$

with zero boundary conditions: $\varphi_{-1} = \varphi_0 = \varphi_N = \varphi_{N+1} := 0$.

We allow for a large choice of the *potential* $V(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ appearing in (2): we only require that the function $x \mapsto \exp(-V(x))$ is bounded and continuous with $\exp(-V(0)) > 0$ and that it satisfies the following integrability conditions:

$$\int_{\mathbb{R}} e^{-V(x)}\,\mathrm{d}x \;<\; \infty \qquad \sigma^2 := \int_{\mathbb{R}} x^2 e^{-V(x)}\,\mathrm{d}x \;<\; \infty \qquad \int_{\mathbb{R}} x\, e^{-V(x)}\,\mathrm{d}x \;=\; 0\,. \tag{3}$$

The typical example is of course $V(x) = x^2$.

2. The main results

In the simple case $\varepsilon = 0$, the measure $\mathbb{P}_{0,N}$ admits a basic random walk interpretation that allows an explicit analysis. It follows in particular that for large N the field under $\mathbb{P}_{0,N}$ is typically at distance $N^{3/2}$, hence very far, from the x–axis: we call this phenomenon *delocalization*. When $\varepsilon > 0$ the law $\mathbb{P}_{\varepsilon,N}$ gives the field a positive reward each time it touches the x–axis. Then the basic question is whether this reward is strong enough to force the field staying close to the x–axis, yielding thus a *localization* scenario, or if delocalization still prevails.

Our results give a precise answer to this question in terms of the path properties of $\mathbb{P}_{\varepsilon,N}$ as $N \to \infty$. We show that our model undergoes a phase transition in ε, namely there exists $\varepsilon_c \in (0, \infty)$ such that the path properties of $\mathbb{P}_{\varepsilon,N}$ are essentially the same as in the case $\varepsilon = 0$ provided $\varepsilon < \varepsilon_c$, while they are radically different (exhibiting distinctive features of localization) as soon as $\varepsilon > \varepsilon_c$.

Let us be more precise: we introduce the rescaled and linearly interpolated field $\{\widehat{\varphi}_N(t)\}_{t\in[0,1]}$, defined by

$$\widehat{\varphi}_N(t) \;:=\; \frac{\varphi_{\lfloor Nt \rfloor}}{\sigma N^{3/2}} \;+\; \big(Nt - \lfloor Nt \rfloor\big)\, \frac{\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}}{\sigma N^{3/2}}\,, \tag{4}$$

and we study the limit in distribution as $N \to \infty$ of the process $\{\widehat{\varphi}_N(t)\}_{t\in[0,1]}$ under $\mathbb{P}_{\varepsilon,N}$ on $C([0,1])$, the space of real valued continuous functions defined on $[0,1]$. Let $\{B_t\}_{t\in[0,1]}$ denote a standard Brownian motion. We define the integrated Brownian motion process $I_t := \int_0^t B_s \,\mathrm{d}s$ and we introduce the Gaussian process

$$\{\widehat{I}_t\}_{t\in[0,1]} \;:=\; \{I_t\}_{t\in[0,1]} \;\text{ conditionally on }\; (B_1, I_1) = (0,0)\,. \tag{5}$$

We can now state our first result.

Theorem 1. *The rescaled field $\{\widehat{\varphi}_N(t)\}_{t\in[0,1]}$ under $\mathbb{P}_{\varepsilon,N}$ converges in distribution on $C([0,1])$ as $N \to \infty$. The limit law is:*

- *If $\varepsilon < \varepsilon_c$, the law of the process $\{\widehat{I}_t\}_{t\in[0,1]}$;*
- *If $\varepsilon \geq \varepsilon_c$, the law concentrated on the constant function $f(t) \equiv 0$.*

By looking on a finer scale, one can show that for $\varepsilon > \varepsilon_c$ the typical paths of the field are really localized close to the x–axis. On the other hand, a more subtle and interesting scenario shows up in the critical regime $\varepsilon = \varepsilon_c$, where the behavior of the field appears to be somewhat intermediate between localization and delocalization. In this regime, by refining the scaling constants $N^{3/2}$ in the definition (4) of the rescaled field, one can extract a non-trivial scaling limit, albeit in a generalized sense. More precisely, we introduce the signed measure $\boldsymbol{\mu}_N$ on $[0,1]$ defined by

$$\boldsymbol{\mu}_N(\mathrm{d}t) \;:=\; \frac{(\log N)^{5/2}}{\sigma N^{3/2}}\, \varphi_{\lfloor Nt \rfloor}\, \mathrm{d}t\,. \tag{6}$$

We look at $\boldsymbol{\mu}_N$ under the critical law $\mathbb{P}_{\varepsilon_c,N}$ as a random element of $\mathcal{M}([0,1])$, the space of finite signed Borel measures on the interval $[0,1]$ equipped with the topology of vague convergence, and we study its convergence in distribution.

Let $\{L_t\}_{t\geq 0}$ denote (a càdlàg version of) the stable symmetric Lévy process of index $2/5$. Since the paths of L are a.s. of bounded variation, we can define path by path the (random) finite signed measure $\mathrm{d}L$ in the Steltjes sense, i.e. $\mathrm{d}L\big((a,b]\big) := L_b - L_a$. We are now ready to state our main result.

Theorem 2. *The random signed measure $\boldsymbol{\mu}_N$ under $\mathbb{P}_{\varepsilon_c,N}$ converges in distribution on $\mathcal{M}([0,1])$ as $N \to \infty$ toward the the random signed measure $\mathrm{d}L$.*

We point out that $\mathrm{d}L$ is a.s. a purely atomic measure, i.e. a sum of Dirac masses. Roughly speaking, the masses of $\mathrm{d}L$ describe the large excursions of the field $\{\varphi_n\}_n$ under $\mathbb{P}_{\varepsilon_c,N}$ for large N.

3. Motivations and a look at the literature

One can interpret $\mathbb{P}_{\varepsilon,N}$ as a model for a homogeneous linear chain attracted to a *defect line*, the x–axis, where the parameter $\varepsilon \geq 0$ tunes the strength of the attraction. The inner structure of the chain is described by the Hamiltonian $\mathcal{H}_N(\cdot)$, which in our case is made up of Laplacian interaction terms $V(\Delta\varphi_n)$. In dimension higher than 1, fields with this type of interactions are used as models for membranes, cf. [5, 7]. While some recent mathematical investigations in high dimensions have been performed, see [6] and [4], the one-dimensional case seem not to have been considered in the mathematical literature.

One-dimensional fields that have been more studied are those with *gradient interaction*, that is with Hamiltonian $\sum_n V(\nabla\varphi(n))$ where $(\nabla\varphi)(n) := \varphi_n - \varphi_{n-1}$. These can be viewed as effective models for $(1+1)$-dimensional interfaces, cf. [3, 2, 1]. We point out that for this kind of models the pinning term $\varepsilon \cdot \delta_0(\mathrm{d}\varphi_i)$ induces a trivial transition ($\varepsilon_c = 0$), i.e. an arbitrarily small reward is able to localize the field. Therefore the non-trivial transition ($\varepsilon_c > 0$) that we have found in the Laplacian case is non-obvious a priori. Heuristically, we could say that the Laplacian interaction describes a stiffer chain, more rigid to bending, and therefore Laplacian models require a stronger reward in order to localize.

Acknowledgements

We are very grateful to Yvan Velenik and Ostap Hryniv for suggesting the present problem to us.

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Critical properties of directed polymers interacting with a random defect line

Fabio Lucio Toninelli
(joint work with Giambattista Giacomin)

Consider a renewal process, with generic configuration $\tau \subset \mathbb{N} \cup \{0\}$ defined as follows: $\tau_0 = 0$ and $\{\tau_i - \tau_{i-1}\}_i$ are IID random variables. We call $\mathbb{P}$ the law of the renewal and we set $K(n) := \mathbb{P}(\tau_1 = n)$, for $n \in \mathbb{N} \cup \{\infty\}$. Of course, $\sum_{n\in\mathbb{N}} K(n) + K(\infty) = 1$. We will make the folliwing important assumption:

$$K(n) \sim n^{-\alpha} \tag{1}$$

for $n \to \infty$, for some $1 \leq \alpha < \infty$. (For a more precise condition, see for instance [2].) Note that if $K(\infty) > 0$ then the renewal process is transient: τ contains only a finite (but random) number of points.

Now we modify the law $\mathbb{P}$ to define *pinning measures* $\mathbb{P}_{N,\omega}$ as follows:

$$\frac{d\mathbb{P}_{N,\omega}}{d\mathbb{P}}(\tau) = \frac{e^{\sum_{n=1}^{N}(\beta\omega_n - h)\mathbf{1}_{\{n\in\tau\}}}}{Z_{N,\omega}}\mathbf{1}_{\{N\in\tau\}}. \tag{2}$$

Here, $\beta \geq 0$, $h \in \mathbb{R}$, $N \in \mathbb{N}$ and ω_n are IID random variables with law $\mathbb{P}$ and such that $\mathbb{E}\,\omega_1 = 0$, $\mathbb{E}\,\omega_1^2 = 1$. Note that sites n where $(\beta\omega_n - h) > 0$ favor the event $n \in \tau$ and viceversa.

For the results below to hold, some technical condition on $\mathbb{P}$ are reaquired. We refer to the original papers for details.

The model is motivated by physical and biological applications, e.g., disordered wetting phenomena in $(1+1)$ dimensions and the modelization of the DNA denaturation transition (cf. [3] and references therein). Typically, one should keep in mind the situation where τ_i are the return times of some Markov Chain S_n, i.e., $\tau = \{n : S_n = 0\}$. S should be thought of as the configuration of a directed polymer interacting with a defect line ($S = 0$) where random charges ω_n are placed.

The free energy, defined as

$$F(\beta, h) = \lim_N \frac{1}{N} \log Z_{N,\omega},$$

is known to exist and to be almost-surely constant. Moreover, one proves easily that $F(\beta, h) \geq 0$. One decomposes the phase diagram (β, h) into two regions: the localized phase $\mathcal{L} = \{(\beta, h) : F(\beta, h) > 0\}$ and the delocalized phase $\mathcal{D} = \{(\beta, h) : F(\beta, h) = 0\}$. The nomenclature refers to the fact that in $\mathcal{L}$ the "contact fraction"

$$L_N := \frac{1}{N}|\tau \cap \{1, \ldots, N\}|$$

is non-zero in the thermodynamic limit, while the opposite is true in $\mathcal{D}$.

For a given $\beta \geq 0$, the critical point is defined as

$$h_c(\beta) := \inf\{h : F(\beta, h) = 0\}.$$

We are interested in investigating the way the free energy vanishes for h approaching the critical point $h_c(\beta)$ from below. This is of course a question on the nature of the phase transition. In the homogeneous situation ($\beta = 0$) it is known that the transition is of first order ($F(0,h) \sim (h_c(0) - h)$) if

$$\sum_{n \in \mathbb{N}} nK(n) < \infty,$$

while in the opposite situation, say if $\alpha < 2$, the transition is smooth: in particular,

$$F(0,h) \sim (h_c(0) - h)^{1/(\alpha-1)}.$$

Our main result is the following [2] [3]:

Theorem 1. *For every $\beta > 0$ there exists a finite $c(\beta)$ such that*

$$0 < F(\beta,h) \leq \alpha c(\beta)(h - h_c(\beta))^2 \tag{3}$$

for $h < f_c(\beta)$.

In other words, an arbitrarily weak amount of disorder is enough to smoothen the transition. The proof of the theorem is based on an energy-entropy argument inspired by Ref. [1], plus large-deviation type estimates on the probability of finding rare but very favorable regions where the disorder is atypically favorable to polymer-line contacts.

In particular, the above theorem implies that at the critical point the order parameter vanishes in the thermodynamic limit:

$$\lim_{N \to \infty} \mathbb{E}_{N,\omega}(L_N) = 0 \tag{4}$$

almost surely, for $\beta > 0$ and $h = h_c(\beta)$. In the same situation, we can give finer, finite-size type estimates on the order parameter [4]:

$$\mathbb{E}_{N,\omega}(L_N) = O(N^{-1/3} \log N). \tag{5}$$

This result is a consequence of the previous theorem, plus some concentration of measure ideas.

A natural and interesting open question concerns the true order of the transition in presence of disorder. There is rather general consensus that for β small and $\alpha < 3/2$ the transition is of the same order as in the pure case, while for $\alpha \geq 3/2$ predictions in the physics literature are rather contradictory.

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Linear polymers subject to a force: Directed walk models

S. G. Whittington

The development of experimental techniques such as atomic force microscopy and optical tweezers has made it possible to apply a force to an individual polymer molecule. This paper is about simple combinatorial models which describe the response of a linear polymer to an applied force.

Lattice models of linear polymers capture many of the large scale (universal) features of the physical system and several different models can be used to describe the conformational and thermodynamic properties. The polymer can be modelled as a random walk, a directed walk or a self-avoiding walk, for instance. Self-avoiding walks have the advantage that they mimic the excluded volume effect (the fact that one monomer takes up space to the exclusion of other monomers) but they are notoriously difficult to handle. This is especially true when modelling a polymer subject to a force since we know very little about the path properties of self-avoiding walks, especially in low dimensions. Here we use directed walk models which can be solved by combinatorial techniques.

We consider three situations: (i) a polymer adsorbed at an impenetrable surface and pulled off the surface by a force, (ii) a collapsed polymer pulled into an expanded form by a force, and (iii) a polymer confined between two parallel lines or planes.

A linear polymer in solution can adsorb at an impenetrable surface and a simple model of this phenomenon is a Dyck path where we keep track of the number of vertices in the distinguished line. Consider the square lattice Z^2 with coordinate system (x_1, x_2) where x_1 and x_2 are integers. A Dyck path is a directed walk on Z^2 with two kinds of directed edges $(1,1)$ and $(1,-1)$. The walk starts at the origin, ends on the line $x_2 = 0$ and has no vertices with negative x_2-coordinate. If we write $d_n(v)$ for the number of Dyck paths with n edges and with $v+1$ vertices in the line $x_2 = 0$ we can define the generating function

$$D(x,z) = \sum_{v,n} d_n(v) x^v z^n$$

and D satisfies the equation

$$D(x,z) = 1 + xz^2 D(1,z) D(x,z).$$

The thermodynamic properties of the model are determined by the singularities of D. A force can be incorporated by allowing the walk to end at any $x_2 \geq 0$ and keeping track of the x_2-coordinate of the nth vertex. If we write $d_n(v,h)$ for the number of such walks with n edges, $v+1$ vertices in $x_2 = 0$ and with the x_2-coordinate of the last vertex being h we can define the generating function

$$F(x,y,z) = \sum_{n,v,h} d_n(v,h) x^v y^h z^n$$

and F satisfies the relation

$$F(x,y,z) = D(x,z)[1 + yzF(1,y,z)].$$

The function F has three singularities, $z = z_1 = 1/2$, corresponding to a desorbed phase, $z = z_2(x)$, corresponding to an adsorbed phase and $z = z_3(y)$, corresponding a desorbed phase where the desorption is due to the applied force. From this one can construct the critical force-temperature curve which describes the force needed to cause desorption as a function of temperature [3]. The model can be extended to Motzkin paths and to partially directed walks.

The conformational properties of a polymer depend on the quality of the solvent in which it is dissolved. Typically at high temperatures the solvent is *good* and the polymer is an expanded coil. At low temperatures the solvent is *poor* and the polymer is a fairly compact ball, tending to minimize its surface area to avoid unfavourable monomer-solvent interactions. A simple model of this phenomenon is a partially directed walk on Z^2 with vertex-vertex interactions [1]. A partially directed walk is a walk with no West steps and in which a North step cannot follow a South step, and vice versa. In addition, for technical reasons, the first step is an East step. If two vertices of the walk are unit distance apart but are not incident on a common edge of the walk, these two vertices define a *contact.* Let $b_n(k)$ be the number of n-edge partially directed walks with k contacts. We define the generating function

$$G(x, z) = \sum_{k,n} b_n(k) x^k z^n.$$

A method developed by Temperley [5] can be used to derive recurrence relations which determine the generating function $G(x, z)$. The singularities of G determine the thermodynamics of the model and it is known [1] that the model displays a phase transition from a compact to an expanded form.

The model can be extended to incorporate a force in the x_1-direction which tends to pull the polymer into an expanded form. (This is joint work with Jennifer Lee.) One needs to keep track of the span s of the walk in the x_1-direction and form the generating function

$$H(x, y, z) = \sum_{k,s,n} b_n(k, s) x^k y^s z^n$$

where x is conjugate to the number of contacts, y is conjugate to the span and z is conjugate to the number of edges. Again using Temperley's approach, recurrence relations can be derived which determine H and the singularities of H determine the thermodynamic behaviour. We have investigated the singularity structure and used this to derive the critical force-temperature behaviour. The force-temperature curve agrees with that derived by Rosa *et al* [4] using a transfer matrix argument. Our approach allows us to say more about the singularity structure but the order of the transition in the presence of a force is an open question.

The third situation considered here is a polymer confined between two parallel planes. This is a crude model of the steric stabilization of a dispersion by polymer molecules weakly adsorbed on the surfaces of colloidal particles. As the particles approach one another the polymer loses entropy and this results in an entropic repulsive force. This can be modelled by a Dyck path confined between two lines, say $x_2 = 0$ and $x_2 = w$. If there are no other interactions then it is easy to see that the entropy is strictly monotone increasing in w in the infinite n limit. If the Dyck path has attractive interactions with the confining lines then the net effect can be an attractive or repulsive force between the confining lines [2]. This is a crude model of sensitized flocculation of dispersions, where adsorbed polymers can destabilize a colloidal dispersion.

Suppose that we consider Dyck paths with maximum x_2-coordinate less than or equal to w. Suppose that vertices in $x_2 = 0$ contribute a weight a and vertices in $x_2 = w$ contribute a weight b. The generating function $L_w(a, b, z)$ clearly depends on w and satisfies the relation

$$L_{w+1}(a, b, z) = L_w\left(a, \frac{1}{1 - bz^2}, z\right)$$

for $w \geq 1$. It is easy to see that

$$L_1(a, b, z) = \frac{1}{1 - abz^2}.$$

The model can be solved completely at certain special values of a and b. In the (a, b)-plane $(a, b > 0)$ the curve $ab = a + b$ corresponds to zero force in the $w \to \infty$ limit (where $n \to \infty$ first and then $w \to \infty$). The asymptotics (ie large w) can be worked out everywhere in the (a, b)-plane $(a, b > 0)$.

The model can be extended in various ways. We have looked at Motzkin paths and partially directed walks (joint work with Gary Iliev, Richard Brak and Andrew Rechnitzer) and we have some partial results for the corresponding self-avoiding walk model (joint work with Enzo Orlandini and Buks van Rensburg).

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Fluctuations of the front in a one-dimensional $X + Y \to 2X$ reaction

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(joint work with Francis Comets and Jeremy Quastel)

Consider the following model of a combustive or epidemic reaction on the integer lattice: There are two types of particles; X particles, which move as independent, continuous-time, symmetric, nearest neighbor random walks of total jump rate 2, and Y particles which do not move. Initially the Y particles occupy sites $1, 2, \ldots$, with a fixed number $a \geq 1$ of Y particles at each site. Initially there is at least one X particle at 0, and any distribution of X particles at sites $\ldots, -2, -1$, such that $\sum_{x \leq 0} \eta(0, x) e^{\theta x} < \infty$, where $\theta > 0$ is a number that will be chosen small and $\eta(0, x)$ is the number of X particles at site x at time 0. We want to study the long time asymptotics of the rightmost site r_t, visited by the X particles up to time t, which we call the *front*. We adopt the convention that initially each one of the sites $\ldots, -2, -1, 0$ has already been visited by an X particle so that $r_0 = 0$. We will show that regeneration time methods as developed in the context of transient Random Walks in Random Environments (for example [3] or [7]), can give interesting information about the behavior of the front.

Let $\eta(x, t)$ be the number of X particles at site x at time t. Note that we can just think of an X particles as branching into $a + 1$ particles each time it jumps to a $r_t + 1$ with the result that there are $a + 1$ particles at site $r_t + 1$ and without keeping track of the Y particles.

Let $\mathbb{S} := \{(r, \eta) : \eta \in \mathbb{N}^{\ldots, r-r, r}\}$ and consider the state space,

$$\mathbb{S}'_\theta := \{(r, \eta) \in \mathbb{S} : \sum_{x \leq r} e^{\theta(x-r)} \eta(x) < \infty\}.$$

If $\mathbb{S}_\theta$ is endowed with the metric $d((r, \eta), (r', \eta')) = |r - r'| + \sum_{x \leq 0} e^{\theta x} |\eta(x + r) - \eta'(x + r')|$ we have the following.

Proposition 1. *Assume that $(r, \eta) \in \mathbb{S}'_\theta$. Then, $(r_t, \eta(t)) \in \mathbb{S}'_\theta$ and the process is Feller.*

Let us call $T_{x,y}$ the first time the front r_t is in the position y, given that the initial condition of the process was $r_0 = x$ and $\eta_0 = a\delta_x$, the configuration with a particles at site x and no particles at sites $y < x$. It is possible to construct a coupling between the random variables $\{T_{x,y} : y > x \geq 0\}$ so that the subadditivity property $T_{x,y} \leq T_{x,z} + T_{z,y}$ holds for every $x < z < y$. Using Kingman's subadditive ergodic theorem one can then show that there exists a $v > 0$ independent of the initial condition $(0, \eta) \in \mathbb{S}_\theta$ such that,

$$\lim_{t \to \infty} \frac{r_t}{t} = v \qquad a.s.$$

In [6] this was shown for certain initial conditions.

The main results discussed in this report are the following.

Theorem 2 (Central limit theorem)**.** *For $\theta > 0$ small enough, there exists σ^2 nonrandom, $0 < \sigma^2 < \infty$, and independent of the the initial conditions $(0, \eta) \in \mathbb{S}'_\theta$, such that*

$$B_t^\epsilon := \epsilon^{1/2} \left(r_{\epsilon^{-1} t} - \epsilon^{-1} v t \right), \qquad t \geq 0, \tag{1}$$

converges in law as $\epsilon \to 0$ to Brownian motion with variance σ^2.

Theorem 3 (Ergodic theorem). *Consider the process as seen from the front, $\tau_{-r_t}\eta(t)$. For $\theta > 0$ small enough, there exist exactly two invariant measures: One supported on the configuration with no particles, and another, μ_∞. The domain of attraction of the first consists of exactly the configuration with no particles. Any nontrivial configuration in $\mathbb{S}'_\theta$ is in the domain of the second; if we denote by μ_t the distribution of the process $\tau_{-r_t}\eta(t)$, then $\mu_t \to \mu_\infty$ in the sense of weak convergence of probability measures.*

A modified version of this process, depending on a parameter $M \geq a$, where an X particle is annihilates whenever it jumps to a site which already has M particles was studied in [2]. The model of this report has been considered in the physics literature (see [5]). Recently there has been a resurgence of interest in such models due to the experimental detection of strong deviations from mean field behavior. On the other hand, in [1] and [6], the model is studied in arbitrary dimensions starting with an initial configuration with one X particle at 0 and 1 at each other site, proving a shape theorem for the set of visited sites in the ballistic scale. In [4], Kesten and Sidoravicius consider a model where the Y particles move as well. If we call D_X and D_Y the corresponding jump rates, they prove that if $D_X = D_Y$, initially the total number of particles is given by a product Poisson and there is a finite number of X particles, then the asymptotic shape of the set of sites visited by the X particles is also described by a shape theorem.

The proofs of theorems 2 and 3 are based on the use of a renewal structure of the process. Regeneration time methods were already used by Kesten in [3] to study the invariant measure of an i.i.d. environment as seen from a one dimensional random walk on that environment (RWRE). Our approach to define the regeneration times in terms a sequence of stopping times is inspired in the methods presented in [7] for multidimensional RWRE. At a heuristic level, regeneration occurs each time the front moves forward and the particles behind it never catch it up later on. After such a time, the behaviour of the front depends only on the a newly created particles sitting at the front at that time, but not on those behind the front at that time. The idea is to find an increasing sequence $\{\kappa_n : n \geq 1\}$ of regeneration times, having independent increments and such that the probability of the event $\{\kappa_n > t\}$ decreases fast enough as $t \to \infty$ providing good enough integrability conditions. As in [2], in order to estimate the tails of the regeneration times, it is useful to decouple particles initially on the front from those behind it. Nevertheless, a crucial difficulty and difference in the construction of the sequence of stopping times with respect to [2], is that in this model the number of X particles per site is not bounded. This requires a control in terms of some norm of the size of the cloud of particles behind the front. To do so, we introduce at each time $t \geq 0$, an *exponential norm* depending on the parameter θ and on an integer z, which is given by $\sum_{x \leq r_t} e^{\theta(x-r_t)}\eta_z(t,x)$. Here, $\eta_z(t,x)$ is the number of X particles at site x and at time t which originated from some branching (of an X particle) at some site $y \leq z$. This is a measure of the magnitude of the density of particles from r_t to $-\infty$, which originated from some site $y \leq z$. We then define a stopping time S depending on an integer length L, as the first hitting time to a site of the form

$r_0 + jL$, $j \geq 1$, such that the exponential norm of the particles originating to the left of $r_0 + (j-1)L$ is small enough. In [2], the corresponding stopping time was defined simply as the first time the front advances L steps to the right. One of the main difficulties of our proof, is to show that the tails of the law of S provide good enough integrability conditions for the corresponding regeneration times and the associated position of the front. We are able to do this only for small values of θ and large values of L: we obtain polynomially decaying tails of a degree which increases linearly with L for the regeneration times $\{\kappa_n : n \geq 2\}$.

The results presented in this report can be easily extended to the case where the initial configuration of the Y particles is a product Poisson measure of parameter 1 at sites $1, 2, \ldots$. Nevertheless, it remains a challenge to extend them to more general stochastic front dynamics, as for example the case studied in [4] where the Y particles have a positive jump rate $D_Y > 0$.

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Interacting Brownian motions and the Gross-Pitaevskii formula

Wolfgang König

(joint work with Stefan Adams and Jean-Bernard Bru)

Consider a quantum system of N particles in a trap potential $W : \mathbb{R}^d \to [0, \infty]$ with mutually repellent pair interaction potential $v : (0, \infty) \to [0, \infty]$. This system is described by the Hamilton operator

$$\mathcal{H}_N = -\sum_{i=1}^N \Delta_i + \sum_{i=1}^N W(x_i) + \sum_{1 \leq i < j \leq N} v\big(|x_i - x_j|\big), \qquad x_1, \ldots, x_N \in \mathbb{R}^d.$$

The trap potential W is supposed to are continuous, say, and to explode to ∞ at infinity. Examples are $W(x) = |x|^2$ or $W = \infty 1\!\!1_{\Lambda^c}$ for a bounded box Λ, but much more general trap potentials are possible. The interaction potential, v, is supposed to decay quickly at infinity and to explode to ∞ at zero. An example is $v = \infty 1\!\!1_{(0,a^*)}$ for some $a^* > 0$. The main goal is the description of the system at zero or very low temperature in the limit as $N \to \infty$ with the box Λ tending to $\mathbb{R}^d$ coupled with N. In particular, one seeks to mathematically understand *Bose-Einstein condensation*, which states that the N-particle wave function can be asymptotically described by a single-particle wave function.

From now we restrict to $d = 3$ and replace W by the rescaling $W_N(\cdot) = L_N^{-2} W(\cdot\, L_n^{-1})$ for some scale function $L_N \to \infty$. An ambitious goal is to handle rescalings such that the particle density N/L_N^3 stays bounded and bounded away from zero, but in this talk we only consider the a particular dilute system: we shall put $L_N = N$. By a rescaling argument, we can leave W unchanged and replace instead v by $N^2 v(\cdot\, N)$. For the rest of this talk, we keep it like that.

In a series of papers around 2000, see [LSSY05], Lieb, Seiringer and Yngvason identified the large-N behavior of the ground state of $\mathcal{H}_N$,

$$N\chi_N = \min_{h\in H^1(\mathbb{R}^{3N}):\, \|h\|_2=1} \langle h, \mathcal{H}_N h\rangle = \langle h_N^*, \mathcal{H}_N h_N^*\rangle,$$

and its ground state, h_N^*, in the limit $N \to \infty$. They found that χ_N converges towards the *Gross-Pitaevskii formula*,

$$\chi_\alpha^{\mathrm{GP}} = \inf_{\varphi\in H^1(\mathbb{R}^3):\, \|\varphi\|_2=1} \Big(\|\nabla\varphi\|_2^2 + \langle W, \varphi^2\rangle + 4\pi\alpha\|\varphi\|_4^4 \Big),$$

where the parameter $\alpha > 0$ must be taken equal to the *scattering length* of the pair potential v.

The behaviour of the system at *positive* temperature $1/\beta$ is described in terms of the trace of $\mathrm{e}^{-\beta\mathcal{H}_N}$, which in turn can be written as

$$\mathrm{Tr}(\mathrm{e}^{-\beta\mathcal{H}_N}) = \int_{(\mathbb{R}^3)^N} \mathrm{d}x \,\Big[\bigotimes_{i=1}^N \mathbb{E}^\beta_{x_i,x_i} \Big] \big[\mathrm{e}^{-H_{N,\beta}}\big],$$

where $\mathbb{E}^\beta_{x,y}$ is the (non-normalised) expectation with respect to a Brownian bridge that runs from x at time zero to y at time β, and

$$H_{N,\beta} = \int_0^\beta W(B_t^i)\,\mathrm{d}t + \sum_{1\le i<j\le N} \int_0^\beta v\big(|B_t^i - B_t^j|\big)\,\mathrm{d}t.$$

Bosons are described in the same way by a system of symmetrised Brownian bridges, where the i-th Brownian bridge runs from x_i to $x_{\sigma(i)}$, and an expectation is taken over a uniformly distributed random permutation σ of the numbers $1, \dots, N$.

In the present talk, we do not consider this canonical model, but the *Hartree model*, where $H_{N,\beta}$ is replaced by

$$K_{N,\beta} = \int_0^\beta W(B_t^i)\,\mathrm{d}t + \sum_{1\le i<j\le N} \int_0^\beta \frac{1}{\beta} \int_0^\beta v\big(|B_t^i - B_s^j|\big)\,\mathrm{d}t\mathrm{d}s.$$

That is, the particle interaction is replaced by a path interaction. This model is not as 'physical', but it is in the same spirit, and we hope to solve problems in future about the Hartree model that are currently too difficult for the canonical model. We introduce and analyse the Hartree model and variants in our papers [ABK06a], [ABK06b] and [AK06].

One of our first results about the Hartree model is that its free energy (i.e., the large-β exponential rate of its total mass) is described in terms of the ground *product* energy, $N\chi_N^{\otimes}$, of $\mathcal{H}_N$, i.e., by the minimum of $\langle h, \mathcal{H}_N h\rangle$ taken over all normalised functions h of product form. (This variational formula is sometimes called the*Hartree formula* and gave the model its name.)

Furthermore, we analysed the behavior of $\chi_N^{\otimes}$ in the limit $N \to \infty$ under the assumption that the integral $\int v(|x|)\,\mathrm{d}x$ is finite. We obtain the same statement as Lieb et. al. above and the same limiting formula, the Gross-Pitaevskii formula. However, the scattering length must be replaced by that integral.

Furthermore, we study the effect of symmetrisation to the model in general. We do this in any dimension d, but without (trap or pair) interaction. Hence we must replace the Lebesgue measure as the initial distribution of our Brownian bridges by some probability measure $\mathfrak{m}$ on $\mathbb{R}^d$. We show that the mean of the normalised occupation measures of the motions,

$$\mu_N = \frac{1}{N}\sum_{i=1}^N \frac{1}{\beta}\int_0^\beta \mathrm{d}s\,\delta_{B_s^i},$$

satisfies a large-deviation principle on the set of probability measures on $\mathbb{R}^d$. That is, we identify a rate function $I_{\mathfrak{m},\beta}$ such that, formally,

$$\frac{1}{N!}\sum_\sigma \int_{(\mathbb{R}^d)^N} \mathfrak{m}(\mathrm{d}x_1)\cdots\mathfrak{m}(\mathrm{d}x_N)\bigotimes_{i=1}^N \mathbb{P}^\beta_{x_i,x_{\sigma(i)}}(\mu_N \in A) \approx \mathrm{e}^{-N\inf_A I_{\mathfrak{m},\beta}}.$$

Our formula for $I_{\mathfrak{m},\beta}$ is explicit and interpretable, but involved. Luckily, it turns out that, for the important special case of $\mathfrak{m}$ being the Lebesgue measure on some bounded box, $I_{\mathfrak{m},\beta}$ can be identified in much easier terms: it turns out to be equal to β times the energy of the squareroot of the density of the measure, i.e., equal to the map $\varphi^2(x)\,\mathrm{d}x \mapsto \beta\|\nabla\varphi\|_2^2$.

It is known that this function governs the large deviations of the occupation measures of a single Brownian motion in the large-time limit. The fact that this function also governs the ones of the mean of the occupation measures of N symmetrised motions can be interpreted by saying the main contribution comes from those permutations that have so many cycles of length of order N that their lengths sum up to N. Indeed, for such a permutation, the integral over $x_1, \dots, x_N$ splits into independent pieces over the indices belonging to the cycles, and these may be seen as single Brownian bridges of length equal to β times the respective cycle length. Each of these independent pieces represents a single Brownian motion with time length of order N.

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Random walks on the incipient infinite cluster for oriented percolation

M. T. Barlow
(joint work with A. A. Járai, T. Kumagai, G. Slade)

Following recent work in [2, 6, 13, 14] random walks on supercritical percolation clusters in $\mathbb{Z}^d$ are now quite well understood. For the critical case, on the other hand, very little is known. It is believed that at $p = p_c$, all clusters are finite – this is known (for standard bond percolation on $\mathbb{Z}^d$) when $d = 2$ or $d \geq 19$. While it may be possible to formulate interesting questions about random walks on finite clusters, this is less straightforward than considering the asymptotics of random walk on a fixed infinite graph.

In [11] Kesten defined the *incipient infinite cluster (IIC)* for $\mathbb{Z}^2$. Let A_N be the event that $\mathcal{C}(0)$, the cluster containing 0, is connected to the boundary of $[-N, N]^2$. Let

$$\mathbb{P}^{(N)}(\cdot) = \mathbb{P}_{p_c}(\cdot | A_N). \tag{1}$$

Kesten proved that $\mathbb{P}^{(N)}$ converged to a law $\widetilde{P}$; the IIC is $\mathcal{C}(0)$ under $\widetilde{P}$. In [12] Kesten proved that simple random walk on the IIC (for $\mathbb{Z}^2$) is subdiffusive, and also obtained much more detailed results on the SRW on the IIC for the binary tree $\mathbb{B}$.

Let $\Gamma = (G, E)$ be an infinite, connected locally bounded graph. We write $d(x, y)$ for the usual graph distance, and denote balls by

$$B(x, r) = \{y : d(x, y) < r\}.$$

Let μ_x be the degree of the vertex x; we extend μ to a measure on G and write

$$V(x, r) = \mu(B(x, r)),$$

which (following terminology for manifolds) we sometimes call the volume of the ball $B(x, r)$. The simple random walk $X = (X_n, n \geq 0, P^x, x \in G)$ on Γ is defined by

$$P^z(X_{n+1} = y | X_n = x) = P(x, y)$$

where

$$P(x, y) = \begin{cases} 1/\mu_x, & \text{if } y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

The transition density of X, or discrete time heat kernel on Γ is

$$p_n(x, y) = \frac{P^x(X_n = y)}{\mu_y} = p_n(y, x).$$

The *spectral dimension of* Γ is defined by

$$d_s = d_s(\Gamma) = -2 \lim_{n\to\infty} \frac{\log p_n(x,x)}{\log n}$$

(if this limit exists). Note that $d_s(\mathbb{Z}^d) = d$. In [1] Alexander and Orbach conjectured that the spectral dimension of the IIC is $\frac{4}{3}$ in *all* dimensions. (The actual conjecture in [1] is stated less precisely, and in a slightly different way). This is now thought to be false for small d, but it is likely that this is true for $d > d_c = 6$, the upper critical dimension for percolation.

Not enough is yet known about the IIC for standard percolation; but note that the IIC has been constructed for spread-out models for $d > 6$ in [7]. The IIC for (spread-out) oriented percolation in $\mathbb{Z}^d \times \mathbb{Z}_+$, $d > 4$, is better understood. In this case, one considers the oriented graph with vertex set $\mathbb{Z}^d \times \mathbb{Z}_+$, and with (oriented) edges given by

$$(x,n) \to (y,n+1) \quad \text{if and only if} ||x-y||_\infty \le L.$$

Here $L \gg 1$.

Theorem 1. *Let $d > 6$, and $\widetilde{\mathcal{C}}$ be the IIC for spread-out oriented percolation in $\mathbb{Z}^d \times \mathbb{Z}_+$. If $L \ge L_0(d)$ then $d_s(\widetilde{\mathcal{C}}) = 4/3$.*

The proof of this theorem falls into two parts, which use quite different sets of techniques. The first is to consider SRW on a family of random graphs $\Gamma(\omega)$ satisfying suitable hypotheses, and using random walk and heat kernel methods to prove that one has $d_s(\Gamma) = 4/3$, a.s. The second part is to prove that the IIC for oriented percolation satisfies these hypotheses: this is done using the lace expansion.

Let $(\Omega, \mathbb{P})$ be a probability space, carrying a family $\Gamma(\omega)$ of random graphs such that for each $\omega \in \Omega$, $\Gamma(\omega) = (G(\omega), E(\omega))$ is an infinite connected graph with vertex degree bounded by some constant C_1, and there is a marked vertex $0 \in G(\omega)$. Write

$$V(r) = V(r)(\omega) = V_\omega(0,r) = \mu_\omega(B_\omega(0,r)) \tag{2}$$

$$R_{\text{eff}}(r) = R_{\text{eff}}(0, B(0,r)^c). \tag{3}$$

Here R_{eff} denotes the effective resistance when $\Gamma(\omega)$ is considered as an electric network with unit resistors on each edge.

Let $\lambda \in [1,\infty)$. Define $J(\lambda) = J(\lambda)(\omega) \subset [1,\infty)$ by: $r \in J(\lambda)$ ("r is good") if:

$$r^2/\lambda \le V(r) \le \lambda r^2 \quad \text{and } r/\lambda \le R_{\text{eff}}(r) \le \lambda r.$$

Definition. $(\Gamma(\omega))$ satisfies Hypothesis A if there exists $C_2 > 0$, $q > 0$ such that

$$\mathbb{P}(r \in J(\lambda)) \ge 1 - \frac{C_2}{\lambda^q},$$

for each $r \in [1,\infty)$.

We write

$$p_n^\omega(x,y) = P_\omega^x(X_n = y)/\mu^\omega(y), \tag{4}$$

$$\tau_r = \min\{n : d(X_n, 0) > r\}. \tag{5}$$

Theorem 2. *Suppose* $(\Gamma(\omega))$ *satisfies Hypothesis A. Then*
(a) There exists $N_0(\omega)$ *with* $\mathbb{P}(N_0 < \infty) = 1$ *such that for* $n > N(\omega)$,

$$(\log n)^{-\alpha_1} n^{-2/3} \leq p_{2n}^\omega(0,0) \leq (\log n)^{\alpha_1} n^{-2/3}.$$

In particular, $d_s(\Gamma) = \frac{4}{3}$, $\mathbb{P}$*-a.s.*
(b) There exists $R_0(\omega)$ *with* $\mathbb{P}(R_0 < \infty) = 1$ *such that for* $r > R_0(\omega)$,

$$(\log r)^{-\alpha_2} r^3 \leq E_\omega^0 \tau_r \leq (\log r)^{\alpha_2} r^3.$$

Remarks. 1. See [4] for similar results for the IIC for the binary tree. In that case $\mathbb{P}(r \notin J(\lambda) \leq \exp(-\lambda^p)$, and the error terms were of order $\log\log$.
2. This result more or less removes the 'random walk' component from the remaining high dimensional problems for SRW on the IIC. For example, to prove the AO conjecture for ordinary percolation one would need to prove the IIC satisfies Hypothesis A. This means solving (hard) questions about the geometry of the cluster.
3. The proofs use methods developed in [3], which looks at 'strongly recurrent' graphs for which

$$V(x,r) \asymp r^\alpha, \quad x \in G, r \geq 1 \tag{6}$$

$$R_{\text{eff}}(x, B(x,r)^c) \asymp r^\delta \quad x \in G, r \geq 1. \tag{7}$$

4. Usually control of SRW on a graph requires control (e.g. in terms of volume, resistance) of *all* balls. A surprising feature of Theorem 2 is that $p_n(0,0)$ and $E^0\tau_r$ can be bounded just using information on balls $B(0,r)$, $r \geq 1$.
5. See [5] for more detailed results.

For spread-out oriented percolation we have the following result.

Theorem 3. *Let* $d > 6$, *and* $L \geq L_0(d) \gg 1$. *Then the IIC for spread-out oriented percolation satisfies Hypothesis A.*

Remarks. 1. The upper critical dimension for oriented percolation is 4, so one might expect to have this result for $d > 4$. In fact, using the results of [8, 9, 10] the 'volume bounds' part of Hypothesis A does hold when $d > 4$. However, the proof of the lower bound for the resistance required $d > 6$.
2. The situation for $d = 5, 6$ is not clear. Because the SRW on the IIC is a random walk on an unoriented graph, the SRW 'sees' connections which the oriented percolation process misses.

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Invasion percolation on regular trees

Frank den Hollander

(joint work with Omer Angel, Jesse Goodman and Gordon Slade)

We consider invasion percolation on a rooted regular tree. The edges of the tree are assigned i.i.d. uniform $(0,1)$ random variables, and an infinite cluster is grown from the root by successively adjoining the edge in the outer boundary of the cluster that carries the smallest weight. It is known that, with probability 1, $\limsup_{m\to\infty} w_m = p_c$, where w_m is the weight of the m-th edge accepted in the *invasion percolation cluster* (IPC) and p_c is the critical probability for ordinary percolation.

For the IPC, we identify the scaling behaviour of its r-point function for any integer $r \geq 2$, and of its volume both at and below a given height above the root. In addition, we derive scaling estimates for simple random walk on the IPC starting from the root. We find that, while the power laws of the scaling are the same as for the *incipient infinite cluster* (IIC) for ordinary percolation, the scaling functions differ. Thus, somewhat surprisingly, IPC and IIC have *different* scaling limits.

A key ingredient in the proofs is the following representation. The IPC can be viewed as consisting of a uniformly random infinite backbone with, for each $k \in \mathbb{N}_0$ emerging from the k-th vertex on the backbone, independent *supercritical* percolation clusters with parameter $W_k > p_c$ conditioned to stay *finite*, where W_k is the maximal weight of the edges in the backbone *above* the k-th vertex. On the tree, by duality, a supercritical percolation cluster with parameter $p > p_c$ conditioned to stay finite has the same law as a subcritical percolation cluster with a *dual* parameter $\widehat{p} < p_c$. Therefore, the IPC can be viewed as consisting of a uniformly random infinite backbone with, for each $k \in \mathbb{N}_0$ emerging from the k-th vertex on the backbone, independent *subcritical* percolation clusters with parameter $\widehat{W}_k < p_c$.

Since the IIC has the same representation but with all the emerging percolation clusters *critical*, it follows that the IPC is stochastically dominated by the IIC. Since, with probability 1, $\widehat{W}_k \uparrow p_c$ as $k \to \infty$, it also follows that far above the root IPC and IIC have the same law locally. We show that

$$\left(k\left[1 - \widehat{W}_{\lceil kt \rceil}/p_c\right]\right)_{t>0} \Longrightarrow (L(t))_{t>0},$$

where the right-hand side is the lower envelope of the Poisson process with intensity 1 on the positive quadrant. The slow decay of $\widehat{W}_k$ towards p_c causes an anomalous scaling behaviour. For instance, while for the IIC the r-point function only depends on the *total volume* of the spanning tree connecting the root with the $r - 1$ designated vertices, for the IPC the r-point function depends on the *topology* of the spanning tree and the lengths of its *segments*.

Asymptotic behavior of the critical two-point function for Ising ferromagnets above four dimensions

Akira Sakai

We consider the Ising model on the d-dimensional integer lattice $\mathbb{Z}^d$. Let Λ be a finite subset of $\mathbb{Z}^d$ containing the origin $o \in \mathbb{Z}^d$. We denote a spin variable at $x \in \Lambda$ by $\varphi_x \in \{\pm 1\}$. The thermal average of a function f at the inverse temperature $p \geq 0$ is defined by

$$\langle f \rangle_{p;\Lambda} = \frac{2^{-|\Lambda|}}{Z_{p;\Lambda}} \sum_{\varphi \in \{\pm 1\}^\Lambda} f(\varphi)\, e^{-pH_\Lambda(\varphi)}, \tag{1}$$

where $Z_{p;\Lambda}$ is the normalization and $H_\Lambda(\varphi)$ is the Hamiltonian defined as

$$H_\Lambda(\varphi) = -\sum_{\{x,y\} \subset \Lambda} J_{x,y} \varphi_x \varphi_y, \tag{2}$$

where $\{J_{x,y}\}_{x,y\in\mathbb{Z}^d}$ is a collection of spin-spin couplings.

Suppose that the spin-spin couplings are finite-range, translation-invariant and $\mathbb{Z}^d$-symmetric, and that $J_{o,x} \geq 0$ for any $x \in \mathbb{Z}^d$ (i.e., ferromagnetic). Then, for $d \geq 2$, there is a $p_c \in (0,\infty)$ such that the two-point function $G_p(x)$ defined as

$$G_p(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle\varphi_o\varphi_x\rangle_{p;\Lambda}, \tag{3}$$

decays exponentially as $|x| \uparrow \infty$ if $p < p_c$, while $G_p(x)$ is bounded away from 0 if $p > p_c$ (see [1] and references therein). It is generally believed that the critical two-point function decays in powers of $|x|$ as

$$G_{p_c}(x) \approx |x|^{-(d-2+\eta)} \qquad (|x| \uparrow \infty), \tag{4}$$

where η is a critical exponent. It is of interest to investigate this exponent, since we believe in universality: η is insensitive to the precise definition of $J_{o,x} \geq 0$.

Let

$$\tau_{x,y} = \tanh(pJ_{x,y}). \tag{5}$$

In [6] we proved the following lace expansion:

Proposition 1. *For any $p \geq 0$ and any $\Lambda \subset \mathbb{Z}^d$, there exist $\pi^{(j)}_{p;\Lambda}(x)$ and $R^{(j+1)}_{p;\Lambda}(x)$ for $x \in \Lambda$ and $j \geq 0$ such that*

$$\langle\varphi_o\varphi_x\rangle_{p;\Lambda} = \Pi^{(j)}_{p;\Lambda}(x) + \sum_{u,v} \Pi^{(j)}_{p;\Lambda}(u)\,\tau_{u,v}\langle\varphi_v\varphi_x\rangle_{p;\Lambda} + (-1)^{j+1}R^{(j+1)}_{p;\Lambda}(x), \tag{6}$$

where $\Pi^{(j)}_{p;\Lambda}(x) = \sum_{i=0}^{j}(-1)^i\,\pi^{(i)}_{p;\Lambda}(x)$. For the ferromagnetic case, we have

$$\pi^{(j)}_{p;\Lambda}(x) \geq \delta_{j,0}\delta_{o,x}, \qquad 0 \leq R^{(j+1)}_{p;\Lambda}(x) \leq \sum_{u,v} \pi^{(j)}_{p;\Lambda}(u)\,\tau_{u,v}\langle\varphi_v\varphi_x\rangle_{p;\Lambda}. \tag{7}$$

To derive the above lace expansion (with the exact expressions of the expansion coefficients $\pi^{(i)}_{p;\Lambda}(x)$ and the remainder $R^{(j+1)}_{p;\Lambda}(x)$), we exploited the so-called random-current representation and the "source-switching" lemma [3]. We should emphasize that the lace expansion is an identity, not an inequality, so that it holds for any spin-spin coupling. For example, (6) holds for a spin glass.

In [6] we were able to control the expansion coefficients and the remainder only when the spin-spin couplings are translation-invariant, $\mathbb{Z}^d$-symmetric and nonnegative. Two examples that satisfy these properties are the nearest-neighbor interaction (i.e., $J_{o,x} = \mathbb{1}_{\{\|x\|_1=1\}}$) and the following spread-out interaction:

$$J_{o,x} = L^{-d}\mu(L^{-1}x) \qquad (1 \leq L < \infty), \tag{8}$$

where $\mu : [-1,1]^d \setminus \{o\} \mapsto [0,\infty)$ is a bounded probability distribution, which is $\mathbb{Z}^d$-symmetric and and piecewise continuous so that $L^{-d}\sum_{x\in\mathbb{Z}^d}\mu(L^{-1}x)$ approximates $\int_{\mathbb{R}^d} d^dx\;\mu(x) \equiv 1$.

Proposition 2. *Let $\rho = 2(d-4) > 0$. For the nearest-neighbor model with $d \gg 1$ and for the spread-out model with $L \gg 1$, there are finite constants θ and λ such that*

$$(9) \quad |\Pi^{(j)}_{p;\Lambda}(x) - \delta_{o,x}| \leq \theta\delta_{o,x} + \frac{\lambda(1-\delta_{o,x})}{|x|^{d+2+\rho}} \quad (j \geq 1), \quad |R^{(j)}_{p;\Lambda}(x)| \to 0 \quad (j \uparrow \infty),$$

for any $p \leq p_c$ and any $x \in \Lambda \subset \mathbb{Z}^d$.

The proof of these bounds depends on bounds on the expansion coefficients in terms of two-point functions. These diagrammatic bounds are results of counting the number of "edge-disjoint connections," which correspond to applications of the BK inequality in percolation [2].

Let

$$(10) \qquad \tau \equiv \tau(p) = \sum_x \tau_{o,x}, \qquad D(x) = \frac{\tau_{o,x}}{\tau}, \qquad \sigma^2 = \sum_x |x|^2 D(x).$$

Due to (9) uniformly in $\Lambda \subset \mathbb{Z}^d$, there is a limit $\Pi_p(x) \equiv \lim_{\Lambda\uparrow\mathbb{Z}^d} \lim_{j\uparrow\infty} \Pi^{(j)}_{p;\Lambda}(x)$ such that

$$(11) \quad G_p(x) = \Pi_p(x) + (\Pi_p * \tau D * G_p)(x), \quad |\Pi_p(x) - \delta_{o,x}| \leq \theta\delta_{o,x} + \frac{\lambda(1-\delta_{o,x})}{|x|^{d+2+\rho}},$$

for any $p \leq p_c$ and any $x \in \mathbb{Z}^d$, where $(f * g)(x) = \sum_{y\in\mathbb{Z}^d} f(y)\, g(x-y)$. We note that the identity in (11) is similar to the recursion equation for the random-walk Green's function:

$$(12) \qquad S_r(x) \equiv \delta_{o,x} + \sum_{i=1}^{\infty} r^i D^{*i}(x) = \delta_{o,x} + (rD * S_r)(x),$$

where $f^{*i}(x) = (f^{*(i-1)} * f)(x)$. The leading asymptotics of $S_1(x)$ for $d > 2$ is known as $\frac{a_d}{\sigma^2}|x|^{-(d-2)}$, where $a_d = \frac{d}{2}\pi^{-d/2}\Gamma(\frac{d}{2}-1)$ (e.g., [4, 5]). Following the model-independent analysis of the lace expansion in [4, 5], we obtain the following asymptotics of the critical two-point function:

Theorem 3. *Fix any small $\epsilon > 0$, and let $d \geq 5$ and $\tilde{\epsilon}_5 = \epsilon \mathbb{1}_{\{d=5\}}$. For the nearest-neighbor model with $d \gg 1$ and for the spread-out model with $L \gg 1$, we have that, for $x \neq o$,*

$$(13) \qquad G_{p_c}(x) = \frac{A}{\tau(p_c)} \frac{a_d}{\sigma^2|x|^{d-2}} \times \begin{cases} \big(1 + O(|x|^{-2/d+\tilde{\epsilon}_5})\big) & (\textit{NN model}), \\ \big(1 + O(|x|^{-2+\epsilon})\big) & (\textit{SO model}), \end{cases}$$

where constants in the error terms may vary depending on ϵ, and

$$(14) \qquad \tau(p_c) = \bigg(\sum_x \Pi_{p_c}(x)\bigg)^{-1}, \qquad A = \bigg(1 + \frac{\tau(p_c)}{\sigma^2}\sum_x |x|^2 \Pi_{p_c}(x)\bigg)^{-1}.$$

Although we restricted ourselves in [6] to the nearest-neighbor model for $d \gg 4$ and to the spread-out model for $d > 4$ with $L \gg 1$, it is strongly expected that our method can show the same asymptotics of the critical two-point function for any finite-range model (which satisfies the required symmetries) above four dimensions, by taking the coordination number sufficiently large.

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The Airy$_1$ and Airy$_2$ processes in the TASEP

Patrik L. Ferrari

We consider a stochastic interacting particle system, the totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$ in continuous time. At any given time t, every site $j \in \mathbb{Z}$ can be occupied at most by one particle. Thus a configuration of the TASEP can be described by $\eta = \{\eta_j, j \in \mathbb{Z} | \eta_j \in \{0,1\}\}$. η_j is called the *occupation variable* of site j, which is defined by $\eta_j = 1$ if site j is occupied and $\eta_j = 0$ if site j is empty.

The dynamics of the TASEP is defined as follows. Particles jumps on the neighboring right site with rate 1 provided that the site is empty. This means that jumps are independent of each other and occur after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty.

On a macroscopic scale the density of particles $u(x,t)$ evolves deterministically according to the Burger's equation $\partial_t u + \partial_x(u(1-u)) = 0$ [15]. Therefore it is natural to focus on fluctuations properties and large deviations, which have some interesting and unexpected features. The observables analyzed in our recent works [3, 2] are the positions of given particles, which are closely related to integrated particle currents. It turns out that the observables fluctuation depends on the initial condition. Thus the natural question is to analyze which kind of initial conditions leads to a common limit distribution and limit process.

The first result in this direction has been obtained with *step initial conditions*. To be precise, let us denote by $x_k(t)$ the position at time t of the particle with label k. Then step initial condition means $x_k(0) = -k$, $k \in \mathbb{N}$, which is studied by Johansson [8, 9] in terms of a corner growth model. The positions of particles fluctuate on a $t^{1/3}$-scale while two particles are (in this scale) non-trivially correlated if they are at a distance of order $t^{2/3}$. For example,

$$\lim_{t\to\infty} \frac{x_{[t/4+u(t/2)^{2/3}]}(t) - (-2u(t/2)^{2/3} + u^2(t/2)^{1/3})}{-(t/2)^{1/3}} = \mathcal{A}_2(u) \tag{1}$$

where $\mathcal{A}_2$ is the Airy$_2$ process (usually simply called Airy process), first discovered in the polynuclear growth (PNG) model under droplet growth [13]. The 1/3 and 2/3 exponents are the one of the KPZ universality class [10]. The Airy$_2$ process is the marginal of the determinantal point process with extended Airy kernel. $\mathcal{A}_2$ appears also in Dyson's Brownian Motion [4], where the motion of the largest eigenvalue properly rescaled converges to the Airy$_2$ process [9]. In particular, the one-point distribution of $\mathcal{A}_2$ is the GUE Tracy-Widom distribution [19]. The same result holds if one focuses around $k \sim \alpha t$, $\alpha \in (0,1)$, but with different numerical factors.

Besides the step-initial condition explained above, two other situations are of particular interest. One is the *stationary* initial condition, where the one-point distribution has been obtained in [7]. The second are *deterministic* initial conditions leading to a macroscopically uniform density profile, thus called *flat initial conditions*. The simplest realization is obtained by setting $x_k(0) = -2k$, $k \in \mathbb{Z}$.

In [16] an important new result has been discovered, allowing the analysis of such initial conditions. First of all, as expected by universality, the fluctuations of the position of a particle is governed by the GOE Tracy-Widom distribution, F_1 [20]. This result is a combination of [16, 6], that is,

$$\lim_{t\to\infty} \mathbb{P}(x_{[t/4]}(t) \leq -st^{1/3}) = F_1(2s). \tag{2}$$

More importantly, for flat initial condition, the analogue of the Airy$_2$ process has been determined, which we denote by $\mathcal{A}_1$ and call Airy$_1$ process. It is the marginal of the determinantal point measure with the extended kernel K_{F_1} given as follows. Let $B_0(x,y) = \mathrm{Ai}(x+y)$ and Δ the one-dimensional Laplacian, then

$$K_{\mathrm{F}_1}(u_1,s_1;u_2,s_2) = -(e^{(u_2-u_1)\Delta})(s_1,s_2)\mathbf{1}(u_2>u_1) + (e^{-u_1\Delta}B_0 e^{u_2\Delta})(s_1,s_2). \tag{3}$$

The process $\mathcal{A}_1$ has m-point joint distributions at $u_1 < u_2 < \ldots < u_m$ given by a Fredholm determinant (regarded as its Fredholm series)

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_1(u_k) \leq s_k\}\Big) = \det(\mathbf{1} - \chi_s K_{\mathrm{F}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})} \tag{4}$$

where $\chi_s(u_i,x) = \mathbf{1}(x > s_i)$.

In [3] we analyze the continuous-time TASEP with $x_k(0) = -2k$, $k \in \mathbb{Z}$, and show that the joint distributions of particle positions are given by a Fredholm determinant of a kernel. Then in the appropriate scaling limit we obtain pointwise convergence of the kernel to K_{F_1}. The analysis starts from a determinantal formula of the joint distributions of particle position obtained by Schütz [17]. In [2] we consider the discrete-time TASEP with sequential update for which the corresponding of Schütz formula has been determined in [14]. The analogue formula for parallel update has been obtained in a recent work [11], but whether a similar approach as in [3, 2] can be applied has still to be investigated. There are other update rules introduced in the literature, but we will not discuss them. For a review, see [18].

Instead of restricting to density $1/2$ (the $d = 2$ case) we consider a more general set of initial conditions: for any integer $d \geq 2$, we take $x_k(0) = -dk$, $k \in \mathbb{Z}$. By universality it is expected that the limit process is independent of d (if $d \geq 2$). This is proven in [2], where we show convergence of Fredholm determinants too, thus convergence in the sense of finite-dimensional distributions to the Airy$_1$ process. The final result, rewritten for continuous-time TASEP, is

$$\lim_{t\to\infty} \frac{x_{\lfloor \alpha t + \mu t^{2/3}\rfloor}(t) + d\mu u t^{2/3}}{-\kappa t^{1/3}} = \mathcal{A}_1(u) \tag{5}$$

with $\kappa = \frac{2^{1/3}(d(d-1))^{2/3}}{d}$, $\alpha = \frac{d-1}{d^2}$, and $\mu = \frac{2^{5/3}(d(d-1))^{1/3}}{d^2}$.

As briefly discussed in [3], the TASEP can also be reinterpreted as a stochastic growth model, a directed last passage percolation, and a directed polymer model. Step initial conditions corresponds to point-to-point directed polymers [8, 9] and corner growth [13]. There the Airy$_2$ process appears. Flat initial condition translates into growth on a flat substrate [12, 1, 5] and point-to-line directed polymers. In particular, $d \geq 3$ is growth on a *flat but tilted surface*, and to our knowledge, the analysis of the limit distribution and/or limit process has not been carried out before for models in the KPZ class.

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An expansion for self-interacting random walks

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(joint work with Remco van der Hofstad)

1. Introduction

We introduce an expansion that is valid for random walks that are self-interacting in the sense that the transition probabilities are allowed to depend on the history of the walk, including such models as *reinforced random walks.* To be precise, for a path $\vec{\omega}_i = (\omega_0, \dots, \omega_i)$, we write $p^{\vec{\omega}_i}(x,y)$ for the conditional probability that the walk steps from $\omega_i = x$ to y, given the history of the walk $\vec{\omega}_i$.

Let $\mathbb{Q}$ denote the law of the self-interacting random walk, i.e.

$$\mathbb{Q}(\vec{\omega}_n = (x_0, x_1, \dots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{\omega}_i}(x_i, x_{i+1}).$$

The goal is to investigate the *two-point function*

$$c_n(x) = \mathbb{Q}(\omega_n = x).$$

2. Expansion

We define the concatenation of two walks into a single walk in the obvious way:

$$(\vec{\eta}_j \circ \vec{\omega}_m)(i) = \begin{cases} \eta(i) & \text{when } 0 \le i \le j, \\ \omega(i-j) & \text{when } j \le i \le m+j. \end{cases}$$

Then $c_{n+1}(x)$ can be written as

$$\sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_n^{(1)} : \omega_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}),$$

where $D(x) = p^{\emptyset}(0,x)$. Now

$$\prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}} = \prod_{i=0}^{n-1} \left[p^{\vec{\omega}_i^{(1)}} + \left(p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}} - p^{\vec{\omega}_i^{(1)}} \right) \right].$$

Applying the following expansion many times

$$\prod_{i=0}^{n-1}(a_i+b_i)=\prod_{i=0}^{n-1}a_i+\sum_{j=0}^{n-1}\Big(\prod_{i=0}^{j-1}(a_i+b_i)\Big)b_j\Big(\prod_{i=j+1}^{n-1}a_i\Big)$$

we obtain $c_{n+1}(x) = (D * c_n)(x) + \sum_{m=2}^{\infty}(\pi_m * c_{n+1-m})(x)$, where π_m is some quantity involving differences in transition probabilities

$$\Delta^{(i)}_{j_i+1} = \big(p^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}\circ\vec{\omega}^{(i)}_{j_i}} - p^{\vec{\omega}^{(i)}_{j_i}}\big)(\omega^{(i)}_{j_i}, \omega^{(i)}_{j_i+1}).$$

We then proceed with an inductive analysis of the Fourier transform of the recursion equation for $c_{n+1}(\cdot)$. The remaining ingredient of the analysis is model specific and involves estimating diagrams that arise as bounds for the "coefficients" π_m of the recursion.

3. Models and results

We apply the expansion to two models, namely *excited random walk* (ERW) in high dimensions and *once edge-reinforced random walk with drift* (OERRWd), when the excitement/reinforcement parameter is sufficiently small. Excited random walk Initially there is one cookie at each site in $\mathbb{Z}^d$. Random walker prefers step e_1 to $-e_1$ whenever he eats a cookie, i.e. the first time he visits each site. Therefore

$$p^{\vec{\omega}_i}(y, y+x) = \frac{1+\beta(e_1\cdot x)I[y\notin\vec{\omega}_{i-1}]}{2d}I[|x|=1]$$

Benjamini and Wilson [1] showed that this model has a positive drift (in the sense of $\liminf_{n\to\infty}\omega_n/n > c$) when $d > 4$, while Kozma proved this in dimensions 2 and 3 ([4] and [3]). We use the fact that the ERW is a simple random walk in the remaining $d-1$ directions in the model dependent part of the analysis to prove the following CLT for ERW.

Theorem 1. *Fix $d > 8$. There exist $\beta_0 > 0$ such that for $\beta \le \beta_0$ $\theta \in [-1,1]^d$, and Σ ,*

(a) $\mathbb{E}_\beta[\omega_n] = \theta n[1 + O\left(n^{-1}\right)]$.
(b) $\mathrm{Var}_\beta(\omega_n) = \Sigma n[1 + O\big(\frac{\log n}{n^{1\wedge\frac{d-7}{2}}}\big)]$
(c) $\frac{\omega_n-\theta n}{\sqrt{n}} \overset{d}{\Longrightarrow} \mathcal{N}(0,\Sigma)$.

Zerner [5] investigates more general classes of ERW, and his methods may enable an alternative proof of the above theorem.

The expansion is also used to prove a CLT for the OERRWd in $\mathbb{Z}^d$ (all d) in which the edges are directed and have initial weights that are translation invariant but induce a drift, and the reinforcement parameter is sufficiently small. In this case the drift remains after reinforcement and large deviations estimates can be used for the model dependent part of the analysis.

4. Remarks

Our methods also work for the boundedly-ERRWd case provided the supremum of the reinforcement is sufficiently small, and for undirected edges where the drift is in the definition of the walk itself rather than the edge weights. We could also prove a CLT for a model in $d_R + d_N$ dimensions where the edges in d_R of the dimensions are reinforced and edges in the remaining d_N dimensions are not reinforced, for $d_N > 7$.

The condition $d > 8$ in the CLT for excited random walk is practically meaningless, and we expect that we can lower this dimension by proving more appropriate diagrammatic bounds and improving the inductive analysis of the recursion equation. A major improvement in this inductive analysis would be required before our methods might yield a (self-contained) proof of a CLT for OERRW in $\mathbb{Z}^d$.

In [2] the authors show that OERRW on a tree has linear speed, and it may be possible to prove monotonicity in the speed for this model using our methods, as the drift is given explicitly in terms of π_m using our expansion.

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Rigid interfaces in systems with continuous symmetry

Senya Shlosman

(joint work with Yvon Vignaud)

The first example of a pure state describing the coexistence of phases, separated by an interface, was discovered by R. Dobrushin in 1972, [D]. There he was studying the low temperature 3D Ising model. He was considering the Ising spins in a cubic box V_N with $(\pm)$-*boundary condition* $\sigma^{\pm}$, i.e. all spins of $\sigma^{\pm}$ are $(+)$ in the upper half-space and $(-)$ in the lower half-space. Such a boundary condition forces an interface Γ into V_N, separating $(+)$-phase from the $(-)$-phase. Dobrushin have shown that in the thermodynamic limit $N \to \infty$ the distribution of Γ goes to a proper limit (in contrast with 2D case) and describes the coexistense of the $(+)$- and $(-)$-phases. His method of analysis was what is now called the cluster expansion, based on Pirogov-Sinai Contour Functional theory . Later on this approach was applied to other discrete models in [HKZ, GG].

The phenomenon of coexistense of phases of systems with continuous symmetry was not described in the literature. The purpose of the talk is to show that in dimension $d = 3$ some systems with continuous symmetry in the multyphase regime also have non-translation-invariant states, describing the coexistence of states, with the rigid interface separating them.

The talk is based on a joint work with Yvon Vignaud, where we establish the existence of the interfaces by using the restricted Reflection Positivity method.

In it, we study the so-called non-linear σ-model, considered recently in [ES1, ES2]. Its Hamiltonian is given by

$$H(\varphi) = -\sum_{\substack{x,y \in \mathbb{Z}^3, \\ |x-y|=1}} \left(\frac{1+\cos(\varphi_x - \varphi_y)}{2} \right)^p,$$

with $\varphi_x \in \mathbb{S}^1$. For p large enough this model exhibits the following behavior: at high temperatures it has unique Gibbs state (the chaotic state), while at low temperatures it presumably has infinitely many ordered Gibbs states, indexed by magnetization. Moreover – and that is the main result of [ES1] – there exists a critical temperature $T_c = T_c(p)$, at which we have the coexistence of the chaotic state and the ordered states. (Of course, all these states are translation-invariant.)

The above statement is valid in any dimension $d \geq 2$. The purpose of the talk is to explain that in dimension $d = 3$ at the critical temperature T_c the system has also non-translation-invariant states, describing the coexistance of ordered states and chaotic state, with the rigid interface, separating them.

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Birth of the critical droplet and related topics

Ostap Hryniv

(joint work with D. Ioffe and R. Kotecký)

In a recent paper by Biskup, Chayes and Kotecký [1], the authors study the phenomenon of formation/dissolution of equilibrium droplets in finite systems at values of parameters corresponding to phase coexistence. More specifically, they consider the 2D Ising model in volumes of size L^2, at inverse temperature $\beta > \beta_{\mathsf{cr}}$, and overall magnetisation conditioned to take the value $m^* L^2 - 2m^* v_L$, where β_{cr}^{-1} is the critical temperature, $m^* = m^*(\beta)$ is the spontaneous magnetisation and v_L is a certain sequence of positive numbers. The authors show that the critical scaling for droplet formation/dissolution occurs when $v_L L^{-4/3}$ converges to a definite limit. In particular, they identify a dimensionless parameter Δ proportional to this limit, a non-trivial critical value Δ_{cr} and a function λ_Δ such that the following holds: For $\Delta < \Delta_{\mathsf{cr}}$, there are no droplets beyond the $\mathcal{O}(\log L)$ scale, while for $\Delta > \Delta_{\mathsf{cr}}$ there is a single, Wulff-shaped droplet of the opposite phase with the total spin inside contributing a fraction $\lambda_\Delta \geq \lambda_{\mathsf{cr}} = 2/3$ of the overall magnetisation shift $2m^* v_L$ and there are no other droplets beyond the $\mathcal{O}(\log L)$ scale. Moreover, the parameters λ_Δ and Δ are related via a universal relation that apparently is system-independent.

The purpose of this talk is to report on a joint work with D. Ioffe and R. Kotecký [7], where the above phenomenon of emergence of the critical droplet is studied on a much finer scale. Our main results read as follows.

Theorem 1. *In the setup of the paper* [1]*, define*

$$\delta = \lim_{L\to\infty} \frac{v_L - \Delta_c\, L^{4/3}}{L^{2/3}\log L}$$

then there is a constant $\delta_{\mathsf{cr}} \in (0,\infty)$ such that in the limit $L \to \infty$ one has (a.s.):

if $\delta < \delta_{\mathsf{cr}}$: no droplets beyond the $\mathcal{O}(\log L)$ scale;

if $\delta > \delta_{\mathsf{cr}}$: a single droplet of volume

$$\lambda_\Delta v_L \geq \lambda_{\mathsf{cr}} v_L \equiv 2v_L/3$$

of "correct" shape and <u>no other</u> *droplets beyond the $\mathcal{O}(\log L)$ scale.*

Theorem 2. *In the setup of Theorem 1, define*

$$\rho = \lim_{L\to\infty} \frac{v_L - \Delta_c\, L^{4/3} - \delta_{\mathsf{cr}}\, L^{2/3}\log L}{L^{2/3}}\,;$$

then, as $L \to \infty$, the probability of observing a single droplet under the magnetisation constraint $m^* L^2 - 2m^* v_L$ is given by $\bigl(1 + a\, e^{-b\rho}\bigr)^{-1}$ with positive constants a, b determined through basic characteristics of the system.

In addition, the function λ_Δ has an expansion similar to that of $v_L\, L^{-4/3}$ with the leading term $\lambda_{\mathsf{cr}} = 2/3$.

It is instructive to compare the results above with their analogues for random graphs [5, 9]: The result of [1] as well as that of Theorem 1 are similar in spirit to the famous random graph theorem by Erdös and Rényi [5], whereas Theorem 2 is analogous to the famous "birth of the giant component" result [9].

We introduce the main ideas of the proof on the example of a simpler Self-Avoiding Polygon model, where, in addition to the sharp large deviation result for SA polygons centred at the origin and enclosing large area, one can also derive an analogue of the invariance principle for the fluctuations around the limiting shape [6].

The approach used in the proof of the main results—Theorems 1 and 2—is based upon an accurate evaluation of the contributions of fluctuations of the total spin as well as the statistics of phase boundaries [8, 2]. As a result, we obtain a sharp asymptotics of large and moderate deviations for the magnetisation of the 2D Ising model, that were previously obtained only on the logarithmic scale (cf. [3, 10, 4, 8]). Our results are valid in the whole "droplet regime" for arbitrary $\beta > \beta_{\mathsf{cr}}$ and various boundary conditions (constant, free, periodic).

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Fluctuation theory of connectivities for subcritical random cluster measures

Dmitry Ioffe

(joint work with Massimo Campanino and Yvan Velenik)

We develop a fluctuation theory of connectivities for subcritical random cluster models. The theory is based on a comprehensive non-perturbative probabilistic description of long connected clusters in terms of essentially one-dimensional chains of irreducible objects. Statistics of local observables, eg displacement, over such chains obey classical limit laws, and our construction leads to an effective random walk representation of percolation clusters.

The results include a derivation of a sharp Ornstein-Zernike type asymptotic formula for two point functions, a proof of analyticity and strict convexity of inverse correlation length and a proof of an invariance principle for connected clusters under diffusive scaling.

In two dimensions duality considerations enable a reformulation of these results for supercritical nearest-neighbour random cluster measures, in particular for nearest-neighbour Potts models in the phase transition regime. Accordingly, we prove that in two dimensions Potts equilibrium crystal shapes are always analytic and strictly convex and that the interfaces between different phases are always diffusive. Thus, no roughening transition is possible in the whole regime where our results apply.

Our results hold under an assumption of exponential decay of finite volume wired connectivities in rectangular domains that is conjectured to hold in the whole subcritical regime; the latter is known to be true, in any dimensions, when $q = 1$, $q = 2$, and when q is sufficiently large. In two dimensions our main Assumption holds whenever there is an exponential decay of connectivities in the infinite volume measure. By duality this includes all supercritical nearest-neighbour Potts models with positive surface tension between ordered phases.

CLEs

Wendelin Werner
(joint work with Scott Sheffield)

Oded Schramm has defined SLEs (Schramm-Loewner Evolutions) in [3], arguing that these are the only possible conformally invariant scaling limits of interfaces for various two-dimensional lattice models from statistical physics. This had led to numerous work that improved the mathematical understanding of these questions. Recall that an SLE is a random curve (with no self-crossings) joining two prescribed boundary points of a prescribed simply connected domain. SLEs are the only such curves that combine a conformal invariance property and a "domain Markov property". The discrete counterpart of this last property is satisfied by many discrete models, and should therefore still be valid in the scaling limit (provided that this limit exists).

Motivated by the desire to further understand the possible conformally invariant models arising in statistical physics (and for instance their relations with conformal field theory), we would like to understand not only the law of one interface, but the joint law of all interfaces in a domain. When the boundary conditions on the domain are "monochromatic" the interfaces form a family of loops. This leads us to define a property for random families of loops defined in a domain, that we call the CLE (conformal loop-ensemble) property, in the following way:

- For each simply connected domain D, a simple loop-ensemble in D is a collection $(\gamma_j, j \in J)$ of mutually disjoint self-avoiding loops in D.
- We furthermore assume that two loops can not be nested (no loop disconnect another loop from ∂D).
- Suppose now that the law P_{D_0} of a random loop-ensemble in a given simply connected domain D_0 (that is not equal to the entire complex plane) is invariant under any fixed conformal map from D_0 onto itself. Then, this allows to define for any other simply connected domain D (that is not equal to the entire complex plane) the law P_D of a conformal loop-ensemble in D, as the image of P_{D_0} under a conformal map that maps D_0 onto D. By definition, the family (P_D) then satisfies that for any D and any conformal

Figure 1. CLE in D

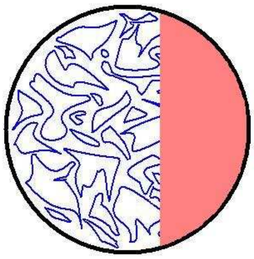

Figure 2. CLE in D intersected with D'

map $\Phi : D \to \Phi(D)$, $\Phi \circ P_D = P_{\Phi(D)}$. We say that such a random loop-ensemble is conformally invariant.

- Suppose that (P_D) is a conformally invariant random loop-ensemble. Suppose that $D' \subset D$ are two simply connected domains and that $\partial D \cap \partial D'$ contains a point (i.e. a prime end) A. Suppose that $(\gamma_j, j \in J)$ is a CLE in D. There exists two sorts of loops: Those that stay in D' (that we call $(\gamma_j, j \in J'))$ and those that do not stay in D' (that we call $(\gamma_j, j \in I)$). We call $\tilde{D}$ the connected component of $D' \setminus \cup_{j \in I} \gamma_j$ for which $A \in \partial \tilde{D}$. We say that (P_D) satisfies the CLE property if (for all such D and D'), the law of $(\gamma_j, j \in J')$ given $(\gamma_j, j \in I)$ is $P_{\tilde{D}}$.

This property is expected to hold for the scaling limits of the outermost loops of critical $O(N)$ models, Ising model on the triangular lattice etc. This property is rather simple to sketch on this sequence of three pictures...

The main result that I want to present is that:

Theorem[Sheffield-W. [6]]. *There exists exactly a one-parameter family of random simple loop-ensembles that satisfy the CLE property. More precisely, for*

Figure 3. CLE in $\tilde{D}$

each $d \in (4/3, 3/2]$ there exists exactly one CLE such that a.s. the loops have all a Hausdorff dimension d (and are SLE$_\kappa$-loops with $\kappa \in (8/3, 4]$ such that $d = 1 + \kappa/8$). And these are the only random simple loop-ensembles that satisfy the CLE property.

The proof of this theorem has two quite different parts:

- First, we show that if a family satisfies the CLE property, then the loops in the CLE can be defined as SLE$_\kappa$ loops for some $\kappa \in (8/3, 4]$. To do this, one proceeds in several steps that we sketch in the talk. The first one is to show that the CLE-family is characterized by an (infinite) measure supported on the set of loops (here one has just one loop) that touch the boundary of a given D at just one given point z_0. Then, we show that the CLE property implies that this infinite measure is the excursion measure of some SLE process for a parameter $\kappa \in (8/3, 4]$.
- The second part is to construct explicitly a one-parameter family of CLEs. This is done by means of the fractal percolation procedure with Brownian loops, as described in [7]. More precisely, consider the outermost boundaries of clusters of loops of a Poisson point process of loops in D defined under the intensity $c\mu_D$ where μ_D is the measure on self-avoiding loops in D as defined in [8] (or equivalently via the Brownian loop-measure defined in [2]) and $c > 0$. It can be shown that for each subcritical c (i.e. for each c such that the above procedure does construct loop-ensembles, and this is true for small enough c), this random loop-ensemble does satisfy the CLE property. Hence, by the first step, it corresponds to one of the SLEs. But the distortion properties of SLEs as derived in [1], and further considerations on this loop-soup yield that the relation between c and κ is $c(\kappa) = (6 - \kappa)(3\kappa - 8)/2\kappa$ and that all $c \leq 1$ are subcritical.

This allows to show that the exploration trees as defined in [5] via the SLE$(\kappa, \kappa-6)$ processes define also random loop-ensembles that satisfy the CLE property, and

–combined with the upcoming work of Schramm and Sheffield [4]– it could also show that these CLEs can be found in Gaussian Free Fields.

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Critical percolation in "other" 2D lattices and 2D Incipient Infinite Cluster estimates

Vincent Beffara

In recent years, the mathematical understanding of critical two-dimensional systems in statistical mechanics has improved dramatically with the introduction by Schramm of SLE processes. However, the number of such models for which a formal proof of convergence to SLE in the scaling limit is still rather limited (the only cases where a general proof is known are loop-erased random walks, uniform spanning trees and level lines of Gaussian free fields).

It was a spectacular result by Smirnov [5] that crossing probabilities of critical site-percolation on the regular triangular lattice do indeed converge, in the scaling limit, to values predicted by Cardy, and this constitutes the main step in a formal proof that the percolation exploration path converges to the trace of an SLE_6 process (see [2]).

Smirnov's proof is quite miraculous, and uses features of the geometry of the triangular lattice several times. This dependence of the proof itself on the details of the model can be weakened a little, but nevertheless it is very difficult to extend it to other cases; and the one such extension that was achieved by Chayes and Lei (see [3] and the next abstract in the present volume), while extremely involved, still uses a lattice with the same symmetries as the triangular lattice.

The main question is then that of *universality*, *i.e.* determining whether the scaling limit of critical percolation in other 2D lattices (exists and) is the same, and in particular whether Cardy's formula holds in more generality. The aim of this talk is to present a possible approach to this question, relating a partial universality result for a family of lattice models to a specific estimate for the rate of convergence in the construction of an IIC-like object.

Start with Bernoulli bond-percolation on the square lattice $\mathbb{Z}^2$. We want to relate it to site-percolation on a regular triangulation of the plane; let Λ_0 be the *covering graph* of $\mathbb{Z}^2$, defined as follows: The vertex set of Λ_0 is the set of edges of $\mathbb{Z}^2$, and two such edges are connected by an edge of Λ_0 if they share an endpoint. Λ_0 is a square lattice on which every second face is replaced by a complete 4-vertex graph (see Fig. 1); it is very easy to check that bond-percolation on $\mathbb{Z}^2$ and site-percolation on Λ_0 are exactly equivalent.

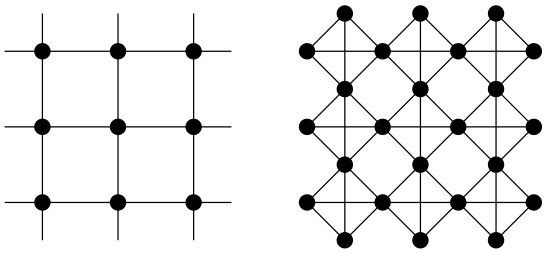

FIGURE 1. The graph $\mathbb{Z}^2$ and its covering graph Λ_0 — apparent edge intersections that are not marked by a black disc are not vertices of Λ_0, which is not planar.

Since we like to work on planar triangulations, add a vertex at the center of each of the squares of Λ_0, to get the usual square lattice with centered faces Λ. Let P_q be the product of Bernoulli measures over the vertices of Λ, each site having parameter $1/2$ if it corresponds to an original edge of $\mathbb{Z}^2$, q if it is at the center of one of the empty squares of Λ_0 (we will call such a vertex a site of type II), and $1-q$ otherwise (and the remaining vertices we call sites of type III).

The measure P_q is critical, in the sense that one gets Russo-Seymour-Welsh *a priori* estimates for crossing probabilities; clearly P_0 is exactly equivalent to the initial bond-percolation probability measure, while $P_{1/2}$ is usual critical site-percolation on a regular triangulation of the plane and lends itself to at least part of Smirnov's approach.

Now fix $a, b > 0$ and let B_n be an $an \times bn$ box; let $f_n(q)$ be the probability, under measure P_q, that there is a chain of open vertices connecting the two sides of B_n having length an. The partial version of universality we are interested in is then the following: Assume that the sequence of functions (f_n) has a limit f; we say that we have partial universality if f is constant. One way to ensure this is to show that f'_n goes to 0 uniformly in q.

Luckily, we have a version of Russo's formula for f'_n, namely:

$$\partial_q f_n(q) = \sum_{z \in V(\Lambda) \cap B_n} \varepsilon_z P(z \text{ is pivotal}),$$

where ε is $+1$ for sites of type II, -1 for sites of type III and 0 otherwise. Here as usual, a pivotal site is a site z whose state, conditionally to the rest of the system, determines whether the crossing event occurs (if z is opened) or not (if z is closed). This is equivalent to the existence, starting from the 4 neighbors of z, of 4 disjoint chains of vertices, connecting them to the 4 sides of B_n, and each being composed of either open sites (for the two going to the sides of length an), or of closed sites (for the other two). Such an event is commonly referred to as the existence of 4 *arms.*

In order to get local cancellations in the above sum, one needs to compare a typical pivotal site of type II to one of site III far away from the box boundary. This is closely related to the following problem. Let ν_n be the measure obtained by conditioning P_q to producing 4 such arms between the origin and a square of diameter $2n$ centered at 0. For every cylindrical event A, it is possible to show, following Kesten ([4]), that $\nu_n(A)$ has a limit $\nu(A)$ as n goes to infinity, and that ν extends to a probability distribution on configuration in the whole plane; we call it the four-armed incipient infinite cluster, or IIC_4 for short.

Let $\mathcal{F}_n$ be the σ-field of events depending only on the state of vertices within the square of diameter $2n$ centered at 0. From Kesten's constructions follows that, as $m \to \infty$,

$$\sup_{A \in \mathcal{F}_n} |\nu_m(A) - \nu(A)| \leqslant C\, e^{-c\sqrt{\log(m/n)}}.$$

This is not small enough to ensure cancellation in Russo's formula; indeed, one would need an estimate of the form

$$\sup_{A \in \mathcal{F}_n} |\nu_m(A) - \nu(A)| \leqslant C\, \left(\frac{n}{m}\right)^{\zeta_4}$$

with ζ_4 large enough.

As it turns out, such an estimate would follow if one knew that crossing probabilities converged to Cardy's formula, for one could then use SLE methods. More specifically, in the case of conditioning by the existence of 2 arms instead of 4, the corresponding SLE event would be that the trace of a radial SLE_6 in the disc touches a small circle around 0 before disconnecting 0 from the unit circle. Then ζ_2 would be expressed in terms of the difference between the first two eigenvalues of the generator of the following diffusion, with Dirichlet boundary conditions on $[0, 2\pi]$:

$$\mathrm{d}X_t = \sqrt{6}\,\mathrm{d}B_t + \mathrm{cotan}\frac{X_t}{2}\,\mathrm{d}t.$$

These eigenvalues can be exactly computed, and one gets

$$\lambda_k = \frac{(3k+2)^2 - 1}{12}.$$

Those are the exponents describing the decay of the existence probabilities for $3k+2$ arms, in other words the error term when approximating the IIC measure by a finitely conditioned one is given in terms of the probability of a percolation event. Understanding why this is the case would certainly lead to an SLE-free proof, and this in turn would imply that partial universality as described earlier does in fact hold in this particular setup. However, such a combinatorial interpretation is still missing.

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A Universality Result for 2D Percolation Models

Lincoln Chayes

(joint work with Helen K. Lei)

This abstract summarizes the talk concerning [1] that was delivered by one of us (L.C.) at the 2006 Oberwolfach meeting *Spacial Random Processes and Statistical Mechanics.* The starting point of the research – from a certain perspective – is the seminal result of S. Smirnov [4]. In this work, Smirnov showed that the so–called *Carleson–Cardy functions*, functions related to crossing probabilities in a (conformal) triangle, had a particular scaling limit: They are harmonic functions obeying certain boundary conditions on any triangle. These boundary conditions are asymptotically satisfied in the percolation problem defined on the discretization of any triangle and hence, if some version of harmoniticity or analyticity can be established at the discrete level, conformal invariance follows. Smirnov addressed this problem for the critical site percolation model on the triangular lattice. For this geometry, an approximate version of the Cauchy–Riemann equations among the triple of the Cardy–Carleson functions was demonstrated by exploiting the exact color symmetry of the random hexagon tiling realization of this problem.

But the purported "power" of scaling theory for critical models is the notion that *all* models of a similar type should have the same scaling behavior – universality. Till now, there has been little substantive progress in this direction; the talk concerned a (modest) example where this form of universality was demonstrated to be the case.

From our perspective, this work began a while ago with the investigation of bond percolation – and general q–state Potts and random cluster models – on the triangular *bond* lattice [2]. The relevant ingredient from [2] is a straightforward perspective on the somewhat mysterious duality relation for the bond problem on this lattice. The key is to abandon independence on any single triangle while keeping disjoint up–pointing triangles independent. Thus, as far as connectivity is concerned, there are only five (rather than eight) relevant configurations a single such triangle: full, empty, and three single bond events. Parametrizing the model in terms of the respective probabilities of these events a, e, and s, with $a+e+3s=1$, it is not hard to see that the duality/criticality condition is simply $a=e$. (Much of this had been realized in the physics community, especially in the context of spin systems where the local correlations are represented by three–body interactions. See [5] and the references therein.) The desirable feature of this perspective is that the breakdown maps directly into a modified hexagon tiling of the plane. The "a" and "e" configurations correspond to *yellow* and *blue* hexagons while the three "s" configurations correspond to three half yellow and half blue configurations in which the hexagon is split at the midpoints of two opposing edges. It is noted that *not* all of the possible splits appear, ergo the model does *not* enjoy full color reversal symmetry. This and other features render the full bond problem too hard, at least for the present, and we have to make certain modifications.

The model we treat will be described presently; first let us first fix some parlance. We call a hexagon surrounded by 6 other hexagons a *flower*. We call the middle hexagon an *iris* and the six surrounding hexagons *petals*. Here we first restrict attention to configurations where the hexagons allowed to exhibit the mixed configuration are insulated from each other and second we introduce local correlations forbidding certain configurations on the flowers. Explicitly, our model is defined as follows: Tile the plane with hexagons, some of which are designated to be irises in such a way that flowers are disjoint. The iris can be yellow, mixed or blue with the appropriate probability, petals and all other hexagons are only allowed to be pure blue or pure yellow. Finally, in certain *triggering* configurations, when there are exactly three blue petals, exactly two of which are contiguous, we forbid the iris from exhibiting the mixed configuration.

What has been accomplished with the flower model is the restoration of a strictly local version of self–duality: E.g., if one asks that a designated subset of petals be blue and blue connected inside the flower, then that has the same probability as if one asked the question in yellow. Furthermore, the model enjoys many of the advantages afforded by a hexagon tiling of the plane. The work therefore divided into two episodes which in practice were not disjoint: First establishing that the model as described display the standard features of a lattice percolation model at criticality and second using these critical ingredients and the underlying hexagonal/triangular geometry in an adaptation of the methods of [4].

The crucial device, for both of the episodes, is an object that is termed a *path designate*, which, in a statistical sense, replaces the notion of a microscopic path. A path designate is, as a geometric object, a collection of paths. These paths all agree with each other (and the microscopic definition) on the complement of flowers. However, within the flowers, only the entrance and exit petals are specified along with the requirement that they be somehow connected within. Therefore, two paths in the same path designate which goes through the same flower may exhibit *different* microscopic connections within the flower. The path designation event, which comes in two colors, is that some microscopic realization of the path designate is monochrome. Since the existence of a path designation event is equivalent to the existence of an actual microscopic connection, a preliminary result which goes a long way towards ingredients necessary for both episodes is the following:

Theorem 1. *Consider the model as described above with arbitrary placement of the flowers. Let* $\mathbf{r}$, $\mathbf{r}'$ *denote points (hexagons) which are not irises. Then the probability of a connection between* $\mathbf{r}$ *and* $\mathbf{r}'$ *is the same in blue as it is in yellow.*

Theorem 1 goes much of the distance to proving the critical ingredients, but two additional difficulties had to be overcome: The RSW lemmas and the FKG property. With regards to the former, which demonstrates a weak scale invariance of crossing probabilities, we are forced into periodic arrangements of flowers so as to use the standard arguments [3] to establish these properties. An input to the RSW lemmas is a set of correlation inequalities known as the FKG property. For independent percolation models, these inequalities hold for all increasing events, a property that was irrevocably lost when we introduced the triggers. However, under the assumption $ae \geq 2s^2$ (a condition that was borrowed from [2]), we are able to prove a curtailed but sufficient version of the FKG property, which holds for path type events.

As for episode two, the fundamental difficulty in extending the result of [4] to other lattices is to establish the so–called Cauchy–Riemann relations which relate pieces of the discrete derivatives of the Cardy–Carleson functions to one another. These relations were established in [4] via the ability, at the microscopic level, to switch colors without changing the probabilities of events. In particular, the needed extension of Theorem 1 is to establish that probabilities of connections are the same for both colors in the (conditioned) presence of paths of fixed color. It is easy to demonstrate that this is patently false for the model at hand. New machinery has to be developed.

Conditional color symmetry in our model is disrupted on the level of single flowers: In particular this means that the probability of, say, a connection from one petal to another in the presence of conditioned petals has a bigger probability in blue than it does in yellow. (The conditioned petals are fixed and may be blue or yellow; ostensibly they belong to another path.) If the conditioned petals are blue, then we may forbid, as a random event, any blue path from touching them. The probability of this event is set so as to lower the overall probability of a blue "transmission" down to that of a yellow. On the other hand, if the conditioned petals are yellow, we could allow a yellow path to share the conditioned petals, again with the appropriate probability. Ultimately what is needed is a set of random variables, whose distribution is strongly coupled with the underlying percolation configurations, which provide or deny "permissions" for transmissions in these conditioned situations. We call these $*$–rules. The upshot is there is a theorem along the lines of Theorem 1, but now the "connection" takes place in the presence of pre–conditioned paths. As is clear from the above discussion, the definition of connection, disjoint, etc. has to be rethought with the notion of these permissions taken into account.

The penultimate step is to substitute the $*$–ruled versions of path events in the definition of the Cardy–Carleson functions. Some work is required to show that these $*$–functions obey the appropriate Cauchy–Riemann relations (boundary conditions are still satisfied) but they are not, after all, exactly the objects of interest. However, a detailed analysis of the configurations which lie in the symmetric difference of the starred and unstarrred versions show that the difference between the two vanishes (in the L^1 sense) as the mesh scale tends to zero. The desired result is therefore recovered for the original Cardy–Carleson functions.

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Percolation-like problems in the study of forest-fires

J. van den Berg
(joint work with R. Brouwer and B. Vágvölgyi)

1. Background and motivation

The self-destructive percolation model on the square lattice is described as follows: First we perform independent site percolation on this lattice: we declare each site *occupied* with probability p, and *vacant* with probability $1-p$, independent of the other sites. We write $\{V \leftrightarrow \infty\}$ for the event that there is an infinite occupied path starting at V.

Let, as usual, $\theta(p)$ denote the probability that a given site, say $O = (0,0)$, belongs to an infinite occupied cluster. It is well-known that there is a critical value $0 < p_c < 1$ such that $\theta(p) > 0$ for all $p > p_c$, and $\theta(p) = 0$ for all $p \leq p_c$. Now suppose that, by some catastrophe, the infinite occupied cluster (if present) is destroyed; that is, each site in this cluster becomes vacant. Further suppose that after this catastrophe we give the sites independent 'enhancements', as follows: Each site that was already vacant in the beginning, or was *made* vacant by the catastrophe, becomes occupied with probability δ, independent of the others. Let $\mathcal{P}_{p,\delta}$ be the distribution of the final configuration.

We use the notation $\theta(p,\delta)$ for the probability that, in the final configuration, O is in an infinite occupied cluster:

$$\theta(p,\delta) := \mathcal{P}_{p,\delta}(O \leftrightarrow \infty).$$

Note that O is occupied in the final configuration if and only if the above mentioned enhancement was successful, or O belonged initially (before the catastrophe) to a non-empty but finite occupied cluster. This gives

$$\mathcal{P}_{p,\delta}(O \text{ is occupied }) = \delta + (1-\delta)(p - \theta(p)).$$

Also note that, in the case that $p \le p_c$, nothing happens in the above catastrophe, so that in the final configuration the sites are independently occupied with probability $p + (1-p)\delta$. In particular, if $p \le p_c$, then

$$\theta(p_c,\delta) = \theta(p_c + (1-p_c)\delta) > 0, \tag{1}$$

for each $\delta > 0$.

It turns out (see Proposition 3.1 of [1]) that, if $p > p_c$, a 'non-negligible' enhancement is needed after the catastrophe to create again an infinite occupied cluster. More precisely, for each $p > p_c$ there is a $\delta > 0$ with $\theta(p,\delta) = 0$. A much more difficult question is whether the needed enhancement goes to 0 as $p \downarrow p_c$. By (1) one might be tempted to reason intuitively that this is indeed the case. In [1] it was shown that for the analogous model on the binary tree this is correct. However, in [1] a conjecture is presented which says, in particular, that for the square lattice (and other planar lattices) there is a $\delta > 0$ for which $\theta(p,\delta) = 0$ for all $p > p_c$. In Section 4 of [1] and in [2] we showed remarkable consequences for certain forest-fire models.

Note that, since $\theta(p_c,\delta) > 0$, the above conjecture says that the function $\theta(\cdot,\cdot)$ has discontinuities at points of the form (p_c,δ) with δ sufficiently small. This naturally raises the question whether this function is continuous in the complement of a region of such form: is there a $\delta > 0$ such that $\theta(\cdot,\cdot)$ is continuous outside the set $\{p_c\} \times [0,\delta]$? We have proved in [3] that this is indeed the case. The next section gives a formal statement of this result and some remarks about the proof.

2. A (partial) continuity theorem and some remarks on its proof

The conjecture mentioned in the previous subsection raises the natural question whether $\theta(\cdot,\cdot)$ *is* continuous outside the indicated 'suspected' region. The following theorem states that this is indeed the case.

Theorem 1. *There is a $\delta \in (0,1)$ such that the function $\theta(\cdot,\cdot)$ is continuous outside the segment $\{p_c\} \times (0,\delta)$.*

A rough outline of the proof of Theorem 1 is as follows: First we show that if $\theta(\cdot,\cdot)$ is strictly positive in some open region, then it is continuous on this region. It is also shown that if $\theta(p,\delta) = 0$, then $\theta(\cdot,\cdot)$ is continuous at (p,δ). These two results reduce the proof of Theorem 1 to showing that if $\theta(p,\delta) > 0$ and $p \neq p_c$, then $\theta(p,\delta) > 0$ in an open neighborhood of (p,δ). This in turn requires a suitable finite-size criterion for sdp. Finally, to obtain such criterion one needs a result which, roughly speaking, says that if for each n the probability of crossing an $n \times n$ square is large, then also the probability of (for instance) crossing a given $2n \times n$ square (in the 'difficult' direction) is large. For ordinary percolation such result exists since the late seventies: it is (a version of) the RSW theorem. Unfortunately, if one tries to adapt the proof of that theorem to our self-destructive model, very serious difficulties appear, due to the dependencies in the model. However, we could reach our goal (and complete the proof of Theorem 1) by modifying (and slightly strengthening) a recent box-crossing theorem of Bollobás and Riordan (see [4], which they had used for the so-called Voronoi percolation model.

The finite-size criterion we obtain differs in a subtle way from the one for ordinary percolation: the box involved must have a size depending on p, and it turns out that the required size goes to infinity as p approaches p_c from above. This is why Theorem 1 does not work for $p = p_c$ (when δ is small).

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Dynamical models for circle covering: Brownian motion and Poisson updating

Johan Jonasson

(joint work with Jeffrey Steif)

1. The classical (static) circle covering model.

Let C denote the circle with circumference 1 and consider a decreasing sequence $\{\ell_n\}_{n\geq 1}$ of positive numbers approaching 0. Let $\{U_n\}_{n\geq 1}$ be a sequence of independent random variables each of which is uniformly distributed on C. Let I_n be the open arc of C with center point U_n and length ℓ_n. Put $E := \limsup_n I_n$ and $F := E^c$. It follows immediately from the Borel-Cantelli Lemma that for each $x \in C$, $P(x \in E) = 1$ if and only if $\sum_{n=1}^{\infty} \ell_n = \infty$. Fubini's Theorem yields immediately that in this case F has Lebesgue measure 0 a.s. In 1956, Dvoretzky (see [1]) raised the question of whether in the $\sum_n \ell_n = \infty$ case it was possible that F was nonempty and gave examples where this occurred. There were a number of various contributions to this question with the final result proved by Shepp (1972):

Theorem 1 (L. Shepp)**.** *$P(F = \emptyset) = 1$ if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{\ell_1 + \ldots + \ell_n} = \infty.$$

In particular if $\ell_n = c/n$ for all n, then $P(F = \emptyset) = 1$ if and only if $c \geq 1$.

The special cases $\ell_n = c/n$ for a constant c were known earlier. For background, proofs and other results (concerning among other things the Hausdorff dimension of F), see Kahane [2].

2. The dynamical circle covering model.

In this work, we consider two dynamical variants of the above problem. In the first of these models, each of the centers U_n perform independent Brownian motions on C, each with variance 1. In the second model, we associate independent Poisson processes with the different intervals, where the Poisson process associated with the nth interval has intensity $\ell_n^{-\alpha}$ for some real parameter α. At the times of the Poisson process associated to the nth interval, I_n is given a new center, chosen uniformly on C, independent of everything else We refer to the two models as the Brownian model and the Poisson model respectively.

We then ask for each of these two models if there are exceptional times at which we see different "covering behavior" from that which is seen in the static model. We have potentially four (or even more) different types of exceptional times, depending on the ℓ_n's and which of the two models we are looking at.

- (I) times when a fixed point is not covered even though $\sum_n \ell_n = \infty$,
- (II) times when the circle is not fully covered even though $\sum_n e^{\ell_1+\ell_2+\ldots+\ell_n}/n^2 = \infty$,
- (III) times when a fixed point is covered i.o. even though $\sum_n \ell_n < \infty$,
- (IV) times when the circle is fully covered i.o. even though $\sum_n e^{\ell_1+\ell_2+\ldots+\ell_n}/n^2 < \infty$,

Types (III) we consider less important and for type (IV) we have no results so far. Therefore this talk is focused on exceptional times of type (I) and (II). For all our results we assume that the arc lengths decrease reasonably nicely, i.e.

$$\exists M_0, M_1 : 0 < M_0 \leq M_1 < \infty : \forall n : M_0 \leq n\ell_n \leq M_1. \tag{1}$$

For a fixed point $x \in C$, it follows immediately from the Borel-Cantelli Lemma that $P(x \in F) = 0$ and hence for any of the dynamical models, by Fubini's Theorem, $\{t : x \in F_t\}$ has Lebesgue measure 0 a.s. The question we address first is when there are exceptional times t at which x is covered by only finitely many of the $I_{n,t}$'s; i.e., $x \in F_t$. See [2] for the definition of Hausdorff dimension which we denote here by HD.

Theorem 2. *Assume that (1) holds. Consider the Brownian model. Fix $x \in C$ and let $u_n := \prod_{k=1}^{n}(1 - \ell_k)$.*
(i). If $\liminf_n n^2 u_n < \infty$, then $P(\exists t \in [0,1] : x \in F_t) = 0$. In particular, if $\ell_n = c/n$ for all n, then this holds if $c \geq 2$.

(ii). If $\sum_{n=1}^{\infty} e^{\ell_1+\ell_2+\ldots+\ell_n}/n^3 < \infty$, *then* $P(\exists t \in [0,1] : x \in F_t) = 1$. *In particular, if* $\ell_n = c/n$ *for all* n, *then this holds if* $c < 2$.
(iii). Let

$$\beta_0 := \inf\{\beta : \sum_{n=1}^{\infty} \frac{e^{\ell_1+\ell_2+\ldots+\ell_n}}{n^{1+\beta}} < \infty\}.$$

Then

$$\mathrm{HD}(\{t \in [0,1] : x \in F_t\}) = (1 - \frac{\beta_0}{2}) \wedge 0 \quad a.s.$$

In particular, in the case $\ell_n = c/n$ *for all* n *with* $c \leq 2$, *we have*

$$\mathrm{HD}(\{t \in [0,1] : x \in F_t\}) = 1 - \frac{c}{2} \quad a.s.$$

The Poisson model turns out to be very amenable to our analysis and we obtain an exact condition for having exceptional times of type (I).

Theorem 3. *Assume that (1) holds. Consider the Poisson model with parameter* $\alpha > 0$. *Fix* $x \in C$.
(i). Then $P(\exists t \in [0,1] : x \in F_t) = 1$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{e^{\ell_1+\ell_2+\ldots+\ell_n}}{n^{1+\alpha}} < \infty.$$

In particular, if $\ell_n = c/n$ *for all* n, *then this holds if and only if* $c < \alpha$.
(ii). Let, as in Theorem 2,

$$\beta_0 := \inf\{\beta : \sum_{n=1}^{\infty} \frac{e^{\ell_1+\ell_2+\ldots+\ell_n}}{n^{1+\beta}} < \infty\}.$$

Then

$$\mathrm{HD}(\{t \in [0,1] : x \in F_t\}) = (1 - \frac{\beta_0}{\alpha}) \wedge 0 \quad a.s.$$

In particular, in the case $\ell_n = c/n$ *for all* n *with* $c \leq \alpha$, *we have*

$$\mathrm{HD}(\{t \in [0,1] : x \in F_t\}) = 1 - \frac{c}{\alpha} \quad a.s.$$

The next two results deal with the question of exceptional times of type (II).

Theorem 4. *Assume that (1) holds. Consider the Brownian model and let* u_n *be defined as in Theorem 2.*
(i). If $\liminf_n n^3 u_n < \infty$, *then*

$$P(\exists t \in [0,1] : F_t \neq \emptyset) = 0.$$

In particular if $\ell_n = c/n$ *for all* n, *then this holds if* $c \geq 3$.

(ii). If $\sum_{n=1}^{\infty} e^{\ell_1+\ell_2+\ldots+\ell_n}/n^4 < \infty$*, then*

$$P(\exists t \in [0,1] : F_t \neq \emptyset) = 1.$$

In particular if $\ell_n = c/n$ *for all* n*, then this holds if* $c < 3$*.*
(iii). Let, as in Theorem 2,

$$\beta_0 := \inf\{\beta : \sum_{n=1}^{\infty} \frac{e^{\ell_1+\ell_2+\ldots+\ell_n}}{n^{1+\beta}} < \infty\}.$$

Then a.s.

$$\text{(a)}\ \ \mathrm{HD}(\{(t,x) : x \in F_t\}) = \begin{cases} 2 - \frac{\beta_0}{2} & \textit{if } 0 \le \beta_0 \le 2 \\ 3 - \beta_0 & \textit{if } 2 \le \beta_0 \le 3 \\ 0 & \textit{if } \beta_0 \ge 3 \end{cases}$$

$$\text{(b)}\ \ \mathrm{HD}(\{x : \exists t : x \in F_t\}) \begin{cases} = 1 & \textit{if } 0 \le \beta_0 < 2 \\ \le 3 - \beta_0 & \textit{if } 2 \le \beta_0 \le 3 \\ = 0 & \textit{if } \beta_0 \ge 3 \end{cases}$$

$$\text{(c)}\ \ \mathrm{HD}(\{t : F_t \neq \emptyset\}) \begin{cases} = 1 & \textit{if } 0 \le \beta_0 < 1 \\ \le \frac{3-\beta_0}{2} & \textit{if } 1 \le \beta_0 \le 3 \\ = 0 & \textit{if } \beta_0 \ge 3 \end{cases}$$

In particular, in the case $\ell_n = c/n$ *for all* n *and* $c < 3$*, then the dimension bounds are simply obtained by plugging in* c *for* β_0*.*

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On the limiting velocity of high dimensional random walk in random environment

Noam Berger

Let $d \ge 1$. A Random Walk in Random Environment (RWRE) on $\mathbb{Z}^d$ is defined as follows: Let $\mathcal{M}^d$ denote the space of all probability measures on $\{\pm e_i\}_{i=1}^d$ and let $\Omega = \left(\mathcal{M}^d\right)^{\mathbb{Z}^d}$. An *environment* is a point $\omega \in \Omega$. Let P be a probability measure on Ω. For the purposes of this paper, we assume that P is an i.i.d. measure, i.e.

$$P = Q^{\mathbb{Z}^d}$$

for some distribution Q on $\mathcal{M}^d$ and that P is *uniformly elliptic*, i.e. there exist $\epsilon > 0$ s.t. for every $e \in \{\pm e_i\}_{i=1}^d$,

$$Q(\{d : d(e) < \epsilon\}) = 0.$$

For an environment $\omega \in \Omega$, the *Random Walk* on ω is a time-homogenous Markov chain with transition kernel

$$P_\omega\left(X_{n+1} = z + e \middle| X_n = x\right) = \omega(z, e).$$

The **quenched law** P_ω^z is defined to be the law on $\left(\mathbb{Z}^d\right)^{\mathbb{N}}$ induced by the kernel P_ω and $P_\omega^z(X_0 = z) = 1$. We let $\mathbf{P} = P \otimes P_\omega^0$ be the joint law of the environment and the walk, and the **annealed** law is defined to be its marginal

$$\mathbb{P} = \int_\Omega P_\omega^0 dP(\omega).$$

We consider the limiting velocity

$$S = \lim_{n\to\infty} \frac{X_n}{n}.$$

Based on the work of Zerner [4] and Sznitman and Zerner [3] we know that S exists $\mathbb{P}$-a.s. Further more, there is a set A of size at most two such that almost surely $S \in A$.

Zerner and Merkl [5] proved that in dimension two a 0-1 law holds and therefore the set A is of size one, i.e. a law of large numbers hold in dimension two (see also [2] for a continuous version).

The main result presented is as follows:

Theorem 1. *For $d \geq 5$, there is at most one non-zero limiting velocity, i.e. if $A = \{S_1, S_2\}$ and $S_1 \neq 0$ then $S_2 = 0$.*

Theorem 1 has the following immediate corollary:

Corollary 2. *For $d \geq 5$, if the distribution is distributionally symmetric, then the limiting velocity is an almost sure constant.*

Based on Theorem 1, we can also establish various connections between some of the important open problems in the field.

The talk is based on the paper [1].

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Majorizing multiplicative cascades for directed polymers in random media

Francis Comets
(joint work with Vincent Vargas)

Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting at 0, defined on a probability space $(\Omega, \mathcal{F}, P)$. We also consider a sequence $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on another probability space $(H, \mathcal{G}, Q)$ with finite exponential moments. The path ω represents the directed polymer and η the random environment. The polymer is weighted according to the total reward it collects on his way,

$$H_n(\omega) \stackrel{\text{def.}}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

For any $n > 0$, we define the polymer partition function by

$$Z_n = P[\exp(\beta H_n(\omega))]$$

where $\beta \in \mathbb{R}^+$ is the inverse temperature. We use the notation $P[X]$ for the expectation of a random variable X. The free energy of the polymer is defined as the limit

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln(Z_n(\beta)/Q[Z_n(\beta)]) \tag{1}$$

where the limit exists Q-a.s. and in L^p for all $p \geq 1$ and is non-random. An application of Jensen's inequality to the concave function $\ln(\cdot)$ yields $p(\beta) \leq 0$. As shown in [1], there exists a $\beta_c \in [0, \infty]$ such that

$$p(\beta) \quad \begin{cases} = 0 & \text{if} \quad \beta \in [0, \beta_c], \\ < 0 & \text{if} \quad \beta > \beta_c. \end{cases}$$

Determining the critical value β_c is an important question in the study of directed polymers. In fact, for $\beta > \beta_c$ (and only in this case), the polymer localizes in narrow corridors with positive probability. It is not known how to characterize directly these corridors, and therefore this criterion for the transition localization/delocalization is rather efficient since it does not require any knowledge on them. Hence, it is of primer importance to get good upper bounds on p in order to spot the transition.

In this work, we find a family of upper bounds for p, given by the free energies $p_m^{\text{tree}}(\beta)$ of models on trees depending on an integer parameter m ($m \geq 1$). These trees are deterministic and regular, with random weights, they fall in the scope of the generalized multiplicative cascades [3] or smoothing transformations [2] which are well known generalizations of the random cascades introduced by Mandelbrot for a statistical description of turbulence. When the environment variables have nice concentration properties – e.g., gaussian or bounded η's –, we prove that the polymer free energy is the infimum over m of the one of the m-tree model.

An important corollary is:

In dimension $d = 1$, $\beta_c = 0$.

There is a clear consensus on this fact in the physics literature, but the present proof seems to be the first one.

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Quenched invariance principle for random walks in random environment admitting a bounded cycle representation

Jean-Dominique Deuschel

(joint work with Holger Kösters)

We consider a class of random walks admitting a bounded cycle decomposition. That is, a random walk $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ such that

$$p_1(x,y) = P(X_{n+1} = y | X_n = x) = \frac{1}{\pi(x)} \sum_{\gamma\in\mathcal{C}} w(\gamma) 1_{(x,y)\in\gamma},$$

where $\mathcal{C} = \{\gamma = (x_0, x_1, ..., x_n = x_0), x_i \in \mathbb{Z}^d\}$ is a family of oriented cycles, $w(\gamma) \geq 0$ is the weight of the cycle γ and $\pi(x) > 0, x \in \mathbb{Z}^d$ is the so-called centering measure. Assuming the cycles are of bounded length and diameter, also that the chain is uniformly irreducible, we show a gaussian bound for the transition probabilities $p_n(x,y) = P(X_n = y|X_0 = x)$:

$$p_n(x,y) \leq \frac{c_1}{n^{d/2}} \exp(-\frac{\|x-y\|^2}{c_2 n})$$

for some $c_1, c_2 > 0$. This estimate is based on Mathieu's result, cf. [M], and a Nash inequality.

Next we consider a model of random walk in random environment, with transitions depending on randomness $\omega \in \Omega \longrightarrow p_1(x,y)(\omega)$ which is ergodic with respect to the shift $(\theta_z)_{z\in\mathbb{Z}^d}$ such that as usual

$$p_1(x+z, y+z)(\omega) = p_1(x,y)(\theta_z\omega).$$

We assume the above bounded cycle decomposition. Note that the reversible random conductances model with trivial two points cycles is a particular case, see [S], thus our model extends to the non reversible situation. Assuming uniform irreducibility, we prove a quenched invariant principle for the rescaled process

$$t \in \mathbb{R}^+ \longrightarrow \beta^N(t) = X_{[Nt]}/N^{1/2}$$

plus polygonal interpolation. That is, for almost all environments ω, β^N converges weakly as $N \to \infty$ to a Brownian motion with non-degenerate deterministic covariance matrix.

A corresponding annealed CLT result (that is averaged with respect to the environment) has been proved recently in the special case of two-fold walks by Komorovski and Olla in [K]. We adapt the quenched proof of Sidoravicius and Sznitman, [S], to the non reversible case using corrector, the sector condition and the above heat kernels upper bounds.

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Multiscale analysis of exit distributions for random walks in random environments

Ofer Zeitouni

(joint work with Erwin Bolthausen)

We present a multiscale analysis for the exit measures from large balls in $\mathbb{Z}^d$, $d \geq 3$, of random walks in certain i.i.d. random environments which are small perturbations of the fixed environment corresponding to simple random walk. Our main assumption is an isotropy assumption on the law of the environment, introduced by Bricmont and Kupiainen. Under this assumption, we prove that the exit measure of the random walk in a random environment from a large ball, approaches the exit measure of a simple random walk from the same ball, in the sense that the variational distance between smoothed versions of these measures converges to zero. We also prove the transience of the random walk in random environment. The analysis is based on propagating estimates on the variational distance between the exit measure of the random walk in random environment and that of simple random walk, in addition to estimates on the variational distance between smoothed versions of these quantities.

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