Mathematisches Forschungsinstitut Oberwolfach

Report No. 55/2006

Infinite-Dimensional Lie Theory

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December 10th – December 16th, 2006

Abstract. The workshop focussed on recent developments in infinite-dimensional Lie theory. The talks covered a broad range of topics, such as structure and classification theory of infinite-dimensional Lie algebras, geometry of infinite-dimensional Lie groups and homogeneous spaces and representation theory of infinite-dimensional Lie groups, Lie algebras and Lie-superalgebras.

Mathematics Subject Classification (2000): 22E, 17B, 20G, 11E.

Introduction by the Organisers

Nowadays infinite-dimensional Lie theory is a core area of modern mathematics, covering a broad range of branches, such as the structure and classification theory of infinite-dimensional Lie algebras, geometry of infinite-dimensional Lie groups and their homogeneous spaces, and representation theory of infinite-dimensional Lie groups, Lie algebras and Lie-superalgebras. The focus of this workshop was on recent developments in all of these areas with particular emphasis on connections with other branches of mathematics, such as algebraic groups and Galois cohomology.

The meeting was attended by 52 participants from many European countries, Canada, the USA, Brazil, Japan and Australia. The meeting was organized around a series of 23 lectures each of 50 minutes duration representing the major recent advances in the area. We feel that the meeting was exciting and highly successful. The quality of the lectures, several of which surveyed recent developments, was outstanding. The exceptional atmosphere of the Oberwolfach Institute provided an optimal environment for bringing people working in algebraically, geometrically or analytically oriented areas of infinite-dimensional Lie theory together, and to create an atmosphere of scientific interaction and cross-fertilization.

Without going too much into detail, let us mention some important new developments that were showcased during the workshop. In the structure theory of infinite-dimensional Lie algebras, the classification of extended affine Lie algebras, based on the notion of a Lie torus has now reached a mature state (Neher). In the classification theory of infinite-dimensional Lie algebras, several deep results were obtained recently with Galois cohomology methods exhibiting exciting connections between forms of multiloop algebras and the Galois theory of forms of algebras over rings (Allison, Gille, Chernousov). This branch of structure theory is complemented by the connection between the classification of generalized Kac–Moody algebras and automorphic forms (Scheithauer).

In the representation theory of infinite-dimensional Lie algebras, the most exciting new developments concern various kinds of categories of representations of current algebras and Kac–Moody–Lie (super-)algebras (Benkart, Chari, Futorny, Gorelik, Kumar, Littelmann, Serganova). Another interesting, recently very active direction of Kac–Moody theory are Kac–Moody groups over finite fields, which leads to a new class of infinite simple groups (Caprace).

On geometric and analytic Lie theory, we had exiting talks on new geometric directions in the representation theory of Banach–Lie groups, related to Banach–Lie–Poisson spaces (Ratiu), and applications of heat kernel measures in the representation theory of loop groups (Pickrell). On the opposite side of the spectrum of Lie group theory, namely direct limit theory, crucial progress has been made on direct limits of unitary representations, as well as in the context of direct limits of infinite-dimensional groups (Wolf, Glöckner). We further had several contributions dealing with geometric aspects such as Chern forms, gerbes and generalized projective geometries (Paycha, Schweigert, Bertram).

Finally, we had several exciting talks about several more particular results, dealing with vertex operator algebras, polyzeta values and quantization (Billig, Mathieu, Omori). More specific information is contained in the abstracts which follow in this volume.

Workshop: Infinite-Dimensional Lie Theory

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Abstracts

Coadjoint orbits and the beginnings of a geometric representation theory

Tudor S. Ratiu
(joint work with D. Beltiţă and A.B. Tumpach)

This talk presents several results in the newly introduced theory of Banach Lie-Poisson spaces. The notion of a Lie-Poisson space is as old as the concept of a Lie algebra and both were introduced simultaneously by S. Lie. A Lie-Poisson space is a Poisson vector space with the property that its dual is invariant under the Poisson bracket, which is equivalent to the statement that the Poisson bracket is linear. In the finite dimensional case the notions of Lie algebras and Lie-Poisson spaces are equivalent in the sense that for any Lie algebra $\mathfrak{g}$ its dual $\mathfrak{g}^*$ is a Lie-Poisson space and, conversely, given a Lie-Poisson space its dual is a Lie algebra. This is so because finite dimensional vector spaces are reflexive, the operation of taking the dual defining an isomorphism between these two categories. To generalize this to infinite dimensions, it is reasonable to assume that a Lie-Poisson space is a Banach space $\mathfrak{b}$ endowed with a Poisson bracket $\{\cdot,\cdot\}$ such that the bracket of any two linear continuous functions on $\mathfrak{b}$ is again a linear continuous function. This implies that $(\mathfrak{b}^*, \{\cdot,\cdot\})$ is a Banach Lie algebra. In order to preserve the correspondence between Banach Lie-Poisson spaces and Banach Lie algebras it is necessary to restrict to those Banach Lie algebras $(\mathfrak{g}, [\cdot,\cdot])$ that admit a predual $\mathfrak{g}_*$ and satisfy in addition the condition that $\mathrm{ad}^*_{\mathfrak{g}} : \mathfrak{g}^* \to \mathfrak{g}^*$ preserves the predual $\mathfrak{g}_*$. Thus, in the infinite dimensional case, Banach Lie-Poisson spaces form a subcategory of the category of Banach Lie algebras.

This new class of infinite-dimensional linear Poisson manifolds is remarkable in several respects: it includes all the preduals of W^*-algebras, thus establishing a bridge between Poisson geometry and the theory of operator algebras (a crucial example is the Banach space $\mathfrak{S}_1(\mathcal{H})$ of linear trace class operators on a separable complex Hilbert space $\mathcal{H}$ which is predual to the Banach Lie algebra $\mathcal{B}(\mathcal{H})$ of all linear bounded operators on $\mathcal{H}$), and hence it provides links with algebraic quantum theories (see [17]); it interacts in a fruitful way with the theory of extensions of Lie algebras (see [12]); and finally, there exist large classes of Banach Lie-Poisson spaces which share with the finite-dimensional Poisson manifolds the fundamental property that the characteristic distribution is integrable, the corresponding integral manifolds being in addition Poisson submanifolds which are symplectic and, in several important situations, are even weak *Kähler* manifolds (see [3]). More precisely, preduals of Banach Lie algebras which are invariant under the coadjoint representation of the underlying Banach Lie group are Banach Lie-Poisson spaces. If the coadjoint isotropy subgroup of an element in the predual is a Banach Lie subgroup (in the sense that it is closed and an embedded submanifold), then the coadjoint orbit is a weakly immersed (the inclusion is smooth and has injective derivative, but no splitting condition, or even a closed range condition on the derivative, usually imposed in the definition of an immersion, holds) weak symplectic manifold and is a symplectic leaf of the Banach Lie-Poisson structure on the predual. Thus, the search for weak symplectic coadjoint orbits in preduals of Banach Lie algebras is motivated. In addition, one would like to find examples of such orbits that are weakly Kähler and tie these to geometric representation theory.

The restricted Grassmannian (see [14]) is a remarkable infinite-dimensional weakly *Kähler* manifold that plays an important role in many areas of mathematics and physics. It is related to the integrable system defined by the KP hierarchy (see [14]) and to the fermionic second quantization (see [17]).

Using the method of central extensions one can construct a certain Banach Lie-Poisson space $\widetilde{\mathfrak{u}}_2$ whose characteristic distribution is integrable and one of the integral manifolds of this distribution is symplectomorphic to the connected component $\mathrm{Gr}^0_{\mathrm{res}}$ of the restricted Grassmannian. Using a similar method, the restricted Grassmannian is realized as a symplectic leaf in another Banach Lie-Poisson space $(\mathfrak{u}_{\mathrm{res}})_*$, the predual to a 1-dimensional central extension of the restricted Lie algebra $\mathfrak{u}_{\mathrm{res}}$. This second construction is closely related to the theory of extensions initiated in [11, 12].

By the very construction of the Banach Lie-Poisson space $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$, the predual $(\mathfrak{u}_{\mathrm{res}})_*$ appears as a Poisson submanifold of $(\tilde{\mathfrak{u}}_{\mathrm{res}})_*$ and carries a natural structure of Banach Lie-Poisson space. The integrability of the characteristic distribution of $(\mathfrak{u}_{\mathrm{res}})_*$ is raised into question by several unpleasant properties of the restricted $*$-algebra $\mathcal{B}_{\mathrm{res}}$: its unitary group is unbounded, its natural predual is not spanned by its positive cone, and a conjugation theorem for its maximal Abelian $*$-subalgebras fails to be true. Despite these unpleasant properties, it turns out that the characteristic distribution of $(\mathfrak{u}_{\mathrm{res}})_*$ has numerous smooth integral manifolds which are, in particular, *smooth* coadjoint orbits of the restricted unitary group $\mathrm{U}_{\mathrm{res}}$.

The study of geometric properties of state spaces is a basic topic in operator algebras. The GNS construction produces representations of operator algebras out of states. A method to do this is to proceed as in the theory of geometric realizations of Lie group representations (see e.g., [8]) and to try to build the representation spaces as spaces of sections of certain vector bundles. The basic ingredient in this construction is the reproducing kernel Hilbert space. This method can be applied to the case of group representations obtained by restricting GNS representations to unitary groups of C^*-algebras. For these representations, one can construct one-to-one intertwining operators from the representation spaces onto reproducing kernel Hilbert spaces of sections of certain Hermitian vector bundles. The construction of these vector bundles is based on a choice of a sub-C^*-algebra that is related in a suitable way to the state involved in the GNS construction.

In the case of normal states of W^*-algebras there is a natural choice of the subalgebra (namely the centralizer subalgebra), and the base of the corresponding vector bundle is just one of the symplectic leaves studied in [3]. Since the corresponding symplectic leaves are just unitary orbits of states, the geometric representation theory initiated in [4] provides, in particular, a geometric interpretation of the result in [7], namely the equivalence class of an irreducible GNS representation only depends on the unitary orbit of the corresponding pure state.

In [6] and references therein one can find several interesting results regarding the classification of unitary group representations of various operator algebras. The point of [4] is to show that some of these representations (namely the ones obtained by restricting GNS representations to unitary groups) can be realized geometrically following the pattern of the classical Borel-Weil theorem for compact groups. This raises the challenging problem of finding geometric realizations of more general representations of unitary groups of operator algebras.

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Heat kernel measures and regular representations for loop groups

Doug Pickrell

An abstract Wiener group is an inclusion of a separable Hilbert Lie group into a Banach Lie group, $\mathbf{H} \subset \mathbf{G}$, such that the corresponding inclusion of Lie algebras, $\mathbf{h} \subset \mathbf{g}$, is an abstract Wiener space in the sense of Leonard Gross. A fundamental example is the Sobolev inclusion

$$\mathbf{H} = W^s(X, G_0) \subset \mathbf{G} = C^0(X, G_0) \tag{1}$$

where X is a compact Riemannian manifold, G_0 is a finite dimensional real Lie group with fixed inner product on its Lie algebra, W^s denotes an L^2 Sobolev space with degree of smoothness s, and, most significantly, $s > dim(X)/2$, the critical degree. The basic idea of an abstract Wiener group appears in [2], but I am unaware of any further development of the idea in a nonabelian context.

Let $\nu_t^{\mathbf{h}\subset\mathbf{g}}$ denote the convolution semigroup of Gaussian measures on $\mathbf{g}$ with Cameron-Martin space $\mathbf{h}$. There is a corresponding convolution semigroup of "heat kernel measures" on $\mathbf{G}$, defined as a weak limit

$$\nu_t^{\mathbf{H}\subset\mathbf{G}} = \lim_{N\to\infty} \left(exp_*(\nu_{t/N}^{\mathbf{h}\subset\mathbf{g}}) * .. * exp_*(\nu_{t/N}^{\mathbf{h}\subset\mathbf{g}}) \right). \tag{2}$$

For $\mathbf{H} \subset \mathbf{G}$ as in (1) above, if V denotes a finite number of points in X, then there is a projection

$$\mathbf{G} = C^0(X, G_0) \to G_0^V : g \to (g(v))_{v\in V}.$$

The image of $\nu_t^{\mathbf{H}\subset\mathbf{G}}$ is the heat kernel semigroup which corresponds to the left invariant metric on G_0^V defined on ${\mathfrak{g}_0}^V$ by

$$\langle (x_v), (y_v) \rangle = \sum_{v\in V} c_{v,w} \langle x_v, y_v \rangle$$

where the matrix $(c_{v,w})$ is inverse to the "covariance matrix"

$$(c^{v,w} = (1 + \Delta_X)^{-s}(v, w))_{v,w\in V}.$$

For the linear Gaussian measures, $\nu_t^{\mathbf{h}\subset\mathbf{g}}$, $t > 0$, the translations which fix the measure class correspond to vectors in $\mathbf{h}$ (the "Cameron-Martin theorem"). For the measures $\nu_t^{\mathbf{H}\subset\mathbf{G}}$, $t > 0$, the analogous question is apparently open. If $\nu_t^{\mathbf{H}\subset\mathbf{G}}$ is left translation quasiinvariant by an element $h \in \mathbf{H}$, then it is also right and conjugation quasiinvariant by h, because $\nu_t^{\mathbf{H}\subset\mathbf{G}}$ is inversion invariant. This suggests (to me, but does not prove) that $\nu_t^{\mathbf{h}\subset\mathbf{g}}$ should be $Ad(h)$-quasiinvariant, and this would imply that $Ad(h)$ satisfies the condition $AA^* - 1$ is a Hilbert-Schmidt operator on $\mathbf{h}$. Conjecturally this condition is necessary and sufficient for quasiinvariance. For the fundamental example (1), assuming that K is compact, $Ad(h)$

is a pseudodifferential operator of order -1, and the condition holds if and only if $dim(X) = 1$ (and fails in a critical way for $dim(X) = 2$).

In the context of example (1), with $X = S^1$ and $G_0 = K$, a compact Lie group, Driver and others, inspired by Malliavin, have proven a number of remarkable results (which I cannot adequately review; see the survey [3] and references there). In particular, in this case the measures are $\mathbf{H}$-translation quasiinvariant (for $s > 1/2$, $t > 0$; it is notable that the Radon-Nikodym derivative involves a regularization of the Ricci curvature for the loop space, due to Freed). Also, for $s = 1$, the heat kernel measures are equivalent to (but not equal to) the corresponding Wiener measures.

The $W^s(S^1, K)$-quasiinvariance of the measures $\nu_t^{\mathbf{H}\subset\mathbf{G}}$ implies the existence of corresponding unitary "regular representations"

$$W^s(S^1, K \times K) \to U(L^2(d\nu_t^{\mathbf{H}\subset\mathbf{G}}))$$

For a discussion of this representation, when $s = 1$ (using the equivalence with Wiener measures), see chapter 4 of [1].

A basic question concerns the behavior of the heat kernel measures when either $t \to \infty$ or $s \downarrow 1/2$, the critical degree for $X = S^1$. To formulate this question in a useful way, we need a "distributional completion" of the loop space $C^0(S^1, K)$. In one approach, strongly suggested by physics and Kac-Moody theory, one is led to adopt the coefficients of the Riemann-Hilbert factorization of a loop, rather than the values of a loop, as the fundamental variables. One is thus led to consider the inclusion

$$C^0(S^1, K) \to \mathbf{L}G,$$

where G is the complexification of K and $\mathbf{L}G$ denotes the socalled formal completion of the G loop space. When K is simply connected, it is known that there exists a $L_{pol}K$-biinvariant probability measure on $\mathbf{L}G$, which we denote by μ. There is also a deformation by quasi-invariant measures, which have heuristic expressions in terms of Toeplitz determinants. We will denote this deformation by μ_l ([4]).

Conjecturally

$$\nu_t^{\mathbf{H}\subset\mathbf{G}} \to \tilde{\mu}_l$$

as $s \downarrow 1/2$, where $l = 1/t$, the inner product on $\mathfrak{k}$ has been normalized in a standard way, and $\tilde{\mu}_l$ denotes a measure equivalent to μ_l. This is true when K is a torus. When $t \to \infty$, the heat kernel measures are asymptotically invariant in a sense introduced by Malliavin. Furthermore, when K is simply connected,

$$\nu_t^{\mathbf{H}\subset\mathbf{G}} \to \mu,$$

as $t \to \infty$.

In the context of the example (1), when $dim(X) = 2$, the critical behavior of the measures as $s \downarrow 1$ is undoubtedly related to the (ill-understood) critical behavior of the "energy representations" in two dimensions; see section 3.5 of [1].

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Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions

Peter Littelmann
(joint work with Ghislain Fourier)

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. There are several natural ways to make out of $\mathfrak{g}$ an infinite-dimensional algebra: we have the current algebra $\mathcal{C}\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}[t]$, the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$ and its central extension, the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, and the quantized versions of these algebras.

The theory of finite dimensional representations of the loop algebra $\mathcal{L}\mathfrak{g}$, its quantized loop algebra $U_q(\mathcal{L}\mathfrak{g})$ and its current algebra $\mathcal{C}\mathfrak{g}$ have been the subject of many articles in the recent years. See for example [1], [2], [3], [4], [6], [7], [9], [10], [11], [12], [14], [18], [19], [20], [21], [23], [25] for different approaches and different aspects of this subject.

The notion of a Weyl module in this context was introduced by Chari and Pressley in [10] for the affine Kac-Moody algebra and its quantized version. These modules can be described in terms of generators and relations, and they are characterized by the following universal property: any finite dimensional highest weight module which is generated by a one dimensional highest weight space, is a quotient of a Weyl module. This notion can be naturally extended to the category of finite dimensional representations of the current algebra ([6], [16]). Another intensively studied class of modules are the Kirillov-Reshetikhin modules, a name that originally refers to evaluation modules of the Yangian. In [5] Chari gave a definition of these modules for the current algebra in terms of generators and relations.

The current algebra is a subalgebra of a maximal parabolic subalgebra of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$. Let Λ be a dominant weight for $\widehat{\mathfrak{g}}$ and denote by $V(\Lambda)$ the associated (infinite dimensional) irreducible $\widehat{\mathfrak{g}}$-representation. Another natural class of finite dimensional representations of the current algebra are provided by certain Demazure submodules of $V(\Lambda)$. Of particular interest for this paper are the (possibly twisted) $\mathcal{C}\mathfrak{g}$-stable Demazure submodules $D(m, \lambda)$ of $V(m\Lambda_0)$, where Λ_0 is the fundamental weight associated to the additional node of the extended Dynkin diagram of $\mathfrak{g}$.

If $\mathfrak{g}$ is simply laced, then we can identify the weight and the coweight lattice, so the Weyl modules as well as the twisted $\mathcal{C}\mathfrak{g}$-stable Demazure submodules of $V(m\Lambda_0)$ are classified by dominant weights $\lambda \in P^+$.

Theorem A *For a simple complex Lie algebra of simply laced type, the Weyl module $W(\lambda)$ and the Demazure module $D(1,\lambda)$ are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.*

Also the Demazure modules of higher level are related to an interesting class of finite dimensional modules for $\mathcal{C}\mathfrak{g}$. Let $\mathfrak{g}$ be an arbitrary simple complex Lie algebra, the $\mathcal{C}\mathfrak{g}$-stable Demazure modules $D(m,\lambda^\vee)$ are classified by dominant coweights $\lambda \in \check{P}^+$.

Theorem B *For a fundamental coweight $\omega_i^\vee$ let $d_i = 1, 2$ or 3 be such that $d_i\omega_i = \nu(\omega_i^\vee)$. The Kirillov-Reshetikhin module $KR(d_i m\omega_i)$ is, as $\mathcal{C}\mathfrak{g}$-module, isomorphic to the Demazure module $D(m,\omega_i^\vee)$. In particular, in the simply laced case all Kirillov-Reshetikhin modules are Demazure modules.*

The fact that $D(m,\omega_i^\vee)$ is a quotient of a Kirillov-Reshetikhin module has been already pointed out in [8]. In the same paper Chari and Moura have shown that $D(m,\omega_i^\vee)$ is isomorphic to $KR(d_i m\omega_i)$ for all classical groups using character calculations. Our proof is independent of the type of the algebra.

To stay inside the class of cyclic highest weight modules, the tensor product of cyclic $\mathcal{C}\mathfrak{g}$-modules is often replaced by the fusion product of modules [15].

Theorem C *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\lambda^\vee = \lambda_1^\vee + \ldots + \lambda_r^\vee$ be a decomposition of a dominant coweight as a sum of dominant coweights. Then $D(m,\lambda^\vee)$ and the fusion product $D(m,\lambda_1^\vee) * \cdots * D(m,\lambda_r^\vee)$ are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.*

The theorem shows in particular that the fusion product of Demazure modules of the same level is associative and independent of the parameters used in the fusion construction. In [2] it is shown that the fusion product of Kirillov-Reshetikhin modules of arbitrary levels is independent of the parameters.

As a consequence we obtain for the Weyl module $W(\lambda)$ in the simply laced case:

Corollary A *Suppose $\mathfrak{g}$ is of simply laced type. Let $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ be a decomposition of a dominant weight $\lambda \in P^+$ as a sum of fundamental weights. Then the Weyl module $W(\lambda)$ for the current algebra is the fusion product of the fundamental Weyl modules:*

$$W(\lambda) \simeq \underbrace{W(\omega_1) * \cdots * W(\omega_1)}_{a_1} * \cdots * \underbrace{W(\omega_n) * \cdots * W(\omega_n)}_{a_n}.$$

The Weyl modules for the loop algebra are classified by n-tuples $\pi = (\pi_1,\ldots,\pi_n)$ of polynomials $\pi_j \in \mathbb{C}[u]$ with constant term 1 [10]. The associated dominant weight is $\lambda_\pi = \sum_i \deg \pi_i \omega_i$. Similarly, the Weyl modules for the quantized loop algebra are classified by n-tuples $\pi_q = (\pi_{q,1},\ldots,\pi_{q,n})$ of polynomials $\pi_{q,j} \in \mathbb{C}(q)[u]$ with constant term 1, the associated weight λ_{π_q} is defined as above.

It was conjectured in [10] (and proved in the $\mathfrak{sl}_2$-case) that the dimension of the Weyl modules depend only λ_{π_q} respectively λ_π. More precisely, they conjectured that the dimension is the (appropriate) product of the dimension of the "fundamental modules". As Hiraku Nakajima has pointed out to us, the dimension conjecture can be deduced using the theory of global basis. The results of Kashiwara [21, 22] imply that the Weyl modules are specializations (the q = 1 limit) of certain finite-dimensional quotients of the extremal weight modules for the quantum affine algebra. The results of Beck and Nakajima [3, 25, 26] imply that these quotients (and hence their specializations) have the correct dimension.

A different approach to prove the dimension conjecture was suggested in [10, 11]. In fact, using the specialization and dimension arguments outlined there, in the simply laced case the dimension formula is an immediate consequence of Theorem A and Theorem C:

Corollary B *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, let $\lambda = \sum m_i\omega_i$ be a dominant weight (for $\mathfrak{g}$), let π (resp. π_q) be an n-tuple of polynomials in $\mathbb{C}[u]$ (resp. in $\mathbb{C}(q)[u]$) with constant term 1 such that $\lambda = \lambda_\pi = \lambda_{\pi_q}$. Then*

$$\dim W(\lambda) = \dim W(\pi) = \dim W_q(\pi_q) = \dim D(1, \lambda^\vee) = \prod_i (\dim W(\omega_i))^{m_i}$$

For $\mathfrak{g} = \mathfrak{sl}_n$, the connection between Demazure modules in $V(\Lambda_0)$ and Weyl modules had been already obtained by Chari and Loktev in [6]. The isomorphism between the Weyl module $W(\lambda)$ and the Demazure module $D(1, \lambda)$ has been conjectured in [14].

For a dominant weight $\lambda = \sum m_i\omega_i$ let $\pi_{\lambda,a}$ be the tuple having $(1 - au)^{m_i}$ as i-th entry. The quantum Demazure module $D_q(m, \lambda)$ has an associated crystal graph which is a subgraph of the crystal graph of the corresponding irreducible $U_q(\widehat{\mathfrak{g}})$-representation. We conjecture that by adding appropriate label zero arrows, one gets the graph of an irreducible $U_q(\widehat{\mathfrak{g}})$-representation. In the simply laced case and level one we have:

Proposition *The crystal graph of $D_q(1, \lambda)$ is obtained from the crystal graph of $W_q(\pi_{q,\lambda,1})$ by omitting certain label zero arrows. More precisely, let $B(\lambda)_{cl}$ be the path model for $W_q(\pi_{q,\lambda,1})$ described in* [29]*, then the crystal graph of the Demazure module is isomorphic to the graph of the concatenation $\pi_{\Lambda_0} * B(\lambda)_{cl}$.*

In the simply laced case, the restriction of the loop Weyl module $W(\pi_{\lambda,a})$ to $\mathcal{C}\mathfrak{g}$ is (up to a twist by an automorphism) the Weyl module $W(\lambda)$. It follows:

Corollary C *The Demazure module $D(m, \lambda)$ of level m can be equipped with the structure of a cyclic $U(\mathcal{L}\mathfrak{g})$-module such that the $\mathfrak{g}$-module structure coincides with the natural $\mathfrak{g}$-structure coming from the Demazure module construction.*

Let $V(m\Lambda_0)$, $m \in \mathbb{N}$, be the irreducible highest weight module of highest weight $m\Lambda_0$ for the affine Kac-Moody $\widehat{\mathfrak{g}}$. In [17] we gave a description of the $\mathfrak{g}$-module structure of this representation in terms of a semi-infinite tensor product. Using Theorem C, we are able to lift this result to the level of modules for the current algebra. The theorem holds in a much more general setting, but for the convenience of a uniform presentation, let Θ be the highest root of the root system of $\mathfrak{g}$.

Theorem D *Let $D(m, n\Theta) \subset V(m\Lambda_0)$ be the Demazure module of level m corresponding to the translation at $-n\Theta$. Let $w \neq 0$ be a $\mathcal{C}\mathfrak{g}$-invariant vector of $D(m, \Theta)$. Let V_m^∞ be the direct limit*

$$D(m, \Theta) \hookrightarrow D(m, \Theta) * D(m, \Theta) \hookrightarrow D(m, \Theta) * D(m, \Theta) * D(m, \Theta) \hookrightarrow \ldots$$

where the inclusions are given by $v \mapsto w \otimes v$.
Then $V(m\Lambda_0)$ and V_m^∞ are isomorphic as $U(\mathcal{C}\mathfrak{g})$-modules.

The semi-infinite fusion construction can be seen as an extension of the construction of Feigin and Feigin [13] ($\mathfrak{g} = \mathfrak{sl}_2$) and Kedem [23] ($\mathfrak{g} = \mathfrak{sl}_n$) to arbitrary simple Lie algebras. We conjecture that, as in [13] and [23], the semi-infinite fusion construction works for arbitrary dominant weights and not only for multiples of Λ_0.

Naito and Sagaki [27], [28], [29] gave a path model for the Weyl modules $W(\omega)$ for all fundamental weights and $\mathfrak{g}$ of arbitrary type. Since the Weyl modules coincide with the level-one Demazure modules provided $\mathfrak{g}$ is simply-laced, the semi-infinite limit construction above gives on the combinatorial side a combinatorial limit path model for the representation $V(\Lambda_0)$ as a semi-infinite concatenation of a finite path model, extending in this sense the approach of Magyar in [24].

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Gerbes and loop groups

Christoph Schweigert
(joint work with Jürgen Fuchs, Urs Schreiber, and Konrad Waldorf)

Throughout this contribution, let G be a compact, connected Lie group which, for simplicity, we assume to be simply connected. We denote by M an arbitrary smooth manifold, not necessarily with a group structure; finally, LM stands for the loop space of M.

Central extensions of loop groups

$$0 \to U(1) \to \widehat{LG} \to LG \to 0$$

play a crucial role in the representation theory of loop groups and affine Kac-Moody algebras. It is well known that they are classified by $H^3(G,\mathbb{Z})$. The corresponding class is an important datum in conformal field theory as well, where it is called the *level*. The guiding question of the talk is to describe a geometric object on G – i.e. in the framework of finite-dimensional geometry – that describes the central extension of LG.

1. Bundle gerbes

A hint comes from cohomology in one degree less: it is well known that $H^2(M,\mathbb{Z})$ classifies line bundles on a manifold M. At this point, we shift our attention from line bundles to hermitian line bundles with connection. For the latter, a notion of holonomy around closed loops exists; a similar notion of holonomy, but for closed surfaces, is crucial in applications to conformal field theory.

The local data describing a hermitian bundle gerbe are a generalization of those describing hermitian line bundles. Correspondingly, they are classified by an appropriate class in Deligne hypercohomology. There is a group homomorphism from this hypercohomology group to $H^3(M,\mathbb{Z})$; its image for a given gerbe $\mathcal{G}$ is called the Dixmier-Douady class $\mathrm{dd}(\mathcal{G})$ of the gerbe.

In geometric terms, a hermitian bundle gerbe consists of a surjective submersion $\pi : Y \to M$ with a hermitian line bundle $L \to Y^{[2]} := Y \times_M Y$ and an associative isomorphism of line bundles on $Y^{[3]}$

$$\mu : \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L \,.$$

The final datum is a two-form $C \in \Omega^2(Y)$ such that $\mathrm{curv}(L) = \pi_2^*(C) - \pi_1^*(C)$. It is easy to see that gerbes pull back under maps $f : M \to N$.

Gerbes on compact connected Lie groups have been constructed by several authors [5, 3, 4]. Here we emphasize the point that such simple and simply-connected Lie groups are not only Riemannian manifolds (with a metric given by the Killing form) but carry equally natural structure gerbes. (For more general compact Lie groups, several choices of gerbes can exist.)

2. From a gerbe on M to a line bundle on LM

To any $\rho \in \Omega^2(M)$ one can construct a gerbe I_ρ with connection: put $Y = M$ and $C = \rho$ and $L \to Y^{[2]} = M$ the trivial line bundle. The Dixmier-Douady class $\mathrm{dd}(I_\rho)$ vanishes.

Next, we describe morphisms of gerbes: a stable morphism $\mathcal{G} \to \mathcal{G}'$ consists of line bundle $A \to Z := Y \times_M Y'$ with $\mathrm{curv}(A) = p^*(C) - (p')^*(C')$ and an isomorphism of line bundles on $Z^{[2]}$

$$\alpha : p^*(L) \otimes (p')^*(L')^* \otimes \pi_2^*(A) \to \pi_1^*(A)$$

that is compatible with the gerbe multiplication. We also introduce morphisms

$$\beta : \mathcal{A}_1(A_1, \alpha_1) \Rightarrow \mathcal{A}_2(A_2, \alpha_2)$$

between stable morphisms: these are morphisms of line bundles $A_1 \to A_2$ on Z compatible with the α_i. Gerbes thus naturally form a 2-category. A *trivialization* of a gerbe is a morphism

$$\mathcal{T} : \mathcal{G} \to I_\rho \,.$$

Trivializations only exist, if the Dixmier-Douady class $\mathrm{dd}(\mathcal{G})$ vanishes; isomorphism classes of trivializations with given two-form ρ form a torsor over the group $Pic_0(M)$ of flat line bundles on M.

We are now ready to construct a line bundle over the loop space LM for a given gerbe $\mathcal{G}$ on M. Its fibre over a loop

$$\gamma : S^1 \to M$$

consists of isomorphism classes of trivializations

$$\mathcal{T} : \gamma^*\mathcal{G} \to I_0$$

(which exist since $H^3(S^1, \mathbb{Z}) = 0$). They indeed form a torsor over $Pic_0(S^1) \cong U(1)$. Given these fibres, one shows that they fit together into a line bundle $P(\mathcal{G})$ with a smooth structure and with a connection that is compatible with transgression. The association $\mathcal{G} \to P(\mathcal{G})$ preserves isomorphisms, pullback and duality. Finally, the holonomy of the line bundle $P(\mathcal{G})$ on LM is related to the surface holonomy of the gerbe $\mathcal{G}$ for the corresponding map $S^1 \times S^1 \to M$.

3. Applications to conformal field theory

Our own work has been motivated by the applications of gerbes to two-dimensional σ-models, whose configuration space are spaces of maps

$$\Phi : \Sigma \to (M, g, \mathcal{G})$$

where Σ is a two-dimensional manifold and M a manifold with metric g and gerbe $\mathcal{G}$. (Such a triple is, at the present stage of our knowledge, the mathematical structure that deserves best the name σ-model background. Compact connect Lie groups are examples.) It is a crucial insight of [2, 3] that the so-called Wess-Zumino term in the action of such models is just the holonomy of the pullback $\Phi^*(\mathcal{G})$ on Σ.

Many notions of conformal field theory now lead to analogous constructions for bundle gerbes. For example, a *gerbe module* consists of a rank N hermitian vector bundle $E \to Y$ with an isomorphism

$$\rho : \; L \otimes p_2^* E \to p_1^* E$$

on $Y^{[2]}$ such that on $Y^{[3]}$ the relation $\rho \circ (\mu \otimes \mathrm{id}) = \rho \circ (\mathrm{id} \otimes \rho)$ holds. Gerbe modules only exist if the Dixmier-Douady class is pure torsion. D-branes describing boundary conditions for Σ correspond to submanifolds $\iota : Q \to M$ with a gerbe module of $\iota^*(\mathcal{G})$. Prominent examples of such submanifolds, in the case when $M = G$ is a compact Lie group, are (twisted) conjugacy classes.

Our own contribution to the subject concerns two points:

- One can endow a bundle gerbe with additional structure so that a notion of holonomy exists for unoriented surfaces as well [6]. Such a Jandl-structures consists of a smooth action of $K = \{1, k\} \cong \mathbb{Z}_2$ on M, with a stable isomorphism $(A, \alpha) : k^*\mathcal{G} \to \mathcal{G}^*$. The action of K on M can be lifted to the base space of the line bundle $A \to Z := Y_k \times_M Y$. The last datum of a Jandl structure is a k-equivariant structure

$$\varphi : k^*A \to A$$

that is compatible with α. It can be shown, in particular, that Jandl structures pull back, that there is a natural equivalence relation on Jandl structure such that different representatives give physically equivalent results and that the equivalence classes form a torsor over the group of flat K-equivariant line bundles. All results agree with computations in an approach to conformal field theories that is based on separable symmetric Frobenius algebras in modular tensor categories.
- There is a natural notion of bimodules for gerbes as well [1]. A bimodule of the pullback of a gerbe to a submanifold describes topological defects in conformal field theories. Important examples, when $M = G$ is a compact Lie group are (twisted) bi-conjugacy classes:

$$\mathcal{C}(g_1, g_2) = \{(h_1 g_1 h_2^{-1}, h_1 g_2 h_2^{-1}) \,|\, h_i \in G\} \subset G \times G\,.$$

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Chern Weil forms in infinite dimensions

Sylvie Paycha

The aim of this talk, based on joint work with various coauthors [PR], [PS], [MP], is to describe an analog in infinite dimensions of the classical Chern-Weil formalism in finite dimensions. The basic idea is to replace the trace on matrices used in the finite dimensional setup by regularised traces on pseudodifferential operators. In some infinite dimensional geometric situations, such as for the complexified tangent bundle to the loop group [F], or for the infinite rank vector bundle associated with a family of Dirac operators [BF], [BGV], a regulator can be chosen to get closed

forms and thereby de cohomology classes by mimicking the Chern-Weil formalism with the help of the corresponding regularised traces.
Let us briefly recall the classical Chern-Weil formalism. Let $G \subset Gl_n(\mathbb{C})$ be a Lie group and $P \to B$ a principal bundle with structure group G. Then the adjoint bundle $\mathrm{ad}P = P \times_{\mathrm{ad}} \mathrm{Lie}(G)$ locally looks like $U \times \mathrm{Lie}(G)$ on an open subset U of B. Since $\mathrm{Lie}(G) \subset gl_n(\mathbb{C})$, the ordinary trace on matrices extends to a map $\mathrm{tr} : \Omega\,(B, \mathrm{adP}) \to \Omega(B)$ which sends ad P-valued form on B to ordinary forms on B, implementing a fibrewise trace. Given a map

$$\begin{array}{rcl} \mathcal{C}(P) & \to & \Omega\,(B, \mathrm{ad}P) \\ \nabla & \mapsto & f(\nabla) \end{array}$$

which sends a connection on P to an ad P-valued form on B, one can therefore build a form $\mathrm{tr}(f(\nabla))$ on B.
Using the fact that $[\nabla, f(\nabla)] = 0$ (which boils down to the Bianchi identity) combined with essential ingredients, namely

(1) $d \circ \mathrm{tr} = \mathrm{tr} \circ d$,
(2) $\partial \mathrm{tr} = 0$

where ∂ is the Hochschild coboundary, one shows that $\mathrm{tr}(f(\nabla))$ is closed and that the corresponding de Rham class is independent of the choice of connection.
The two infinite dimensional situations mentioned above (loop groups, families of Dirac operators) lead to forms $f(\nabla)$ in $\Omega(B, \mathcal{A})$ where $\mathcal{A}$ is a bundle over some infinite dimensional manifold B, which locally over an open subset U of B, looks like $U \times \mathrm{C}\ell(M, \mathbb{C}^n)$, with $\mathrm{C}\ell(M, \mathbb{C}^n)$ the algebra of classical pseudodifferential operators acting on smooth $\mathbb{C}^n$-valued functions on a closed manifold M (the circle in the loop group case, a spin manifold in the family of Dirac operators case). To mimic the above constructions, we therefore need an Ersatz for the trace on matrices with properties 1) and 2). Freed in [F] introduced a conditioned trace which was later reinterpreted as a weighted trace in [CDMP]. In both setups, one can choose an adequate weight (or regulator) for these weighted (or regularised traces) to build closed Chern forms similar to $f(\nabla)$; an ad invariant Laplacian in the first setup, the squared superconnection associated with the Dirac operators in the second setup.
Focusing on the family of Dirac operators setup, we express these Chern-Weil forms in terms of local expressions reminiscent of the Chern character for families of Dirac operators [MP].

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An introduction to extended affine Lie algebras

Erhard Neher

Like Kac-Moody Lie algebras, extended affine Lie algebras are a generalization of affine Lie algebras and of simple finite-dimensional complex Lie algebras. But contrary to arbitrary Kac-Moody algebras, extended affine Lie algebras have concrete realizations. In this report we will give some rough ideas about their structure theory. The main examples of extended affine Lie algebras are toroidal Lie algebras.

0. A pivotal example: Untwisted affine Lie algebras. All algebras will be over a field F of characteristic 0. Let $\mathfrak{g}$ be a split simple Lie algebra $\mathfrak{g}$, let $L = \mathfrak{g} \otimes F[t^{\pm 1}]$ be the associated loop algebra, $K = L \oplus Fc$ its universal central extension, and put $E = K \oplus Fd$ (semidirect product), where d is the degree derivation of K sending $x \otimes t^n$ to $n(x \otimes t^n)$ and annihilating Fc. We thus have a diagram of 3 related Lie algebras (see below), where E or sometimes even K is referred to as an untwisted affine Kac-Moody Lie algebra. In the theory of extended affine Lie algebras, this diagram is generalized by replacing the loop algebra L by a centreless Lie torus, the algebra K by a central extension of L, which is also a Lie torus, and E by an extended affine Lie algebra, abbreviated EALA:

$$\begin{array}{lll} K & \hookrightarrow & E \\ \downarrow & & \\ L & & \end{array} \qquad \rightsquigarrow \qquad \begin{array}{lll} \text{Lie torus } K & \hookrightarrow & \text{EALA } E \\ \downarrow & & \\ \text{centreless} & & \\ \text{Lie torus } L & & \end{array}$$

1. Extended affine Lie algebras. Due to space limitations we will not give the precise definitions. Rather the reader is referred to [1] for a definition of an EALA over $\mathbb{C}$ and to [14] for a definition over a field F of characteristic 0. Closely related Lie algebras are considered in [7] and [8]. While these definitions do not agree in general, not even over $F = \mathbb{C}$, the following most important features (EA1)-(EA3) of an extended affine Lie algebra E are present in all approaches.

(EA1): E has a nondegenerate invariant symmetric bilinear form $(.|.)$.

(EA2): E contains a nontrivial finite-dimensional self-centralizing and ad-diagonalizable subalgebra H.

To prepare the axiom (EA3) note that by (EA2) the algebra E has a root space decomposition $E = \bigoplus_{\xi \in H^*} E_\xi$ with $E_0 = H$, where, as usual, $E_\xi = \{e \in E : [h, e] = \xi(h)e$ for all $h \in H\}$. The invariance of $(.|.)$ implies that $(E_\xi | E_\zeta) = 0$ for $\xi + \zeta \neq 0$. It follows that $(.|.)$ restricted to $H \times H$ is nondegenerate. Hence, every $\xi \in H^*$ is represented by a unique $t_\xi \in H$ via $(t_\xi | h) = \xi(h)$ for all $h \in H$. The subalgebra E_c of E, generated by $\{E_\xi : (t_\xi | t_\xi) \neq 0\}$ is called the *core of* E.

(EA3): For $\xi \in R^{\mathrm{an}}$ and $x_\xi \in E_\xi$, the endomorphism $\operatorname{ad} x_\xi \in \operatorname{End}_F E$ is locally nilpotent.

Depending on the authors, several other axioms are added to (EA1)–(EA3). They mostly concern the nature of the root system R of an EALA.

2. Lie tori. As indicated in the diagram above, the loop algebra in the construction of an untwisted affine Lie algebra is replaced by a Lie torus. The essential properties of a Lie torus L are the following: L has two compatible gradings, one by the abelian group $\mathbb{Z}^n$ and one by the root lattice of a finite irreducible root system Δ. With respect to the second grading, the only non-zero homogeneous spaces of L have degrees in $\Delta \cup \{0\}$. In addition, one requires the existence of enough $\mathfrak{sl}_2$-triples. The precise definition is given in [13], see [16] for a different approach.

Due to the efforts of many people, one now has a complete and precise classification of centreless Lie tori: [10] for $\Delta = \mathrm{A}_l$, $l \geq 3$, and $\Delta = \mathrm{D}_l$, $l \geq 4$ and $\Delta = \mathrm{E}_l$, $l = 6, 7, 8$; [11] for $\Delta = \mathrm{A}_2$; [15] for $\Delta = \mathrm{A}_1$; [3] for Δ reduced, but not simply-laced; [6] and [5] for $\Delta = \mathrm{BC}_1$; [12] for $\Delta = \mathrm{BC}_2$; and finally [4] for $\Delta = \mathrm{BC}_l, l \geq 3$. For example, if $\mathfrak{g}$ is a split simple finite-dimensional Lie algebra of type Δ the associated multiloop algebra $L = \mathfrak{g} \otimes F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is always a centreless Lie torus. If Δ is of type D or E, this covers all possibilities. However, already for Δ of type A, more general algebras do occur. Many of them are related to quantum tori. It is amazing that in order to classify Lie tori one needs all important classes of nonassociative algebras, namely alternative algebras for $\Delta = \mathrm{A}_2$, Jordan algebras for $\Delta = \mathrm{A}_1$, structurable algebras for $\Delta = \mathrm{BC}_1, \mathrm{BC}_2$, and a mixture of these for the other types.

The first step in understanding the structure of extended affine Lie algebras is the following.

3. Proposition. ([1, 3, 14, 17]) *Let E be an extended affine Lie algebra. Then its core E_c is a Lie torus, and $E_c/Z(E_c)$ is a centreless Lie torus.*

The proposition above begs the question: Does every centreless Lie torus L arise from an extended affine Lie algebra E? If yes, how can one reconstruct E from L? These questions will be answered in Th. 5 below. As can already be seen from the papers [10] and [11] there are in general infinitely many extended affine Lie algebras associated to a given centreless Lie torus.

4. Construction. Let L be a centreless Lie torus. As explained above, L is $\mathbb{Z}^n$-graded, say $L = \bigoplus_{\lambda \in \mathbb{Z}^n} L^\lambda$. Let ∂_i, $1 \leq i \leq n$ be the ith degree derivation, $\partial_i x = \lambda_i x$ for $x \in L^\lambda$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ and let $\mathcal{D} = \operatorname{span}_F\{\partial_1, \ldots, \partial_n\}$. Also, let

$C = \{\chi \in \mathrm{End}_F L : [\chi, \mathrm{ad}\, x] = 0$ for all $x \in L\}$ be the centroid of L. It is shown in [9] that $C = \bigoplus_{\xi\in\Xi} C^\xi$ is graded by a subgroup Ξ of $\mathbb{Z}^n$. We note that $C\mathcal{D}$ is a subalgebra of derivations, the so-called *skew-centroidal derivations* of L. It is also known that L has an essentially unique invariant nondegenerate symmetric bilinear form $(.|.)$ [17]. We denote by SCDer$L$ the subalgebra of $C\mathcal{D}$ consisting of the skew-symmetric derivations in $C\mathcal{D}$. It is a Ξ-graded algebra.

As a second ingredient of our construction, let $D = \bigoplus_{\xi\in\Xi} D^\xi$ be a graded subalgebra of SCDerL with the property that D^0 induces the $\mathbb{Z}^n$-grading of L, i.e., the L^λ are the joint eigenspaces of D^0. The graded dual $D^{\mathrm{gr}*}$ is canonically a D-module. Associated to D and the bilinear form $(.|.)$ is a 2-cocycle $\sigma : L\times L \to D^{\mathrm{gr}*}$, given by $\sigma(x, y)(d) = (dx|y)$ for $x, y \in L$ and $d \in D$. Thus, we have a central extension $K = L \oplus D^{\mathrm{gr}*}$ with product $[x \oplus \phi\, ,\, y \oplus \psi]_K = [x, y]_L \oplus \sigma(x, y)$, where $x, y \in L$ and $\phi, \psi \in D^{\mathrm{gr}*}$. We can also form the semidirect product $E = K \oplus D$. While this will be an extended affine Lie algebra, it is not general enough. Rather, one can twist the product on E by a special 2-cocycle $\tau : D \times D \to D^{\mathrm{gr}*}$. We denote the corresponding Lie algebra by $E(L, D, \tau)$.

5. Theorem ([14]) *The Lie algebra $E(L, D, \tau)$ is an extended affine Lie algebra. Conversely, every extended affine Lie algebra E arises in this way with $L = E_c/Z(E_c)$ and appropriate choices of D and τ.*

An important point in the proof of this theorem is

6. Theorem ([13]) *Let L be a centreless Lie torus of type $\Delta \neq \mathrm{A}$. Then L is finitely generated as a module over its centroid.*

As an immediate corollary of this result, it follows from [2] that over an algebraically closed field of characteristic 0 every centreless Lie torus of type $\Delta \neq \mathrm{A}$ is a so-called multiloop algebra.

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Multiloop algebras and extended affine Lie algebras of nullity 2

Bruce Allison

(joint work with Stephen Berman and Arturo Pianzola)

This report outlines some recent joint work with Stephen Berman and Arturo Pianzola on multiloop algebras and extended affine Lie algebras. Proofs of the results stated here will appear elsewhere.

Throughout the report we will assume that k is the field of complex numbers, although it is very likely that the same results hold over any algebraically closed field of characteristic 0. All algebras are assumed to be algebras over k. We set $\zeta_m = \exp(\frac{2\pi i}{m}) \in k$ for $m \geq 1$.

An *extended affine Lie algebra* (EALA) $\mathcal{E}$ possesses a nontrivial finite dimensional ad-diagonalizable subalgebra and a nondegenerate symmetric invariant bilinear form that satisfy a list of natural axioms that are modeled after the properties possessed by affine Kac-Moody Lie algebras [1, 8].

If $\mathcal{E}$ is an extended affine Lie algebra, the *nullity* of $\mathcal{E}$ is the rank of the group generated by the isotropic roots. For example, EALA's of nullity 0 are precisely finite dimensional simple Lie algebras, and EALA's of nullity 1 are precisely affine Kac-Moody Lie algebras [3]. The *type* Δ of $\mathcal{E}$ is the type of the finite irreducible root system (possibly not reduced) that is obtained by factoring out the isotropic roots from the root system of $\mathcal{E}$. The *core* $\mathcal{E}_{\mathrm{c}}$ of $\mathcal{E}$ is the subalgebra of $\mathcal{E}$ generated by the root spaces of $\mathcal{E}$ corresponding to non-isotropic roots, and the *centreless core* $\mathcal{E}_{\mathrm{cc}}$ of $\mathcal{E}$ is $\mathcal{E}_{\mathrm{c}}/Z(\mathcal{E}_{\mathrm{c}})$. For example, if $\mathcal{E}$ is of nullity 0 then $\mathcal{E}_{\mathrm{cc}} = \mathcal{E}$, whereas if $\mathcal{E}$ is of nullity 1 then $\mathcal{E}_{\mathrm{cc}} = [\mathcal{E},\mathcal{E}]/Z([\mathcal{E},\mathcal{E}])$. We call the centreless core of an EALA of nullity n simply a *centreless core of nullity n*.

Centreless cores can be described axiomatically as *centreless Lie tori* [10, 8]. They play a central role in the theory of EALA's since there is a general construction that yields the family of all EALA's with a given centreless core [8].

One method of describing the structure of centreless cores is *coordinatization*. This method, which constructs centreless cores using *matrix constructions* from (in general nonassociative) coordinate algebras, has been very successful, providing a great deal of detailed structural information about centreless cores of all types. (See for example [7] and the references therein.) It has however one drawback: the matrix construction used to build the centreless cores depends on the type.

In this report we discuss a more uniform approach using twisted multiloop algebras. Recall that if $\sigma_1, \ldots, \sigma_n$ are commuting automorphisms of a Lie algebra $\mathfrak{g}$ with $\sigma_i^{m_i} = 1$, then the (twisted) *n-fold multiloop algebra* based on $\mathfrak{g}$ and determined by $\sigma_1, \ldots, \sigma_n$ is the Lie algebra

$$L(\mathfrak{g}, \sigma_1, \ldots, \sigma_n) = \textstyle\sum_{(\ell_1,\ldots,\ell_n)\ \in \mathbb{Z}^n} \mathfrak{g}^{(\bar{\ell}_1,\ldots,\bar{\ell}_n)} \otimes z_1^{\ell_1} \ldots z_n^{\ell_n}$$

in $\mathfrak{g} \otimes k[z_n^{\pm 1}, \ldots, z_n^{\pm 1}]$, where $(\bar{\ell}_1, \ldots, \bar{\ell}_n)$ is the canonical image of $(\ell_1, \ldots, \ell_n)$ in $\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}$ and $\mathfrak{g}^{(\bar{\ell}_1,\ldots,\bar{\ell}_n)} = \{x \in \mathfrak{g} \mid \sigma_i(x) = \zeta_{m_i}^{\ell_i} x \text{ for } 1 \le i \le n\}$. (One can easily show that the isomorphism class of $L(\mathfrak{g}, \sigma_1, \ldots, \sigma_n)$ does not depend on the choice of the periods m_i.) In particular, when $n = 1$, $L(\mathfrak{g}, \sigma_1)$ is the *loop algebra* determined by σ_1.

Classically V. Kac showed (in the language of EALA's) that any centreless core of nullity 1 can be constructed as a (twisted) loop algebra based on a centreless core of nullity 0. It is natural then to ask whether any centreless core of nullity 2 can be constructed as either a 2-fold multiloop algebra based on a centreless core of nullity 0 or as a loop algebra based on a centreless core of nullity 1. This report describes our results on these questions.

Our first theorem gives characterizations of 2-fold multiloop algebras based on centreless cores of nullity 0.

Theorem 1. *Let $\mathcal{L}$ be a Lie algebra over k. The following statements are equivalent:*

(1) *$\mathcal{L}$ has a $\mathbb{Z}^2$-grading so that $\mathcal{L}$ is graded-central-simple,* $\dim(\mathcal{L}^\lambda) < \infty$ *for $\lambda \in \mathbb{Z}^2$ and $\mathbb{Z}^2/\Gamma(\mathcal{L})$ is finite. (Here $\Gamma(\mathcal{L})$ denotes the support of the centroid of $\mathcal{L}$.)*
(2) *$\mathcal{L}$ is isomorphic to a 2-fold multiloop algebra based on a finite dimensional simple Lie algebra.*
(3) *$\mathcal{L}$ is isomorphic to $L(\mathfrak{g}_{cc}, \tau)$, where $\mathfrak{g}$ is an affine Lie algebra and τ is a finite order automorphism of the first kind of $\mathfrak{g}_{cc}$.*
(4) *$\mathcal{L}$ is isomorphic to $L(\mathfrak{g}_{cc}, \tau)$, where $\mathfrak{g}$ is an untwisted affine Lie algebra and τ is a diagram automorphism of $\mathfrak{g}_{cc}$.*

We note that the equivalence of (1) and (2) is true more generally for $\mathbb{Z}^n$-graded Lie algebras and n-fold multiloop algebras. This is proved in our joint work with John Faulkner [2, Theorem 8.3.2].

Let $\mathfrak{M}_2$ be the class of Lie algebras that satisfy conditions (1)–(4) of the preceding theorem. It is important to give necessary and sufficient conditions for two algebras in $\mathfrak{M}_2$ to be isomorphic. This problem can be approached by viewing algebras in $\mathfrak{M}_2$ as 2-fold multiloop algebras algebras as in (2) or by viewing them as loop algebras of affine centreless cores as in (4). The first of these approaches is adopted in work of P. Gille and A. Pianzola described in another report to this workshop. Here we adopt the second approach and obtain the following classification of algebras in $\mathfrak{M}_2$ up to isomorphism:

Theorem 2.
(a) *Let $\mathfrak{g}_i$ be an untwisted affine Lie algebra and let τ_i be a diagram automorphism of $(\mathfrak{g}_i)_{cc}$ for $i = 1, 2$. If $L((\mathfrak{g}_1)_{cc}, \tau_1) \simeq L((\mathfrak{g}_2)_{cc}, \tau_2)$, then $\mathfrak{g}_1 \simeq \mathfrak{g}_2$.*
(b) *Suppose that $\mathfrak{g}$ is an untwisted affine Lie algebra and τ_1 and τ_2 are diagram automorphism of $\mathfrak{g}_{cc}$. Then $L(\mathfrak{g}_{cc}, \tau_1) \simeq L(\mathfrak{g}_{cc}, \tau_2)$ if and only if τ_1 is conjugate to τ_2 in the automorphism group of the Dynkin diagram of $\mathfrak{g}$.*

We need some terminology for the next theorem. First, if $\mathcal{L}$ is any algebra, we denote the *centroid* of $\mathcal{L}$ by $C(\mathcal{L})$. We say that $\mathcal{L}$ is *fgc* if $\mathcal{L}$ is finitely generated as a module over $C(\mathcal{L})$. If $\mathcal{L} \in \mathfrak{M}_2$ then $\mathcal{L}$ is fgc, $C(\mathcal{L})$ is an integral domain, and we define the *central closure* of $\mathcal{L}$ to be the finite dimensional central simple Lie algebra $\widetilde{\mathcal{L}} = \widetilde{C(\mathcal{L})} \otimes_{C(\mathcal{L})} \mathcal{L}$ over the quotient field $\widetilde{C(\mathcal{L})}$ of $C(\mathcal{L})$. We say that $\mathcal{L}$ is *isotropic* if $\widetilde{\mathcal{L}}$ is isotropic, which means that $\widetilde{\mathcal{L}}$ contains a nonzero ad-nilpotent element.

Let $\mathfrak{C}_2$ be the class of Lie algebras that are isomorphic to centreless cores of nullity 2. The next theorem compares the classes $\mathfrak{C}_2$ and $\mathfrak{M}_2$.

Theorem 3. *Let $\mathcal{L}$ be a Lie algebra over k. The following statements are equivalent:*

(1) *$\mathcal{L} \in \mathfrak{M}_2$ and $\mathcal{L}$ is isotropic.*
(2) *$\mathcal{L}$ is isomorphic to $L(\mathfrak{g}_{cc}, \tau)$, where $\mathfrak{g}$ is an untwisted affine Lie algebra and τ is a diagram automorphism of $\mathfrak{g}_{cc}$ that is not transitive on the vertices of the Dynkin diagram.*
(3) *$\mathcal{L} \in \mathfrak{C}_2$ and $\mathcal{L}$ is fgc.*

Remarks:
(a) Combining Theorems 2 and 3 we have a classification up to isomorphism of fgc centreless cores of EALA's of nullity 2.
(b) There is only one family of nonfgc algebras in $\mathfrak{C}_2$. These are the Lie algebras $\mathrm{sl}_r(\mathcal{Q})$, where $r \geq 2$ and $\mathcal{Q}$ is a nonfgc quantum torus in two variables. This fact follows using a theorem of E. Neher [9] (any nonfgc centreless core is of type A) and using the classification of centreless cores of type A.
(c) There is only one family of anisotropic (not isotropic) algebras in $\mathfrak{M}_2$. These are the Lie algebras $\mathrm{sl}_1(\mathcal{Q})$, where $\mathcal{Q}$ is an fgc quantum torus in two variables.
(d) The proofs of Theorems 1, 2 and 3 use among other things the results from our earlier papers on loop algebras (also called covering algebras) [4, 5, 6].

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Loop algebras as forms of Lie algebras

Philippe Gille

(joint work with A. Pianzola)

Throughout $k = \mathbf{C}$ be the field of complex numbers. We will be interested on certain Lie algebras over the ring $R_n = k[t_1^{\pm 1}, \cdots t_n^{\pm 1}]$ of Laurent polynomial in n variables with coefficients in k.

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra. We say that an R_n-Lie algebra $\mathfrak{L}$ is a *form* of the toroidal Lie algebra $\mathfrak{g} \otimes_k R_n$ if $\mathfrak{L} \otimes_{R_n} S \cong \mathfrak{g} \otimes_k S$ for some ring extension S/R_n which is faithfully flat and of finite presentation. We discuss here only objects up to R_n-isomorphisms (the difference between k-isomorphisms and R_n-isomorphisms is controlled by centroid considerations). Descent theory shows that the R_n-isomorphism classes of forms of $\mathfrak{g} \otimes_k R_n$ are classified by the cohomology pointed set $H^1_{fppf}(R, \mathbf{Aut}(\mathfrak{g}))$, where $\mathbf{Aut}(\mathfrak{g})$ stands for the algebraic group associated to $\mathfrak{g}$ (see [SGA3], and also [K]).

A very important class of forms are the (multi) loop algebras, which are defined as follows. We are given an n-tuple $\sigma = (\sigma_1, ..., \sigma_n)$ of commuting automorphisms of $\mathfrak{g}$ which are of finite order. Fix a common period $m > 0$ for all the σ_i, i.e. $\sigma_i^m = 1$ for all $i = 1, ..., n$ We then have a group homomorphism $\phi : H := (\mathbf{Z}/m\mathbf{Z})^n \to \mathbf{Aut}(\mathfrak{g})(k)$, and the underlying simultaneous eigenspaces of the σ_i's gives rise to an H-grading on $\mathfrak{g}$, namely

$$\mathfrak{g}_{\overline{j}_1, \dots, \overline{j}_n} = \Big\{ \; X \in \mathfrak{g} \;\; | \;\; \sigma_i(X) = \zeta_m^{j_i} \, X \text{ for } i = 1, ..., n \Big\}$$

for $(\overline{j}_1, ..., \overline{j}_n) \in H$, where $\zeta_m = e^{2\pi i/m}$. We define then the loop algebra

$$\mathfrak{L}(\mathfrak{g}, \sigma) := \bigoplus_{(j_1, \dots, j_n) \in \mathbf{Z}^n} \mathfrak{g}_{\overline{j}_1, \dots, \overline{j}_n} \otimes t_1^{\frac{j_1}{m}} ... t_n^{\frac{j_n}{m}} \subset \mathfrak{g} \otimes_k k[t_1^{\pm \frac{1}{m}}, \cdots t_n^{\pm \frac{1}{m}}].$$

This is an R_n–Lie algebra (independent of the choice of period m), and we have a natural $R_{n,m}$–Lie algebra isomorphism

$$\mathfrak{L}(\mathfrak{g}, \sigma) \otimes_{R_n} R_{n,m} \simeq \mathfrak{g} \otimes_k R_{n,m}.$$

where $R_{n,m} = k[t_1^{\pm\frac{1}{m}}, \cdots t_n^{\pm\frac{1}{m}}]$. In terms of descent, $\mathfrak{L}(\mathfrak{g},\sigma)$ is the twist of $\mathfrak{g}\otimes_k R_n$ by the Galois cocycle $\phi^{-1} : \mathrm{Gal}(R_{n,m}/R_n) \cong H \to \mathbf{Aut}(\mathfrak{g})(k) \subset \mathbf{Aut}(\mathfrak{g})(R_{n,m}))$.

The case $n = 1$ is fully understood and permits to recover the classification of affine Kac-Moody Lie algebras by purely cohomological considerations (see [P1] and [P2]). For $n = 2$ the (inner) loop algebras for $n = 2$ are classified by a finite invariant [GP2]. The general case is due to Allison, Berman and Pianzola in a series of papers on covering algebras (see Allison's talk). The higher nullity cases are open.

The talk has been mainly devoted to the definition of the Witt-Tits invariant of a loop algebra. This invariant is crucial for our approach to the topic. We have shown that all k-parabolic subalgebras of $\mathfrak{g}$ which are normalized by σ and are minimal with respect to this property, are of the same type. This defines the Witt-Tits type $I(\sigma) \subset \Delta$, where Δ is the Dynkin diagram of $\mathfrak{g}$. The most delicate point is the uniqueness of $I(\sigma)$, which is established by showing that $I(\sigma)$ coincides with the Witt-Tits type (see [T]) of the semisimple $k(t_1, ..., t_n)$–algebraic group $\mathbf{Aut}(L(\mathfrak{g},\sigma) \otimes_{R_n} k(t_1, ..., t_n))^0$.

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Generalized Lie groups and quantization problems

Hideki Omori

1. μ-regulated algebra

A complete topological associative algebra $(\mathcal{A}, *)$ /$\mathbb{C}$ or /$\mathbb{R}$ is called a μ-**regulated algebra**, iff it satisfies (A.1)$\sim$ (A.4):

- **(A.1):** There is a special element $\mu \in \mathcal{A}$, called the **regulator** such that $[\mu, \mathcal{A}] \subset \mu*\mathcal{A}*\mu$, where $[a, b] = a*b - b*a$.
- **(A.2):** $[\mathcal{A}, \mathcal{A}] \subset \mu*\mathcal{A}$.
- **(A.3):** There is a closed linear subspace B such that $\mathcal{A} = B \oplus \mu*\mathcal{A}$.
- **(A.4):** $\mu* : \mathcal{A} \to \mu*\mathcal{A}$, $*\mu : \mathcal{A} \to \mathcal{A}*\mu$ defined by $a \to \mu*a$, $a \to a*\mu$ are linear isomorphism.

By these we see $\mu*\mathcal{A} = \mathcal{A}*\mu$, and that $\mu*\mathcal{A}$ is a closed two-sided ideal of $\mathcal{A}$. Thus, by (A.2) the product $*$ defines a commutative associative product on the space $B \cong \mathcal{A}/\mu*\mathcal{A}$. This product will be denoted by $a \cdot b$. Furthermore, successive use of (A.3) gives a decomposition of $\mathcal{A}$

$$\mathcal{A} = B \oplus \mu*B \oplus \cdots \oplus \mu^{N-1}*B \oplus \mu^N*\mathcal{A}. \tag{1}$$

By (A.4), one can define $[\mu^{-1}, a] = -\mu^{-1}*[\mu, a]*\mu^{-1}$ as a derivation of $(\mathcal{A}, *)$, which will be denoted by $\mathrm{ad}(\mu^{-1})a$. We see also that $\mu^{-1}*\mathcal{A}$ forms a Lie algebra under the bracket product.

$\Phi : (\mathcal{A}, *, \mu) \to (\mathcal{A}, *, \mu)$ is called an **automorphism**, iff $\Phi : (\mathcal{A}, *) \to (\mathcal{A}, *)$ is an algebra isomorphism, and $\Phi(\mu^{-1}) = a*\mu^{-1}$, $a \in \mathcal{A}_\times$. $\mathcal{D}(\mathcal{A}, \mu)$ is the group of all automorphisms.

A typical example μ-regulated algebra is the algebra $\psi DO(M)$ of all pseudo-differential operators of order 0 on a closed manifold M, where μ is given by $\sqrt{1+\Delta}^{-1}$ as a pseudo-differential operator of order -1, and Δ is the Laplacian. If this is the case, $\mathcal{D}(\mathcal{A}, \mu)$ is given as the group of all invertible Fourier integral operators of order 0. This group is known to be an infinite dimensional Lie group. In particular, this is a Fréchet manifold.

2. Generalized Lie groups

Even though $\mathcal{D}(\mathcal{A}, \mu)$ is an infinite dimensional Lie group, we have to treat various subgroups fixing several elements to construct an effective quantum theory. However, it is in general very difficult to make such subgroups Lie subgroups. In particular, it is hard to give an infinite dimensional manifold structure.

At this stage, we need a new convenient notion of infinite dimensional Lie groups with several properties which are easy to transfer to closed subgroups and quotient groups.

Recall that the most fundamental notion of mathematics is calculus (differentiation and integration).

Recall also that the most useful facts in the theory of finite dimensional Lie groups is the following:
(**A**): Any closed subgroup is a Lie group.
(**B**): Any factor group by a closed normal subgroup is a Lie group.

A *generalized Lie group* is a minimal notion of groups, where the multiplicative version of calculus is defined, and two theorems (**A**), (**B**) hold.
To be precise, the multiplicative version of differentiation is the notion of logarithmic derivative, and the multiplicative version of integration is the notion of product integrals.

A manifold structure is **not** requested. However, a generalized Lie group is not a group itself, but a pair $(G, \mathfrak{g})$ of a topological group and a Fréchet Lie algebra.

Lie algebra $\mathfrak{g}$ is more important than G, for if $(G, \mathfrak{g})$ is a generalised Lie group, then for every G' containing G as a subgroup, $(G', \mathfrak{g})$ is also a generalized Lie group. Given a generalized Lie group $(G, \mathfrak{g})$, there is the "minimal" generalized Lie group $(\tilde{G}, \mathfrak{g})$ such that $\tilde{G}$ is continuously embedded in G.

A topological Lie algebra $\mathfrak{g}$ is *enlargeable*, if exists a generalized Lie group $(G, \mathfrak{g})$.

Direct limits of infinite-dimensional Lie groups

Helge Glöckner

Many infinite-dimensional Lie groups of interest can be written as a union $G = \bigcup_{n\in\mathbb{N}} G_n$, where $G_1 \subseteq G_2 \subseteq \cdots$ are (finite- or infinite-dimensional) Lie groups and the inclusion maps $G_n \to G_{n+1}$ and $G_n \to G$ are smooth homomorphisms. In this report, we first recall some facts concerning the case where each G_n is finite-dimensional (which is by now well understood). We then describe various examples where already the groups G_n can be infinite-dimensional, and survey some results concerning such groups G. Frequently, G fails to be the direct limit $G = \varinjlim G_n$ in the categories of topological spaces and smooth manifolds, i.e., there exists a map $f\colon G \to X$ to a suitable topological space (resp., smooth manifold) X which is discontinuous (resp., not smooth) although $f|_{G_n}$ is continuous (resp., smooth) for each $n \in \mathbb{N}$. Surprisingly, in all examples inspected here, such pathologies do not occur if f is a *homomorphism* to a topological group (or Lie group) X, and thus $G = \varinjlim G_n$ holds in the category of topological groups and in the category of Lie groups. For certain ascending unions $G = \bigcup_{n\in\mathbb{N}} G_n$ of Lie groups, we also discuss regularity (in Milnor's sense) and the (non-)existence of small subgroups.

1. Direct limits of finite-dimensional Lie groups

If $G_1 \subseteq G_2 \subseteq \cdots$ is an ascending sequence of finite-dimensional Lie groups such that the inclusion maps are smooth homomorphisms, then the union $G = \bigcup_n G_n$ can be given a Lie group structure modelled on the locally convex direct limit $\varinjlim L(G_n)$, which makes G the direct limit $\varinjlim G_n$ in the categories of Lie groups, topological groups, smooth manifolds, and topological spaces (see [4]; in special cases, the Lie group structure has already been constructed in [9]). For $G = \bigcup_n G_n$ as before, many other properties are known. For example, it is known that G has no small subgroups, i.e., G has an identity neighbourhood which does not contain any non-trivial subgroup of G (see [5]). Furthermore, every closed subgroup H of G is a Lie subgroup and G/H admits a manifold structure making the quotient map $G \to G/H$ a smooth principal bundle with structure group H (see [4]). It was also shown in [4] that G is a regular Lie group in Milnor's sense, i.e., the initial value problem $\eta(0) = 1 \in G$, $\eta'(t) = \eta(t) \cdot \gamma(t)$ (using multiplication in the tangent group TG) has a (necessarily unique) smooth solution $\eta = \eta_\gamma \colon [0,1] \to G$ for each smooth curve $\gamma \colon [0,1] \to L(G)$, and the evolution map

$$\mathrm{evol}_G \colon C^\infty([0,1], L(G)) \to G\,, \quad \gamma \mapsto \eta_\gamma(1)$$

is smooth with respect to the C^∞-topology on $C^\infty([0,1], L(G))$.

2. Examples of ascending unions of infinite-dimensional Lie groups

Lie groups of compactly supported diffeomorphisms. Let M be a non-compact, σ-compact finite-dimensional smooth manifold of positive dimension and $M = \bigcup_n K_n$ with compact sets K_n such that $K_n \subseteq K_{n+1}^0$. We consider the group $\mathrm{Diff}_c(M)$ of all smooth diffeomorphisms $\gamma \colon M \to M$ which are compactly supported in the sense that the closure of $\{x \in M \colon \gamma(x) \neq x\}$ is compact. Then $\mathrm{Diff}_c(M)$ and the groups $\mathrm{Diff}_{K_n}(M)$ of diffeomorphisms supported in K_n are Lie groups (cf. [8]), and $\mathrm{Diff}_c(M) = \bigcup_n \mathrm{Diff}_{K_n}(M)$.

Test function groups. Similarly, if H is a (non-discrete) Lie group and $M = \bigcup_n K_n$ is as before, we can consider the so-called test function group $C_c^\infty(M, H)$ of smooth maps $\gamma \colon M \to H$ such that the closure of $\{x \in M \colon \gamma(x) \neq 1\}$ is compact. It is a Lie group and is a union $C_c^\infty(M,H) = \bigcup_n C_{K_n}^\infty(M,H)$, where $C_{K_n}^\infty(M,H)$ is the Lie group of smooth H-valued maps on M which are supported in K_n (see [1]).

Weak direct products of Lie groups. If $(H_n)_{n\in\mathbb{N}}$ is a sequence of Lie groups, we can form the weak direct product $\prod^*_{n\in\mathbb{N}} H_n$ consisting of all $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} H_n$ such that $x_n = 1$ for all but finitely many n. Then $\prod^*_{n\in\mathbb{N}} H_n$ can be made a Lie group modelled on the locally convex direct sum $\bigoplus_{n\in\mathbb{N}} L(H_n)$, and $\prod^*_{n\in\mathbb{N}} H_n = \bigcup_{n\in\mathbb{N}} \prod_{k=1}^n H_k$ (see [2]).

Ascending unions of unit groups of Banach algebras. Let $A_1 \subseteq A_2 \subseteq \cdots$ be an ascending sequence of unital Banach algebras such that the inclusion maps are continuous homomorphisms of unital algebras. Then the locally convex direct limit topology makes $A := \bigcup_{n\in\mathbb{N}} A_n$ a topological algebra with open unit group $A^\times$. If A is Hausdorff (e.g., if each inclusion map $A_n \to A_{n+1}$ is a topological embedding), then $A^\times$ is a Lie group (see [6]). Furthermore, $A^\times = \bigcup_{n\in\mathbb{N}} A_n^\times$ is a union of Banach-Lie groups.

Groups of germs of analytic Lie group-valued mappings. Let H be a complex Banach-Lie group, X be a metrizable, complex locally convex space and $K \subseteq X$ be a non-empty, compact subset. Let $\mathrm{Germ}(K,H)$ be the group of germs around K of complex analytic H-valued maps on open neighbourhoods of K in X. Then $\mathrm{Germ}(K,H)$ is a complex Lie group in a natural way, and its identity component is a union $\mathrm{Germ}(K,H)_0 = \bigcup_{n\in\mathbb{N}} G_n$ of certain Banach-Lie groups G_n (see [3] and [6]). Here $U_1 \supseteq U_2 \supseteq \cdots$ is a basis of open neighbourhoods of K in X, and G_n is the subgroup of $\mathrm{Germ}(K,H)$ generated by all germs of the form $[\exp_H \circ \gamma]$, where $\gamma\colon U_n \to L(H)$ is bounded and complex analytic.

Groups of germs of analytic diffeomorphisms. Let $X = \mathbb{C}^d$ now for some $d \in \mathbb{N}$, $K \subseteq X$ be compact and $U_1 \supseteq U_2 \supseteq \cdots$ be as before. Let $\mathrm{GermDiff}(K,X)$ be the group of all germs around K of complex analytic diffeomorphisms γ between open neighbourhoods of K in X such that $\gamma|_K = \mathrm{id}_K$. Then $\mathrm{GermDiff}(K,X)$ is a Lie group in a natural way, and it is a union $\mathrm{GermDiff}(K,X) = \bigcup_{n\in\mathbb{N}} M_n$ of Banach *manifolds* M_n (see [6]). Here M_n consists of all germs of bounded, complex analytic functions $\gamma\colon U_n \to X$ such that $\gamma|_K = \mathrm{id}_K$ and $\gamma'(K) \subseteq \mathrm{GL}(X)$.

3. Direct limit properties of the main examples

The following table (compiled from [6]) shows whether $G = \varinjlim G_n$ (resp., $G = \varinjlim M_n$) holds in the indicated category. In some cases, the answer depends on $\vec{G}$.

categ. \ group	$C^\infty_c(M,H)$	$\mathrm{Diff}_c(M)$	$A^\times$	$\mathrm{Germ}(K,H)_0$	$\mathrm{GermDiff}(K,X)$	$\prod^*_n H_n$
Lie groups	yes	yes	yes	yes	–	yes
top. groups	yes	yes	yes	yes	–	yes
smooth mfds	no	no	dep.*	dep.**	yes	dep.†
top. spaces	no	no	dep.*	dep.**	yes	dep.†

* "Yes" if the inclusion maps $A_n \to A_{n+1}$ are compact operators. "No" if the inclusion maps are topological embeddings, each A_n is infinite-dimensional and $A_n \neq A_{n+1}$. Other cases unknown.

** "Yes" if both X and H are finite-dimensional; otherwise unknown.

† "Yes" if each H_n is finite-dimensional or a k_ω-space (i.e., a hemicompact k-space). "No" if each H_n is modelled on an infinite-dimensional Fréchet space (which we assume nuclear when dealing with the category of smooth manifolds). Other cases unknown.

4. Regularity in Milnor's sense

Given $k \in \mathbb{N}_0$, let us say that a Lie group G is C^k*-regular* if G is regular and evol_G is smooth with respect to the C^k-topology on $C^\infty([0,1],L(G))$. It can be shown that finite-dimensional Lie groups (and their countable direct limits) are C^1-regular [4]. If a Lie group H is C^k-regular for some $k \in \mathbb{N}_0$, then $C^\infty_c(M,H)$ is C^k-regular (and hence regular). Furthermore, $\prod^*_n H_n$ is C^k-regular (and hence regular) if each H_n is so, for a fixed $k \in \mathbb{N}_0$. The following result from [7] might lead to the existence of non-regular Lie groups.

Proposition. *Suppose that, for each $n \in \mathbb{N}$, there exists a Lie group H_n such that* evol_{H_n} *is discontinuous at* 0 *with respect to the* C^n*-topology on* $C^\infty([0,1], L(H_n))$*. Then the Lie group* $\prod^*_{n\in\mathbb{N}} H_n$ *is not regular in Milnor's sense.*

The author does not know examples of Lie groups H_n with the required properties.

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Infinite-dimensional projective geometries and symmetric spaces

Wolfgang Bertram

In my talk I presented results of joint work with K.-H. Neeb ([BN04], [BN05]) on the construction of infinite-dimensional geometries via Jordan theory, and I explained some recent developments in this area.

The definition of symmetric spaces as quotient spaces of the form $M = G/H$ of a Lie group by some closed subgroup becomes cumbersome in the general infinite-dimensional theory as soon as one goes beyond the context of Banach-Lie groups. The elegant algebraic approach to symmetric spaces by O. Loos [Lo69] is much better suited for generalization to the infinite-dimensional situation: in this approach, a symmetric space is a smooth manifold M together with a family of symmetries $(\sigma_x)_{x\in M}$ such that some natural axioms are satisfied. Such spaces still have rich symmetry groups, but beyond the context of Banach manifolds one no longer has transitive Lie group actions on topological connected components.

For this reason, we return to the "old-fashioned" way of constructing manifolds directly by constructing a suitable atlas, studying its transition functions and proving that they are smooth. This strategy needs some stronger input than usually given by Lie theory: we need some more specific information on the geometry which enables us to select good charts and relates the transition functions to geometric data. For instance, in the case of Grassmann geometries such an atlas is given by the natural affine parts, whose transition functions are the well-known fractional linear transformations. In the purely algebraic paper [BN04], we generalize this observation to a large class of geometries called "generalized projective geometries". It turns out that the transition functions are "fractional quadratic" in a suitable sense – in fact, they are best described by *Jordan pairs* which correspond to the "stronger input" mentioned above. (See [Be00] for the finite-dimensional case, and [Be02] for the axiomatic approach to such geometries.)

In the subsequent paper [BN05] we study the smoothness properties of the transition functions and give sufficient and necessary conditions in order to obtain by our construction smooth manifolds and smooth symmetric spaces. A major advantage of this approach is that it leads to much more general results: we not only construct *real* manifolds and symmetric spaces, but our construction works over any (non-discrete) topological field and even over certain topological base rings. In fact, this was one of the starting points of the joint work with H. Glöckner and K.-H. Neeb [BGN04] where a differential calculus over such rings and fields is defined and which lead to new developments which are interesting in their own right (concerning analysis, see work of H. Glöckner on the arXiv, and concerning differential geometry see [Be05]). I think that the finite- and infinite-dimensional symmetric spaces over general base fields and rings thus constructed provide a rich ground for further research. Some new examples and problems are discussed in the recent preprints [Be06] and [BL06].

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Descent of line bundles to GIT quotients of flag varieties

Shrawan Kumar

Let G be a connected semisimple complex algebraic group with a maximal torus T and let P be a parabolic subgroup containing T. We denote their Lie algebras by the corresponding Gothic characters. The following theorem is our main result.
Theorem. Let $\mathcal{L}(\lambda)$ be a homogeneous ample line bundle on the flag variety $X = G/P$. Then, the line bundle $\mathcal{L}(\lambda)$ descends to a line bundle on the GIT quotient $X^{ss}(\lambda)//T$ (i.e., there exists a line bundle $\mathcal{L}$ on $X^{ss}(\lambda)//T$ whose pull-back to $X^{ss}(\lambda)$ is the restriction of $\mathcal{L}(\lambda)$) if and only if for all the semisimple subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ containing $\mathfrak{t}$ (in particular, rank $\mathfrak{s}$ = rank $\mathfrak{g}$),

$$\lambda \in \sum_{\alpha \in \triangle_+(\mathfrak{s})} \mathbb{Z}\alpha,$$

where $\triangle_+(\mathfrak{s})$ is the set of positive roots of $\mathfrak{s}$.

As a consequence of the above theorem, we get precisely which line bundles descend to the geometric quotients $X^{ss}(\lambda)//T$.

In the following Q (resp., Λ) is the root (resp., weight) lattice and we follow the indexing convention as in Bourbaki.
(9) Theorem. Let G be a connected, simply-connected simple algebraic group, $P \subset G$ a parabolic subgroup and let $\mathcal{L}(\lambda)$ be a homogeneous ample line bundle on the flag variety $X = G/P$. Then, the line bundle $\mathcal{L}(\lambda)$ descends to a line bundle on the GIT quotient $X^{ss}(\lambda)//T$ if and only if λ is of the following form depending upon the type of G.

- a) G of type $A_\ell\,(\ell \geq 1) : \lambda \in Q$
- b) G of type $B_\ell\,(\ell \geq 3) : \lambda \in 2Q$
- c) G of type $C_\ell\,(\ell \geq 2) : \lambda \in \mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell$
- d1) G of type $D_4 : \lambda \in \{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_i : n_i \in \mathbb{Z}$ and $n_1 + n_3 + n_4$ is even$\}$.
- d2) G of type $D_\ell\,(\ell \geq 5) : \lambda \in \{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell, n_i \in \mathbb{Z}$ and $n_{\ell-1} + n_\ell$ is even $\}$.
- e) G of type $G_2 : \lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$.
- f) G of type $F_4 : \lambda \in \mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$.
- g) G of type $E_6 : \lambda \in 6P$.
- h) G of type $E_7 : \lambda \in 12P$
- i) G of type $E_8 : \lambda \in 60Q$.

Current algebras, highest weight categories and quivers

Vyjayanthi Chari
(joint work with Jacob Greenstein)

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be its polynomial current algebra. Then $\mathfrak{g}[t]$ is a $\mathbb{Z}_+$-graded Lie algebra, the grading being given by powers of t. The object of our study is the category $\mathcal{G}$ of graded finite-dimensional representations of $\mathfrak{g}[t]$. The main motivation for the study of this category is the existence of several interesting families of indecomposable finite dimensional graded (and, in general, non-simple) $\mathfrak{g}[t]$-modules, namely: the Demazure modules arising from the positive level representations of the affine algebra, the fusion products of finite-dimensional representations of $\mathfrak{g}$ defined in [6], the Kirillov-Reshetikhin modules studied in [2, 3] and the Weyl modules introduced in [4] and studied in [1, 7].
The isomorphism classes of simple objects in $\mathcal{G}$ are indexed by the set $\Lambda = P^+ \times \mathbb{Z}_+$ where P^+ is the set of dominant integral weights of $\mathfrak{g}$. The set Λ can be identified in a natural way with a subset of the lattice of integral weights $\widehat{P}$ of the untwisted affine Lie algebra associated to $\mathfrak{g}$. We define a locally finite partial order $\preccurlyeq$ on Λ which is a refinement of the usual order on $\widehat{P}$ and show that $\mathcal{G}$ is a highest weight category, in the sense of [5], with the poset of weights $(\Lambda, \preccurlyeq)$. To do this, we study first the category $\widehat{\mathcal{G}}$ of graded $\mathfrak{g}[t]$-modules with finite-dimensional graded pieces. This category has enough projectives and the graded character of the projective modules can be described explicitly in terms of finite dimensional representations of $\mathfrak{g}$. Namely, given $(\lambda, r) \in \Lambda$, let $V(\lambda, r)$ be a representative of the corresponding isomorphism class of simple modules and $P(\lambda, r)$ be its projective cover in the category $\widehat{\mathcal{G}}$. Note that $V(\lambda, r)$ is the simple finite dimensional $\mathfrak{g}$-module $V(\lambda)$ of highest weight $\lambda \in P^+$ with the action of $\mathfrak{g}[t]$ given by evaluation at zero. Then the graded piece of $P(\lambda, r)$ of degree s is zero if $s < r$ and is isomorphic, as a $\mathfrak{g}$-module, to $S^{(s-r)}(\mathfrak{g}) \otimes V(\lambda)$, where

$$S^{(k)}(\mathfrak{g}) := \bigoplus_{(r_1,\dots,r_k)\in\mathbb{Z}_+^k\,:\,\sum_{j=1}^k jr_j = k} S^{r_1}(\mathfrak{g}) \otimes \cdots \otimes S^{r_k}(\mathfrak{g}).$$

Then, using a certain duality, we are able to show that the category $\mathcal{G}$ has enough injectives and we compute the graded character of the injective envelope $I(\lambda, r)$ in $\mathcal{G}$ of $V(\lambda, r)$. We then prove that $\mathcal{G}$ is a directed highest weight category by computing the extensions between simple objects.

We study algebraic structures associated with Serre subcategories of $\mathcal{G}$. Let Γ be an interval closed subset of Λ, that is, if $(\lambda, r) \prec (\mu, s) \in \Gamma$, then all $(\nu, k) \in \Lambda$ such that $(\lambda, r) \prec (\nu, k) \prec (\mu, s)$ are also in Γ. Let $\mathcal{G}[\Gamma]$ be the full subcategory of $\mathcal{G}$ consisting of objects whose simple constituents are parametrized by elements of Γ and let $I(\Gamma)_\Gamma$ be the injective cogenerator of $\mathcal{G}[\Gamma]$. It is well-known that there is an equivalence of categories between $\mathcal{G}[\Gamma]$ and the category of finite-dimensional right $\mathfrak{A}(\Gamma) = \operatorname{End}_{\mathcal{G}[\Gamma]} I(\Gamma)_\Gamma$-modules. Moreover $\mathfrak{A}(\Gamma)$ is a quotient of the path algebra of its Ext quiver $Q(\Gamma)$ and has a compatible grading. By using the character formula for the injective envelopes, we show that $Q(\Gamma)$ can be computed quite explicitly in terms of finite dimensional representations of $\mathfrak{g}$. In particular, the number of paths from (λ, r) to (μ, s) in $Q(\Gamma)$ can only be non-zero if $s < r$ and is equal to the $\mathfrak{g}$-module multiplicity of $V(\mu)$ in $\mathfrak{g}^{\otimes(r-s)} \otimes V(\lambda)$.

We consider various interesting examples of algebras $\mathfrak{A}(\Gamma)$. We show that for all $\mathfrak{g}$ (in some cases one has to exclude $\mathfrak{sl}_2$ or $\mathfrak{sp}_{2n}$, $n > 1$), there exists interval closed finite subsets Γ such that the corresponding algebra $\mathfrak{A}(\Gamma)$ is hereditary and $Q(\Gamma)$ is (a) a generalized Kronecker quiver; (b) a quiver of type $\mathbb{A}_\ell$, $\mathbb{D}_\ell$; (c) an affine quiver of type $\tilde{\mathbb{D}}_\ell$; (d) any star shaped quiver with three branches. Another interesting example arises from the theory of Kirillov-Reshetikhin modules for $\mathfrak{g}[t]$ where $\mathfrak{g}$ is of type D_n, $n > 6$. In this case the algebra $\mathfrak{A}(\Gamma)$ is not hereditary, but is still of tame representation type, and the Kirillov-Reshetikhin module is the unique indecomposable projective-injective module in the category $\mathcal{G}[\Gamma]$. In general, for $\mathfrak{g}$ of classical type, it can be shown that for any $\lambda \in P^+$ there is a natural interval closed subset Γ of Λ such that the Kirillov-Reshetikhin module $KR(\lambda)$ is the unique projective-injective module in $\mathcal{G}[\Gamma]$.

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Lower bounds for essential dimensions via orthogonal representations

Vladimir Chernousov

(joint work with J.-P. Serre)

Let k be an algebraically closed field and let $K = k(x_1, \cdots, x_n)$ be a pure transcendental extension of k. Given a simple Lie algebra $\mathfrak{g}$ over K, one can ask whether it "lives" over a subfield $L \subset K$ of K of smaller transcendence degree over k, i.e. whether there exists a Lie algebra $\mathfrak{g}'$ over L such that $\mathfrak{g} = \mathfrak{g}' \otimes_L K$. Since all forms of $\mathfrak{g}$ are classified by the non-abelian Galois cohomology set $H^1(K, G)$ where $G = \mathrm{Aut}\,(\mathfrak{g}_0)$ and $\mathfrak{g}_0$ is a split simple Lie algebra over k of the same type as $\mathfrak{g}$, this question naturally leads us to a notion of the essential dimension of an algebraic group which was first invented by J. Buhler and Z. Reichstein for finite groups, and then by Z. Reichstein for arbitrary algebraic groups.

Let us recall the definition of essential dimension. Let G be an arbitrary algebraic group over k. Let F/k be an extension, and let ξ be an element of $H^1(F,G)$. If E is a field with $k \subset E \subset F$ we say that ξ *comes from* E if it belongs to the image of $H^1(E,G) \to H^1(F,G)$. The *essential dimension* $\mathrm{ed}\,(\xi)$ of ξ is the minimum of the transcendence degrees E/k, for all E with $k \subset E \subset F$ such that ξ comes from E. One has $\mathrm{ed}(\xi) \leq \mathrm{tr.\,deg.}\,F$. The *essential dimension* $\mathrm{ed}\,(G)$ *of* G is

$$\mathrm{ed}\,(G) = \max\,\{\mathrm{ed}\,(\xi)\},$$

the maximum being taken over all pairs (F,ξ) with $k \subset F$ and $\xi \in H^1(F,G)$. Thus the essential dimension of $\xi \in H^1(F,G)$ tell us how far ξ is from being split and the essential dimension of G is the minimal number of parameters that we need to describe all G-torsors.

In the general case we know little about $\mathrm{ed}(G)$. For example, even for $G = \mathrm{PGL}_p$ where $p > 3$ is prime, we do not know whether $\mathrm{ed}(G) \geq 3$. It is worth mentioning that a long standing conjecture of Albert asserts that every central simple algebra of prime degree is cyclic. This conjecture, if true, would imply $\mathrm{ed}(\mathrm{PGL}_p) = 2$. On the other hand, the inequality $\mathrm{ed}(\mathrm{PGL}_p) \geq 3$ would disprove Albert's conjecture.

In 2000 Z. Reichstein and B. Youssin developed a method of computing lower bounds of essential dimensions based on consideration of G-torsors which come from proper finite subgroups of G whose centralizer is finite. This method allowed them to show that for many simple groups defined over fields of characteristic zero, one has $\mathrm{ed}(G) \geq r+1$ where r is the rank of G. Their proof, however, uses resolution of singularities, hence does not apply when $\mathrm{char}(k) \neq 0$.

In my work with J.-P. Serre we found another general method of computing lower bounds of essential dimensions of algebraic groups via orthogonal representations which works for all fields of characteristic $\neq 2$. Namely, given a group G satisfying some additional assumptions, we construct an orthogonal irreducible representation $\phi : G \to \mathbf{O}(V,q)$ with the following property: the nonzero weights of a maximal torus $T \subset G$ are the short roots and they have multiplicity 1. This morphism ϕ gives rise to a canonical functorial mapping from the set of isomorphism classes of G-torsors over a field K, to the set of isomorphism classes of quadratic forms over K of the same dimension as q. We next noticed that under this correspondence some G-torsors correspond to special quadratic forms which we call monomial. We introduced the notion of the rank of a monomial quadratic form and proved that the essential dimension of a monomial form is equal to its rank. This allows us to conclude immediately that for most of groups studied by Z. Reichstein and B. Youssin the lower bound of G is at least $r+1$. For instance:

(1.1) *If G is semisimple of adjoint type, and -1 belongs to the Weyl group, then*

$$\mathrm{ed}(G) \geq \mathrm{rank}(G) + 1.$$

Note that it implies

$$\mathrm{ed}(G_2) \geq 3,\ \mathrm{ed}(F_4) \geq 5,\ \mathrm{ed}(E_8) \geq 9 \quad \text{and} \quad \mathrm{ed}(E_7) \geq 8.$$

(1.2) $\mathrm{ed}\,(\mathbf{Spin}_n) \geq [n/2]$ *for* $n > 6,\ n \neq 10$, *the inequality being strict if* $n \equiv -1, 0$ *or* $1 \pmod 8$.

(1.3) $\mathrm{ed}\,(\mathbf{HSpin}_n) > n/2$ *if* $n \geq 8,\ n \equiv 0 \pmod 8$.

N.B. In practice, many of the lower bounds in the above list are obtained from torsors corresponding to multiloop algebras.

Infinite-dimensional Lie algebras and bases of simple Lie algebras

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(joint work with Paul Terwilliger)

1. Introduction

This talk featured a certain remarkable basis, termed the *equitable basis*, for the Lie algebra $\mathfrak{sl}_2$ of 2×2 traceless matrices over a field $\mathbb{F}$ of characteristic zero. This basis plays an essential role in the study of tridiagonal pairs of linear transformations (see [6], [4]) and in unraveling the structure of the three-point loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}, (t-1)^{-1}]$ in ([2], [5]). It exhibits many striking properties including connections with the theory of Kac-Moody Lie algebras and with Pythagorean triples. Further details and proofs of the results quoted here can be found in [3].

The *equitable basis* $\{x, y, z\}$ for $\mathfrak{sl}_2$ consists of the matrices

$$x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad z = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix},$$

whose products satisfy

$$[x,y] = 2x + 2y, \qquad [y,z] = 2y + 2z, \qquad [z,x] = 2z + 2x. \tag{1.1}$$

From this it is apparent that all basis elements are "created equal", and there is a Lie algebra automorphism ϱ of $\mathfrak{sl}_2$ of order 3 such that $\varrho(x) = y,\ \varrho(y) = z$, and $\varrho(z) = x$.

Relative to the equitable basis, the matrix of the trace form $(u, v) = \mathrm{tr}(uv)$ is given by

$$\mathcal{A} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}. \tag{1.2}$$

Assume $x^*, y^*, z^* \in \mathfrak{sl}_2$ are dual to the equitable basis in the sense that $(u, v^*) = 2\delta_{u,v}$ for all $u, v \in \{x, y, z\}$. Then x^*, y^*, z^* are the nilpotent matrices given by

$$x^* = -\frac{1}{2}(y+z), \quad y^* = -\frac{1}{2}(z+x), \quad z^* = -\frac{1}{2}(x+y), \tag{1.3}$$

so that ϱ cyclically permutes them also: $\varrho(x^*) = y^*$, $\varrho(y^*) = z^*$, and $\varrho(z^*) = x^*$.

2. Connections with the modular group and braid group $\mathcal{B}_3$

Let G denote the subgroup of the automorphism group $\mathrm{Aut}_{\mathbb{F}}(\mathfrak{sl}_2)$ generated by $X := \exp(\operatorname{ad} x^*)$, $Y := \exp(\operatorname{ad} y^*)$, and $Z := \exp(\operatorname{ad} z^*)$. We prove the following:

Proposition 2.1.
(i) $\varrho = XY = YZ = ZX \in G$;
(ii) $G = \langle \varrho, \tau_x \rangle = \langle \varrho, \tau_y \rangle = \langle \varrho, \tau_z \rangle$, *where* $\tau_x := YZY = ZYZ$, $\tau_y := ZXZ = XZX$, and $\tau_z := XYX = YXY$;
(iii) $G \cong \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\langle \pm \mathrm{I} \rangle$.

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is a free product $\mathbb{Z}_3 * \mathbb{Z}_2$ of a cyclic group of order 3 and a cyclic group of order 2. The result in (iii) is established by taking ϱ to be the generator of the cyclic group of order 3, and either τ_x, τ_y, or τ_z to be the generator of the cyclic group of order 2.

The lattice $L := \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z = \left\{ \begin{bmatrix} p & q \\ r & -p \end{bmatrix} \in \mathfrak{sl}_2(\mathbb{Z}) \,\middle|\, q, r \equiv 0 \mod 2 \right\}$ is a Lie algebra analogue of the congruence subgroup. We show that the group $\mathrm{AAut}_{\mathbb{Z}}(L)$ of automorphisms and antiautomorphisms of L satisfies $\mathrm{AAut}_{\mathbb{Z}}(L) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes G$, where the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $-\mathrm{I}$ and the transposition $(x\, y)$. The orbit of x under G has the following description:

$$Gx = \{ u \in L \mid (u, u) = 2 \} \overset{1-1}{\longleftrightarrow} G \overset{1-1}{\longleftrightarrow} \mathrm{PSL}_2(\mathbb{Z}). \tag{2.2}$$

Let $\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be the braid group on three strands. Any irreducible representation $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ affords an irreducible representation $\mathcal{B}_3 \to \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ of $\mathcal{B}_3$ given by $\sigma_1 \mapsto \exp(x^*) \mapsto \exp(\varphi x^*)$, $\sigma_2 \mapsto \exp(y^*) \mapsto \exp(\varphi y^*)$ (any pair of elements x^*, y^*, z^* could be used). The adjoint representation $\varphi = \mathrm{ad}$ (with $V = \mathfrak{sl}_2$) provides an example of this (see Proposition 2.1 (ii) above). The center of $\mathcal{B}_3$ is generated by $(\sigma_1\sigma_2)^3$, and this must map to a scalar multiple of the identity by Schur's lemma. In the adjoint example, $(\sigma_1\sigma_2)^3$ maps to $\varrho^3 = \mathrm{I}$.

Suppose now that V is the (unique up to isomorphism) irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$, and let $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be the corresponding irreducible representation of $\mathrm{SL}_2(\mathbb{Z})$. Then V has a basis $v_0, v_1, \ldots, v_n$, where $v_k = (1/k!)(y^*)^k.v_0$ and $x.v_k = (n-2k)v_k$ for all $k = 0, 1, \ldots, n$. Setting $P = \exp(\varphi x^*)\exp(\varphi y^*)$, we have the following:

Theorem 2.3.
(i) $Pv_0, Pv_1, \ldots, Pv_n$ *is an eigenbasis of* V *for* y, *and* $P^2v_0, P^2v_1, \ldots, P^2v_n$ *is an eigenbasis of* V *for* z;
(ii) $P^3 = (-1)^n\mathrm{I}$;
(iii) $\sum_{i=0}^n v_i = Pv_0$, $\sum_{i=0}^n Pv_i = P^2v_0$, *and* $\sum_{i=0}^n P^2v_i = (-1)^n v_0$;
(iv) $\mathrm{span}_{\mathbb{F}}\{v_0, v_1, \ldots, v_k\} = \mathrm{span}_{\mathbb{F}}\{Pv_n, Pv_{n-1}, \ldots, Pv_{n-k}\}$ *for* $k = 0, 1, \ldots, n$.

3. Connections with Kac-Moody Lie algebras

The matrix $\mathcal{A}$ of the trace form in (1.2) is the Cartan matrix of a hyperbolic Kac-Moody algebra. One of the most remarkable features of the equitable basis is its connections with the roots Δ, weights, and Weyl group W of this Lie algebra. Indeed, we may identify the set of simple roots with the equitable basis $\{x, y, z\}$ and the root lattice $Q = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z$ with the Lie algebra L. The real roots $\Delta^{\mathrm{re}} = Wx \cup Wy \cup Wz$ are in bijection with the orbit Gx, hence with $\mathrm{PSL}_2(\mathbb{Z})$. The isotropic roots $\Delta^0 = \{u \in \Delta \mid (u, u) = 0\}$ correspond to nilpotent matrices in L, and the remaining imaginary roots $\Delta^{\mathrm{i}m} \setminus \Delta^0 = \{u \in \Delta \mid (u, u) < 0\}$ correspond to matrices in L with purely imaginary eigenvalues. For the automorphism group $\mathrm{aut}_{\mathbb{Z}}(Q) = \{\psi \in \mathrm{GL}(Q) \mid (\psi v, \psi w) = (v, w) \text{ and } \psi(\Delta) \subseteq \Delta\}$ of the root lattice Q, we argue that

$$\mathrm{aut}_{\mathbb{Z}}(Q) \cong \pm S_3 \ltimes W \cong \mathrm{AAut}_{\mathbb{Z}}(L) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes G,$$

where the reflection $r_x : u \mapsto u - (x, u)x$ is identified with the map $(y\,z)\tau_x$, and similarly, r_y with $(z\,x)\tau_y$, and r_z with $(x\,y)\tau_z$.

Under these identifications, the fundamental weights are matched with $(1/2)x^*$, $(1/2)y^*$, $(1/2)z^*$. Every element $u \in Wx^* \cup Wy^* \cup Wz^*$ has an expression $u = -1/2(a^2x + b^2y + c^2z)$ as a linear combination of x, y, z, where $a, b, c \in \mathbb{Z}_{\geq 0}$ (not all 0), and one of the following relations holds: $a + b = c$, $b + c = a$, or $c + a = b$. The matrix product

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tag{3.1}$$

always yields a triple (α, β, γ) of integers satisfying the Pythagorean relation $\alpha^2 + \beta^2 = \gamma^2$ when $a + b = c$. (When $b + c = a$ or $c + a = b$, just permute the rows of the matrix and α, β, γ to obtain a Pythagorean triple.) What results is a simple procedure for generating Pythagorean triples different from (though related to) the dynamical system in [7] (compare also [1]).

We have explored analogous bases for other simple Lie algebras and for other hyperbolic Kac-Moody Lie algebras. An interesting problem is to understand how the characters of the Kac-Moody Lie algebra representations are related to the simple Lie algebra under the various correspondences.

4. Acknowledgments

Support from NSF grant #DMS–0245082 is gratefully acknowledged.

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Kac-Moody superalgebras and their representations

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Kac-Moody Lie superalgebras were introduced by V. Kac.

Let $I = \{1, \dots, n\}$, $p : I \to \mathbb{Z}_2$ and $A = (a_{ij})$ be a matrix. Fix a vector space $\mathfrak{h}$ of dimension $n + \mathrm{rk}\,(A)$, linearly independent $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ and $h_1, \dots, h_n \in \mathfrak{h}$ such that $\alpha_j\,(h_i) = a_{ij}$, define a superalgebra $\bar{\mathfrak{g}}\,(A)$ by generators $X_1, \dots, X_n, Y_1, \dots, Y_n$ and $\mathfrak{h}$ with relations

$$[h, X_i] = \alpha_i\,(h)\,X_i,\ \ [h, Y_i] = -\alpha_i\,(h)\,Y_i,\ \ [X_i, Y_j] = \delta_{ij} h_i. \tag{1}$$

Here we assume that all elements of $\mathfrak{h}$ are even and $p\,(X_i) = p\,(Y_i) = p\,(i)$.

By $\mathfrak{g}\,(A)$ (or $\mathfrak{g}$ when the Cartan matrix is fixed) denote the quotient of $\bar{\mathfrak{g}}\,(A)$ by the unique maximal ideal which intersects $\mathfrak{h}$ trivially. It is clear that if a matrix $B = DA$ for some invertible diagonal D then $\mathfrak{g}\,(B) \cong \mathfrak{g}\,(A)$. Indeed an isomorphism can be obtained by mapping h_i to $d_{ii} h_i$. Therefore without loss of generality we may assume that $a_{ii} = 2$ or 0.

The Lie superalgebra $\mathfrak{g} = \mathfrak{g}\,(A)$ has a *root* decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Clearly, one can define $p : \Delta \to \mathbb{Z}_2$ by putting $p\,(\alpha) = 0$ or 1 whenever $\mathfrak{g}_\alpha$ is even or odd, respectively. By $\Delta_0\,(\Delta_1)$ we denote the set of even (odd) roots. Every root is a positive or negative linear combination of $\alpha_1, \dots, \alpha_n$. According to this we call a root positive or negative and have the decomposition $\Delta = \Delta^+ \cup \Delta^-$. The roots $\alpha_1, \dots, \alpha_n$ are called *simple roots.* Sometimes instead of a_{ij} we will write $a_{\alpha\beta}$ where $\alpha = \alpha_i$, $\beta = \alpha_j$.

One sees easily that there are the following possibilities for each simple root $\alpha = \alpha_i$:

(1) if $a_{\alpha\alpha} = 0$, then $[X_\alpha, X_\alpha] = [Y_\alpha, Y_\alpha] = 0$ and X_α, Y_α and h_α generate the subalgebra isomorphic to $\mathrm{sl}\,(1|1)$;
(2) if $a_{\alpha\alpha} = 2$ and $p\,(\alpha) = 0$, then X_α, Y_α and h_α generate the subalgebra isomorphic to $\mathrm{sl}\,(2)$;
(3) if $a_{\alpha\alpha} = 2$ and $p\,(\alpha) = 1$, then $[X_\alpha, X_\alpha] = [Y_\alpha, Y_\alpha] \neq 0$ and X_α, Y_α and h_α generate the subalgebra isomorphic to $\mathrm{osp}\,(1|2)$, in this case $2\alpha \in \Delta$.

In the first case we say that α is *isotropic* and by definition any isotropic root is odd. In other cases a root is called *non-isotropic.* A simple root α is *regular* if for any other simple root β, $a_{\alpha\beta} = 0$ implies $a_{\beta\alpha} = 0$. Otherwise a simple root is called *singular.*

A superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ has a natural $\mathbb{Z}$-grading $\mathfrak{g} = \oplus\mathfrak{g}_n$ if we put $\mathfrak{g}_0 := \mathfrak{h}$ and $\mathfrak{g}_1 = \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_n}$. This grading is called *principal.* We say that $\mathfrak{g}$ is of *finite growth* if $\dim \mathfrak{g}_n$ grows polynomially depending on n.

We say that $\mathfrak{g}(A)$ is *integrable* if ad_{X_i} are locally nilpotent for all $i \in I$. In this case ad_{Y_i} are also locally nilpotent.

Theorem 1. *(Hoyt, Serganova) Let A be non-singular. If $\mathfrak{g}(A)$ has finite growth, then $\mathfrak{g}(A)$ is integrable.*

Let $a_{ii} = 0$ and $p(i) = 1$. Define $r_i(\alpha_j)$ by the following formula

$$r_i(\alpha_i) = -\alpha_i,$$

$$r_i(\alpha_j) = \alpha_j \text{ if } a_{ij} = a_{ji} = 0, i \neq j,$$

$$r_i(\alpha_j) = \alpha_i + \alpha_j \text{ if } a_{ij} \neq 0 \text{ or } a_{ji} \neq 0.$$

Let $X_j^{'} \in \mathfrak{g}_{r_i(\alpha_j)}$, $Y_j^{'} \in \mathfrak{g}_{r_i(-\alpha_j)}$ and $h_j' = c\left[X'_{r_i(\alpha_j)}, Y'_{r_i(\alpha_j)}\right]$ such that that $\left[h_j', X_j'\right] = 2X_j'$ or 0.

Lemma 2. *The roots $r_i(\alpha_1), \ldots, r_i(\alpha_n)$ are linearly independent. The elements $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ together with $h_1', \ldots, h_n'$ satisfy* (1)*. Moreover if α_i is regular, then $\mathfrak{h}$ and $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ generate $\mathfrak{g}$.*

We see from Lemma 2 that if one has a matrix A and some regular isotropic simple root α_i one can construct another matrix A' such that $\mathfrak{g}(A') \cong \mathfrak{g}(A)$. In this case we say that A' is obtained from A and a base $\Pi' = \{\alpha_1', \ldots, \alpha_n'\}$ is obtained from $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ by the *odd reflection* with respect to α_i.

Symmetrizable Kac-Moody superalgebras of finite growth were classified by J. van de Leur in 1986. As in the Lie algebra case they are central extensions of loop superalgebras or twisted loop superalgebras.

In what follows we assume that A is indecomposable and $a_{ij} \neq 0$ implies $a_{ji} \neq 0$.

Theorem 3. *(Hoyt, Serganova) A non-symmetrizable superalgebras of finite growth is isomorphic to $q(n)^{(2)}$ or $S(1,2;a)$ described below.*

Description of $q(n)^{(2)}$. The Lie superalgebra $q(n)^{(2)}$ is an extension of twisted loop algebra of the simple Lie superalgebra $\mathfrak{t} = q(n)$ by the involutive automorphism ϕ such that $\phi_{|\mathfrak{t}_0} = \mathrm{id}$, $\phi_{|\mathfrak{t}_1} = -id$. The Lie superalgebra $q(n)^{(2)}$ is isomorphic to

$$\mathfrak{t}_0 \otimes \mathbb{C}\left[t^2, t^{-2}\right] \oplus \mathfrak{t}_1 \otimes t\mathbb{C}\left[t^2, t^{-2}\right] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with $d = t\frac{\partial}{\partial t}$ and c be the central element. For any $x, y \in \mathfrak{t}$, the commutator is defined by the formula

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \delta_{m,-n}(1 - (-1)^m)\operatorname{tr}(xy)\,c.$$

Description of $S(1,2;a)$. Consider commutative associative Lie superalgebra $R = \mathbb{C}\left[t, t^{-1}, \xi_1, \xi_2\right]$ with even generator t and two odd generators ξ_1, ξ_2. By $\mathcal{W}(1,2)$ we denote the Lie superalgebra of derivations of R, *i. e* all linear maps $d : R \to R$ such that

$$d(fg) = d(f)\,g + (-1)^{p(d)p(f)}\,fd(g)\,.$$

An element $d \in W(1,2)$ can be written as

$$d = f\frac{\partial}{\partial t} + f_1\frac{\partial}{\partial \xi_1} + f_2\frac{\partial}{\partial \xi_2}$$

for some $f, f_1, f_2 \in R$. It is easy to see that the subset of all $d \in \mathcal{W}(1,2)$ satisfying the condition

bft^{-1} $+ \frac{\partial f}{\partial t} = (-1)^{p(d)}\,\frac{\partial f_1}{\partial \xi_1} + \frac{\partial f_2}{\partial \xi_2}$

form a subalgebra $\mathcal{S}_b$ of $\mathcal{W}(1,2)$. and $\mathcal{S}_b$ is simple if $b \notin \mathbb{Z}$. If $b \in \mathbb{Z}$, then $\mathcal{S}'_b = [\mathcal{S}_b, \mathcal{S}_b]$ has codimension 1, more precisely

$$\mathcal{S}_b = \mathbb{C}\xi_1\xi_2 t^{-b}\frac{\partial}{\partial t} \oplus \mathcal{S}'_b.$$

Assume that $b \neq 0$ and set

$$X_1 = \frac{\partial}{\partial \xi_1},\ X_2 = -\xi_1\xi_2 t^{-1}\frac{\partial}{\partial \xi_1} + \frac{\xi_2}{b}\frac{\partial}{\partial t},\ X_3 = \xi_1\frac{\partial}{\partial \xi_2},$$

$$Y_1 = \frac{b+1}{b}\xi_1\xi_2\frac{\partial}{\partial \xi_2} + \frac{t}{b}\frac{\partial}{\partial t},\ Y_2 = t\frac{\partial}{\partial \xi_2},\ Y_3 = \xi_2\frac{\partial}{\partial \xi_1},$$

$$H_1 = \frac{b+1}{b}\xi_2\frac{\partial}{\partial \xi_2} + \frac{t}{b}\frac{\partial}{\partial t},\ H_2 = \xi_1\frac{\partial}{\partial \xi_1} + \frac{\xi_2}{b}\frac{\partial}{\partial \xi_2} + \frac{t}{b}\frac{\partial}{\partial t},\ H_3 = \xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2}.$$

Then X_i, Y_i, H_i, $i = 1,2,3$, generate $\mathcal{S}_b$ or $\mathcal{S}'_b$ if $b \in \mathbb{Z}$. They satisfy the relations (1) with Cartan matrix $S(1,2;a)$ with $a = -\frac{b+1}{b}$. Hence The contragredient Lie superalgebra $S(1,2;a)$ can be obtained from $\mathcal{S}_b\,(\mathcal{S}'_b)$ by adding the element $d = t\frac{\partial}{\partial t}$ and taking a central extension. This superalgebra appears in the list of conformal superalgebras classifies by Kac and van de Leur.

Representation theory of Kac-Moody Lie superalgebras is interesting although not much is known. In symmetrizable case Kac and Wakimoto have an interesting conjecture on a character formula which provides new combinatorial identities. In non-symmetrizable case there are some completely new phenomena. For example, as we proved with M. Gorelik any Verma module over $q(n)^{(2)}$ is reducible.

Multiplicities in direct limits of principal series representations

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(joint work with Ivan Penkov)

The representation theory of infinite dimensional Lie groups is still in a beginning state. So it seems worthwhile to look at the simplest cases and to try to develop them from a general viewpoint. Here I take the viewpoint of Harish–Chandra theory; I will not discuss the Olshanskii approach, the operator–theoretic approach, nor the Banach–Lie group approach.

The first case, the Bott–Borel–Weil Theorem for direct limits of compact and complex reductive finite dimensional Lie groups, was developed from the analytic viewpoint in [NRW] and from the algebraic viewpoint in [DPW]. The second case (though chronologically the first), limits of holomorphic discrete series representations, was studied by Natarajan [N]. Further studies of limits of discrete series representations were made by A. Habib [H]. At that point it seemed appropriate to look at direct limits of representations from the various tempered series of Harish–Chandra. The first general results there were worked out for direct limits of principal series representations [W]. Here we consider applications to multiplicities in the restrictions of direct limit principal series representations.

Recall that if G is a real reductive (e.g. semisimple) Lie group of Harish–Chandra class, K is a maximal compact subgroup of G, and π is an irreducible unitary representation, then $\pi|_K$ is a discrete direct sum of irreducible representations, $\pi|_K = \sum_{\kappa \in \widehat{K}} m(\kappa,\pi)\kappa$ where the multiplicity $m(\kappa,\pi) \leqq \dim \kappa$. Now suppose that $G = \varinjlim G_n$ is a strict direct limit of real reductive Lie groups of Harish–Chandra class, and $K = \varinjlim K_n$ is the limit of their maximal compact subgroups. Given an irreducible (perhaps one should also require unitary) representation π of G one can ask (i) whether $\pi|_K$ is a discrete direct sum of irreducible representations, (ii) whether some or all of the multiplicities are finite, and (iii) whether there is a systematic bound on the finite multiplicities. More modestly, we can ask these questions when π is a direct limit $\varinjlim \pi_n$ of irreducible representations of the G_n, or even when further restrictions are placed on (a) the groups G_n, (b) the sort of direct limit, and (c) the representations π_n. That, in fact, is exactly what we do here.

In the representation theory of finite dimensional Lie groups it is natural to consider unitary representations for application to harmonic analysis, and to consider Fréchet space representations for applications to complex function theory. For infinite dimensional Lie groups the "best" choice of representations is not so clear, and in particular it is not so clear that one should concentrate on unitary representations. However, for direct limit groups one naturally ends up with projective limit representations, and that problem is avoided by working in the category of unitary representations, where direct and inverse limits coincide.

In order to discuss limit principal series representations for the limit $G = \varinjlim G_n$ of the strict direct system $\{G_n, \phi_{m,n})$ we need coherent systems of Iwasawa decompositions $G_n = K_n A_n N_n$ and minimal parabolic subgroups $P_n = M_n A_n N_n$.

Lemma 1. *The Iwasawa decompositions of the $G_n = K_n A_n N_n$ can be chosen so that*

$$\phi_{m,n}(K_n) \subset K_m; \text{ so we have } K = \varinjlim K_n,$$
$$\phi_{m,n}(A_n) \subset A_m; \text{ so we have } A = \varinjlim A_n, \text{ and}$$
$$\phi_{m,n}(N_n) \subset N_m; \text{ so we have } N = \varinjlim N_n.$$

In that notation K, A and N are closed subgroups of G and $(k,a,n) \mapsto kan$ defines a diffeomorphism of $K \times A \times N$ onto G. In particular we have an Iwasawa–type decomposition $G = KAN$.

The groups $M_n = Z_{K_n}(A_n)$ do not nest automatically because in principle M_{n+1} can have more to centralize that does M_n. However this is not a problem unless $A_n \subsetneqq A_{n+1}$. Thus we have

Lemma 2. *Suppose that G has finite real rank. Then $\phi_{m,n}(M_n) \subset M_m$ for $m \geqq n$ and n sufficiently large, so we have $M = \varinjlim M_n$ and $P = \varinjlim P_n$. Further, M and P are closed subgroups of G, and $(m,a,n) \mapsto man$ defines a diffeomorphism of $M \times A \times N$ onto P. Thus we have a Langlands–type decomposition $P = MAN$.*

Examples 3. Here are the classical limit groups that have finite real rank, so that Lemma 2 applies to them.

Example 3.a. Let $G_n = SU(p,n)$, the special unitary group for the hermitian form $h(u,v) = \sum_1^p u_i \bar{v}_i - \sum_1^n u_{p+i}\bar{v}_{p+i}$, with embeddings $G_n \hookrightarrow G_{n+1}$ given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\varinjlim G_n$ is the classical direct limit group $SU(p,\infty)$ of real rank p.

Example 3.b. Let $G_n = SO(p, n_0 + 2n)$, the special orthogonal group for the bilinear form $h(u,v) = \sum_1^p u_i v_i - \sum_1^n u_{p+i} v_{p+i}$, with embeddings $G_n \hookrightarrow G_{n+1}$ given by $x \mapsto \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Then $\varinjlim G_n$ is the classical direct limit group $SO(p,\infty)$ of real rank p. We do this with two 1's for reference later. Note that this direct system consists either of groups of type B (when $p + n_0 + 2n$ is odd) or of type D (when $p + n_0 + 2n$ is even), but not both, but those two cases result in isomorphic direct limit groups $SO(p,\infty)$.

Example 3.c. Let $G_n = Sp(p,n)$, the unitary group for the quaternion–hermitian form $h(u,v) = \sum_1^p u_i \bar{v}_i - \sum_1^n u_{p+i}\bar{v}_{p+i}$, with embeddings $G_n \hookrightarrow G_{n+1}$ given by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$. (We are using quaternionic matrices.) Then $\varinjlim G_n$ is the classical direct limit group $Sp(p,\infty)$ of real rank p.

In these examples suppose $n \geqq p$. Then K_n consists of all $\left(\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right)$ in $SO(p) \times SO(2n)$, $S(U(p) \times U(n)$ or $Sp(p) \times Sp(n)$, respectively, and we can take $\mathfrak{a}_n$ to consist of all $\left(\begin{smallmatrix} 0 & a & 0 \\ {}^t a & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ where a is a $p \times p$ real diagonal matrix. Thus M consists of all $\left(\begin{smallmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & k \end{smallmatrix}\right)$ where

$SU(p,n)$ case: $m \in U(p)$ is diagonal, $k \in U(n-p)$ and $\det(m)^2 \det(k) = 1$.
$SO(p, n_0 + 2n)$ case: $m \in O(p)$ is diagonal and $k \in SO(n_0 + 2n - p)$.
$Sp(p,n)$ case: $m \in Sp(m)$ is diagonal and $k \in Sp(n-p)$.

This is consistent with the alignment of Iwasawa decompositions as described in Lemma 1, and it gives us the additional condition $\phi_{m,n}(M_n) \subset M_m$ of Lemma 2.

A projective limit of spherical principal series representations of the G_n is defined by a quasi–character $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$. There σ defines $\sigma_n = \sigma|_{\mathfrak{a}_{n,\mathbb{C}}}$, and σ_n defines the spherical principal series representation $\pi_{\sigma,n} = \operatorname{Ind}_{P_n}^{G_n}(m_n a_n n_n \mapsto e^{\sigma_n(\log a_n)})$ of G_n. Further, $\pi_\sigma := \operatorname{Ind}_P^G(man \mapsto e^{\sigma(\log a)})$ is equivalent to $\varprojlim \pi_{\sigma,n}$.

The condition for unitarity of $\pi_{\sigma,n}$ is $\sigma_n \in i\mathfrak{a}_n^* + \rho_{n,\mathfrak{g},\mathfrak{a}}$ where

$$\rho_{n,\mathfrak{g},\mathfrak{a}} = -\tfrac{1}{2} \sum_{\gamma_n \in \Sigma(\mathfrak{g}_n,\mathfrak{a}_n)^+} (\dim \mathfrak{g}_n^{\gamma_n})\gamma_n\,,$$

half the sum (with multiplicities) of the negative restricted roots of $\mathfrak{g}_n$.

Proposition 4. *Let $\mathfrak{g}_n \subset \mathfrak{g}_m$ be real semisimple Lie algebras. Choose a Cartan involution θ of $\mathfrak{g}_m$ that preserves $\mathfrak{g}_n$ and a maximal abelian subspace $\mathfrak{a}_n$ of $\{\xi \in \mathfrak{g}_n \mid \theta(\xi) = -\xi\}$, and enlarge $\mathfrak{a}_n$ to a maximal abelian subspace $\mathfrak{a}_m$ of $\{\xi \in \mathfrak{g}_m \mid \theta(\xi) = -\xi\}$. Suppose that $\mathfrak{a}_m = \mathfrak{a}_n \oplus \mathfrak{a}_{m,n}$ where $\mathfrak{a}_{m,n}$ centralizes $\mathfrak{g}_n$, in other words that $\mathfrak{g}_n \oplus \mathfrak{a}_{m,n}$ is a subalgebra of $\mathfrak{g}_m$. Then following conditions are equivalent.*

1. *The restriction $\rho_{m,\mathfrak{g},\mathfrak{a}}|_{\mathfrak{a}_n} = \rho_{n,\mathfrak{g},\mathfrak{a}}$.*
2. *$(\mathfrak{g}_n + \mathfrak{m}_m) \oplus \mathfrak{a}_{m,n}$ is the centralizer of $\mathfrak{a}_{m,n}$ in $\mathfrak{g}_m$.*
3. *Modulo $\mathfrak{m}_m$, the algebra $\mathfrak{g}_n$ is the semisimple component of a real parabolic subalgebra of $\mathfrak{g}_m$ that contains $\mathfrak{a}_m$.*

The strict direct system $\{G_n\Phi_{m,n}\}$ is **weakly parabolic** if these three conditions hold for every $n \leqq m$. In that case, given $\sigma' \in i\mathfrak{a}^*$ we have $\sigma \in \mathfrak{a}^*_{\mathbb{C}}$ such that every $\sigma_n \in i\mathfrak{a}_n^* + \rho_{n,\mathfrak{g},\mathfrak{a}}$. Thus all the $\pi_{\sigma,n}$ are unitary and we can take the projective limit π_σ in the category of unitary representations. Then it is equivalent to a direct limit $\varinjlim \pi_{\sigma,n}$. That constructs the unitary spherical principal series representations of weakly parabolic strict direct systems.

In order to decompose $\pi_\sigma|_K$ we define

$$\widehat{\widehat{K}} = \{[\kappa] \in \widehat{K} \mid \kappa \text{ has form } \varprojlim \kappa_n \text{ where } [\kappa_n] \in \widehat{K_n} \text{ for } n \ggg 0\}. \tag{5}$$

Align by highest weight vector. Let $E_{\kappa,n}$ is the representation space of $\kappa_n \in \widehat{K_n}$, so $E_\kappa := \varprojlim E_{\kappa,n}$ is the representation space of $\kappa = \varprojlim \kappa_n \in \widehat{K}$, where limits are taken in the category of Hilbert spaces and unitary representations. Using orthocomplementation on the finite level one checks that

$$\widehat{\widehat{K}} = \{[\kappa] \in \widehat{K} \mid \kappa \text{ has form } \varinjlim \kappa_n \text{ where } [\kappa_n] \in \widehat{K_n} \text{ for } n \ggg 0\}. \tag{6}$$

The point is

Proposition 7. *Let $\{G_n, \phi_{m,n}\}$ be weakly parabolic. Suppose that $\sigma \in i\mathfrak{a}^* + \rho$, or equivalently that there is an index n_0 such that $\sigma_n \in i\mathfrak{a}_n^* + \rho_{n,\mathfrak{a}}$ for all $n \geqq n_0$. Then the representation space $L^2(G,K,\sigma)$ of π_σ has K–module structure $\sum_{\kappa \in \widehat{\widehat{K}}} E_\kappa \widehat{\otimes} E_\kappa^{*M}$, independent of choice of σ.*

Corollary 8. *The trivial representation of K occurs in $\pi_\sigma|_K$, and there it has multiplicity* 1.

Now we look at the rank 1 case. The starting point is

Proposition 9. *Let G_n be a finite dimensional connected simple real Lie group. Then (K_n, M_n) is a commutative pair (Gelfand pair) if and only if G_n has real rank* 1*. All cases are given by*

	G_n	K_n	M_n	K_n/M_n
1	$SO(1,n)$	$SO(n)$	$SO(n-1)$	S^{n-1}
2	$SU(1,n)$	$S(U(1)\times U(n))$	$U(1)\times U(n-1)$	S^{2n-1}
3	$Sp(1,n)$	$Sp(1)\times Sp(n)$	$Sp(1)\times Sp(n-1)$	S^{4n-1}
4	F_{4,B_4}	$Spin(9)$	$Spin(7)$	S^{15}

These are the cases where each finite level multiplicity $\dim E_{\kappa_n}^{M_n} \leqq 1$. The resulting direct limit groups are the $S0(1,\infty) = \varinjlim SO(1,n)$, the $SU(1,\infty) = \varinjlim SU(1,n)$, and the $Sp(1,\infty) = \varinjlim Sp(1,n)$. A careful analysis of the restrictions $\kappa_{n+1}|_{K_n}$ and an explicit determination of their highest weight vectors leads to the corresponding multiplicity $\leqq 1$ result in the limit:

Theorem 10. *Let G be one of $SO(1,\infty)$, $SU(1,\infty)$ or $Sp(1,\infty)$. (So K is $SO(\infty)$, $U(\infty)$ or $Sp(1)\times Sp(\infty)$, respectively.) Let π be a direct limit spherical principal series representation of G. Then π is K–multiplicity free.*

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On polyzeta values

Olivier Mathieu

Reporter's comment: *This abstract is a typed version of Mathieu's handwritten entry in the abstract book.*

Bailey, Borwein and Plouffe proved a quickly convergent formula for π:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right).$$

We show a similar formula for polyzeta values, e.g.

$$\zeta(3) = \sum_{0<n\leq m\leq \ell} \frac{1}{nm\ell}\,2^{-\ell} + 2\sum_{0<n<m\leq \ell} \frac{1}{nm\ell}\,2^{-\ell}.$$

The combinatorics used in the formula is described in terms of an infinite-dimensional symmetric space.

On the generic Verma module at the critical level

Maria Gorelik

Let $\hat{\mathfrak{g}}$ be an affine Lie algebra. The characters of irreducible modules of a generic highest weight at the critical level over $\hat{\mathfrak{g}}$ are given by Kac-Kazhdan formula. This was proven by different methods in different cases in [Wk], [Wl], [H], [GW]; for the non-twisted affine Lie algebras by B. Feigin and E. Frenkel ([FF], [F]) and recently by T. Arakawa [Ar]; in finite characteristic by O. Mathieu [M]. For an arbitrary affine Lie algebra (including the twisted case) the formula was proven by J.-M. Ku [Ku] and recently reproven by M. Szczesny [Sz]. The approach of B. Feigin, E. Frenkel and M. Szczesny is based on the explicit realization of the irreducible module $L(\lambda)$: they show that if λ is a generic highest weight at the critical level then $L(\lambda)$ is isomorphic to a Wakimoto module, which is a representation of $\hat{\mathfrak{g}}$ in a Fock module over some infinite-dimensional Heisenberg algebra. The Heisenberg algebra here corresponds to the set of *real* roots of $\hat{\mathfrak{g}}$.

Another approach to Kac-Kazhdan formula appeared in a paper of J.-M. Ku; this approach is based on a study of singular vectors in the Verma module $M(\lambda)$. It can be interpreted in terms of an infinite-dimensional Heisenberg algebra which corresponds to the set of *imaginary* roots of $\hat{\mathfrak{g}}$ (this Heisenberg algebra is a subalgebra of $\hat{\mathfrak{g}}$ spanned by imaginary root spaces), see §3 below.

1. Affine Lie algebras are contragredient Lie algebras of a finite growth. The Cartan algebras of affine Lie algebras are symmetrizable. An affine Lie algebra can be described in terms of a finite-dimensional contragredient Lie algebra and its finite order automorphism.

The superalgebra generalization of contragredient algebras was introduced in [K]. Call a contragredient superalgebra *affine* if it has a finite growth and *symmetrizable* if it has a symmetrizable Cartan matrix. In [vdL],[S] the affine symmetrizable Lie superalgebras are described in terms of finite-dimensional contragredient superalgebras and their finite order automorphisms.

A symmetrizable affine Lie superalgebra has a Casimir element. As a consequence, a Verma module $M(\lambda)$ is irreducible unless λ belongs to the union of countably many hyperplanes. Among these hyperplanes one is rather special: $M(\lambda)$ has an infinite length if λ lies on this hyperplane. This hyperplane is the set of critical weights and its equation is $K(\lambda) = -h^\vee$, where K is the canonical central element of $\hat{\mathfrak{g}}$ and $h^\vee$ is the dual Coxeter number.

Affine Lie superalgebras were classified in a recent paper [HS]. It turns out that non-symmetrizable affine Lie superalgebras consist of 4 series. One of this series is $q(n)^{(2)}$; these algebras are twisted affinizations of "strange" Lie superalgebras $q(n)$.

In my recent preprint [G] I studied a Verma module with a generic highest weight at the critical level over a symmetrizable affine Lie superalgebra $\hat{\mathfrak{g}} \neq A(2k,2l)^{(4)}$. In particular, I proved Kac-Kazhdan conjecture describing the character of a generic simple module on the critical level for these superalgebras. My approach in [G] is close to one of J.-M. Ku.

2. Let $\hat{\mathfrak{g}}$ be a symmetrizable affine Lie superalgebra. Let $\mathcal{H}$ (resp., $\mathcal{H}_-$) be the sum of positive (resp., negative) imaginary root spaces. Then $\mathfrak{l} := \mathcal{H}_- \oplus \mathbb{C}K \oplus \mathbb{C} \oplus \mathcal{H}$ is a Lie subalgebra of $\hat{\mathfrak{g}}$; let V^k be a Verma $\mathfrak{l}$-module of the central charge $k \in \mathbb{C}$ (i.e. K acts by kid). Identify V^k with $\mathcal{S} = \mathcal{U}(\mathcal{H}_-)$ and say that $v \in V^k$ is singular if $\mathcal{H}v = 0$ and v is a D-eigenvector.

Let $\hat{\mathfrak{g}} \neq A(2k,2l)^{(4)}$. Then $\mathcal{H}_-, \mathcal{H}$ are commutative and $\mathcal{H}_- \oplus \mathbb{C}K \oplus \mathcal{H}$ is a countably dimensional Heisenberg algebra. The structure of V^k is well-known: it irreducible if and only if $k \neq 0$; in V^0 any D-eigenvector is singular and the Jantzen filtration of V^0 identifies with the adic filtration of $\mathcal{S}$ (here $\mathcal{S} = \mathcal{U}(\mathcal{H}_-)$ is a polynomial algebra of countably many variables).

3. Let $\hat{\mathfrak{g}}$ be a symmetrizable affine Lie superalgebra and $\hat{\mathfrak{g}} \neq A(2k,2l)^{(4)}$. The structure of a generic Verma module at the critical level is described as follows:

$M(\lambda)$ with a generic highest weight at the critical level looks like V^0. More precisely, there exists a linear map $HC_+ : M(\lambda) \to \mathcal{S}$ which maps singular vectors to the singular ones and induces a bijection between the submodules of $M(\lambda)$ and of V^0. This bijection is compatible with the Jantzen filtrations.

Kac-Kazhdan character formula immediately follows from the above description of $M(\lambda)$.

4. I believe that a similar result holds for $\hat{\mathfrak{g}} = A(2k,2l)^{(4)}$. A difficulty in this case is that $\mathcal{H}_-$ is not commutative and so $\mathcal{H}_- \oplus \mathbb{C}K \oplus \mathcal{H}$ is not a Heisenberg algebra. In particular, it is not true that any submodule of V^0 (and of $M(\lambda)$) is generated by singular vectors.

In my joint work with V. Serganova [GS] a result similar to §3 is proven for the non-symmetrizable affine Lie superalgebra $q(n)^{(2)}$ for an arbitrary level. In this case $\mathcal{H}_- \oplus \mathbb{C}K \oplus \mathcal{H}$ is not a Heisenberg algebra. The Verma module V^k over $\mathcal{H}_- \oplus \mathbb{C}K \oplus \mathcal{H}$ is reducible for all $k \in \mathbb{C}$ and $M(\lambda)$ is reducible for all $\lambda \in \hat{\mathfrak{h}}^*$.

A very interesting question is whether there exists a Kac-Moody Lie algebra such that all its Verma modules are reducible.

5. *The projection* HC_+. The projection HC_+ can be described as follows. The Lie superalgebra $\hat{\mathfrak{g}}$ has a standard triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}$ and $\hat{\mathfrak{n}}_-$ admits a triangular decomposition $\hat{\mathfrak{n}}_- = \mathcal{N}_-^- \oplus \mathcal{H}_- \oplus \mathcal{N}_-^+$, where $\mathcal{H}_-$ consists of the elements of imaginary weights (for a non-twisted case, $\mathcal{H}_- = \mathcal{L}\mathfrak{h} \cap \hat{\mathfrak{n}}_-$ and $\mathcal{N}_-^{\pm} = \mathcal{L}\mathfrak{n}_{\pm} \cap \hat{\mathfrak{n}}_-$, where $\mathcal{L}$ stands for the loop space of a given subalgebra of $\mathfrak{g}$). Recall that $\mathcal{S} := \mathcal{U}(\mathcal{H}_-)$ and introduce the projections $\mathrm{HC}_{\pm} : \mathcal{U}(\hat{\mathfrak{n}}_-) \to \mathcal{S}$, where

$$\operatorname{Ker}\mathrm{HC}_+ = \mathcal{U}(\hat{\mathfrak{n}}_-)\mathcal{N}_-^- + \mathcal{N}_-^+\mathcal{U}(\hat{\mathfrak{n}}_-), \quad \operatorname{Ker}\mathrm{HC}_- = \mathcal{U}(\hat{\mathfrak{n}}_-)\mathcal{N}_-^+ + \mathcal{N}_-^-\mathcal{U}(\hat{\mathfrak{n}}_-).$$

For each $\lambda \in \hat{\mathfrak{h}}^*$ define $\mathrm{HC}_\pm : M(\lambda) \to \mathcal{S}$ via the natural identification of $M(\lambda)$ with $\mathcal{U}(\hat{\mathfrak{n}}_-)$. The projection HC_+ appeared in [Ku] and [Ch]. The statement similar to one in §3 holds for HC_- as well.

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Simplicity of Kac-Moody lattices

Pierre-Emmanuel Caprace
(joint work with Bertrand Rémy)

We highlight an infinite family of pairwise non-isomorphic finitely generated infinite simple groups, many of which are finitely presented and enjoy Kazhdan's property (T). The proof of simplicity is obtained in an axiomatic setting which applies in particular to all split or almost Kac-Moody groups over finite fields of order larger than the rank. Details and related results are to be found in the reference [CR06].

1. The axiomatic setting: root data

Let (W, S) be a Coxeter system and $\Phi = \Phi(W, S)$ be its abstract root system. A set of roots Ψ is called **prenilpotent** if there exist $w, w' \in W$ such that $w.\Psi \subset \Phi_+$ and $w'.\Psi \subset \Phi_-$. Given a prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, we denote by $[\alpha, \beta]$ the set consisting of all those $\gamma \in \Phi$ such that $w.\gamma > 0$ for all $w \in W$ such that $\{w.\alpha, w.\beta\} \subset \Phi_+$ and $w'.\gamma < 0$ for all $w' \in W$ such that $\{w'.\alpha, w'.\beta\} \subset \Phi_-$.

Let now Λ be a group endowed with a family $(U_\alpha)_{\alpha\in\Phi}$ of subgroups indexed by Φ. We assume that the axioms of a **root datum** are satisfied. These axioms read as follows:

(RD0): For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.
(RD1): For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $\langle U_\gamma | \ \gamma \in [\alpha, \beta] \backslash \{\alpha, \beta\}\rangle$.
(RD2): For each $\alpha \in \Phi$ and each $u \in U_\alpha \backslash \{1\}$, there exists $\mu_u \in U_{-\alpha} u U_{-\alpha}$ such that $\mu_u U_\beta \mu_u^{-1} = U_{r_\alpha(\beta)}$ for each $\beta \in \Phi$.
(RD3): For each $\alpha \in \Phi_+$, the group $U_{-\alpha}$ is not contained in $U_+ = \langle U_\beta \mid \beta \in \Phi_+\rangle$.
(RD4): $G = \langle U_\alpha | \ \alpha \in \Phi\rangle$.

When these axioms hold, the U_α's are called **root subgroups** of Λ.

Amongst groups endowed with a root datum are the following examples:

- Let k be a field and $\mathbf{G}$ be a semisimple algebraic k-isotropic k-group. Any maximal k-split torus $\mathbf{T}$ of $\mathbf{G}$ yields a root system indexing a family of root subgroups for the group $\mathbf{G}^+(k)$, which satisfy the axioms above.
- (Minimal) split or almost split Kac-Moody groups over arbitrary fields are all endowed with a family of root subgroups satisfying the axioms, see [Rém02].
- There are also several known ways of producing examples of root data which are not of algebraic origin in the sense that there is no underlying Kac-Moody algebra. B. Mühlherr [Müh99] is able to integrate certain local data, called *Moufang foundations*, in order to produce root data in which Ree groups ${}^2F_4$ arise as rank 2 subgroups. When the Coxeter system (W, S) is right-angled (i.e. for all $s, t \in S$, either s and t commute or $\langle s, t\rangle$

is infinite), there is a lot of freedom to produce even more exotic examples, see [RR06]. For instance, rank one subgroups need not be of Lie type, but could be any sharply 2-transitive group such as the affine group $k \rtimes k^\times$ where k is an arbitrary field.

2. Buildings and topological groups

Let Λ and $(U_\alpha)_{\alpha\in\Phi}$ be as above. We set $T = \bigcap_{\alpha\in\Phi} N_\Lambda(U_\alpha)$, $B_+ = TU_+$, $B_- = T\langle U_\alpha \mid \alpha \in \Phi_-\rangle$ and $N = T\langle \mu_u \mid \mu_u \in U_{-\alpha}uU_{-\alpha},\ \alpha \in \Phi,\ u \in U_\alpha\backslash\{1\}\rangle$ where μ_u is as in (RD2). Then the pairs $\{B_+, N\}$ and $\{B_-, N\}$ are both BN-pairs for Λ. We denote the corresponding buildings respectively by X_+ and X_-, and the natural actions of Λ by $\varphi_\pm : \Lambda \to \mathrm{Aut}(X_\pm)$. The groups $\mathrm{Aut}(X_\pm)$ are canonically endowed with a structure of topological groups given by the bounded-open topology. This topology is discrete whenever W is finite, in which case the buildings $X_\pm$ are bounded. In general it is nontrivial and totally disconnected. We set $G_\pm = \overline{\varphi_\pm(\Lambda)}$.

Tits' proof of abstract simplicity for simple algebraic groups works in the context of BN-pairs. When adapting the arguments in the present context, one obtains the following:

Proposition 1. *Assume that (W, S) is irreducible, that U_α is solvable for each $\alpha \in \Phi$ and that $[G_\pm, G_\pm]$ is dense in $G_\pm$. Then $G_\pm$ is topologically simple.*

The condition that $[G_\pm, G_\pm]$ is dense in $G_\pm$ is known to hold for all examples of twin root data described in the published literature (except for the smallest finite Chevalley groups, which are solvable).

3. Lattices

Suppose now that the root subgroups are all finite. Then the buildings $X_\pm$ are locally compact, the canonical topology on $\mathrm{Aut}(X_\pm)$ is the compact-open topology, which makes these groups locally compact and second countable. In this case $\mathrm{Aut}(X_\pm)$ admit a Haar measure. This allows one to consider *lattices*, which are by definition discrete subgroups of finite covolume. The following existence result is due to B. Rémy:

Proposition 2. *Assume that $|S| < U_\alpha < \infty$ for all $\alpha \in \Phi$. Then $\varphi_+ \times \varphi_-(\Lambda)$ is an irreducible lattice in $G_+ \times G_-$.*

4. Simplicity

Lattices in semisimple Lie groups have been extensively studied; numerous breakthroughs in this area are due to G. Margulis. It is remarkable that many properties of semisimple Lie groups of higher rank (with no compact factors) are reflected in some way in properties of their irreducible lattices. For instance these enjoy the normal subgroup property: any normal subgroup is either finite central or of finite index. This very strong statement, which reflects the semisimplicity of the ambient group, is optimal in the sense that lattices do possess many finite index normal subgroups: in fact, they are residually finite, i.e. the intersection of all finite index (normal) subgroups is trivial. Our main result shows that, however, in the context of lattices coming from root data, residual finiteness is an exception:

Theorem 3. *Assume that (W, S) is irreducible and that $[G_\pm, G_\pm]$ is dense in $G_\pm$. Suppose also that, for all $\alpha \in \Phi$, the group U_α is nilpotent and $|S| < U_\alpha < \infty$. If W is not virtually abelian, then Λ is virtually simple modulo its center. Moreover the finitely many quotients of Λ are all nilpotent.*

Note that Λ is finitely generated; moreover $\Lambda/Z(\Lambda)$ is infinite whenever W is infinite. The condition that W is not virtually abelian amounts to requiring that the Coxeter system (W, S) is not of affine type. Note that affine Kac-Moody groups over finite fields are (S-)arithmetic groups (e.g. $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$). In particular they are residually finite.

Simplicity is obtained in two steps: one first shows that the normal subgroup property holds (this part is due to U. Bader and Y. Shalom [BS06] and B. Rémy [Rém05]) and confronts this to strong restrictions on finite quotients [CR06]. A similar scheme of proof was previously developed by M . Burger and Sh. Mozes [BM01] to construct simple groups.

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On the classification and realization of generalized Kac-Moody algebras

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Kac-Moody algebras fall into three classes, those of finite, affine and indefinite type. While the first two classes are well understood, the knowledge of the indefinite Kac-Moody algebras is quite restricted. Affine Kac-Moody algebras have found various applications, for example in conformal field theory, the theory of Jacobi forms and recently in the geometric Langlands correspondence. The relevance of the affine Kac-Moody algebras is due to the fact that they can be classified and admit simple realizations as central extensions of loop algebras. Generalized Kac-Moody algebras are defined essentially in the same way as Kac-Moody algebras, i.e. by generators and relations. The only difference is that they are allowed to have imaginary simple roots. As for Kac-Moody algebras the theory of generalized Kac-Moody algebras is still similar to the finite dimensional case. In particular there is a character formula for irreducible highest weight modules and a denominator identity.

In this talk we described a class of generalized Kac-Moody algebras which can be classified and admit realizations as bosonic strings.

Let L be an even lattice of signature $(n, 2)$ with $n > 2$ and V the underlying real vector space. Then the set D of two dimensional negative definite subspaces of V is a hermitean symmetric space acted on by $O^+(V)$. The space D has a realization as tube domain H in $\mathbb{C}^n$ and the action of $O^+(V)$ on D induces an action on H by fractional linear transformations. A meromorphic function f on H is called automorphic form of weight k for a discrete subgroup Γ of finite index in $O^+(L)$ if $f(MZ) = j(M, Z)^k f(Z)$ for all M in Γ. Borcherds' singular theta correspondence [B2] is a map from vector valued modular forms on $SL_2(\mathbb{Z})$ with poles at cusps to automorphic forms on orthogonal groups with divisors $D_\lambda = \{Z \in D \,|\, Z \perp \lambda\}$ where λ is a primitive vector of positive norm in L. These automorphic forms have nice product expansions and therefore are called automorphic products. An automorphic product is reflective if its divisors are orthogonal to roots of L and are zeros of order one. A reflective automorphic product is called completely reflective if all roots of L give zeros of order one.

Recall that the Mathieu group M_{23} acts on the Leech lattice Λ. Let g be an element of order N in M_{23}. Then g has characteristic polynomial $\prod_{k|N}(x^k - 1)^{b_k}$ as automorphism of Λ. Define $\eta_g(\tau) = \prod_{k|N} \eta(k\tau)^{b_k}$.

Theorem 1. *Let g be an element of order N in M_{23}. Then g corresponds naturally to a reflective automorphic product Ψ of singular weight on the lattice $\Lambda^g \oplus II_{1,1} \oplus II_{1,1}(N)$. Let $m||N$. Then the Fourier expansion of Ψ at the level m cusp is given by*

$$\sum_{w\in W} \det(w)\, \eta_g((w\rho, Z))$$

where W is a reflection group of $L = \Lambda^g \oplus II_{1,1}$ and ρ is a primitive norm zero vector in $II_{1,1}$. This is the denominator function of a generalized Kac-Moody algebra G. Suppose N is squarefree. Then the expansion of Ψ at any cusp is

$$e((\rho, Z)) \prod_{k|N} \prod_{\alpha\in(L\cap kL')^+} \big(1 - e((\alpha, Z))\big)^{[1/\eta_g](-\alpha^2/2k)} = \sum_{w\in W} \det(w)\, \eta_g((w\rho, Z))$$

where W is the full reflection group of L. This is the denominator identity of G.

Theorem 2. *Let L be an even lattice of signature $(n, 2)$ with $n > 2$ and squarefree level N. Suppose L splits $II_{1,1} \oplus II_{1,1}(N)$. Let G be a real generalized Kac-Moody algebra whose denominator identity is a completely reflective automorphic product of singular weight on L. Then G can be constructed from an element of order N*

in M_{23}.

Theorem 3. *The cohomology group of ghost number one of the BRST-operator* Q *acting on the vertex superalgebra* $V_\Lambda \otimes V_{II_{1,1}} \otimes V_{\mathbb{Z}}$ *gives a natural realization of the generalized Kac-Moody algebra corresponding to the identity in* M_{23}.

Theorem 4. *Suppose the vertex operator algebra* V *of central charge* 24*, trivial fusion algebra and spin-1 current* $\hat{A}^r_{q,p}$ *where* $p = 2, 3, 5$ *or* 7*,* $q = p - 1$ *and* $r = 48/q(p + 1)$ *exists and has a real form. Then the cohomology group of ghost number one of the BRST-operator* Q *acting on* $V \otimes V_{II_{1,1}} \otimes V_{\mathbb{Z}}$ *gives a natural realization of the generalized Kac-Moody algebra corresponding to the elements of order* p *in* M_{23}*. For* $p = 2$ *the existence of* V *and a real form is proven.*

The theorems show that there is nice class of generalized Kac-Moody algebras which can be classified and have simple realizations. This is similar to the affine case in Kac-Moody theory.

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Wakimoto type modules for affine Lie algebras

Vyacheslav Futorny

(joint work with Ben Cox)

Modules induced from the *natural Borel subalgebra* were first introduced by H. Jakobsen and V. Kac in their study of unitarizable highest weight representations of affine Kac-Moody algebras ([JK]). They were studied in [F] under the name of *imaginary Verma modules.* A Fock space realization of the imaginary Verma modules for $\hat{\mathfrak{sl}}(2)$ were constructed by D. Bernard and G. Felder in [BF] and then extended in [C2] to the case of $\hat{\mathfrak{sl}}(n)$. These realizations are given generically by certain Wakimoto type modules. In his effort to prove the Kac-Kazhdan conjecture on the characters of irreducible representations of affine Kac-Moody algebras at the critical level, M. Wakimoto discovered a remarkable boson realization of $\hat{\mathfrak{sl}}(2)$ on the Fock space $\mathbb{C}[x_i, y_j \,|\, i \in \mathbb{Z}, j \in \mathbb{N}]$ ([W]). Wakimoto modules for general affine Lie algebras were introduced by B. Feigin and E. Frenkel in [FF1] by a homological characterization, [FF2] which play an important role in the conformal field theory providing a new bosonization rule for the Wess-Zumino-Witten models. Wakimoto modules have a geometric interpretation as certain sheaves on a semi-infinite flag manifold [FF3]. They belong to the category $\mathcal{O}$ and generically are isomorphic to corresponding Verma modules. Explicit formulae for these realizations for $\hat{\mathfrak{sl}}(2)$ are given in [FF1] and for general affine Lie algebras they are given in [BoF] and [PRY]. Affine Lie algebras admit Verma type modules associated with non-standard Borel subalgebras ([JK]). All Borel subalgebras form a finite number of conjugacy orbits, these orbits are parametrized by parabolic subalgebras of the underlined simple Lie algebra. In [CF] suitable boson type realizations for all Verma type modules over $\hat{\mathfrak{sl}}(n+1)$ were found. These realizations, *intermediate Wakimoto modules*, depend on the parameter $0 \leq r \leq n$. If $r = n$ this construction coincides with the boson realization of Wakimoto modules in [FF1], and when $r = 0$ this is a realization described in [C2]. Intermediate Wakimoto modules form a family of representations with certain weight spaces being infinite dimensional. Using the realization of the intermediate Wakimoto module for $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$ given in [CF] we show that generically Verma type modules and intermediate Wakimoto modules are isomorphic, which is an analog of the classical relation between Verma and Wakimoto modules. Moreover, when intermediate Wakimoto modules are in general position (but not necessarily isomorphic to Verma type module) we completely describe their submodule structure. Namely, in this case intermediate Wakimoto module is generated by the subspace consisting of the finite-dimensional weight spaces. Moreover this subspace is isomorphic to a classical Wakimoto module W_r for an affine Lie algebra $\hat{\mathfrak{sl}}(r+1, \mathbb{C})$. In addition, any submodule (and hence any irreducible subquotient) of the intermediate Wakimoto module is generated by its intersection with the Wakimoto module W_r (respectively by a subquotient of W_r). Finally, any generic intermediate Wakimoto module is isomorphic to a generalized Verma module induced from W_r by parabolic induction.

Wakimoto modules can be obtained from the classical Verma modules by an infinite number of twistings. The same twisting can be applied to a Verma type module using the reflections corresponding to the roots of $\hat{\mathfrak{sl}}(r+1, \mathbb{C})$. We will say that in this case the module is obtained by *real twisting*. Clearly, imaginary Verma modules do not admit any real twisting, while on the other hand any intermediate Wakimoto module is obtained from the corresponding Verma type module by an an infinite number of real twistings. Hence, all boson type realizations associated with the natural Borel subalgebra correspond to infinite (or empty in the imaginary case) real twistings of corresponding Verma type modules. Note that we do not get realizations of Verma type modules this way. In order to construct boson type realizations for these modules one needs to start with the Borel subalgebra different from the standard one or the natural one and consider a corresponding "flag variety". Its "cells" will produce boson type realizations for Verma type modules, their contragradient analogs and finite real twistings.

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Solving non-commutative differential equations in vertex algebras

Yuly Billig

(joint work with Alexander Molev, Ruibin Zhang)

In this talk we will show how one can use the technique of vertex algebras in order to study a class of irreducible modules for the Lie algebra $\mathcal{D}$ of vector fields on a 2-dimensional torus.

The Lie algebra $\mathcal{D}$ is spanned by the elements $t_0^j t_1^k d_0$ and $t_0^j t_1^k d_1$ with $j, k \in \mathbb{Z}$, where d_i is the degree derivation $t_i \frac{\partial}{\partial t_i}$, $i = 0, 1$.

We view $\mathcal{D}$ as a $\mathbb{Z}$-graded Lie algebra by degree in t_0. For the zero component of this grading, we consider a family of modules, $T(\alpha, \beta, \gamma)$ with basis

$$\{v(m) | m \in \gamma + \mathbb{Z}\},$$

and the action of $\mathcal{D}_0$ defined in the following way:

$$(t_1^k d_1) v(m) = (m - \alpha k) v(m + k),$$

$$(t_1^k d_0) v(m) = \beta v(m + k).$$

We construct the generalized Verma module $M(\alpha,\beta,\gamma)$ over $\mathcal{D}$ in the standard way, by making $\mathcal{D}_+$ act on T trivially, and letting $M(\alpha,\beta,\gamma)$ be the induced module $U(\mathcal{D}_-)\otimes T$. The generalized Verma module M has a unique irreducible quotient $L = L(\alpha,\beta,\gamma)$.

The module L inherits a $\mathbb{Z}^2$-grading from the Lie algebra $\mathcal{D}$. It is known from [1] that the graded components of L are finite-dimensional, however nothing else was previously known about the structure of these modules.

The Lie algebra $\mathcal{D}$ contains a loop subalgebra $sl_2(\mathbb{C}[t_0,t_0^{-1}])$, spanned by the elements

$$\{t_0^j t_1^k d_1 | j\in\mathbb{Z}, k=-1,0,1\}.$$

It turns out that this subalgebra plays an important role here. The picture is analogous to the case of the basic module for affine Lie algebras, where the reduction from the affine algebra to the principal Heisenberg subalgebra leaves the basic module irreducible.

For the Lie algebra of vector fields on a 2-dimensional torus we prove the following result:

Theorem. Let $\alpha\notin\mathbb{Q}$, and let $\beta=\frac{\alpha(\alpha+1)}{2}$.
(a) Viewed as a module over the loop subalgebra, $L(\alpha,\beta,\gamma)$ remains irreducible.
(b) Moreover, $L(\alpha,\beta,\gamma)\cong U(sl_2(t_0^{-1}\mathbb{C}[t_0^{-1}])\otimes T$, which yields the character of L.

The proof of this theorem uses the technique of vertex algebras. First, we note that $\mathcal{D}$ is a vertex Lie algebra, which allows us to construct the corresponding vertex algebra $V_{\mathcal{D}}$ in the usual way. Next, we define a certain quotient vertex algebra, $\overline{V}_{\mathcal{D}}$. We can calculate the kernel of the projection $V_{\mathcal{D}}\to\overline{V}_{\mathcal{D}}$.

Each element a of this kernel yields a relation $Y(a,z)=0$ in $\overline{V}_{\mathcal{D}}$. We can show that the same relations will also hold in L. In this way we can obtain a number of interesting and non-trivial relations holding in L. Some of these relations are actually operator-valued differential equations. Solving these equations is not easy, since they are highly non-commutative, nonetheless it can be done. The structure of the solutions shows that the action of the Lie algebra $\mathcal{D}$ can be completely described in terms of the action of the loop subalgebra, which proves the Theorem.

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