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Konvexgeometrie

Organised by
Keith Ball (London)
Paul R. Goodey (Norman)
Peter M. Gruber (Wien)

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ABSTRACT. The geometry of convex domains in Euclidean space plays a central role in several branches of mathematics: functional and harmonic analysis, the theory of PDE, linear programming and, increasingly, in the study of other algorithms in computer science. High-dimensional geometry, both the discrete and convex branches of it, has experienced a striking series of developments in the past 5 years. Several examples were presented at this meeting, for example the work of Naor on the non-linear Dvoretzky theorem, that of Paouris on the distribution of the Euclidean norm on a convex domain and the results of Rudelson on the singular values of random matrices.

Introduction by the Organisers

The meeting *Konvexgeometrie* organised by K. M. Ball, P. Goodey and P. M. Gruber, was held from December 17 to December 23, 2006. The meeting was attended by some 40 participants working in all areas of convex geometry. The program involved 10 plenary lectures of one hour's duration and about 15 shorter lectures. Some highlights of the program were as follows.

Grigoris Paouris explained his proof that if K is an isotropic convex body of volume 1 in $\mathbf{R}^n$ and the random variable X is uniformly distributed on K, then for some absolute constant C,

$$\mathrm{P}(|X| > C\sigma t) \le e^{-\sqrt{n}t}$$

for all $t > 1$, where σ^2 is the variance of X. The estimate is optimal apart from the value of C. Olivier Guédon then explained joint work with Fleury and Paouris, showing how the method of Paouris yields the central limit theorem for convex bodies, conjectured in 1996 by Ball and recently proved by Klartag.

Assaf Naor described his new results with Manor Mendel which now give a complete picture of the non-linear Dvoretzky Theorem. 20 years ago, Bourgain, Figiel and Milman proved that any n-point metric space has a subset of size about $\log n$ which can be embedded in Hilbert space with a constant distortion. In 2003, Bartal, Linial, Mendel and Naor discovered a remarkable threshold phenomenon: that if we allow distortion larger than a factor 2, there are subsets of size a power of n which are embeddable in Hilbert space. In the recent work the authors determine exactly the correct dependence of the power on the distortion: for each $\epsilon > 0$ there are subsets of size $n^{1-\epsilon}$ that are $O(1/\epsilon)$ embeddable.

Mark Rudelson described his recent estimates for the smallest singular values of almost square random (Gaussian) matrices. Considerably sharpening earlier work of Litvak, Pajor, himself, Tomczak and Vershynin and (in a slightly different direction) Candes and Tao, he established strong bounds for the probability that a random $N \times n$ matrix maps a point of the unit sphere in $\mathbf{R}^n$ to a point of small ℓ_1^N norm. This is equivalent to understanding the maximum radius of an almost full-dimensional random section of the cross-polytope. The passage from such estimates to singular numbers uses standard techniques.

Ralph Howard spoke about his recent work (joint with Paul Goodey) on bodies of constant brightness. This follows Howard's recent solution of the problem (dating back to 1926) of whether there exist bodies in 3-space which are of constant width and constant brightness but which are not Euclidean balls.

Richard Gardner gave an account of several new algorithms for the reconstruction of convex bodies from their x-rays in a small number of directions. The new algorithms are robust in that they can accommodate noisy data and are sufficiently simple that convergence proofs are rendered quite straightforward.

Matthias Reitzner gave a survey of the well-developed theory of random polytopes focussing on deviation estimates for the numbers of faces of given dimension. This covers work of himself, Bárány, Vu and others. Ryabogin and Zvavitch described their joint work with Nazarov solving a conjecture of Weil about the characterisation of zonoids.

Semyon Alesker gave an impromptu evening presentation of the recent work of Greg Kuperberg who has given a very short and self-contained proof of the Bourgain-Milman theorem on the product of the volumes of a convex body and its polar. Mahler conjectured that this volume product is minimised by a simplex: the Bourgain-Milman theorem proves this up to a factor of $(\text{constant})^n$ in n-dimensions which is what is needed for most applications.

Workshop: Konvexgeometrie

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Random Polytopes
Matthias Reitzner

Let $K \subset \mathbb{R}^d$ be a compact convex set of volume one. Assume $X_1, \ldots, X_n$ is a random sample of n independent, uniform points from K. The *random polytope* P_n is the convex hull of these points: $P_n = [X_1, \ldots, X_n]$.

We are interested in the $\boldsymbol{f}$-vector of P_n, where $\boldsymbol{f}(P_n) = (f_0(P_n), \ldots f_{d-1}(P_n))$, and $f_s(P_n)$ is the number of s–dimensional faces of P_n, and further in the intrinsic volumes $V_i(P_n)$, where, for instance, V_d is the volume, $2V_{d-1}$ the surface area and V_1 is a multiple of the mean width. Most results concern random points chosen uniformly either in a smooth convex body or in a convex polytope.

For recent surveys see [11] and [4].

1. Mean values

The expectation of the $\boldsymbol{f}$-vector was investigated in a series of papers. Let K be a smooth convex body (for our purposes it is sufficient to assume that the boundary is of differentiability class $\mathcal{C}^3$ with Gaussian curvature $\kappa(x) > 0$ for all boundary points.) Wieacker [16], Bárány [3], and Reitzner [10] obtained that

$$\mathbb{E}\boldsymbol{f}(P_n) = \boldsymbol{c}_d \, \Omega(K) \, n^{\frac{d-1}{d+1}}(1 + o(1))$$

as $n \to \infty$ where $\boldsymbol{c}_d$ is a constant vector. Here $\Omega(K)$ denotes the affine surface area of the convex body K. The result for $f_0(P_n)$ also holds for general convex sets which was proved by Schütt [13], yet the corresponding result for $f_s(P_n)$, $s = 1, \ldots, d-1$, seems to be open. If K is a polytope, then

$$\mathbb{E}\boldsymbol{f}(P_n) = \overline{\boldsymbol{c}}_d \, T(K) \ln^{d-1} n(1 + o(1))$$

which follows from work of Affentranger and Wieacker [1], Bárány and Buchta [5], and Reitzner [10]. Here $T(K)$ denotes the number of chains $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ of i–dimensional faces F_i of K.

To determine the mean values of the intrinsic volumes turns out to be more delicate. For smooth convex bodies it was proved by Schneider and Wieacker [12], Bárány [3], and Reitzner [8] that

$$(1) \qquad V_i(K) - \mathbb{E}V_i(P_n) = c_i \Omega_i(K) \, n^{-\frac{2}{d+1}}(1 + o(1))$$

where $\Omega_d(K) = \Omega(K)$ is the affine surface area, and in general $\Omega_i(K)$ is a certain integral of the mean curvatures on the boundary of K. For polytopes it is known that $V_d(K) - \mathbb{E}V_d(P_n)$ asymptotically equals a constant times $n^{-1} \ln^{d-1} n$, and $V_1(K) - \mathbb{E}V_1(P_n)$ a constant times $n^{-1/d}$. For $i = 2, \ldots, d-1$ a conjecture of Bárány [2] states that $V_i(K) - V_i(P_n)$ should behave like $n^{-1/(d-i+1)}$, but a rigorous proof still is missing.

An essential ingredient for proving these results is the floating body of K. Let K be an arbitrary convex body. For a point $z \in K$ denote by $v(z)$ the minimal

volume of the caps of K containing z. The *floating body* with parameter t is the set $K_{[t]} = \{z \in K : v(z) \geq t\}$. Schütt and Werner proved that $V_d(K) - V_d(K_{[t]})$ asymptotically equals a constant times $\Omega(K)t^{2/(d+1)}$. That there is a connection between the random polytope P_n and the floating body was observed by Bárány and Larman. They proved that P_n is with high probability close to $K_{[1/n]}$. Combining the (geometric) result of Schütt and Werner with the observation of Bárány and Larman gives some insight in the case $i = d$ of (1).

Of high interest is the question to determine the extremal convex sets minimizing or maximizing the mean values mentioned above. Important results were obtained by Blaschke, Dalla and Larman, Giannopoulos, Groemer, and Hartzoulaki and Paouris. In particular it would be of interest to prove that under all convex sets with given intrinsic volume $V_i(K)$, the ball is an extremal body for the intrinsic volumes $V_i(P_n)$ which is known only for $i = d$.

2. Central limit theorems and large deviation inequalities

The last years have seen a lot of new results on the asymptotic distribution of the random variables $V_d(P_n)$ and $f_s(P_n)$. In work of Reitzner [9], Vu [15], and Bárány and Reitzner [6] the central limit theorem was proved for $V_d(P_n)$ and $f_s(P_n)$, $s = 0, \ldots, d-1$. If K is either smooth or a polytope, then

$$\mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E}V_d(P_n)}{\sqrt{\mathrm{Var}V(P_n)}} \leq x\right) \to \Phi(x)$$

and

$$\mathbb{P}\left(\frac{f_s(P_n) - \mathbb{E}f_s(P_n)}{\sqrt{\mathrm{Var}f_s(P_n)}} \leq x\right) \to \Phi(x).$$

In the planar case the results have been obtained by Groeneboom, and Cabo and Groeneboom. It would be of interest to prove a CLT also for intrinsic volumes. For the surface area the method used in the papers mentioned above possibly works, but no general results are known.

Recently, Vu [14] and Calka and Schreiber [7] succeeded in proving large deviation inequalities in important cases. If K is smooth, then

$$\mathbb{P}\left(\left|\frac{f_0(P_n) - \mathbb{E}f_0(P_n)}{\sqrt{\mathrm{Var}f_0(P_n)}}\right| \geq t\right) \leq 2e^{-ct^2} + e^{-cn^{\frac{d-1}{3d+5}}},$$

and

$$\mathbb{P}\left(\left|\frac{V_d(P_n) - \mathbb{E}V_d(P_n)}{\sqrt{\mathrm{Var}V_d(P_n)}}\right| \geq t\right) \leq 2e^{-ct^2} + e^{-cn^{\frac{d-1}{3d+5}}}$$

for $t^2 \leq n^{\frac{(d-1)(d+3)}{(d+1)(3d+5)}}$. If K is a polytope, then a result for $f_0(P_n)$ is missing, but Vu proved that

$$\mathbb{P}\left(\left|\frac{V_d(P_n) - \mathbb{E}V_d(P_n)}{\sqrt{\mathrm{Var}V_d(P_n)}}\right| \geq t\right) \leq 2e^{-ct^2 \ln^{-2d} n} + n^{-\alpha}$$

for $t^2 \leq \ln^{3d} n$. All these results follow from a large deviation inequality for general convex sets which was proved by Vu [14].

Large deviation inequalities for $f_s(P_n)$, $s = 1, \ldots, d-1$ and $V_i(P_n)$, $i = 1, \ldots d-1$ are maybe even harder to prove than the cases mentioned above, and are still missing.

An essential geometric ingredient for these distributional results is the *visibility region* of a point $z \in K$ with respect to the floating body with parameter t. The visibility region $S(z, t)$ consists of all points $y \in K$ such that the segment $[y, z]$ is disjoint from the interior of $K_{[t]}$. It turns out to be of importance that $S(z, t)$ is small compared to $K \backslash K_{[t]}$.

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The Steiner polynomial and successive radii

María A. Hernández Cifre

(joint work with Martin Henk)

For a convex body $K \subset \mathbb{R}^n$, a positive real number ρ and the n-dimensional unit ball B_n, the Minkowski sum $K_\rho = K + \rho B_n$ is called the *outer parallel body* of K at distance ρ. The well-known *Steiner polynomial* states that the volume of the outer parallel body can be expressed as a polynomial of degree n in the parameter ρ,

$$(1) \qquad V(K_\rho) = V(K + \rho B_n) = \sum_{i=0}^{n} \kappa_{n-i} V_i(K) \rho^{n-i},$$

where the coefficients $V_i(K)$ so defined are called the *intrinsic volumes* of K. In particular, $V_n = V$ is the volume, $2V_{n-1} = F$ the surface area, $2\kappa_{n-1}/(n\kappa_n)V_1 = b$ the mean width and $V_0 = 1$ the Euler characteristic of K.

In [5] Teissier investigated the Steiner polynomial and its relation to problems arising in Algebraic Geometry. He suggested to replace ρ by $-\rho$ in (1), and to look for relations between the zeros of the so called *alternating Steiner polynomial*, $\sum_{i=0}^{n} \kappa_{n-i} V_i(K)(-\rho)^{n-i}$, and the inradius r and circumradius R of K. In the plane the alternating Steiner polynomial is the left-hand side of the well-known Bonnesen inequality:

$$A(K) - P(K)\lambda + \pi\lambda^2 \leq 0 \quad \text{if} \quad r \leq \lambda \leq R.$$

In particular, this inequality implies that the roots of the alternating Steiner polynomial $\lambda_1 \leq \lambda_2$ must satisfy $\lambda_1 \leq r \leq R \leq \lambda_2$. In [4] and [5] (see also [2, p. 103] and [3, p. 65]), an extension of this fact is conjectured for arbitrary dimension; we formulate it in terms of the Steiner polynomial:

Conjecture 1. *Let $K \in \mathcal{K}^n$. If $a_1 \leq \cdots \leq a_n$ are the real parts of the roots of the Steiner polynomial, then $a_1 \leq -R \leq -r \leq a_n \leq 0$.*

Let $-\gamma_i$, $i = 1, \ldots, n$, be the roots of the Steiner polynomial of K. Then,

$$\sum_{i=0}^{n} \kappa_{n-i} V_i(K) \rho^{n-i} = \kappa_n \prod_{i=1}^{n} (\rho + \gamma_i),$$

and clearly, $\kappa_{n-i}/\kappa_n V_i(K)$ is the i-th elementary symmetric function of the γ_i:

$$\frac{\kappa_{n-i}}{\kappa_n} V_i(K) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} \gamma_{j_1} \cdot \ldots \cdot \gamma_{j_i}.$$

Here we are interested in bounds on the intrinsic volumes $V_i(K)$ in terms of functionals involving the inradius and the circumradius of the convex body, which will provide bounds on the elementary symmetric functions of the γ_j. These functionals are called the *inner and outer successive radii* of the body K, and they are

defined in the following way:

$$R_\pi^i(K) = \max_{L \in \mathcal{L}_i^n} R(K|L), \qquad R_\sigma^i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} R\big(K \cap (x + L)\big),$$

$$r_i^\pi(K) = \min_{L \in \mathcal{L}_i^n} r(K|L; L), \qquad r_i^\sigma(K) = \min_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} r\big(K \cap (x + L); x + L\big).$$

It is easy to see that

$$R(K) = R_\pi^n(K) = R_\sigma^n(K), \qquad D(K)/2 = R_\pi^1(K) = R_\sigma^1(K),$$

$$r(K) = r_n^\pi(K) = r_n^\sigma(K), \qquad \omega(K)/2 = r_1^\pi(K) = r_1^\sigma(K),$$

where D and ω denote, respectively, the diameter and the minimal width of K. Replacing the first max-condition by a min-condition and viceversa, we get four other series of successive radii, which go from half of the diameter (half of the width) to the circumradius (inradius).

In [1] the above functionals and some other related families of functionals associated to a convex body were studied, as well as their relations to the volume. For instance, it was shown:

Theorem 2. *Let $K \subset \mathbb{R}^n$ be a convex body. Then,*

$$\kappa_n r_1^\pi(K) \cdot \ldots \cdot r_n^\pi(K) \le V(K) \le \kappa_n R_\pi^1(K) \cdot \ldots \cdot R_\pi^n(K),$$

$$\kappa_n r_1^\sigma(K) \cdot \ldots \cdot r_n^\sigma(K) \le V(K) \le \kappa_n R_\sigma^1(K) \cdot \ldots \cdot R_\sigma^n(K).$$

If K has non-empty interior, the equality holds if and only if K is a ball.

Here we have proved similar bounds for the i-th intrinsic volume V_i of a convex body in terms of the i-th elementary symmetric function of inner and outer radii. For instance, we obtain results of the following type:

Theorem 3. *Let $K \subset \mathbb{R}^n$ be a convex body. Then,*

$$V_i(K) \le \frac{\kappa_n}{\kappa_{n-i}} \sum_{1 \le j_1 < \cdots < j_i \le n} R_\pi^{j_1}(K) \cdot \ldots \cdot R_\pi^{j_i}(K),$$

$$V_i(K) \ge \frac{\kappa_n}{\kappa_{n-i}} \sum_{1 \le j_1 < \cdots < j_i \le n} r_{j_1}^\pi(K) \cdot \ldots \cdot r_{j_i}^\pi(K).$$

In both cases equality holds if and only if K is a ball.

As a consequence of these results, we can obtain bounds for the i-th elementary symmetric function of the roots of the Steiner polynomial in terms of the i-th elementary symmetric function of the inner and outer radii. For instance:

Corollary 4. *Let $K \subset \mathbb{R}^n$ be a convex body and $-\gamma_j$ be the roots of its Steiner polynomial. Then,*

$$\sum_{1 \le j_1 < \cdots < j_i \le n} \gamma_{j_1} \cdot \ldots \cdot \gamma_{j_i} \le \sum_{1 \le j_1 < \cdots < j_i \le n} R_\pi^{j_1}(K) \cdot \ldots \cdot R_\pi^{j_i}(K).$$

$$\sum_{1 \le j_1 < \cdots < j_i \le n} \gamma_{j_1} \cdot \ldots \cdot \gamma_{j_i} \ge \sum_{1 \le j_1 < \cdots < j_i \le n} r_{j_1}^\pi(K) \cdot \ldots \cdot r_{j_i}^\pi(K).$$

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Examples and Structure of Smooth Convex Bodies of Constant k-Brightness

RALPH HOWARD

(joint work with P. Goodey)

Denote by $\mathbf{Gr}_k(\mathbf{R}^n)$ the Grassmannian of all k-dimensional linear subspaces of $\mathbf{R}^n$. If K is a convex body in $\mathbf{R}^n$ and $P \in \mathbf{Gr}_k(\mathbf{R}^n)$ let $K|P$ be the orthogonal projection of K onto P. The convex body K has **constant k-brightness** iff there is a constant β, the **k-brightness** of K, such that $V_k(K|P) = \beta$ for all $P \in \mathbf{Gr}_k(\mathbf{R}^n)$. When $k = 1$ these are the sets of **constant width**; there are non-spherical examples going back to Euler. When $k = n - 1$ and $n \geq 3$ these are the bodies of **constant brightness**. Here the first non-spherical examples are due to Blaschke [1]. Both in the case of bodies of constant width and constant brightness, the set of examples is "large" (infinite dimensional) and there are examples that have no symmetry.

In the case of $2 \leq k \leq n - 2$ the first examples of non-spherical sets of k-brightness are due to Firey [2]. His examples have rotational symmetry about a line and to the best of our knowledge all examples in the literature have this symmetry. One goal of our work is to give examples that do not have any such symmetry.

If $a \in \mathbf{R}^n$ and $r > 0$ let $B(a, r)$ be the closed ball of radius r with center a. If $k \geq 2$ is an integer then a convex body is of class C_+^k iff its boundary ∂K is a C^k submanifold of $\mathbf{R}^n$ and the Gauss-Kronecker curvature of ∂K is everywhere positive.

Definition 1. *A convex body K in $\mathbf{R}^n$ is **channel body** iff there is a curve $c \colon [a, b] \to \mathbf{R}^n$ and a function $\rho \colon [a, b] \to (0, \infty)$ such that ∂K is the envelope of the one parameter family of spheres $\{\partial B(c(s), \rho(s)) : s \in [a, b]\}$. This implies*

$$(1) \qquad K := \bigcup_{s \in [a,b]} B(c(s), \rho(s)).$$

The following gives some known elementary properties of channel bodies.

Proposition 2. *With the notation of Definition 1, if K is a channel body of class C_+^2, then*

   (1) *ρ has a unique maximum at some $s_* \in (a,b)$.*
   (2) *For any $s \in (a,b)$ there is an affine hyperplane P_s of $\mathbf{R}^n$ such that*

$$\partial K \cap \partial B(c(s), \rho(s)) = P_s \cap \partial B(c(s), \rho(s)) = P_s \cap \partial K.$$

Definition 3. *If K is a channel body of class C_+^2, let $H_{s_*}^+$ and $H_{s_*}^-$ be the two closed half-spaces of $\mathbf{R}^n$ determined by P_{s_*}. Then the two **halves** of K are $K^+ := K \cap H_{s_*}^+$ and $K^- := K \cap H_{s_*}^-$.*

Theorem 4. *Let $2 \le k \le n-2$ and assume K is a channel body of class C_+^3, then, for any sufficiently large β, the half K^+ can be "capped off" to form a channel body L with constant k-brightness β. More precisely, let K be given by (1). Then there is a $b_1 > s_*$ a curve $c_1 \colon [a,b_1] \to \mathbf{R}^n$, a function $\rho_1 \colon [a,b_1] \to (0,\infty)$ such that the restrictions of these to $[a,s_*]$ satisfy $c_1\big|_{[a,s_*]} = c\big|_{[a,s_*]}$, $\rho_1\big|_{[a,s_*]} = \rho\big|_{[a,s_*]}$ and the channel hypersurface $L := \bigcup \{B(c_1(s), \rho_1(s)) : s \in [a,b_1]\}$ has constant k-brightness β.*

This gives new examples of convex bodies of constant k-brightness. A structure theorem describing all (sufficiently smooth) convex bodies of constant k-brightness is desirable. For some values of n and k, such a theorem exists in the form of a weak converse to the last theorem.

Theorem 5. *Let $2 \le k \le n-3$ and let K be a non-spherical convex body of constant k-brightness in $\mathbf{R}^n$. Then, at each point $x \in \partial K$, there are only two distinct radii of curvature and one of these, ρ, has multiplicity at least $n-2$. Then there is a C^1 function $f \colon \partial K \to \mathbf{R}^n$ such that*

   (1) *At each point x of ∂K the differential, df, of f has rank at most one.*
   (2) *The function ρ is constant on each of the level sets $f^{-1}[f(x)]$ of f. Therefore if $y \in f[\partial K]$, then $\rho(y)$ can be defined as $\rho(x)$ where $f(x) = y$ and this will be independent of the choice of x.*
   (3) *The convex body K is given by $K = \displaystyle\bigcup_{y \in f[\partial K]} B(y, \rho(y))$.*

The rank one condition on the differential of f implies that the image $f[\partial K]$ is one dimension in the sense that it has Hausdorff dimension one. If df has rank one at all but two points, then the rank theorem implies that the image is a C^1 curve and Theorem 5 implies K is a channel body. If f has only a finite number of points where df has rank zero, or more generally if $\{x \in \partial K : df(x) = 0\}$ only has finite many connected components and f is constant on each of these, then the image, $f[\partial K]$ will be a graph in the sense that it is a finite collection of points (vertices) in $\mathbf{R}^n$ with C^1 curves (edges) connecting some pairs of the vertices. In this case ∂K is the envelope of the spheres $\{\partial B(y, \rho(y)) : y \in f[\partial K]\}$, which is a natural generalization of a channel body. More complicated geometry is possible, but the notion that, when $2 \le k \le n-3$, all C_+^3 convex bodies of constant k-brightness are generalized channel bodies is basically correct.

The proofs are based on a detailed analysis of the support functions of convex bodies of constant k-brightness and rely on results from papers [3] and [4] which characterize C_+^2 convex bodies by a system of non-linear second order differential equation for the support functions. The spherical Hessian $L := \nabla^2 f + fI$ of a C^3 function $f \colon \mathbf{S}^{n-1} \to \mathbf{R}$ is a field self-adjoint linear maps (that is a tensor field of type $(1,1)$) acting on the tangent spaces to $\mathbf{S}^{n-1}$. Such fields of linear maps are characterized as those that satisfy the ***Codazzi equation*** $(\nabla_X L)(Y) = (\nabla_L Y)X$. Our analysis of the equations for the Hessians of the support functions is greatly simplified by working directly with the Codazzi tensors, rather than with the derivatives of the support functions. Codazzi tensors are not a standard tool in convexity, as they require smoothness of the bodies, but the study of smooth convex bodies is really a part of differential geometry and using differential geometric techniques, such as Codazzi tensors, is natural.

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A. D. Alexandrov's problem, hyperbolic virtual polytopes and related topics

GAIANE PANINA

We discuss various results motivated by recently discovered counterexamples to the following conjecture (A.D. Alexandrov's problem):

Uniqueness conjecture for smooth convex surfaces.
Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant C, at every point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

For a long time mathematicians were certain about correctness of the conjecture but obtained only some partial results. Recently, Y. Martinez-Maure [6] has given a counterexample. First, he demonstrated that each smooth *hyperbolic hérisson* generates a desired counterexample. Next, he presented such an example. It is a smooth hyperbolic surface with four *horns* (i.e., singular non-saddle points), given by an explicit formula.

This counterexample is not unique: a series of counterexamples was given by the author of the paper (see [6], [8]). She used a different technique based on the theory of hyperbolic virtual polytopes.

Figuratively speaking, hyperbolic virtual polytopes relate to convex ones in the same way as convex surfaces relate to saddle ones. As is known, there exist no

closed saddle polytopal surfaces. Still non-trivial hyperbolic virtual polytopes do exist, and this is probably the most remarkable fact which is known about them.

I. Virtual polytopes and hyperbolic virtual polytopes. Convex polytopes in $\mathbb{R}^3$ form a semigroup $\mathcal{P}$ with respect to the Minkowski addition $\otimes$. The semigroup $\mathcal{P}$ is isomorphic to the semigroup of continuous convex piecewise linear (with respect to a *fan*) functions defined on $\mathbb{R}^3$. The isomorphism maps a convex polytope to its support function.

(A necessary reminding: the support function of a polytope is piecewise linear with respect to some conical tiling of the $\mathbb{R}^3$. To visualize the tiling, we intersect it with the unit sphere centered at O and get the *spherical fan* of the polytope.)

Passing to the Grothendieck group $\mathcal{P}^*$ (it is the group of formal Minkowski differences of convex polytopes) which is called the *group of virtual polytopes* (introduced originally by A. Pukhlikov and A. Khovanskij in [5]), only the convexity property disappears. Thus we get a group isomorphism

virtual polytope in $\mathbb{R}^3 \longleftrightarrow$ continuous piecewise linear (with respect to a fan) function defined on $\mathbb{R}^3$.

A virtual polytope can be represented geometrically as a polytopal function [5] or as a closed polytopal surfaces [8]. Recall that the support function of a convex polytope is convex, i.e. its graph is a convex surface (it is reasonable to consider either the spherical graph or the collection of affine graphs [8,9]).

Definition. A virtual polytope is *hyperbolic* if the graph of its support function is a saddle surface.

Here is a way of constructing (unexpectedly diverse) counterexamples to the above conjecture [8].

- Construct a hyperbolic polytope (this is the most difficult step).
- Smoothen its support function h (preserving saddle property).
- Add to h the support function of a ball (which is sufficiently large to make the sum convex). The result is the support function of a counterexample to the conjecture.

Theorem [8, 10]. For any number $N \geq 4$, there exists a hyperbolic virtual polytope with N horns and its smooth version - a hyperbolic hérisson with N horns. □

II. A.D. Alexandov's uniqueness theorem for convex polytopes and its refinements. A.D. Alexandrov claimed that the following assertion for convex polytopes is parallel to the above conjecture:

Uniqueness theorem for convex polytopes [1]. Let K, M be 3-dimensional convex polytopes. If for any pair of their parallel faces, none of the faces can be placed strictly into another via a translation, then the polytopes coincide up to a translation. □

This theorem admits a natural interpretation in terms of hyperbolic polytopes [9]. In the below refinements, we replace the condition of the theorem by milder ones.

Theorem [9]. There exist two different 3-dimensional convex polytopes K and L such that for any pair of their parallel faces, there is **at most** one translation which inserts one of them strictly into another. □

The example is far from trivial. For its construction, we need a hyperbolic polytope H with an additional property: the fan of H admits a regular triangulation without Steiner points. It seems that none of the hyperbolic polytopes known before (see [8], [10]) possesses this property, so we need some advanced technique to construct hyperbolic polytopes. To present H, we construct the graph of its support function. It is a (spherically) saddle surface in the 3-dimensional sphere, spanned by some special linkage of 8 great semicircles.

Theorem [9]. Let K, M be 3-dimensional convex polytopes. Suppose that for each pair of parallel faces, the two assertions are valid:

  (1) There exists **at most one** translation t placing the face of K into the face of L.
  (2) There exists **no** translation t placing the face of L into the face of K.

Then the polytopes coincide up to a translation. □

III. Two non-isotopic hyperbolic polytopes with 4 horns. Each hyperbolic hérisson H generates an object with non-trivial combinatorics: an arrangement of disjoint great semicircles on the unite sphere S^2. More precisely, there is a natural one-to-one correspondence

"semicircles of the arrangement $\leftrightarrow$ horns of the hérisson $\leftrightarrow$ inflection arches of the graph of the support function h_H".

The latter correspondence recalls the Möbius theorem on inflection points of a curve in the projective plane.

Theorem [10]. There exist two non-isotopic smooth hérissons (and non-isotopic hyperbolic virtual polytopes), both with 4 horns. One of them is already known - it is the hérisson presented by Martinez-Maure [6]. The generated arrangements of great semicircles for these polytopes are non-isotopic. □

To construct the second hyperbolic polytope, we construct a saddle surface spanned by some special linkage on the 3-dimensional sphere.

IV. Hyperbolic polytopes and pointed tilings. The theory of hyperbolic polytopes interacts nicely with the theory of pointed tilings. The interaction is motivated by the following lemma.

Lemma [9]. The fan of a virtual polytope K is pointed $\Rightarrow$ the polytope K is hyperbolic. □

The theory of **pointed pseudo-triangulations** was useful for implicit solution of the carpenter's rule problem ([3], [13]) and proved later to give a nice tool for explicit graph embeddings ([4]). Uniting the methods, we get some results for both theories [12]. On the one hand, passing from planar pseudo-triangulations to spherical pseudo-tilings, we get more freedom for pointed embeddings.

Theorem [12]. Each Laman-plus-one graph admits an embedding in the 2-dimensional sphere S^2 generating a pointed tiling of the sphere such that each tile is either a pseudo triangle or a pseudo di-gon (a spherical polygon with just two convex angles). $\square$

On the other hand, the difficult problem of hyperbolic polytopes constructing can be reduced to finding spherically embedded graphs.

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Slicing convex sets and measures by a hyperplane

Imre Bárány

(joint work with A. Hubard and J. Jerónimo)

A well known result in elementary geometry states that there is a unique sphere which contains a given set of $d+1$ points in general position in $\mathbb{R}^d$. A similar thing happens with d-pointed sets and hyperplanes. What happens if we consider convex bodies instead of points?

Results answering this question were obtained by Kramer and Németh [4], and Klee, Lewis, and B. Von Hohenbalken [3] for balls, and by Cappell et al. [2] for hyperplanes. They all need the following definition: A family $\mathcal{F}$ of connected sets in $\mathbb{R}^d$ is said to be *well separated*, if for any $k \le d+1$ distinct elements, $K_1, \ldots, K_k$, of $\mathcal{F}$ and for any choice of points $x_i \in K_i$, the set $\mathrm{aff}\{x_1, \ldots, x_k\}$ is a $(k-1)$-dimensional flat. Let $[k]$ stand for the set $\{1, \ldots, k\}$. It is proved in [3] that if $\mathcal{F}$ is a well separated family of $d+1$ convex compact sets in $\mathbb{R}^d$, and if I, J is a partition of $[d+1]$, then there exists a unique Euclidean sphere that touches each K_k in such a way that K_i is inside the sphere for each $i \in I$ and K_j is outside of the sphere for each $j \in J$. Similarly, it is shown in [2] that if $\mathcal{F}$ is a well separated family of d convex compact sets in $\mathbb{R}^d$, and if I, J is a partition of $[d]$, then there are exactly two hyperplanes h with the following property. h touches each K_k in such a way that K_i is on one side of h for each $i \in I$ and K_j is on the other side of h for each $j \in J$. (In fact, the result in [2], much more generally, describe the topology of the set of hyperplanes that are simultaneously tangent to each member of a well separated family of k compact convex sets in R^d when $k \in \{2, \ldots, d\}$.)

Now let $Q^d = [0,1]^d$ be the unit cube in $\mathbb{R}^d$. Thus $\alpha \in Q^d$ means $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \in [0,1]$. A sphere S is said *transversal* to a family $\mathcal{F}$ if S intersects each member of $\mathcal{F}$. In [1] the following result is proved.

Theorem 1. Assume $\mathcal{F} = \{K_1, \ldots, K_{d+1}\}$ is a well separated family of convex compact sets in $\mathbb{R}^d$, and $\alpha \in Q^{d+1}$. Then there exists a unique Euclidean ball B such that its boundary is a transversal to $\mathcal{F}$ and $\mathrm{Vol}\,(B \cap K_i) = \alpha_i \mathrm{Vol}\,K_i$ for each $i \in [d+1]$.

Next, let $\mathcal{F} = \{K_1, \ldots, K_d\}$ be a family of convex compact sets in $\mathbb{R}^d$. A transversal hyperplane, h, to $\mathcal{F}$ is a hyperplane intersecting each K_i. There are two unit normal vectors to h, v and $-v$. It is not difficult to see that one can choose between v and $-v$ in such a way that this choice, $v(h)$ say, is a continuous function of h, and $v(h)$ only depends on h and $\mathcal{F}$. Thus a transversal hyperplane h defines a unique halfspace H whose bounding hyperplane is h and whose outer normal is $v(h)$. Call such a halfspace *an oriented transversal halfspace*.

Theorem 2. Assume $\mathcal{F} = \{K_1, \ldots, K_d\}$ is a well separated family of convex compact sets in $\mathbb{R}^d$, and $\alpha \in Q^d$. Then there exists a unique oriented transversal halfspace H such that its boundary is a transversal to $\mathcal{F}$ and $\mathrm{Vol}\,(H \cap K_i) = \alpha_i \mathrm{Vol}\,K_i$ for each $i \in [d]$.

When all $\alpha_i = 1/2$, the existence of such a halfspace is guaranteed by Borsuk's theorem, even without the condition of convexity or $\mathcal{F}$ being well separated. (Connectivity of the sets implies that the halving hyperplane is a transversal to $\mathcal{F}$.) The case of general α_i, however, needs some extra condition as the following two examples show. If all K_i are equal, then each oriented hyperplane section cuts off the same amount from each K_i, so $\alpha_1 = \cdots = \alpha_d$ must hold. The second example consists of d concentric balls with different radii, and if the radius of the first ball is very large compared to those of the others. If α_1 is too small, then a hyperplane cutting off α_1 fraction of the first ball is disjoint from all other balls. Thus no hyperplane transversal exists that cuts off an α_1 fraction of the first set.

Theorem 2 can be extended from convex bodies to "nice" measures, if their supports are well separated. For the precise statement see [1]. The proof of these results uses some convex geometry and the Brouwer fixed point theorem.

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Curvature, integrability and concentration

Franck Barthe

(joint work with A. Kolesnikov)

We are interested in isoperimetric inequalities or concentration properties of simple metric probability spaces. If μ is a Borel probability measure on a metric space (M, d), one studies for $h > 0$ the best function R_h such that any Borel subset $A \subset M$ verifies

$$\mu(A_h) \geq R_h(\mu(A)),$$

where $A_h := \{x;\ d(x, A) \leq h\}$ is the h-neighborhood of A. The function $1 - R_{1/2}$ is often called the concentration function, it measures how sets of measure one half are filling the space in the sense of measure. When h tends to zero, the limiting problem is equivalent to minimizing an appropriate notion of boundary measure among sets of given probability. We refer to the books [4, 8] for more background and applications.

Such properties can often be understood thanks to Sobolev type inequalities. Classically if a measure μ say on a Riemannian manifold M (with geodesic distance d) satisfies a logarithmic Sobolev inequality, it satisfies a Gaussian type concentration inequality: if $\mu(A) \geq 1/2$ then $\mu(A_h) \geq 1 - e^{-ch^2}$ where c depends on the numerical constant in the logarithmic Sobolev inequality. Recall that μ satisfies a

logarithmic Sobolev inequality means that there exists $C > 0$ so that every locally Lipschitz function $f : M \to \mathbb{R}$ verifies

$$\mathbf{Ent}_\mu(f^2) := \int f^2 \log\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq C \int |\nabla f|^2 d\mu.$$

Such inequalities are also crucial in the study of the dynamical properties of certain Markov processes with invariant measure μ.

We are interested in simple natural conditions which ensure that a measure satisfies a logarithmic Sobolev inequality or variants. First let us point out that if a measure satisfies a logarithmic Sobolev inequality then by the Gaussian concentration inequality it follows that there exists $\varepsilon > 0$ such that $\int e^{\varepsilon d(x_0,x)^2} d\mu(x) < +\infty$ for any $x_0 \in M$. Another necessary condition of more local nature is that there cannot be holes in the support of μ, otherwise one may build non-constant functions that vary only outside of the support of μ and thus with nul gradient μ-a.s.

A few sufficient conditions of such flavor exist. The first ones are related to strict convexity. The Bakry-Emery criterion [1] asserts that a probability measure $d\mu = e^{-V} d\mathrm{Vol}$ on a Riemannian manifold M such that for some $\lambda > 0$ and all $x \in M$, $\mathrm{Hess}_x V + \mathrm{Ricci}_x \geq \lambda \mathrm{Id}$ satisfies a logarithmic Sobolev inequality. In the same spirit a celebrated result of Gromov and Milman [6] ensures that if X is a finite dimensional uniformly convex normed space, then its unit sphere with the distance of the norm, satisfies a concentration inequality.

When strict convexity is not assumed integrability conditions seem to be needed. Our goal was to bridge and possibly extend the following results. The first one was discovered by Wang [10]: let $d\mu = e^{-V} d\mathrm{Vol}$ is a probability measure on a Riemannian manifold M. If for all $x \in M$, $\mathrm{Hess}_x V + \mathrm{Ricci}_x \geq -\lambda \mathrm{Id}$ where $\lambda > 0$ and if there exists $\varepsilon > 0$ such that

$$\int e^{\frac{\lambda+\varepsilon}{2} d(x,y)^2} d\mu(x) d\mu(y),$$

then μ satisfies a logarithmic Sobolev inequality.

In the particular case of log-concave measures $d\mu(x) = e^{-V(x)} dx$ (V convex) in Euclidean space, Bobkov [3] provided a more flexible approach to the above statement based on a specific isoperimetric inequalities. We could adapt his approach in [2] to show that a log-concave measure as above, which also satisfies $\int e^{\varepsilon |x|^p} d\mu(x) < +\infty$ ($p \in (1,2]$ but actually $p > 1$ works) satisfies up to constant the same isoperimetric inequality as the measure $d\nu_p(t) = e^{-|t|^p} dt / (2\Gamma(1 + 1/p))$, and therefore inherits of functional inequalities satisfied by ν_p. These techniques were pushed further in [7].

Wang's result is proved via a beautiful Harnack type inequality for the Heat equation attached to μ; this approach seems to lead to conditions on the exponential integrability of $d(x,y)^2$ only. Since our goal is to bridge the above two statements, and in particular to exploit hypotheses as $\int e^{cd(x,y)^p} d\mu(x) d\mu(y) < +\infty$, another approach was needed. Using optimal transportation techniques with non-quadratic cost, in the spirit of e.g. [9, 5] we could prove the following result. By definition for $x \in \mathbb{R}^n$, $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$.

Theorem. *Let $p \in (1, 2]$ and $q = p/(p-1) \in [2, +\infty)$ be its dual exponent. Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^n$. Assume that there exists $\lambda \geq 0$ such that the function $x \mapsto V(x) + \lambda\|x\|_p^p$ is convex and that there exists $\varepsilon > 0$ such that*

$$\int_{\mathbb{R}^{2n}} e^{(\lambda 2^{2-p}+\varepsilon)\|x-y\|_p^p} d\mu(x)d\mu(y) < +\infty.$$

Then there exists constants K_1, K_2 such that for every nonnegative smooth function g it holds

$$\mathbf{Ent}_\mu(g^q) \leq K_1 \int \|\nabla g\|_q^q d\mu + K_2 \int g^q d\mu.$$

The latter inequality is not tight, i.e. it is not an equality for constant functions. Under certain assumptions, as for example a spectral gap, it can be upgraded to a tight inequality.

The above statement allows to deal with potentials V which need not be convex, but for which the defect of convexity is controlled. Note that the defect of convexity and the integrability condition are measured in terms of the same function $\| \cdot \|_p^p$. For this reason conditions on the Hessian of V as in Wang's theorem are related to exponential integrability of $d(x, y)^2$. Finally for log-concave densities, since $\lambda = 0$, any functional in the "convexity" hypothesis may be chosen, which explains the freedom to use a variety of integrability conditions as in the developments of Bobkov's method.

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Dvoretzky's theorem in metric spaces
ASSAF NAOR

Abstract. Let $f : X \to Y$ be an embedding of the metric spaces (X, d_X) into (Y, d_Y). The *distortion* of f is defined by

$$\text{dist}(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \cdot \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_X(x, y)}{d_Y(f(x), f(y))}.$$

We denote by $c_Y(X)$ the least distortion with which X may be embedded in Y. When $Y = L_2$ we use the simpler notation $c_2(X) = c_{L_2}(X)$. The parameter $c_2(X)$ is known in the literature as the Euclidean distortion of X.

The classical Dvoretzky theorem is equivalent to the following statement: For every $\alpha > 1$ there exists a constant $c(\alpha)$ such that every n-dimensional normed space has a linear subspace Y such that $c_2(Y) \leq \alpha$ and $\dim Y \geq c(\alpha) \log n$ (the optimal logarithmic dependence on the dimension n is due to Milman). The purpose of this talk is to discuss the best known results on a natural variant of this theorem which makes sense in the context of arbitrary finite metric spaces. Namely, we study the following problem that was introduced by Bourgain, Figiel and Milman in 1986: Given a finite metric space X and a target distortion $\alpha > 1$, what is the largest subset $Y \subseteq X$ such that $c_2(Y) \leq \alpha$. Formally let $R(\alpha, n)$ be the largest integer m such that *every* n-point metric space X has a subset $Y \subseteq X$ with $c_2(Y) \leq \alpha$ and $|Y| \geq m$. Bourgain, Figiel and Milman proved that for all $\alpha > 1$ we have $R(\alpha, n) \geq c(\alpha) \log n$, while there exists $\alpha_0 \approx 1.023$ such that $R(\alpha_0, n) = O(\log n)$. In 2003 Bartal, Linial, Mendel and Naor proved that the behavior of $R(\alpha, n)$ exhibits a phase transition at $\alpha = 2$. Namely they showed that for every $\alpha > 1$ there exist constants $c, C, c', C', K > 0$ depending only on α such that $0 < c' < C' < 1$ and for every integer n:

a) If $1 < \alpha < 2$ then $c \log n \leq R(\alpha, n) \leq C \log n$.

b) If $\alpha > 2$ then $n^{c'} \leq R_2(\alpha, n) \leq K n^C$.

This talk is focused on the "isomorphic" case, i.e. the case $\alpha \to \infty$. Bartal, Linial, Mendel and Naor showed that for every $\varepsilon \in (0, 1)$, any n-point metric space has a subset of size $n^{1-\varepsilon}$ which embeds into L_2 with distortion $O\big(\frac{\log(2/\varepsilon)}{\varepsilon}\big)$. Moreover, they showed that this result is optimal up to the $\log(2/\varepsilon)$ factor, i.e. there exists arbitrarily large n-point metric spaces, every subset of which of size $n^{1-\varepsilon}$ incurs distortion $\Omega(1/\varepsilon)$ in any embedding into Hilbert space. In this talk we will present a complete proof of the following recent theorem of Mendel and Naor which closes this gap.

Theorem 1. *Let (X, d_X) be an n-point metric space and $\varepsilon \in (0, 1)$. Then there exists a subset $Y \subseteq X$ with $|Y| \geq n^{1-\varepsilon}$ such that $c_2(Y) = O(1/\varepsilon)$.*

The proof is a probabilistic argument which considers special distributions over random partition trees, which are defined as follows. Let (X, d_X) be a metric space. For $x \in X$ and $r \geq 0$ we let $B_X(x, r) = \{y \in X : d_X(x, y) \leq r\}$ be the

closed ball of radius r centered at x. Given a partition $\mathscr{P}$ of X and $x \in X$ we denote by $\mathscr{P}(x)$ the unique element of $\mathscr{P}$ containing x. For $\Delta > 0$ we say that $\mathscr{P}$ is Δ-bounded if for every $C \in \mathscr{P}$, $\mathrm{diam}(C) \leq \Delta$. A *partition tree* of X is a sequence of partitions $\{\mathscr{P}_k\}_{k=0}^{\infty}$ of X such that $\mathscr{P}_0 = \{X\}$, for all $k \geq 0$ the partition $\mathscr{P}_k$ is $8^{-k}\mathrm{diam}(X)$-bounded, and $\mathscr{P}_{k+1}$ is a refinement of $\mathscr{P}_k$ (the choice of 8 as the base of the exponent in this definition is convenient, but does not play a crucial role here). For $\beta, \gamma > 0$ we shall say that a probability distribution Pr over partition trees $\{\mathscr{P}_k\}_{k=0}^{\infty}$ of X is completely β-padded with exponent γ if for every $x \in X$,

$$\Pr\left[\forall\, k \in \mathbb{N},\ B_X\left(x, \beta \cdot 8^{-k}\mathrm{diam}(X)\right) \subseteq \mathscr{P}_k(x)\right] \geq |X|^{-\gamma}.$$

We prove the following two lemmas.

Lemma 2. *Let (X, d_X) be an n-point metric space which admits a distribution over partition trees which is completely β-padded with exponent γ. Then there exists a subset $Y \subseteq X$ with $|Y| \geq n^{1-\gamma}$ such that $c_2(Y) \leq 8/\beta$.*

Lemma 3. *Let (X, d_X) be a finite metric space. Then for every $\Delta > 0$ there exists a probability distribution Pr over Δ-bounded partitions of X such that for every $0 < t \leq \Delta/8$ and every $x \in X$,*

$$(1) \qquad \Pr\left[B_X\left(x, t\right) \subseteq \mathscr{P}(x)\right] \geq \left(\frac{|B_X(x, \Delta/8)|}{|B_X(x, \Delta)|}\right)^{\frac{16t}{\Delta}}.$$

¿From Lemma 2 we see that it is enough to show that for every $\alpha > 1$, every finite metric space (X, d_X) admits a completely $1/\alpha$ padded random partition tree with exponent $16/\alpha$. This follows from Lemma 3 via the following argument. Without loss of generality we may assume that $\mathrm{diam}(X) = 1$. We construct a partition tree $\{\mathscr{E}_k\}_{k=0}^{\infty}$ of X as follows. Set $\mathscr{E}_0 = \{X\}$. Having defined $\mathscr{E}_k$ we let $\mathscr{P}_{k+1}$ be a partition as in Lemma 3 with $\Delta = 8^{-k}$ and $t = \Delta/\alpha$ (the random partition $\mathscr{P}_{k+1}$ is chosen independently of the random partitions $\mathscr{P}_1, \ldots, \mathscr{P}_k$). Define $\mathscr{E}_{k+1}$ to be the common refinement of $\mathscr{E}_k$ and $\mathscr{P}_{k+1}$, i.e.

$$\mathscr{E}_{k+1} := \{C \cap C' :\ C \in \mathscr{E}_k,\ C' \in \mathscr{P}_{k+1}\}.$$

The construction implies that for every $x \in X$ and every $k \geq 0$ we have $\mathscr{E}_{k+1}(x) = \mathscr{E}_k(x) \cap \mathscr{P}_{k+1}(x)$. Thus one proves inductively that

$$\forall\, k \in \mathbb{N},\ B_X\left(x, \frac{8^{-k}}{\alpha}\right) \subseteq \mathscr{P}_k(x) \implies \forall\, k \in \mathbb{N},\ B_X\left(x, \frac{8^{-k}}{\alpha}\right) \subseteq \mathscr{E}_k(x).$$

From Lemma 3 and the independence of $\{\mathscr{P}_k\}_{k=1}^{\infty}$ it follows that

$$
\begin{aligned}
\Pr\left[\forall\, k \in \mathbb{N}, B_X\left(x, \frac{8^{-k}}{\alpha}\right) \subseteq \mathscr{E}_k(x)\right] &\geq \Pr\left[\forall\, k \in \mathbb{N},\ B_X\left(x, \frac{8^{-k}}{\alpha}\right) \subseteq \mathscr{P}_k(x)\right] \\
&= \prod_{k=1}^{\infty} \Pr\left[B_X\left(x, \frac{8^{-k}}{\alpha}\right) \subseteq \mathscr{P}_k(x)\right] \\
&\geq \prod_{k=1}^{\infty}\left[\frac{|B_X(x, 8^{-k-1})|}{|B_X(x, 8^{-k})|}\right]^{\frac{16}{\alpha}} \\
&= |B_X(x, 1/8)|^{-\frac{16}{\alpha}} \geq |X|^{-\frac{16}{\alpha}}.
\end{aligned}
$$

The complete details of the proofs of Lemma 2 and Lemma 3, together with applications of these results to theoretical computer science, can be found in [1].

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Some recent advances on the covariogram problem
GABRIELE BIANCHI

Let K be a convex body in $\mathbb{R}^n$. The *covariogram* g_K of K is the function

$$
g_K(x) = vol(K \cap (K + x)),
$$

where $x \in \mathbb{R}^n$ and *vol* denotes volume in $\mathbb{R}^n$. This functional, which was introduced by Matheron in his book [Mat75] on random sets, is also called the *set covariance* and it coincides with the *autocorrelation* of the characteristic function of K, that is $g_K = 1_K * 1_{(-K)}$. The covariogram g_K is clearly unchanged by a translation or a reflection (in a point) of K. Matheron [Mat86] and, independently, Adler and Pyke [AP97] asked the following question.

Covariogram problem. *Does the covariogram determine a convex body, among all convex bodies, up to translations and reflections?*

Matheron conjectured a positive answer for the case $n = 2$ but this conjecture has not been completely settled. What follows is an update on some very recent advances regarding this problem.

New motivations from X-ray crystallography. The original motivations of the problem come from stochastic geometry and image analysis. A recently discovered one comes from the problem of determining the atomic structure of a *quasicrystal* starting from its X-ray diffraction image. A convenient way of describing many important examples of quasicrystals is via the "cut and project scheme". Here to the atomic structure of the material, represented by a discrete set S contained in $\mathbb{R}^n$, is associated a lattice N in a higher dimensional space $\mathbb{R}^n \times \mathbb{R}^d$ and a "window" $W \subset \mathbb{R}^d$. In this setting S coincides with the projection

on $\mathbb{R}^n$ of the points of the lattice N which belong to $\mathbb{R}^n \times W$. In many examples the lattice N can be determined by the diffraction image. To determine S it is however necessary to know W: the covariogram problem enters at this point, since the covariogram of W can be obtained by the diffraction image; see Baake and Grimm [BG06]. According to M. Baake, in many important cases the window is a 2-, 3- or 4-dimensional convex polytope. The paper [BG06] also constructs two different model sets with equal diffraction image. This construction uses as windows two nonconvex polygons with equal covariogram discovered in [GGZ05].

Complete solution of the problem in the class of convex polytopes. The first answer to the covariogram problem was a positive one for the class of convex polygons [Nag93]. On the other hand, in any dimension $n \geq 4$ there are convex polytopes for which the answer is negative; see [Bia05]. Recently it has been possible to understand the three-dimensional case.

Theorem 1 ([Bia06]). *Let $P \subset \mathbb{R}^3$ be a convex polytope with nonempty interior. Then g_P determines P, in the class of convex bodies in $\mathbb{R}^3$, up to translations and reflections.*

The construction of the counterexamples in higher dimension is related to the possibility of decomposing a convex body into direct summands whose dimension is at least two. The answer to the covariogram problem for a specific convex polytope $P \subset \mathbb{R}^n$ depends, when $n \geq 4$, on whether or not P has a nontrivial decomposition, and a complete understanding of the problem for polytopes depends ultimately on understanding the situation for indecomposable bodies.

Redundancy of the covariogram data in the planar case. Two results are relevant for this point. It is possible to indicate some subsets of the support of the covariogram, with arbitrarily small Lebesgue measure, such that the covariogram restricted to those subsets suffices to identify, up to translations and reflections, any C_+^2 planar convex body; see [AB06b]. In general this restricted data identifies certain geometric properties of the body.

In a different spirit, it can be proved that the knowledge of the *cross-covariogram*, a generalisation of the covariogram where two unknown bodies are involved, determines in certain cases both bodies. In order to be precise, the cross covariogram $g_{H,K}$ of the convex bodies H and K in $\mathbb{R}^n$ is the function

$$g_{H,K}(x) = vol(H \cap (K + x)),$$

where $x \in \mathbb{R}^n$. It is proved that, when H and K are polygons, $g_{H,K}$ determines both H and K, up to certain inherent ambiguities, except for certain exceptions; see [Bia06]. These families of exceptions are made of pairs of parallelograms with the same cross-covariogram.

Generalised covariogram, where another functional replaces the volume. What happens if, in the plane, the area in the definition of g_K is replaced by the perimeter, or more generally by any strictly monotone, translation invariant valuation on the space of convex bodies? This author and G. Averkov believe

that, for the class of convex polygons, the corresponding covariogram problem has a positive answer; see [AB06a].

Some open problems. Many aspects of this problem have still to be understood. We mention here some of them.

(1) Are all planar convex bodies determined by their covariogram, as conjectured by G. Matheron? This conjecture has been confirmed for C^2 convex bodies, non-strictly convex bodies, and convex bodies that are not C^1; see [Bia05]. It is also true for planar convex bodies whose boundary contains two arcs which are reflections of each other; see [AB06b].
(2) Which four-dimensional polytopes are determined by their covariogram?
(3) The only known examples of convex sets not determined by their covariogram are cartesian products. Find other examples.
(4) Is the answer to the covariogram problem positive for all C^2_+ convex bodies in $\mathbb{R}^3$?

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Some functional inequalities on log-concave functions

Matthieu Fradelizi

(joint work with M. Meyer)

In this talk, I presented some works that were obtained in collaboration with Mathieu Meyer on functional forms of Blaschke-Santaló inequality and its inverse. The results concerning Blaschke-Santaló inequality are already published in [6], the ones on the inverse form are still in preparation ([7]).

I. Blaschke-Santaló. Recall that if K is a centrally symmetric set then the Blaschke-Santaló inequality ([3], [10]) asserts that

$$|K||K^*| \leq |B_2^n|^2.$$

1) unconditional case. The next proposition is a form of Prékopa-Leindler inequality for the geometric mean due to Borell ([5]), Ball ([2]), Uhrin ([11]). This result is well known and follows from the usual Prékopa-Leindler inequality. This proposition gives a first functional form of Blaschke-Santaló inequality.

Proposition 1. *Let f_1, f_2, $f_3 : \mathbb{R}^n \to \mathbb{R}_+$ be unconditional measurable functions such that*

$$f_1(x_1, \ldots, x_n) f_2(y_1, \ldots, y_n) \leq f_3(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n})^2$$

for every $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n) \in \mathbb{R}_+^n$. Then

$$\int_{\mathbb{R}^n} f_1(x) dx \int_{\mathbb{R}^n} f_2(y) dy \leq \left(\int_{\mathbb{R}^n} f_3(z) dz \right)^2$$

As a corollary, it implies that for all sets $A, B \subset \mathbb{R}_+^n$ if we define $\sqrt{A.B} := \{(\sqrt{x_i y_i})_{1 \leq i \leq n} \; ; \; x \in A, y \in B\}$ then $|A||B| \leq |\sqrt{A.B}|^2$. It also gives the usual Blaschke-Santaló inequality $|K||K^*| \leq |B_2^n|^2$ for unconditional sets $K \subset \mathbb{R}^n$.

2) symmetric case. The next result is K. Ball's functional version of the Blaschke-Santaló inequality for symmetric sets ([2]).

Proposition 2. *Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}_+$ be measurable even functions such that*

$$f_1(x) f_2(y) \leq \rho^2(\langle x, y \rangle) \text{ for every } x, y \in \mathbb{R}^n .$$

then

$$\int_{\mathbb{R}^n} f_1(x) dx \int_{\mathbb{R}^n} f_2(y) dy \leq \left(\int_{\mathbb{R}^n} \rho(|x|^2) dx \right)^2 .$$

As a corollary, it follows that for every even function $\varphi : \mathbb{R}^n \to \mathbb{R}$ one has

$$\int_{\mathbb{R}^n} e^{-\varphi(x)} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi(x)} dx \leq (2\pi)^n.$$

The preceding inequality was noticed by Artstein, Klartag and Milman in [1] where the authors also proved a non-symmetric version and established the equality case. With Mathieu Meyer, in [6], we proved a non-symmetric version of Ball's theorem with its equality case.

II. Inverse Santaló. In the geometric case, the inverse of Santaló inequality is still a conjecture, known as Mahler's conjecture. It asserts in its symmetric version that for every symmetric convex set

$$|K||K^*| \geq |B_1^n||B_\infty^n| = \frac{4^n}{n!}.$$

In 1987, Bourgain and Milman ([4]) proved that $|K||K^*| \geq \left(\frac{c}{n}\right)^n$. In 2006, Kuperberg ([8]) gave another proof of the same result.

Saint Raymond proved in [9] that for unconditional convex sets the inequality $|K^+||(K^*)^+| \geq \frac{1}{n!}$ holds true. With Mathieu Meyer in [7] we established the functional version of Saint Raymond's theorem.

Theorem 3. *Let* $f, g : \mathbb{R}^n_+ \to \mathbb{R}_+$ *non-increasing. Let* $h : \mathbb{R}^n_+ \to \mathbb{R}_+$ *defined by* $h(z) = \sup_{z=x.y} f(x)g(y)$. *Then*

$$\int_{\mathbb{R}^n_+} f(x)dx \int_{\mathbb{R}^n_+} g(x)dx \geq \int_{\mathbb{R}^n_+} h(x)dx.$$

As a corollary, using a suitable functional Lozanovskii's theorem, we get the known results that for every decreasing sets $A, B \subset \mathbb{R}^n_+$ one has $|A||B| \geq |A.B|$ and we also get that for every $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$ increasing and convex

$$\int_{\mathbb{R}^n} e^{-\varphi(x)}dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi(x)}dx \geq 1.$$

The method of proof of the Theorem follows Saint Raymond's ideas. After a change of variables to get integrals over $\mathbb{R}^n$ and an inverse Laplace transform, the result follows from the following theorem which is proved by induction on n.

Proposition 4. *Let* $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ *be an increasing function. Then*

$$\int_{\mathbb{R}^n} F(x, z-x)dx \geq \int_{\{w \leq z\}} \sup_{x \in \mathbb{R}^n} F(x, w-x)dw.$$

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Convex bodies are concentrated within a thin shell
Olivier Guédon
(joint work with B. Fleury and G. Paouris)

In this talk, I have presented the results of [6] where we studied how the volume of a symmetric convex body concentrates within a very thin Euclidean shell. Let K be an isotropic convex body in $\mathbb{R}^n$ i.e. a symmetric convex body of volume 1 such that for some fixed $\rho > 0$,

$$\forall \theta \in S^{n-1}, \int_K \langle x, \theta \rangle^2 dx = \rho^2.$$

It is known that every symmetric convex body has an affine image which is isotropic. We denote by $|x|_2$ the Euclidean norm of $x \in \mathbb{R}^n$. Several results about the concentration of the volume of a convex body within a Euclidean ball of radius $c\rho\sqrt{n}$ are known since the 70's. Namely, studying for every $t \geq 1$ the behaviour of

$$h(t) = \{x \in K, \ |x|_2 \geq ct\rho\sqrt{n} \ \}|$$

where $|A|$ denotes the volume of a subset $A \in \mathbb{R}^n$ and c is a universal fixed constant, it is known from classical log-concavity results of Borell [4] that $h(t) \leq e^{-t}$. Alesker in [1] improved the argument showing that $h(t) \leq e^{-t^2}$. Very recently, Bobkov and Nazarov in [3] proved that for unconditional convex bodies $h(t) \leq e^{-t\sqrt{n}}$ and Paouris in [11] proved that this last estimate is valid for every isotropic convex body. It is known that any isotropic convex body is contained in a Euclidean ball of radius $c'\,n\rho$ (where c' is a universal constant), therefore the estimate

$$(1) \qquad \{x \in K, \ |x|_2 \geq ct\rho\sqrt{n} \ \}| \leq e^{-ct\sqrt{n}}$$

is better than all the previous one, and it can be observed that it is optimal when $K = \frac{1}{|B_1^n|}B_1^n$.

In the paper [2], Anttila, Ball and Perissinaki asked if every isotropic convex body in $\mathbb{R}^n$ satisfies an ε-concentration hypothesis namely:

Concentration hypothesis. Does there exist ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$ and such that for every isotropic convex body $K \subset \mathbb{R}^n$

$$\left| \left\{ x \in K, \left| \frac{|x|_2}{\rho\sqrt{n}} - 1 \right| \geq \varepsilon_n \right\} \right| \leq \varepsilon_n?$$

It was proved in [2] and in [5] that the concentration hypothesis implies some type of central limit theorem. The conjecture about a central limit theorem for convex sets stated by Anttila, Ball, Perissinaki [2] and Brehm, Voigt [5] has been recently proved by Klartag [7]. The approach of Klartag is also based on the proof of the Concentration Hypothesis. In this talk, I have presented a different approach to the result of Klartag [7] proving that the Concentration Hypothesis holds true for

$$\varepsilon_n = \frac{(\log \log n)^2}{(\log n)^{1/6}}.$$

The proof is based on the study of the L_q norm of the function $x \mapsto |x|_2$ with respect to the probability measure uniformly distributed on the convex body K. These techniques were used by Paouris [11] to prove that for every $2 \le q \le c\sqrt{n}$,

$$\left(\int_K |x|_2^q dx \right)^{1/q} \le C \left(\int_K |x|_2^2 dx \right)^{1/2} = C\sqrt{n}\,\rho.$$

This immediately implies inequality (1). The main theorem that I have presented in the talk is the following

Theorem 1. *There exists c and c' such that for every isotropic convex body K in $\mathbb{R}^n$, and every $p \le (\log n)^{1/3}$,*

$$1 \le \left(\int_K |x|_2^p dx \right)^{1/p} \Big/ \left(\int_K |x|_2 dx \right) \le 1 + c\, p/(\log n)^{1/3}.$$

In particular, for every $\varepsilon \in (0,1)$,

$$(2) \qquad \left| \left\{ x \in K, \left| \frac{|x|_2}{\sqrt{n}\rho} - 1 \right| \ge \varepsilon \right\} \right| \le 2 e^{-\,c\sqrt{\varepsilon}(\log n)^{1/12}}.$$

Very recently, Klartag [8] has proved that inequality (2) is still true with polynomial dependance in n (instead of $\log n$).

The main tool of the proof is the study of L_p-centroid bodies introduced by Lutwak and Zhang [9]. To any star shaped body with respect to the origin, $L \subset \mathbb{R}^n$, we associate its L_p-centroid body $Z_p(L)$ which is a symmetric convex body defined by its support function:

$$\forall y \in \mathbb{R}^n, h_{Z_p(L)}(y) = \left(\int_L |\langle x, y \rangle|^p dx \right)^{1/p}.$$

The main result of this paper compares the mean width of the L_p-centroid bodies of an isotropic convex body to the mean width of the L_p-centroid bodies of the Euclidean ball of volume 1.

Theorem 2. *There exists a constant c such that for any n, for every isotropic convex body K in $\mathbb{R}^n$, if $\widetilde{D}$ denotes the Euclidean ball in $\mathbb{R}^n$ of volume 1, for every $p \le (\log n)^{1/3}$*

$$\frac{W_1(Z_p(K))}{W_1(Z_1(K))} \frac{W_1(Z_1(\widetilde{D}))}{W_1(Z_p(\widetilde{D}))} \le 1 + cp/(\log n)^{1/3}.$$

The proof of this result involves also a new stability result for the L_p-centroid bodies.

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Geometrization of Probability

Vitali Milman

The framework of the subject we will discuss in this talk involves very high dimensional spaces (normed spaces, convex bodies) and accompanying asymptotic (by increasing dimension) phenomena.

The starting point of this direction was the open problems of Geometric Functional Analysis (in the '60s and '70s). This development naturally led to the Asymptotic Theory of Finite Dimensional spaces (in '80s and '90s). See the books [MSch86], [P89] and the survey [LM93] where this point of view still prevails.

During this period, the problems and methods of Classical Convexity were absorbed by Asymptotic Theory (including geometric inequalites and many geometric, i.e. "isometric" as opposed to "isomorphic" problems).

As an outcome, we derived a new theory: Asymptotic Geometric Analysis. (Two surveys, [GM01] and [GM04] give a proper picture of this theory at this stage.)

One of the most important points of already the first stage of this development is a change in intuition about the behavior of high-dimensional spaces. Instead of the diversity expected in high dimensions and chaotic behavior, we observe a unified behavior with very little diversity. We refer the reader to [M00] for some examples which illustrate this.

Extension of the Category of Convex Bodies to the Category of Log-Concave Measures. Let us first define the class of log-concave measures and functions.

Definitions. A Borel measure μ on $\mathbb{R}^n$ is log-concave iff for any $0 < \lambda < 1$ and any A and $B \subset \mathbb{R}^n$ such that all involved sets $(A, B, \lambda A + (1-\lambda)B)$ are measurable

$$\mu\big(\lambda A + (1 - \lambda)B\big) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Here λA is a homothety and $+$ is the Minkowski sum, i.e. $\lambda A + (1 - \lambda)B = \{\lambda x + (1 - \lambda)y \mid x \in A$ and $y \in B\}$.

A few very important examples of log-concave measures:

(i) The standard volume on $\mathbb{R}^n$, $\mu(K) = \operatorname{Vol} K$ (by Brunn–Minkowski inequality).

(ii) The restriction of volume on a convex set K: $\mu_K(A) = \operatorname{Vol}(K \cap A)$, K-convex.

(iii) Marginals of volume restricted to a convex set.

A function $f(x) \geq 0$ is called log-concave if $\log f$ is concave, i.e. $f(x) = e^{-\varphi(x)}$ and φ is convex.

The connection between log-concavity of measures and functions was established by C. Borell [B74]: Let the support of a measure μ, $\operatorname{Supp} \mu$, not belong to any affine hyperplane. Then μ is log-concave iff μ is absolutely continuous on $\operatorname{Supp} \mu$ and the density f is a log-concave function.

Log-concavity was used in Convexity Theory already from the '50s (Henstock–MacBeath) and later, say, Prékopa–Leindler extension of Brunn–Minkowsky inequality, or the use of log-concave functions to study volume of sections of ℓ_p^n by Meyer–Pajor. But a purely geometric study of log-concavity waited until the end of the '80s, and was initiated by K. Ball [Ba86], who extended the study of some geometric problems of convexity to a larger category of log-concave measures. In particular, he studied isotropicity of such measures and connected it with isotropicity of convex bodies. He also considered some important geometric inequalities in the extended framework of log-concave measures ("functional versions" of geometric inequalities). However, just recently it was observed that such an extension is much broader than we thought, and is needed to understand and to solve some problems of asymptotic theory of high dimensional convexity proper.

Three features characterize this extension.

(i) On the one hand, important geometric inequalities (and other kinds of geometric statements) are interpreted, extended and proved for log-concave measures.

(ii) On the other hand, some typical probabilistic results (and thinking) are interpreted and proved in a geometric framework (say, *Central Limit Theorem for convex bodies and log-concave measures*, which was recently proved by B. Klartag [K07b], [K07c]).

(iii) And most importantly, an extension of the geometric approach to the log-concave category is needed to solve some central problems of a purely geometric nature.

The goal of this talk was to demonstrate examples of results to confirm this picture. These are mainly results of B. Klartag from [K07a].

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Polytopal approximation of smooth convex bodies

KÁROLY J. BÖRÖCZKY

How well a polytope of restricted complexity can approximate a smooth convex body in $\mathbb{R}^d$? This natural question has attracted the attention of mathematicians of various background since the middle of the 20th century. In this extended abstract, polytopes are always inscribed, and restricted complexity mostly means restricting the number of vertices of the polytope. In addition distance from the smooth convex body is mostly measured by affine invariant notions like the Banach-Mazur distance or the volume difference.

Concerning notation, we write B^d to denote the Euclidean unit ball of $\mathbb{R}^d$. We recall that the Banach-Mazur distance $\delta_{\mathrm{BM}}(K, M)$ of the convex bodies K and M in $\mathbb{R}^d$ is the minimal $\lambda \geq 1$ such that $K - x \subset \Phi(M - y) \subset \lambda(K - x)$ for some $\Phi \in \mathrm{GL}(d)$ and $x, y \in \mathbb{R}^d$. In the case if K and M are o-symmetric then $x = y = o$ can be assumed.

Let me start with A.M. Macbeath's classical result in [29]. It says that ellipsoids are worst approximable among convex bodies by inscribed polytopes in terms of volume. For any convex body K in $\mathbb{R}^d$ and $n \geq d+1$, let $V(K, n)$ be the maximal volume of polytopes with n vertices inscribed into K. According to [29], if E is an ellipsoid in $\mathbb{R}^d$ with $V(E) = V(K)$ then

$$(1) \qquad V(K, n) \leq V(E, n).$$

From now on, problems of approximation by polytopes of "low complexity" and of "high complexity" are discussed separately. In both cases I only present very few results which I feel typical.

I. Polytopes of few vertices. Here the main question is whether an inscribed polytope can reasonably well approximate the convex body at all. As (1) suggests, the convex body is the ball (ellipsoid) in these problems. Few vertices means that the number of vertices is at most exponential in the dimension d for Banach-Mazur distance, and at most $d^{d/2}$ for volume approximation.

Let $P_n \subset B^d$ be a polytope of n vertices. In high dimensions Bárány, Füredi [4], Gluskin [17] and Carl, Pajor [10] obtained independently the following result (all the three papers appeared in 1988!): If $n \geq 2d$ then

$$(2) \qquad \sqrt[d]{\frac{V(P_n)}{V(B^d)}} \leq \sqrt{\frac{c \ln \frac{n}{d}}{d}}$$

for some absolute constant $c > 0$. We note that if n is at most exponential in d then the estimate of (2) is optimal. Bárány, Füredi [4] also show that to get a polytope P_n with $V(P_n) > \frac{1}{2} V(B^d)$, one needs approximately $d^{d/2}$ vertices.

If $d + 1 \leq n \leq 2d$ then the estimate $\sqrt[d]{V(P_n)/V(B^d)} \leq \sqrt{c/d}$ resulting from (2) is optimal, as it is shown by the example of the inscribed regular simplex. If $n = d + 1$ then Steiner symmetrization (see Steiner [31]) shows that the regular simplex is optimal.

Turning to the Banach-Mazur distance, (2) yields that if $n \geq 2d$ then

$$(3) \qquad \delta_{\mathrm{BM}}(P_n, B^d) \geq \sqrt{\frac{d}{c \ln \frac{n}{d}}}$$

This estimate is optimal if n is at most exponential in d. In particular if $\delta_{\mathrm{BM}}(P_n, B^d) \leq 2$ then n is at least exponential in d, and on the other hand this property can be achieved using exponentially many vertices.

Related Problems:

(1) I conjecture that (3) also holds for any n with $d + 1 \leq n \leq 2d$. More precisely if $n = d + k$ for $k = 1, \ldots, d$ then

$$\delta_{\mathrm{BM}}(P_n, B^d) \geq \frac{\tilde{c}d}{\sqrt{k}}$$

for some absolute constant $\tilde{c} > 0$. This estimate would be optimal as the following (conjecturally optimal) polytopes exhibit. Take the convex hull of k pairwise orthogonal regular simplices of circumradius one and of dimensions either $\lceil \frac{d}{k} \rceil$ or $\lfloor \frac{d}{k} \rfloor$.

(2) It is a long standing open problem whether the mean width of P_{d+1} is maximal for the inscribed regular simplex (see Gritzmann, Klee [18] for history, especially for a list of wrong proofs that have been published).

The polytopes conjectured to be extremal in the first problem are known to be extremal in the following cases. If $k = 1$ or $k = 2$ then Steiner symmetrization (see Steiner [31]) yields the results in any dimension (see Böröczky, Jr., Wintsche [9]). In addition the optimality the cross polytope ($k = d$) is known if $d = 3$ (see Fejes Tóth [14]) or $d = 4$ (see Dalla, Larman, Mani-Levitska, Zong [12]).

The second problem has been solved by Linhart [26] if $d = 3$. His argument is based on the spherical Moment Theorem of Fejes Tóth [14]. Actually the spherical Moment Theorem of Fejes Tóth also yields the following results in the three dimensional case. If $n = 6$ or $n = 12$ then the optimal P_n with respect to volume approximation and the Banach-Mazur distance is the regular octahedron and icosahedron, respectively.

It follows by (2) that the volume of a convex body cannot be well approximated by polytopes of polynomial many vertices in d. However there exists algorithm polynomial in d that estimates well the volume with high probability according to Dyer, Frieze, Kannan [13]. The high degree in [13] has been brought down in a series of papers, culminating in an essentially degree four bound of Lovász, Vempala [27]. In addition A.R. Barron [2] and G. Cheang, A.R. Barron [11] (see also Artstein-Avidan, Friedland, Milman [1]) construct a non-convex body X with linear complexity in d such that $\frac{1}{2} B^d \subset X \subset B^d$.

II. Best approximation with many vertices. Let K be a convex body in $\mathbb{R}^d$. We discuss approximation of K by polytopes of say n vertices where n tends to infinity. For much broader surveys on the subject, consult P.M. Gruber [22] and [25].

We note that the Gauss-Kronecker curvature $\kappa(x)$ can be defined at most points $x \in \partial K$, hence the affine surface area

$$A(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, dx$$

is well-defined (see Schütt, Werner [32]). In addition a flag of a polytope P in $\mathbb{R}^d$ is a sequence $F_0 \subset \ldots \subset F_{d-1}$ where F_i is an i-face of P. Using random polytopes, if $A(K) > 0$ and n is large then Bárány [3] proved the existence of a polytope $P \subset K$ with at most n flags such that

$$V(K \backslash P) \leq \gamma(d) A(K)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}}$$

where $\gamma(d) > 0$ depends only on d. This estimate is optimal up to the value of $\gamma(d)$ according to Böröczky, Jr. [6]. If ∂K is C^2 and $P_n \subset K$ is a polytope with n vertices that has maximal volume then we even have the asymptotic formula

$$(4) \qquad V(K \backslash P) \sim \frac{\mathrm{del}_{d-1}}{2} A(K)^{\frac{d+1}{d-1}} n^{\frac{-2}{d-1}}$$

as n tends to infinity. Here $\mathrm{del}_2 = \frac{1}{2\sqrt{3}}$ (see Gruber [19]), and $\mathrm{del}_d \sim \frac{d}{2\pi e}$ as d tends to infinity (see P. Mankiewicz, C. Schütt [30]). The formula (4) was conjectured by Fejes Tóth [15] if $d = 3$, and proved by Gruber [21] if $\kappa(x)$ is positive for any d. The restriction $\kappa(x) > 0$ was removed by Böröczky, Jr. [5]. Generalizing results in Glasauer, Gruber [16], [5] also showed that the vertices of P_n are uniformly distributed on ∂K with respect to the density function $\kappa(x)^{\frac{1}{d+1}}$. In the three dimensional case, following Gruber [23], Böröczky, Tick, Wintsche [8] proved that the typical faces of P_n are asymptotically regular triangles in a suitable sense. Now if ∂C is C^3 with positive curvature then Böröczky, Jr. [7] even estimated the error term in (4), which estimate was substantially improved by Gruber [24].

Next let ∂K be C^2, and let P_n be a polytope with n vertices such that $\delta_{\text{BM}}(K, P_n)$ is minimal. Combining ideas in Gruber [20] and Böröczky, Jr. [5], one can prove the following. Writing u_x to denote the exterior unit normal at $x \in \partial K$, K can be translated in a way such that $o \in \text{int} K$, and

$$(5) \qquad \delta_{\text{BM}}(K, P_n) - 1 \sim \frac{1}{2} \left(\frac{\vartheta_{d-1}}{\kappa_{d-1}} \right)^{\frac{2}{d-1}} \left(\int_{\partial K} \frac{\kappa(x)^{\frac{1}{2}}}{\langle x, u_x \rangle^{\frac{d-1}{2}}} \, dx \right)^{\frac{2}{d-1}} n^{\frac{-2}{d-1}}$$

as n tends to infinity where κ_m is the volume of the unit m-ball, and ϑ_m is the minimal density of coverings of $\mathbb{R}^m$ by unit balls. Here the integral in the parentheses is the so called centro-affine surface area.

Related Problems:

  (1) Prove (4) or (5) if $d \geq 3$ and ∂K is not C^2 but still $A(K) > 0$.
  (2) Prove the analogue of (4) or (5) if not the number of vertices is restricted but the number of k-faces where $1 \leq k \leq d - 2$ and $d \geq 4$.

The first problem was solved in the plane by Ludwig [28]. For the second problem, if the number of facets is restricted ($k = d - 1$) or $d = 3$ and $k = 1$ then the analoguous results are known.

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An infinitesimal version of Brunn-Minkowski type inequalities

Andrea Colesanti

This report is an informal summary of the paper [3]. Our aim is to present a connection between the Brunn-Minkowski inequality and the Poincaré inequality.

The Brunn-Minkowski inequality asserts that the n-dimensional volume V_n raised to the power $1/n$ is concave in the set of n-dimensional convex bodies. Assume that we manage to give a meaning to the *second variation* of $V_n^{1/n}$, then this has to be negative semi-definite. We will see that this information is in turn equivalent to a family of functional inequalities, similar to the Poincaré inequality, which can be read either on the unit sphere $\mathbf{S}^{n-1}$ of $\mathbf{R}^n$ or on the boundary of convex bodies.

The idea of the argument that I will present originates from a paper by Jerison [4] where the first and second variation of the Newton capacity of convex bodies are computed.

Let $K \subset \mathbf{R}^n$ be a convex body of class C_+^2, i.e. $\partial K \in C^2$ and the Gauss curvature is everywhere strictly positive on ∂K. We denote by h the support function of K, defined on $\mathbf{S}^{n-1}$ (and not on the whole space). Then we have that $h \in C^2(\mathbf{S}^{n-1})$ and the following matrix inequality holds

$$(h_{ij} + h\delta_{ij}) > 0 \quad \text{on } \mathbf{S}^{n-1}.$$

Here h_{ij}, $i, j = 1, \ldots n - 1$, denote the (second) covariant derivatives of h and (δ_{ij}) is the identity matrix. The volume of K can be computed according to the formula

$$V_n(K) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} h \det(h_{ij} + h\delta_{ij}) \, dH^{n-1}.$$

Hence we define the family of functions $\mathcal{C} = \{h \in C^2(\mathbf{S}^{n-1}) : (h_{ij} + h\delta_{ij}) > 0 \text{ on } \mathbf{S}^{n-1}\}$ and the functional

$$F : \mathbf{S}^{n-1} \to \mathbf{R}_+, \quad F(h) := \frac{1}{n} \int_{\mathbf{S}^{n-1}} h \det(h_{ij} + h\delta_{ij}) \, dH^{n-1}.$$

By the Brunn-Minkowski inequality we have that: *the functional F raised to $1/n$ is concave in $\mathcal{C}$.*

Our next step is to compute the first and second variation of F. Let us fix $h \in \mathcal{C}$ and $\phi \in C^\infty(\mathbf{S}^{n-1})$; note that $h + s\phi \in \mathcal{C}$ for $|s|$ sufficiently small. It can be shown that

$$\frac{d}{ds} F(h + s\phi)|_{s=0} = \int_{\mathbf{S}^{n-1}} \phi \det(h_{ij} + h\delta_{ij}) \, dH^{n-1}.$$

This equality is well-known (note that if ϕ is the support function of a convex body, then the above derivative is a mixed volume). Then we can write

$$\frac{d}{ds} F(h + s\phi)|_{s=0} = (F'(h), \phi),$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\mathbf{S})^{n-1}$ and $F'(h) := \det(h_{ij} + h\delta_{ij})$ is the first variation of F at h.

In a similar way we will now compute the second variation of F. We set $(c_{ij}) := \det(h_{ij} + h\delta_{ij}) (h_{ij} + h\delta_{ij})^{-1}$, i.e. (c_{ij}) is the *cofactor matrix* of $(h_{ij} + h\delta_{ij})$. For $h \in \mathcal{C}$ and $\phi \in C^\infty(\mathbf{S}^{n-1})$ we have

$$(1) \qquad\qquad \frac{d}{ds} F'(h + s\phi)|_{s=0} = \sum_{ij} c_{ij}(\phi_{ij} + \phi\delta_{ij}).$$

The proof of this equality is a consequence of the following fact: if (γ_{ij}) is the cofactor matrix of $A = (a_{ij})$, then, for every i, j, γ_{ij} is the partial derivative of $\det(A)$ with respect to a_{ij}. The right hand-side term of (1) is the second variation of F at h applied to ϕ:

$$F''(h)\phi := \sum_{ij} c_{ij}(\phi_{ij} + \phi\delta_{ij}).$$

The concavity of $F^{1/n}$ implies that for every $s \geq 0$ the set $\{h \in \mathcal{C} \,:\, F^{1/n}(h) \geq s\}$ is concave and then $\{h \in \mathcal{C} \,:\, F(h) \geq s\}$ is also concave. In turn this yields that for every $h \in \mathcal{C}$ and $\phi \in C^\infty(\mathbf{S}^{n-1})$

$$(2) \qquad (F'(h), \phi) = 0 \;\Rightarrow\; (F'(h)\phi, \phi) \leq 0\,.$$

There is an heuristic explanation of (2): as the super-level sets of F are convex, for every $h \in \mathcal{C}$, F restricted to $F'(h)^\perp$ (i.e. the tangent space to the level set of F through h) has a maximum at h. Note also that it can be shown that (2) is equivalent to (and not just a consequence of) the concavity of $F^{1/n}$.

Using (2) and the expressions of F' and F'' that we have found before we can write:

$$\int_{\mathbf{S}^{n-1}} \phi \det(h_{ij} + h\delta_{ij})\, dH^{n-1} = 0 \quad \Rightarrow \quad \int_{\mathbf{S}^{n-1}} \sum_{ij} c_{ij}(\phi_{ij} + \phi\delta_{ij})\phi\, dH^{n-1} \leq 0\,.$$

From the last inequality we obtain

$$\int_{\mathbf{S}^{n-1}} \operatorname{tr}(c_{ij})\phi^2\, dH^{n-1} \leq -\int_{\mathbf{S}^{n-1}} \sum_{ij} c_{ij}\phi_{ij}\phi\, dH^{n-1} = \int_{\mathbf{S}^{n-1}} \sum_{ij} c_{ij}\phi_i\phi_j\, dH^{n-1}$$

where in the last equality we used the divergence theorem and the following property of the cofactor matrix:

$$\operatorname{div}_i c_{ij} = 0 \quad \text{for every fixed } j.$$

Therefore we have the following result.

Theorem 1. *Let $h \in C^2(\mathbf{S}^{n-1})$ be such that $(h_{ij} + h\delta_{ij}) > 0$ on $\mathbf{S}^{n-1}$. Then for every $\phi \in C^\infty(\mathbf{S}^{n-1})$*

$$\int_{\mathbf{S}^{n-1}} \phi \det(h_{ij} + h\delta_{ij})\, dH^{n-1} = 0$$

$$\implies \int_{\mathbf{S}^{n-1}} \operatorname{tr}(c_{ij})\phi^2\, dH^{n-1} \leq \int_{\mathbf{S}^{n-1}} \sum_{ij} c_{ij}\phi_i\phi_j\, dH^{n-1}.$$

Let K be the convex body which has h as support function. Performing the change of variable given by the Gauss map of K we obtain, after some computation, an equivalent form of Theorem 1.

Theorem 2. *Let K be a convex body of class C^2_+ in $\mathbf{R}^n$ and let $\nu : \partial K \to \mathbf{S}^{n-1}$ be the Gauss map of K. Then for every $\psi \in C^\infty(\partial K)$*

$$(3) \quad \int_{\partial K} \psi\, dH^{n-1} = 0 \;\Rightarrow\; \int_{\partial K} \operatorname{tr}(D\nu)\phi^2\, dH^{n-1} \leq \int_{\partial K} ((D\nu)^{-1}\nabla\phi, \nabla\phi)\, dH^{n-1}\,.$$

Here $D\nu$ denotes the differential of ν, i.e. the Weingarten map.

Remark 1. Choosing K as the unit ball we obtain for all $\psi \in C^\infty(\mathbf{S}^{n-1})$

$$\int_{\mathbf{S}^{n-1}} \psi\, dH^{n-1} = 0 \;\Rightarrow\; (n-1)\int_{\mathbf{S}^{n-1}} \psi^2\, dH^{n-1} \leq \int_{\mathbf{S}^{n-1}} |\nabla\psi|^2\, dH^{n-1}\,.$$

This is the classical Poincaré inequality on $\mathbf{S}^{n-1}$ with the sharp constant.

Remark 2. If $\psi(x) = (\nu(x), u)$, $x \in \partial K$, where $u \in \mathbf{R}^n$ is fixed, then equality holds in (3).

Remark 3. Let K be as in Theorem 2 and $\alpha > 0$ be such that all the principal curvatures are greater than or equal to α at any point of ∂K. Then, as the eigenvalues of $D\nu$ coincides with the principal curvatures, from (3) it follows that

$$(4) \qquad \int_{\partial K} \psi \, dH^{n-1} = 0 \;\Rightarrow\; (n-1)\alpha^2 \int_{\partial K} \phi^2 \, dH^{n-1} \leq \int_{\partial K} |\nabla \phi|^2 \, dH^{n-1}$$

for every $\psi \in C^\infty(\mathbf{S}^{n-1})$. In particular (4) provides a lower bound for the first eigenvalue $\lambda(\partial K)$ of the Laplace operator on ∂K:

$$\lambda(\partial K) \geq (n-1)\alpha^2 \,.$$

This a special case of a theorem by Lichnerowicz's in Riemannian geometry (see for instance [2]).

Remark 4. Assume that $f = e^{-u}$, $u \in C^2(\mathbf{R}^n)$, $D^2 u > 0$ in $\mathbf{R}^n$, $\lim_{|x| \to \infty} u(x) = \infty$, and denote by μ the measure on $\mathbf{R}^n$ such that $d\mu = f \, dx$. Then it is proved in [1] that

$$(5) \quad \int_{\mathbf{R}^n} \psi \, d\mu = 0 \;\Rightarrow\; \int_{\mathbf{R}^n} \psi^2 \, d\mu \leq \int_{\mathbf{R}^n} ((D^2 u)^{-1} \nabla \psi, \nabla \psi) \, d\mu, \quad \forall \psi \in C_c^\infty(\mathbf{R}^n) \,.$$

One can see an analogy between (3) and (5) which is strengthen by the following observation: inequality (5) can be deduced by the Prékopa-Leindler inequality (which is considered to be the functional form of Brunn-Minkowski inequality) in the same way as we deduced (3) from the Brunn-Minkowski inequality.

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A Solution to Hammer's X-ray Reconstruction Problem

Richard J. Gardner

(joint work with Markus Kiderlen)

In 1963, before the first CAT scanner was built, Hammer [7] posed the following problem.

Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord along an X-ray line. (No diffusion, please.) How many pictures must be taken to permit exact reconstruction of the body if:

a. The X-rays issue from a finite point source?

b. The X-rays are assumed parallel?

If K is a convex body in $\mathbb{R}^n$, the *parallel X-ray of K in the direction $u \in S^{n-1}$* is the function giving the lengths of the chords of K parallel to u. Although it is clear from the question that Hammer had in mind a method for reconstruction, early work concerned uniqueness. The first result was that of Gardner and McMullen [6], who proved that a planar convex body is determined, among all planar convex bodies, by its parallel X-rays in a finite set U of directions if and only if U is not a subset of the directions of edges of an affinely regular polygon. An example is any set U of four directions in S^1 such that the cross-ratio of the slopes is a transcendental number. Later, Gardner and Gritzmann [4] showed that more practical choices of sets U of four directions are possible; if each direction in U has rational slope and these slopes are arranged in increasing order, then to guarantee uniqueness one has only to avoid the cross-ratios 4/3, 3/2, 2, 3, and 4. A specific example is the set U specified by the vectors $(1,0)$, $(0,1)$, $(2,1)$, and $(-1,2)$. As an aside, we mention that the following question is open (see [3, Problem 2.1]).

Question. *Are convex bodies in $\mathbb{R}^3$ determined uniquely by parallel X-rays in any set of seven directions in general position?*

An example depicted in [3, Figure 2.1] shows that in general six directions in general position are insufficient.

Again, let K be a convex body in $\mathbb{R}^n$. The *point X-ray of K at a point $p \in R^n$* is the function giving the lengths of all the chords of K lying on lines through p. The uniqueness aspect of Hammer's question (a) is not completely solved even in the plane, but Volčič [9] proved that a planar convex body is determined uniquely among all planar convex bodies by its X-rays taken at any set of four points in general position.

For a fairly complete account of uniqueness results on parallel and point X-rays of convex bodies, see Chapters 1, 2, and 5 of [3].

It is clear from its phrasing that Hammer's problem is directed not just to issues of uniqueness, but also to the actual reconstruction of an unknown convex body from its X-rays taken in a finite set of directions or at a finite set of points that guarantees a unique solution. As far as we know, three such algorithms have been proposed for parallel X-rays. The first, a discretization method due to Kölzow,

Kuba, and Volčič [8], suffers from some serious deficiencies (see the discussion in [8] and [3, Note 1.2]). The second algorithm, proposed independently by Gardner and by Volčič (see Chapter 1 of [3]), makes some restrictive assumptions about the convex body and lacks a proof of convergence even under these assumptions. Finally, Brunetti and Daurat [1] suggest approximating convex bodies by convex lattice sets, but point out that no efficient algorithm is known for reconstructing convex lattice sets from their discrete parallel X-rays when these are only known approximately. (In [2], these authors study a similar approach to reconstructing the more general Q-convex bodies, but do not prove their algorithm converges.)

The new work, detailed in [5], presents new algorithms for reconstructing planar convex bodies from their parallel or point X-rays, in situations that guarantee a unique solution when the data is exact. The algorithms are inspired by a least-squares optimization procedure used previously for reconstructing homogeneous objects from noisy X-ray data in a program developed by an electrical engineer, A. S. Willsky, and his students, from the early 1980's.

The mainly Fourier-transform-based algorithms of computerized tomography produce an approximate image of a density function. Of course, in practise one can only measure a finite number of values of each X-ray, and increasing this number will improve the image. However, for the class of density functions, there is a fundamental lack of uniqueness that in general also requires X-rays to be taken in more directions to enhance the image. This lack of uniqueness remains even in the class of compact sets. Algorithms in the papers arising from Willsky's program often reconstruct planar convex bodies, but they do not exploit the uniqueness results that hold for this restricted class.

Unlike all algorithms previously proposed for solving Hammer's problem, ours still work when the data is noisy, and our convergence proofs apply also in this case. To be more specific, the algorithms take as input k equally spaced noisy X-ray measurements of the unknown planar convex body K_0 in each of the fixed directions or at each of the fixed points, and produce a convex polygon P_k that, almost surely, converges in the Hausdorff metric to K_0 as $k \to \infty$. The noise is modeled in the traditional way by adding independent $N(0, \sigma^2)$ random variables. The main tools in the proof of convergence are some geometric estimates involving inner parallel bodies and a version of the strong law of large numbers that applies to a triangular family, rather than a sequence, of independent random variables.

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Random ε nets and embeddings in ℓ_∞^N

YEHORAM GORDON

(joint work with A. E. Litvak, A. Pajor and N. T. Jaegermann)

In this note we show that, given an n-dimensional normed space X a sequence of $N = (8/\varepsilon)^{2n}$ independent random vectors $(X_i)_{i=1}^N$, uniformly distributed in the unit ball of X^*, with high probability forms an ε net for this unit ball. Thus the random linear map $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ defined by $\Gamma x = (\langle x, X_i \rangle)_{i=1}^N$ embeds X in ℓ_∞^N with at most $1 + \varepsilon$ norm distortion. In the case $X = \ell_2^n$ we obtain a random $1 + \varepsilon$ embedding into ℓ_∞^N with asymptotically best possible relation between N, n, and ε.

Let $X = (\mathbb{R}^n, \| \cdot \|)$ be an arbitrary n-dimensional normed space with unit ball K. It is well known that, for any $0 < \varepsilon < 1$, X can be $1 + \varepsilon$ embedded into ℓ_∞^N, for some $N = N(\varepsilon, n)$, depending on ε and n, but independent of X. In this note we investigate $1 + \varepsilon$ isomorphic embeddings which are random with respect to some natural measure, depending on K. We first show that for $N = (8/\varepsilon)^{2n}$, a sequence of N independent random vectors $(X_i)_{i=1}^N$, uniformly distributed in the unit ball K^0 of the dual space X^*, forms an ε net for K^0, with high probability. Thus, with high probability, the random linear map $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ defined by $\Gamma x = (\langle x, X_i \rangle)_{i=1}^N$ embeds X in ℓ_∞^N with at most $1 + \varepsilon$ norm distortion.

The important case is $X = \ell_2^n$. In this case it is more natural to consider random vectors X_i uniformly distributed on the sphere S^{n-1}. Such vectors also form an ε-net on the sphere hence they determine a random $1 + \varepsilon$ embedding Γ of ℓ_2^n into ℓ_∞^N. We also show that $\sqrt{n/N}\,\Gamma$ is a $1 + \varepsilon$ isometry from ℓ_2^n into ℓ_2^N, with high probability.

The case $X = \ell_2^n$ is connected with Dvoretzky's theorem ([D]). Milman found a new proof ([M]), using the Levy isoperimetric inequality on the sphere, that there exists a function $c(\varepsilon) > 0$ such that for all $n \le c(\varepsilon) \log N$, ℓ_2^n can be $1 + \varepsilon$ embedded into any normed space Y of dimension N. His proof gives $c(\varepsilon) \sim \varepsilon^2 / \log(2/\varepsilon)$. Later a new approach was found in ([Go]) by using random Gaussian embeddings. It yields that $c(\varepsilon) \sim \varepsilon^2$ is sufficient. Milman raised the question what is the best behavior of $c(\varepsilon)$, as $\varepsilon \to 0$, in the above estimates. Recently Schechtman showed in [S1] that one may take $c(\varepsilon) \sim \varepsilon / (\log(2/\varepsilon))^2$, however his approach is not random.

Since in this paper we deal with embeddings into ℓ_∞^N, we shall restrict our attention to this case only. When $Y = \ell_\infty^N$, it is well known that there exists an

embedding with $c(\varepsilon) \sim 1/\log(2/\varepsilon)$. It is also known that this behavior of $c(\varepsilon)$ as $\varepsilon \to 0$ cannot be improved. The standard embedding relies on the existence of ε-nets of appropriate cardinalities. It is therefore natural to ask whether this embedding can be randomized.

In this paper we provide a positive answer to this question. Namely, we show that for the random embedding Γ determined by independent uniformly distributed vectors on S^{n-1}, with large probability one may achieve $c(\varepsilon) \sim \frac{1}{\log(2/\varepsilon)}$, which is the best possible as mentioned above. The precise statement is:

Theorem 1. *Let $n \geq 1$, $0 < \varepsilon \leq 1$, and $N = (4/\varepsilon)^{2n}$. Let $X_1, \ldots, X_N$ be independent random variables uniformly distributed on a symmetric convex body K in $\mathbb{R}^n$. Then with a probability larger than $1 - \exp\left(-(8/\varepsilon)^n/2\right)$ the set $\mathcal{N} = \{X_1, \ldots, X_N\}$ forms an ε-net in K.*

We would like to note that such result is not valid in the setting of the Haar measure on Grassman manifold (equivalently, for embeddings defined by Gaussian matrices). Indeed, Schechtman recently showed ([S2]) that if "most" $n = c'(\varepsilon) \log N$ dimensional subspaces of ℓ_∞^N are $1 + \varepsilon$ Euclidean then $c'(\varepsilon) \sim \varepsilon$.

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On the local equatorial characterization of zonoids and intersection bodies

DMITRY RYABOGIN AND ARTEM ZVAVITCH

(joint work with F. Nazarov)

A *zonoid* in $\mathbb{R}^n$ is an origin symmetric convex body that can be approximated (in the Hausdorff metric) by finite Minkowski sums of line segments. It turns out that zonoids appear in many different contexts in convex geometry, physics, optimal control theory, and functional analysis (we refer the reader to [B], [BL], [BLM], [Ga2], [GW2], [P], [Sc1], [Sc2], [ScW]). One of the equivalent definitions of zonoids, useful in convex geometry, leads to a notion of a projection body. An origin symmetric convex body L in $\mathbb{R}^n$ is called a *projection body* if there exists another origin symmetric convex body K such that the support function of L in

every direction is equal to the volume of the hyperplane projection of K orthogonal to this direction: for every $\xi \in S^{n-1}$,

$$h_L(\xi) = \text{Vol}_{n-1}(K|\xi^\perp),$$

$\xi^\perp = \{y \in \mathbb{R}^n : \xi \cdot y = 0\}$. The support function $h_L(\xi) = \max_{x \in L} \xi \cdot x$ is equal to the dual norm $\|\xi\|_{L^*}$ where L^* stands for the polar body of L. From the above definition and Cauchy formula (see [K], page 25), we immediately derive the following analytic definition, which will be useful for us in this paper: An origin symmetric convex body $L \subset \mathbb{R}^n$ is a zonoid if and only if

$$h_L(\xi) = Cos\mu(\xi) := \int_{S^{n-1}} |\xi \cdot \theta| d\mu(\theta)$$

with some even positive measure μ on S^{n-1}. Finally, a functional analytic definition shows that an origin symmetric convex body $L \subset \mathbb{R}^n$ is a zonoid if and only if it is a polar body to the unit ball of a subspace of L_1.

It is well known that every origin symmetric convex body in $\mathbb{R}^2$ is a projection body, but this is no longer true in $\mathbb{R}^n$ for $n \geq 3$ (see [Sc2], [K]). It is an interesting question how to determine if a given convex body is a zonoid or not. It is very reasonable to assume that one can provide a strictly local characterization of zonoids. This question was posed repeatedly (see [Sc2] for the history of the problem), however W. Weil showed [W] that a *local* characterization of zonoids does not exist. In particular, he showed that *there exists an origin-symmetric convex C^∞ body $K \subset \mathbb{R}^n$, $n \geq 3$, that is not a zonoid but has the following property: for every $u \in S^{n-1}$ there exists a zonoid Z_u centered at the origin and a neighborhood $U_u \subset S^{n-1}$ of u such that the boundaries of K and Z_u coincide at all points where the exterior unit normal vectors belong to U_u.* Thus, no characterization of zonoids that involves only arbitrarily small neighborhoods of boundary points is possible.

In 1977, W. Weil (see [W]) proposed the following conjecture about *local equatorial* characterization of zonoids. *Let $L \subset \mathbb{R}^n$ be an origin-symmetric convex body and assume that for any equator $\sigma \subset S^{n-1}$, there exists a zonoid Z_σ and a neighborhood E_σ of σ such that the boundaries of L and Z_σ coincide at all points where the exterior unit vector belongs to E_σ; then L is a zonoid.* *Affirmative* answers for *even* dimensions were given independently by G. Panina [Pan] in 1988 and Goodey and Weil [GW] in 1993, but the question was left open in odd dimensions. That was a consequence of the fact that the inversion formulas for the cosine transform are not local in odd dimensions.

We show that the answer to the conjecture in *odd* dimensions is *negative*. We prove that in both cases (for odd and even dimensions) the answer can be obtained as a consequence of the characterization of zonoids in terms of sections of the polar body, given in [KRZ]. In even dimensions the answer follows directly from the geometric inversion formula for the Cosine transform [KRZ]. The odd dimensional case, on the other hand, requires much more tricky and detailed analysis of the behavior of the inverse Cosine transform.

Our main tool is the Fourier analytic inversion formula from [GKS2]. It allows to obtain the results for zonoids together with the results about the intersection

bodies. The notion of an *intersection body of star body* was introduced by E. Lutwak [Lu]. K is called the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction: $\forall \xi \in S^{n-1}$,

$$\rho_K(\xi) = \mathrm{Vol}_{n-1}(L \cap \xi^\perp),$$

where $\rho_K(\xi) = \max\{a : a\xi \in K\}$ is the radial function of the body K. Passing to polar coordinates in $\xi^\perp$, we derive the following analytic definition of an *intersection body of star body*: K is called the intersection body of L if

$$\rho_K(\xi) = \frac{1}{n-1}\Re\rho_L^{n-1}(\xi) := \frac{1}{n-1}\int\limits_{S^{n-1}\cap\xi^\perp} \rho_L^{n-1}(\theta)d\theta.$$

Here $\Re$ stands for the spherical Radon transform.

A more general class of *intersection bodies* was defined by R. Gardner [Ga1] and G. Zhang [Zh] as the closure of intersection bodies of star bodies in the radial metric $d(K,L) = \sup_{\xi\in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|$. We will consider only C^∞ smooth intersection bodies: a body K is an intersection body if there exists an even nonnegative function f on S^{n-1}, such that the radial function of K is a spherical Radon transform $\Re f$ of f. Since we can always define $L: \rho_L^{n-1}(\theta) = (n-1)f(\theta)$, we will not distinguish between *intersection bodies of star bodies* and *intersection bodies*.

We prove that the local equatorial characterization of intersection bodies is not possible in *odd* dimensions. Namely, we show that one can construct *an origin-symmetric convex body $L \subset \mathbb{R}^n$, $n \geq 5$ is odd, such that for any equator $\sigma \subset S^{n-1}$, there exists an intersection body I_σ and a neighborhood E_σ of σ such that the boundaries of L and I_σ coincide at all points of E_σ (i.e. $\rho_L(\xi) = \rho_{I_\sigma}(\xi)$ for all $\xi \in E_\sigma$); but nevertheless, L is not an intersection body.* On the other hand, we show that the local equatorial characterization of intersection bodies is possible in *even* dimensions.

We also extend the result of W. Weil [W] to the class of intersection bodies by proving that there is no local characterization of those bodies in odd and even dimensions. We prove that *there exists an origin-symmetric convex C^∞ body $K \subset \mathbb{R}^n$, $n \geq 5$, that is not an intersection body, but has the following property: for each $u \in S^{n-1}$ there exists an intersection body I_u centered at the origin and a neighborhood $U_u \subset S^{n-1}$ of u such that the boundaries of K and I_u coincide on U_u.* In odd dimensions this is a consequence of the lack of a local equatorial characterization of intersection bodies mentioned above but we give an independent proof that does not distinguish between even and odd dimensions.

Our proofs for zonoids and intersection bodies are very similar, they are based on almost identical Fourier analytic inversion formulas for the Cosine and Radon transforms. This is one more indication of the remarkable duality between sections and projections (see [KRZ1]).

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Improved Stability Results in Geometric Tomography

Markus Kiderlen

A celebrated result of Aleksandrov [1] states that a full-dimensional origin-symmetric compact convex subset K of $\mathbb{R}^n$ is determined by its brightness function $u \mapsto V_{n-1}(K|u^\perp)$. Here, V_{n-1} stands for $(n-1)$-dimensional volume and $K|u^\perp$ is the orthogonal projection of K onto the hyperplane $u^\perp$ with unit normal u. Bourgain and Lindenstrauss [3] showed a corresponding stability result. We write $\mathcal{K}(r, R)$ for the set of all convex bodies (compact convex sets in $\mathbb{R}^n$) containing an origin-symmetric ball of radius $r > 0$ and contained in a concentric ball of radius R. For any $r, R, \gamma > 0$ there is a constant $c_1 = c_1(n, r, R, \gamma) > 0$ such that

$$(1) \qquad \delta(K, K') \leq c_1 \|V_{n-1}(K|u^\perp) - V_{n-1}(K'|u^\perp)\|^{2/(n(n+4))-\gamma}$$

holds for all origin-symmetric $K, K' \in \mathcal{K}(r, R)$. Here, $\delta(K, K')$ is the Hausdorff distance between K and K' and $\|\cdot\|$ denotes the L^2-metric on the unit sphere (with respect to u). Independently of Bourgain and Lindenstrauss, Campi [4] has shown a similar stability result in the special case $n = 3$ with Hölder exponent arbitrarily close to $1/9$, which is better than $2/21$ in (1). The original motivation for the present work was the question whether Campi's improved stability result can be extended to higher dimensions. It turns out that a combination of Campi's method with the Poisson integral estimates exploited by Bourgain and Lindenstrauss leads to an improved exponent in (1) for arbitrary $n \geq 2$: $2/(n(n+4))$ can be replaced by $2/(n(n+1))$, which yields $1/6$ in $\mathbb{R}^3$.

Following Hug and Schneider [12] this can be generalized to tomographic data, which can be written as a *multiplier transform w.r.t. the $(n-1)$-st surface area measure $S_{n-1}(K, \cdot)$* of K. We write $S_{n-1}(K, \cdot) \sim \sum_{k=0}^\infty s_k(\cdot)$ to denote the (formal and condensed) spherical harmonic expansion of $S_{n-1}(K, \cdot)$ (cf. Groemer [G], Kiderlen [14]). The observed tomographic data $F(K, \cdot)$ is assumed to be a function on the unit sphere and to satisfy

$$F(K, \cdot) \sim \sum_{k=0}^\infty a_k s_k(\cdot)$$

for some real sequence $a_1, a_2, \ldots$ of so-called *multipliers*. Assume further that this sequence is slowly decreasing: there are positive constants b and β such that

(2)
$$|a_k| \geq bk^{-\beta}, \qquad \text{for all } k \text{ with } a_k \neq 0.$$

The size of the parameter β indicates how smooth the function $F(K, \cdot)$ is. Thus, stability results for tomographic data with large β will be weaker than those for $F(K, \cdot)$ with small β. In the following we state a result for $\beta \geq 3/2$, a stronger version exists for smaller β. Let $\mathcal{K}_F(r, R)$ be the family of all convex bodies $K \in \mathcal{K}(r, R)$ with Steiner point at the origin such that

$$S_{n-1}(K, \cdot) \sim \sum_{\substack{k=0 \\ a_k \neq 0}}^{\infty} s_k(\cdot).$$

Any $K \in \mathcal{K}_F(r, R)$ is uniquely determined by its tomographic data $F(K, \cdot)$. In the important special case, where exactly the even multipliers of $F(K, \cdot)$ are non-zero, $\mathcal{K}_F(r, R)$ is the family of all origin-symmetric $K \in \mathcal{K}(r, R)$.

Theorem 1. *Let $R \geq r > 0$, $n \geq 2$, and $\gamma > 0$ be given. If $F(K, \cdot)$ is a multiplier transformation w.r.t. the $(n-1)$-st surface area measure such that its multipliers satisfy (2) with $\beta \geq 3/2$, then there is a constant $c_2 = c_2(n, b, \beta, \gamma, r, R)$ with*

(3)
$$\delta(K, K') \leq c_2 \|F(K, \cdot) - F(K', \cdot)\|^{\frac{1}{n(\beta - 1/2)} - \gamma}$$

for all $K, K' \in \mathcal{K}_F(r, R)$.

This result is extended to multiplier transformations w.r.t. lower order surface area measure (cf. [12]), w.r.t. a power of the radial function and w.r.t. a power of the support function. This yields better Hölder exponents for many of the classical stability results (cf. [2], [5], [6], [7], [8], [10], [11], [13], [16], [17]) for tomographic data derived from sections, projections and certain averages of them. It is, however, an open problem to find the best possible Hölder exponents in stability inequalities. For instance, the Hölder exponent in (1), and also the improved exponent $2/(n(n+1))$, depend quadratically on $1/n$, although the dependence on $1/n$ of the best possible exponent is conjectured to be linear, cf. [3].

These results will be published in [15].

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Inequalities of the Khinchin type and sections of L_p-balls

Alexander Koldobsky

(joint work with A. Pajor and V. Yaskin)

The slicing problem asks whether there exists a universal constant C such that for every origin symmetric convex body in $\mathbb{R}^n$ the following inequality holds

$$(1) \qquad (\text{vol}(K))^{(n-1)/n} \le C \max_{\xi \in S^{n-1}} \text{vol}(K \cap \xi^{\perp}),$$

where $\xi^{\perp}$ is the central hyperplane orthogonal to ξ. In other words, does there exist a universal constant such that every convex origin symmetric body of volume one has a hyperplane section of volume greater than this universal constant?

The problem still remains open. Bourgain [Bo] proved the inequality above with $O(n^{1/4} \log n)$ in place of C, and very recently Klartag [Kl] removed the logarithmic term in this estimate. However there are many classes of bodies for which the slicing problem holds true with a constant independent of dimension (see e.g [Ba], [BKM], [KMP], [MP]). In particular the slicing problem is solved for the unit balls of quotients of L_p, $p > 1$ by Junge [J], for the unit balls of subspaces of L_p, $1 \le p \le 2$ by Ball [Ba] and for the unit balls of subspaces of L_p, $p > 2$ by E. Milman [M]. As $p \to \infty$ the latter would have solved the problem, hadn't the constant behaved at infinity as $\sqrt{p}$.

We try a different approach, considering negative values of p. The concept of embedding in L_{-p} with $0 < p < n$ was introduced in [Ko1], and it was proved that

a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\cdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$. We will call unit balls of such spaces *p-intersection bodies* or L_{-p}-*balls*. For example, L_{-1}-balls are intersection bodies and L_{-k} balls are k-intersection bodies; see [Ko2].

We would like to know whether the statement of the slicing problem is true for L_p-balls with p negative. Of course, if one could show this for $p \in (-n, -n+3]$, then one would solve the slicing problem completely, since for any origin-symmetric convex body $K \subset \mathbb{R}^n$, the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p for such values of p; see [Ko3, Section 4.2]. In this paper we show that the slicing problem is true for L_p-balls, $p > -2$. The proof of the following theorem is based on the extension of Khinchin's inequalities for linear functionals to the exponents greater than -2. Note that Khinchin's inequalities of this type were previously known only for the exponents greater than -1; see [MP], [G].

Theorem 1. *Let* $0 < p < 2$, *if* K *is an origin-symmetric convex p-intersection body in* $\mathbb{R}^n$, *then*

$$(\mathrm{vol}(K))^{(n-1)/n} \le C(p) \max_{\xi \in S^{n-1}} \mathrm{vol}_{n-1}(K \cap \xi^\perp),$$

where

$$C(p) = \begin{cases} \left(\frac{\pi^{1-p/2}}{\Gamma(p/2)\Gamma(2-p)\sin(\pi p/2)} \right)^{1/p}, & \text{if } 0 < p \le 1, \\ \left(\frac{\pi^{1-p/2}}{\Gamma(p/2)\sin(\pi p/2)} \right)^{1/p}, & \text{if } 1 < p < 2. \end{cases}$$

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Random matrices and applications to convexity
Mark Rudelson

This is a report on the paper [9]. We consider random matrices with independent identically distributed (i.i.d.) entries. Such matrices appear in convex geometry as one of the standard tools to construct "nice" sections of convex bodies. In these applications it is usually important that a random matrix does not distort too much the Euclidean metric. More precisely, let A be an $N \times n$ matrix with $N > n$. Denote by $s_k(A)$ the k-th singular value of the matrix A, i.e. the k-th largest eigenvalue of the matrix $(AA^*)^{1/2}$. Then the distortion (*condition number*) can be written as

$$D(A) = \frac{s_1(A)}{s_n(A)} = \frac{\max_{x \in A} \|Ax\|}{\min_{y \in A} \|Ay\|}.$$

To bound the distortion, one has to estimate the first singular value from above, and the n-th one from below.

We assume that the entries of a random matrix A have mean 0 and satisfy the subgaussian tail estimate. More precisely, a random variable β is called subgaussian if for any $t > 0$, $\mathbb{P}(|\beta| > t) \leq b_1 \exp(-b_2 t^2)$ for some constants $b_1, b_2 > 0$. Let C, C', c etc. denote constants depending only on b_1, b_2, whose value can change from line to line. The class of subgaussian random variables includes many types of variables, which naturally arise in applications, such as normal variables, Bernoulli variables etc. For matrices with subgaussian entries the first singular value $s_1(A) = \|A\|$ is strongly concentrated about $(1 + \sqrt{\alpha}) \cdot \sqrt{N}$, where $\alpha = n/N$, see [3]. Estimating the smallest singular value $s_n(A)$ is a much more delicate problem. In [2] Bai and Yin proved that if we consider a sequence of $N \times n$ random matrices A_n, where $n \to \infty$, while $\alpha = n/N < 1$ remains constant, then the smallest singular value of $A_n/\sqrt{N}$ converges almost surely to $1 - \sqrt{\alpha}$. This result, however does not provide the estimates for fixed N and n. The paper [6] contains estimates for $s_n(A)$ in the case when $\delta = (N - n)/n \geq c/\log n$, but theses estimates are exponential in $1/\delta$. Later Artstein-Avidan, Friedman, Milman and Sodin proved the concentration for Bai–Yin result for random ± 1 matrices if $\delta \geq cn^{-1/6}$ (see [1]).

In [9] we obtain a bound for the smallest singular value, which is valid for all values of δ and all subgaussian random variables. While such estimate does not involve any geometry, the proof relies on a combination of probabilistic and geometric ideas. Instead of the ℓ_2-norm we consider the ℓ_1 norm, and the geometry of the unit ball of ℓ_1 plays a crucial role. We prove the following main

Theorem 1. *Let n, N be natural numbers such that $n < N < 2n$. Denote $\delta = (N - n)/n$. Let β be a centered subgaussian random variable of variance 1. Let $A = A(\omega)$ be an $N \times n$ matrix, whose entries are independent copies of β. Then for any t such that $\bar{C}n^{-3/2} < t < \bar{c}\delta$*

$$\mathbb{P}\left(\omega \mid \exists x \in S^{n-1} \|Ax\|_1 < t\delta n\right) \leq C\exp(-cn) + (t/\bar{c}\delta)^{\delta n}.$$

Here $\bar{C} > 1$ and $\bar{c} < 1$ are constants depending only on b_1, b_2 from the definition of the subgaussian random variable. Notice that the definition of δ implies $\delta \geq 1/n$. In the case $N \geq 2n$ ($\delta \geq 1$) an estimate for the minimum of $\|Ax\|_1$ over $x \in S^{n-1}$ follows from the results of [7], [8].

To derive applications of Theorem 1 we need the following standard lemma (see e.g. [3] or [10], Lemma 2.3).

Lemma 2. *Let n, N, A be as above. Then $\mathbb{P}\left(\|A\| > C\sqrt{N}\right) \leq \exp(-cN)$.*

Combining Theorem 1 and Lemma 2 we show that

(1) $$\forall x \in \mathbb{R}^n \quad t\delta n\|x\|_2 \leq \|Ax\|_1 \leq \sqrt{N}\|Ax\|_2 \leq C'n\|x\|_2.$$

with probability greater than $1 - C\exp(-cn) - (t/\bar{c}\delta)^{\delta n}$. This immediately yields the following

Corollary 3. *Let n, N, A, t be as above. Then the smallest singular number of A is bounded below by $t\delta \cdot \sqrt{n}$ with probability at least $1 - C\exp(-cn) - (t/\bar{c}\delta)^{\delta n}$.*

Another application of Theorem 1 is related to Kashin's sections of the cross-polytope. A celebrated theorem of Kashin [5] states that a random section of the cross-polytope B_1^n of dimension $m \sim n$ is close to the section of the inscribed ball $(1/\sqrt{n})B_2^n$. The optimal estimates for the diameter of a random section of the octahedron were obtained by Garnaev and Gluskin [4]. Recently the attention was attracted to the question whether the almost spherical sections of the octahedron can be generated by simple random matrices, in particular by a random ± 1 matrix. A general result proved in [7], [8] implies that if $N = (1 + \delta)n$ with $\delta \geq c/\log n$, then a random $N \times n$ matrix with independent subgaussian entries generates a section of the octahedron B_1^N which is not far from the ball with probability exponentially close to 1. For random ± 1 matrices this result was improved by Artstein-Avidan, Friedland and Milman and Sodin [1], who proved a polynomial type estimate for the diameter of a section for $\delta \geq Cn^{-1/6}$. Using (1) we obtain a polynomial estimate for the diameter of sections for all values of δ.

Corollary 4. *Let n, N be natural numbers such that $n < N < 2n$. Denote $\delta = (N - n)/n$. Let β be a centered subgaussian random variable of variance 1. Let $A = A(\omega)$ be an $N \times n$ matrix, whose entries are independent copies of β and let $E = A\mathbb{R}^n$. Then for any t such that $\bar{C}n^{-3/2} \leq t \leq \bar{c}\delta$*

$$\mathbb{P}\left(\omega \mid \forall y \in E, \ \|y\|_1 \leq \sqrt{N}\|y\|_2 \leq \frac{c}{t\delta}\|x\|_1\right) \geq 1 - C\exp(-cn) - (t/\bar{c}\delta)^{\delta n}.$$

Notice that in the case $\delta \geq Cn^{-1/6}$ this improves both the diameter and the probability estimates of [1].

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Curvature and q-strict convexity

Evangelia Samiou

(joint work with L. Dalla)

Let $\mathcal{C}$ be the set of nonempty compact convex subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric and the induced topology. By $\mathcal{C}^k$ we denote the subset of $\mathcal{C}$ of those convex sets whose boundary is a hypersurface of class C^k. Furthermore let $\mathcal{S} \subset \mathcal{C}$ be the set of strictly convex subsets of $\mathbb{R}^d$, i.e. of those $K \subset \mathbb{R}^d$ whose boundary ∂K does not contain a line segment. It is proved in [4, 5], see also [3], that $\mathcal{C} \setminus (\mathcal{C}^1 \cap \mathcal{S})$ is a F_σ-subset of first category and that $\mathcal{C}^2$ is of first category in $\mathcal{C}$. We are concerned with analogous questions within the spaces $\mathcal{C}^k$, $k \geq 2$.

A set $K \in \mathcal{C}^q$ is q-strictly convex if at each point $p \in \partial K$ the tangent hyperplane $T_p \partial K$ has contact of order at most $q - 1$ with ∂K. In terms of defining functions this can be rephrased as follows.

Definition 1. *Let* $K = \rho^{-1}((-\infty, 0])$ *with* $\rho \in C^q(\mathbb{R}^d)$ *and* $d_x\rho \neq 0$ *for each* $x \in \partial K = M$. *Then* K *is* q-strictly convex if for each $x \in M$ and each $u \in T_x M$, $u \neq 0$, *there is* $l \leq q$ *such that* $d_x^l \rho(u) > 0$ *(where* $d_x^l \rho(u) = d_x^l \rho(u, \dots, u)$ *is the lth derivative of* ρ*).*

We will denote by $\mathcal{S}_q$ the subspace of $\mathcal{C}^q$ consisting of q-strictly convex sets. Clearly,

$$\mathcal{C}^{q+1} \cap \mathcal{S}_q \subset \mathcal{S}_{q+1} \ .$$

It was shown in [1] that strictly convex compact sets with real analytic boundary are q-strictly convex for some q.

For $y \in \mathbb{R}^d$, $n \in \mathbb{R}^d \setminus \{0\}$, $q \in \mathbb{N}$ let

$$y_n = \langle y \mid n \rangle \in \mathbb{R} \quad \text{and} \quad y_{n^\perp} = y - \frac{y_n}{\|n\|^2} n \in \mathbb{R}^d$$

denote the projections. The "q-cone" at $x \in \mathbb{R}^d$ in direction of n is then defined as

$$C_q(x, n) := \{y \in \mathbb{R}^d \mid (y - x)_n \geq \|(y - x)_{n^\perp}\|^q\} .$$

For $K \in \mathcal{C}$ and $x \in M = \partial K$ we define the "q-curvature" of M at x by

$$\kappa^q(x) = \sup\{\|n\|^{-1} \mid K \cap B_\epsilon(x) \subset C_q(x, n) \text{ for some } \epsilon > 0\} .$$

In the case $q = 2$, $\kappa^2(x)$ is the minimal principal curvature of M at x. If $\kappa^p(x) > 0$ at some $x \in M$ then $\kappa^q(x) = \infty$ for all $q > p$.

Theorem 2. *A set $K \in \mathcal{C}^q$ is q-strictly convex if and only if the q-curvature of ∂K is positive, i.e. for each $x \in \partial K = M$ there are $n_x \in T_x M^\perp$, $n_x \neq 0$, such that $K \subset C_q(x, n_x)$.*

The notion of q-strict convexity of a set $K \in \mathcal{C}^q$ is related to intrinsic curvature properties of its boundary ∂K. We prove that an estimate from below on the sectional curvature of ∂K implies q-strict convexity. The minimal sectional curvature of M at $x \in M$ is defined as

$$K(x) := \min\{K(\sigma) \mid \sigma \subset T_x M, \dim \sigma = 2\}$$

where $K(\sigma)$ denotes the sectional curvature of the plane σ. A set $K \in \mathcal{C}^2$ is 2-strictly convex if and only if its boundary has positive sectional curvature.

Theorem 3. *Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $d_x \rho \neq 0$ for all $x \in M := \rho^{-1}(0)$. Assume that each $x \in M$ has a neighbourhood $U \subset M$ such that on U the sectional curvature K of M satisfies $K(x') \geq C d_M(x', x)^m$ with some constant $C = C(U) > 0$ independent of x'. Then for each component M_0 of M one of the two components of $\mathbb{R}^d \setminus M_0$ is strictly $(m + 2)$-convex.*

There is no characterization of q-strict convexity, $q > 2$, by an isotropic growth condition for the sectional curvature as in the assumption of the theorem. For example the function $\rho \colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\rho(x, y, z) = x^{2k} + y^{2l} + z$$

for $k \geq l > 2$ describes a $2k$-strictly convex set contained in the half space $\{z \leq 0\}$ in $\mathbb{R}^3$. The sectional curvature of the surface $\{\rho = 0\}$ vanishes on the lines $\{x = 0\}$ and $\{y = 0\}$. In particular, there is no estimate $K(x, y, z) \geq C d((x, y, z), 0)^m$ with $C > 0$.

Among the $\mathcal{S}_q$, $\mathcal{C}^q$ we have for $q \geq 2$ inclusions

$$\mathcal{S}_2 \subset \mathcal{S}_q \subset \mathcal{C}^q \subset \mathcal{C}^2 .$$

It is shown in [1] that $\mathcal{S}_2 \subset \mathcal{C}^2$ is dense. Hence all these inclusions are dense as well. In contrast to the results in [4, 5] for $\mathcal{C}$ we obtain here that analytic strict convexity is rather exceptional.

Theorem 4. $\mathcal{S}_q \subset \mathcal{C}^q$ *is a F_σ-set of first category.*

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A Fourier type transform on translation invariant valuations
Semyon Alesker

Let V be an n-dimensional real vector space. Let $\mathcal{K}(V)$ denote the class of convex compact subsets of V. Let $Val(V)$ denote the space of all continuous valuations on $\mathcal{K}(V)$. Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, $Val(V)$ is a Banach space.

Observe that the group $GL(V)$ of invertible linear transformations acts continuously on $Val(V)$ as follows:

$$(g(\phi))(K) = \phi(g^{-1}K) \text{ for any } g \in GL(V), \phi \in Val(V), K \in \mathcal{K}(V).$$

Definition 1. *A valuation $\phi \in Val(V)$ is called smooth if the map $GL(V) \to Val(V)$ given by $g \mapsto g(\phi)$ is C^∞-differentiable.*

Let us denote by $Val^\infty(V)$ the set of smooth valuations. It is a basic and a general fact from representation theory that $Val^\infty(V)$ is a linear $GL(V)$-invariant subspace dense in $Val(V)$. Moreover it carries a natural linear topology which is stronger than that induced from $Val(V)$ and which makes it a Fréchet space.

We have to remind the definition of the product on smooth valuations as it was introduced by the speaker in [2].

Theorem 2 ([2]). *There exists a bilinear map*

$$Val^\infty(V) \times Val^\infty(V) \to Val^\infty(V)$$

which is uniquely characterized by the following two properties:
  1) continuity;
  2) if $\phi(\bullet) = vol_n(\bullet + A)$, $\psi = vol_n(\bullet + B)$ then

$$(\phi, \psi) \mapsto vol_{2n}(\Delta(\bullet) + (A \times B))$$

where $\Delta \colon V \to V \times V$ is the diagonal imbedding.
  This bilinear map defines a product making $Val^\infty(V)$ a commutative associative algebra with unit (which is the Euler characteristic).

Let $D(V)$ denote the (one dimensional) space of complex valued Lebesgue measures on V. Recently Bernig and Fu [4] have defined a convolution product of $Val^\infty(V) \otimes D(V)^*$. They proved the following result.

Theorem 3 ([4]). *There exists a bilinear map*

$$Val^\infty(V) \otimes D(V)^* \times Val^\infty(V) \otimes D(V)^* \to Val^\infty(V) \otimes D(V)^*$$

which is uniquely characterized by the following two properties:
  1) continuity;
  2) if $\phi(\bullet) = vol_n(\bullet + A) \otimes vol_n^{-1}$, $\psi = vol_n(\bullet + B) \otimes vol_n^{-1}$ then

$$(\phi, \psi) \mapsto vol_n(\bullet + A + B) \otimes vol_n^{-1}.$$

  This bilinear map defines a product making $Val^\infty(V) \otimes D(V)^*$ a commutative associative algebra with unit (which is equal to $vol_v \otimes vol_n^{-1}$).

The main result of the talk is the following recent theorem due to the speaker.

Theorem 4 ([3]). *There exists an isomorphism of linear topological spaces (called a Fourier type transform)*

$$\mathbb{F}_V \colon Val^\infty(V) \to Val^\infty(V^*) \otimes D(V)$$

which satisfies the following properties:
  1) $\mathbb{F}_V$ commutes with the natural action of the group $GL(V)$ on both spaces;
  2) $\mathbb{F}_V$ is an isomorphism of algebras when the source space is equipped with the product from Theorem 2, and the target space is equipped with the convolution from Theorem 3.
  3)(analogue of the Plancherel inversion formula) Consider the composition $\mathcal{E}_V$

$$Val^\infty(V) \overset{\mathbb{F}_V}{\to} Val^\infty(V^*) \otimes D(V) \overset{\mathbb{F}_{V^*} \otimes Id_{D(V)}}{\to} Val^\infty(V) \otimes D(V^*) \otimes D(V) = Val^\infty(V).$$

  This composition $\mathcal{E}_V$ satisfies

$$(\mathcal{E}\phi)(K) = \phi(-K).$$

Remark 5. *The operator $\mathbb{F}_V$ was first introduced in the even case by the speaker in [1] under a different name and notation.*

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On the L_p-affine surface area of convex bodies

Alina Stancu

Motivated by the characterization of ellipsoids obtained via floating bodies [3], we investigated recently a similar characterization employing illumination bodies. A first result along this line concluded the following.

Theorem 1 ([4])**.** *Let $K \subset \mathbb{R}^{n+1}$ be a strictly convex body with boundary of class $\mathcal{C}^{\geq 4}$ and let K^δ denote the illumination body of K of factor δ. There exists a positive constant $\delta(K)$ such that the following holds:*

K is homothetic to K^δ, for some $\delta \in (0, \delta(K))$, with respect to the same center of homothety if and only if K is an ellipsoid.

Recall that the illumination body K^δ is defined as the set

$$(1) \qquad K^\delta = \{x \in \mathbb{R}^{n+1} : Vol\,(co[x,K] \setminus K) \leq \delta\},$$

where $co[x,K]$ is the convex hull of x and K, [5].

If K is a convex body of elliptic type, then there exists a convex body L, called the *curvature image* of K, [1], whose support function is $h_L(u) = f_K(u)^{-1/(n+2)}$. Here f_K denotes the curvature function of K as a function on the unit sphere $\mathbb{S}^n$. Interestingly, in this case, the illumination bodies of K for small factors δ can be described, up to an error term, as the Minkowski sum $K + \delta^{2/(n+2)} c_n L$, where c_n is a constant depending only on the dimension.

It seems then natural to ask what happens if one considers, up to small error terms, Firey sums, or Blaschke sums, of K with small homothetic copies of its curvature image body. For Firey sums, this reasoning led to the definition of weighted, respectively p, illumination bodies where the ellipticity of K is no longer required.

Definition 2. *Let μ be the Lebesgue measure on $\mathbb{R}^{n+1}$ and let $\rho : \overline{Ext(K)} \to \mathbb{R}$ be an integrable function, positive μ-a.e. Then*

$$K^{\delta,\rho} = \{x \in \mathbb{R}^{n+1} : \int_{co[x,K] \setminus K} \rho \, d\mu \leq \delta\}$$

is called the weighted δ-illumination body of K.

Furthermore,

Definition 3. *Let $p > 1$. We will call a p-illumination body of K, and denote it by $K^{\delta,p}$, if it is a weighted illumination body $K^{\delta,\rho}$ with*

$$(2) \qquad \rho(y) = \left((<y,u>)^{n+2} f_K(u) \right)^{\frac{(p-1)(n+1)}{2(n+1+p)}}$$

for $y \in \partial K$, where u is the normal of the hyperplane supporting ∂K at y, and extended continuously on $Ext(K)$ such that it decreases exponentially at infinity.

Consequently, for convex bodies with $\mathcal{C}^2$ boundaries, we obtain a new geometric interpretation of the p-affine surface area introduced by Lutwak [2] as

$$(3) \qquad \Omega_p(K) = \int_{\mathbb{S}^n} (f_{K,p}(u))^{\frac{n+1}{n+p+1}}\, d\mu_{\mathbb{S}^n}(u),$$

where $f_{K,p} = h^{1-p} f_K$ is the L_p-curvature function of K.

Theorem 4 (Interpretation of the L_p-affine surface area, [4])**.** *Let $K \subset \mathbb{R}^{n+1}$ be a strictly convex body with boundary of class $\mathcal{C}^{\geq 2}$ and let $p > 1$ a real number. Then, the L_p-affine surface area satisfies*

$$(4) \qquad \Omega_p(K) = \frac{1}{c_n} \lim_{t \to 0^+} \frac{Vol(K^{t,p}) - Vol(K)}{t},$$

where c_n is a constant depending only on the dimension.

We should mention that other interpretations of the L_p-affine surface area were also investigated by Hug, Ludwig-Reitzner, Lutwak, Meyer-Werner and others.

To conclude, two more characterizations of ellipsoids, similar in spirit to Theorem 1, were derived, one of them using p-illumination bodies, [4].

Theorem 5 ([4])**.** *Let $K \subset \mathbb{R}^{n+1}$ be a strictly convex body with boundary of class $\mathcal{C}^{\geq 4}$ and let $p > 1$ a real number. There exists a positive number $\delta(K)$ such that $K^{\delta,p}$ is homothetic to K, for some $\delta < \delta(K)$, with respect to the same center, if and only if K is an ellipsoid.*

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Shaken False Centre Theorems
Luis Montejano

Let K be a convex body and let p_0 be a point. Suppose that every section of K through p_0 is centrally symmetric, then Rogers proved in [3] that K is centrally symmetric, although p_0 may not be the centre of K. If this is the case, Aitchison, Petty and Rogers [1] and Larman [2] proved that K must be an ellipsoid. Suppose now that for every direction we can choose continuously a section of K that is centrally symmetric, if K is strictly convex, then we proved that K must be centrally symmetric. Consider now the following example: Let D be a solid sphere centered at the origin from which two symmetric caps are deleted. Then, D is centrally symmetric with respect the origin and has a lot of circular sections whose center is not the origin. In fact, we can choose continuously, for every direction, a section of D which is centrally symmetric in such a way that not all these sections pass through the origin. Nevertheless, no matter how we choose these sections, there are always many of them that necessarily pass through the origin. For those sections, of course, we have not imposed any condition which explains the fact that D is not a quadric elsewhere.

Let K be a centrally symmetric convex body and suppose that for every direction we can choose continuously a section of K which either does not pass through the centre of K and it is centrally symmetric or it passes through the centre of K and it is an ellipsoid. The purpose of this talk is to let you know that K is an ellipsoid.

Suppose now that in every direction we can choose continuously a section of K which is a translated copy of the corresponding parallel section of K through p_0. If except for a topological curve of directions all these sections are different, then K is an ellipsoid.

Note that if p_0 is a point of euclidean 3-space R^3 and if in every direction we choose continuously a plane parallel to this direction, then the collection of directions, as a subset of RP^2, for which the corresponding planes pass through the point p_0 contains always a curve which represents a non zero element of $\pi_1(RP^2)$. Of course, for any convex body K and any point $p_0 \in \mathrm{int} K$, we can always choose continuously a section of K which is a translated copy of the corresponding parallel section of K through p_0, by choosing trivially the same section through p_0, but our theorem states that if we choose as much sections as we can, which are translated copies of the corresponding parallel section of K through p_0, but which does not pass through p_0, then K is an ellipsoid.

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The Distribution of the Number of Vertices of the Convex Hull of Random Points

CHRISTIAN BUCHTA

This is an outlook to results which can be obtained using the main theorem in [4], which provides the distribution of the number of vertices of a random convex chain:

Theorem. *Assume that n points $P_1, \ldots, P_n$ are distributed independently and uniformly in the triangle with vertices $(0,1), (0,0)$, and $(1,0)$. Consider the convex hull of $(0,1), P_1, \ldots, P_n$, and $(1,0)$. Denote by N_n the number of those points $P_1, \ldots, P_n$ which are vertices. Let $p_k^{(n)}$ $(k = 1, \ldots, n)$ be the probability that $N_n = k$. Then*

$$p_k^{(n)} = 2^k \sum \frac{1}{i_1(i_1 + i_2) \ldots (i_1 + \ldots + i_k)}$$
$$\cdot \frac{i_1 \ldots i_k}{(i_1 + 1)(i_1 + i_2 + 1) \ldots (i_1 + \ldots + i_k + 1)},$$

where the sum is taken over all $i_1, \ldots, i_k \in \mathbb{N}$ such that $i_1 + \ldots + i_k = n$.

When $k = n$, the equation $i_1 + \ldots + i_n = n$ is fulfilled if and only if $i_1 = \ldots = i_n = 1$. Hence the sum representing $p_n^{(n)}$ consists of a single summand, and we immediately see that

$$p_n^{(n)} = \frac{2^n}{n!(n+1)!}.$$

This special case of the Theorem was obtained earlier by Bárány, Rote, Steiger, and Zhang [1]. Equivalently,

$$p_n^{(n)} = \frac{2}{n(n+1)} p_{n-1}^{(n-1)},$$

with $p_0^{(0)} = 1$. The Theorem implies that

$$p_k^{(n)} = \frac{2}{n(n+1)} \sum_{j=k-1}^{n-1} (n-j) p_{k-1}^{(j)},$$

with $p_0^{(0)} = 1$ and $p_0^j = 0$ for $j \in \mathbb{N}$. The recurrence formula can be used to determine the expected value and the variance of the random variable N_n:

$$EN_n = \frac{1}{3} \left(2 \sum_{k=1}^{n} \frac{1}{k} + 1 \right),$$

$$\text{var} \, N_n = \frac{1}{27} \left(10 \sum_{k=1}^{n} \frac{1}{k} + 12 \sum_{k=1}^{n} \frac{1}{k^2} - 28 + \frac{12}{n+1} \right).$$

The asymptotic version of the first formula, i.e. $EN_n \sim \frac{2}{3} \log n$ as n tends to infinity, is a classical result due to Rényi and Sulanke [7]. The asymptotic version of the second formula, i.e. $\text{var} \, N_n \sim \frac{10}{27} \log n$ as n tends to infinity, is due

to Groeneboom [5]. (For comments on Groeneboom's paper [5] see [3], and for Groeneboom's answer to these comments see [6].)

Now assume that C is a convex polygon. Consider two adjacent edges and those two of n points chosen at random from C which have the smallest distances to these edges. (The possibility that one and the same point has the smallest distances to both edges has to be dealt with separately.) Clearly, both points are vertices of the convex hull of the chosen points. The knowledge of the number of points which are also vertices of the convex hull and are situated — in an obvious sense — "between" the two points will give rise to the knowledge of the total number $N_n(C)$ of vertices of the convex hull.

Associating — in regard to affine invariance — the two points with the smallest distances to the considered adjacent edges with the points $(0,1)$ and $(1,0)$, associating the two lines which pass through one of the points and are parallel to the respective edges with the co-ordinate axes, and hence associating their intersection point with the point $(0,0)$, we are led to the question answered by the Theorem. It will be described in a future paper how the Theorem can in fact be used to obtain the exact distribution of the number $N_n(C)$ of vertices of the convex hull of n random points in C. The details are intricate. Here we only state a resulting formula for illustration: In the case of a triangle T it is well known that the expected value of $N_n(T)$ is given by

$$EN_n(T) = 2 \sum_{k=1}^{n-1} \frac{1}{k};$$

cf. [2]. The corresponding expression for the variance will be shown to be

$$\mathrm{var}\, N_n(T) = \frac{10}{9} \sum_{k=1}^{n-1} \frac{1}{k} - \frac{4}{3} \sum_{k=1}^{n-1} \frac{1}{k^2}.$$

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Geometry of the Cone of Positive Quadratic Forms
Peter M. Gruber

Among the special convex sets which play a role outside convex geometry, are balls, ellipsoids, regular polytopes, space fillers, zonoids, the polytope of doubly stochastic matrices, and the cone of positive definite quadratic forms. In [7] we study geometric properties of the latter.

A (real) quadratic form on Euclidean d-space $\mathbb{E}^d$, say

$$x \to \sum_{i,k=1}^{d} a_{ik} x_i x_k \ \text{ for } x = (x_1, \ldots, x_d)^T \in \mathbb{E}^d,$$

may be identified with its symmetric $d \times d$ *coefficient matrix* $A = (a_{ik})$ and also with its *coefficient vector* $(a_{11}, \ldots, a_{1d}, a_{22}, \ldots, a_{2d}, \ldots, a_{dd})^T \in \mathbb{E}^{\frac{1}{2} d(d+1)}$. Considering this identification, we write also $A \in \mathbb{E}^{\frac{1}{2} d(d+1)}$. The family of all positive definite quadratic forms on $\mathbb{E}^d$ then can be identified with an open convex cone $\mathcal{P}^d$ in $\mathbb{E}^{\frac{1}{2} d(d+1)}$ with apex at the origin O, the *cone of positive definite quadratic forms* on $\mathbb{E}^d$. The family of all positive semidefinite quadratic forms on $\mathbb{E}^d$ can be identified with the closure $\mathcal{Q}^d$ of $\mathcal{P}^d$.

At least since the fundamental contributions of Voronoĭ [12, 13, 14] to the geometric theory of positive definite quadratic forms at the beginning of the 20th century, this identification plays an important role in the geometric theory of positive definite quadratic forms, including lattice packing of balls. It also provides a better insight into reduction theory. See, e.g. Ryshkov [10], Ryshkov and Baranovskiĭ [11], Gruber and Lekkerkerker [8], Erdös, Gruber and Hammer [4] and Engel and Syta [3]. More recent are applications to John type and minimum position problems in the context of the local theory of normed spaces due to the author [5, 6] and the author and Schuster [9]. In all these applications, problems on positive definite quadratic forms, ellipsoids, lattice packing of balls, and linear transformations (up to rigid motions) in $\mathbb{E}^d$ are transformed into more accessible geometric problems dealing with subsets of $\mathcal{P}^d$, $\mathcal{Q}^d$, or $\mathbb{E}^{\frac{1}{2} d(d+1)}$. In many cases these subsets are convex.

The cones $\mathcal{P}^d$ and $\mathcal{Q}^d$ thus appear natural objects of investigation. To our surprise we found only a few pertinent articles: Ryshkov [10], Ryshkov and Baranovskiĭ [11], Bertraneu and Fichet [1] and El Kadiri [2]. Ryshkov and Baranovskiĭ show the simple fact that the group of linear symmetries of $\mathcal{P}^d$ is transitive. In the article of Bertraneu and Fichet it is proved that each face of $\mathcal{Q}^d$ is exposed and the face lattice of $\mathcal{Q}^d$ is isomorphic to the lattice of linear subspaces of $\mathbb{E}^d$ and, thus, a modular lattice. The article of El Kadiri deals with more general situations.

We hope to convince the reader that the cones $\mathcal{P}^d$ and $\mathcal{Q}^d$ carry rich geometric structure. To achieve this, the above results are first reproved in a geometric way. Next, extending well known notions for convex polytopes, flag transitivity of the group of isometries and neighborliness properties of the convex cone $\mathcal{Q}^d$ are studied. Then we investigate singularity properties of boundary points and faces of $\mathcal{Q}^d$

and show the simple fact that the cone $\mathcal{Q}^d$ is self dual. Finally, a characterization of the group of isometries of $\mathcal{Q}^d$ is given. It turns out that each isometry of $\mathcal{Q}^d$ is generated by an orthogonal transformation of $\mathbb{E}^d$. A conjecture deals with a related characterization of the linear symmetries of $\mathcal{Q}^d$. The two characterizations provide examples for a principle on structure preserving mappings in convex geometry.

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L_p **Intersection Bodies and the Nonsymmetric** L_p **Busemann-Petty Problem**

CHRISTOPH HABERL

Let $\mathcal{S}^n$ denote the set of star bodies in Euclidean n-space $\mathbb{R}^n$. The intersection body operator assigns to each star body $K \in \mathcal{S}^n$ the star body IK with radial function

$$\rho(\mathrm{I}K, u) = \mathrm{vol}(K \cap u^{\perp}), \quad u \in S^{n-1},$$

where vol denotes $(n-1)$-dimensional volume and $u^\perp$ is the hyperplane orthogonal to u. For $p < 1$, $p \neq 0$ and a star body $K \in \mathcal{S}^n$, let the body $\mathrm{I}_p^+ K$ be defined by

$$\rho(\mathrm{I}_p^+ K, u)^p = \frac{1}{\Gamma(1-p)} \int_{K \cap u^+} |x \cdot u|^{-p} \, dx, \quad u \in S^{n-1},$$

where $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$. In [2] it was shown that from a valuation theoretic point of view the body $\mathrm{I}_p^+ K$ is the nonsymmetric analogue of the classical intersection body of K within the dual L_p Brunn-Minkowski theory. The L_p intersection body $\mathrm{I}_p K$ itself (which is, up to normalization, the polar L_{-p} centroid body) is the L_p radial sum of $\mathrm{I}_p^+ K$ and $\mathrm{I}_p^+(-K)$.

First analogies between the operators I_p and I were established for example in [5] and [3]. In the following, we will present other relations between these two operators and point out significant differences between I_p and I_p^+.

The next two results can be found in [1]. First, every intersection body $\mathrm{I}K$ of a convex body K which contains the origin in its interior can be approximated by L_p intersection bodies with respect to the radial metric. This can be used to derive results for intersection bodies from their L_p analogues. An example will be given in the next paragraphs.

Second, Hensley's result on intersection bodies of convex bodies in isotropic position also holds for L_p intersection bodies. This further confirms the strong relation between intersection bodies an their L_p analogues.

The operator I_p^+ is closely related to a transformation of functions on the sphere, namely

$$\mathrm{C}_{-p}^+ f(u) = \int_{S^{n-1} \cap u^+} |v \cdot u|^{-p} f(v) \, dv, \quad u \in S^{n-1}.$$

This is a special case of the generalized Minkowski-Funk transform. By results of Rubin [4] it follows for $p < 1$ and $p \notin -\mathbb{N} \cup \{0\}$, that the operator I_p^+ is injective on star bodies. For convex bodies, a stability version of this result holds (see [1]). The approximation result mentioned before therefore yields stability results for intersection bodies. Moreover, Rubin [4] proved an inversion formula for C_{-p}^+ on $C^\infty(S^{n-1})$ for certain values of p.

The classical Busemann-Petty problem asks whether the implication

$$\mathrm{I}K \subset \mathrm{I}L \implies V(K) \leq V(L)$$

holds for arbitrary origin symmetric convex bodies K and L. The nonsymmetric L_p Busemann-Petty problem is the question whether

$$(1) \qquad\qquad \mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L \implies V(K) \leq V(L)$$

is true for convex bodies K and L containing the origin in their interiors. For $-1 < p < 1$ and symmetric bodies K, L the latter was solved in [5]. The next theorem (see [1]) shows again a strong resemblance between the L_p situation and the original context on intersection bodies. The proof is partly based on the in- and bijectivity result from above.

Theorem 1. *Suppose* $0 < p < 1$. *If* K, L *are star bodies such that* K *is a nonsymmetric* L_p *intersection body, i.e. contained in* $\mathrm{I}_p^+ \mathcal{S}^n$, *then*

$$\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L,$$

implies

$$V(K) \leq V(L).$$

For smooth star bodies L *which are not nonsymmetric* L_p *intersection bodies, this is no longer true.*

Thus (1) can be true for nonsymmetric star bodies K which is in contrast to the original Busemann-Petty problem and its L_p analogue in [5].

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Roots of Ehrhart polynomials

MARTIN HENK

(joint work with C. Bey and J. M. Wills)

In 1962 Ehrhart [3] showed that for $k \in \mathbb{N}$ and a lattice polytope $P \subset \mathbb{R}^n$ the lattice point enumerator $\mathrm{G}(k\,P) = \#(k\,P \cap \mathbb{Z}^n)$ is a polynomial of degree n in k:

$$\mathrm{G}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P)\,k^i,$$

where the coefficients $\mathrm{G}_i(P)$, $0 \leq i \leq n$, depend only on P. Two of the $n+1$ coefficients $\mathrm{G}_i(P)$ are obvious, namely, $\mathrm{G}_0(P) = 1$ and $\mathrm{G}_n(P) = \mathrm{vol}(P)$, where $\mathrm{vol}()$ denotes the n-dimensional volume. Also the second leading coefficient admits a simple geometric interpretation as normalized surface area of P. To this end let $F_1, \ldots, F_m$ be the facets of P, then

$$\mathrm{G}_{n-1}(P) = \frac{1}{2} \sum_{i=1}^{m} \frac{\mathrm{vol}_{n-1}(F_i)}{\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^n)},$$

where $\mathrm{vol}_{n-1}()$ denotes the $(n-1)$-dimensional volume and $\det(\mathrm{aff}\,F_i \cap \mathbb{Z}^n)$ denotes the determinant of the $(n-1)$-dimensional sublattice of $\mathbb{Z}^n$ contained in the affine hull of the facet F_i. All other coefficients $\mathrm{G}_i(P)$, $1 \leq i \leq n-2$, have no such direct geometric meaning, except for special classes of polytopes.

A sometimes more convenient representation of $G(k\,P)$ is given by a change from the monomial basis $\{x^i : i = 0, \ldots, n\}$ to the basis $\{\binom{x+n-i}{n} : i = 0, \ldots, n\}$:

$$G(k\,P) = \sum_{i=0}^{n} a_i(P) \binom{k+n-i}{n}.$$

Then

$$a_0(P) = 1, \quad a_1(P) = G(P) - (n+1), \quad a_n(P) = G(\operatorname{int}(P)),$$
$$a_0(P) + a_1(P) + \ldots + a_n(P) = n!\operatorname{vol}(P),$$

and all $a_i(P)$ are integers. Due to Stanley's famous non-negativity theorem [5] they are also non-negative, in contrast to the $G_i(P)$'s which might be negative.

In recent years the Ehrhart polynomial was not only regarded as a polynomial for integers k, but as a formal polynomial of a complex variable $s \in \mathbb{C}$ (cf. [1, 4]). Therefore, for $P \in \mathcal{P}^n$ and $s \in \mathbb{C}$ we set

$$G(s, P) = \sum_{i=0}^{n} G_i(P)\,s^i = \prod_{i=1}^{n} \left(1 + \frac{s}{\gamma_i(P)} \right),$$

where $-\gamma_i(P) \in \mathbb{C}$, $1 \le i \le n$, are the roots of the Ehrhart polynomial $G(s, P)$. In particular, for their geometric and arithmetic mean we have

$$(1) \qquad \left(\prod_{i=1}^{n} \gamma_i(P) \right)^{1/n} = (1/\operatorname{vol}(P))^{1/n}, \quad \frac{1}{n} \sum_{i=0}^{n} \gamma_i(P) = \frac{1}{n} \frac{G_{n-1}(P)}{\operatorname{vol}(P)}.$$

Here we are interested in geometric properties of the roots and in their size. To this end we definie for an integer $l \in \mathbb{N}$ a simplex $S_n(l)$ by

$$S_n(l) = \operatorname{conv}\left\{ e_1, \ldots, e_n, -l \sum_{i=1}^{n} e_i \right\},$$

where e_i denotes the i-th unit vector. Observe that $G(\operatorname{int}(S_n(l))) = l$ and that $\operatorname{vol}(S_n(l)) = (n\,l + 1)/n!$.

Theorem 2. *Let P be a lattice polytope. Then*

$$(2) \qquad \operatorname{vol}(P) \ge \frac{n\,G(\operatorname{int} P) + 1}{n!}.$$

The bound is best possible for any number of interior lattice points. For $G(\operatorname{int} P) = 1$ equality holds if and only if P is unimodular isomorphic to the simplex $S_n(1)$.

Hence by (1) we get an upper bound on the geometric mean of the roots. In the case $G(\operatorname{int} P) > 1$ the extremal cases in (2) are not necessarily unimodular equivalent. We remark, however, that all extremal cases have the same Ehrhart polynomial.

Proposition 3. *Let P be a lattice polytope with $G(\operatorname{int} P) = l \ge 1$ and $\operatorname{vol}(P) = (n\,l + 1)/n!$. Then $a_i(P) = a_i(S_n(l)) = l$, $1 \le i \le n$.*

For 0-symmetric lattice polytopes P there is a classical upper bound on the volume due to Blichfeldt and van der Corput

$$\mathrm{vol}(P) \leq 2^{n-1} \left(\mathrm{G}(\mathrm{int}P) + 1 \right).$$

As an analogue to Theorem 2 in the 0-symmetric case we conjecture

Conjecture 4. *Let P be a 0-symmetric lattice polytope. Then*

$$\mathrm{vol}(P) \geq \frac{2^{n-1}}{n!} \left(\mathrm{G}(\mathrm{int}P) + 1 \right).$$

In fact we believe that the following stronger inequalities among the coefficients $\mathrm{a}_i(P)$ are valid

$$\mathrm{a}_i(P) \geq \binom{n}{i} + \binom{n-1}{i-1} \left(a_n(P) - 1 \right), \quad i = 0, \ldots, n.$$

These inequalities have been verified for 0-symmetric crosspolytopes and in the case $G(\mathrm{int}P) = 1$ which, in particular, implies $\mathrm{a}_i(P) \geq \binom{n}{i}$ for any 0-symmetric lattice polytope.

The simplex $S_n(1)$ seems also to be extremal with respect to the norm of the roots.

Theorem 5. *All roots of the polynomial $\mathrm{G}(s, S_n(1))$ have real part $-1/2$. If α_n is a root of $\mathrm{G}(s, S_n(1))$ with maximal norm, then*

$$\left| \alpha_n + \frac{1}{2} \right| = \frac{n(n+2)}{2\,\pi} + o(n),$$

as n tends to infinity.

In a recent paper Braun [2] proved that the roots of an Ehrhart polynomial lie inside the disc with center $-1/2$ and radius $n(n-1)/2$. The above theorem shows that this bound is essentially tight and improves on the former best known bound of order n [1, Theorem 1.3].

Looking at geometric properties of lattice polytopes P whose roots have all real part $-1/2$ leads immediately to the class of reflexive lattice polytopes. Here a lattice polytope P with $0 \in \mathrm{int}P$ is called reflexive if the polar polytope of P is again a lattice polytope.

Proposition 6. *Let P be a lattice polytope. If all roots of $\mathrm{G}(s, P)$ have real part $-1/2$ then, up to an unimodular translation, P is a reflexive polytope of volume $\mathrm{vol}(P) \leq 2^n$.*

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A Classification of SL(n) invariant Valuations

MONIKA LUDWIG

(joint work with M. Reitzner)

Let $\mathcal{K}^n$ denote the set of convex bodies (that is, compact convex sets) in $\mathbb{R}^n$. A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is called a *valuation* if it satisfies the inclusion-exclusion relation

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{K}^n$. Valuations are a central notion in geometry and probably the most famous result on valuations is the Hadwiger characterization theorem.

Theorem 1 (**Hadwiger** [1]). *A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is a continuous and rigid motion invariant valuation if and only if there are constants $c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that*

$$\Phi(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Here $V_0(K), \ldots, V_n(K)$ are the intrinsic volumes of $K \in \mathcal{K}^n$. In particular, $V_0(K)$ is the Euler characteristic (that is, $V_0(K) = 1$ for $K \neq \emptyset$ and $V_0(\emptyset) = 0$) and $V_n(K)$ is the volume of K.

Prior to Hadwiger, Blaschke proved that every continuous, translation and SL(n) invariant valuation on $\mathcal{K}^3$ is a linear combination of volume and the Euler characteristic. This also follows immediately from the Hadwiger characterization theorem. However, if continuity is weakened to upper semicontinuity, there are more examples and the authors obtained the following result.

Theorem 2 ([3]). *A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous, translation and SL(n) invariant valuation if and only if there are constants $c_0, c_1 \in \mathbb{R}$ and $c_2 \geq 0$ such that*

$$\Phi(K) = c_0 \, V_0(K) + c_1 \, V_n(K) + c_2 \, \Omega(K)$$

for every $K \in \mathcal{K}^n$.

The 'new' valuation $\Omega(K)$ in this characterization theorem is the *affine surface area* of a convex body K in $\mathbb{R}^n$. It is defined by

$$\Omega(K) = \int_{\partial K} \kappa(K, x)^{\frac{1}{n+1}} \, dx,$$

where $\kappa(K, x)$ is the generalized Gaussian curvature of ∂K at x.

Theorem 2 shows that within the theory of valuations, Ω is the natural notion of surface area for the equi-affine group. This raises the question to obtain the natural notion of surface area for affine groups without assuming translation invariance. In view of Theorem 2, the question therefore is:

Is it possible to classify all $SL(n)$ or $GL(n)$ invariant valuations on $\mathcal{K}_0^n$?

Here $\mathcal{K}_0^n$ denotes the space of convex bodies that contain the origin in their interiors.

A complete answer for the centro-affine group $GL(n)$ is contained in the following theorem.

Theorem 3 ([4]). *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $GL(n)$ invariant valuation if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that*

$$\Phi(K) = c_0\, V_0(K) + c_1\, \Omega_c(K)$$

for every $K \in \mathcal{K}_0^n$.

This theorem establishes a characterization of the *centro-affine surface area* $\Omega_c(K)$. For a convex body $K \in \mathcal{K}_0^n$, the centro-affine surface area is defined by

$$\Omega_c(K) = \int_{\partial K} \kappa_0(K, x)^{\frac{1}{2}}\, d\mu_K(x).$$

Here $\kappa_0(K, x) = \kappa(K, x)/(x \cdot u(K, x))^{n+1}$, where $x \cdot u$ denotes the standard inner product of $x, u \in \mathbb{R}^n$, $u(K, x)$ is the exterior normal unit vector to K at $x \in \partial K$, and $d\mu_K(x) = x \cdot u(K, x)\, dx$ is the cone measure on ∂K.

The classification of $SL(n)$ invariant valuations leads to a much richer class of examples – even in the case of homogeneous valuations. Here a functional Φ is called *homogeneous of degree* q, $q \in \mathbb{R}$, if $\Phi(t\, K) = t^q\, \Phi(K)$ for every $t > 0$, $K \in \mathcal{K}_0^n$. Let $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for } y \in K\}$ denote the polar body of $K \in \mathcal{K}_0^n$.

Theorem 4 ([4]). *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $SL(n)$ invariant valuation that is homogeneous of degree q if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that*

$$\Phi(K) = \begin{cases} c_0\, V_0(K) + c_1\, \Omega_n(K) & \text{for } q = 0 \\ c_1\, \Omega_p(K) & \text{for } -n < q < n,\ q \neq 0, \\ c_0\, V_n(K) & \text{for } q = n \\ c_0\, V_n(K^*) & \text{for } q = -n \\ 0 & \text{for } q < -n \text{ or } q > n \end{cases}$$

for every $K \in \mathcal{K}_0^n$ where $p = n(n - q)/(n + q)$.

The 'new' valuation $\Omega_p(K)$ in this characterization theorem is the L_p *affine surface area* of K. For $p > 1$, L_p affine surface area was defined by Lutwak [5] as the notion corresponding to affine surface area in the L_p Brunn Minkowski theory. Note that

$\Omega = \Omega_1$ and $\Omega_c = \Omega_n$, that is, affine and centro-affine surface areas are just special L_p affine surface areas.

In the background of these results is a rather general theorem which establishes a complete classification of $SL(n)$ invariant valuations on $\mathcal{K}_0^n$ which vanish on polytopes. Combined with [2], where a classification of all Borel measurable, homogeneous and $SL(n)$ invariant valuations on polytopes is obtained, this result implies Theorems 3 and 4. Let $\mathcal{P}_0^n$ denote the set of convex polytopes that contain the origin in their interiors.

Theorem 5 ([4]). *A functional $\Phi : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and $SL(n)$ invariant valuation that vanishes on $\mathcal{P}_0^n$ if and only if there is a concave function $\phi : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0} \phi(t) = \lim_{t \to \infty} \phi(t)/t = 0$ such that*

$$\Phi(K) = \int_{\partial K} \phi(\kappa_0(K, x)) \, d\mu_K(x)$$

for every $K \in \mathcal{K}_0^n$.

This theorem shows that each of these 'L_ϕ affine surface areas' is a natural choice of an $SL(n)$ invariant surface area on $\mathcal{K}_0^n$.

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Uniqueness and stability results in geometric tomography

CARLA PERI

The topic of my research concerns uniqueness and stability problems in geometric tomography, the area which deals with "the retrieval of information about a geometric object from data about its sections, or projections, or both" (see [5]). Motivated by genuine applications in the material sciences, these type of inverse problems have been studied recently within discrete tomography, a new area of geometric tomography which focuses on determination of finite sets of the integer lattice by means of their discrete parallel X-rays (see [6]). In this field, important and deep results have been obtained by R. Gardner and P. Gritzmann in [4].

a) Together with P. Dulio and R. Gardner, we started a systematic study of discrete point X-rays, with an emphasis on uniqueness results and subsets of the integer lattice (see [1]). We proved that for discrete point X-rays, there is a general lack of uniqueness and we then focused on convex lattice sets. We provided a rather complete analysis when discrete point X-rays are taken at two sources; in

fact, no open problems remain, in contrast to the continuous case. We also proved uniqueness results for discrete point X-rays at collinear sources, analogous to the corresponding results for discrete parallel X-rays due to Gardner and Gritzmann. Somewhat surprisingly, we showed that for non-collinear sources some results obtained by Volčič in [9] for continuous point X-rays do not hold in the discrete case.

b) Uniqueness results for discrete parallel (point) X-rays hinge on the non-existence of special lattice polygons called lattice U-polygons (P-polygons, respectively). Together with P. Dulio we studied the geometric structure of these polygons by extending some characterizations of affinely regular polygons (see [2]). We introduced the notion of class of a U-polygon to obtain a geometric proof of a week version of an important uniqueness result due to Gardner and Gritzmann [4].

c) Questions of stability are important for any inverse problem (see [7]). One significant result in this context is due to Volčič (see [8]), who proved that the reconstruction of a convex body from parallel X-rays is well posed when the set of directions guarantees uniqueness. However, the known uniqueness results are, unfortunately, unstable in the sense that a small perturbation of a finite set of directions providing uniqueness may cause the uniqueness property to be lost. Together with P. Dulio, M. Longinetti and A. Venturi, we obtained an upper bound for the distance between two convex bodies with the same parallel X-rays which provides affine stability estimates for the reconstruction of a convex body which are optimal in the constant and in the order (see [3]).

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Rotation Equivariant Minkowski Valuations

Franz Schuster

(joint work with R. Schneider)

Let $\mathcal{K}^n$ denote the space of convex bodies in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), endowed with the Hausdorff metric. The *projection body* ΠK of K is the convex body whose support function is defined by

$$h(\Pi K, u) = \mathrm{vol}(K|u^{\perp}) \qquad \text{for } u \in S^{n-1}.$$

Here, $\mathrm{vol}(K|u^{\perp})$ denotes the $(n-1)$-dimensional volume of the image of K under orthogonal projection to the subspace orthogonal to u.

We focus on the fact that the operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ is a *Minkowski valuation*, i.e., a valuation with respect to Minkowski addition on $\mathcal{K}^n$. In general, a mapping $\varphi : \mathcal{K}^n \to A$ into an abelian semigroup $(A, +)$ is called a *valuation* if

$$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M)$$

whenever $K, M, K \cup M \in \mathcal{K}^n$. Among all continuous, translation invariant valuations from $\mathcal{K}^n$ to $\mathcal{K}^n$, the projection body operator has recently been characterized in [1], up to a factor, by its $SL(n)$ contravariance. In the following, we will consider continuous, translation invariant valuations $\Phi : \mathcal{K}^n \to \mathcal{K}^n$, but we will replace the strong assumption of $SL(n)$ contravariance by the Euclidean condition of rotation equivariance, i.e., the property that, for all $K \in \mathcal{K}^n$ and every ϑ in the rotation group $SO(n)$ of $\mathbb{R}^n$,

$$\Phi \vartheta K = \vartheta \Phi K.$$

The projection body operator is no longer characterized by these properties. Simple further examples are the trivial maps $\mathcal{I}$ and $-\mathcal{I}$ given by

$$\mathcal{I}(K) = K - s(K) \quad \text{and} \quad (-\mathcal{I})(K) = -K + s(K) \qquad \text{for } K \in \mathcal{K}^n.$$

Here, $s : \mathcal{K}^n \to \mathbb{R}^n$ denotes the Steiner point map, defined by

$$(1) \qquad s(K) = n \int_{S^{n-1}} h(K, u) u \, \mathrm{d}u,$$

where the integration is with respect to the rotation invariant probability measure on the sphere.

The main object of our work was to find an additional assumption which suffices to single out among the large class of continuous, translation invariant and rotation equivariant valuations the combinations of the projection body operator Π and the mappings $\mathcal{I}$ and $-\mathcal{I}$. This additional assumption will be the property that polytopes are mapped to polytopes, see [4].

Theorem 1. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation. If Φ maps polytopes to polytopes, then*

$$\Phi = c_1 \Pi + c_2 \mathcal{I} + c_3 (-\mathcal{I})$$

with constants $c_1, c_2, c_3 \geq 0$.

Remarks. In the plane, where the rotation group is abelian, the assertion has to be modified. Let $\Phi : \mathcal{K}^2 \to \mathcal{K}^2$ be a continuous, translation invariant and rotation equivariant valuation. If the image of Φ contains some polygon with more than one point, then there are rotations $\vartheta_1, \ldots, \vartheta_r$ of $\mathbb{R}^2$ and positive numbers $\lambda_1, \ldots, \lambda_r$ such that

$$\Phi K = \lambda_1 \vartheta_1 [K - s(K)] + \cdots + \lambda_r \vartheta_r [K - s(K)]$$

for all $K \in \mathcal{K}^2$. This was proved in [3].

An operator φ from $\mathcal{K}^n$ to $\mathcal{K}^n$ is called *homogeneous of degree j* if $\varphi(\lambda K) = \lambda^j \varphi(K)$ for $K \in \mathcal{K}^n$ and $\lambda \geq 0$. Among a subclass (including the even ones) of the valuations homogeneous of degree $n-1$, the projection body operator was characterized (up to a factor) in [5], by the assumption that it maps some n-dimensional convex body to a polytope. The wish to generalize this characterization has led us to the following result.

Theorem 2. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation. If Φ maps bodies of dimension $n-2$ to $\{0\}$ and maps some n-dimensional convex body to a polytope, then $\Phi = c\Pi$ with some constant $c \geq 0$.*

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Reporters: Christoph Haberl, Franz Schuster

Participants

Dr. Semyon Alesker
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, P.O. Box 39040
Tel Aviv 69978
ISRAEL

Prof. Dr. Keith M. Ball
Department of Mathematics
University College London
Gower Street
GB-London, WC1E 6BT

Prof. Dr. Imre Barany
Department of Mathematics
University College London
Gower Street
GB-London, WC1E 6BT

Prof. Dr. Franck Barthe
Laboratoire de Statistique et
Probabilites
Universite Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 4

Prof. Dr. Gabriele Bianchi
Dipartimento Matematica ”U.Dini”
Universita degli Studi
Viale Morgagni, 67/A
I-50134 Firenze

Prof. Dr. Karoly Böröczky, Jr.
Alfred Renyi Institute of
Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Prof. Dr. Christian Buchta
Fachbereich Mathematik
Universität Salzburg
Hellbrunnerstr. 34
A-5020 Salzburg

Dr. Andrea Colesanti
Dipartimento di Matematica ”U.Dini”
Universita di Firenze
Viale Morgagni 67/A
I-50134 Firenze

Matthieu Fradelizi
Equipe d’Analyse et Math. Appliq.
Universite de Marne-la-Vallee
Cite Descartes, 5 BD Descartes
Champs-sur-Marne
F-77454 Marne-La-Vallee Cedex

Prof. Dr. Richard J. Gardner
Dept. of Mathematics
Western Washington University
Bellingham, WA 98225-9063
USA

Prof. Dr. Paul R. Goodey
Dept. of Mathematics
University of Oklahoma
601 Elm Avenue
Norman, OK 73019-0315
USA

Prof. Dr. Yehoram Gordon
Department of Mathematics
Technion - Israel Institute of
Technology
Haifa 32000
ISRAEL

Prof. Dr. Peter M. Gruber
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedener Hauptstr. 8 - 10
A-1040 Wien

Dr. Olivier Guedon
Institut de Mathematiques
Projet Analyse Fonctionnelle
Boite 186, Universite Paris 6
4, place Jussieu
F-75005 Paris

Dipl.Ing. Christoph Haberl
Institut für Diskrete Mathematik
und Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8-10/104
A-1040 Wien

Prof. Dr. Martin Henk
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg

Prof. Dr. Maria A. Hernandez Cifre
Departamento de Matematicas
Facultad de Matematicas
Universidad de Murcia
E-30100 Murcia

Prof. Dr. Ralph Howard
Dept. of Mathematics
University of South Carolina
Columbia, SC 29208
USA

Dr. Daniel Hug
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Dr. Markus Kiderlen
Matematisk Institut
Aarhus Universitetet
Ny Munkegade
DK-8000 Aarhus C

Prof. Dr. Alexander Koldobsky
Dept. of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia, MO 65211
USA

Prof. Dr. David G. Larman
Department of Mathematics
University College London
Gower Street
GB-London, WC1E 6BT

Dr. Monika Ludwig
Institut für Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8 - 10
A-1040 Wien

Prof. Dr. Mark W. Meckes
Department of Mathematics
Cornell University
Malott Hall
Ithaca NY 14853-4201
USA

Prof. Dr. Mathieu Meyer
Equipe d'Analyse et de
Mathematiques Appliquees
Universite de Marne-la-Vallee
5, bd Descartes
F-77454 Marne-la-Vallee Cedex 2

Dr. Emanuel Milman
Dept. of Mathematics
Weizmann Institute
Rehovot 76100
ISRAEL

Prof. Dr. Vitali D. Milman
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, P.O. Box 39040
Tel Aviv 69978
ISRAEL

Prof. Dr. Luis Montejano
Instituto de Matematicas
U.N.A.M.
Circuito Exterior
Ciudad Universitaria
04510 Mexico, D.F.
MEXICO

Prof. Dr. Assaf Naor
Microsoft Research
One Microsoft Way
Redmond, WA 98052-6399
USA

Prof. Dr. Alain Pajor
Equipe d'Analyse et de
Mathematiques Appliquees
Universite de Marne-la-Vallee
5, bd Descartes
F-77454 Marne-la-Vallee Cedex 2

Dr. Gajane Panina
Steklov Institute of Mathematics
Academy of Sciences
Fontanka 26
191011 St. Petersburg
RUSSIA

Prof. Dr. Grigoris Paouris
Marne-la-Vallee University
Faculte de Mathematiques
5, Blvd. Descartes
Champs-sur-Marne
F-77454 Marne-la-Vallee Cedex 2

Prof. Dr. Carla Peri
Istituto di Matematica generale,
finanziaria ed economica
Universita Cattolica d.Sacro Cuore
Largo Gemelli 1
I-20123 Milano

Dr. Matthias Reitzner
Institut für Diskrete Mathematik
und Geometrie
TU Wien
Wiedner Hauptstr. 8 - 10
A-1040 Wien

Prof. Dr. Mark Rudelson
Dept. of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia, MO 65211
USA

Prof. Dr. Dmitry Ryabogin
Department of Mathematics
Kansas State University
Manhattan, KS 66506-2602
USA

Dr. Evangelia Samiou
Department of Mathematics
and Statistics
University of Cyprus
P.O.Box 20357
1678 Nicosia
CYPRUS

Prof. Dr. Rolf Schneider
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Dr. Franz Schuster
Institut für Diskrete Mathematik
und Geometrie
TU Wien
Wiedner Hauptstr. 8 - 10
A-1040 Wien

Dr. Carsten Schütt
Mathematisches Seminar
Christian-Albrechts-Universität
Kiel
24098 Kiel

Prof. Dr. Alina Stancu
Department of Mathematics
Concordia University
1455 De Maisonneuve Blvd. West
Montreal, QC H3G 1M8
CANADA

Prof. Dr. Aljosa Volcic
Department of Mathematics
University of Calabria
P.Bucci cubo 30b
I-87036 Rende

Prof. Dr. Wolfgang Weil
Institut für Algebra u. Geometrie
Universität Karlsruhe
76128 Karlsruhe

Dr. Elisabeth Werner
Dept. of Mathematics and Statistics
Case Western Reserve University
10900 Euclid Avenue
Cleveland, OH 44106-7058
USA

Prof. Dr. Artem Zvavitch
Dept. of Mathematical Sciences
Kent State University
Kent OH 44242-0001
USA