Mathematisches Forschungsinstitut Oberwolfach

Report No. 52/2006

Mini-Workshop: Logic, Combinatorics and Independence Results

Organised by
Andrey Bovykin (Liverpool), Lorenzo Carlucci (Rome), Andreas Weiermann (Gent)

November 25th – December 2nd, 2006

Abstract. This is a collection of extended abstracts of a mini-workshop "Logic, Combinatorics and Independence results" that took place on November 25 – December 2, 2006 in Oberwolfach. The mini-workshop was attended by 16 researchers from 11 countries and concentrated on one unifying theme: independence results in mathematics. The workshop brought together researchers from different fields (Reverse Mathematics, Proof Theory, WQO theory, Models of Arithmetic, weak arithmetics and Ramsey Theory) with the purpose of building bridges and re-vitalizing existing connections between these fields. The workshop consisted of 16 one-hour talks, two open problem sessions and many informal small group discussions.

Mathematics Subject Classification (2000): 03D20 (recursive functions and relations, subrecursive hierarchies), 03F15 (recursive ordinals and ordinal notations), 03D80 (applications of computability and recursion theory), 03F30 (first-order arithmetic and fragments), 03C62 (models of arithmetic and set theory), 03H15 (nonstandard models of arithmetic), 03E35 (consistency and independence results), 03F03 (proof theory, general), 03F50 (metamathematics of constructive systems), 03F35 (second- and higher-order arithmetic and fragments), 03F45 (provability logics and related algebras), 05D10 (Ramsey theory), 05C05 (trees), 05A18 (partitions of sets), 05D40 (probabilistic methods), 05C83 (graph minors), 11M99 (zeta and L-functions: analytic theory), 11U99 (number theory: connections with logic), 37B10 (dynamical systems: symbolic dynamics).

Introduction by the Organisers

The mini-workshop "Logic, Combinatorics and Independence results" took place on November 25 – December 2, 2006. The workshop was organized by Andrey Bovykin, Lorenzo Carlucci and Andreas Weiermann and attended by 16 participants:

- Lev Beklemishev (Moscow)
- Andrey Bovykin (Liverpool)
- Wilfried Buchholz (Munchen)
- Lorenzo Carlucci (Rome)
- Lev Gordeev (Tübingen)
- Henryk Kotlarski (Warszawa)
- Alberto Marcone (Udine)
- Joseph Mileti (Chicago)
- Antonio Montalbán (Wellington)
- Eran Omri (Be'er-Sheva)
- Michael Rathjen (Leeds)
- Sergei Tupailo (Tallinn)
- Stanley Wainer (Leeds)
- Andreas Weiermann (Gent)
- Alan Woods (Crawley)
- Konrad Zdanowski (Warszawa)

There were 16 one-hour talks, two problem sessions and many one-to-one and small group discussions. The workshop brought together researchers specialising in several connected disciplines: Reverse Mathematics (Alberto Marcone, Joseph Mileti, Antonio Montalbán), Proof Theory (Lev Beklemishev, Wilfried Buchholz, Lev Gordeev, Michael Rathjen, Sergei Tupailo, Stanley Wainer, Andreas Weiermann), WQO theory (Alberto Marcone, Antonio Montalbán, Lev Gordeev, Andreas Weiermann), Models of Arithmetic (Andrey Bovykin, Henryk Kotlarski, Konrad Zdanowski, Alan Woods), weak arithmetics (Lev Beklemishev, Alan Woods, Konrad Zdanowski), logical aspects of finite Ramsey Theory (Andrey Bovykin, Lorenzo Carlucci, Henryk Kotlarski, Joseph Mileti, Eran Omri, Andreas Weiermann). However, the central theme of the workshop was first-order unprovable statements and statements of large logical strength. The subject originated in the late 1970s in the work of several mathematicians, most notably Jeff Paris and Harvey Friedman and attracted a large community of researchers at that time. The discoveries of the Paris-Harrington Principle and unprovability of Kruskal's Theorem provided, fourty years after Gödel's theorems, the first examples of mathematically natural unprovable statements. Since then, many other examples were found in Ramsey Theory, Graph Theory, well-quasi-order theory and other subjects. One of the main objectives of the workshop was to revive research in this area, especially in view of some spectacular recent developments. These developments revealed deep connections between the study of logical strength and several mathematical disciplines: Analytic Combinatorics, Graph Theory, Tauberian Theory, Number Theory, Dynamical Systems. Another objective of the workshop was to stimulate communication and joint research between researchers from different sub-areas of the subject (Ordinal Analysis, Reverse Mathematics, Models of Arithmetic). All these different areas were represented at the workshop by leading researchers. The workshop was very successful in setting grounds for comparison and interaction of methods from these areas. The two open problem sessions resulted in a list of problems of common interest and in better understanding of possible directions for future research. The talks varied from reports on recent results and proposals of new general approaches to discussions of new strategies to tackle long-standing open problems. Time allowed for free informal discussion and research in small groups that will eventually result in publications.

We would like to thank the Oberwolfach Institute for the wonderful opportunity to hold a meeting there and for providing NSF travel grants to some participants. We also thank all the staff working in the institute for the pleasant experience we all had during our week in Oberwolfach Institute.

Mini-Workshop: Logic, Combinatorics and Independence Results

Table of Contents

Abstracts

Representing worms as a term rewriting system

Lev D. Beklemishev

Since term rewriting systems are universal models of computation, it is clear that there must exist some systems for which termination is hard to prove, in particular, whose termination is not provable in PA. Such examples, however, would prima facie involve heavy coding in the spirit of Gödel's incompleteness theorem.

The question arises, whether there are natural, simple to formulate and memorable examples of such systems. Combinatorial independent principles like the Hydra battle offer an obvious line of approach. There are several papers reformulating the Hydra battle as a term rewrite system.

A system formulated by N. Dershowitz [2] has only recently been fully analyzed by G. Moser [3]. The analysis turned out to be rather complicated.[1]

Alternative systems have been offered by Touzet [4]. They have nicer behavior and are simpler to analyze, but they only capture parts of the full Hydra battle. In particular, the termination for each individual system is provable in PA. The strength of these systems approximates PA from below, but the arity of functions involved in the systems increases ad infinitum.

The Worm principle is one of the simplest combinatorial principles which looks almost as a statement of termination for a particular string rewrite system. It does lead to some natural examples of term rewrite systems whose termination is not provable in PA. We present three such systems here. Of those three, especially the second one is memorable and extremely simple, but it is infinite. The third system, obtained by adapting the second one, is finite and has a simple analysis, but it is less memorable. In some respects it is closer to the systems by Touzet than to the one of Dershowitz.

Our main result can be summarized as follows.

Theorem 1. If $\mathcal{W}$ is any of the three systems $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ formulated below, $\mathcal{W}$ is terminating. The termination of $\mathcal{W}$ is not provable in PA.

An infinite string rewrite system. The following system very closely approximates the original Worm principle. Consider an infinite alphabet $\Sigma = \{a_0, a_1, a_2, \dots\}$. Let S_n denote the set of words in the sub-alphabet $\{a_i : i \geq n\}$. The system $\mathcal{W}_1$ has the following rules:

$$a_{n+1}\alpha \longrightarrow (a_n\alpha)^k, \quad \alpha \in S_{n+1},\ k \geq 1.$$

Here α runs through all words from S_{n+1}. Hence, this is essentially a schema of rules.

[1] At this workshop, W. Buchholz presented a simplification of a system of Dershowitz which also admits a simplified analysis. It is interesting to compare it with the systems given in this paper.

Computations in this system differ from the Worm sequence in the following aspects: 1) Reductions can occur anywhere within a word, not only at the beginning; 2) The symbol a_0 is never erased; thus, a long sequence of a_0's, rather than an empty word, is computed; 3) k does not depend on the step of the process, but is chosen freely; 4) α need not be the longest possible part of the word to which the rule is applicable.

A nicer system. We consider a signature with a constant 0, a binary function symbol $\cdot$, and a unary function symbol f. The system $\mathcal{W}_2$ has three rules (the second and the third one schematic):

$$\begin{cases} (x \cdot y) \cdot z \longrightarrow x \cdot (y \cdot z) \\ f(0) \longrightarrow 0^m, \quad m \geq 1; \\ f(0 \cdot x) \longrightarrow (0 \cdot f(x))^m, \quad m \geq 1. \end{cases}$$

Here x^m means $x \cdot (\cdots (x \cdot (x \cdot x)) \cdots)$ (m times).

Intuitively, $\cdot$ and f denote certain operations on words from Σ: $\cdot$ corresponds to concatenation and $f(x)$ is the result of replacing all letters a_i in x by a_{i+1}. Hence, we can define a function mapping every word $\alpha \in S_0$ to a ground term $\alpha^{\#}$ as follows. If $\alpha = a_0^n$ then $\alpha^{\#} := 0^n$. Otherwise, write $\alpha = \alpha_1 a_0 \alpha_2 a_0 \cdots a_0 \alpha_n$, where all $\alpha_i \in S_1$. Then

$$\alpha^{\#} := f((\alpha_1^-)^{\#}) \cdot 0 \cdot f((\alpha_2^-)^{\#}) \cdot 0 \cdots 0 \cdot f((\alpha_n^-)^{\#}),$$

where we assume the right association of brackets.

Given a worm α and stage m of the game let $\alpha[m]$ denote the next worm.[2] The following lemma is straightforward.

Lemma 1. Suppose α does not begin with a_0. Then $\alpha^{\#} \longrightarrow_* \alpha[m]^{\#}$ in $\mathcal{W}_2$.

Termination of $\mathcal{W}_2$ is easy to obtain by lexicographic path ordering: let 0 precede $\cdot$ precede f. Unprovability in PA follows from the fact that $\mathcal{W}_2$ simulates the Worm sequence by the stated lemma.

A finite system. In order to make $\mathcal{W}_2$ finite, we have to consider step m of the game as part of the data so that a term encoding (m, α), where α is a worm, rewrites to a term encoding $(m+1, \alpha[m])$.

We introduce new unary function symbols a, b, b_1 and c. (m, α) will be represented by a term of the form $ca^m(t)$, where t represents a worm as in $\mathcal{W}_2$. The rules of the system enable the markers a to move to a place within t where a reduction is to be made. They perform the reduction and become the markers b_1. After the last of the markers a acts, a symbol b appears that eats up all the letters b_1 and they all move to the beginning of the term where c transforms them back into a.

[2]The worms here are symmetric to the worms in [1].

The rules of the system $\mathcal{W}_3$ are as follows:

(1) (associativity) $(x \cdot y) \cdot z \longrightarrow x \cdot (y \cdot z)$
(2) (a moves downwards)
$$a(f(x)) \longrightarrow f(a(x)); \qquad a(x \cdot y) \longrightarrow a(x) \cdot y; \qquad a(b_1(x)) \longrightarrow b_1(a(x))$$
(3) (b moves upwards and subsumes b_1)
$$f(b(x)) \longrightarrow b(f(x)); \qquad b(x) \cdot y \longrightarrow b(x \cdot y); \qquad b_1(b(x)) \longrightarrow b(b(x))$$
(4) (copying)
$$a(f(0 \cdot x)) \longrightarrow b_1(f(0 \cdot x) \cdot (0 \cdot f(x))); \qquad a(f(0)) \longrightarrow b_1(f(0) \cdot 0)$$
(5) (reduction)
$$f(0 \cdot x) \longrightarrow b(0 \cdot f(x)); \qquad f(0) \longrightarrow b(0)$$
(6) $c(b(x)) \longrightarrow c(a(x)); \qquad a(b(x)) \longrightarrow b(a(x))$.

It is easy to see that $\mathcal{W}_3$ emulates the Worm sequence. Termination of $\mathcal{W}_3$ can be proved by relating it to the system $\mathcal{W}_2$.

References

[1] L.D. Beklemishev. The Worm principle. In Z. Chatzidakis, P. Koepke, and W. Pohlers, editors, *Lecture Notes in Logic 27, Logic Colloquium '02*, pages 75–95. A.K. Peters, 2006.

[2] N. Dershowitz and Jouannaud J. P. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages 243–320. North-Holland, Amsterdam, 1990.

[3] G. Moser. The Hydra battle revisited. Unpublished manuscript. `http://cl-informatik.uibk.ac.at/~georg/publications/hydra.pdf`, 2006.

[4] H. Touzet. Encoding the hydra battle as a rewrite system. In *Proceedings 23rd International Symposium on Mathematical Foundations of Computer Science 1998, vol. 1450 of Lecture Notes in Computer Science*, pages 267–276. Springer, 1998.

New results and visions of unprovability and logical strength

Andrey Bovykin

I talked about several directions in the subject of Unprovability and Logical Strength that I am currently working on and about visions of some future developments that are already within reach.

1. Reverse Mathematics

Reverse mathematics is the study of logical strength and consistency strength of second-order arithmetical statements. Originally, model-theoretic and recursion-theoretic approaches in the subject complemented each other. Nowadays, recursion-theoretic analysis is dominating in this area. Interconnections of reverse mathematics with the topic of this workshop (first-order combinatorial independence results) can be best illustrated by the fact that the proof of unprovability of PH was originally a modification of the (model-theoretic) proof that RT^3_2 implies all of Peano Arithmetic (in [9]). Since then many higher-order strong statements inspired the discovery of first-order 'miniaturizations' that preserved a lot of their strength.

My contributions that may have some interest from the point of view of reverse mathematics are as follows.

- Model-theoretic re-proof of Joseph Mileti's Theorem on the strength of CanRT^2, the Canonical Ramsey Theorem for pairs [6]. The proof uses strong initial segments in models of arithmetic.
- Development, together with Andreas Weiermann in [6], of the model-theoretic method of 'densities' to approximate logical strength of second-order statements by strengths of their first-order iterations. Successful results so far include the treatment of RT^2_2, RegRT^2, CanRT^2 in [6] and of Hilbert's basis Theorem, Kruskal's Theorem, Infinitary Erdös-Moser principle and a few other statements in [7]. Several assertions (e.g., Hindman's Theorem) resisted density treatment so far.

There are several beautiful theorems in modern Infinite Ramsey Theory and geometric functional analysis that are the next candidates for application of these methods (to establish their strength), e.g. Pudlak-Rödl theorem, Gowers' theorems, some statements about blocks and barriers, about strategically Ramsey sets and about oscillation stability. Possibly, a version of Dvoretzki's theorem in geometric functional analysis will have some strength too.

2. Building upon the sine-principle

The sine-principle says: "for all m, n $(n < m)$, there is N such that whenever $a_1 < a_2 < \ldots < a_N$ is an increasing sequence of rational numbers, there is $H \subset N$ of size m such that for any $i_1 < i_2 < \cdots < i_n$ and $i_1 < k_2 < \cdots < k_n$ in H,

$$|\sin(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_n}) - \sin(a_{i_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_n})| < 2^{-i_1}\text{''}.$$

This PA-unprovable statement was proposed by Harvey Friedman in the internet forum FOM. An important feature of this principle is that there is no quantifier over all colourings as in PH or KM. A proof, using the Rhin-Viola theorem from number theory, can be found in [2]. The core of this result is the following lemma ('every function can be approximated by sine on a subset'): for any $\varepsilon > 0$ and any dimension n, any number K and any function $g\colon [K]^n \to [-1, 1]$, there is a set of rational numbers $A = \{a_1, a_2, \ldots, a_K\}$ such that for any $i_1 < i_2 < \cdots < i_n \le K$, we have

$$|g(i_1, i_2, \ldots, i_n) - \sin(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_n})| < \varepsilon.$$

It is not difficult to prove that every function that satisfies this lemma gives us an independence result of a similar shape. So, an interesting investigation would be to catalogue a few other examples of this kind and convert them into shapes that are interesting in the corresponding mathematical disciplines. Recently, Andreas Weiermann found another proof of the lemma, using an effective version of Kronecker's result on simultaneous diophantine approximation. This led him to a new result that $\sin(x_1 \cdot \ldots \cdot x_n)$ can be replaced by $\{x_1 \cdot \ldots \cdot x_n\}$, the fractional part of the product, and (using results on distribution of zeros of the Riemann zeta-function on vertical lines), that sine can be replaced by the Riemann zeta-function.

It was noticed by the author that sine can also be replaced by the complex exponent, and a new result with a large class of complex functions in place of sine is forthcoming. Also, Andreas Weiermann proved an independence result about the logistic mapping in the theory of chaotic dynamical systems using the same ideas in combination with symbolic dynamics.

3. Ramseyan unprovable statements

Despite the recent spectacular independence results that do not look exactly WQO-theoretic or Ramsey-theoretic, I still think that Ramsey Theory is the easiest source of entirely new future independence results. I suggest that more connections with modern Ramsey Theory should be developed, with the idea of constructing the right sets of indiscernibles out of certain homogeneous sets, for example:

- make use of sparseness conditions ($\forall x < y$ in H, $2^x < y$) as in [1] and in some old papers from the 1980s by the Founders of the subject;
- try to use ascending and descending waves instead of arithmetic progressions to build unprovable versions of van der Waerden's Theorem;
- try to formulate a new game of Noughts and Crosses to establish an unprovable version of the Hales-Jewett Theorem;
- combine sieve methods with indiscernibles (a very rudimentary combination is mentioned in the end of [1]).

Another interesting idea, first mentioned by Alan Woods in his PhD thesis, is to try to prove unprovability of Hypothesis H: "for any finite collection of irreducible polynomials $F_1(x), F_2(x), \ldots, F_n(x)$ with integer coefficients and such that $\prod_{i \leq n} F_i$ has no fixed prime divisor, there exist infinitely-many integers m such that for all $i \leq n$, $F_i(m)$ are prime". This conjecture is extremely strong and its formulation already provides some necessary ingredients for independence proofs (e.g. the enumeration of all Δ_0-formulas can probably be extracted), so it begs to become an independence result.

4. Braids

Several independence results can be extracted from Dehornoy's left-invariant ordering of positive braids as ω^{ω^ω}. Some results are translations of transfinite induction up to ω^{ω^ω}, using a correspondence between braids and ordinals, another result (by L. Carlucci and the author) in [5] is a braid-theoretic version of the hydra battle (for definitions, see L. Carlucci's abstract on page 3102). The possibility of such results was originally suggested to Andreas Weiermann by Patrick Dehornoy and this is currently a fast-developing topic with new results being obtained by L. Carlucci, P. Dehornoy, A. Weiermann and the author.

There is some hope to translate statements about braids into geometrical, topological statements, even with some amount of physical meaning, e.g. via Artin's Braid Theorem.

5. Weiermann's program of finding thresholds

A. Weiermann's (now well-known) threshold results for the Paris-Harrington Principle in [10] and for Kruskal's theorem in [11] started a new program of research: to introduce a threshold parameter and find a gap between provability and unprovability of the statement in terms of the value of the parameter. Several new results have emerged recently in this direction, e.g. a theorem by P. Dehornoy and A. Weiermann on a threshold value for braid principles.

My contributions to this program are: a model-theoretic proof of threshold results for PH and KM in [4] and a not yet exact threshold result for the Graph Minor theorem in [3]. For a function f, let GM_f be the following statement: "for every K there is N such that for any sequence of simple graphs $G_1, G_2, \ldots, G_N$ such that $|G_i| < K + f(i)$, there are $i < j \leq N$ such that G_i is isomorphic to a minor of G_j". So far we have:

(1) For any $r \leq \sqrt{2}$ and $f(i) = r \cdot \sqrt{\log i}$, the statement GM_f is provable in $I\Delta_0 + \exp$.
(2) For $f(i) = 7 \log i$, the statement GM_f is unprovable in PA.

The proofs rely on Pólya theory and a translation between Graph Minor Theorem and Kruskal's Theorem in [8]. We conjecture that the final exact result will be:

(1) For any $r \leq \sqrt{2}$, the statement $\mathrm{GM}_{r\sqrt{\log}}$ is provable in $I\Delta_0 + \exp$.
(2) For any $r > \sqrt{2}$, $\mathrm{GM}_{r\sqrt{\log}}$ is unprovable in PA.

References

[1] Bovykin, A. (2005). Several proofs of PA-unprovability. *Contemporary Mathematics series of the AMS*, volume 380, pp. 21-36.
[2] Bovykin, A. (2006). Unprovability of sharp versions of Friedman's sine-principle. To appear in *Proceedings of the AMS*.
[3] Bovykin, A. (2006). Exact unprovability results for the Graph Minor Theorem. *Draft.* Downloadable at `http://logic.pdmi.ras.ru/~andrey/research.html`.
[4] Bovykin, A. (2005). Model-theoretic proofs of threshold results for PH. (2005). To appear in *Logic Colloquium 2006* collection.
[5] Bovykin, A., Carlucci, L. (2006). Long games on braids. Draft. Downloadable from `http://logic.pdmi.ras.ru/~andrey/research.html`.
[6] Bovykin, A., Weiermann, A. (2006). The strength of infinitary ramseyan statements can be accessed by their density. *Submitted to APAL.*
[7] Bovykin, A., Weiermann, A. (2007). The strength of infinitary statements can be approximated by their densities. In preparation.
[8] Friedman, H., Robertson, N., Seymour, P. (1987). The metamathematics of the graph minor theorem. *Contemporary Mathematics*, vol. 65, pp.229-261.
[9] Kirby, L., Paris, J. (1977). Initial segments of models of Peano's axioms. In: *Set theory and hierarchy theory, V (Proc. Third Conf., Bierutowice, 1976)*, pp. 211–226. Lecture Notes in Math., Vol. 619. Springer, Berlin.
[10] A. Weiermann, (2004). A classification of rapidly growing Ramsey functions, *Proceedings of the AMS*, 132 , no. **2**, 553–561.
[11] A. Weiermann (2003). An application of graphical enumeration to PA, *Journal of Symbolic Logic* 68, no. **1**, 5–16.

Another rewrite system for the standard hydra battle

Wilfried Buchholz

In their 1990 Handbook article [1] on term rewriting Dershowitz and Jouannaud presented a certain rewrite system based on the Kirby-Paris hydra battle [2], and claimed that the system is terminating. In 2004 Dershowitz gave a corrected version of this system[1] which describes the hydra battle more faithfully. Only quite recently G. Moser [3] proved that both these systems are terminating. In our talk we present a variant $\mathcal{Q}$ of the 2004 system which is even closer to the hydra battle and for which termination can be proved in a rather simple way.

Let $\mathcal{F}$ denote the signature consisting of the constant $\mathsf{0}$ and the binary function symbols $\mathsf{g}, \mathsf{d}, \mathsf{h}$. Let $\mathbf{T} := \mathcal{T}(\mathcal{F}, \mathcal{V})$ be the set of all terms over $\mathcal{F}$ and a countably infinite set $\mathcal{V}$ of variables. Let $\mathbf{H} := \mathcal{T}(\{\mathsf{0}, \mathsf{g}\})$ be the set of all closed terms over $\{\mathsf{0}, \mathsf{g}\}$. The elements of $\mathbf{H}$ can be seen as representations of finite trees (*hydras*) or, equivalently, nested finite sequences: $\mathsf{0}$ stands for the empty sequence, and $\mathsf{g}(a, b)$ represents $a * \langle b \rangle$. Now working in $\mathbf{H}$, a single step in the (standard) hydra battle can be described as the transition from $(a, n) \in \mathbf{H} \times \mathbb{N}$ to $(a[n], n{+}1) \in \mathbf{H} \times \mathbb{N}$, where $a[n]$ is defined by recursion on the build up of a. Since we want to model the hydra battle in an ordinary (single sorted) term rewriting system, we identify $\mathbb{N}$ with the subset $\{\underline{0}, \underline{1}, \underline{2}, \ldots\}$ of $\mathbf{H}$, where $\underline{0} := \mathsf{0}$ and $\underline{n{+}1} := \mathsf{g}(\underline{n}, \mathsf{0})$, and define $a[c]$ for arbitray $a, c \in \mathbf{H}$ such that $a[\underline{n}]$ corresponds to $a[n]$.

Notation. In the sequel the letters a, b, c always denote elements of $\mathbf{H}$ (hydras), while q, r, s, t denote arbitrary terms from $\mathbf{T}$. By $\mathrm{lh}(t)$ we denote the number of symbols of t.

Definition of $a[c] \in \mathbf{H}$ for $a, c \in \mathbf{H}$

$$\mathsf{0}[c] := \mathsf{0}$$

$$\mathsf{g}(a,b)[c] := \begin{cases} a & \text{if } b = \mathsf{0} \\ a & \text{if } b = \mathsf{g}(b_0, \mathsf{0}) \ \& \ c = \mathsf{0} \\ \mathsf{g}(\mathsf{g}(a,b)[c_0], b_0) & \text{if } b = \mathsf{g}(b_0, \mathsf{0}) \ \& \ c = \mathsf{g}(c_0, c_1) \\ \mathsf{g}(a, b[c]) & \text{if } b = \mathsf{g}(b_0, b_1) \text{ with } b_1 \neq \mathsf{0} \end{cases}$$

The **rewrite system** $\mathcal{Q}$ over $\mathbf{T}$ is given by the following rules:

($\mathcal{Q}0$) $\mathsf{h}(\mathsf{g}(x,y),z) \to \mathsf{h}(\mathsf{d}(\mathsf{g}(x,y),z),\mathsf{g}(z,\mathsf{0}))$
($\mathcal{Q}1$) $\mathsf{d}(\mathsf{0},z) \to \mathsf{0}$
($\mathcal{Q}2$) $\mathsf{d}(\mathsf{g}(x,\mathsf{0})),z) \to x$
($\mathcal{Q}3$) $\mathsf{d}(\mathsf{g}(x,\mathsf{g}(y,\mathsf{0}))),\mathsf{0}) \to x$
($\mathcal{Q}4$) $\mathsf{d}(\mathsf{g}(x,\mathsf{g}(y,\mathsf{0})),\mathsf{g}(z,v)) \to \mathsf{g}(\mathsf{d}(\mathsf{g}(x,\mathsf{g}(y,\mathsf{0})),z),y)$
($\mathcal{Q}5$) $\mathsf{d}(\mathsf{g}(x,\mathsf{g}(y,\mathsf{g}(u,v))),z) \to \mathsf{g}(x,\mathsf{d}(\mathsf{g}(y,\mathsf{g}(u,v)),z))$

[1] https://listes.ens-lyon.fr/wws/arc/rewriting

The rules ($\mathcal{Q}1$)-($\mathcal{Q}5$) model exactly the recursive definition of $\mathsf{d}(a,c) := a[c]$, while ($\mathcal{Q}0$) models (one step of) the hydra battle.

Remark.
(a) If $t \in \mathbf{T}(\{\mathsf{0},\mathsf{g},\mathsf{d}\})$ is $\mathcal{Q}$-irreducible then $t \in \mathbf{H}$.
(b) If $t = \mathsf{h}(a,c)$ is irreducible then $a = \mathsf{0}$.

Definition.
$W^0 := \emptyset$, $W^{n+1} := \{t : \forall t'(t \rightarrow_{\mathcal{Q}} t' \;\Rightarrow\; t' \in W^n)\}$;
$W := \bigcup_{n\in\mathbb{N}} W^n$.

We will now prove that $\mathcal{Q}$ is terminating, i.e. $\forall t(t \in W)$.

The following Lemma 1 holds, since for each t there are only finitely many t' such that $t \rightarrow_{\mathcal{Q}} t'$.
Lemma 1. $t \in W \;\Leftrightarrow\; \forall t'(t \rightarrow_{\mathcal{Q}} t' \;\Rightarrow\; t' \in W)$.

Lemma 2.
(a) $s \in W^m \;\&\; t \in W^n \;\Rightarrow\; \mathsf{g}(s,t) \in W^{m+n}$;
(b) $\mathsf{g}(s,t) \in W^n \;\Rightarrow\; s,t \in W^n$;
(c) $s \in W^m \;\&\; r \in W^n \;\Rightarrow\; \mathsf{d}(s,r) \in W$.
Proof.
(a) and (b) are obvious, since $\mathcal{Q}$ contains no rule with g as leading symbol. Part (c) is proved with the help of (a) and (b) by induction on $\omega\cdot(m+n)+\mathrm{lh}(s)+\mathrm{lh}(r)$.

Definition of $\mathrm{v}(t) \in \mathbf{H}$
1. $\mathrm{v}(0) := \mathrm{v}(x) := 0$;
2. $\mathrm{v}(\mathsf{g}(s,t)) := \mathsf{g}(\mathrm{v}(s),\mathrm{v}(t))$;
3. $\mathrm{v}(\mathsf{d}(t,r)) := \mathrm{v}(t)[\mathrm{v}(r)]$;
4. $\mathrm{v}(\mathsf{h}(t,r)) := \mathsf{0}$.

Lemma 3. $q \rightarrow_{\mathcal{Q}} q' \implies \mathrm{v}(q') = \mathrm{v}(q)$.
Proof by straightforward induction on $\mathrm{lh}(q)$.

Definition of $o(a) \in On$
$o(\mathsf{0}) := 0$ and $o(\mathsf{g}(a,b)) := o(a)\#\omega^{o(b)}$.

Lemma 4. $a \neq \mathsf{0} \;\Rightarrow\; o(a[c]) < o(a)$.
Proof by straightforward induction on $\mathrm{lh}(a)$ using elementary ordinal arithmetic.

Definition. $\mathbf{o}(t) := o(\mathrm{v}(t))$.

Lemma 5. $\forall t,r \in W(\mathsf{h}(t,r) \in W)$.
Proof by induction on $\mathbf{o}(t)$.
Assume: $\tilde{t} \in W \;\&\; \forall t,r \in W(\mathbf{o}(t) < \mathbf{o}(\tilde{t}) \Rightarrow \mathsf{h}(t,r) \in W)$ (MIH)
To prove: $\forall r \in W(\mathsf{h}(\tilde{t},r) \in W)$.
By side induction on $m+n$ we prove:
$\forall t,r(\mathbf{o}(t) \leq \mathbf{o}(\tilde{t}) \;\&\; t \in W^m \;\&\; r \in W^n \Rightarrow \mathsf{h}(t,r) \in W)$.
Assumptions: (1) $\mathbf{o}(t) \leq \mathbf{o}(\tilde{t})$, (2) $t \in W^m$, (3) $r \in W^n$.
Let $\mathsf{h}(t,r) \rightarrow_{\mathcal{Q}} q$. To prove: $q \in W$.

We only consider the following two cases.
1. $q = \mathsf{h}(t', r)$ with $t \to_{\mathcal{Q}} t'$: Then, by Lemma 3, $\mathrm{v}(t') = \mathrm{v}(t)$ and therefore, by (1), $\mathbf{o}(t') = \mathbf{o}(t) \leq \mathbf{o}(\tilde{t})$. We also have $t' \in W^{m-1}$ and $r \in W^n$. Hence $q \in W$ by the side induction hypothesis.
2. $t = \mathsf{g}(t_0, t_1)$ and $q = \mathsf{h}(\mathsf{d}(t, r), \mathsf{g}(r, \mathsf{0}))$:
By Lemma 4 and (1) we get (4) $\mathbf{o}(\mathsf{d}(t, r)) < \mathbf{o}(t) \leq \mathbf{o}(\tilde{t})$.
By Lemma 2 from (2) and (3) we get $\mathsf{d}(t, r), \mathsf{g}(r, \mathsf{0}) \in W$.
Hence $q = \mathsf{h}(\mathsf{d}(t, r), \mathsf{g}(r, \mathsf{0})) \in W$ by (4) and MIH.

Theorem. $\forall t (t \in W)$.
Proof by induction on $\mathrm{lh}(t)$ using Lemma 2a,c and Lemma 5.

Remark. The above termination proof can be "locally" formalized in PA.

References

[1] N. Dershowitz and J. P. Jouannaud, *Rewrite systems.* In: Handbook of Theoretical Computer Science B: Formal Methods and Semantics, J. van Leeuwen (ed.) North-Holland 1990, pp. 243–320.
[2] L. Kirby and J. Paris, *Accessible independence results for Peano Arithmetic.* Bull. London Math. Soc. **4** 1982, pp. 285–293.
[3] G. Moser, *The Hydra Battle revisited.* Notes of a talk given at the Workshop on Proof Theory and Rewriting 2006, Obergurgl, September 5-9, 2006.

Some results on theorems unprovable in Peano Arithmetic and its subsystems

Lorenzo Carlucci

Some results are presented about 'mathematically natural' theorems independent from Peano Arithmetic and its subsystems.

1. Phase transition for Regressive Ramsey functions

A contribution to Weiermann's programme of phase transitions for independence results (see Andreas' abstract on page 3121) is presented: the full classification of the Kanamori-McAloon principle for fixed arbitrary dimension $d+1$ with respect to the $I\Sigma_d$ subsystem of Peano Arithmetic PA. The results of this section are joint with Lee and Weiermann [4].

A colouring F of the set of the d-tuples of non-zero natural numbers is called f-regressive (f a number-theoretic function) if for each d-tuple σ with minimum s we have $F(\sigma) < f(s)$. The Kanamori-McAloon principle in dimension d and parameter f (KM^d_f) says that for all k, there exists an ℓ so large that for every f-regressive colouring F of the d-tuples of elements of $\{1, \ldots, \ell\}$, there exists an ℓ-element set $H \subseteq \{1, \ldots, \ell\}$ such that F assigns the same colour to all d-tuples from H with the same minimum (such a set H is called *min-homogeneous* for F). $KM :\equiv (\forall d) KM^d_{identity}$ is often considered one of the most 'natural' examples of a theorem independent from PA, and $KM^{d+1}_{identity}$ is independent of $I\Sigma_d$.

The Schwichtenberg-Wainer Hierarchy is defined as follows.

$$F_0(x) := x + 1;\;\; F_{\alpha+1}(x) := F_\alpha^x(x);\;\; F_\alpha(x) := F_{\alpha[x]}(x).$$

It is well-known that F_{ω_d} dominates all $I\Sigma_d$-provably total functions, where ω_d denotes a tower of d ω's, and that $d \mapsto F_{\omega_d}$ dominates the provably-total functions of PA.

Let $\log_d$ denote the d-th iteration of the binary logarithm.

Theorem 1 (Carlucci, Lee, Weiermann, 2005)**.** Let

$$f_\alpha^d(i) = \lfloor \sqrt[F_\alpha^{-1}(i)]{\log_d(i)} \rfloor.$$

Then, for all $d > 0$,

$$\mathrm{I}\Sigma_d \vdash KM^{d+1}_{f_\alpha^{d-1}} \text{ iff } \alpha < \omega_d.$$

Corollary 2. $\mathrm{I}\Sigma_d \nvdash KM^{d+1}_{\lfloor \log_c \rfloor}$ iff $c \leq d - 1$.

For the upper bound (provability part), a non-trivial adaptation of a proof by Erdös and Rado is used. The Skolem function of $KM^{d+1}_{f_\alpha^{d-1}}$ is shown to be primitive recursive in F_α, for $\alpha < \omega_d$.

For the lower bound (unprovability), a key ingredient is to force the existence of min-homogeneous sets whose elements are spread out with respect to some relevant function. Non-trivial adaptations of methods from [7] are used.

The case $d = 1$ was already proved by Kojman, Lee, Omri and Weiermann (see Omri's abstract on page 3115). Surprisingly, the phase transition for the Kanamori-McAloon principle with *unbounded* dimensions with respect to full Peano Arithmetic has a different form, and is the same as the phase transition for the Paris-Harrington principle (as established by Weiermann [9]).

The above results were used by Bovykin in his recent work on Friedman's Sine Principle in [2].

2. Long games on positive braids

Two rewriting games on positive braid words are introduced, much in the spirit of Kirby-Paris' Hydra Game. Game 1 is played on positive braids with three strands and has an Ackermannian length. Game 2 is played on arbitrary positive braids and its termination is unprovable in $I\Sigma_2$. These results are part of an ongoing investigation joint with A. Bovykin, P. Dehornoy and A. Weiermann. The eventual goal is to obtain natural algebraic independence results on the braid group, a natural and well-studied mathematical structure.

The n-strand braid group B_n is a group with the following presentation:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1}; \sigma_i\sigma_j = \sigma_j\sigma_i \;\text{ for } |i-j| \geq 2, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \;\text{ for } |i-j| = 1\rangle.$$

A braid is called positive if it has a representation without σ_i^{-1} for any i. We denote the set of positive n-strand braids by B_n^+.

Dehornoy defined in [5] a left-invariant linear ordering of braids. Laver later showed [8] that Dehornoy's ordering is a well-ordering when rescricted to positive braids. Burckel [3] showed that the order-type is ω^{ω^ω} (the reason for this order-type of braids essentially coming from Higman's Lemma).

Rules of Game 1

For each word $a = b_n^{m_n} b_{n-1}^{m_{n-1}} \dots b_1^{m_1} \in \{1,2\}^*$ and $n \in \mathbf{N}$ we define the reduct $a[n]$ of a as follows.

- If $a = a_0 1$, then $a[n] = a_0$.
- Otherwise, let $b_i^{m_i}$ be the first block of a with more than two letters. Put

$$a[n] = b_n^{m_n} \dots b_i^{m_i - 1} b_{i-1}^{m_{i-1}+n} \dots b_1^{m_1}.$$

Proposition 3. Game 1 terminates but the termination is unprovable in $I\Sigma_1$.

If $a \leq b$, we let $w_{[a\uparrow b]} := a(a+1)^2 \dots (b-1)^2 b^2 (b-1)^2 \dots (a+1)^2 a$, and $w_{[b\downarrow a]} := b(b-1)^2 \dots (a+1)^2 a^2 (a+1)^2 \dots (b-1)^2 b$. We call these words *waves* from a to b and from b to a respectively.

Let $w = *ba*$. We define $\hat{b}$, the *first crossing* of b in w as follows: (1) if $a > b$, then $\hat{b}$ is the first occurrence of a letter $c \leq b$ in $a*$; (2) if $a < b$, then $\hat{b}$ is the first occurrence of a letter $c \geq b$ in $a*$; (3) if $a* = \epsilon$ or there is no occurrence as in 1, 2, then $\hat{b} = \epsilon$. For example, in 3124, 4 is the first crossing of 3. In 25342, the rightmost 2 is the first crossing of the leftmost 2.

Rules of Game 2

Given a word w let we define its reduct $w[k]$ for every $k > 0$ as follows.

- If the last letter of w is 1 then $a[w]$ is obtained from w by deleting this 1.
- Otherwise, let $b_i^{m_i}$ be the first block in w with more than two letters. Let w be of the form $w = *b_i^{m_i} w_0 \hat{b_i} *$, where $\hat{b_i}$ is the first crossing of b_i in w (if $\hat{b_i} = \epsilon$, then $w = *b_i^{m_i} w_0$). Let m be the minimal letter in w_0 and M be the maximal letter in w_0.

  Then, $w[k]$ is defined by the following substitution:

$$*b_i^{m_i} w_0 \hookrightarrow \begin{cases} b_i^{m_i-1}(b_i+1)^2 (w_{[(b_i+1)\uparrow M]})^k (b_i+1) & \\ & \text{if } \epsilon \neq w_0 = aw_1, a > b_i, \\ b_i^{m_i-1}(b_i-1)^2 (w_{[(b_i-1)\downarrow m]})^k (b_i-1) & \\ & \text{if } \epsilon \neq w_0 = aw_1, a < b_i. \\ b_i^{m_i-1}(b_i-1)^2 (w_{[(b_i-1)\downarrow 1]})^k (b_i-1) & \\ & \text{if } w_0 = \epsilon. \end{cases}$$

Note that Game 2 restricted to B_3 is essentially the same as Game 1.

Proposition 4. Game 2 terminates but the termination is unprovable in $I\Sigma_2$.

Proofs using ordinals and comparison with fast-growing hierarchies are available, as well as model-theoretic proofs developed by Bovykin using Indicator Theory, see [1].

References

[1] A. Bovykin and L. Carlucci, (2006). Long games on positive braids. Manuscript.

[2] A. Bovykin (2006). Unprovability of sharp versions of Friedman's sine-principle. To appear in *Proc. Amer. Math. Society.*
[3] S. Burckel, (1997). The wellordering on positive braids. *Journal of Pure and Applied Algebra* 120, pp. 1-17.
[4] L. Carlucci, G. Lee and A. Weiermann, (2005). Classifying the phase transition for the regressive Ramsey functions. Submitted to *Journal of the American Mathematical Society.*
[5] P. Dehornoy, (1994). Braid groups and left distributive operations. *Transactions of the American Mathematical Society* 345(1), pp. 115-151.
[6] A. Kanamori and K. McAloon, (1987). On Gödel incompleteness and finite combinatorics, *Ann. Pure Appl. Logic*, 33, no. 1, 23–41.
[7] J. Ketonen and R. Solovay, (1981). Rapidly growing Ramsey functions, *Ann. of Math.*, 113, no. 2, 267–314.
[8] R. Laver. (1996). Braid group actions on left distributive structures and well-orderings in the braid groups. *Journal of Pure and Applied Algebra*, 108(1), pp. 81-98.
[9] A. Weiermann, (2004). A classification of rapidly growing Ramsey functions, *Proc. Amer. Math. Soc.*, 132, no. 2, 553–561.

Strong WQO phase transitions

L. Gordeev
(joint work with A. Weiermann)

1. Summary

We investigate phase transitions for well-quasi-ordering (also referred to as *well-partial-ordering*, abbr.: *wpo*) results with respect to nested finite sequences and nested finite trees under the homeomorphic embedding with symmetrical gap condition. We consider three *wpo* spaces:

(1) nested finite sequences,
(2) finite trees labeled by nested finite sequences,
(3) nested finite trees.

With every *wpo* $\mathcal{W}$ in question we correlate a natural PA-extension T (below $\Pi^1_1\mathsf{TR}_0$), that proves the corresponding 2-order sentence $\mathsf{WPO}(\mathcal{W})$. Furthermore, we consider the appropriate parametrized 1-order *slow well-partial-ordering* sentence $\mathsf{SWP}(\mathcal{W}, \cdots, x)$ with x ranging over computable reals and actually compute a real number λ and prove that the following hold:

(1) if $x < \lambda$ then $\mathsf{SWP}(\mathcal{W}, \cdots, x)$ is provable in PA,
(2) if $x > \lambda$ then $\mathsf{SWP}(\mathcal{W}, \cdots, x)$ is not provable in T.

Such (uniquely determined) λ is called *phase transition* for $\mathsf{SWP}(\mathcal{W}, \cdots, x)$. In limit cases we replace computable reals r by computable functions $f : \mathbb{N} \to \mathbb{R}$ and prove analogous theorems. These results strengthen familiar Kruskal-Friedman-Gordeev-Kriz wpo theorems and Weiermann's phase transitions concerning Kruskal-Friedman-Schütte-Simpson cases.

2. Preliminaries

2.1. **Partial and linear well orderings.** By $\trianglelefteq$ and $\leq$ we denote partial and linear countable well orderings (abbr.: *wpo* and *wo*), respectively. A *wo* $\mathcal{O} = (W, \leq)$ is called a *linearization* of a *wpo* $\mathcal{W} = (W, \trianglelefteq)$ iff $(\forall x, y \in W)(x \trianglelefteq y \rightarrow x \leq y)$. A *wpo* $\mathcal{W} = (W, \trianglelefteq)$ is called *enumerated* iff it is supplied with a bijection, also called *enumeration*, $\nu : \mathbb{N} \rightarrow W$. For any enumerated *wpo* $\mathcal{W} = (W, \nu, \trianglelefteq)$ we fix its lexicographical linearization $\mathcal{W}_\nu = (W, \leq_\nu)$ that is defined as follows

$$\begin{aligned} W \times W \quad \ni \quad & x \leq_\nu y :\Leftrightarrow (\forall i \in \mathbb{N})(\nu(i) \trianglelefteq x \longleftrightarrow \nu(i) \trianglelefteq y) \vee (\exists i \in \mathbb{N}) \\ & (\nu(i) \ntrianglelefteq x \wedge \nu(i) \trianglelefteq y \wedge (\forall j < i)(\nu(j) \trianglelefteq x \longleftrightarrow \nu(j) \trianglelefteq y)) \end{aligned}$$

2.2. **Partial orderings on labeled sequences and trees.**

Definition 1. A labeled finite tree T is *embeddable* with symmetrical gap condition into a labeled finite tree T' (abbr.: $T \trianglelefteq T'$) iff there is a homeomorphic embedding $h : T \rightarrow T'$ such that $(\forall x \in T)(x \leq h(x))$ and $\min_{\leq}(x, x') \leq y$ holds for any two neighbors x, x' in any path $P \subset T$ and any $y \in h(P) \subset T'$. Here $\leq$ stands for the underlying *wo* on the set of labels occurring in T, T'. Note that $\trianglelefteq$ is a *wpo*. 1-D trees, i.e. trees without branchings, are called labeled sequences. Embedding of labeled finite sequences is an obvious 1-D specialization of general definition for trees. All finite structures in question are supplied with natural enumerations (ν) and norm functions $(\#)$.

Remark 2. Without loss of generality we can just as well replace vertex-labeled trees by the corresponding edge-labeled trees.

2.3. **Nested sequences and trees.** By Seq^d we denote the set of nested finite sequences, i.e. finite sequences, finite sequences labeled by finite sequences, etc., where d stands for the depth of nesting in question. By $\trianglelefteq_d^1$ and $\leq_d^1$ we denote the corresponding embedding relation and its linearization, respectively, which are defined by simultaneous recursion on d. Thus $\mathrm{Seq}^1 \cong \mathbb{N}$ and Seq^2 corresponds to the set of finite sequences with labels from $\mathbb{N}$.

By Ts^d, $\trianglelefteq_d^2$ and $\leq_d^2$ we denote the set of finite trees with labels from Seq^d, the corresponding embedding relation and its linearization, respectively.

By Tt^d we denote the set of nested finite trees, i.e. finite trees, finite trees labeled by finite trees, etc., where d stands for the depth of nesting in question; the corresponding *wpo* $\trianglelefteq_d^3$ and *wo* $\leq_d^3$ are defined by simultaneous recursion on d. In particular $\mathrm{Tt}^0 = \mathrm{Ts}^0$ is the set of plain (unlabeled) finite trees.

3. Basic results

For any *wpo* $\mathcal{W} = (W, \trianglelefteq)$ let $\mathsf{WPO}(\mathcal{W})$ be an abbreviation of "$\mathcal{W}$ *is a wpo*" in the form $(\forall f : \mathbb{N} \rightarrow W)(\exists i < j \in \mathbb{N})(f(i) \trianglelefteq f(j))$. For any *wpo* $\mathcal{W} = (W, \trianglelefteq)$, $0 < d \in \mathbb{N}$ and $0 \leq r \in \mathbb{Q}$, let $\mathsf{SWP}(W, \trianglelefteq, d, r)$ be the corresponding first order refinement of $\mathsf{WPO}(\mathcal{W})$

$$\begin{aligned} & (\forall K \in \mathbb{N})(\exists M \in \mathbb{N})(\forall x_0, \cdots, x_M \in W) \\ & ((\forall i \leq M)(\#(x_i) \leq K + r \cdot \lceil \log_d(i+1) \rceil) \rightarrow (\exists i < j \leq M)(x_i \trianglelefteq x_j)) \end{aligned}$$

Theorem 3. (1) If $r < 1 \leq d$ then $\mathsf{PA} \vdash \mathsf{SWP}\left(\text{Seq}^2, \trianglelefteq_d^1, d, r\right)$.
(2) If $r > 1$ then $\mathsf{PA} \nvdash \mathsf{SWP}\left(\text{Seq}^2, \trianglelefteq_2^1, 2, r\right)$ and $\Delta_1^1\mathsf{CA} \nvdash \mathsf{SWP}\left(\text{Seq}^3, \trianglelefteq_3^1, 3, r\right)$.

Theorem 4. For any computable $f : \mathbb{N} \to \mathbb{Q}$ and $I(x) := x$, the following holds.
(1) If $f \prec^\infty I$ then $\forall k\,(\exists d > k)\,\mathsf{PA} \vdash \mathsf{SWP}\left(\text{Seq}^d, \trianglelefteq_d^1, d, f(d)\right)$.
(2) If $I \prec^\infty f$ then:
(a) $(\exists k)\,\mathsf{ATR}_0 \nvdash (\forall d > k)\,\mathsf{SWP}\left(\text{Seq}^d, \trianglelefteq_d^1, d, f(d)\right)$,
(b) $\mathsf{ATR} \vdash (\forall d > 0)\,\mathsf{SWP}\left(\text{Seq}^d, \trianglelefteq_d^1, d, f(d)\right)$.

Let $\rho(d) := \frac{1}{8}\left(-d + \sqrt{d^2 + 16}\right)$ and $\ell(d) := -\log_{\rho(d)}(2)$. Note that $\ell(1) \approx .7369095552$, $\ell(2) \approx .5902344834$, while $\ell(d) \to 0$ as $d \to \infty$. For brevity we also identify $\ell(1)$ with constant function $\ell(1) : \mathbb{N} \ni x \mapsto \ell(1) \in \mathbb{R}$. By $\mathsf{ID}_{<\omega}^{(d)}$ we denote a formal system that is defined by recursion $\mathsf{ID}_{<\omega}^{(1)} := \mathsf{ID}_{<\omega}$, $\mathsf{ID}_{<\omega}^{(d+1)} := \mathsf{ID}_{<o\left(\mathsf{ID}_{<\omega}^{(d)}\right)}$. Note that $\mathsf{ID}_{<\omega} = \Pi_1^1\mathsf{CA}_0$ and $\mathsf{ID}_{<\omega}^{(d)} \to \Pi_1^1\mathsf{TR}_0$ as $d \to \infty$.

Theorem 5. For any $d > 1$ the following holds, where $\varphi_0(0) := 1$, $\varphi_1(0) := \omega$, $\varphi_2(0) := \varepsilon_0$, $\varphi_{d+1}(0) := \varphi(\varphi_d(0), 0)$; thus $\varphi_d(0) \to \Gamma_0$ as $d \to \infty$.
(1) If $r \leq \ell(d)$ then $\mathsf{PA} \vdash \mathsf{SWP}\left(\text{Ts}^d, \trianglelefteq_d^2, 2, r\right)$.
(2) If $r > \ell(d)$ then $\mathsf{ID}_{<\varphi_d(0)} \nvdash \mathsf{SWP}\left(\text{Ts}^d, \trianglelefteq_d^2, 2, r\right)$.

Theorem 6. For any $d \geq 0$ the following holds.
(1) If $r \leq \ell(1)$ then $\mathsf{PA} \vdash \mathsf{SWP}\left(\text{Tt}^d, \trianglelefteq_d^3, 2, r\right)$.
(2) If $r > \ell(1)$ then $\mathsf{ID}_{<\omega}^{(d+1)} \nvdash \mathsf{SWP}\left(\text{Tt}^d, \trianglelefteq_d^2, 2, r\right)$.

Theorem 7. For any computable $f : \mathbb{N} \to \mathbb{R}$ the following holds.
(1) If $f \prec^\infty \ell(1)$ then $\forall k\,(\exists d > k)\,\mathsf{PA} \vdash \mathsf{SWP}\left(\text{Tt}^d, \trianglelefteq_d^3, 2, f(d)\right)$.
(2) If $\ell(1) \prec^\infty f$ then:
(a) $(\exists k)\,\Pi_1^1\mathsf{TR}_0 \nvdash (\forall d > k)\,\mathsf{SWP}\left(\text{Tt}^d, \trianglelefteq_d^3, 2, f(d)\right)$,
(b) $\Pi_1^1\mathsf{TR} \vdash (\forall d > 0)\,\mathsf{SWP}\left(\text{Tt}^d, \trianglelefteq_d^3, 2, f(d)\right)$.

A survey of full satisfaction classes

Henryk Kotlarski

The goal of my lecture[1] is to give a survey of the theory of full satisfaction classes. I follow my paper [7]. I hope that this sort of problems is of some logical interest.

Arithmetize the language L of Peano arithmetic. It is convenient to pick a formalization such that no function symbols are allowed, but we have names for all natural numbers. By $\bar{z}$ we denote the name for z. Say that D is a *full satisfaction class* if it satisfies the usual condition on truth given by A. Tarski, i.e.,

[1]This is my lecture on satisfaction classes, given in Oberwolfach, during Workshop on Logic, Combinatorics and Independence Results, in November 2006. It is not intended to be published.

(1) If $D(\varphi)$, then $\text{Sent}(\varphi)$;
(2) if φ is of the form $\bar{a} + \bar{b} = \bar{c}$, then $D(\varphi)$ iff $a + b = c$;
(3) the same for other atomic formulas in L;
(4) $D(\neg(\varphi))$ iff not $D(\varphi)$;
(5) $D(\varphi \wedge \psi)$ iff $D(\varphi)$ and $D(\psi)$;
(6) $D(\exists x \varphi(x))$ iff $D(\varphi(\bar{m}))$ for some m.

This definition is a single sentence in $L \cup \{D\}$. Think of it as follows. Let M be a model of PA which has a full satisfaction class D. Then D is just a notion of truth for all objects in M of which M thinks that they are Gödel numbers of sentences (including nonstandard ones). Thus, D determines a semantics for the language in the sense of M.

Robinson [15] was the first to treat seriously nonabsoluteness of finiteness in the very definitions of the language. Krajewski [11] found the appropriate definitions and proved that satisfaction for the language in the sense of a given model M of PA is not uniquely determined by M.

Theorem 1. There exists a model M of PA which has many full satisfaction classes. Precisely, if D_0 is a full satisfaction class for a countable model M for PA such that (M, D_0) is recursively saturated, then D_0 has continuum many automorphic images, i.e., $\{D \subset \text{Sent}^M : \exists g \in \text{Aut}(M)\, D = g * D_0\}$ is of power $2^{\aleph_0}$.

This result follows (nowadays) immediately from the countable version of the Chang–Makkai theorem and Tarski's theorem on undefinability of truth. Some strengthenings of this result are known, cf. [4].

Knowing that semantics for L in the sense of M need not be unique for some models M for PA we ask the question when does such a semantics exist? That is, which models M admit a full satisfaction class? This question was answered by Krajewski, Lachlan and the present author [8].

Theorem 2. Every countable recursively saturated model M for PA has a full satisfaction class.

Corollary 3. The theory "PA+ D is a full satisfaction class" is conservative over PA.

This result seems to be slightly surprising. The reason is that in many cases one proves consistency of some theory T is a theory T_1 essentially by constructing in T_1 a full satisfaction class for T. For example, this is one of the reasons for which the second order arithmetic proves consistency of PA. The reason for which consistency does not follow from the existence of a full satisfaction class is that we do not have induction in the language $L \cup \{D\}$, where D is given by the proof of theorem 2. Thus, we cannot prove by induction on the length of a given proof $d \in M$ that all of its items are D–true.

In fact, induction in the language $L \cup \{D\}$ fails in a much more surprising manner. Let A_i be the disjunction of $i + 1$ copies of the obviously true sentence $0 = 0$. But we need to distribute parentheses in this disjunction, so we define: A_0 is $(0 = 0)$ and A_{i+1} is $(A_i \vee A_i)$.

Theorem 4. If M is a countable recursively saturated model of PA, then it admits a full satisfaction class making some A_i false.

The essential reason for which theorem 2, corollary 3 and theorem 4 hold is lemma 5, which says that there is no uniform bound for heights of proofs of sentences A_i in the following version of ω–logic. Let Γ_0 be the set of all true atomic sentences and negated false atomic ones. Let Γ_{i+1} be Γ_i together with the set of all sentences which can be derived from elements of Γ_i by a single application of a rule of predicate calculus or by a single application of the ω–rule: infer $\psi \vee \forall x\, \varphi(x)$ from all sentences $\psi \vee \varphi(\bar{m})$.

Lemma 5. For every k there exists r such that $A_r \notin \Gamma_k$.

This seems to be an innocent result, but its proof is rather torturous. V. Halbach [3] gave another proof of theorem 2.

Alistair Lachlan [12] proved that theorem 2 has the following converse.

Theorem 6. If M is a nonstandard model of PA which admits a full satisfaction class, then M is recursively saturated.

Let me remark that the proof of theorem 2 gives a full satisfaction class which makes all instances of induction true (of course, including nonstandard ones). It turns out that if we require all theorems of PA be true, then we obtain satisfaction classes which are not that pathological. I studied this phenomenon in [6].

Theorem 7. The following theories are equal:

(1) PA+ "D is a full satisfaction class" + Δ_0 induction in $L \cup \{D\}$;
(2) PA+ "D is a full satisfaction class" + $\forall\varphi\, [\mathrm{Pr}_{\mathrm{PA}}(\varphi) \Rightarrow D(\varphi)]$.

It follows for example that if M admits a full Δ_0 inductive satisfaction class, then it must satisfy $\mathrm{Con}_{\mathrm{PA}}$ and much more, for example iterated uniform reflection (iterated $< \omega$ times). Let me say something more in this direction. In the system of ω–logic described above the use of each rule increased the height of the proof. Change it to the system in which only the use of the ω–rule increases the measure of complexity of the proof. More exactly, we let $\Gamma_0(\varphi)$ be $\mathrm{Pr}_{\mathrm{PA}}(\varphi)$, $\Gamma_{n+\frac{1}{2}}(\varphi)$ be "φ is of the form $\eta \vee \forall z \psi(z)$ and for all z $\Gamma_n(\eta \vee \psi(\bar{z}))$, and $\Gamma_{n+1}(\varphi)$ be $\exists \langle \xi_0, \cdots \xi_{r-1}\rangle \forall i < r\, \Gamma_{n+\frac{1}{2}}(\xi) \wedge \mathrm{Pr}_{\mathrm{PA}}(\bigwedge\!\!\!\bigwedge_{i<r} \to \varphi)$.

Theorem 8. Let M be a countable recursively saturated model of PA. Then M has a full Δ_0–inductive satisfaction class iff for all $n \in \mathbb{N}$, $M \models \neg\Gamma_n(0=1)$.

But this system of ω–logic also may be inconsistent:

Theorem 9. Let M be a countable recursively saturated model of PA which admits a full Δ_0–inductive satisfaction class. Then it admits a full Δ_0–inductive satisfaction class which makes $\Gamma_j(0=1)$ true for some $j \in M$.

Rather than giving more details here (see [7]), let me state a more familiar proof theoretic characterization of the theory with a full satisfaction class. It is taken from [10]. Fix a system of notations for ordinals below $\varepsilon_{\varepsilon_0}$. Let $\mathrm{TI}(\alpha)$ denote the scheme of transfinite induction over α. Let $\omega_0(\alpha) = \alpha$ and $\omega_{m+1}(\alpha) = \omega^{\omega_m(\alpha)}$.

Theorem 10. Fix $m \in \mathbb{N}$. Let M be a countable recursively saturated model of PA. Then admits a full Σ_m–inductive satisfaction class iff for all $k \in \mathbb{N}$, $M \models \mathrm{TI}(\varepsilon_{\omega_m(k)})$.

Similarly for full induction in the language with D. Let $\omega_m = \omega_m(\omega)$.

Theorem 11. Let M be a countable recursively saturated model of PA. Then admits a full inductive satisfaction class iff for all $k \in \mathbb{N}$, $M \models \mathrm{TI}(\varepsilon_{\omega_m})$.

Ratajczyk [14] studied inductive full satisfaction classes from the point of view of partition properties in the Paris–Harrington style and gave a sentence independent from "PA+ D is an inductive full satisfaction class". Smith [16, 18, 17] showed that the countability assumption is essential in theorem 2. Engström shows that it is possible to work in the language with function symbols. Kossak and the present author [5] show more concrete sentences in the sense of the model under consideration made true by some full inductive satisfaction class and made false by another one. Ratajczyk and the author [9] give a description of first order consequences of the existence of a full inductive satisfaction class in terms of consistency of some version of ω–logic iterated to the transfinite. In his recent [1] Cieśliński shows that if D is a full satisfaction class closed under provability in predicate calculus, then it is already Δ_0–inductive, so a weaker assumption of making logic true suffices in theorem 7.

A weaker notion, that of a *partial* inductive satisfaction class was used to study recursively saturated models of PA.

The following is a list of all known to me references on full satisfaction classes.

References

[1] Cieśliński, C., *Truth, conservativeness and provability*, to appear.

[2] Engström. F., *Satisfaction classes in nonstandard models of first order arithmetic*, thesis for the degree of licentiate of philosophy, University of Göteborg, 2002.

[3] Halbach, V., *Conservative theories of classical truth*, Studia Logica 62(3), pp. 353–370.

[4] Kossak, R., *A note on satisfaction classes*, Notre Dame Jour. of Formal Log. 26, 1985, pp. 1–8.

[5] Kossak, R., Kotlarski, H., *Game approximations of satisfaction classes and the problem of rather classless models*, Zeitschr. Math. Log. 38, 1992, pp. 21–26.

[6] Kotlarski, H., *Bounded induction and satisfaction classes*, Zeitschr. Math. Log. 32, 1986, pp. 531–544.

[7] Kotlarski, H., *Full satisfaction classes: a survey*, Notre Dame Journal of Formal Logic 32 No 4, 1991, pp. 573–579.

[8] Kotlarski, H., Krajewski, S., and Lachlan, A. H., *Construction of satisfaction classes for nonstandard models*, Canad. Math. Bull. 24, 1981, pp. 283–293.

[9] Kotlarski, H., Ratajczyk, Z., *Inductive full satisfaction classes*, Annals of Pure and Applied Logic 47, 1990, pp. 199–223.

[10] Kotlarski, H., Ratajczyk, Z., *More on induction in the language with a satisfaction class*, Zeitschrift Math. Log. 36(5), 1990, 441–454.

[11] Krajewski, S., *Nonstandard satisfaction classes*, pp. 121–145 in Set Theory and Hierarchy theory, Springer Lecture Motes in Mathematics 537, 1976.

[12] Lachlan, A. H., *Full satisfaction classes and recursive saturation*, Canad. Math. Bull. 24, 1981, pp. 295–297.

[13] Murawski, R., *Semantics for nonstandard languages*, Reports on Math. Log., 22, 1988, 105–114.
[14] Ratajczyk, Z., *Satisfaction classes and sentences independent from* PA, Zeitschr. Math. Log. 28, 1982, pp. 149–165.
[15] Robinson, A., *On languages based on nonstandard arithmetic*, Nagoya Math. Jour. 22, 1963, pp. 83–117.
[16] Smith, S. T., *Nonstandard syntax and semantics and full satisfaction classes for models of arithmetic*, Ph. D. thesis, Yale University, 1984.
[17] Smith, S. T., *Nonstandard charicterization of recursive saturation and resplendency*, Journ. Symb. Log., 52(3), 1987, pp. 842–863.
[18] Smith, S. T., *Nonstandard definability*, Ann. Pure and Appl. Log. 42, 1989, pp. 21–43.

On Fraïssé's conjecture for linear orders of finite Hausdorff rank

Alberto Marcone
(joint work with Antonio Montalbán)

Let LO be the class of countable linear orders. If $L, L' \in \mathsf{LO}$ let $L \preceq L'$ mean that there exists an order preserving embedding of L into L'. $L \sim L'$ abbreviates $L \preceq L'$ and $L' \preceq L$. In this case we say that L and L' are *equimorphic*.

Fraïssé's conjecture (FC) is the statement that LO is well-quasi-ordered by $\preceq$, i.e. that there are neither infinite descending chains nor infinite antichains. Fraïssé formulated this conjecture in 1948 ([1]). Laver ([4]) established FC in 1971 by proving a stronger statement using Nash-Williams' notion of better quasi-order ([8]). *Laver's Theorem* states that LO is better-quasi-ordered by $\preccurlyeq$ (actually Laver proved even more, extending the result beyond LO). All known proofs of Fraïssé's conjecture actually establish Laver's theorem.

It is easy to state FC in the language of second order arithmetic, and it has been a longstanding open problem in reverse mathematics ([11] is the main reference for this research program) to establish its exact axiomatic strength (see [5] for a survey of the area). Laver's proof can be carried out within the strong system $\mathbf{\Pi}^1_2$-CA_0, and Shore ([9]) showed that FC implies ATR_0. Since FC is a $\mathbf{\Pi}^1_2$ statement, standard model theoretic considerations show that FC does not imply $\mathbf{\Pi}^1_1$-CA_0 over ATR_0. More recently Montalbán ([7]) showed that FC is equivalent over RCA_0 to other statements about linear orders.

An easy observation is that to establish FC it suffices to consider *scattered* linear orders, i.e. lineal orders $L \in \mathsf{LO}$ such that $\mathbb{Q} \npreceq L$ (all non-scattered countable linear orders are equimorphic). Scattered linear orders were first studied by Hausdorff a century ago ([2]), and his results lead to the notion of Hausdorff rank.

Definition 1. If L is a linear order we define for every ordinal α an equivalence relation $\approx_\alpha$ on L such that $\approx_\alpha$-equivalence classes are intervals in L:

- $\approx_0$ is equality;
- if $\alpha > 0$ then $x \approx_\alpha y$ iff there exists $\beta < \alpha$ such that there are finitely many $\approx_\beta$-equivalence classes between x and y.

Let the Hausdorff rank of L, $\mathrm{rk_H}(L)$, to be the least α such that there exists finitely many $\approx_\alpha$-equivalence classes, if such an α exists.

Hausdorff essentially showed that $\mathrm{rk_H}(L)$ exists if and only if L is scattered. Moreover it is easy to see that if L is countable scattered then $\mathrm{rk_H}(L)$ is a countable ordinal.

Let α be an ordinal and let $\mathsf{LO}_\alpha = \{ L \in \mathsf{LO} : \mathrm{rk_H}(L) < \alpha \}$ and FC_α be the statement that LO_α is well-quasi-ordered by $\preceq$. Our goal is to gather information on the strength of FC by looking at the strength of various FC_α. The first step in this project is the study of FC_ω, i.e. of Fraïssé's conjecture for linear orders of finite Hausdorff rank.

It is well-known that a quasi-order $(Q, \leq)$ is a well quasi-order if and only if all its linear extensions are well-orders.

Definition 2. If $(Q, \leq)$ is a well quasi-order let

$$o(Q, \leq) = \sup\{ \alpha : \alpha \text{ is the order type of a linear extension of } (Q, \leq) \}.$$

De Jongh and Parikh ([3]) showed that this sup is actually attained.

The starting point of our reverse mathematics results is the computation of $o(\mathsf{LO}_\omega, \preceq)$. Recall that the Veblen functions give an ordinal notation system for ordinals below Γ_0. They are defined by letting $\varphi_0(\alpha) = \omega^\alpha$ and, for $\beta > 0$, $\varphi_\beta(\alpha) =$ the α-th common fixed point of all φ_γ with $\gamma < \beta$. In particular φ_1 enumerates the ε-numbers, i.e. the critical points of the exponential function.

Theorem 1. *$o(\mathsf{LO}_\omega, \preceq) = \varphi_2(0)$, i.e. the least fixed point of the ε function.*

The ordinal $\varphi_2(0)$ is the proof-theoretic ordinal of the system ACA_0^+ which consists of RCA_0 plus the statement "for every X, $X^{(\omega)}$ (the arithmetic jump of X) exists" (see [10] for recent results about ACA_0^+).

The proofs of FC make use of the notion of indecomposable linear order.

Definition 3. A linear order L is *indecomposable* if whenever $L = L_1 + L_2$ then $L \preceq L_1$ or $L \preceq L_2$.

It follows from FC that every scattered linear order is the finite sum of indecomposable linear orders. Montalbán ([7]) showed that the latter statement is indeed equivalent to FC over RCA_0.

We use Montalbán's signed trees to study indecomposable scattered linear orders.

Definition 4. A *signed tree* ia a pair (T, s_T) where $T \subseteq \omega^{<\omega}$ is a nonempty well-founded tree and $s_T : T \to \{+, -\}$. Let ST be the set of all signed trees.

If $(T, s_T), (T', s_{T'}) \in \mathsf{ST}$ a map $f : T \to T'$ is a *homomorphism* if

- $\sigma \subset \tau$ implies $f(\sigma) \subset f(\tau)$ for every $\sigma, \tau \in T$;
- $s_{T'}(f(\sigma)) = s_T(\sigma)$ for every $\sigma \in T$.

If there exists a homomorphism from (T, s_T) to $(T', s_{T'})$ we write $(T, s_T) \preceq (T', s_{T'})$. (T, s_T) and $(T', s_{T'})$ are *equimorphic* (and we write $(T, s_T) \sim (T', s_{T'})$) if $(T, s_T) \preceq (T', s_{T'})$ and $(T', s_{T'}) \preceq (T, s_T)$.

To each $(T, s_T) \in \mathsf{ST}$ we associate a countable linear order $\text{lin}(T, s_T)$ so that:

(1) each $\text{lin}(T, s_T)$ is scattered indecomposable and each scattered indecomposable countable linear order different from $\mathbf{1}$ is equimorphic to $\text{lin}(T, s_T)$ for some $(T, s_T) \in \mathsf{ST}$;
(2) $(T, s_T) \preceq (T', s_{T'})$ (in ST) if and only if $\text{lin}(T, s_T) \preceq \text{lin}(T', s_{T'})$ (in LO), and in particular $(T, s_T) \sim (T', s_{T'})$ if and only if $\text{lin}(T, s_T) \sim \text{lin}(T', s_{T'})$;
(3) $\text{rk}_{\text{H}}(\text{lin}(T, s_T))$ and the height of the tree T differ by at most one.

Details of the definition of lin can be found in [7] and [6].

In particular indecomposable linear orders of finite Hausdorff rank are represented by signed trees of finite height. Since each of these trees is easily seen to be equimorphic to a finite signed tree, we study LO_ω by looking at ST_ω, the set of finite signed trees.

Theorem 2. *RCA_0 proves that the following are equivalent:*
(1) ST_ω is well-quasi-ordered by $\preceq$;
(2) $\varphi_2(0)$ is well-ordered.

Theorem 3. *RCA_0 proves that FC_ω implies that ST_ω is well-quasi-ordered by $\preceq$.*

Theorem 4. *ACA_0^+ proves that if ST_ω is well-quasi-ordered by $\preceq$ then FC_ω holds.*

Theorems 2 and 4 yield an upper bound for the complexity of FC_ω:

Theorem 5. *ACA_0^+ + "$\varphi_2(0)$ is well-ordered" proves FC_ω.*

To obtain a lower bound for the complexity of FC_ω we start by noticing that some of the intermediate steps in Shore's proof that FC implies ATR_0 ([9]) show that RCA_0 proves that FC_ω implies ACA_0. Shore's arguments can be refined to yield:

Theorem 6. *RCA_0 proves that FC_ω implies ACA_0'.*

Here the system ACA_0' consists of RCA_0 plus the statement "for every X and n, $X^{(n)}$ exists". ACA_0' is properly weaker than ACA_0^+ and properly stronger than ACA_0.

Theorem 6 cannot be strengthened by replacing ACA_0' with ACA_0^+. In fact FC_ω holds in the ω-model consisting of all arithmetic sets, where ACA_0^+ fails.

Theorems 2, 3 and 6 yield:

Theorem 7. *RCA_0 proves that FC_ω implies ACA_0' + "$\varphi_2(0)$ is well-ordered".*

References

[1] Roland Fraïssé. Sur la comparaison des types d'ordres. *C. R. Acad. Sci. Paris*, 226:1330–1331, 1948.
[2] F. Hausdorff. Grundzüge einer Theorie der geordneten Mengen. *Math. Ann.*, 65(4):435–505, 1908.
[3] Dick H. J. de Jongh and Rohit Parikh. Well-partial orderings and hierarchies. *Nederl. Akad. Wetensch. Proc. Ser. A* **80**=*Indag. Math.*, 39(3):195–207, 1977.
[4] Richard Laver. On Fraïssé's order type conjecture. *Ann. of Math. (2)*, 93:89–111, 1971.

[5] Alberto Marcone. Wqo and bqo theory in subsystems of second order arithmetic. Lecture Notes in Logic, pages 303–330. Assoc. Symbol. Logic, La Jolla, CA, 2005.
[6] Antonio Montalbán. On the equimorphism types of linear orderings. *Bull. Symb. Log.* to appear.
[7] Antonio Montalbán. Equivalence between Fraïssé's conjecture and Jullien's theorem. *Ann. Pure Appl. Logic*, 139(1-3):1–42, 2006.
[8] C. St. J. A. Nash-Williams. On better-quasi-ordering transfinite sequences. *Proc. Cambridge Philos. Soc.*, 64:273–290, 1968.
[9] Richard A. Shore. On the strength of Fraïssé's conjecture. In John N. Crossley, Jeffrey B. Remmel, Richard A. Shore, and Moss E. Sweedler, editors, *Logical methods*, pages 782–813. Birkhäuser, Boston, MA, 1993.
[10] Richard A. Shore. Invariants, Boolean algebras and ACA_0^+. *Trans. Amer. Math. Soc.*, 358(3):989–1014, 2006.
[11] Stephen G. Simpson. *Subsystems of second order arithmetic.* Springer-Verlag, Berlin, 1999.

The Strength of the Rainbow Ramsey Theorem

Joseph Mileti
(joint work with Barbara Csima)

The Rainbow Ramsey Theorem is essentially an "anti-Ramsey" theorem which roughly states that certain colorings always have a large heterogeneous section. To begin, we first recall Ramsey's Theorem.

Definition 1. Given a set Z and an $n \in \mathbb{N}^+$, we let $[Z]^n = \{x \subseteq Z : |x| = n\}$.

Theorem 2 (Ramsey's Theorem)**.** Suppose that $n, k \in \mathbb{N}^+$ and that $f\colon [\mathbb{N}]^n \to k$. There exists an infinite set H such that f is constant on $[H]^n$. Such an H is called *homogeneous* for f. We denote the formal statement of this fact in second-order arithmetic by RT^n_k.

The Rainbow Ramsey Theorem puts a finiteness conditions on the number of times each color may appear instead of the total number of possible colors.

Definition 3. A function $f\colon [\mathbb{N}]^n \to \mathbb{N}$ is called *k-bounded* if $|f^{-1}(c)| \leq k$ for all $c \in \mathbb{N}$.

Theorem 4 (Rainbow Ramsey Theorem - Galvin)**.** Suppose that $n, k \in \mathbb{N}^+$ and that $f\colon [\mathbb{N}]^n \to \mathbb{N}$ is k-bounded. There exists an infinite set R such that f is injective on $[R]^n$. Such an R is called a *rainbow* for f. We denote the formal statement of this fact in second-order arithmetic by RRT^n_k.

Surprisingly, one can prove RRT^n_k from RT^n_k by a clever but simple argument which is easily formalized from RCA_0, the standard base theory of reverse mathematics. In particular, any computability-theoretic upper bounds of solutions to computable instances of Ramsey's Theorem will apply to computable instance of the Rainbow Ramsey Theorem. The following fundamental result characterizes the location of homogeneous sets for computable $f\colon [\mathbb{N}]^n \to k$ in terms of the arithmetical hierarchy.

Theorem 5 (Jockusch)**.** For every computable $f\colon [\mathbb{N}]^n \to k$, there exists a Π^0_n set homogeneous for f. Furthermore, for each $n \geq 2$, there exists a computable $f\colon [\mathbb{N}^n \to 2$ such that no Σ^0_n set is homogeneous for f.

As a consequence, we get the same upper bounds for the Rainbow Ramsey Theorem, and using similar diagonalization techniques we obtain the same lower bounds.

Theorem 6. For every computable k-bounded $f\colon [\mathbb{N}]^n \to \mathbb{N}$, there exists a Π^0_n set which is a rainbow for f. Furthermore, for each $n \geq 2$, there exists a computable 2-bounded $f\colon [\mathbb{N}]^n \to \mathbb{N}$ such that no Σ^0_n set is a rainbow for f.

Thus, we can diagonalize at the same level for both theorems. However, if we want to compare strengths in the sense of reverse mathematics, an analysis of what can be coded into the Turing degrees of solutions to computable instances should be our goal. To obtain a distinction in this sense between Ramsey's Theorem and the Rainbow Ramsey Theorem, we need to give a proof which does not filter through Ramsey's Theorem. As with most partition theorems, we do this by a sequence of finite approximations which leave infinitely many points left to continue the construction.

Definition 7. Let $f\colon [\mathbb{N}]^2 \to \mathbb{N}$.

- A finite set F is *heterogeneous for* f if f is injective on $[F]^2$.
- If F is heterogeneous for f, we let
  $$Viab_f(F) = \{a \in \mathbb{N} : F \cup \{a\} \text{ is heterogeneous for } f\}$$
  We call $Viab_f(F)$ the set of *viable* numbers for F.
- If F is heterogeneous for f, we say that F is *admissible for* f if $Viab_f(F)$ is infinite.

The fundamental fact is that if F is admissible for f, then there exists at most $|F|$ many elements of $Viab_f(F)$ which destroy admissibility when added to F. This allows one to recursively build a rainbow by avoiding "bad" elements. However, this simple upper bound on the number of bad elements gives a lot more information. If we choose to continue by picking an element of $Viab_f(F)$ in a sufficiently random manner, then with very high probability we should be able to continue adding elements to build a rainbow. This idea can be used to prove the following result.

Theorem 8. Suppose that $f\colon [\mathbb{N}]^2 \to \mathbb{N}$ is 2-bounded and computable. Let X be 2-random. There exists an X-computable rainbow for f.

Using the relativization of this result together with van-Lambalgen's Theorem, we can in fact build entire models below 2-randoms.

Theorem 9. Suppose that X is 2-random. There exists an ω-model of $RCA_0 + RRT^2_2$ in which every set is X-computable.

This is sharp contrast to the situation for Ramsey's Theorem.

Theorem 10 (Mileti)**.** There exists a computable $f\colon [\mathbb{N}]^2 \to 2$ such that

$$\mu(\{X \in 2^{\mathbb{N}} : X \text{ computes a set homogeneous for } f\}) = 0$$

Combining these results, we conclude that RRT^2_2 is in fact strictly weaker than RT^2_2.

Corollary 11. $RCA_0 + RRT^2_2 \nvdash RT^2_2$.

The question of whether RRT^n_2 implies RT^n_2 for any $n \geq 3$ is still open, although we can show that $RCA_0 + RRT^2_2 \nvdash RRT^3_2$. One conjecture is that n-randomess suffices for RRT^n_2, which would imply that that RRT^n_2 is strictly weaker than RRT^{n+1}_2 for all n (thus providing a strictly ascending chain of combinatorial statements below ACA_0).

Sharp thresholds for Ackermannian Ramsey Numbers

Eran Omri[1]

(joint work with Menachem Kojman, Gyesik Lee, Andreas Weiermann)

Omri, Eran

Suppose $g : \mathbb{N} \to \mathbb{N}$ is a function. A nonempty set $X \subseteq \mathbb{N}$ is *g-large* if $|X| \geq g(\min X)$. A coloring $C : [\mathbb{N}]^2 \to \mathbb{N}$ is *g-regressive* if $C(\{m, n\}) \leq g(\min\{m, n\})$ for all $\{m, n\} \subseteq \mathbb{N}$.

The *g-large Ramsey number* of k and c, denoted $R^*_g(k, c)$, is the least N so that

$$N \xrightarrow{*}_g (k)_c.$$

This symbol means: for every coloring C of $[N]^2$ by c colors there is a g-large C-homogeneous subsets of N of cardinality at least k.

The *g-regressive Ramsey number* of k, denoted $R^{\text{reg}}_g(k)$, is the least N so that

$$N \xrightarrow{\min} (k)_g.$$

This symbol means: for every g-regressive coloring $C : [N]^2 \to \mathbb{N}$ there exists a *min-homogeneous* $H \subseteq N$ of size at least k, that is, the color $C(m, n)$ of a pair $\{m, n\} \subseteq H$ depends only on $\min\{m, n\}$.

We compute below the sharp thresholds on g at which g-large and g-regressive Ramsey numbers cease to be primitive recursive and become Ackermannian. We prove:

Theorem 1. *Suppose $g : \mathbb{N} \to \mathbb{N}$ is nondecreasing and unbounded. Then $R^*_g(k, c)$ is bounded by some primitive recursive function in k and c if and only if for every $t > 0$ there is some $M(t)$ so that for all $n \geq M(t)$ it holds that*

$$g(n) < \log n/t$$

and $M(t)$ is primitive recursive in t.

[1]Supported by the Lynn and William Frankel center for Computer Sciences.

Theorem 2. *Suppose $g : \mathbb{N} \to \mathbb{N}$ is nondecreasing and unbounded. Then $R_g^{\mathrm{reg}}(k)$ is bounded by some primitive recursive function in k if and only if for every $t > 0$ there is some $M(t)$ so that for all $n \geq M(t)$ it holds that*

$$g(n) < n^{1/t}$$

and $M(t)$ is primitive recursive in t.

We also identify the threshold below which g-regressive colorings have usual Ramsey numbers, that is, admit homogeneous, rather than just min-homogeneous sets, and give a lower bound of

$$A_{53}(2^{2^{274}})$$

on the Id-regressive Ramsey number of $k = 82$, where A_{53} is the 53-d approximation of Ackermann's function.

References

[1] H. L. Abbott, *A note on Ramsey's theorem*, *Canad. Math. Bull.*, 15:9–10, 1972.

[2] P. F. Blanchard, *On regressive Ramsey numbers*, *J. Combin. Theory Ser. A*, 100(1):189–195, 2002.

[3] C. Calude, *Theories of computational complexity*, volume 35 of *Annals of Discrete Mathematics*. North-Holland, Amsterdam, 1988.

[4] P. Erdös, A. Hajnal, A. Máté, and R. Rado, *Combinatorial set theory: partition relations for cardinals*, volume 106 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1984.

[5] P. Erdös and G. Mills, *Some bounds for the Ramsey-Paris-Harrington numbers*, *J. Combin. Theory Ser. A*, 30(1):53–70, 1981.

[6] P. Erdös and R. Rado, *Combinatorial theorems on classifications of subsets of a given set*, *Proc. London Math. Soc. (3)*, 2:417–439, 1952.

[7] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, *Monatshefte f. Math. u. Phys.*, 38:173–198, 1931.

[8] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey theory*, Second edition. John Wiley & Sons Inc., New York, 1990.

[9] A. Kanamori and K. McAloon, *On Gödel incompleteness and finite combinatorics*, *Ann. Pure Appl. Logic*, 33(1):23–41, 1987.

[10] J. Ketonen and R. Solovay, *Rapidly growing Ramsey functions*, *Ann. of Math. (2)*, 113(2):267–314, 1981.

[11] M. Kojman and S. Shelah, *Regressive Ramsey numbers are Ackermannian*, *J. Combin. Theory Ser. A*, 86(1):177–181, 1999.

[12] G. Lee, *Phase transitions in axiomatic thought*, Ph.D. Thesis. University of Münster, Germany, 2005.

[13] H. Lefmann and V. Rödl, *On canonical Ramsey numbers for coloring three-element sets*, In *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, volume 411 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 237–247. Kluwer Acad. Publ., Dordrecht, 1993.

[14] E. Omri, *Thresholds for Regressive Ramsey numbers.* M.Sc. Thesis. Ben-Gurion University, Israel, 2004.

[15] J. B. Paris and L. Harrington, A mathematical incompleteness in Peano arithmetic. In *J. Barwise, ed., Handbook of Mathematical Logic*, volume 90 of *Studies in Logic and the Foundations of Mathematics*, pages 1133–1142. North-Holland, 1977.

[16] J. B. Paris, Some independence results for Peano arithmetic. *J. Symbolic Logic*, 43(4):725–731, 1978.

[17] R. Péter, *Recursive functions.* Third edition. Academic Press, New York, 1967.

[18] H. J. Prömel, W. Thumser, and B. Voigt, Fast growing functions based on Ramsey theorems. *Discrete Math.*, 95(1-3):341–358, 1991.
[19] F. P. Ramsey, On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–285, 1930.
[20] A. Weiermann, A classification of rapidly growing Ramsey functions. *Proc. Amer. Math. Soc.*, 132(2):553–561, 2004.

The natural numbers in constructive set theory

Michael Rathjen

1. Introduction

In our forthcoming monograph on constructive set theory (see [1]), Peter Aczel and the author develop constructive mathematics within the framework of intuitionistic set theory. Many a times the monograph strives to single out the weakest set-theoretic axioms required for developing a specific part of mathematics. Being the mother structure of mathematics, the set of natural numbers deserves a lot of attention. The weakest set theory considered in [1] is *Elementary Constructive Set Theory* ($ECST$). $ESCT$ has an Axiom of Infinity which allows one to prove the existence of the von Neumann natural numbers ω as well as the Peano axioms for the successor function. Experience with $ECST$, however, induced us to conjecture that $ECST$ is not strong enough to prove the existence of the addition function on ω. As a result, we isolated a new axiom, dubbed *Finite Powers Axiom* (FPA), to ensure the existence of all primitive recursive functions on ω. The objective of this abstract is to confirm the hunch that $ECST$ is insufficient to capture the structure $(\mathbb{N}; 0, 1, +, \times)$.

2. Elementary Constructive Set Theory

$ECST$ is based on the following axioms and axiom schemes: **Extensionality**, **Pairing**, **Union**, **Replacement**, **Δ_0-Separation**, and **Strong Infinity**:

$$\exists a[Ind(a) \wedge \forall b[Ind(b) \rightarrow \forall x \in a(x \in b)]],$$

where we use the following abbreviations.

- $Empty(y)$ for *"y is the empty set"*,
- $Succ(x, y)$ for $y = x \cup \{x\}$,
- $Ind(a)$ for $(\exists y \in a)Empty(y) \wedge (\forall x \in a)(\exists y \in a)Succ(x, y)$.

3. Main Result

Theorem $ECST$ does not prove the existence of the addition function on ω.

Proof sketch: Let $\mathfrak{M}$ be an elementary recursively saturated extension of $\mathfrak{N}_L :=(\mathbb{N}; 0, S, <)$. Let $\mathbb{HYP}_{\mathfrak{M}}$ be the least admissible set above $\mathfrak{M}$ (see [2], Ch.II,sect.5). $\mathbb{HYP}_{\mathfrak{M}}$ is a model of Kripke-Platek set theory with urelement structure $\mathfrak{M}$. Let M be the domain of $\mathfrak{M}$. Since $\mathfrak{M}$ is recursively saturated it follows from [2], II.7.2

that every relation on $\mathfrak{M}$ in $\mathbb{HYP}_{\mathfrak{M}}$ is first-order definable on $\mathfrak{M}$. In consequence of this,

$$(1) \qquad \mathbb{HYP}_{\mathfrak{M}} \not\models \exists f\,[f : M \times M \to M \ \wedge\ \forall x \in M\, f(x,0) = x \\ \wedge\ \forall x, y \in M\, f(x, S(y)) = S(f(x,y))]$$

for otherwise an addition relation would be definable in $\mathfrak{M}$ and thus, as $\mathfrak{N}_L \prec \mathfrak{M}$, addition would be definable in $\mathfrak{N}_L$, which is impossible. Any subset of $\mathbb{N}$ definable in $\mathfrak{N}_L$ is either finite or has finite complement.

Next we shall utilize the fact that $\mathbb{HYP}_{\mathfrak{M}}$ is a model of Kripke-Platek set theory. By transfinite recursion in $\mathbb{HYP}_{\mathfrak{M}}$ we define the class of M-sets, $\in_M$-elementhood, and an equivalence relation $\approx_M$ on M-sets as follows: M is an M-set with $x \in_M M$ iff $x \in M$. Each $a \in M$ is an M-set with $x \in_M a$ iff $x <_{\mathfrak{M}} a$. A set b is an M-set iff all its elements are M-sets. For M-sets a, b, $a \approx_M b$ holds iff $\forall x \in a\, \exists y \in b\, x \approx_M y$ and $\forall y \in b\, \exists x \in a\, x \approx_M y$.

In a further step we view the class of M-sets equipped with $\in_M$ and $\approx_M$ (notated $(\mathcal{M}; \in_M, \approx_M)$) as a (non-wellfounded) set-theoretic universe wherein the role of ω is played by the M-set M. We wish to interpret the theory $ECST$ in $(\mathcal{M}; \in_M, \approx_M)$. This, however, requires a further step in which we graft onto $(\mathcal{M}; \in_M, \approx_M)$ a type-theoretic structure $\mathcal{T}_{\mathcal{M}}$ and subject $ECST$ to a functional interpretation in $\mathcal{T}_{\mathcal{M}}$. For intuitionistic set theories with $\in$-induction this has been carried out by W. Burr in his dissertation (see [3]). We also have to ensure that in $\mathcal{T}_{\mathcal{M}}$ one cannot define new relations on $\mathbb{HYP}_{\mathfrak{M}}$. This follows from the fact that the functional terms in the interpretation do not involve recursors and is ultimately to be proved via a normalization theorem. As a result of (1) it then follows that the existence of the addition function on ω is not provable in $ECST$. $\Box$.

Conjecture. $ECST$ has a finitistic consistency proof (i.e., the consistency of $ECST$ is provable in **PRA**).

References

[1] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences, 2001). *http://www.ml.kva.se/preprints/archive2000-2001.php*

[2] J. Barwise: *Admissible Sets and Structures* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[3] W. Burr: *Functionals in Set Theory and Arithmetic*, PhD thesis, University of Münster, 1998.

Set Theory, Automorphisms, and External Forcing

Sergei Tupailo

I am based on Boffa's 1988 theorem [1] that **NF** is consistent if there is a model M of **ZF**, a cardinal $\kappa \in M$, and an $\in$-automorphism σ of M such that $2^\kappa = \sigma(\kappa)$. Boffa has conjectured that such a model does exist, but has given no indication how this might be shown. Can we prove Boffa's conjecture, or whatever variation can be taken instead of it?

A non-trivial (external) automorphism σ of a model of **ZF** exists by the Ehrenfeucht-Mostowski theorem. With a little more effort, for example using consistency of **ZF** with $\mathbf{V} = \mathbf{L}$, one can achieve $\sigma(\kappa) > \kappa$, for κ being a regular cardinal. But Boffa's theorem asks more: it wants $2^\kappa = \sigma(\kappa)$. Can we arrange for this??

The idea which immediately springs into mind is to use forcing to achieve this goal. Further reflections have shown that the only way to go is through fine structure of Easton's iterated forcing.

References

[1] M. Boffa. ZFJ and the consistency problem for NF. *Jahrbuch der Kurt Gödel Gesellschaft* (Wien), pp. 102–106, 1988

Predicative Arithmetic and Slow Growing Bounds

S.S. Wainer

1. The Theory EA(I;O)

We describe a weak system of arithmetic developed in Ostrin and Wainer, analogous to PA, but with two kinds of variables – induction (or "input") variables x, y, z and quantifier (or "output") variables a, b, c, which play roles corresponding to the normal/safe recursion variables of Bellantoni and Cook. Our theory is closely related to the ramified, intrinsic theories developed previously by Leivant, but the formalism here is a particularly simple and straightforward revision of full arithmetic, codifying basic principles of Nelson's "Predicative Arithmetic". Thus two fundamental ideas guide the theory: (i) proofs are parameterized by their numerical inputs, which control the lengths of inductions and, once introduced, cannot be quantified; (ii) quantifiers range over values defined or computed from the inputs. The induction axioms are

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow A(x)$$

or equivalently, and more explicitly,

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow \forall a \leq x.A(a)\ .$$

Thus the theory is really a theory of input–bounded induction, and its computational strength turns out to be that of $I\Delta_0 + \exp$.

The principal logical restriction which must be applied to such theories concerns the ∃-introduction and (dually) ∀-elimination rules. Only "basic" terms: variables or 0 or their successors or predecessors, may be used as witnesses. The effect is that an arbitrary term may be used to witness an existential quantifier, but only when it has been proven to be defined. Thus, provided we formulate the theory carefully enough, Σ_1–proofs will correspond to computations, and bounds on proof-size will yield complexity measures.

A numerical function $f(\vec{x})$ is *provably recursive* if it has a provable Σ_1 definition $\exists a B(\vec{x}, a)$ with inputs $\vec{x}$.

Theorem 1.1. Every elementary (E^3) function is provably recursive in the theory EA(I;O), and every sub-elementary (E^2 or equivalently LINSPACE) function is provably recursive in the fragment which allows induction only on Σ_1 formulas.

One can obtain upper bounds by embedding EA(I;O) into an appropriate infinitary system analogous to that for full PA. However, because of the "pointwise–at–x" nature of induction, the ordinal assignment to each rule is now of a particularly simple kind: if β is assigned to a premise and α is assigned to the conclusion, then the restriction is that $\beta <_x \alpha$ for the fixed input x (where $<_x$ is the transitive closure of $\beta <_x \beta + 1$ and $\lambda_x <_x \lambda$). This means that each proof with ordinal bound α can be collapsed to a proof of finite height $G_\alpha(x)$ where G is the slow–growing hierarchy. Since, for ordinals $\alpha < \varepsilon_0$ the functions G_α are exponential polynomials, the bounds on Σ_1 proofs are elementary. Consequently,

Theorem 1.2. The provably recursive functions of EA(I;O) are exactly the elementary (E^3) functions, and the provably recursive functions of its Σ_1 inductive fragment are subelementary (E^2).

By adding axioms for (iterated) inductive definitions to EA(I;O) one obtains theories $ID_1(I;O)$, $ID_2(I;O)$ etc. Since the ordinal of ID_1 is the Bachmann–Howard ordinal, and since below this, the slow–growing functions exhaust the provably recursive functions of PA, $ID_1(I;O)$ has computational strength equivalent to full PA, and $ID_2(I;O)$ has strength that of classical ID_1 etcetera. In his Leeds thesis (2004), Williams gives a detailed analysis of the $ID_n(I;O)$ for $n < \omega$. As shown also by Wirz in his Bern thesis (2005), provable transfinite induction in such theories is of the following weak form (similar to one first considered by Schmerl):

$$A(0) \wedge \forall\beta(A(\beta) \rightarrow A(\beta+1)) \wedge \forall\lambda(\forall a \leq x.A(\lambda_a) \rightarrow A(\lambda)) \rightarrow \forall\beta <_x \alpha A(\beta)\,.$$

This enables one to prove termination of the slow–growing function $G_\alpha(x)$ but nothing more.

References

[1] S. Bellantoni and S. Cook, *A new recursion theoretic characterization of the polytime functions*, Computational Complexity Vol. 2 (1992) pp. 97 - 110.

[2] D. Leivant, *Intrinsic theories and computational complexity* , in D. Leivant (Ed) Logic and Computational Complexity, Lecture Notes in Computer Science Vol. 960, Springer-Verlag (1995) pp. 177 - 194.
[3] E. Nelson, *Predicative arithmetic*, Princeton University Press (1986).
[4] G.E. Ostrin and S.S. Wainer, *Elementary arithmetic*, Annals of Pure and Applied Logic 133 (2005) pp. 275 - 292.
[5] U. Schmerl, *Number theory and the Bachmann–Howard ordinal*, in J. Stern (Ed) Logic Colloquium '81, North–Holland Amsterdam (1982) pp. 287 - 298.
[6] S.S. Wainer and R.S. Williams, *Inductive definitions over a predicative arithmetic*, Annals of Pure and Applied Logic 136 (2005) pp. 175 - 188.
[7] R.S. Williams, *Finitely iterated inductive definitions over a predicative arithmetic*, Ph.D. thesis Leeds (2004).
[8] M. Wirz, *Wellordering two sorts: a slow-growing proof theory for variable separation*, Ph.D. thesis Bern (2005).

Phase transitions in logic and combinatorics

Andreas Weiermann

1. Introduction

Phase transition is a type of behaviour wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behaviour of the system, such shifts being usually marked by a sharp 'threshold point'. (Everyday life examples of such thresholds are ice melting and water boiling temperatures.) This kind of phenomena nowadays occurs throughout many mathematical and computational disciplines: statistical physics, evolutionary graph theory, percolation theory, computational complexity, artificial intelligence etc.

The last few years have seen an unexpected series of achievements that bring together independence results in logic, analytic combinatorics and Ramsey Theory. These achievements can be intuitively described as phase transitions from provability to unprovability of an assertion by varying a threshold parameter [13, 16]. Another face of this phenomenon is the transition from slow-growing to fast-growing computable functions [15, 18].

2. Phase transitions for Kruskal's theorem

To fix the context let us define a finite tree to be finite partial order $\langle B, \leq_B \rangle$, such that for every $b \in B$ the set $\{b' \in B : b' \leq_B b\}$ is linearly (i.e. totally) ordered trough $\leq_B$ and such that $\mathbb{B}$ contains a minimum, the root. For two given vertices $b, b' \in B$ there exists an infimum, which we denote by $b \wedge_{\mathbb{B}} b'$. (If we go from b and b' to the root the infimum is the first vertex where the path´s meet.) We say that a tree $\mathbb{B}$ is embeddable into a tree $\mathbb{B}'$ (this situation is denoted by $\mathbb{B} \trianglelefteq \mathbb{B}'$) if there exists an one to one mapping $h : B \to B'$ such that $h(b \wedge_{\mathbb{B}} b') = h(b) \wedge_{\mathbb{B}'} h(b')$ for all $b, b' \in B$.

Let us denote the number of nodes in a tree $\mathbb{B}$ with $\text{lh}\mathbb{B}$. For a given function $F : \mathbb{N} \to \mathbb{N}$ let $\text{FKT}(F)$ be the assertion: For every $K \in \mathbb{N}$ exists an $M \in \mathbb{N}$, such that for every finite sequence $(\mathbb{B}_i)_{i=0}^{M}$ of finite trees satisfying $(\forall i \le M)[\mathrm{lh}\mathbb{B}_i \le K + F(i)]$ there exist indices $i, j \le M$ with $i < j$ and $\mathbb{B}_i \trianglelefteq \mathbb{B}_j$. For any $F : \mathbb{N} \to \mathbb{N}$ the assertion $FKT(F)$ is true but concerning unprovability the following result holds.

Theorem 1 (Friedman, Matousek and Loebl)**.**

(1) If $F(i) = i$ then PA *does not prove* FKT(F) *(cf* [11]*).*
(2) Let lh$i$ *denote the binary length of i. Let $F_\alpha(i) := \alpha \cdot \mathrm{lh}i$.*
  (a) If $\alpha \le \frac{1}{2}$ then PA *proves* FKT(F_α) *(cf.* [8]*).*
  (b) If $\alpha \ge 4$ then PA *does not prove* FKT(F_α) *(cf.* [8]*).*

It is an immediate question to ask for the threshold function resp. the threshold value for α. The surprising answer is as follows (cf. [13]).

Theorem 2. *Let $T(z) := \sum_{n=0}^{\infty} t_n \cdot z^n$ be a power series such that $T(z) = z \cdot \exp(\sum_{i=1}^{\infty} \frac{T(z^i)}{i})$. Let ρ be the radius of convergence of T. Then $1 > \rho > 0$. Let $c := -\frac{1}{\log_2(\rho)}$.*

(1) $\alpha \le c$ then PA *does prove* FKT(F_α)
(2) If $\alpha > c$ then PA *does not prove* FKT(F_α).

Things change drastically if we switch to (planar) binary trees. For a given function $F : \mathbb{N} \to \mathbb{N}$ let FKTB(F) be the assertion: For every $K \in \mathbb{N}$ exists an $M \in \mathbb{N}$, such that for every finite sequence $(\mathbb{B}_i)_{i=0}^{M}$ of finite binary trees satisfying $(\forall i \le M)[\mathrm{lh}\mathbb{B}_i \le K + F(i)]$ there exist indices $i, j \le M$ with $i < j$ and $\mathbb{B}_i \trianglelefteq \mathbb{B}_j$.

It is again an immediate question to ask for the threshold function resp. the threshold value. The answer is as follows. [20].

Theorem 3. *Let $c := \frac{1}{2}$.*

(1) $\alpha \le c$ then PA *does prove* FKT(F_α)
(2) If $\alpha > c$ then PA *does not prove* FKT(F_α).

Surprisingly there is an extremely sharp phase transition from feasibility to unfeasibility in case of (planar) binary trees. For a given function $F : \mathbb{N} \to \mathbb{N}$ let B(F)(K) be the least $M \in \mathbb{N}$, such that for every finite sequence $(\mathbb{B}_i)_{i=0}^{M}$ of finite binary trees satisfying $(\forall i \le M)[\mathrm{lh}\mathbb{B}_i \le \mathrm{lh}K + F(i)]$ there exist indices $i, j \le M$ with $i < j$ and $\mathbb{B}_i \trianglelefteq \mathbb{B}_j$.

Theorem 4. *Let $c := \frac{1}{2}$.*

(1) $\alpha < c$ then B(F_α) *is bounded by a polynomial.*
(2) $\alpha = c$ then B(F_α) *is bounded by a polynomial time computable function, but not by a polynomial.*
(3) If $\alpha > c$ then B(F_α) *eventually dominates every provably recursive function of* PA.

It would be interesting to see whether a similar result holds for nonplanar trees at the corresponding threshold point. There are some weak indications that things will behave differently.

3. Phase transitions for ε_0

We code ordinals below ε_0 in the following way: Let p_i be the i-th prime for $i \geq 1$. Let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ and let $\lceil 0 \rceil := 1$ und $\lceil \alpha \rceil := p_{\lceil \alpha_1 \rceil} \cdot \ldots \cdot p_{\lceil \alpha_n \rceil}$, if $\alpha =_{NF} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$. Then the mapping $\lceil \cdot \rceil$ is a bijection between ε_0 and $\mathbb{N}^+$ and we may induce an ordering which we denote with $\prec$ on $\mathbb{N}^+$. It can easily be seen that the induced ordering can be defined by a formula in the language of PA.

The scheme $TI(\prec, P)$ can be written as follows:

$$TI(\prec, P) := (\forall n \in \mathbb{N}^+)[(\forall m \prec nP(m)) \to P(n))] \to \forall n \in \mathbb{N}^+ P(n).$$

The ordinal ε_0 is characteristic for an ordinal-theoretic phase transition for PA in so far as PA proves the transfinite induction for every initial segment of ε_0. More explicitly, PA proves for every $k \in \mathbb{N}^+$ the assertion
$\forall m \prec k[(\forall n \prec mP(n)) \to P(m)] \to \forall m \prec kP(m)$.

Via a compactness argument $\mathbb{N} \models TI(\prec, P)$ yields for every function $F : \mathbb{N} \to \mathbb{N}$ the truth of the following assertion FWO(F):
$(\forall K)(\exists M)(\forall m_1, \ldots, m_n \in \mathbb{N}^+)\big[(\forall i \leq M) \lceil m_i \rceil \leq K + F(i) \to (\exists i < M) m_i \prec m_{i+1}\big]$.

Theorem 5. *(1) Let* $F(i) = 2^i$. *Then* PA *does not prove* FWO(F) *(cf.* [10, 4]*)* .

(2) Let $\mathrm{lh}i_1 := \mathrm{lh}i$ *and* $\mathrm{lh}i_{d+1} := \mathrm{lhlh}i_d$. *If* $F(i) = 2^{\mathrm{lh}i \cdot \mathrm{lh}i_d}$ *then* PA *does not prove* FWO(F).

(3) Let $\log^*(i) := \min\{d : \mathrm{lh}i_d \leq 2\}$. *If* $F(i) = 2^{\mathrm{lh}i \cdot \log^*(i)}$ *then* PA *proves* FWO(F).

¿From the viewpoint of analytic number theory the last phase transition result refers to a multiplicative norm on ordinals. It is a natural question to investigate phase transitions in the additive setting. To this end we define a norm $N : \varepsilon_0 \to \mathbb{N}$ as follows. $N(0) := 0$ and $N\alpha := n + N\alpha_1 + \ldots + N\alpha_n$, if $\alpha =_{NF} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$. Via the natural isomorphis between ε_0 and $\mathbb{N}^+$ the norm can be extended to $\mathbb{N}^+$.

As before a compactness argument applied to $TI(\prec, P)$ yields for every function $F : \mathbb{N} \to \mathbb{N}$ the truth of the following assertion FWON(F):
$(\forall K)(\exists M)(\forall m_1, \ldots, m_n \in \mathbb{N}^+)\big[(\forall i \leq M) N m_i \leq K + F(i) \to (\exists i < M) m_i \prec m_{i+1}\big]$.

Theorem 6. *(1) If* $F(i) = i$ *then* PA *does not prove* FWON(F) *(cf.* [10, 4]*).*

(2) If $F(i) = \mathrm{lh}i \cdot \mathrm{lh}i_d$ *then* PA *does not prove* FWON(F).

(3) If $F(i) = \mathrm{lh}i \cdot \log^*(i)$ *then* PA *proves* FWON(F).

The optimal phase transition in this case has been obtained by Arai[1].

Remark: Sharp phase transition thresholds have been obtained in the meantime for the hydra games, the Goodstein principle, the Paris Harrington principle, the Kananmori McAloon principle, the Friedman-style Ramsey principle, and various parameterized hierarchies of recursive functions (including the Grzegorczyk- and extended Grzegorczyk hierarchy). (Several related results have been obtained over the last years in joint work with the workshop coorganizers Bovykin and Carlucci, and with Kojman, Lee and Omri.)

References

[1] T. Arai, *On the slowly well orderedness of ε_0*, *Math. Log. Q.*, **48** (2002), 125–130.
[2] A. Bovykin and A. Weiermann. *The strength of infinitary rameyan principles can be accessed by their densities*. Preprint 2006. Submitted to APAL.
[3] L. Carlucci, G. Lee and A. Weiermann *Classifying the phase transition threshold for regressive Ramsey functions.* Preprint 2006 (submitted to JAMS).
[4] H. Friedman, M. Sheard: *Elementary descent recursion and proof theory.* Ann. Pure Appl. Logic 71 (1995), no. **1**, 1–45. MR1312428 (96c:03110)
[5] A. Kanamori and K. McAloon, *On Gödel incompleteness and finite combinatorics*, Ann. Pure Appl. Logic, **33** (1987), no. 1, 23–41.
[6] M. Kojman, G. Lee, E. Omri and A. Weiermann, *Sharp thresholds for the Phase Transition between Primitive Recursive and Ackermaniann Ramsey Numbers.* Preprint 2005. Submitted to JCTA.
[7] G. Lee, *Phase Transitions in Axiomatic Thought*, PhD thesis (written under the supervision of A. Weiermann), Münster 2005, 121 pages.
[8] M. Loebl and J. Matouv(s)ek. On undecidability of the weakened Kruskal theorem. In *Logic and combinatorics (Arcata, Calif., 1985)*, 275-280. Amer. Math. Soc., Providence, RI, 1987.
[9] H. Schwichtenberg, *Eine Klassifikation der ε_0-rekursiven Funktionen*, Z. Math. Logik Grundlagen Math., **17** (1971), 61–74.
[10] S. G. Simpson: *Nonprovability of certain combinatorial properties of finite trees.* Harvey Friedman's research on the foundations of mathematics, 87–117, Stud. Logic Found. Math., **117**, North-Holland, Amsterdam, 1985.
[11] R L. Smith: *The consistency strength of some finite forms of the Higman and Kruskal theorems.* In *Harvey Friedman's Research on the Foundations of Mathematics*, L. A. Harrington et al. (editors), (1985), pp. 119–136.
[12] A. Weiermann, *How to characterize provably total functions by local predicativity*, J. Symbolic Logic, **61** (1996), no.1, 52–69.
[13] A. Weiermann, *An application of graphical enumeration to* PA, J. Symbolic Logic 68 (2003), no. **1**, 5–16.
[14] A. Weiermann, *An application of results by Hardy, Ramanujan and Karamata to Ackermannian functions*, Discrete Mathematics and Computer Science, **6** (2003), 133–142.
[15] A. Weiermann: *A very slow growing hierarchy for* Γ_0. Logic Colloquium '99, 182–199, Lect. Notes Log., **17**, Assoc. Symbol. Logic, Urbana, IL, 2004.
[16] A. Weiermann, *A classification of rapidly growing Ramsey functions*, Proc. Amer. Math. Soc., 132 (2004), no. **2**, 553–561.
[17] A. Weiermann, *Analytic Combinatorics, proof-theoretic ordinals and phase transitions for independence results*, Ann. Pure Appl. Logic, **136** (2005), 189–218.
[18] A. Weiermann:*An extremely sharp phase transition threshold for the slow growing hierarchy.* Mathematical Structures in Computer Science **16**, (2006), 925-946.
[19] A. Weiermann, *Classifying the phase transition for Paris Harrington numbers*, *Preprint*, 2005.
[20] A. Weiermann: *Phase transitions thresholds for some Friedman style independence results* Preprint 2006, MLQ **53** (1), (2007), 4-18.

Combinatorial Principles in Bounded Arithmetic.

Alan Woods

$I\Delta_0$ is the axiom system similar to Peano Arithmetic, but with induction hypotheses restricted to being Δ_0 formulas, i.e., arithmetic formulas with only *bounded quantifiers*. There is a long list of theorems of elementary number theory for which no proof in $I\Delta_0$ is known, and yet which are provable if one adds "new" functions from the Grzegorczyk class $\mathcal{E}^2$. As such functions do not exceed polynomial growth, these theorems are candidates for independence results not relying on fast growing functions. Typically the theorems in question can in fact be proved in $I\Delta_0$ augmented by some suitable combinatorial "counting principle", examples of which include the *Pigeonhole Principle* and the *Equipartition Principle*. So these principles may also turn out to be independent of $I\Delta_0$.

When m is a function $m(n)$ of n, let $PHP_n^m(f)$ denote a sentence in the language of arithmetic, augmented by an "extra" function symbol f, expressing that for all n, f does not map $\{1,\ldots,m\}$ one-to-one into $\{1,\ldots,n\}$. $PHP_n^m(\Delta_0)$ will denote the axiom schema asserting this principle for those functions f which are definable by Δ_0 formulas.

Open Problem 1. Does $I\Delta_0 \vdash PHP_n^{n+1}(\Delta_0)$, or even $I\Delta_0 \vdash PHP_n^{n^2}(\Delta_0)$?

The obvious way to try to show $I\Delta_0 \vdash PHP_n^{n+1}(\Delta_0)$, a problem which incidently goes back to Macintyre [4], is to start "let f be a function ... ". This does *not* work, because Ajtai [1] proved that

$$I\Delta_0(f) \nvdash PHP_n^{n+1}(f).$$

In fact he proved more, namely that the natural propositional tautologies PHP_n^{n+1} expressing the pigeonhole principle for $n+1$ objects, do not have constant depth LK proofs of size bounded by some fixed polynomial in n. The connection is that à proof of $PHP_n^m(f)$ in $I\Delta_0(f)$ would translate into a very uniform family of polynomial size proofs of the corresponding tautalogies PHP_n^m in propositional calculus. The constant quantifier depth in the single predicate calculus proof, translates into constant depth (of alternation of connectives) in the propositional proofs. In [3] and [7] it was shown that constant depth proofs of PHP_n^{n+1} require roughly exponential size, dramatically strengthening Ajtai's theorem.

Notice also, that we get stonger theorems if we state provability results in terms of weak axiom systems such as $I\Delta_0(f)$, and unprovability results in terms of propositional proofs of the appropriate size and constant depth.

The weak pigeonhole principle $PHP_n^{2n}(f)$ is of particular interest because it is known that from $I\Delta_0$ enhanced by the addition of $PHP_n^{2n}(f)$ for Δ_0 definable functions f, the existence of quadratic nonresidues for all odd primes [4], the existence of arbitrarily large prime numbers [6], and Lagrange's theorem on sums of four squares [2] can all be proved, but no proofs for these using only $I\Delta_0$ have been found [4], [8].

Open Problem 2. Does $I\Delta_0(f) \vdash PHP_n^{2n}(f)$, or even $I\Delta_0(f) \vdash PHP_n^{n^2}(f)$?

Reformulated in terms of propositional proofs, the analogous problem is:

Open Problem 3. Do PHP_n^{2n} and $\mathrm{PHP}_n^{n^2}$ have polynomial size, constant depth LK proofs?

It is known ([9] or [5]) that if $\mathcal{E}^2$ cardinality functions $|\{n \leq x : \theta(n, \mathbf{y})\}|$ counting the elements $n \leq x$ satisfying certain Δ_0 formulas $\theta(n, \mathbf{y})$ (with parameters $\mathbf{y}$) together with their recursive definitions, are added to $I\Delta_0$, and if these functions are allowed to be used in induction hypotheses, then it becomes possible to prove each instance of $PHP_n^{n+1}(\Delta_0)$. It is of some interest to know whether "natural" instances of θ suffice for consequences of $I\Delta_0 + PHP(\Delta_0)$ such as the existence of arbitrarily large primes. Let $\pi(x)$ denote the number of primes less than or equal to x and let $I\Delta_0(\pi)$ denote $I\Delta_0$ augmented by π together with its natural recursive definition. Twenty five years ago, I conjectured in [9] that:

Conjecture 4. $I\Delta_0(\pi) \vdash$ There exist arbitrarily large prime numbers.

The best result I know in this direction appears in a recent paper by Cornaros and myself [10]. Recall that *Bertrand's Postulate* (a theorem of Chebyshev) asserts that for every $n \geq 1$ there is some prime p satisfying $n < p \leq 2n$. Define $\xi(x, y, e)$ to be the number of primes less than or equal to x for which the integer part $[y/p^e]$ is odd. Let $I\Delta_0(\xi)$ be $I\Delta_0$ augmented by ξ and its obvious recursive definition.

Theorem 1. $I\Delta_0(\xi) \vdash$ Bertrand's Postulate .

This theorem is a step in the direction of the conjecture, because $\pi(x)$ can be defined, and its recursive definition proved, in $I\Delta_0(\xi)$.

References

[1] M. Ajtai. *The complexity of the pigeonhole principle.* 29th Annual Symposium on Foundations of Computer Science, IEEE, 1988.

[2] A. Berarducci and B. Intrigila. *Combinatorial principles in elementary number theory.* Ann. Pure Appl. Logic, **55** (1991) 35–50.

[3] J. Krajív(c)ek, P. Pudlák, and A. Woods. *Exponential lower bound to the size of bounded depth Frege proofs of the pigeonhole principle.* Random Structures and Algorithms **7** (1995) 15–39.

[4] A. J. Macintyre. *The strength of weak systems.* In *Logic, Philosophy of Science and Epistemology: Proceedings 11th International Wittgenstein Symposium, Kirchberg/Wechsel, Austria, 1986*, pages 43–59. Hölder-Pichler-Tempsky, Vienna, 1987.

[5] J. B. Paris and A. J. Wilkie. *Counting problems in bounded arithmetic.* In *Methods in Mathematical Logic: Proceedings of the 6th Latin American Symposium on Mathematical Logic 1983*, Lecture Notes in Mathematics, **1130** (1985) 317–340.

[6] J. B. Paris, A. J. Wilkie, and A. R. Woods. *Provability of the pigeonhole principle and the existence of infinitely many primes.* J. Symbolic Logic **53** (1988) 1235–1244.

[7] T. Pitassi, P. Beame, and R. Impagliazzo. *Exponential lower bounds for the pigeonhole principle.* Comput. Complexity, **3** (1993) 97–140.

[8] A. J. Wilkie. *Some results and problems on weak systems of arithmetic.* In A. Macintyre, L. Pacholski, and J. Paris, editors, *Logic Colloquium '77*, pages 237–248. North-Holland, 1978.

[9] A. R. Woods. *Some Problems in Logic and Number Theory, and their Connections.* Ph.D. Thesis, University of Manchester, 1981.

[10] A. R. Woods, and Ch. Cornaros. *On bounded arithmetic augmented by the ability to count certain sets of primes.* Preprint, December 2006.

Upper and lower bounds for proving Herbrand consistency in weak arithmetics

Konrad Zdanowski
(joint work with Zofia Adamowicz)

We consider the family of arithmetics of the form $I\Delta_0 + \Omega_i$, where $I\Delta_0$ is the induction restricted to bounded formulas only and Ω_i is the axiom stating that the function ω_i is total. Let $|x| = \ulcorner \log(x+1) \urcorner$. We define $\omega_1(0) = 0$ and $\omega_1(x) = 2^{(|x|-1)^2}$, for $x > 0$. Similarly, $\omega_{i+1}(0) = 0$ and $\omega_{i+1}(x) = 2^{\omega_i(|x|-1)}$, for $x > 0$. For other basic notions we refer to [HP93].

One of the main methods of showing that one set of axioms, say T, is strictly stonger than the other one, say $S \subseteq T$, is to show that $T \vdash \mathrm{Con}_S$. However, as was shown by Wilkie and Paris in [WP87] this method does not work for bounded arithmetic theories if we use the usual Hilbert style provability predicate. Indeed, they proved that even the strong arithmetic $I\Delta_0 + \exp$ does not prove the Hilbert style consistency of Robinons's arithmetic Q, that is $I\Delta_0 + \exp$ does not prove that there is no Hilbert prove of $0 \neq 0$ from Q. Thus, if we hope to differentiate various bounded arithmetics we should use some other provability notions, like tableux or Herbrand provability. Indeed, for these notions it is usually easier to show that a given theory is consistent since the Herbrand proofs are of bigger size than Hilbert ones. Only when we know that the iterated exponentiation function is total we can prove the equivalence of these notions of provability.

For some time it was even unknown whether the second Gödel incompleteness theorem holds for arithmetics $I\Delta_0 + \Omega_1$ and Herbrand style provability predicate. Adamowicz and Zbierski in [AZ01] proved the second incompleteness theorem for $I\Delta_0 + \Omega_i$, for $i \geq 2$ and Herbrand notion of consistency and later Adamowicz in [A01] proved this result for $I\Delta_0 + \Omega_1$. Lately, Kołodziejczyk showed in [K06] a strengthening of these results. He proved that there is a finite fragment S of $I\Delta_0 + \Omega_1$ such that no theory $I\Delta_0 + \Omega_i$ proves Herbrand consistency of S. Thus, if one wants to differentiate bounded arithmetics by means of provability of Herbrand consistency one should consider Herbrand proofs restricted to some cuts of a given model of a bounded arithmetic. So, let $\mathrm{HCons}(T, I)$ be an arithmetical statement saying that for any set Λ of terms of depths in I from the skolemization of the theory T there is a boolean valuation p on pairs of terms from Λ such that p makes all axioms from T true.

The following theorem is known.

Theorem 1. $I\Delta_0 + \Omega_i$ proves its Herbrand consistency restricted to the terms of depth not greater than $\log^{i+3}$ that is $I\Delta_0 + \Omega_i \vdash \mathrm{HCons}(I\Delta_0 + \Omega_i, \log^{i+3})$.

The proof of theorem 1 relies on the fact that terms of the skolemized $I\Delta_0 + \Omega_i$ of depths in $\log^{i+3}$ denotes only the elements from the logarithm of a given model

$M \models I\Delta_0 + \Omega_i$. Thus, one can use the truth definition for Δ_0 formulas to interpret all these terms in M and to obtain a suitable boolean valuation.

On the other hand we have the following theorem.

Theorem 2 (AZ06)**.** Let $T = I\Delta_0 + \Omega_i$. Then, for any $\varepsilon > 0$, T does not prove its Herbrand consistency restricted to terms of depth not greater than $(1+\varepsilon)\log^{i+2}$ that is $T \nvdash \mathrm{HCons}(T, (1+\varepsilon)\log^{i+2})$.

It is tempting to close the gap by proving, at least for some $i \geq 1$, either that

$$I\Delta_0 + \Omega_i \vdash \mathrm{HCons}(I\Delta_0 + \Omega_i, \log^{i+2}) \tag{1}$$

or

$$I\Delta_0 + \Omega_i \nvdash \mathrm{HCons}(I\Delta_0 + \Omega_i, A\log^{i+3}), \text{ for some } A \in \mathbb{N}. \tag{2}$$

Indeed both conjectures would have interesting consequences for bounded arithmetics. If (1) holds then $I\Delta_0 + \Omega_{i+1}$ would not be Π_1–conservative over $I\Delta_0 + \Omega_i$.

On the other hand, if (2) holds this would mean that we cannot mimic the proof of theorem 1 for the cut $A\log^{i+3}$. We could conclude that there is no Δ_0–truth definition which uses only Ω_i function and has properties provable in $I\Delta_0 + \Omega_i$.

References

[A01] Z. Adamowicz, *On tableaux consistency in weak theories*, prepring 618, Institute of Mathematics of the Polish Academy of Sciences, 2001.

[AZ01] Z. Adamowicz and P. Zbierski, *On Herbrand consistency in weak arithmetics*, in Archive for Mathematical Logic, 40(2001), pp. 399–413.

[AZ06] Z. Adamowicz and K. Zdanowski, *Lower bounds in for proving Herbrand consistency in weak arithmetics*, submitted to Archive for Mathematical Logic.

[HP93] P. Hájek and P. Pudlák, *Metamathematics of first–order arithmetic*, Springer–Verlag, 1993.

[K06] L. A. Kołodziejczyk, *On the Herbrand notion of consistency for finitely axiomatizable fragments of bounded arithmetic theories*, in Journal of Symbolic Logic 71(2006), pp. 624-638.

[WP87] A. J. Wilkie and J. B. Paris,*On the scheme of induction for bounded arithmetical formulas*, in Annals of Pure and Applied Logic, 35(1987), pp. 261–302.

Reporter: Andrey Bovykin

Participants

Prof. Dr. Lev D. Beklemishev
V.A. Steklov Institute of
Mathematics
Russian Academy of Sciences
8, Gubkina St.
119991 Moscow GSP-1
RUSSIA

Dr. Andrey I. Bovykin
Department of Computer Science
Liverpool University
Chadwick Building
Peach Street
GB-Liverpool L69 3BX

Prof. Dr. Wilfried Buchholz
Mathematisches Institut
Universität München
Theresienstr. 39
80333 München

Dr. Lorenzo Carlucci
John Cabot University, Via della Lungara
233 00165 - Rome, Italy.

Dr. Lew Gordeew
WSI
Universität Tübingen
Sand 13
72076 Tübingen

Prof. Dr. Henryk Kotlarski
Faculty of Mathematics
Cardinal Stefan Wyszynski Univers.
ul. Dewajtis 5
01-815 Warszawa
Poland

Prof. Dr. Alberto Marcone
Dipartimento di Matematica e
Informatica
Universita di Udine
Via delle Scienze 206
I-33100 Udine

Prof. Dr. Joseph R. Mileti
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
USA

Prof. Dr. Antonio Montalban
Department of Mathematics
University of Chicago
5734 S. University Avenue
Chicago,Il 60637
USA

Dr. Eran Omri
Dept. of Math. and Computer Science
Ben Gurion University of the Negev
PO BOX 653
84105 Beer Sheva
ISRAEL

Prof. Dr. Michael Rathjen
School of Mathematics
University of Leeds
GB-Leeds, LS2 9JT

Dr. Sergei Tupailo
Institute of Cybernetics
Akadeemia tee 21
EE-12618 Tallinn
ESTONIA

Prof. Dr. Stanley S. Wainer
School of Mathematics
University of Leeds
GB-Leeds, LS2 9JT

Dr. Andreas Weiermann
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
NL-3508 TA Utrecht

Prof. Dr. Alan Woods
Department of Mathematics
University of Western Australia
Crawley 6009 W.A.
Australia

Prof. Dr. Konrad Zdanowski
Mathematical Institute
Polish Academy of Sciences
ul. Sniadeckich 8
00-956 Warsaw
POLAND