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Mini-Workshop: Dirac Operators in Differential and Noncommutative Geometry

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Abstract.

This mini-workshop brought together mathematicians and physicists working either on classical or on noncommutative differential geometry. Our aim was to show current interests, methods and results within each group and to open the possibility for interaction between the two groups. The first three days were devoted to expository presentations. The remaining two days were devoted to talks on advanced current research problems and results.

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Introduction by the Organisers

The mini-workshop, organized by Christian Bär and Andrzej Sitarz, had a very special character. The participating scientists came from two different mathematical communities: differential geometry (working mainly on problems related to the Dirac operator on spin manifolds) and noncommutative geometry (working mainly on concepts of Dirac operators in the framework of spectral geometry as postulated by Alain Connes).

Spin geometry has become an established and very active subfield of Differential Geometry, after Lichnerowicz observed that the Index Theorem yields a topological obstruction against the existence of metrics with positive scalar curvature. The Dirac operator plays a key role in the deep work of Gromov, Lawson, Rosenberg, Stolz and others on manifolds admitting metrics with positive scalar curvature.

The birth of noncommutative geometry offered completely new possibilities for extending some notions of differential geometry into the realm of operator algebras. In Connes' notion of spectral triples the Dirac operator was used to define a (possibly noncommutative) geometry itself rather than being an object derived from a geometry. Since then many interesting examples of noncommutative spaces

and Dirac operators were studied. The equivalence theorem, allowing reconstruction of a spin manifold from a spectral geometry of a commutative algebra was proved only recently and the proof was presented at the workshop.

The aim of the workshop was twofold: to show current interests, methods and results within each group and open the possibility for interaction between two groups. Due to the character of the meeting, first three days were devoted to the expository presentations, when we tried to cover the possibly broadest scope of topics from one subject presented for the participants from the other group. The remaining two days were devoted to talks on advanced current research problems and results, which had closer links to the topics of both groups. During problem sessions in the evenings various open questions were discussed some of which were solved during the week.

Mini-Workshop: Dirac Operators in Differential and Noncommutative Geometry

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Part I: A mini-course in spin geometry

Introduction to spin geometry

Christian Bär

We give a brief introduction to the basic concepts of spin geometry. Detailed expositions can found in [1, 2, 3].

For $n \geq 3$ the spin group $\mathrm{Spin}(n)$ is defined as the universal covering group of $\mathrm{SO}(n)$. Furthermore, $\mathrm{Spin}(1) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Spin}(2) = \mathrm{SO}(2)$. In any case we get a central extension

$$\{1\} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \{1\}. \tag{1}$$

Let $P \to M$ be an $\mathrm{SO}(n)$-principal bundle. The exact sequence on Čech cohomology induced by (1) yields the obstruction class $w_2(P) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$, known as *second Stiefel-Whitney class*, against the existence of lifts of P to a $\mathrm{Spin}(n)$-bundle $Q \to P$. Such a lift is called a *spin structure* for P. There is a natural notion of equivalence of spin structures. This notion is finer than just equivalence as $\mathrm{Spin}(n)$-bundles since it also has to take the covering $Q \to P$ into account. In case a spin structure exists, i. e. $w_2(P) = 0$, then $H^1(M, \mathbb{Z}/2\mathbb{Z})$ acts simply transitively on the set of all equivalenc classes of spin structures of P.

For example, if $M = T^2$ and $P = T^2 \times \mathrm{SO}(2)$ is the trivial $\mathrm{SO}(2)$-bundle, then since $H^1(T^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there are exactly 4 inequivalent spin structures. They are all trivial as $\mathrm{Spin}(2)$-bundles however.

Similarly, one defines the group $\mathrm{Spin}^c(n)$ as a central extension

$$\{1\} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1) \to \{1\}.$$

Given an $\mathrm{SO}(n)$-bundle $P \to M$ and a $\mathrm{U}(1)$-bundle $L \to M$ existence of a Spin^c-*structure* for (P, L) is equivalent to $w_2(P) = c_1(L) \bmod 2$ where $c_1(L)$ is the (first) Chern class of L. The bundle L is called the *determinant line bundle* of the Spin^c-structure. Hence an $\mathrm{SO}(n)$-bundle admits a Spin^c-structure if and only if $w_2(P)$ is the reduction modulo 2 of an integral cohomology class $c \in H^2(M, \mathbb{Z})$.

The *Clifford algebra* $\mathrm{Cl}(n)$ is the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty} \bigotimes^k \mathbb{R}^n$ by the 2-sided ideal generated by the elements of the form $v \otimes w + w \otimes v + 2\langle v, w\rangle \cdot 1$ where $v, w \in \mathbb{R}^n$. Its complexification is denoted by $\mathbb{C}\mathrm{l}(n) := \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$. The groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}^c(n)$ can be realized concretely in the Clifford algebras,

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2r} \in \mathrm{Cl}(n) \,|\, v_j \in \mathbb{R}^n, \|v_j\| = 1, r \in \mathbb{N}\}$$

and

$$\mathrm{Spin}^c(n) = \{z \cdot v_1 \cdots v_{2r} \in \mathbb{C}\mathrm{l}(n) \,|\, z \in \mathbb{C}, |z| = 1, v_j \in \mathbb{R}^n, \|v_j\| = 1, r \in \mathbb{N}\}.$$

Multiplication in $\mathbb{C}\mathrm{l}(n)$ yields an action of $\mathrm{Spin}(n)$ and $\mathrm{Spin}^c(n)$ on $\mathbb{C}\mathrm{l}(n)$. We decompose $\mathbb{C}\mathrm{l}(n)$ into irreducibles as $\mathrm{Spin}(n)$-modules (or as $\mathrm{Spin}^c(n)$-modules what amounts to the same thing) and we obtain

$$\mathbb{C}\mathrm{l}(n) = \underbrace{\Sigma_n^+ \oplus \Sigma_n^- \oplus \ldots \oplus \Sigma_n^+ \oplus \Sigma_n^-}_{2^{n/2} \text{ copies each}}$$

for n even and

$$\mathbb{C}\mathrm{l}(n) = \underbrace{\Sigma_n \oplus \ldots \oplus \Sigma_n}_{2^{(n-1)/2} \text{ copies}}$$

for odd n. In the even dimensional case we put $\Sigma_n := \Sigma_n^+ \oplus \Sigma_n^-$ and the elements of Σ_n are called *spinors* in either case.

If we fix a spin or Spin^c-structure Q for the frame bundle $P_{\mathrm{SO}}(M)$ of an oriented Riemannian manifold the spinor module allows us to define the *spinor bundle*

$$\Sigma M := Q \times_{\mathrm{Spin}^{(c)}(n)} \Sigma_n.$$

The spinor bundle is a hermitian vector bundle of rank $2^{[n/2]}$. In the spin case the Levi-Civita connection induces a connection form on Q and hence a connection on ΣM. In the Spin^c-case one has to choose a connection on L and one then obtains a connection on ΣM. The fact that the spinor modules are actually restrictions of modules for the Clifford algebra allows us to define *Clifford multiplication* $TM \otimes \Sigma M \to \Sigma M$, $X \otimes \phi \mapsto X \cdot \phi$, satisfying the Clifford relations

$$X \cdot Y \cdot \phi + Y \cdot X \cdot \phi + 2 \langle X, Y \rangle \cdot \phi = 0.$$

Clifford multiplication is parallel, $\nabla_Y (X \cdot \phi) = X \cdot \nabla_Y \phi + (\nabla_Y X) \cdot \phi$, and skew symmetric, $\langle X \cdot \phi, \psi \rangle = - \langle \phi, X \cdot \psi \rangle$.

We define the *Dirac operator*

$$D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M), \quad D\phi := \sum_{j=1}^n e_j \cdot \nabla_{e_j} \phi,$$

where $e_1, \ldots, e_n$ is a local orthonormal basis of TM. The Dirac operator is a linear differential operator of first order with principal symbol given by Clifford multiplication. Hence the principal symbol is invertible for nonzero covectors, in other words, D is elliptic. From the properties of the spinor bundle one sees directly that in even dimensions the Dirac operator interchanges *chiralities*, i. e. with respect to the splitting $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ the operator takes the block form

$$D = \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}.$$

A simple computation shows that D is formally self-adjoint. This means that D is a symmetric operator in $L^2(M, \Sigma M)$, the Hilbert space of square-integrable spinors, when given the domain $C_c^\infty(M, \Sigma M)$, the space of smooth spinors with compact support.

The question arises whether D is also essentially self-adjoint. It is not hard to see that the answer is yes if the manifold M is complete, in particular, if it is

compact. In general, it is not essentially self-adjoint. For example, if $M = (0, 1)$ and $D = i\frac{d}{dt}$, then D has more than one self-adjoint extension while on $M = (0, \infty)$ it has no self-adjoint extensions at all.

The square of the Dirac operator is a Laplace type operator and the connection ∇ on ΣM also yields the Laplacian $\nabla^*\nabla$. In the Spin^c case they compare as follows:

$$D^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{Scal} + \frac{1}{2}F^L$$

where Scal denotes the scalar curvature of M and F^L Clifford multiplication by the curvature form of L. In the spin case the term $\frac{1}{2}F^L$ disappears. This formula is often called the *Lichnerowicz formula* [4] but was already proven by Schrödinger in 1932, see [5].

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The local index theorem for twisted Dirac operators

Alexander Strohmaier

The aim of the talk is to give a short outline of the proof of the local index theorem for twisted Dirac operators on manifolds. Details of the exposition can be found in [1, 2, 3].

Let $E \to X$ be a hermitian vector bundle with compatible connection ∇ over a complete manifold X of bounded geometry. Then one may define a Sobolev space of sections $H^s(X; E)$ in E as the domain of the selfadjoint operator $\Delta^{s/2} = (\nabla^*\nabla + 1)^{s/2}$. Rellichs lemma is the statement that any section of $C_0^\infty(X; E)$ defines a compact map from $H^s(X; E) \to H^{s'}(X; E)$ if $s > s'$. As an application of the Lichnerowicz formula it follows immediately that the Dirac operator on a complete spin manifold, which has scalar curvature bounded from below outside a compact set by a positive constant, has a finite dimensional kernel.

Suppose now that X is a compact Riemannian spin manifold and E is a hermitian vector bundle. Let D be the twisted Dirac operator. Then $D_+ : H^{s+1}(X; S^+ \otimes E) \to H^s(X; S^- \otimes E)$ is a Fredholm operator and its index may be

expressed by the McKean-Singer formula

$$\mathrm{ind}D_+ = \mathrm{Tr}(e^{-D_+^*D_+t} - e^{-D_+D_+^*t}) = \mathrm{Tr}_s(e^{-D^2t}).$$

Since the heat kernels $K_t = e^{-D_+^*D_+t}$ and $K_t' = e^{-D_+D_+^*t}$ are smoothing they have a smooth integral kernel and their trace may be expressed as the integral over the diagonal.

For a formally selfadjoint differential operator A of order m with scalar principal symbol $a(x,\xi) > 0$ for $\xi \neq 0$ the operator $A - \lambda$ may be understood as an elliptic parameter-dependent classical pseudodifferential operator with parameter in some angle $\Lambda := \{z \in \mathbb{C} \mid |\arg(z)| > \pi/13\}$. It follows from the calculus of pseudodifferential operators with parameter that there is a parametrix and thus the resolvent $(A-\lambda)^{-1}$ is again a pseudodifferential operator with parameter. In local coordinates the symbol $r(x,\xi,\lambda)$ of the resolvent kernel is therefore asymptotic to

$$r(x,\xi,\lambda) \sim \sum_{k=0}^{\infty} r_{-n-k}(x,\xi,\lambda),$$

where r_j is jointly homogeneous in ξ and λ in the sense that

$$r_j(x,\alpha\xi,\alpha^m\lambda) = \alpha^j r_j(x,\xi,\lambda),$$

for $\alpha > 1$ and $||\xi|| + \lambda^{1/m} > 1$. By the method of construction the r_m depend on a finite number of derivatives of the coefficients of A only.

From the spectral calculus it follows that we can write

$$e^{-At} = \frac{1}{2\pi i}\int_\gamma e^{-\lambda t}(A-\lambda)^{-1}d\lambda$$

if γ is a suitable curve around the spectrum of A. In particular we may choose γ to be invariant under dilations. Let $K_t(x,y)$ be the integral kernel of e^{-At}. Then we get from the above expansion of $r(x,\xi,\lambda)$:

$$K_t(x,x) = \int_{\mathbb{R}^n}\int_\gamma e^{-\lambda t} r(x,\xi,\lambda)d\xi d\lambda \sim$$

$$\sum_{k=0}\int_{\mathbb{R}^n}\int_\gamma e^{-\lambda t} r_{-m-k}(x,\xi,\lambda)d\xi d\lambda \sim \sum_{k=0} t^{\frac{-n+k}{m}}\int_\gamma\int_{\mathbb{R}^n} e^{-\lambda} r_{-m-k}(x,\xi,\lambda)d\xi d\lambda$$

$$\sim \sum_{k=0} t^{\frac{-n+k}{m}}\alpha_{n-k}(x)$$

where we used the homogeneity properties of r_k in the last step. Here the $\alpha_j \in C^\infty(X, \mathrm{End}(E))$ are determined by a finite number of derivatives in the coefficients of A. In particular the index of a twisted Dirac operator is an integral over a local object.

Getzlers ingenious trick allows now to obtain the local index formula

$$\mathrm{tr}_x(\Gamma e^{-D^2t}) = (\hat{A}(X)\cdot \mathrm{ch}(E))^{(n)}(x) + \mathrm{O}(t),$$

where the right hand side is given by Chern-Weil theory (and still depends on the curvature of the connections whereas the integral does not).

The central observation is that the supertrace $\mathrm{tr}_s(\cdot) = \mathrm{tr}(\Gamma\cdot)$ is non-vanishing only on the top order of the Clifford algebra. One now introduces a rescaling which makes the top order term the term of highest degree. The so called Getzler scaling associates to Clifford multiplication with a vector field and to covariant derivations order 1, whereas multiplication by scalars has order 0. In the scaling limit at a point $p \in X$ the Dirac operator becomes

$$D_p = -\sum_i \left(\partial_i + \frac{1}{4}\sum_j R_{ij}x^j\right)^2 + F$$

where R_{ij} is the Riemann curvature at p and F is the curvature of E at p both thought of as two forms. The x^j are geodesic local coordinates around p. The fundamental solution $K_t(0,x)$ at 0 of the heat equation associated with the operator D_p can be solved explicitely using Mehler's formula. The result is

$$(4\pi t)^{-\frac{1}{2}}\left(\det(\frac{tR/2}{\sinh tR/2})\right)^{1/2}\exp\left(-\frac{1}{4t}\left\langle\frac{tR}{2}\coth\frac{tR}{2}x,x\right\rangle\right)\exp(-tF).$$

The top order term (which is the constant term of the t-expansion) of this explicit fundamental solution is given by

$$\left(\det(\frac{R/2}{\sinh R/2})\right)^{1/2}\exp(-F).$$

Together with the observation that only the top order term is non-vanishing under the supertrace this gives the local index theorem.

The calculus for parameter-dependent classical pseudodifferential operators may be found in [1]. Details on how to construct the expansion for the heat kernel from the resolvent expansion may be found in [2]. Getzler's trick is treated in a systematic manner in [3].

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A short survey on eigenvalue estimates for the Dirac operator on compact Riemannian spin manifolds

Nicolas Ginoux

As its name indicates, this talk does not contain any new result on the spectrum of the Dirac operator. It is based on [4] which is itself inspired from [2] and [6]. Its aim is to give an actualized overview for non spin geometers.

Denote by $D := \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}$ the fundamental Dirac operator acting on sections of the spinor bundle of a Riemannian spin manifold (M^n, g). If M is closed the spectrum of D is a discrete unbounded subset of $\mathbb{R}$ and is in general not explicitely known.

There is however a class of manifolds on which the spectrum can be theoretically computed: that of homogeneous spaces. Indeed the space of L^2-sections of ΣM then splits into a direct Hilbert sum of finite dimensional subspaces, each of those being left invariant by D. Determining the spectrum of D turns out to compute the spectrum of the restriction of D to each of those subspaces, task which belongs to elementary linear algebra but still may be hard. We give without proof the spectrum of the round sphere as well as a table showing all homogeneous spaces (up to the knowledge of the author) for which the spectrum of D has already been computed.

In the second part of the talk we concentrate on *a priori* geometric estimates for the eigenvalues of D on any compact Riemannian spin manifold (M^n, g). We first consider lower bounds, separating the case $\partial M = \emptyset$ from the other one.
In the case of a closed manifold M we recall and give a short proof of the most general sharp lower bound obtained for any eigenvalue of D^2, namely that of T. Friedrich [3] in terms of the scalar curvature S of (M^n, g). The proof relies on the Schrödinger-Lichnerowicz formula $D^2 = \nabla^*\nabla + \frac{S}{4}\mathrm{Id}$ and on a fine orthogonal decomposition of the covariant derivative of a spinor field. Although this eigenvalue estimate is sharp (there exist manifolds for which the inequality is an equality for the smallest eigenvalue of D^2, e.g. the round sphere) there are still cases where the equality cannot be attained, so that Friedrich's lower bound should be improved in those cases. For example if M admits a non-trivial parallel k-form with $1 \leq k \leq n-1$ then A. Lichnerowicz and O. Hijazi showed that equality in Friedrich's inequality cannot hold. This was the starting point for proving better estimates on such manifolds, and we cite just two of them, namely K.-D. Kirchberg's estimate on Kähler spin manifolds and Hijazi-Milhorat/Kramer-Semmelmann-Weingart one on quaternionic Kähler spin manifolds (see [4] for references).
Returning then to the general case, we recall a fundamental property of D: its conformal covariance. Using this property we give a short proof of two important lower eigenvalue estimates which improve Friedrich's inequality: O. Hijazi's lower

bound [5] by the first eigenvalue of the conformal Laplacian (which implies the existence of a conformal lower bound for the spectrum of D) in dimension $n \geq 3$ and C. Bär's lower bound [1] on S^2 in dimension $n = 2$.
We describe a third improvement of Friedrich's inequality, namely in the case where M bounds a Riemannian spin manifold $(\overline{M}^{n+1}, g)$. In this situation we recall and give a sketch of proof of a lower bound proved by O. Hijazi, S. Montiel and X. Zhang (see [4] for references) for any nonnegative eigenvalue of D in terms of the mean curvature of M in $\overline{M}$. This lower bound is derived solving a boundary value problem on $\overline{M}$.
Boundary value problems are precisely what one should deal with when looking at the spectrum of D on manifolds with non-empty boundary. In that case we recall the necessity of fixing elliptic boundary conditions in order to be able to define the spectrum of D. We give lower bounds proved by Hijazi-Montiel-Zhang/Hijazi-Montiel-Roldán for the spectrum of D under four different boundary conditions we briefly define (see [4] for more details and references).

Turning to upper bounds on closed manifolds, we recall two different methods employed to obtain such estimates. The first one is due to C. Vafa and E. Witten and consists in comparing D with another Dirac-type operator $\widehat{D}$ which has a kernel for index-theoretical reasons, then by bounding the norm of the (zero-order) difference $D - \widehat{D}$ by a geometric quantity C; the method allows (with some work) to conclude that there are then a topologically determined number of eigenvalues of D that are bounded by C. We give an example where this method applies, namely H. Baum's upper bound in terms of the sectional curvature (see [4] for more details) on any even-dimensional closed spin manifold with positive sectional curvature.
Last but not the least, we illustrate an application of a second well-known method to obtain upper bounds, namely the min-max principle. We give and partially prove an upper bound for the smallest eigenvalue of D^2 on a closed hypersurface of a spaceform in terms of the mean curvature of the immersion.

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The Dirac spectrum on open manifolds and harmonic spinors

Uwe Semmelmann

Part I: The Dirac spectrum on open manifolds

In the first part of the talk we consider the Dirac operator on complete open manifolds. Here the Dirac operator is essentially self adjoint and has a real spectrum. It may be written as the disjoint union of discrete and essential spectrum, or of point and continous spectrum.

Explicit calculations

The simplest example is $\mathbb{R}^n$ with the euclidean metric. Here the point spectrum is empty and the continous spectrum coincides with $\mathbb{R}$. The same statement holds for the real hyperbolic space [2]. The continuous spectrum of the other symmetric spaces of rank 1 is computed in [3]. The question whether or not point spectrum exists on arbitrary symmetric spaces is settled in [4]. The result is the following:

Let $M = G/K$ be a Riemannian symmetric space of non–compact type. The point spectrum of the Dirac operator is non-empty if and only if it consists of zero, which is the case if and only if $M = U(p+q)/U(p) \times U(q)$ with $p+q$ odd.

The decomposition principle

The essential spectrum is stable under compact perturbations. As a consequence the decomposition principle states that two complete Riemannian spin manifolds, which are isometric outside a compact set, have the same essential spectrum. As an interesting application we cite the following result of Ch. Bär [5]:

Let M be a hyperbolic spin manifold of finite volume. If the spin structure is trivial along one cusp then the discrete spectrum of the Dirac operator is empty and the essential spectrum is all of $\mathbb{R}$. If the spin structure is non-trivial along all cusps then the essential spectrum is empty.

Part II: Harmonic spinors

In the second part of the talk we consider the Dirac operator on compact Riemannian spin manifolds. Here the spectrum is discrete and possibly contains the zero. Harmonic spinors are by definition spinors in the kernel of the Dirac operator. The name is chosen in analogy to harmonic forms, which are differential forms in the kernel of the Laplace operator. It is well known that the kernel of the Laplace operator and in particular its dimension is a topological invariant. In contrast the dimension of the space of harmonic spinors is only a conformal invariant [9]. However the Atiyah-Singer index theorem implies that the absolute value of the A-roof genus, i.e. a topological invariant, gives a lower bound for this dimension. In particular, if $\hat{A}(M) \neq 0$, then any metric on M admits harmonic spinors.

Harmonic spinors and scalar curvature

If g is a metric with positive scalar curvature then the corresponding Dirac operator has no harmonic spinors. This is a consequence of the Lichnerowicz Schrödinger formula. Note that on the torus there is no metric of positive scalar curvature, nevertheless one can show that there are metrics without harmonic spinors. It is also possible that the A-roof genus is zero but the space of harmonic spinors for some metric is non-empty. The simplest example is provided by certain Berger spheres. Parallel spinors are automatically harmonic. However the existence of parallel spinors leads to strong restrictions on the geometry of the underlying manifolds. Such manifolds have holonomy $SU(n), Sp(n), G_2$ or $Spin_7$ and in particular vanishing scalar curvature.

Harmonic spinors on surfaces and spheres

Harmonic spinors on surfaces were considered in [9] by N. Hitchin and by Ch. Bär and P. Schmutz . It turns out that there are no harmonic spinors on the sphere S^2. On surfaces of genus 1 or 2 the dimension of the space of harmonic spinors does not depend on the metric but only on the spin structure. For a suitable choice one has harmonic spinors. For surfaces of genus greater than 2 the dimension of the space of harmonic spinors depends on the metric and the spin structure and again there are harmonic spinors for suitable choices.

N. Hitchin showed in [9] that there are harmonic spinors on S^3. Ch. Bär showed in [6] that there are harmonic spinors on S^{4k+3}. In both cases one has 1-parameter families of Berger metrics for which the dimension of the space of harmonic spinors becomes arbitrary large. L. Seeger showed the existence of harmonic spinors on S^{4n} and recently M. Dahl showed in [8] the existence on all spheres.

Existence of harmonic spinors

The following conjecture is due to Ch. Bär: On any closed spin manifold of dimension greater than 2 there exists a metric with harmonic spinors. The conjecture was proved to be true in all dimensions congruent 0,1 or 7 modulo 8 by N. Hitchin in [9] and in dimensions congruent 3 modulo 4 by Ch. Bär in [6]. The result of Ch. Bär is based on a gluing theorem, which states that the Dirac spectrum of a connected sum of two closed spin manifolds is in some sense close to the disjoint sum of the two seperate spectra. Another important ingredient of the proof is the explicit calculation of the Dirac spectrum on the Berger spheres.

A second conjecture of Ch. Bär claims that on a compact spin manifold the dimension of the space of harmonic spinors for a generic metric is not greater than it is forced to be by the index theorems. This conjecture was proved by S. Maier [10] in dimension 4, by Ch. Bär and M. Dahl [7] for simply connected manifolds in dimensions larger than 4 and recently by B. Ammann, M. Dahl and E. Humbert [1] in complete generality.

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The enlargeability obstruction to positive scalar curvature

Bernd Ammann

In this talk we explain the enlargeability obstruction to positive scalar curvature due to Gromov and Lawson [2], [3], [6, IV, 5+6]. One of the main goals of this talk is to show that tori (of any dimension) belong to a large class of manifolds that do not admit metrics of positive scalar curvature.

A compact riemannian manifold M of dimension n is said to be enlargeable if there is for any $\epsilon > 0$ an orientable riemannian covering $\widehat{M} \to M$ together with an ϵ-contracting map $f : \widehat{M} \to S^n$ which is constant at infinity and of non-zero degree. One easily sees that this definition is an invariant of the homotopy type of M, in particular, it is independent of the metric on M. The simplest examples of enlargeable manifolds are tori. It is not very hard to see that compact manifolds with non-positive sectional curvature are enlargeable. With some more effort, it can be shown that compact quotients of a solvable Lie group by a discrete subgroup are enlargeable.

Theorem (Gromov, Lawson [2, 3])
An enlargeable spin manifold cannot carry a metric of positive scalar curvature.

The proof of this remarkable theorem is based on the index theorem. We argue by contradiction and assume that M is an enlargeable manifold equipped with a metric of positive scalar curvature. We can assume (by possibly taking the product with S^1) that M has even dimension n. The enlargeability condition provides the existence of a riemannian covering $\widehat{M} \to M$ together with an ϵ-contracting map $f : \widehat{M} \to S^n$. For any complex vector bundle E over S^n, let $D_E : \Gamma(\Sigma\widehat{M} \otimes f^*E) \to \Gamma(\Sigma\widehat{M} \otimes f^*E)$ be the Dirac operator on $\widehat{M}$ acting on spinors twisted by f^*E. As explained in the talk of C. Bär, the completeness of

$\widehat{M}$ implies that D_E has a self-adjoint extension, and we have the Lichnerowicz formula

$$D_E^2 = \nabla_E^* \nabla_E + \frac{1}{4}\mathrm{scal} + \frac{1}{2}F^{f^*E},$$

where F^{f^*E} is the curvature of f^*E acting on twisted spinors by Clifford action.

In particular, if E_1 is a trivial bundle (with a trivial connection), then the Dirac operator D_{E_1} has no spectrum in the interval $(-\sqrt{s}/2, \sqrt{s}/2)$ where $s := \min \mathrm{scal} > 0$. If E_2 is a non-trivial bundle over S^n then the curvature term $F^{f^*E_2}$ is bounded by a constant times ϵ^2, and hence for small $\epsilon > 0$ it is dominated by the scalar curvature term. Hence, in this case we obtain for small $\epsilon > 0$ that D_{E_2} has no spectrum in $(-\sqrt{s}/4, \sqrt{s}/4)$. In particular, the indices of (the positive parts of) D_{E_1} and D_{E_2} vanish.

As mentioned in the talk of A. Strohmaier, usual index theory extends to the possibly non-compact manifold $\widehat{M}$. In particular, the Dirac operators are Fredholm and a relative version of the index theorem holds:

$$\mathrm{ind}(D_{E_1}) - \mathrm{ind}(D_{E_2}) = \{(\mathrm{ch}\, f^*(E_1) - \mathrm{ch}\, f^*(E_2)) \cup \hat{A}(TM)\}[M]. \tag{1}$$

We have already seen that the left hand side is zero. We can choose E_2 such that $\alpha := c_{n/2}(E_2) \neq 0$ and such that the fibers of E_1 and E_2 have the same dimension. This implies that $\mathrm{ch}(E_1) - \mathrm{ch}(E_2) \in H^*(S^n)$ is trivial in all degrees except in degree n where it is $-\frac{1}{((n/2)-1)!}\alpha \neq 0$. Hence, on the right hand side of equation (1), $\hat{A}(TM)$ only contributes in order zero, and we obtain

$$\begin{aligned}\{(\mathrm{ch}\, f^*(E_1) - \mathrm{ch}\, f^*(E_2)) \cup \hat{A}(TM)\}[M] &= -\frac{1}{((n/2)-1)!}(f^*\alpha)[M] \\ &= -\frac{1}{((n/2)-1)!}(\deg f)\alpha[S^n] \neq 0.\end{aligned}$$

The theorem follows from this contradiction.

The theorem as presented here was generalized by Gromov and Lawson in many directions. At first the condition that f has non-trivial degree can be weakened to the condition that f has non-trivial $\hat{A}$-degree. Here we define the $\hat{A}$-degree of $f : M^n \to S^{n-4k}$ as $\hat{A} - \deg(f) := \hat{A}(f^{-1}(p))$ where p is a generic point of S^{n-4k}, and as the $\hat{A}$-numbers are bordism invariants, the $\hat{A}$-degree does not depend on the choice of p. As before, Gromov and Lawson prove that $\hat{A}$-enlargeable manifolds do not carry metrics of positive scalar curvature.

The argument still works if we replace the fact that f is contractible by the weaker condition

$$|f^*(\omega)| \leq \epsilon^2 |\omega| \qquad \forall \omega \in \Lambda^2 T^* S^{n-4k},$$

and we obtain the notion of $\hat{A}$-area-enlargeability. In a similar way as before, $\hat{A}$-area-enlargeable manifolds do not carry metrics of positive scalar curvature.

For a further generalization, so-called "weak enlargeability" which applies to non-compact manifolds, we refer directly to the literature [3], [6, Chap. IV §6].

Another spinorial obstruction to positive scalar curvature is the Mishchenko-Fomenko index theorem. In this index theorem one twist spinors by an appropriate

infinite dimensional bundle, and the index of the resulting Dirac operator is an element of $KO_n(C^*_{\max}\pi_1(M))$, see for example [9] for an introduction. If the compact manifold M admits a positive scalar curvature metric, then this index has to vanish [7]. Thus, this index is an obstruction to positive scalar curvature as well. In two recent articles [4, 5] B. Hanke and T. Schick prove that area-enlargeable spin manifolds have non-trivial Mishchenko-Fomenko index. It is believed that the same construction also works for $\hat{A}$-area-enlargeability, but the construction has not yet been carried out. Hence, the enlargeability obstruction to positive scalar curvature can be reduced in some cases to the Mishchenko-Fomenko obstruction.

However, if one wants to prove that a given manifold does not admit a metric of positive scalar curvature, then in some cases the enlargeability condition is simpler to verify than the calculation of the Mishchenko-Fomenko index. Hence, enlargeability provides an efficient mean for finding examples with non-vanishing Mishchenko-Fomenko index. Using fiber bundle techniques, one can then construct new classes of manifolds that do not admit positive scalar curvature metrics [4, Section 6].

Finally, we want to mention that Schoen and Yau [8] have developed an alternative method to prove the non-existence of positive scalar curvature metrics on another class of manifolds containing tori. Their method is based on the construction of a minimal hypersurface in a riemannian manifold of positive scalar curvature and is strongly linked to their proof of the positive mass conjecture. As such minimal hypersurfaces may have singularities in codimension ≥ 7, the Schoen-Yau method does not extend directly to manifolds of dimension ≥ 7. However, in a recent preprint Christ and Lohkamp [1] explain, how to overcome this difficulty.

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Geometric applications of index theory

Helga Baum

We give a short survey on some geometric applications of Index Theory for non-experts. Thereby we cover the following topics:

- Integrality and divisibility theorems
- Immersions of manifolds and existence of global vector fields
- Group actions on manifolds

Geometric applications of the kind mentioned above are consequences of the index theorem for differential operators associated to G-structures. Let P be a G-structure on M, $E_i = P \times_G V_i$ associated vector bundles and $D : \Gamma(E_1) \longrightarrow \Gamma(E_2)$ an differential operator associated to the G-structure. Then the topological index of D is given by

$$t - index(D) = f_P^* \left(\frac{ch(V_1) - ch(V_2)}{\prod_j (-\omega_j)} \right) Todd\,(TM^{\mathbb{C}})\,[M]$$

where $f_P : M \longrightarrow B_G$ is the classifying map for P, ω_j are the weights of the G-representation $\mathbb{R}^n$, and $ch(V_i)$ the Chern characters of the G-representations V_i. Typical applications use the index of differential operators for special G-structures adapted to the geometric problem in question to obtain obstructions. Details can be found in [3] or [4].

1. Integrality Theorems

As an typical example we mention a special integrality theorem of K.H.Mayer ([1]). Let M^{2n} be a compact oriented manifold, ξ^k a real oriented vector bundle, η^l a complex vector bundle on M and $d \in H^2(M, \mathbb{Z})$ such that $d \equiv w_2(M) + w_2(\xi)$ mod 2. Let $k = 2s$ or $k = 2s + 1$. Then

$$t(d, \xi, \eta) := 2^s \{ e^{\frac{d}{2}} ch(\eta) C(\xi) \hat{A}(M) \}[M]$$

is an integer. Under additional conditions this integer is even.

If we take $\xi = TM$, $d = 0$ and $\eta = 0$ we obtain as special case the signature

$$t(0, TM, 0) = L(TM)[M] = \sigma(M).$$

If in addition M is spin, then $t(0, 0, 0) = \hat{A}(M)[M]$. In dimension 4 this yields Rochlin's theorem ('50), that the signature of a smooth 4-dimensional manifold M with $w_1(M) = w_2(M) = 0$ is divisible by 16. The generalization of this theorem for dimension $8k + 4$ was proven by Ochianin in 1981. Note, that there are topological spin manifolds with signature 8.

2. Vector field problems and immersions of manifolds

Let M be a compact manifold and let us denote by $span(M)$ the number of linearely independent global vector fields on M. We are looking for a lower bound for $span(M)$. It is well known, the $span(S^n) = 2^c+8d-1$, where $n+1 = 2^{4d+c}(2s+1)$. If we apply Mayer's Theorem for a manifold M^{4m} with $span(M) \geq k$, taking $TM = \xi \oplus \mathbb{R}^k$, we obtain that the signature $sign(M)$ has to be divisible by b_k, where

k	1	2	3	4	5	6	7	8
b_k	2	4	8	16	16	16	16	32

und $b_{k+8} = 16b_k$

In a similar way one obtains obstructions for the existence of immersions and embeddings. If M^n is a smooth n-dimensional manifold, then by Whitney's theorem M can always be immersed into $\mathbb{R}^{2n-1}$ and be embedded into $\mathbb{R}^{2n}$. We are interested in the smallest number k such that there is an immersion (embedding) of M^n into $\mathbb{R}^{n+k}$. Obstructions can be obtained for example using the normal bundle of the immersion for ξ in Mayer's Theorem. For details and special cases see [1] or [2].

3. Group actions on manifolds

We mention here two central and typical results:

1. Let M be a compact spin amnifold which admit an S^1-action, then $\hat{A}(M) = 0$. (Atiyah/Hirzebruch 1970).
2. Let M be a compact spin manifold which admit an S^3-action, then $\hat{\mathbf{A}}(M) = 0$. (Lawson/Yau 1974).

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Part II: A mini-course in noncommutative geometry

On noncommutative geometry

José M. Gracia-Bondía

"When physicists talk about the importance of beauty and elegance in their theories, the Dirac equation is often what they have in mind. Its combination of great simplicity and surprising new ideas, together with its ability both to explain previously mysterious phenomena and predict new ones [spin], make it a paradigm for any mathematically inclined theorist" [1].

The Dirac equation naturally lives on spin manifolds, and these constitute the crucial paradigm, too, for Connes' program of research (and unification) of mathematics. There is no general theory of noncommutative spaces. The practitioners' tactics has been that of multiplying the examples, whereas trying to anchor the generalizations on the more solid ground of ordinary (measurable / topological / differentiable / Riemannian...) spaces. This is what we try to do here, within the limitations imposed by the knowledge of the speaker.

Thus the first task is to learn to think of ordinary spaces in noncommutative terms. Arguably, this goes back to the Gelfand–Naĭmark theorem (1943), establishing that the information on any locally compact Hausdorff topological space X is fully stored into the commutative algebra $C(X)$ of continuous function over it, vanishing at ∞. This is both a way to recognize the importance of C^*-algebras, and to think of them as of locally compact Hausdorff noncommutative spaces. If we just ask for the functions to be measurable and bounded, we are led to von Neumann algebras. Vector bundles are identified through their spaces of sections, which algebraically are projective modules of finite type over the algebra of functions associated to the base space —this is the Serre–Swan theorem (1962). In this way, we come to think of noncommutative vector bundles. Under the influence of quantum physics, the general idea is then to forget about sets of points and obtain all information from classes of functions; e.g. open sets in X are replaced by ideals.

The rules of the game would then seem to be: (1) find a way to express a mathematical category through algebraic conditions, and then (2) relinquish commutativity. This works wonders in group theory, which is replaced by Hopf algebra theory, relinquishing (co)commutativity. However, that kind of generalization quickly runs into sands, for two reasons: (i) Some mathematical objects, like differentiable manifolds, and de Rham cohomology, are reluctant to direct noncommutative generalization. The same is true of Riemannian geometry; after all, all smooth manifolds are Riemann. (ii) Genuine noncommutative phenomena are missed.

For instance, in the second respect, in many geometrical situations the associated set is very pathological, and a direct examination yields no useful information. The set of orbits of a group action, such as the rotation of a circle by multiples of an irrational angle θ, is generally of this type. In such cases, when we examine

the matter from the algebraic point of view, we are sometimes able to obtain a perfectly good operator algebra that holds the information we need; however, this algebra is generally not commutative.

One can situate the beginning of noncommutative geometry (NCG) in the 1980 paper by Connes, where the 'noncommutative torus' $\mathbb{T}_\theta$ was studied [2]. Not only this algebra is able to answer the question mentioned above, but one can decide what are the smooth funcions on this noncommutative space, what vector bundles and connections on $\mathbb{T}_\theta$ are and, decisively, how to construct a Dirac operator on it. Even now, the importance of this early example in the development of the theory can hardly be underestimated. The noncommutative torus provides a simple but nontrivial example of *spectral triple* (A, H, D) or 'noncommutative spin manifold', the algebraic apparatus with which Connes eventually managed to push aside the obstacles to the definition of noncommutative Riemannian manifolds.

The more advanced rules of the game would now seem to be: (1) Escape of the difficulties 'from above' by finding the algebraic means of describing a richer structure. If we reformulate algebraically what a spin manifold is, we can describe its de Rham cohomology, its Riemannian distance and the like, algebraically as well. Choice of a Dirac operator D means imposing a metric. However, there is the risk that the link to the commutative world is obscured. Therefore: (2) Make sure that the link is kept. In other words, prove that a noncommutative spin manifold is in fact a spin manifold in the everyday sense (!) when the underlying algebra is commutative. In point of fact, the second desideratum only received a satisfactory, partial answer a few weeks ago [3].

Thus the irony is in that, first, "Mathematicians were much slower to appreciate the Dirac equation and it had little impact on mathematics at the time of its discovery. Unlike the case with the physicists, the equation did not immediately answer any questions that mathematicians had been thinking about" [1]. The situation changed only forty years later, with the Atiyah–Singer theory of the index. Second, now that spin manifold theory is an established and respectable line of mathematical business, its community of practitioners seems oblivious to the fact it underpins a whole new branch/paradigm/method of doing mathematics.

Now come the *informal rules* for noncommutative geometers —rules which in any society are the most binding. These seem to be: (1) Keep close to physics, and in particular to quantum field theory. There is no doubt that Connes came to his 'axioms' for noncommutative manifolds by thinking of the Standard Model of particle physics as a noncommutative space. (2) Try to intepret and solve most problems conceivably related to noncommutative geometry by use of spectral triple theory. This of course is not to everyone's taste, and a cynic could say: "Who is good with the hammer, thinks everything is a nail"; moreover it is of course literally impossible, as the world teems with virtual objects for which complete taxonomy is an impossible task. It has proved surprisingly rewarding, however.

About (2): there is an underlying layer of index theory and K-theory, which is a deep way of addressing quantization. But even there, when you need to compute

K-theoretic invariants, you are led back to smoother structures where you have more tools, like (A, H, D).

During the discussions, I mentioned some of my favourite neglected problems:

- Lie algebroids, Lie–Rinehart algebras and the like. It is a little mystery why, while groupoids play a central role in NCG, their infinitesimal version does not seem to play any role. All the more so because the algebraic version of Lie algebroids, the theory of Lie–Rinehart algebras, which seems to be the good algebraic framework for BRST theories, has very much the flavour of NCG, and is quite able to deal with many singular spaces [4]. (This question was in part beautifully answered by Erik van Erp during the meeting.)
- Algebraic K-theory, noncommutative geometry and field theory. The role of the two first functors of algebraic K-theory in QFT with external fields is 'well-known'; Connes has dabbled on this, but it has not pursued the subject. To this writer it still seems extremely promising.
- Rota–Baxter operators and skewderivations. A poor man's path to the nc world (akin to the one taken by some quantum group theorists) is to try to generalize the usual derivative/integral pair. This is elementary stuff with many ramifications. A skewderivation of weight $\theta \in \mathbb{R}$ is a linear map $\delta : A \to A$ fulfilling the condition

$$\delta(ab) = a\delta(b) + \delta(a)b - \theta\delta(a)\delta(b). \tag{1}$$

We may call skewdifferential algebra a double $(A, \delta; \theta)$ consisting of an algebra A and a skewderivation δ of weight θ. A *Rota–Baxter map* R of weight $\theta \in \mathbb{R}$ on a not necessarily associative algebra A, commutative or not, is a linear map $R : A \to A$ fulfilling the condition

$$R(a)R(b) = R(R(a)b) + R(aR(b)) - \theta R(ab), \qquad a, b \in A. \tag{2}$$

When $\theta = 0$ we obtain the integration-by-parts rule. The triple $(A, \delta, R; \theta)$ will denote an algebra A endowed with a skewderivation δ and a corresponding Rota–Baxter map R, both of weight θ, such that $R\delta a = a$ for any $a \in A$ such that $\delta a \neq 0$, as well as $\delta R a = a$ for any $a \in A, Ra \neq 0$. One easily checks consistency of conditions (1) and (2) imposed on δ, R. Rota–Baxter operators have proved their worth in probability theory and combinatorics, and in the Connes-Kreimer approach to renormalization; but their range of applications is much wider.

- General Moyal theory. Given the high number of nc spaces that turn out to be related to Moyal quantization (plus the usefulness of Moyal quantization in proofs, for instance of Bott periodicity in the algebraic context), it is surprising that few nc geometers seem interested in general Moyal theory. The latter would run as follows. Let X be a phase space, μ a Liouville measure on X, and H the Hilbert space associated to (X, μ). A *Moyal quantizer* for (X, μ, H) is a mapping Ω of X into the space of selfadjoint operators on H, such that $\Omega(X)$ is weakly dense in $B(H)$, and

verifying

$$\mathrm{Tr}\Omega(u) = 1,$$
$$\mathrm{Tr}\big[\Omega(u)\Omega(v)\big] = \delta(u-v),$$

in the distributional sense. (Here $\delta(u-v)$ denotes the reproducing kernel for the measure μ.) Moyal quantizers, if they exist, are unique, and ownership of a Moyal quantizer solves in principle all quantization problems: *quantization* of a (sufficiently regular) function or "symbol" a on X is effected by

$$a \mapsto \int_X a(u)\Omega(u)\, d\mu(u) =: Q(a),$$

and *dequantization* of an operator $A \in B(H)$ is achieved by

$$A \mapsto \mathrm{Tr}A\Omega(.) =: W_A(.).$$

Indeed, it follows that $1_H \mapsto 1$ by dequantization, and also

$$\mathrm{Tr}Q(a) = \int_X a(u)\, d\mu(u).$$

Moreover, using the weak density of $\Omega(X)$, it is clear that:

$$W_{Q(a)}(u) = \mathrm{Tr}\left[\left(\int_X a(v)\Omega(v)\, d\mu(v)\right)\Omega(u)\right] = a(u),$$

so Q and W are inverses. In particular, $W_{Q(1)} = 1$ says that $1 \mapsto 1_H$ by quantization, and this amounts to the reproducing property. Finally, we also have

$$\mathrm{Tr}[Q(a)Q(b)] = \int_X a(u)b(u)\, d\mu(u).$$

This is the key property. Most interesting cases occur in an equivariant context ; that is to say, there is a (Lie) group G for which X is a symplectic homogeneous G-space, with μ then being a G-invariant measure on X, and G acts by a projective unitary irreducible representation U on the Hilbert space H. A Moyal quantizer for the combo (X, μ, H, G, U) is a map Ω taking X to selfadjoint operators on H that satisfies the previous defining equations and the equivariance property

$$U(g)\Omega(u)U(g)^{-1} = \Omega(g.u),$$

for all $g \in G,\ u \in X$. The question is: how to find the quantizers? The fact that the solution in flat spaces leads to (bounded) parity operators points out to the framework of *symmetric spaces* as the natural one to find Moyal quantizers by interpolation. This heuristic parity rule was found to work for orbits of the Poincaré group [5]. Noncompact symmetric spaces should provide a wealth of noncompact spectral triples (the compact case is somewhat pathological). In general they will not be not isospectral manifolds in the original narrow sense, even if the Dirac operator can stay undeformed —which remains to be seen.

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A short introduction to K-theory and K-homology

Adam Rennie

This is part one of a shared talk with U. Kraehmer. Its aim is to define K-theory and K-homology of C^*-algebras and to briefly explain their role in noncommutative geometry.

The abelian group $K^0(X)$, X a compact Hausdorff space, is defined to be the Grothendieck group of the semigroup of isomorphism classes of (complex) vector bundles over X with respect to direct sum. The Serre-Swan theorem allows to replace this by equivalence classes of projections over $C(X)$ (where p over $C(X)$ means $p \in M_n(C(X))$ for some n). In each case we find stable homotopy classes (of vector bundles or projections) in the resulting abelian group, but the second definition is preferable because it extends immediately to any C^*-algebra A.

Likewise $K^1(X)$ is defined as stable homotopy classes of unitaries over $C(X)$, and this definition extends to all C^*-algebras A. Since the Gel'fand functor exchanging spaces and algebras is contravariant, the C^*-algebra K-theory groups are denoted

$$K_0(A), \quad K_1(A).$$

These groups give covariant functors on the category of all C^*-algebras (and $*$-morphisms), and they satisfy Bott periodicity

$$K_j(A) \cong K_{j+n}(C_0(R^n) \otimes A), \quad j,\, j+n \ \text{ taken } \mod 2.$$

This periodicity reduces the usual long exact sequences to periodic six-term sequences, and similarly we obtain six term Mayer-Vietoris sequences etc etc. An important work which popularised the K-approach for operator algebraists was the paper by Taylor, [5], which remains an excellent introduction to the ideas.

The dual theory, K-homology, took longer to develop. The first attempt came from Atiyah in 1969, [1], and he gave a reasonable description of the basic cycles, though the equivalence relations remained unknown. Working from a completely different direction (the classification of essentially normal operators), Brown, Douglas and Fillmore, [2], discovered odd K-homology. All these developments, as well as the previous work in K-theory, were substantially generalised into one bivariant functor, called KK-theory, by G. G. Kasparov in the early 1980's, [4]. We will follow Kasparov's definitions, but in the special case of K-homology, as in [3].

Definition A Fredholm module $(\mathcal{H}, F)$ for a C^*-algebra A is a Hilbert space $\mathcal{H}$ on which A is represented (not necessarily faithfully) and an operator $F \in \mathcal{B}(\mathcal{H})$ such that for all $a \in A$ the operators

$$a(F - F^*), \quad a(F^2 - 1), \qquad [F, a]$$

are compact operators. The Fredholm module is even if there exists an operator $\Gamma \in \mathcal{B}(\mathcal{H})$ with $\Gamma = \Gamma^2$, $\Gamma^* = \Gamma$, $\Gamma a = a\Gamma$ and $\Gamma F + F\Gamma = 0$. Otherwise it is odd.

Roughly speaking, the group $K^0(A)$ (respectively $K^1(A)$) is the group of stable homotopy classes of even (respectively odd) Fredholm modules under direct sum.

Example If M is a compact Riemannian spin^c manifold, and D is the Dirac operator of the spin^c structure, then

$$\mathcal{H} = L^2(M, S), \quad F = D(1 + D^2)^{-1/2}$$

is a Fredholm module for $A = C(M)$, whee $S \to M$ is the spinor bundle. This Fredholm module is even if and only if the dimension of M is even, with the grading being given by Clifford multiplication by the complex volume form.

The duality pairing is given explicitly in terms of representatives of classes. For the even pairing, let $p \in M_n(A)$ be a projection and $(\mathcal{H}, F, \Gamma)$ be an even Fredholm module. Then the pairing is

$$\langle [p], [(\mathcal{H}, F, \Gamma)] \rangle = \text{Index}\left(p\left(\frac{(1-\Gamma)}{2} F \frac{(1+\Gamma)}{2} \otimes 1_n \right) p : p\mathcal{H}^n \to p\mathcal{H}^n \right).$$

Here Index denotes the usual Fredholm index. In the odd case we let $u \in M_n(A)$ be unitary, and $(\mathcal{H}, F)$ be an odd Fredholm module. The pairing is then given by

$$\langle [u], [(\mathcal{H}, F)] \rangle = \text{Index}\left(PuP \oplus -(1-P) : \mathcal{H}^n \to \mathcal{H}^n \right),$$

where $P = \chi_{[0,\infty)}(F) \otimes 1_n$ is the non-negative spectral projection of $F \otimes 1_n$.

The challenge in many circumstances is to compute the pairing, and this is typically very difficult. In classical cases one can apply Chern character maps to translate the problem into de Rham theory, and the Atiyah-Singer theorem then provides a geometric formula for the index pairing. In noncommutaive geometry, the Chern characters map to cyclic theory (see U. Kraehmer's talk below), but even to define them we need very nice representatives of K-homology classes (K-theory Chern classes are easier). Suppose then that $(\mathcal{H}, F)$ satisfies

$$F = F^*, \quad F^2 = 1, \quad [F, a] \in \mathcal{L}^{p+1}(\mathcal{H}),$$

for all $a \in \mathcal{A} \subset A$, a dense subalgebra. Here $\mathcal{L}^{p+1}(\mathcal{H})$ is the $p+1$-th Schatten ideal. In this case we can define the (degree p representative of the) Chern character of $(\mathcal{H}, F)$ to be the multilinear functional on $\mathcal{A}$ defined by

$$Ch_F^p(a^0, a^1, \ldots, a^p) = C_p \text{Trace}(\Gamma F[F, a^0][F, a^1] \cdots [F, a^p]).$$

Here C_p is a constant designed to make Ch_F^{p+2n} represent the same class as Ch_F^p in periodic cyclic theory. This definition allows one to show, quite explicitly, that

$$\langle [p], [(\mathcal{H}, F, \Gamma)] \rangle = \langle [Ch(p)], [Ch_F] \rangle$$

as well as homotopy invariance of the pairing etc etc.

Typically however, this Chern character is often still difficult to compute, and using other representatives of K-homology classes, such as spectral triples, can lead to more computable representatives of the Chern character. This is discussed in some of the other talks.

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An introduction to Hochschild and cyclic homology

Ulrich Krähmer

This is part two of a shared talk with A. Rennie. Its aim is to define Hochschild and cyclic homology and to explain their role in noncommutative geometry.

Let A be a unital associative algebra over a commutative ring k and M be an A-bimodule. Then $C_\bullet(A,M) := \bigoplus_{n\geq 0} C_n(A,M)$, $C_n(A,M) := M \otimes A^{\otimes n}$ (where $\otimes$ denotes the tensor product of k-modules) becomes a chain complex through

$$\begin{aligned} b : a_0 \otimes a_1 \otimes \ldots \otimes a_n \quad \mapsto \quad & a_0a_1 \otimes a_2 \otimes \ldots \otimes a_n - a_0 \otimes a_1a_2 \otimes a_3 \otimes \ldots \otimes a_n \\ & + \ldots + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}. \end{aligned}$$

Here a_0a_1 and a_na_0 denote the left and right action of $a_1, a_n \in A$ on $a_0 \in M$, respectively, and a_ia_{i+1}, $i \neq 0, n$, is the product of $a_i, a_{i+1} \in A$. This complex defines the Hochschild homology of A with coefficients in M,

$$H_\bullet(A,M) = \bigoplus_{n\geq 0} H_n(A,M) := \ker b / \operatorname{im} b.$$

For $M = A$ with the obvious bimodule structure one wrties $H_\bullet(A,A) =: HH_\bullet(A)$.

This homology theory was introduced by Hochschild in 1945 and generalises both group and Lie algebra homology. It fits into the general pattern of simplicial homology theories and admits a derived functor interpretation in the category of A-bimodules. In a dual fashion, one defines Hochschild cohomology $H^\bullet(A,M)$.

In 1962, Hochschild, Kostant and Rosenberg studied the case of coordinate rings $A = k[X]$ of affine varieties X over perfect fields k. In particular, they identified $HH_n(A)$ for smooth X with the Kähler differentials $\Omega^n(A)$. Later, this was generalised to other classes of commutative algebras and in particular by Connes in a topological setting to smooth functions on manifolds. Thus $HH_n(A)$ can be viewed as generalistion of differential forms for arbitrary algebras.

Cyclic homology arises from the action of the cyclic group $\mathbb{Z}_{n+1}$ on $C_n := C_n(A, A)$ which is given by Connes' cyclic permuter

$$t : a_0 \otimes \ldots \otimes a_n \mapsto (-1)^n a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

Defining $N := 1 + t + t^2 + \ldots + t^n$ and b' to be b with the last term $(-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}$ omitted, this action yields Tsygan's bicomplex $CC_{\bullet\bullet}(A)$:

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
C_1 & \xleftarrow{1-t} & C_1 & \xleftarrow{N} & C_1 & \xleftarrow{1-t} & C_1 & \xleftarrow{N} & C_1 & \xleftarrow{1-t} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
C_0 & \xleftarrow{1-t} & C_0 & \xleftarrow{N} & C_0 & \xleftarrow{1-t} & C_0 & \xleftarrow{N} & C_0 & \xleftarrow{1-t} & \cdots
\end{array}$$

Cyclic homology $HC_\bullet(A)$ can now be defined as the homology of its total complex $\bigoplus_{n\geq 0}(\bigoplus_{p+q=n} CC_{pq})$.

This definition is the correct one as well for nonunital algebras A, but in the unital case the columns with differential $-b'$ are in fact exact which allows to eliminate them in order to get a simpler bicomplex, Connes' (b, B)-bicomplex that will be explained in A. Sitarz's talk. This is especially useful for explicit computations and yields in the commutative case (say $A = k[X]$ as above):

$$HC_n(A) \simeq \Omega^n(A)/\operatorname{im} d \oplus H^{n-2}_{\mathrm{deRham}}(X) \oplus H^{n-4}_{\mathrm{deRham}}(X) \oplus \ldots,$$

where d is the de Rham coboundary. Thus cyclic homology provides a noncommutative geometry substitute for de Rham cohomology.

If k is on the other hand a field of characteristic 0, then the rows of $CC_{\bullet\bullet}$ are exact, and their elimination identifies $HC_\bullet(A)$ with the homology of Connes' complex $C^\lambda_\bullet := C_\bullet/\operatorname{im}(1-t)$. This is for example useful to develop the link to K-theory and differentially graded algebras over A.

At the end we discuss very briefly the Connes-Chern character

$$K_0(A) \to \bigoplus_{n\geq 0} HC_{2n}(A).$$

For an idempotent $p = p^2 \in A$, let c_n be simply $p \otimes \ldots \otimes p \in C_n$. Then

$$b(c_n) = \begin{cases} c_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad (1-t)(c_n) = \begin{cases} 0 & n \text{ even} \\ 2c_n & n \text{ odd} \end{cases}$$

Thus c_n is a Hochschild cycle for n odd, but its class in $HH_n(A)$ is zero. Hoewever, since $b(c_n) \in \operatorname{im}(1-t)$ for n even, the class of c_n in C^λ_n is a cyclic cycle, and its homology class in $HC_\bullet(A)$ is in general nonzero and depends only on the class of p in $K_0(A)$ (see A. Rennie's talk for $K_0(A)$). This defines up to normalisation the Connes-Chern character. For classes in $K_0(A)$ that are represented by matrix idempotents $(p_{ij}) \in M_N(A)$ one replaces c_n by $\sum_{i_0,\ldots,i_n} p_{i_0 i_1} \otimes p_{i_1 i_2} \otimes \ldots \otimes p_{i_n i_0}$. The odd degree Chern character is defined similarly.

For more information, proofs and references to the original literature, we refer to the standard textbooks on the subject.

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An introduction to Hopf algebras

Ulrich Krähmer

We first explain the notion of coalgebra as a formal dualisation of that of a (unital associative) algebra (say over a field k): A coalgebra is a vector space C equipped with two maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ that are assumed to satisfy two axioms. These precisely say that the dual maps $\mu := \Delta^* : (C \otimes C)^* \to C^*$ and $\eta : k = k^* \to C^*$ turn $C^* = \mathrm{Hom}_k(C, k)$ into an algebra with product $ab = \mu(a \otimes b)$ and unit $\eta(1)$. The full dual of an algebra (A, μ, η) is, however, not a coalgebra in general since $\mu^* : A^* \to (A \otimes A)^* \supsetneq A^* \otimes A^*$. On the other hand,

$$A^\circ := \{\phi \in A^* \mid \exists I \subset \ker \phi, I \text{ ideal}, \dim_k(A/I) < \infty\}$$

becomes a coalgebra called the dual coalgebra of A.

A bialgebra is an algebra and coalgebra, Δ, ε being algebra homomorphisms. If G is e.g. a finite set and A is the algebra of all functions $G \to k$ with pointwise addition and multiplication, then $A \otimes A$ is the algebra of functions on $G \times G$ and bialgebra structures on it correspond bijectively to semigroup structures on G via

$$\Delta(f)(x, y) = f(x \cdot y), \quad \varepsilon(f) = f(e).$$

Here $\cdot$ is the semigroup operation and e is the unit element.

Hopf algebras are bialgebras with an additional map $S : A \to A$ called the antipode that links the algebra and coalgebra structure in a stronger way. In the above example, its defining property

$$\varepsilon(a) = \sum_i S(a_i')a_i'' = \sum_i a_i' S(a_i''), \text{ where } \sum_i a_i' \otimes a_i'' = \Delta(a)$$

is tantamount to G being a group (with $S(a)(x) = a(x^{-1})$). Similarly, Hopf algebra structures on coordinate rings of affine varieties correspond to algebraic group structures. Thus if algebras are considered as "noncommutative spaces", then Hopf algebras play the role of groups in this picture.

Classical examples of noncommutative Hopf algebras are group rings kG and universal enveloping algebras $U(\mathfrak{g})$, but only the discovery of quantum groups in the 1980's (essentially due to Drinfeld, Jimbo and Woronowicz) gave full support

to the philosophy outlined above. The simplest example is $\mathbb{C}_q[SL(2)]$, $q \in \mathbb{C} \setminus \{0\}$, which can be defined in terms of generators a, b, c, d and relations

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc,$$
$$bc = cb, \quad ad - da = (q - q^{-1})bc, \quad ad - qbc = 1.$$

For $q = 1$ this is the commutative coordinate ring of $SL(2, \mathbb{C})$, and it turns out that its Hopf structure can be deformed to $\mathbb{C}_q[SL(2)]$ with $\mathbb{C}_q[SL(2)] \simeq \mathbb{C}[SL(2)]$ as coalgebra (for q not root of unity). Thus $\mathbb{C}_q[SL(2)]$ "quantises" $SL(2, \mathbb{C})$ leaving its group structure in a sense untouched - its a "coordinate ring of a quantum group". Similarly, there exists $\mathbb{C}_q[G]$ for any complex reductive Lie group G, and its algebraic properties often reflect geometric ones of a distinguished Poisson structure on G that is obtained as semiclassical limit for $q \to 1$.

In the second half of the talk we consider Hopf algebras A that are cosemisimple, that is, those whose finite-dimensional comodules (a notion obtained by formal dualisation of that of a module) are completely reducible. We present several equivalent definitions that all extend classical representation-theoretic approaches to reductive groups such as the existence of a Peter-Weyl-type vector space basis of A or of a functional $h : A \to k$ which generalises integration over a compact group with respect to Haar measure. Woronowicz's compact quantum groups are special cosemisimple Hopf algebras over $k = \mathbb{C}$ that admit as algebras a completion to a C^*-algebra with h becoming a faithful state. In particular, $\mathbb{C}_q[G]$ is cosemisimple (for q not a root of unity) and for $q \in \mathbb{R}$ a compact quantum group.

At the end we shortly present some recent results about the relevance of Woronowicz's modular automorphism $\sigma \in \mathrm{Aut}(A)$, $A = \mathbb{C}_q[G]$, in Hochschild homology. This automorphism is defined by $h(ab) = h(\sigma(b)a)$, and if A_σ is the A-bimodule which is A as vector space with bimodule structure $a \triangleright b \triangleleft c := ab\sigma(c)$, then $H_{\dim(G)}(A, A_\sigma) \simeq \mathbb{C}$ in contrast to $H_n(A, A) = 0$ for $n > \mathrm{rank}(G)$ [2, 4, 5]. Thus twisting coefficients in Hochschild homology overcomes the "dimension drop" caused by quantisation.

There are many excellent textbooks on quantum groups and Hopf algebras available. For algebraic aspects see e.g. [1, 6], for the relation to Poisson groups and to knot theory [3] and for compact quantum groups [8].

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Dixmier traces and the noncomutative residue

Sylvie Paycha

This is a short summary of an introductory talk to the subject by a non expert.

Let H be a separable Hilbert space, A a compact operator on H, $\{\mu_n(A),\, n \in \mathbb{N}_0\}$ the spectrum of its modulus $|A|$. The operator A is trace-class whenever

$$\sigma_N(A) := \sum_{n=0}^{N} \mu_n(A)$$

converges as $N \to \infty$ in which case the trace of A is defined by $\mathrm{tr}(A) = \sum_{n=0}^{\infty} \lambda_n(A)$ with $\{\lambda_n(A),\, n \in \mathbb{N}\}$ the spectrum of A.
We are concerned here with the case $\sigma_N(A) = O(\log N)$; in particular, when $\frac{\sigma_N(A)}{\log N}$ converges then its limit corresponds to the Dixmier trace of A. However, building a **Diximier trace** [D] does not require convergence of the quotient $\frac{\sigma_N(A)}{\log N}$ as N goes to infinity but only of the Cesaro mean of an interpolation of this quotient; the limiting procedure can then further be weakened to build more general Dixmier traces.

Tauberian theorems are useful to go back and forth from the asymptotics of zeta functions and the trace of the heat semi-group to the asymptotics of $\sigma_N(A)$ [H]. More precisely, a pseudo-differential operator A of order minus the dimension of the underlying closed manifold has the property $\sigma_N(A) = O(\log N)$. In that case, if Δ is a Laplace operator on the manifold, the asymptotics of $\mathrm{tr}\left(A\, e^{-\epsilon\Delta}\right)$ as $\epsilon \to 0$ and of $\mathrm{tr}\left(A\, \Delta^{-z}\right)$ as $z \to 0$ relate via the **Wodzicki residue** $\mathrm{res}(A)$ see [C], [GVF]:

$$-\lim_{\epsilon \to 0} \frac{\mathrm{tr}\left(A\, e^{-\epsilon\Delta}\right)}{\log \epsilon} = \lim_{z \to 0} \left(z \cdot \mathrm{tr}\left(A\, \Delta^{-z}\right)\right) = \mathrm{res}(A).$$

A result by Alain Connes tells us that any elliptic classical operator A of order $-n$ on a closed n-dimensional manifold has a well defined Dixmier trace and

$$\mathrm{tr}_{\mathrm{Dixmier}}(A) = \frac{\mathrm{res}(A)}{n\,(2\pi)^n}.$$

General theorems which enable to compute the Dixmier trace in terms of the asymptotics of the zeta function and the trace of the heat semi-group were recently proved in [CRSS].

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The local index formula of Connes-Moscovici

Andrzej Sitarz

This talks reviews one of the applications of spectral triples, which is the explicit computation of Chern-Connes character of the Fredholm module using the Dirac operator and associated general pseudodifferential calculus.

Most of the definitions presented here are after [3, 4]. For simplicity we treat here only the odd case.

1. The problem

Let $\mathcal{A}$ be an associative algebra and $(\mathcal{A}, \mathcal{H}, F)$ be an odd Fredholm module, representing a K^1-homology class of $\mathcal{A}$. Provided that the Fredholm module is at least p-summable, the Chern-Connes character provides a formula for an odd n-cyclic cocycle $n \geq p$ (as explained by A. Rennie in his talk):

$$\phi(a_0, a_1, \ldots, a_n) = \frac{1}{2}\mathrm{Tr}\ F[F, a_0][F, a_1] \cdots [F, a_n].$$

With a suitable normalization, the cyclic cocycles ϕ_n give the same element of the periodic cyclic cohomology group of $\mathcal{A}$.

If the Fredholm module arises from a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $F = D|D|^{-1}$, the natural problem is: can we construct the Chern-Connes character ϕ using only D and its commutators with the elements of the algebra? A hint that this might be possible is the Hochschild cocycle formula, which states that for a p-summable spectral triple, the following gives a cocycle, which has the same Hochschild cohomology class as ϕ (however, it is not cyclic):

$$\psi(a_0, a_1, \ldots, a_p) = \frac{\Gamma(\frac{p}{2} + 1)}{p\, p!}\mathrm{Tr}_\omega \left(a_0[D, a_1] \cdots [D, a_p]|D|^{-p}\right),$$

where Tr_ω is the Dixmier trace.

2. The setup

To solve the problem we need to assume quite a lot about the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. For simplicity, we restrict to the case with D invertible, we assume that the degree of summability is $p > 0$. We say that the set $S \subset \mathbb{C}$ is the *dimension spectrum* of the spectral triple if the function $\xi_b(z) = \mathrm{Tr}\, b|D|^z$ has a holomorphic extension to $\mathbb{C} \setminus S$ for every $b \in \mathcal{B}$, where $\mathcal{B}$ is the algebra generated by $\delta^n(a)$, $a \in \mathcal{A}, n = 0, 1, \ldots$ and $\delta(a) = [|D|, a]$. We shall further assume that the dimension spectrum is discrete and the poles are at most of order k, for some fixed $k > 0$.

We define the algebra of generalized pseudodifferential operators $\Psi^*(\mathcal{A})$ as an algebra of operators having an expansion:

$$P \sim b_r|D|^r + b_{r-1}|D|^{r-1} + \cdots$$

where $b_i \in \mathcal{B}$. On this algebra we define the functionals τ_k, which extend the Dixmier trace functional to operators of arbitrary order:

$$\tau_k(P) = \mathrm{Res}_{z=0}\left(z^k \operatorname{Tr} P|D|^{-2z}\right).$$

Since the local formula for the Chern-Connes character gives a presentation of the cyclic cocycle as a cocycle in the Connes' (b, B) (normalized) bicomplex, we need to briefly remind its construction. It is the following bicomplex:

$$\begin{array}{ccccc}
 & \uparrow b & & \uparrow b & & \uparrow b & \\
\cdots & \tilde{C}^2(\mathcal{A}) & \xrightarrow{B} & \tilde{C}^1(\mathcal{A}) & \xrightarrow{B} & \tilde{C}^0(\mathcal{A}) \\
 & \uparrow b & & \uparrow b & & & \\
\cdots & \tilde{C}^1(\mathcal{A}) & \xrightarrow{B} & \tilde{C}^0(\mathcal{A}) & & & \\
 & \uparrow b & & & & & \\
 & \tilde{C}^0(\mathcal{A}) & & & & &
\end{array}$$

where $\tilde{C}^n(\mathcal{A})$ denotes the space of $n+1$ linear functionals from $\mathcal{A}$ to $\mathbb{C}$, such that $\phi(a_0, a_1, \ldots, a_n)$ vanishes whenever there exists $1 < i \leq n$ such that $a_i \in \mathbb{C}$. The map b is the Hochschild coboundary and B is the Connes map:

$$B\phi(a_0, a_1, \ldots, a_n) = \sum_{j=0}^{n} (-1)^{nj} \phi(1, a_j, a_{j+1}, \ldots a_n, a_0, \ldots, a_{j-1}).$$

The homology of this bicomplex is equal to the cyclic cohomology of $\mathcal{A}$, hence we can represent the cyclic cocycles as a collection on (b, B) complex cochains: a finite collection of elements (for the odd cocycle, similar formula is valid also in the even case): $(\Phi_1, \Phi_3, \Phi_5, \ldots)$, $\Phi_k \in \tilde{C}^k(\mathcal{A})$, which satisfy:

$$b\Phi_{2i+1} + B\Phi_{2i+3} = 0.$$

3. The formula

Assuming we have an (odd) p-summable spectral triple with discrete dimension spectrum, the following defines a cocycle in the (b, B)-bicomplex:

$$\phi_n(a_0, a_1, \ldots, a_n) = \sum_{q \geq 0} \sum_{k_j \geq 0} C_{n,q,k_j} \tau_q \left(a_0 (da_1)^{(k_1)} \cdots (da_n)^{(k_n)} |D|^{-(n+2\sum_j k_j)} \right),$$

where $da = [D, a]$, $P^{(0)} = P$ and $P^{(n)} = [D^2, P^{(n-1)}]$, and C_{n,q,k_j} are coefficients (of rather complicated form):

$$C_{n,q,k_j} = \sqrt{2\pi i}(-1)^{|k|} \frac{1}{k_1! k_2! \cdots k_n!} \frac{1}{(k_1+1)(k_1+k_2+2)+\cdots(k_1+k_2+\cdots+k_n+n)} \cdot \cdot \sigma_{|k|+\frac{n-1}{2}-q}(|k| + \tfrac{n-1}{2}),$$

where $\sigma_j(k)$ is given through:

$$\prod_{k=0}^{m-1}\left(z+k+\frac{1}{2}\right)=\sum_{j=0}^{m}\sigma_{m-j}(m)z^j.$$

The main theorem of [3] states that the formula gives indeed a cocycle, which is cohomologous to the Chern-Connes character of the associated Fredholm module.

4. The proof and applications

The original paper includes a proof of a part of the formula but for a thorough review and a different proof we recommend [5, 6]. Independent proof (in a slightly different context of von Neumann algebras) and applications are given in [1, 2].

The formula was first derived in the problem of index computation for foliations, it was then used in several explicit computations in noncommutative geometry ([8] for example). For relations between the Connes-Moscovici and Atiyah-Singer index theorem see [7].

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On spectral actions

Bruno Iochum

During the talk, based on collaborations with V. Gayral, C. Levy, A. Sitarz and D. Vassilevich, different aspects of the Connes–Lott action and Chamseddine–Connes spectral action for spectral triples are developed. New results concerning the second action for the noncommutative torus and the triple associated to $SU_q(2)$ are presented.

Since the beginning of noncommutative geometry [6, 15], the notion of action, essential in physics, has had two main evolutions. The first proposed by Connes and Lott [8] was based on the following formula, associated to a spectra triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ of dimension d:

$$YM(\alpha)=\inf_{\eta}\{\ \mathrm{Tr}_{\mathrm{Dixmier}}\big(F^*F|\mathcal{D}|^{-d}\big)\ :\alpha=\pi(\eta)\ \}.$$

Here, π is the representation of $\mathcal{A}$ on $\mathcal{H}$, η is a one-form and $F=\delta\eta+\eta^2$ is its curvature where δ is a formal derivation on $\mathcal{A}$ represented on $\mathcal{H}$ by $\delta(a)\to[\mathcal{D},\pi(a)]$, $a\in\mathcal{A}$. In the commutative case, $YM(\alpha)$ is the usual Yang–Mills action. The point is that, to get a graded differential algebra after the representation, one has to divide by an ideal, so for explicit computations, one has to control that "junk". Moreover the use of the Dixmier trace implies that only the leading term of the poles is concerned. The second step was the appearance of the spectral action proposed by Chamseddine and Connes [3]

$$\mathcal{S}(\mathcal{D},\Lambda,\Phi):=\mathrm{Tr}\big(\Phi(|\mathcal{D}|/\Lambda)\big),$$

where Φ is a positive function and Λ is a scale used for a cut-off purpose. The idea is that if Φ looks like the step function, then this counts the number of eigenvalues of $|\mathcal{D}|$ less than Λ. Its main interest is of course that it depends only of the spectrum of the Dirac operator $\mathcal{D}$, an interesting fact since there exist isometric non isospectral manifolds. It is of course gauge invariant.

Since it is based on a heat kernel approach, it is important to get precise constraints on Φ for applying right holomorphic extensions. It is also possible to a distributional approach like in [10]. However, if one considers the circle of ideas stemming from the heat kernel approach, the zeta function approach or the Dixmier-trace approach, one may note few differences on the hypothesis sufficient to use one of these entrances.

Connes introduced in [7] the notion of spectrum dimension $Sd(\mathcal{A},\mathcal{D},\mathcal{H})$ of a spectral triple by the singularities of the function $\zeta_b(s):=\mathrm{Tr}(b|\mathcal{D}|^{-s})+\mathrm{Tr}(pb)$ when b is in the set of pseudodifferential operators generated by $\mathcal{A}$, $\delta^n(\mathcal{A})$, $\delta^n([\mathcal{D},\mathcal{A}])$ for all $n\in\mathbb{N}$ or powers of $|\mathcal{D}|$, where $\delta(.)=[|\mathcal{D}|,.]$ and p is the projection on the kernel of b. Few remarks:

- When M is a Riemannian compact spin manifold, then

$$Sd(\mathbb{C}^\infty(M),\mathcal{L}^2(M,\text{Spinor bundle}),\text{Dirac operator})\subset\{\ 1,\cdots n\ \}.$$

- It makes sense to define $⨍\, b:=\mathrm{Res}_{s=0}\,\zeta_b(s)$ and $⨍$ is a trace on pseudodifferential operators.

- A way to check that an operator is Dixmier traceable is the following [1]: If T is a positive bounded operator on $\mathcal{H}$ such that T^s is trace-class for $s > 1$ and $l = \lim_{0<\epsilon\to 0} \epsilon \mathrm{Tr}(T^{1+\epsilon})$ exists, then T is measurable and $\mathrm{Tr}_{\mathrm{Dixmier}}(T) = l$.

- The action is $\mathcal{S}(\mathcal{D}, \Lambda, \Phi) \sim \sum_{0\le k\in Sd} \Phi_{n-k} \fint |\mathcal{D}|^{-k} \Lambda^k + \Phi(0)\zeta_{\mathcal{D}}(0) + o(1)$.

Example 1: $\mathcal{A} = \mathcal{A}_\Theta$ is the noncommutative n-torus associated to a skew-symmetric deformation matrix Θ with the Hilbert space $\mathcal{H}$ related to the trace τ on $\mathcal{A}_\Theta$ and the Dirac operator based on the natural derivations δ_μ, $\mu \in \{\, 1 \cdots n \,\}$ such that $\delta_\mu U_k = ik_\mu U_k$ where $\mathcal{A}$ is generated by the unitaries U_k satisfying $U_k U_q = e^{ik.\Theta q} U_q U_k$. The natural reality operator J is associated to $J_0(a) = a^*$. For a selfadjoint one-form $A = \sum_i a_i[\mathcal{D}, b_i]$, a_i, $b_i \in \pi(\mathcal{A})$ (π is the representation on $\mathcal{H}$), the fluctuated Dirac operator is $\mathcal{D}_A := \mathcal{D} + A + JAJ^{-1} = -i\big(\delta_\mu + L(A_\mu) - R(A_\mu)\big) \otimes \gamma^\mu$, with $A_\mu^* = -A_\mu \in \mathcal{A}$ and the γ matrices are selfadjoint.

Using the result $\zeta_{\mathcal{D}_A}(0) - \zeta_D(0) = \sum_{k=1}^n \frac{(-1)^k}{k} \fint (A\mathcal{D})^k$ obtained in [4] and [11], we get by a heat kernel expansion that the spectral action in dimension $n = 4$ is

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 4\pi^2\Phi_0\Lambda^4 - \tfrac{4\pi^2}{3}\Phi(0)\tau(F_{\mu\nu}F^{\mu\nu}) + o(1)$$

where $\Phi_0 := \frac{1}{2}\int_0^\infty x\,\Phi(x)\,dx$ and $F_{\mu\nu} = \delta_\mu(A_\nu) - \delta_\nu(A_\mu) + [A_\mu, A_\nu]$. This result [16] is the same as for the classical 4-torus and was obtained in [14] but as a by-product of more general results under the assumption that Θ satisfies a Diophantine condition.

It is interesting to quote that some of the techniques has been extended to non-compact manifolds [13].

Example 2: The spectral triple introduced in [9, 17] is based on the quantum $SU(2)$: Let $\mathcal{A} = \mathcal{A}(SU_q(2))$ be the $*$-algebra generated polynomially by a and b, subject to the following commutation rules:

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1.$$

with $0 < q < 1$. The spinorial Hilbert space $\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$ has an orthonormal basis consisting of vectors $|j\mu n\uparrow\rangle$ for $j = 0, \frac{1}{2}, 1, \ldots$, $\mu = -j, \ldots, j$ and $n = -j^+, \ldots, j^+$; together with $|j\mu n\downarrow\rangle$ for $j = \frac{1}{2}, 1, \ldots$, $\mu = -j, \ldots, j$ and $n = -j^-, \ldots, j^-$ (here $x^\pm := x \pm \frac{1}{2}$). Using the vector notation $|j\mu n\rangle\!\rangle := (|j\mu n\uparrow\rangle, |j\mu n\downarrow\rangle)$, with the convention that the lower component is zero when $n = \pm(j + \frac{1}{2})$ or $j = 0$, the Dirac operator is chosen the same as in the classical case of a 3-sphere:

$$\mathcal{D}|j\mu n\rangle\!\rangle = \begin{pmatrix} 2j + \frac{3}{2} & 0 \\ 0 & -2j - \frac{1}{2} \end{pmatrix} |j\mu n\rangle\!\rangle.$$

It is sufficient to use the approximate spinorial representation $\underline{\pi}$ of $SU_q(2)$ presented in [17, 9] since all disregarded terms are trace-class and do not influence residue calculations. Moreover, we may replace $\mathcal{D}$ by $|\mathcal{D}|$.

Here $Sd = \{\, 1, 2, 3 \,\}$ so the behavior is totally different from Example 1. First, there exist non vanishing tadpoles $\Psi_1(A) := \fint A\mathcal{D}^{-1}$. In fact, Ψ_1 is a cyclic cocycle with a nontrivial pairing with the generator of K_1 group. One also get
$\fint A_1\mathcal{D}^{-1}A_2\mathcal{D}^{-1}A_3\mathcal{D}^{-1} = \fint A_1A_2A_3\mathcal{D}^{-3}$ or
$\fint a_0[\mathcal{D}, a_1]\mathcal{D}^{-1}a_2[\mathcal{D}, a_3]\mathcal{D}^{-1} = \fint a_0[\mathcal{D}, a_1][\mathcal{D}, a_2[\mathcal{D}, a_3]\mathcal{D}^{-3}$

$$+⨍ a_0[\mathcal{D}, a_1]a_2\mathcal{D}^{-1}[\mathcal{D}, a_3]\mathcal{D}^{-1}.$$

More generally, the inner fluctuations gives for the scale-invariant part of the spectral action given by a universal 3-form $A = a_0 da_1 da_2 da_3$:

$$\zeta_{\mathcal{D}+A}(0) - \zeta_{\mathcal{D}}(0) = \tfrac{1}{2}\int_{\Psi_3}(AdA + \tfrac{2}{3}A^3) + \int_{\Psi_2} A^2 - \int_{\Phi_2} A^2 - \Psi_1(A)$$

with $\Psi_3(a_0, a_1, a_3) = ⨍ a_0[\mathcal{D}, a_1][\mathcal{D}, a_2][\mathcal{D}, a_3]\mathcal{D}^{-1}$, $\Psi_2 = ⨍ a_0[\mathcal{D}, a_1][\mathcal{D}, [\mathcal{D}, a_2]]\mathcal{D}^{-2}$, and $\Phi_2(a_0, a_1, a_2) = ⨍ a_0[\mathcal{D}, a_1][\mathcal{D}, a_2]\mathcal{D}^{-2}$. The first term is of course the Chern-Simons term.

Example 3: Different applications of the spectral action in particle physics can be found since a long time in [2, 3]. A new extension has been made recently in [5] for a case probably appropriate to Lorentzian geometry.

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Some examples of noncommutative spectral geometries

Ludwik Dąbrowski

During the talk the definition of noncommutative spectral triples was recalled together with some additional seven properties (dimension, regularity, projectivity, reality, first order, orientability, Poincare duality) which underlie the concept of noncommutative Riemannian spin geometry due to Alain Connes. Moreover the notion of equivariance under a Hopf algebra symmetry was described.

Next, few examples of equivariant (compact) spectral triples were presented. The examples of noncommutative tori and θ-deformed manifolds (which are equivariant under the torus action) satisfy all the seven properties mentioned above, while the underlying manifold of quantum $SU(2)$ (that has $U_q(su(2)) \otimes U_q(su(2))$ equivariance), quantum Podleś spheres ($U_q(su(2))$-equivariant) and quantum Euclidean S^4 ($U_q(so(5))$-equivariant) satisfy some of them (dimension, regularity) and a modified version of the reality and first order conditions (up to the ideal of infinitesimals of arbitrary order).

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Part III: Research talks

Metrics with few harmonic spinors

Bernd Ammann
(joint work with M. Dahl and E. Humbert)

Let M be a fixed compact manifold with spin structure, $n = \dim M$. For any Riemannian metric g on M we denote by D^g the Dirac operator and by k_g the (complex) dimension of the kernel of D^g acting on complex spinors. The number k_g is finite because of the ellipticity of D^g.
The Atiyah-Singer index theorem provides a lower bound on k_g, namely

$$k_g \geq \begin{cases} |\widehat{A}(M)|, & \text{if } n \equiv 0 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Hence we have a lower bound on k_g in terms of a topological quantity, that only depends on the differential structure of M in the case $n \equiv 0 \mod 4$ and only depends on the differential structure and the spin structure of M in the cases $n \equiv 1$ and $n \equiv 2 \mod 8$.
Motivated by comparing D^g to other generalized Dirac operators as the Gauss-Bonnet-Chern operator, the signature operator or the Dolbeault operator, it is natural to ask whether the dimension k_g has topological significance on its own. Hitchin [8, Prop. 1.3] proved that k_g is a conformal invariant, i.e. $k_g = k_{\tilde{g}}$ for conformal metrics g and $\tilde{g}$. However, k_g depends on the conformal class on M.
To start with an example, let M be a Riemann surface with a spin structure. Because of the quaternionic structure of the spinor bundle which commutes with the Dirac operator, we know that k_g is even, and one can show $k_g/2 \mod 2 = \alpha(M)$. Furthermore, one knows that on a Riemann surface of genus γ there is the bound $k_g \leq \gamma + 1$ [8, 2.1, Remark (4)]. If there is a spin 3-manifold with boundary W such that $\partial W = M$ in the sense of spin manifolds, then $\alpha(M) = 0$; we say M is a spin-boundary, or the spin structure is spin-bounding. If W is not a spin-boundary, one can show that $\alpha(W) = 1 \in \mathbb{Z}/2\mathbb{Z}$. The unique spin structure on S^2 is the boundary of the 3-disk, all other compact Riemann surfaces carry spin-bounding and non-spin-bounding spin structures. Now it is clear that in the spin-bounding case, $k_g = 0$ for $\gamma \leq 2$ and all possible metrics g. Similarly, in the non-spin-bounding case we deduce $k_g = 2$ for $\gamma \in \{1, 2, 3, 4\}$. In all other cases of Riemann spin surfaces there are metrics with large k_g, see [5], but because of the main result of this talk, there are also Riemann surface with $k_g \in \{0, 2\}$, see [9].
We visualize this phenomenon for a Riemann surface M of genus 4. Karsten Große-Brauckmann has constructed a smooth family $F_t : M \to \mathbb{R}^3/\mathbb{Z}^3$, $t \in (-\epsilon, \epsilon)$ of embeddings such that $F_t(M)$ has constant mean curvature h_t, $h_0 = 0$, and $dh_t/dt > 0$, see Figure 1. Furthermore $F_t(M)$ separates the torus $\mathbb{R}^3/\mathbb{Z}^3$ into two

connected components and is the boundary of each of them. The spin structure on M induced from the embedding F_t is hence spin-bounding. Let g_t be the metric induced from F_t, i.e. the maps F_t are isometric. The spinorial Weierstrass correspondence of Kusner-Schmitt in the form of [3] tells us that we can restrict a parallel spinor on $\mathbb{R}^3/\mathbb{Z}^3$ of constant length 1 to $F_t(M) \cong M$, and this restricted spinor $\phi_t \in \Gamma(\Sigma^{g_t} M)$ satisfies $D^{g_t}\phi_t = h_t\phi_t$, $|\phi_t| \equiv 1$. We see $k_{g_0} \neq 0$, and for $t \to 0$ at least one eigenvalue of D^{g_t} converges to 0 which forces k_{g_t} to jump at $t = 0$.
On the other hand we know from the above considerations that k_{g_t} is either 4 or 0. Hence, it is clear that $k_{g_0} = 4$ and $k_{g_t} = 0$ for $t \neq 0$ close to 0. And we have now seen that k_g depends on the conformal structure in this example.

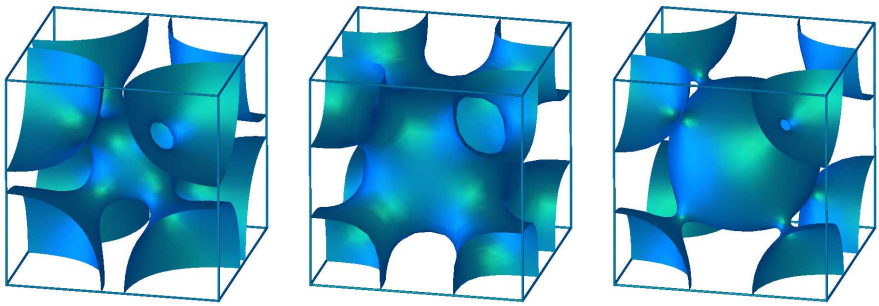

FIGURE 1. The surface $F_t(M)$ for $t < 0$, $t = 0$ and $t > 0$. We thank Karsten Große-Brauckmann for producing these images.

The same arguments would also apply for genus 3 by considering similar deformations of the Schwarz minimal surface.
Other examples for the dependence of k_g on the conformal class are manifolds of dimension $1, 3, 7, 0 \mod 8$ that admit positive scalar curvature metrics. On these manifolds we have $k_{g_1} = 0$ for the positive scalar curvature metric, but on the other hand Hitchin [8] and Bär [2] proved the existence of a metric g_2 with $k_{g_2} > 0$.
These examples suggest to conjecture two statements (for $n \geq 3$).

- For any compact spin manifold M and any $N \in \mathbb{N}$ there is a metric g with $k_g \geq N$.
- For generic metrics g on M one has equality in inequality (1).

Important progress was done during the recent years to confirm cases of the first item [8],[2],[10],[6].
Our main result here is that the second item holds true. In the following we call a metric *D-minimal* if we have equality in (1).
Theorem A (D-minimality theorem, [1]).
Generic metrics on connected compact spin manifolds are D-minimal.
In this theorem, "generic" means that the set of all D-minimal metrics is dense in the C^∞-topology and open in the C^1-topology.

The proof of the theorem has a quite rich and interesting history. As remarked by S. Maier in [9], it suffices to construct one D-minimal metric on M in order to prove the theorem. In the same article Maier used perturbation methods to construct such a D-minimal metric on manifolds on manifolds of dimension at most 4.

The construction of a D-minimal metric is strongly linked to the question of constructing metrics of positive scalar curvature. Important progress in the construction of metrics with positive scalar curvature was achieved by Gromov-Lawson [7] and Stolz [11]. Using techniques from surgery theory, Gromov and Lawson showed that any connected, simply connected non-spin manifold of dimension at least 5 carries a metric of positive scalar curvature. Later Stolz combined these surgery techniques with some very delicate spectral sequence arguments in order to conclude that a connected, simply connected spin manifold of dimension at least 5 carries a metric of positive scalar curvature if and only if the right hand side of inequality (1) vanishes. The Lichnerowicz formula implies that any metric g with positive scalar curvature has $k_g = 0$, and hence it is D-minimal.

A key step in Gromov-Lawson's approach is to prove that if M carries a metric with positive scalar curvature and if $M^\#$ arises from M by surgery of codimension at least 3, then $M^\#$ carries a metric of positive scalar curvature as well. This surgery statement is no longer true for surgery of codimension 2.

A similar surgery theory was then built by C. Bär and M. Dahl [4] for D-minimal metrics, in order to prove the D-minimality theorem for all connected, simply connected manifolds of dimension at least 5. A key step in their article is to prove the following: If M carries a D-minimal metric and if $M^\#$ arises from M by surgery of codimension at least 3, then $M^\#$ carries a D-minimal metric as well. As scalar curvature is local invariant whereas D-minimality is a global invariant, essential steps in the analysis had to be redevelopped. Some other fundamental groups can be treated with the same method.

The breakthrough of [1] is based on a construction, that shows the following theorem:

Theorem B *If M carries a D-minimal metric and if $M^\#$ arises from M by surgery of codimension 2, then $M^\#$ carries a D-minimal metric as well.*

The fact that we can control surgery in codimension 2 admits much stronger conclusions. In particular, if M and N are compact spin-manifold that are spin-bordant, and if N is connected, then the existence of a D-minimal metric on M implies the existence of such a metric on N. In fact, using standard methods in surgery theory, one can simplify a given bordism between M and N into elementary bordisms, that correspond to surgeries in dimensions $0, 1, 2, \ldots, n-2$. Hence starting with a D-minimal metric on M one can successively construct a D-minimal metric on the manifolds after the corresponding surgeries, and finally obtain a D-minimal metric on N.

What remains to show is that any spin bordism class contains a representative with a D-minimal metric. This will be explained here by following arguments of Bär and Dahl in [4].

The first step amounts to finding D-minimal metrics in the following cases

- $M = S^1$ with the non-spin-bounding spin structure $\alpha(M) \neq 0$: the standard metric,
- $M = S^1 \times S^1$ with the non-spin-bounding spin structure $\alpha(M) \neq 0$: the standard metric,
- M is a $K3$ surface ($n = 4$, $\widehat{A}(M) = 2$): a Calabi-Yau metric,
- M is a Bott manifold B ($n = 8$, $\widehat{A}(M) = 1$): a metric with holonomy Spin(7).

Let g_1 (resp. g_2) be D-minimal metrics on M_1 (resp. M_2). The product metric on $M_1 \times M_2$ and the disjoint union metric on $M_1 \dot\cup M_2$ is not always D-minimal, but one easily sees the following:

- If M_1 and M_2 have teh same dimension, if this dimension is divisible by 4, and if $\widehat{A}(M_1) > 0$ and $\widehat{A}(M_2) > 0$, then the disjoint sum $(M_1 \dot\cup M_2, g_1 \dot\cup g_2)$ is D-minimal.
- If M_2 is the Bott manifold, then $(M_1 \times M_2, g_1 \times g_2)$ is D-minimal as well.
- $(-M_1, g_1)$ is D-minimal, where $-M_1$ denotes M_1 with the opposite orientation and the same spin structure.

By induction one obtains a list $\mathcal{L}$ of manifolds with D-minimal metrics.
Now, given an arbitrary connected compact spin manifold M, it is easy to find a manifold $P \in \mathcal{L}$ such that

- $\widehat{A}(M \dot\cup - P) = 0$ if $n \equiv 0 \mod 4$,
- $\alpha(M \dot\cup - P) = 0$ if $n \equiv 1, 2 \mod 8$,
- $P = \emptyset$ otherwise

It follows from a result of Stolz [11] that the manifold $M \dot\cup - P$ is spin-bordant to a manifold T which is the total space of a bundle whose fiber is the 2-dimensional quaternionic projective space. In particular, T carries a metric with positive scalar curvature. This defines a D-minimal metric on $T \dot\cup P$. The bordism from T to $M \dot\cup - P$ can also be seen as a bordism from $T \dot\cup P$ to M, and hence our considerations above imply the existence of a D-minimal metric on M.
More details can be found in [1] and on the website
http://www.berndammann.de/publications.

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Noncommutative Einstein manifolds

Christian Bär

The first part of the talk is devoted to a question that was raised in a problem session of the workshop. To a compact Riemannian spin manifold M one can associate the Fredholm module $(C^\infty(M), L^2(M,\Sigma M), \mathrm{sign}(D))$ where $C^\infty(M)$ is the pre-C^*-algebra of smooth functions, $L^2(M,\Sigma M)$ is the Hilbert space of square integrable spinor fields, and $\mathrm{sign}(D)$ is the sign of the Dirac operator, $\mathrm{sign}(t) = t/|t|$ for $t \neq 0$ and $\mathrm{sign}(0) = 0$. We show

Theorem. *Let M be a compact manifold with a fixed spin structure. Let g and g' be two Riemannian metrics on M. Then the associated Fredholm modules are weakly unitarily equivalent if and only if the metrics are conformally related, i. e., there exists a smooth function $u \in C^\infty(M)$ such that $g' = e^{2u} \cdot g$.*

Here "weakly unitarily equivalent" means that there exists a unitary isomorphism $U : L^2(M,\Sigma_g M) \to L^2(M,\Sigma_{g'} M)$ equivariant with respect to the action of $C^\infty(M)$ such that $\mathrm{sign}(D_{g'}) - U \circ \mathrm{sign}(D_g) \circ U^{-1}$ is a compact operator.

In the second part of the talk we propose a concept of noncommutative Einstein spaces. This is based on the thesis [1]. A Riemannian manifold (M,g) is called *Einstein* if there is a constant $\lambda \in \mathbb{R}$ such that the Ricci curvature satisfies

$$\mathrm{Ric} = \lambda \cdot g.$$

This equation makes, as it stands, no sense for spectral triples because of the lack of a concept of Ricci curvature. But Einstein metrics can also be characterized as being critical for the Einstein-Hilbert functional

$$S_{EH}(g) = \int_M \mathrm{Scal}_g(x)\, \mathrm{dvol}_g(x)$$

among all metrics of fixed volume. Now volume and total scalar curvature are precisely the coefficients of the first two terms in the asymptotic expansion of $\mathrm{Tr}(e^{-tD^2})$. More precisely, with $n = \dim(M)$,

$$\mathrm{Tr}(e^{-tD^2}) = 2^{[n/2]} \cdot (4\pi t)^{-n/2} \cdot \left(\mathrm{vol}(M,g) - \frac{1}{12} \cdot S_{EH}(g) \cdot t + \mathrm{O}(t^2) \right)$$

as $t \searrow 0$. This suggests to look at spectral triples $(\mathcal{A}, \mathcal{H}, D)$ having an asymptotic expansion

$$\mathrm{Tr}(e^{-tD^2}) = (4\pi t)^{-n/2} \left(a_0 + a_1 \cdot t + \mathrm{O}(t^2) \right)$$

as $t \searrow 0$ and then to call it Einstein if the a_1-term is stationary under all variations of permissible D for which a_0 is constant. We propose precise definitions of such variations. We show that a 0-dimensional spectral triple is Einstein if and only if $D = 0$. We also show that the noncommutative 3-torus is Einstein.

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Twisted spectral triples and covariant differential calculi

Ulrich Krähmer

For an abstract algebra A there are no general methods known that allow to construct or classifiy spectral triples (A, H, D) over A. However, one natural invariant is the derivation $\mathrm{d} : A \to \Omega^1 \subset \mathrm{End}(H)$, $a \mapsto [D, a]$, where $\Omega^1 := \{\sum_i a_i \mathrm{d} b_i\}$ is the smallest A-bimodule containing $\mathrm{im}\,\mathrm{d}$, or, when considering spectral triples up to "gauge transformations" $D \mapsto D + \omega$, $\omega \in \Omega^1$, its class [d] in first Hochschild cohomology $H^1(A, \Omega^1)$ (all derivations $A \to \Omega^1$ modulo inner ones).

For the canonical spectral triple over a compact Riemannian spin manifold M, Ω^1 is isomorphic to the bimodule of 1-forms over $A = C^\infty(M)$, and $\mathrm{d}a$ becomes identified with the differential of a function that acts by Clifford multiplication on spinor fields. However, for general A there is no canonical choice of Ω^1.

If A is a Hopf algebra, then it is natural to study spectral triples that take into account the canonical coaction of A on itself given by its coproduct Δ, just as one studies first of all left-invariant metrics on Lie groups. Thus Ω^1 is required to be as well an A-comodule and d to be A-colinear. Under the obvious compatibility assumptions with the bimodule structure Woronowicz called the resulting data (Ω^1, d) a covariant differential calculus over A.

As shown by Woronowicz, the category of such calculi is equivalent to the category of surjective derivations with values in right modules, considered as bimodules with trivial left action $a \triangleright \omega := \varepsilon(a)\omega$ (where ε is the counit of A). Under this equivalence, our cohomology class [d] becomes identified with a class in $\mathrm{Ext}^1_A(k, \Omega^1_{\mathrm{inv}})$. Here k is considered as trivial right A-module.

The main topic in this talk is to review the classification of such calculi for $A = \mathbb{C}_q[SL(2)]$ due to Hecknberger [2] and to point out that any claculus with $\dim_k \Omega^1_{\mathrm{inv}} < \infty$ over any Hopf algebra with bijective antipode can be realised in the form $\mathrm{d}a = D\sigma_+(a) - \sigma_-(a)D$, where $\sigma_\pm$ are two representations on a vector space H and $D \in \mathrm{End}(H)$. If A is a compact quantum group, then H can be completed to a Hilbert space representation of the corresponding C^*-algebra, and the differentials are given by bounded operators. That is, covariant differental calculi essentially lead to twisted spectral triples as studied in [1].

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Lorentzian spectral triples

Mario Paschke
(joint work with Adam Rennie, Andrzej Sitarz and Rainer Verch)

There are several strong heuristic arguments in physics which indicate that spacetime is not a classical manifold but rather a noncommutative space. It therefore seems desirable to define quantum field theories over spectral triples. However, spectral triples only generalize the notion a compact Riemannian manifolds, i.e. compact manifolds with a positive definite metric tensor, but for the physical interpretation of quantum field theory it is essential that the underlying geometry is Lorentzian and noncompact, with signature $n-1$ in $d=n+1$ dimensions, say. I shall describe below how one may carry over the notion of spectral triples to (noncommutative) spaces with a Lorentzian signature.

On a Lorentzian Spin manifold $(\mathcal{M},g)$ the natural scalar product (ψ,ϕ) on sections ψ,ϕ of the spinor bundle, being invariant under the Lorentzian Spin group, is indefinite. If $(\mathcal{M},g)$ is time-orientable, however, on may define a positive definite scalar product $\langle\psi,\phi\rangle_\beta := (\psi,\beta\phi)$, where the operator β is given by Clifford-multiplication with a chosen timelike one-form. In particular $\beta^*=-\beta$ and $\beta^2=-1$. The Hilbert space $\mathcal{H}_\beta$ of square integrable sections of the spinor bundle is then constructed with respect to $\langle\psi,\phi\rangle_\beta$.
The closure of the Lorentzian Dirac-Operator D is then β-symmetric on $\mathcal{H}_\beta$, i.e. $D^*_\beta=\beta D\beta$ on[1] $\mathrm{dom}D=\beta\mathrm{dom}D^*_\beta$. We should stress that D^*_β strongly depends on the choice of β. In particular, the spectrum of D on H_β in general differs for different choices of β. Nevertheless we shall conveniently drop the suffix β in what follows.

A **(real, even) Lorentzian spectral triple** is given as a collection $(\mathcal{A}_c\subset\mathcal{A}_2\subset\tilde{\mathcal{A}},\mathcal{H},D,\beta,J,\gamma)$, in particular subject to the conditions given above. The algebra $\tilde{\mathcal{A}}$ is a unitalization of the pre-C^*-algebra $\mathcal{A}_c$. In the commutative case, $\mathcal{A}_c$ is given as the algebra of smooth functions of compact support, $\mathcal{A}_c=C^\infty_c(\mathcal{M})$, while $\mathcal{A}_2$ is the algebra of smooth square integrable functions, such that all their derivatives are square-integrable.$\tilde{\mathcal{A}}$ is then taken as the algebra of smooth bounded functions, with all their derivatives beeing bounded.

[1]It is a pleasure to thank Helga Baum for many valuable discussion concerning the domain issues for Lorentzian Dirac-Operators during the workshop.

In order to formulate the axioms of dimension and regularity one uses the operator

$$\langle D\rangle := \sqrt{\frac{1}{2}(DD^* + D^*D)}$$

The other axioms are then adjusted in such a way that the following holds:
Lemma [1] *Given any real even Lorentzian spectral triple*

$$(\mathcal{A}_c \subset \mathcal{A}_2 \subset \tilde{\mathcal{A}}, \mathcal{H}, D, \beta, J, \gamma)$$

the data $(\mathcal{A}_c \subset \mathcal{A}_2 \subset \tilde{\mathcal{A}}, \mathcal{H}, D_E, J_E, \gamma)$ *define a (nonunital) real even spectral triple, with*

$$D_E := \frac{1}{2}(D + D^*) + \frac{i}{2}(D - D^*).$$

Using this result one then also proves the
Reconstruction Theorem [2] *Given an irreducible real even Lorentzian spectral triple of finite dimension for a commutative algebra* $\tilde{\mathcal{A}}$*, one can reconstruct a uniquely specified time-oriented smooth Lorentzian spin manifold* $(\mathcal{M}, g, \sigma)$*.*
Vice versa, for any time-orientable smooth Lorentzian spin manifold $(\mathcal{M}, g, \sigma)$ *without boundary,* $\partial\mathcal{M} = \emptyset$*, one may construct – albeit not uniquely – a real even Lorentzian spectral triple.*

Note that one may, upon replacing the triple $\mathcal{A}_c \subset \mathcal{A}_2 \subset \tilde{\mathcal{A}}$ by a single unital algebra $\mathcal{A}$ also define Lorentzian spectral triples for unital algebras, corresponding to compact manifolds in the commuative case. In [3] we gave several noncommutative examples. However, just as in the classical case, there is a topological obstruction to the existence of Lorentzian spectral triples for a unital algebra $\mathcal{A}$:
Theorem [1] *Suppose* $(\mathcal{A}, \mathcal{H}, D, \beta, J, \gamma)$ *is a unital Lorentzian spectral triple, fullfilling Poincaré duality in K-theory, i.e the map*

$$\cdot \otimes_{\mathcal{A}} \mu : K_*(\mathcal{A}) \to K^{*+p}(\mathcal{A})$$

is an isomorphism. Here $\mu = [(\mathcal{A} \otimes \mathcal{A}^{op}, \mathcal{H}, D_E)] \in KK^p(\mathcal{A} \otimes \mathcal{A}^{op}, \mathbb{C})$.
Then $\chi(\mathcal{A}) := \mathrm{rank}K_0(\mathcal{A}) - \mathrm{rank}K_1(\mathcal{A}) = 0$.

This result then implies, for commutative $\mathcal{A}$ the vanishing of the Euler characteristic of the spectrum of $\mathcal{A}$.

The idea of the proof of the above theorem is that one may use β to map even Fredholm-modules, i.e. representatives for the K-homology classes of $\mathcal{A}$, to odd Fredholm modules and vice versa. To this end one splits the operator $F_E = \frac{D_E}{|D_E|}$ as

$$F_E = F_c + F_a = -\frac{1}{2}\beta\{\beta, F_E\} - \frac{1}{2}\beta[\beta, F_E]$$

.
Lemma: *Let* $(\mathcal{A}, \mathcal{H}, D, \beta)$ *be a Lorentzian spectral triple and* $(\mathcal{A}, \mathcal{H}, D_E)$ *the corresponding spectral triple. The following formulae define a Fredholm module* $(\mathcal{A}, \tilde{\mathcal{H}}, \tilde{F_E})$ *of opposite parity:*

$$\tilde{\mathcal{H}} = \mathcal{H},\ \tilde{F_E} = i\beta F_c\gamma + \beta F_a\gamma, \quad (\mathcal{A}, \mathcal{H}, D, \beta, \gamma)\ \textit{even}$$

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H},\ \tilde{F_E} = \begin{pmatrix} F_a & F_c \\ F_c & F_a \end{pmatrix}, \quad (\mathcal{A}, \mathcal{H}, D, \beta)\ \textit{odd},$$

with the grading in the latter case given by $i\beta \oplus (-i\beta)$.

Let us now return to the nonunital case. For physical models it is essential that the Cauchy-problem for the Lorentzian-Dirac operator is well posed. This is the case on globally hyperbolic manifolds (but not on all Lorentzian manifolds). In particular, globally hyperbolic manifolds are of the form $\mathcal{M} = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ can be viewed as the "time-axis".

Definition: Timelike foliated Losts *We call a Lorentzian spectral triple timelike foliated if the operator*

$$\partial_\beta := \frac{1}{2}(D\beta + \beta D)$$

is essentially selfadjoint and if there exists a family of unitary elements

$$\{u^\kappa \in \tilde{\mathcal{A}},\ \ \kappa \in \mathbb{R}\}$$

$u^* = u^{-1}$ *such that*

$$\beta = u^*[D, u].$$

It then follows under rather mild additional assumptions that ∂_β is a derivation on the algebra $\tilde{\mathcal{A}}$. In particular, this is always true for commutative algebras. Note that the requirement that u^κ lies in the algebra for all $\kappa \in \mathbb{R}$ ensures that the spectrum of ∂_β is all of $\mathbb{R}$. Also, it implies the existence of a symmetric, unbounded operator t, affiliated to the algebra, such that $u = e^{it}$, i.e. t might be viewed as "time-coordinate".

In a joint project with Rainer Verch we try to use the one-paramter automorphism groups generated by the ∂_β for the various choices of β, to construct fundamental solutions for the Dirac-operator. We also hope that one may develop a noncommutative version of microlocal analysis with their help.

Finally I would like to thank the organizers for the opportunity to present these results in this workshop, where I learnt a lot that may help me with the projects described above. I enjoyed the open and cordial atmosphere very much.

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(Second) Quantised regularised traces

Sylvie Paycha

Ordinary ζ-regularised traces have been the object of many investigations (see e.g. the works of Grubb and Seeley [GS], Kontsevich and Vishik [KV], Melrose and Nistor [MN], Lesch [Le], and more recent works by Grubb [G1], [G2], [G3], as well as recent papers by Scott and the author [PS1, PS2]). Regularised traces naturally arise in the study of variations of partitions functions in quantum field

theory and provide useful tools in the context of anomalies (see e.g. [BF], [LMR] and [CDP]). They also occur in the framework of index theory, specifically in the local index formula of Connes and Moscovici (from whom we borrow some of the techniques used in this paper) in noncommutative geometry [CM], in the fractional index theory of Mathai, Melrose and Singer [MMS] as well as in the family index theorem see e.g. [Sc], [PS1], [MP].

Whereas regularised traces are not expected to be local [PS2], their variations are. Viewing regularised traces as quantised regularised traces, namely as higher order cochains on the algebra of classical pseudo-differential operators, sheds light on this fact, combining two observations:

(1) variations of regularised traces of level n are regularised traces of level $n+1$,
(2) quantised traces of positive level (i.e. positive order cochains) are local.

The well-known locality property of anomalies for ordinary ζ-regularised traces [KV], [MN], [CDMP], [CDP] then arises as a consequence of the locality of quantised traces of positive level.

Whereas ζ-regularisation and heat-kernel regularisation lead to the same regularised traces only on operators with vanishing noncommutative residue, higher quantised ζ-regularised traces coincide with higher quantised heat-kernel regularised traces on the whole algebra of classical pseudodifferential operators for high enough quantum level n. Heat-kernel regularised traces naturally arise from JLO type cochains.[1] Their analogues in the noncommutative context arise in the work of Connes and Moscovici [CM] (later reformulated by Higson [H]) on the local formula for the Connes-Chern character.

Let us briefly describe the "second quantisation" procedure for ζ-regularised traces. Let $C_\bullet(M,E) := \oplus_{n=0}^{\infty} C_n(M,E)$ with $C_n(M,E) = \otimes^{n+1} \mathrm{C}\ell(M,E)$ be the space of chains associated to the algebra $\mathrm{C}\ell(M,E)$ of classical pseudodifferential operators acting on smooth sections of a vector bundle E over a closed manifold M.

The resolvent $R(\lambda,Q) = (\lambda - Q)^{-1}$ of an operator $Q \in \mathrm{C}\ell(M,E)$ can be quantised to $R_\bullet(\lambda, Q+\theta)$ acting on $C_\bullet(M,E)$, θ being an insertion map $\theta(A) = A$. We set $R_0(\lambda, Q+\theta) = R(\lambda,Q)$ and for $n > 0$

$$\begin{array}{rccl} R_n(\lambda, Q+\theta): & C_{n-1}(M,E) & \to & \mathrm{C}\ell(M,E) \\ & A_1 \otimes \cdots \otimes A_n & \mapsto & R(\lambda,Q)A_1 \cdots R(\lambda,Q)A_n\, R(\lambda,Q). \end{array}$$

We construct quantised functionals $f(Q+\theta)$ using Cauchy integrals

$$f_\bullet(Q+\theta) = \frac{1}{2i\pi} \int f(\lambda)\, R_\bullet(\lambda, Q+\theta)$$

along an adequately chosen contour. We then investigate the behaviour of $R_\bullet(\lambda, Q+\theta)$ under the adjoint action $A \mapsto [C,A]$ of $\mathrm{C}\ell(M,E)$ and under a variation of the weight, from which we derive the corresponding behaviour of the quantised functionals $f(Q+\theta)$.

[1] named after Jaffe, Lesniewski and Osterwalder [JLO].

Quantised functionals are the cornerstones for the quantisation of ζ-regularised traces.

Let us first recall the usual ζ-regularisation procedure. If Q is elliptic with spectral cut, it has well defined complex powers defined by Cauchy integrals. Given an operator $A \in C\ell(M,E)$ and a complex number z with large enough real part, the ζ-regularised operator $R^{Q,\zeta}(A)(z) := A\,Q^{-z}$ is trace class (we assume Q is invertible for simplicity) and the map $z \mapsto \mathrm{tr}(A\,Q^{-z})$ extends to a meromorphic function $z \mapsto \zeta(A,Q,z)|^{mer}$ with simple pole at $z=0$ (see e.g. [KV]). Its finite part at $z=0$ gives rise to a linear form on $\mathrm{tr}^Q : C\ell(M,E)$ which we call the Q-weighted trace. In general, the finite part $\mathrm{tr}^Q(A)$ is not expected to be local [2] in contrast to the complex residue at $z=0$, which is proportional to the noncommutative residue [Wo].

Replacing the resolvent $R(\lambda,Q)$ by the quantised resolvent $R_\bullet(\lambda,Q+\theta)$ boils down to substituting the quantised complex powers $(Q+\theta)^{-z}$ to ordinary complex powers Q^{-z} and leads to a *quantised zeta regularisation* $R^{Q+\theta,\zeta}$ on $C_\bullet(M,E)$ defined in terms of the quantised resolvent by

$$(1)\quad R_n^{Q+\theta,\zeta}\left(A_0\otimes A_1\otimes\cdots\otimes A_n\right)(z) = \frac{1}{2i\pi}\int_\Gamma \lambda^{-z}\,A_0\,R_n(\lambda,Q+\theta)(A_1\otimes\cdots\otimes A_n).$$

Using techniques inspired from [H] and [CM], when Q has *scalar leading symbol*, we show that for any $A_i \in C\ell(M,E), i=0,\cdots,n$ the operator $R_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n)(z)$ is trace class for z with large enough real part. Furthermore, the map $z \mapsto \mathrm{tr}\left(R_n^{Q,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n)\right)(z)$ extends to a meromorphic function with simple pole at $z=0$. It is holomorphic at $z=0$ when $n\neq 0$ and we call (second) *quantised weighted ζ-trace (or quantised Q- weighted trace)* of the chain $A_0\otimes\cdots\otimes A_n$ its value at $z=0$ which we denote by $\mathrm{tr}_n^{Q+\theta}\left(A_0\otimes\cdots\otimes A_n\right)$. In contrast to the ordinary Q-weighted trace, we show that whenever Q has scalar leading symbol, quantised weighted traces $\mathrm{tr}_n^{Q+\theta}$ are local for any positive integer n. We provide a *local formula* expressing them as a finite linear combination of noncommutative residues. When transposed to the noncommutative context, the locality for positive integer n shown here underlies that of the Connes-Moscovici formula for the Connes-Chern character in the case of classical pseudodifferential operators.

When Q has positive leading symbol, if instead of ζ-regularisation we implement heat-kernel regularisation (thus replacing Q^{-z} by $e^{-\epsilon Q}$ for some positive parameter ϵ), a similar construction gives rise to JLO type cochains. The heat-kernel quantised traces one obtains this way (taking finite parts as ϵ tends to 0) coincide for large quantum level with the quantised weighted trace described previously. This again constrasts with the 0 level case. Indeed, the finite part of the heat-kernel regularised trace of an operator $A \in C\ell(M,E)$ only coincides with its weighted trace when the operator A has vanishing residue.

[2] It actually is local if A is a differential operator. In general, it is made of a local piece involving the noncommutative residue and a global piece involving a finite part integral over all the cotangent space [PS2].

Quantised Q-weighted traces are not generally closed in the Hochschild cohomology. When Q has *scalar leading symbol*, we show that their *Hochschild coboundary* is local as a finite linear combination of noncommutative residues. For even cochains, this follows from the fact that the Hochschild coboundary of a quantised regularised trace of level $2p$ is a linear combination of quantised regularised traces of level $2p+1$. [3]

We also express the variation of such quantised weighted traces as the *weight varies* in terms of a linear combination of noncommutative residues. This locality is again a consequence of the fact that the variation of a quantised weighted trace of weight n is a linear combination of weighted traces of weight $n+1$ combined with the locality of quantised regularised traces of any positive level.

Adapting these constructions to the geometric setup of the index theorem for *families* along the lines of [PS1], [MP], Q can be replaced by a pseudodifferential operator-valued even form $\mathbb{Q}$, the exterior diffferentiation by a superconnexion $\mathbb{A}$ and the pseudodifferential operators A_i by pseudodifferential operator-valued forms α_i (the insertion map θ for pseudodifferential operators is replaced by an insertion map Θ for pseudodifferential valued forms) and one gets a local expression for the exterior differential $\left(d\,\mathrm{tr}_n^{\mathbb{Q}+\Theta}\right)$ of quantised regularised traces.

When $\mathbb{A}$ is a superconnection adapted to the zero degree component $Q=\mathbb{Q}_{[0]}$, replacing $\mathbb{Q}$ by $\mathbb{A}^2$ in the above expression yields the expected covariance property for quantised $\mathbb{A}^2$-weighted traces (see Corollary.

To sum up, local formulae for the two types of anomalies mentioned above, lack of traciality and dependence on the weight of the quantised regularised traces, are obtained in the same manner, namely as a combination of the following basic facts:

(1) *Anomalies for quantised regularised traces of level n are linear combinations of quantised regularised traces of level $n+1$.* [4]
(2) *Quantised regularised traces of any positive level are local.*

As mentioned above, quantised regularised traces are tools commonly used in noncommutative geometry; some of the above constructions inspired from techniques used in noncommutative geometry generalise to a noncommutative context. However we feel that even in the present classical setup of classical pseudodifferential operators, the language of quantised regularised traces is well suited to keep track of anomalies. The locality of quantised regularised traces (of any positive level) shown in this paper somewhat clarifies why one is to expect trace anomalies to be local.

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Reconstruction of manifolds in noncommutative geometry

Adam Rennie
(joint work with Joe Varilly)

This talk consisted of a brief sketch of the proof of the reconstruction theorem, focussing on the role the various conditions play in obtaining a C^∞ manifold. Many details, including possible refinements of the theorem, were omitted.

Even more briefly, the proof consists of several steps.

Preliminaries: The main result here is a C^∞ multivariable functional calculus for smooth (aka regular, QC^∞) spectral triples. This allows us to construct partitions of unity in our algebra, and also to locally invert functions in our algebra.

Easy stuff: This is just collecting the immediate consequences of the conditions. The most notable is the formula for the 'Dirac' operator, writing $\underline{\mathrm{d}}a := [\mathcal{D}, a]$,

$$\mathcal{D} = \frac{1}{2}(-1)^{p-1}\Gamma \sum_{\alpha=1}^{n}\sum_{j=1}^{p}(-1)^{j-1} a_\alpha^0\, \underline{\mathrm{d}}a_\alpha^1 \ldots \underline{\mathrm{d}}a_\alpha^{j-1}\, [\mathcal{D}^2, a_\alpha^j]\, \underline{\mathrm{d}}a_\alpha^{j+1} \ldots \underline{\mathrm{d}}a_\alpha^p + \frac{1}{2}(-1)^{p-1}\Gamma\, \underline{\mathrm{d}}\Gamma,$$

where $\Gamma = \sum_{\alpha=1}^{n} a_\alpha^0\, \underline{\mathrm{d}}a_\alpha^1 \ldots \underline{\mathrm{d}}a_\alpha^p$ is the representation of the basic 'orientation' cycle provided by the conditions, and we write $\underline{\mathrm{d}}\Gamma := \sum_{\alpha=1}^{n} \underline{\mathrm{d}}a_\alpha^0\, \underline{\mathrm{d}}a_\alpha^1 \ldots \underline{\mathrm{d}}a_\alpha^n$.

Cotangent bundle: We show that there exists a bundle whose sections are generated by the $\underline{\mathrm{d}}a_\alpha^j$ which plays the role of the cotangent bundle. In fact there is an open cover $\{U_\alpha\}_{\alpha=1}^n$ such that on each 'chart' U_α the $\underline{\mathrm{d}}a_\alpha^j$ for fixed α generate the sections.

Lipschitz Functional Calculus: We develop a Lipschitz functional calculus for elements of our algebra. While all of the results obtained from this could also be obtained from the C^∞ functional calculus, the proof can be generalised to reconstruct locally Lipschitz manifolds. For this reason we have tried to include enough technical machinery to allow the interested reader to do this.

The main consequence of this calculus is that the maps

$$a_\alpha = (a_\alpha^1, \ldots, a_\alpha^p) : U_\alpha \to R^p$$

are open.

Point set topology of the U_α: This lengthy discussion analyses where and by how much the map $a_\alpha : U_\alpha \to R^p$ fails to be one-to-one. Essentially, there may be branch points in U_α where the multiplicity changes, but there is a dense open subset of U_α where a_α is locally one-to-one.

Local structure of the 'Dirac' operator: On open sets where a_α is one-to-one, we show that $\mathcal{D}$ is up to endomorphisms a direct sum of Dirac-type operators, possibly defined with respect to different metrics.

Injectivity of the coordinates: Here we use both weak and strong unique continuation properties for perturbed Euclidean Dirac operators to show that branch

points do not occur in the set U_α, and that in fact U_α is a disjoint union of open sets on each of which the coordinates a_α are homeomorphisms onto there image.

Global Riemannian structure: Here we do the mopping up, showing that the various local results obtained patch together to give a smooth Riemannian manifold.

Poincaré Duality and spin^c structures: Here we show that for a compact oriented Riemannian manifold, satisfying Poincaré Duality in K-theory is equivalent to being spin^c. This allows us to show that our spectral triple is the spectral triple canonically associated to a spin^c manifold and the Dirac operator of the metric and spin^c structure, up to perturbations of $\mathcal{D}$ by endomorphisms.

For detailed statements of the conditions, as well as the full proof, see [3]. For the original discussion of possible reconstruction theorems, see [1], and for a failed proof of the reconstruction theorem, see [2].

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The Atiyah-Singer Formula for Subelliptic Operators on Contact Manifolds

Erik Van Erp

The Atiyah-Singer Index Theorem expresses the index of an elliptic operator on a compact manifold by means of a cohomological formula. The principal symbol of an elliptic differential operator gives rise to a cohomology class, which is most naturally a class in K-theory. The Chern character of this K-theory class is a deRham class, and the Index Formula describes how this class gives rise to an index.

Inherently, this formula cannot be applied to operators that are not elliptic. If the principal symbol is not invertible (i.e., the operator is not elliptic), it does not define a K-theory class, and it's Chern-character is not defined. So the formula does not generalize to nonelliptic operators. However, the index theorem for Toeplitz Operators of Boutet de Monvel shows that the Atiyah-Singer Formula can be applied even in certain non-elliptic cases, if one can work out a correct, but non-standard, notion of what the "symbol" of such a nonelliptic operator is.

There is, in fact, a different notion of "principal part" of an operator. Toeplitz operators are not pseudodifferential operators in the usual calculus, but they are pseudodifferential operators in the so-called Heisenberg calculus. An unusual feature of this calculus is that the principal symbols form a noncommutative algebra.

In our work, we show how to construct the correct K-theory class from the Heisenberg symbol of an arbitrary pseudodifferential operator in the Heisenberg calculus (always on a compact contact manifold). We then prove that the Fredholm index of the operator is calculated as before: take the Chern character of the K-theory class, and apply the Atiyah-Singer formula. The index theorem for Toeplitz operators is shown to be a special case of our formula, but our formula can also be specialized to differential operators. The construction of our K-theory symbol is rather more involved than that for elliptic operators, but we show how, in the special case of subelliptic differential operators, our index formula is easily computed.

The techniques used to prove our index theorem are, in part, a modification and extension of Connes' tangent groupoid proof of the Atiyah-Singer theorem.

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Dirac operator on the standard Podleś sphere

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(joint work with Konrad Schmüdgen)

In 2002, Ludwik Dąbrowski and Andrzej Sitarz constructed an equivariant real even spectral triple on the standard Podleś sphere $\mathcal{O}(\mathrm{S}^2_q)$ [1]. Apart from the so-called "Theta-deformations", it is one of the first known examples of a spectral triple on a quantum homogeneous space arising in the theory of quantum groups. In the meantime, more examples are known, but most of them face problems with the real structure. If a real structure is known at all, typically the "commutant property" and the "first order condition" are satisfied only up to an operator ideal of compact operators. To understand why the standard Podleś sphere behaves much better, it is convenient to study this example by using the representation theory of the Hopf *-algebra $\mathcal{O}(\mathrm{SU}_q(2))$, $q \in (0,1)$. This has been done in a joint work with Konrad Schmüdgen [3].

Starting point of the construction in [3] is the Peter-Weyl theorem for compact quantum groups. It states that $\mathcal{O}(\mathrm{SU}_q(2))$ decomposes into the direct sum of the matrix coefficients of all finite dimensional irreducible corepresentations. From this decomposition, one derives immediately an explicit description of the Haar state on $\mathcal{O}(\mathrm{SU}_q(2))$ with all its invariance properties. Since the Haar state is faithful, one can consider the GNS representation of $\mathcal{O}(\mathrm{SU}_q(2))$ on itself. The GNS representation of $\mathcal{O}(\mathrm{SU}_q(2))$ leads in turn to the so-called left and right regular representations of the dual object, the quantum universal enveloping algebra $\mathcal{U}_q(\mathrm{su}(2))$. The generators of the Hopf *-algebra $\mathcal{U}_q(\mathrm{su}(2))$ are usually denoted by E, $F\,(=E^*)$, $K\,(=K^*)$ and K^{-1}. Decomposing $\mathcal{O}(\mathrm{SU}_q(2))$ into the eigenspaces

of the self-adjoint operator K under the right regular representation, one obtains $\mathcal{O}(\mathrm{SU}_q(2)) = \oplus_{n\in\frac{1}{2}\mathbb{Z}} M_n$, where M_0 is a *-subalgebra of $\mathcal{O}(\mathrm{SU}_q(2))$ isomorphic to $\mathcal{O}(\mathrm{S}^2_q)$, and the M_n's are known as quantum line bundles with winding number $2n$. From $(ab)\triangleleft K = (a\triangleleft K)(b\triangleleft K)$, $a^*\triangleleft K = (a\triangleleft K^{-1})^*$ and $KE = qEK$, it follows that $M_nM_m \subset M_{n+m}$, $M_n^* \subset M_{-n}$, $M_n\triangleleft E \subset M_{n-1}$ and $M_n\triangleleft F \subset M_{n+1}$. In particular, each M_n is an $\mathcal{O}(\mathrm{S}^2_q)$-bimodule.

The last observations are crucial for the definition of an equivariant real even spectral triple on $\mathcal{O}(\mathrm{S}^2_q)$ and for the proof that all requirements are fulfilled. Proceeding exactly along the lines of the classical Dirac operator, one takes the direct sum $S := M_{-1/2} \oplus M_{1/2}$ of line bundles with winding number -1 and 1 as the spinor bundle and its closure with respect to the norm obtained from the GNS-construction as the Hilbert space $\mathcal{H}$. The *-representation of $\mathcal{O}(\mathrm{S}^2_q)$ is given by restricting the GNS-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ to the subalgebra $\mathcal{O}(\mathrm{S}^2_q)$ and the invariant subspace $S \subset \mathcal{O}(\mathrm{SU}_q(2))$. As a consequence, the representation of $\mathcal{O}(\mathrm{S}^2_q)$ is equivariant since the GNS-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ does so.

As in the classical case, the Dirac operator D has the form

$$D = \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix},$$

where E and F act on S via the right regular representation of $\mathcal{U}_q(\mathrm{su}(2))$. The operators from the right regular representation commute with the operators from the left regular representation of $\mathcal{U}_q(\mathrm{su}(2))$, thus the Dirac operator D is equivariant. Using $a\triangleleft K^{-1} = a$ for all $a \in M_0 = \mathcal{O}(\mathrm{S}^2_q)$, an easy calculation gives

$$[D,a] = \begin{pmatrix} 0 & q^{1/2}\, a\triangleleft E \\ q^{-1/2}\, a\triangleleft F & 0 \end{pmatrix}.$$

Since $a\triangleleft E$ and $a\triangleleft F$ are elements in $\mathcal{O}(\mathrm{SU}_q(2))$, and since the GNS-representation of $\mathcal{O}(\mathrm{SU}_q(2))$ is bounded, the commutators $[D,a]$ are obviously bounded. It can easily be shown by using explicit formulas of the left and right regular representations of $\mathcal{U}_q(\mathrm{su}(2))$ that D has eigenvalues $\pm[l+1/2]_q$ with multiplicities $2l+1$, $l = 1/2,\ 3/2,\ \ldots$, where $[n]_q := (q^n - q^{-n})/(q-q{-}1)$. Since $[n]_q^{-1} \sim q^n$, $|D|^{-z}$ is of trace class for all complex numbers with $\Re(z) > 0$.

The equivariant grading operator γ is (up to the sign) uniquely determined by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, $\gamma D = -D\gamma$.

Let T denote the Tomita operator for the GNS-representation of $\mathcal{O}(\mathrm{SU}_q(2))$, i.e. $T(x) := x^*$ for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$, and set

$$J_0 := T|T|^{-1}, \qquad J := J_0\gamma.$$

This definition implies that J is equivariant, $J^2 = -1$ and $J\gamma = -\gamma J$. One also gets $JD = DJ$. Since J_0 is the modular conjugation in Tomita-Takesaki theory, it follows that $JbJ^{-1} = J_0bJ_0^{-1}$ belongs to the commutator of $\mathcal{O}(\mathrm{SU}_q(2))$. Consequently, $[a, JbJ^{-1}] = 0$ and $[[D,a], JbJ^{-1}] = 0$, where the last relation

follows from the fact that $[D,a]$ acts on S by (left) multiplication with functions belonging to $\mathcal{O}(\mathrm{SU}_q(2))$ (see above).

Summarizing, the data $(\mathcal{O}(\mathrm{S}^2_q),\mathcal{H},D,\gamma,J)$ defines an 0-summable equivariant real even spectral triple on the standard Podleś sphere. (For a definition of equivariant spectral triples, see Ludwik Dąbrowski's report.)

A remarkable feature of this spectral triple is that the Dirac operator fits into Woronowicz's theory of covariant differential calculi. A covariant first order differential calculus can be defined by setting

$$\mathrm{d}a := \mathrm{i}[D,a],\ \ a\in\mathcal{O}(\mathrm{S}^2_q),\qquad \Omega^1 := \mathrm{span}\{b\,\mathrm{d}a\,:\,a,b\in\mathcal{O}(\mathrm{S}^2_q)\}.$$

The covariance follows immediately since D commutes with the left regular representation of $\mathcal{U}_q(\mathrm{su}(2))$. A more interesting result concerns the dimension of this calculus: It can be shown that, as a left $\mathcal{O}(\mathrm{S}^2_q)$-module, Ω^1 has rank 2. This is a special case of Ulrich Krähmer's result in [2]. Moreover, the universal higher order differential calculus (obtained by "dividing out junk") has up to multiples a unique invariant volume 2-form ω. The uniqueness allows to define an integral by

$$\int a\,\omega := h(a),\qquad a\in\mathcal{O}(\mathrm{S}^2_q),$$

where h denotes the Haar state (on $\mathcal{O}(\mathrm{SU}_q(2))$). The integral is closed, i.e. $\int \mathrm{d}\eta = 0$ for all $\eta\in\Omega^1$; and $h(ab)=h(\sigma(b)a)$ with the modular automorphism σ. Therefore it defines a σ-twisted cyclic 2-cocycle τ by the formula

$$\tau(a_0,a_1,a_2) := \int a_0\,\mathrm{d}a_1\,\mathrm{d}a_2,\qquad a_0,a_1,a_2\in\mathcal{O}(\mathrm{S}^2_q).$$

It can be shown that

$$a_0\,\mathrm{d}a_1\,\mathrm{d}a_2 = a_0\left((a_1\triangleleft E)(a_2\triangleleft F) - q^2(a_1\triangleleft F)(a_2\triangleleft E)\right)\omega$$

and, for $a\in\mathcal{O}(\mathrm{S}^2_q)$,

$$h(a) = \zeta_q(z)^{-1}\mathrm{Tr}_{M_{\pm 1/2}}\, a|D|^z K^2,\qquad \Re(z)>2,$$

where $\zeta_q(z)$ is a "zeta-type" function. This leads to the following descriptions of the σ-twisted cyclic 2-cocycle:

$$\begin{aligned}\tau(a_0,a_1,a_2) &= h\big(a_0((a_1\triangleleft E)(a_2\triangleleft F) - q^2(a_1\triangleleft F)(a_2\triangleleft E))\big)\\ &= \zeta_q(z)^{-1}\mathrm{Tr}_{\mathcal{H}}\,\gamma_q a_0\,[D,a_1]\,[D,a_2]\,|D|^{-z}\,K^2\\ &= \tfrac{\log q}{q-q^{-1}}\operatorname*{Res}_{z=2}\mathrm{Tr}_{\mathcal{H}}\,\gamma_q a_0\,[D,a_1]\,[D,a_2]\,|D|^{-z}\,K^2,\end{aligned}$$

where γ_q denotes a 2×2-diagonal matrix with entries 1 and $-q^2$.

At present, it is not known if this σ-twisted cyclic 2-cocycle pairs with the K_0-group of $\mathcal{O}(\mathrm{S}^2_q)$.

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