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Affine Algebraic Geometry

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Abstract. Affine geometry deals with algebro-geometric questions of affine varieties that are treated with methods coming from various areas of mathematics like commutative and non-commutative algebra, algebraic, complex analytic and differential geometry, singularity theory and topology. The conference had several main topics. One of them was the famous Jacobian problem, its connections with the Dixmier conjecture and possible algebraic approaches and reductions. A second main theme were questions on Log algebraic varieties, in particular log algebraic surfaces. Thirdly, results on automorphisms of $\mathbb{A}^n$ played a major role, in particular the solution of the Nagata problem, actions of algebraic groups on $\mathbb{A}^n$, Hilbert's $14th$ problem and locally nilpotent derivations. More generally automorphism groups of affine and non-affine varieties, especially in dimension 2 and 3 were treated, and substantial progress on the cancelation problem and embedding problem was presented.

Mathematics Subject Classification (2000): 14Rxx.

Introduction by the Organisers

Affine geometry deals with algebro-geometric questions of affine varieties. In the last decades this area has developed into a systematic discipline with a sizeable international group of researchers, and with methods coming from commutative and non-commutative algebra, algebraic, complex analytic and differential geometry, singularity theory and topology. The meeting was attended by 48 participants, among them the most important senior researchers in this field and many promising young mathematicians.

Especially helpful were the programs for young researchers: the NSF Oberwolfach program, the EU grant and JAMS grant. They allowed to increase the number of young participants considerably.

The conference took place in a very lively atmosphere, made possible by the excellent facilities of the institute. There were 24 talks with a considerable number of lectures given by young researchers at the beginning of their careers. Moreover there were 4 invited lectures that gave an overview over some of the most vivid subfields of Affine Geometry. The program left plenty of time for cooperation and discussion among the participants. We highlight the areas in which new results were presented by the lecturers:

- Jacobian problem, especially its connections with the Dixmier conjecture (Belov-Kanel and Kontsevich) and possible algebraic approaches and reductions.
- Log algebraic varieties; in particular log algebraic surfaces.
- Automorphisms of $\mathbb{A}^n$, in particular tame and wild automorphisms of $\mathbb{A}^3$, Hilbert's $14th$ problem and locally nilpotent derivations.
- Automorphism groups of affine and non-affine varieties, especially in dimension 2 and 3.
- Cancellation problem and embedding problem.

In the first 4 areas there were overview talks given by D. Wright, M. Miyanishi, D. Daigle and Sh. Kaliman. Finally, in a problem session there were presented a number of open questions and problems, which are listed in at the end of this report.

Workshop: Affine Algebraic Geometry

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Abstracts

Describing T-varieties with polyhedral divisors

Klaus Altmann
(joint work with Jürgen Hausen)

We report results obtained together with Jürgen Hausen, cf. [AlHa]. According to them, affine normal varieties X with torus action can be described by divisors $\mathcal{D}$ on their Chow quotients Y. However, this requires the use of rather strange coefficients for $\mathcal{D}$ – they are polyhedra. All coefficients have the same asymptotic behavior – visible as the so-called tail cone of $\mathcal{D}$. This language comprises that of toric varieties as well as the theory of (good) $\mathbb{C}^*$ actions.

To describe some details, let T be an affine torus over an algebraically closed field $\mathbb{K}$ of characteristic 0. It gives rise to the mutually dual free abelian groups $M := \mathrm{Hom}_{\mathrm{algGrp}}(T, \mathbb{K}^*)$ and $N := \mathrm{Hom}_{\mathrm{algGrp}}(\mathbb{K}^*, T)$, and, via $T = \mathrm{Spec}\,\mathbb{K}[M]$, the torus can be recovered from them. Denote by $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding vector space over $\mathbb{Q}$.

Definition 1. If $\sigma \subseteq N_{\mathbb{Q}}$ is a polyhedral cone, then denote by $\mathrm{Pol}(N_{\mathbb{Q}}, \sigma)$ the Grothendieck group of the semigroup

$$\mathrm{Pol}^+(N_{\mathbb{Q}}, \sigma) := \{\Delta \subseteq N_{\mathbb{Q}} \mid \Delta = \sigma + [\text{compact polytope}]\}$$

with respect to Minkowski addition. Moreover, $\mathrm{tail}(\Delta) := \sigma$ is called the tail cone of the elements of $\mathrm{Pol}(N_{\mathbb{Q}}, \sigma)$.

Let Y be a normal and semiprojective (i.e. $Y \to Y_0$ is projective over an affine Y_0) $\mathbb{K}$-variety. A $\mathbb{Q}$-Cartier divisors on Y is called *semiample* if a multiple of it becomes base point free.

Definition 2. An element $\mathcal{D} = \sum_i \Delta_i \otimes D_i \in \mathrm{Pol}(N_{\mathbb{Q}}, \sigma) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y)$ with prime divisors D_i is called a *polyhedral divisor* on (Y, N) with tail cone σ if $\Delta_i \in \mathrm{Pol}^+(N_{\mathbb{Q}}, \sigma)$ and if the evaluations $\mathcal{D}(u) := \sum_i \min\langle \Delta_i, u\rangle D_i$ are semiample for $u \in \sigma^\vee \cap M$ and big for $u \in \mathrm{int}\,\sigma^\vee \cap M$. (Note that the membership $u \in \sigma^\vee := \{u \in M_{\mathbb{Q}} \mid \langle \sigma, u\rangle \geq 0\}$ guarantees that $\min\langle \Delta_i, u\rangle > -\infty$.)

The common tail cone σ of the coefficients Δ_i will be denoted by $\mathrm{tail}(\mathcal{D})$. The positivity assumptions imply that $\mathcal{D}(u) + \mathcal{D}(u') \leq \mathcal{D}(u + u')$, hence $\mathcal{O}_Y(\mathcal{D}) := \oplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u))$ becomes a sheaf of rings. So, our polyhedral divisor $\mathcal{D}$ gives rise to the affine scheme $X := X(\mathcal{D}) := \mathrm{Spec}\,\Gamma(Y, \mathcal{O}(\mathcal{D}))$ over Y_0. The M-grading of its regular functions translates into an action of the torus T on X, and $\mathrm{tail}(\mathcal{D})^\vee$ becomes the cone generated by the weights.

Example 3. In [FlZa], Flenner and Zaidenberg use pairs of divisors (D_+, D_-) with $D_+ + D_- \leq 0$ on affine curves Y to describe hyperbolic $\mathbb{C}^*$-surfaces. In our language, these surfaces are given by $N = \mathbb{Z}$, $\sigma = \{0\}$, and $\mathcal{D} = \{1\} \otimes D_+ - [0,1] \otimes (D_+ + D_-)$.

Note that the modification $\widetilde{X} := \widetilde{X}(\mathcal{D}) := \operatorname{Spec}_Y \mathcal{O}(\mathcal{D})$ of $X(\mathcal{D})$ is a fibration over Y with the toric variety $\mathbb{TV}(\text{tail}(\mathcal{D}), N) := \operatorname{Spec} \mathbb{K}[\text{tail}(\mathcal{D})^\vee \cap M]$ as general fiber.

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_0 \end{array}$$

Special fibers over $y \in Y$ can be reducible: Their components are in a one-to-one correspondence with the vertices of $\Delta_y := \sum_{D_i \ni y} \Delta_i$.

The configuration of T-orbits and their closures is directly encoded in the presentation of X as a polyhedral divisor $\mathcal{D}$. The orbits in $\widetilde{X}$ correspond to pairs (y, F) with $y \in Y$ and faces $F \leq \Delta_y$. Moreover, as is it is known from the toric case, mutually inclusions among orbit closures correspond to opposite inclusions of the corresponding faces. The orbit structure of X may be obtained from that of $\widetilde{X}$ by keeping track of when certain orbits from $\widetilde{X}$ will be identified in X. This happens in relation to the different contractions of Y provided by the semiample divisors $\mathcal{D}(u)$.

Theorem 4 ([AlHa])**.** *This construction of X is, in some sense, functorial and induces* (after some adjustments) *an equivalence of categories.*

Here, morphisms of affine varieties with torus action are induced by the following morphisms $\mathcal{D} \to \mathcal{D}'$ of polyhedral divisors: Let $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ and $\mathcal{D}' = \sum_j \Delta'_j \otimes D'_j$ be polyhedral divisors on (Y, N) and (Y', N') with tail cones σ and σ', respectively. If $\psi : Y \to Y'$ is such that none of the supports of the D'_i contains $\psi(Y)$, and if $F : N \to N'$ is a linear map with $F(\sigma) \subseteq \sigma'$, then the relation

$$\sum_i \Delta'_i \otimes \psi^*(D'_i) =: \psi^*(\mathcal{D}') \leq F_*(\mathcal{D}) := \sum_i \big(F(\Delta_i) + \sigma'\big) \otimes D_i$$

inside $\operatorname{Pol}(N'_{\mathbb{Q}}, \sigma') \otimes_{\mathbb{Z}} \operatorname{CaDiv}(Y)$ gives rise to an equivariant (with respect to $T = N \otimes_{\mathbb{Z}} \mathbb{K}^* \xrightarrow{F \otimes \mathrm{id}} N' \otimes_{\mathbb{Z}} \mathbb{K}^* = T'$) morphism $X(\mathcal{D}) \to X(\mathcal{D}')$.

A special case of this construction is the open embeddings obtained by localization. If $f \in \Gamma(Y, \mathcal{O}(\mathcal{D}(u)))$ is a homogeneous, regular function of degree $u \in M$ on $X = X(\mathcal{D})$, then the open, affine subset $X_f := [f \neq 0] \subseteq X$ is provided by the polyhedral divisor $\text{face}(\mathcal{D}, u) := \sum_i \text{face}(\Delta_i, u) \otimes D_i$, restricted to Y_f. Here, $\text{face}(\Delta, u)$ denotes the face of Δ where the linear form u becomes minimal, and $Y_f := Y \setminus \operatorname{supp}\big(\operatorname{div}(f) + \mathcal{D}(u)\big)$. This construction is very similar to the usage of the polyhedral face relation in the theory of toric varieties and, eventually, leads to a glueing procedure, cf. [AHS].

As an example, the Grassmannian $\operatorname{Grass}(2, n)$ with its $(n-1)$-dimensional torus action can be described as a divisor $\mathcal{S} = \sum_B \mathcal{S}_B \otimes D_B$ on the moduli space $\overline{M}_{0,n}$ of stable rational curves with n marked points. Here, D_B stands for the prime divisor consisting of the two-component curves with a point distribution according to the partition B. In this non-affine situation, the tail cone has been replaced by a so-called tail fan (a coarsified version of the associated system of Weyl chambers), and the $\mathcal{S}_B$ are no longer polyhedra, but polyhedral subdivisions of $N_{\mathbb{Q}}$. They look like their common tail fan – only that the origin has been replaced by a line segment whose direction depends on B, cf. [AlHe].

Returning to the case of an affine T-variety X, one might consider $\Delta := \sum_i \Delta_i$. Its normal fan $\mathcal{N}(\Delta)$ is a subdivision of $\sigma^\vee$; in particular it is contained in $M_{\mathbb{Q}}$ which is an unusual place for a fan. Let $Z := \mathbb{TV}(\mathcal{N}(\Delta), M)$ be the corresponding toric variety. It is projective over $Z_0 := \mathbb{TV}(\sigma^\vee, M)$, and the polytopes Δ_i correspond to semiample divisors E_i on Z. Thus, $\mathcal{D} = \sum_i E_i \otimes D_i$ becomes an element of $\mathrm{CaDiv}_{\mathbb{Q}}(Z) \otimes_{\mathbb{Z}} \mathrm{CaDiv}_{\mathbb{Q}}(Y)$.

On the other hand, the $u \in \sigma^\vee \cap M$ may be interpreted as germs of curves $u : (\mathbb{K}, 0) \to Z_0$ or, after lifting them, as germs of curves $u : (\mathbb{K}, 0) \to Z$. This gives rise to the quasi coherent sheaf $\mathcal{U} := \oplus_{u \in \sigma^\vee M} u_* \mathcal{O}_{\mathbb{K}}$.

Conjecture 5. *The polyhedral divisor $\mathcal{D}$ induces an object $\mathcal{K}_{\mathcal{D}}$ of the derived category $\mathrm{D}^{\mathrm{b}}(Y \times Z)$, and the sheaf $\mathcal{O}_Y(\mathcal{D})$ is obtained from $\mathcal{U} \in \mathrm{D}^{\mathrm{b}}(Z)$ via Fourier-Mukai transformation with kernel $\mathcal{K}_{\mathcal{D}}$.*

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Two dimensional quotients of $\mathbb{C}^n$ by reductive groups

Mariusz Koras

We sketch the idea of proof of the following theorem.

Theorem A. *Let G be a reductive group acting on an affine space $\mathbb{C}^n$.*
If $\dim \mathbb{C}^n//G = 2$ then $V = \mathbb{C}^n//G$ is isomorphic to a quotient $\mathbb{C}^2/\Gamma$ where Γ is a finite group of automorphisms of $\mathbb{C}^2$.

The theorem was proved by R.V.Gurjar in case of linear action of G (it was known as the Wall's conjecture), [Gu1]. In [KR1] Koras and Russell proved the theorem in case $n = 3$ and $G = \mathbb{C}^*$. This was the key point in the proof of the linearization theorem for actions of $\mathbb{C}^*$ on $\mathbb{C}^3$.

It is well known that V is affine and normal. By [KPR] V is contractible. For general $\mathbb{C}^2 \subset \mathbb{C}^3$ the quotient map $\pi \colon \mathbb{C}^3 \to V$ induces a dominant map $\mathbb{C}^2 \to V$. Therefore the Kodaira dimension $\overline{\kappa}(V)$ equals $-\infty$. It is proved in [Gu1] that the singularities are quotient singularities. Such surfaces are classified in [KR2].

Theorem 1,[KR2]. *Let V be a contractible affine surface with only quotient singularities. Assume that the logarithmic Kodaira dimension $\overline{\kappa}(V) = -\infty$. Then $\overline{\kappa}(V - Sing\ V) = -\infty$.*

We have then two cases.

Case 1. $V - Sing\ V$ is not affine ruled.

In this case by the fundamental theorem of Miyanishi and Tsunoda, [MT1], $V - Sing\ V$ contains an open subset U which has a structure of a Platonic fibration. It is known also that $(V - Sing\ V) - U$ is a union of disjoint curves isomorphic to $\mathbb{C}^1$. It is known that the topological Euler characteristic $\chi(U) = 0$. Since $\chi(V - Sing\ V) \leq 0$ we obtain that $V - Sing\ V = U$. It is also known that U is isomorphic to $\mathbb{C}^2 - \{0\}/\Gamma$. It follows easily that $V \simeq \mathbb{C}^2/\Gamma$ in this case.

Case 2. $V - Sing\ V$ is affine ruled.

In this case the structure of V is known, see e.g. [MS, KR2]. It follows that if $Sing\ V$ has only one point then V is isomorphic to $\mathbb{C}^2/Z_a$ for a cyclic group Z_a.

Corollary. *Under assumptions of Thm 1 if $Sing\ V$ consists of at most one point then V is isomorphic to $\mathbb{C}^2/\Gamma$ where Γ is a finite group of automorphisms of $\mathbb{C}^2$. If $Sing\ V$ has more than one points then all singularities of V are cyclic singularities.*
.

It follows that in order to prove the Thm A it suffices to prove the following.

Theorem 2,[Gu2]. *With the notation of Thm A V has at most one singular point.*

Idea of proof of Thm 2. At first we reduce this to the case G is connected. Let G_0 be the connected component of 1 in G. Suppose that $\mathbb{C}^n//G_0$ is isomorphic to $\mathbb{C}^2//\Gamma_0$. We have a finite map

$$\mathbb{C}^2//\Gamma_0 = \mathbb{C}^n//G_0 \to V.$$

It is known [M] that then $V \simeq \mathbb{C}^2/\Gamma$. (one can use also the thm 1). If now G is semisimple then $V \simeq \mathbb{C}^2$. It was proved by Kempf [K], in case G acts linearly on $\mathbb{C}^n$. In the general case it follows quite easily from the theorem 1. For general G we write $G = P \cdot T$ where P is a semisimple group and T is a torus contained in the center of G. Let $W = \mathbb{C}^n//P$. Since P has no non-trivial characters W is UFD although it may be singular. We have $V = W//T$. The key point is the following.

Proposition. *Let q be a singular point in V. Let $p\colon W \to V$ denotes the quotient map. If $p^{-1}(q)$ contains a divisor then it contains a fixed point of T and it is the unique fixed point of T on W. If $p^{-1}(q)$ does not contain a divisor then the analytic local ring of q in V is UFD and hence q is an E_8 singularity.*

Now the theorem 2 follows since E_8 is not a cyclic singularity therefore, by the corollary above, V has at most one singular point.

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Quasihomogeneous affine threefolds

Vladimir L. Popov

[1] Below all algebraic varieties are taken over an algebraically closed field of characteristic zero.

Definition 1. ([G]) An irreducible algebraic variety X is called *quasihomogeneous with respect to an algebraic group* if there exists an algebraic group action on X such that one of the orbits $\mathcal{O}$ is open in X and $X \setminus \mathcal{O}$ is a finite set of points.

In [G] it was proved that all pairwise nonisomorphic *smooth* affine algebraic surfaces quasihomogeneous with respect to an algebraic group are given by the following list:

$$\mathbf{A}^2,\ \ \mathbf{A}^1 \times \mathbf{A}^1_*,\ \ \mathbf{A}^1_* \times \mathbf{A}^1_*,\ \ \mathbf{P}^2 \setminus C,\ \ (\mathbf{P}^1 \times \mathbf{P}^1) \setminus \Delta,$$

where $\mathbf{A}^1_* := \mathbf{A}^1 \setminus \{0\}$, C is a nondegenerate conic, and Δ is the diagonal. In [P_1] this classification was extended to all, *not necessarily smooth*, affine surfaces. Namely, for $n_1, \ldots, n_r \in \mathbf{Z}_{>0}$, let $\mathbf{V}(n_1, \ldots, n_r)$ be the affine cone in $\mathbf{A}^{n_1+1} \times \ldots \times \mathbf{A}^{n_r+1}$ over the image of morphism

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_r}, \quad a \mapsto v_{n_1}(a) \times \ldots \times v_{n_r}(a),$$

where $v_d \colon \mathbf{P}^1 \to \mathbf{P}^d$ is the Veronese embedding. Then all singular affine surfaces quasihomogeneous with respect to an algebraic group are precisely the surfaces

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$\mathbf{V}(n_1,\dots,n_r)$ where $n_i > 1$ for all i, and

$$\mathbf{V}(n_1,\dots,n_r) \simeq \mathbf{V}(m_1,\dots,m_s) \Leftrightarrow \mathbf{Z}_{>0}n_1 + \dots \mathbf{Z}_{>0}n_r = \mathbf{Z}_{>0}m_1 + \dots \mathbf{Z}_{>0}m_s.$$

For *normal* affine surfaces, the classification of [P_1] was recently reproved in [FZ] using other arguments (by [P_1, Prop. 3(d)], $\mathbf{V}(n_1,\dots,n_r)$ is normal if and only if $\mathbf{Z}_{>0}n_1 + \dots + \mathbf{Z}_{>0}n_r = \mathbf{Z}_{>0}d$, where d is the greatest common divisor of $n_1,\dots,n_r$). On the other hand, in [FZ] quasihomogeneity is understood in a more general sense of [P_2], [PV], where the condition of finiteness of $X \setminus \mathcal{O}$ is dropped. Given these recent developments, the problem of classifying affine threefolds quasihomogeneous with respect to an algebraic group in the sense of [P_2], [PV] becomes actual. For quasihomogeneity in the sense of Definition 1, such a classification was obtained in [P_4]. Given the actuality of the aforementioned problem, below we reproduce this classification with some comments concerning consequent developments.

First, introduce some special varieties. For $n_1,\dots,n_s,m_1,\dots,m_s \in \mathbf{Z}_{>0}$, put $N = \prod_{i=1}^{s}\binom{n_i+m_i-1}{m_i}$ and consider the morphism

$$\begin{gathered} v^{m_1,\dots,m_s}_{n_1,\dots,n_s}\colon \mathbf{A}^{n_1}\times\dots\times\mathbf{A}^{n_s} \longrightarrow \mathbf{A}^N, \\ (x_1,\dots,x_{n_1},\dots,z_1,\dots,z_{n_s}) \mapsto (\dots,x_1^{i_1}\cdots x_{n_1}^{i_{n_1}}\cdots z_1^{k_1}\cdots z_{n_s}^{k_{n_s}},\dots), \\ i_1+\dots+i_{n_1} = m_1,\dots,k_1+\dots+k_{n_s} = m_s. \end{gathered}$$

Given a matrix $A = \begin{bmatrix} a_{11}\dots a_{1s} \\ \dots\dots\dots \\ a_{r1}\dots a_{rs}\end{bmatrix}$, where $a_{ij} \in \mathbf{Z}_{>0}$, put

$$\mathbf{V}_{n_1,\dots,n_s}(A) := \big(v^{a_{11},\dots,a_{1s}}_{n_1,\dots,n_s}\times\dots\times v^{a_{r1},\dots,a_{rs}}_{n_1,\dots,n_s}\big)\big(\mathbf{A}^{n_1}\times\dots\times\mathbf{A}^{n_s}\big)$$

(one can show that $\mathbf{V}_{n_1,\dots,n_s}(A)$ is closed).

Identify $\mathrm{Pic}\big((\mathbf{P}^1\times\mathbf{P}^1)\setminus\Delta\big)$ with $\mathbf{Z}$ by an isomorphism φ. Let $\mathbf{X}_n$ be the total space of the one-dimensional vector bundle over $(\mathbf{P}^1\times\mathbf{P}^1)\setminus\Delta$ corresponding to $n \in \mathbf{Z}$, and let $\mathbf{X}_n^*$ be the complement of the zero section in $\mathbf{X}_n$. One can show that $\mathbf{X}_n \simeq \mathbf{X}_{-n}$ and $\mathbf{X}_n^* \simeq \mathbf{X}_{-n}^*$, so actually $\mathbf{X}_n$ and $\mathbf{X}_n^*$ do not depend on φ.

For a nondegenerate conic C in $\mathbf{P}^2$, the group $\mathrm{Pic}\big(\mathbf{P}^2\setminus C\big)$ has order 2. Let $\mathbf{Y}_0$ and $\mathbf{Y}_1$ be the total spaces of, respectively, the trivial and nontrivial one-dimensional vector bundles over $\mathbf{P}^2\setminus C$. Let $\mathbf{Y}_n^*$ be the complement of the zero section in $\mathbf{Y}_n$.

Let $\widetilde{T}$, $\widetilde{O}$, $\widetilde{I}$, and $\widetilde{D}_n$ be, respectively, the binary tetrahedral, octahedral, icosahedral, and dihedral subgroup of order $4n$ in the group SL_2. We put $\mathbf{S}_3 = \mathrm{SL}_2/\widetilde{T}$, $\mathbf{S}_4 = \mathrm{SL}_2/\widetilde{O}$, $\mathbf{S}_5 = \mathrm{SL}_2/\widetilde{I}$, and $\mathbf{W}_n = \mathrm{SL}_2/\widetilde{D}_n$. One can show that $\mathbf{W}_n$ is a quotient of $\mathbf{X}_{2n}^*$ by some involution.

Now the classification of affine threefolds quasihomogeneous with respect to an algebraic group is given by the following two theorems.

Theorem 2. (Classification of smooth threefolds) (i) *Smooth affine threefolds quasihomogeneous with respect to an algebraic group are precisely all the varieties from the following list:*

$$(1)\ \mathbf{X}_n;\ \ (2)\ \mathbf{X}_n^*;\ \ (3)\ \mathbf{W}_n;\ \ (4)\ \mathbf{Y}_0;\ \ (5)\ \mathbf{Y}_0^*;\ \ (6)\ \mathbf{Y}_1^*;\ \ (7)\ \mathbf{S}_3;\ \ (8)\ \mathbf{S}_4;\ \ (9)\ \mathbf{S}_5;$$
$$(10)\ \mathbf{A}^3;\ \ (11)\ \mathbf{A}^2\times\mathbf{A}^1_*;\ \ (12)\ \mathbf{A}^1\times\mathbf{A}^1_*\times\mathbf{A}^1_*;\ \ (13)\ \mathbf{A}^1_*\times\mathbf{A}^1_*\times\mathbf{A}^1_*.$$

(ii) *No two threefolds from different items in the above list are isomorphic to one another.*

$$\mathbf{X}_n\simeq\mathbf{X}_m\Leftrightarrow|n|=|m|;\quad \mathbf{X}_n^*\simeq\mathbf{X}_m^*\Leftrightarrow|n|=|m|;\quad \mathbf{W}_n\simeq\mathbf{W}_m\Leftrightarrow n=m.$$

Theorem 3. (Classification of singular threefolds) (i) *Singular affine threefolds quasihomogeneous with respect to an algebraic group are precisely all the varieties* $\mathbf{V}_{2,2}(A)$ *and* $\mathbf{V}_3(B)$, *where and* $\mathrm{rk}(A)=1$ *and* B *has no entries equal to* 1 . (ii) $\mathbf{V}_{2,2}(A)$ *and* $\mathbf{V}_3(B)$ *are not isomorphic to one another.* $\mathbf{V}_{2,2}(A)$ *and* $\mathbf{V}_{2,2}(A')$ (*respectively,* $\mathbf{V}_3(B)$ *and* $\mathbf{V}_3(B')$) *are isomorphic if and only if the additive semigroup generated by the rows of* A (*respectively,* B) *coincides with that generated by the rows of* A' (*respectively,* B').

Remarks. 1. The equivalence $\mathbf{X}_n\simeq\mathbf{X}_m\Leftrightarrow|n|=|m|$ was conjectured in [P_4, Section 1.7]. F. Bogomolov suggested that it can be proved considering canonical classes. This is indeed so. Namely, by [P_4, Theorem 9], we have $\mathbf{X}_n=\mathrm{SL}_2(|n|)/H_{|n|}$ (the notation is explained in [P_4, Section 6.6]). Whence, by [P_3], $\mathrm{Pic}(\mathbf{X}_n)\simeq\mathbf{Z}$ (we identify these groups by a fixed isomorphism), and the definitions of $\mathrm{SL}_2(|n|)$ and $H_{|n|}$ imply that the weights of a maximal (one-dimensional) torus of $H_{|n|}$ in the isotropy representation of $\mathrm{SL}_2(|n|)/H_{|n|}$ are 2, -2, and $|n|$. Hence $|n|$ is the module of canonical class $K_{\mathbf{X}_n}$ of $\mathbf{X}_n$.

2. In [P_4, §2] an *equivariant* classification of singular affine algebraic varieties of arbitrary dimension that are quasihomogeneous with respect to an algebraic group is obtained: it is proved that such varieties are precisely singular S-varieties $X(\lambda_1,\ldots,\lambda_d)$ in the sense of [PV] such that $\mathbf{Q}\lambda_i=\mathbf{Q}\lambda_j$ for all i,j (see [P_4, Cor. 4 of Theorem 4]). This leads to the problem of classifying S-varieties up to (not necessarily equivariant!) isomorphism. For the special case of HV-varieties in the sense of [PV, §1], a conjectural answer is given by Conjecture 1 in [P_5, Section 2]. This conjecture is still open. On the other hand, using computer, van der Kallen found counterexamples to representation theoretic Conjecture 2 in [P_5, Section 2] that implies Conjecture 1[2] for instance, one obtains such a counterexample taking $G=\mathrm{SL}_5$, $\lambda=5\varpi_3+\varpi_4$, and $\mu=3\varpi_1+5\varpi_1$.

3. The range of dimensions to which methods and results of [P_4], [P_1] can be applied is not restricted by dimensions $\leqslant 3$. For instance, in [KW] they were used for obtaining a partial classification of affine fourfolds quasihomogeneous with respect to an algebraic group.

[2]Using this opportunity fix the following misprints in [P_5]: p. 193, l. 2_-, replace $\mathcal{O}O_\sigma$ by $\mathcal{O}_\sigma$; p. 195, l. 3, replace "is equivalent to" by "follows from".

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Endomorphisms of Weil algebras

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Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. If this mapping is invertible, then it is locally invertible. It follows that the Jacobian matrix of the mapping F is a non-zero constant.

The well-known Jacobian Conjecture (JC_n) says that the converse is also true.This conjecture (posed by O.Keller) is still unsolved since 1934.

Let W_n be the Weil algebra of polynomial differential operators of n variables over the complex number field. This Weil algebra can also more generally be defined over any ring via generators $x_1, \ldots, x_n; \partial_1, \ldots, \partial_n$ and relations $[x_i, x_j] = [\partial_i, \partial_j] = 0$; $[x_i, \partial_j] = \delta_{ij}$, where $[x, y]$ denotes the commutator $xy-yx$ and δ_{ij} the Kronecker delta.

The Dixmier conjecture (DC_n) says that every endomorphism of W_n over a field of characteristic zero is an automorphism. This conjecture is open since 1967.

It was well known that DC_n implies JC_n since a polynomial mapping with Jacobian matrix equal to one induces a homomorphism of the ring of differential operators. The implication $JC_{2n} \to DC_n$ was recently proved by Y.Tsuchimoto and independently by A.J.Belov and M.L.Kontsevich. The main idea of the proof is the following.

In positive characteristic any Weil algebra is finitely dimensional over its center, generated by x_i^p, ∂_i^p. Moreover, one can define Poisson brackets on its center. Let $P, Q \in Z(W_n) \mod p$ and $\hat{P}, \hat{Q}$ denote their lifting to characteristic zero.

Then one can consider

$$\{P, Q\} = \frac{[\hat{P}, \hat{Q}]}{p} \mod p.$$

It is easy to see that these brackets are well-defined and determine a standard symplectic structure. From this it is easy to realize that if we consider positive characteristics, then the restriction of the endomorphism of the Weil algebra to its center preserves Poisson brackets and hence its Jacobian is 1. one can conclude that invertibility of this restriction implies invertibility of the whole endomorphism.

Instead of Poisson brackets Y.Tsuchimoto used another approach, based on p-curvature. This new approach of studying polynomial automorphisms is based on quantization. Let A be an associative commutative algebra endowed with Poisson brackets. The Kontsevich quantization theorem says that there exists a deformation $*$ of multiplication in $A[[h]]$ such that $x * y \equiv xy \mod h$, and

$$\frac{(x * y - y * x)}{h} \equiv x, y \mod h.$$

It is important to notice that we use deformation in arithmetical direction (prime number plays role of the plank constant h).

The Weil algebra is a quantum space and it can be obtained as the quantization of the polynomial ring via standard Poisson bracket. There is a Kontsevich conjecture which says that the group of polynomial automorphisms of $\mathbb{C}^n$ is isomorphical to the group of automorphisms of the Weil algebra which means equivariant properties of quantization.

In the proof of the equivalence between the Jacobian and Dixmier Conjectures the homomorphism between these two groups is constructed.

Let $\varphi : x_i \to P_i; \partial_i \to Q_i$ be an endomorphism of W_n. Then φ sends $x_i^p \to P_i^p = \tilde{P}_i(x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p)$; $\partial_i^p \to Q_i^p = \tilde{Q}_i(x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p)$, since φ sends the center to the center and $Z(W_n)$ is a set of polynomials of x_i^p, ∂_i^p. The degree of $\tilde{P}_i$ and $\tilde{Q}_i$ coincide with the degree of P_i and Q_i respectively, and the vector $(\{\tilde{P}_i\}; \{\tilde{Q}_i\})$ determines a polynomial symplectomorphism. The homomorphism of the group of automorphisms of the Weil algebra and the polynomial symplectomorphisms can be obtained using infinitely large prime.

This is a monomorphism of Ind-schemes. If we suppose that this homomorphism is also an epimorphism, then it is possible to prove that it is independent of the use of the infinitely large prime and unique.

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A new proof on non-tameness of Nagata automorphism from a point of view of Sarkisov Program

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This report is a résumé based on my article [9] and the talk at Oberwolfach Workshop (Affine Algebraic Geometry). In affine algebraic geometry, it is an important problem to understand the structure of the group $G_n := \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[x_1, \cdots, x_n])$ on polynomial rings in n variables over $\mathbb{C}$. We shall, from now on, introduce three sub-groups of G_n. An automorphism τ on $\mathbb{C}[x_1, \cdots, x_n]$ is called *affine* (resp. *de Jonquiére*) if $\tau(x_i) = \sum_{j=1}^{n} a_{ij}x_j + b_i$ with a_{ij}, $b_i \in \mathbb{C}$ and $\det(a_{ij}) \neq 0$ (resp. $\tau(x_i) = a_i x_i + f_i(x_{i+1}, \cdots, x_n)$ with $a_i \neq 0$, $f_i \in \mathbb{C}[x_{i+1}, \cdots, x_n]$ $(1 \leq i < n)$ and $f_n \in \mathbb{C}$). Let A_n (resp. J_n) be the sub-group of G_n consisting of all affine transformations (resp. de Jonquiére transformations). We denote by T_n the sub-group of G_n generated by A_n and J_n. An automorphism τ is said to be *tame* if it is contained in T_n. For the case $n = 2$, it is classically well-known that all automorphisms on the polynomial ring $\mathbb{C}[x_1, x_2]$ are tame, i.e., $G_2 = T_2$ (cf. [14], [1], [13]). On the other hand, for higher-dimensional case $n \geq 3$, we know very little concerning the structure of G_n. In what follows, we shall especially pay attention to the case $n = 3$, and write $\mathbb{C}[x, y, z]$ instead of the original notation $\mathbb{C}[x_1, x_2, x_3]$. In order to indicate complexity of the group G_3, let us consider the following famous automorphism, so-called *Nagata automorphism* (cf. [15, p. 16]):

$$\sigma : \begin{cases} x & \mapsto & x - 2y(xz + y^2) - z(xz + y^2)^2 \\ y & \mapsto & y + z(xz + y^2) \\ z & \mapsto & z. \end{cases}$$

Although the Nagata automorphism σ might seem to be simple, it had remained unknown whether or not σ was tame for the past three decades. Although the structure on G_3 itself can not be analyzed in detail at the present time, I.P. Shestakov and U.U. Umirbaev develop a technique to decide the tameness of a given automorphism on $\mathbb{C}[x, y, z]$ by purely algebraic methods (cf. [16, 17]). As a consequence, they have proved at last that σ is, indeed, not tame, i.e., $\sigma \notin T_3$. However, as mentioned just above, since their method is purely algebraic based on the treatment of poisson algebra, it seems to be complicated and be lack of intrinsic geometry hiding in Nagata automorphism σ. One of the main purposes in this report is to inform the utility of the technique from (Log) Minimal Model Program, especially *Sarkisov Program* (cf. [2, 3], [12], [8]), for the decision of non-tameness of σ, and the further detailed investigation of G_3. Roughly speaking, Sarkisov Program gives us a useful tool for the factorization of a given birational map between 3-dimensional Mori fiber spaces into certain kinds of simple birational maps, so-called *elementary links* (see [2] for the definition of elementary links). But, in general, it is difficult to perform *explicit* factorizations into elementary links. This difficulty mainly seems to result from the lack of classification of 3-dimensional terminal divisorial contractions. (Nevertheless, there exist something remarkable concerning the classification of 3-dimensional terminal divisorial contractions due to Y. Kawamata, M. Kawakita and N. Tziolas and so on; see [7], [5, 6].) We shall make use of an algorithm of Sarkisov Program for the investigation of the Cremona transformation $\Phi_\theta : \mathbb{P}^3 \cdots \to \mathbb{P}^3$ induced by an automorphism $\theta \in G_3$ on $\mathbb{C}^3$. (Note that $\mathbb{P}^3$ is a special kind of 3-dimensional Mori fiber space, and Φ_θ is birational.) The explicit Sarkisov factorization of Φ_θ itself is still difficult to perform, meanwhile, it is possible, in principle, to determine the *maximal center* (cf. [2, 3]) of the first elementary link appearing in the Sarkisov factorization of Φ_θ once θ is given concretely. One of the main theorem asserts that, as far as we are concerned with the (non-)tameness of a given θ, the investigation of a maximal center of the first elementary link gives us a useful information. Namely, we prove the following:

Theorem 1. *Let θ be a tame automorphism on the affine 3-space $\mathbb{C}^3$, and Φ_θ the Cremona transformation on $\mathbb{P}^3$ induced by θ in a natural way. Then, for any Sarkisov factorization of Φ_θ, say:*

$$\Phi_\theta = \chi'_s \circ \cdots \circ \chi'_1,$$

the maximal center of the maximal divisorial blow-up (cf. [2, 3], [12], [8]*) appearing in the first elementary link χ'_1 is either a point or a line on the hyperplane at infinity.*

In order to obtain Theorem 1, we need a mechanism of Sarkisov Program (cf. [2]) and the result due to G. Freudenburg (cf. [4]). Once we obtained Theorem 1, we can prove the following:

Theorem 2. (cf. [16, 17]) *The Nagata automorphism is not tame.*

In fact, the Nagata automorphism σ is naturally extended to the Cremona transformation $\Phi_\sigma : \mathbb{P}^3 \cdots \to \mathbb{P}^3$ as in the following fashion:

$$\Phi_\sigma : \begin{cases} x & \mapsto & xt^4 - 2y(xz+y^2)t^2 - z(xz+y^2)^2 \\ y & \mapsto & \big(yt^2 + z(xz+y^2)\big)t^2 \\ z & \mapsto & zt^4 \\ t & \mapsto & t^5, \end{cases}$$

where the hyperplane at infinity H_∞ is defined by $t = 0$. Fortunately, in our previous paper [8], we succeed in the explicit factorization of Φ_σ into eight elementary links by making use of Sarkisov Program. According to this, we see that the first elementary link starts with a blow-up along a smooth conic on H_∞ to deduce that σ is not tame by Theorem 1. However, in consideration of Theorem 1, it is hopeful to give a proof on the non-tameness of σ, which depends only on the investigation of a maximal center of the first link, for further applications. In [9], we give a proof of Theorem 2 from this viewpoint in combination of Theorem

1 and the technique due to A. Corti, which is one of the conclusions of, so-called, Inversion of Adjunction due to Shokurov and Kollár (cf. [10]).

We shall define a certain class of non-tame automorphism on $\mathbb{C}^3$. For $\theta \in G_3$, we put $\deg\theta := \deg\theta(x) + \deg\theta(y) + \deg\theta(z)$. An automorphism θ is said to be *non-essential* if at least one of $\deg(\tau\circ\theta)$ and $\deg(\theta\circ\tau)$ is strictly smaller than $\deg\theta$ by making use of a suitable tame automorphism $\tau \in T_3$. If θ is not non-essential, then we call it *essential*. Furthermore, if θ is essential and non-tame, then θ is said to be *essentially non-tame*. Since G_3 is generated by T_3 and essentially non-tame automorphisms, it is important to classify essentially non-tame ones, but it seems that we know very little about them at present. For the further investigation of essentially non-tame automorphisms on $\mathbb{C}^3$ and, so that, the group G_3, it is important to consider the following problem:

Problem. Is there a essentially non-tame automorphism θ on the affine 3-space $\mathbb{C}^3$ such that the Sarkisov factorization of the Cremona transformation Φ_θ on $\mathbb{P}^3$ induced by θ starts with a (weighted) blow-up at a point on the hyperplane at infinity ?

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Geometry of open algebraic surfaces

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The talk was to survey the entitled subject in affine algebraic geometry, but covered, for lack of time, only the fibrations as outlined below. The interested readers are recommended to see the author's monograph [10] and article [11]. One purpose to review the developments is to seek the possible directions of generalizing the surface theory to the higher-dimensional cases. To clarify the ideas, we assume that the ground field is the complex field $\mathbb{C}$.

A general philosophy is that an $\mathbb{A}^1$-fibration on an affine surface will play the same role of a $\mathbb{P}^1$-fibration on a projective surface. We consider the $\mathbb{A}^1$-fibration in a broader setting. A morphism $f : X \to Y$ is an *F-fibration* if f is surjective and general fibers are isomorphic to a smooth algebraic variety F. If F is the affine space $\mathbb{A}^n$, f is an *$\mathbb{A}^n$-fibration.* If F is $\mathbb{A}^1_* := \mathbb{A}^1 - \{\text{one point}\}$, then f is an *$\mathbb{A}^1_*$-fibration.* As for the local triviality of the fibrations in the sense of Zariski topology, the following is the fundamental question due to Dolgachev-Weisfeiler.

Question 1. Let $f : X \to Y$ be an A^n-fibration. Assume that f is affine and faithfully flat and all closed fibers are geometrically integral. Is f an $\mathbb{A}^n$-bundle ?

The answer is affirmative in the case Y is normal and $F = \mathbb{A}^1$ by the results of Kambayashi-Miyanishi [7] and Kambayashi-Wright [8]. The following result is essentially due to Sathaye [12].

Theorem 2. *Assume that $F = \mathbb{A}^1$ or $\mathbb{A}^2$ and that the generic fiber X_K is isomorphic to $\mathbb{A}^1_K$ or $\mathbb{A}^2_K$ over the function field K of Y. In the case $F = \mathbb{A}^2$ we assume that Y is a curve. Then there exists a dense open set U of Y such that $f^{-1}(U) \cong U \times F$.*

For an $\mathbb{A}^2$-fibration, there are partial results by Kaliman-Zaidenberg [6] and Kaliman [5].

Theorem 3. *Let $f : X \to Y$ be a dominant morphism of a smooth quasi-projective variety X to a smooth quasi-projective variety Y. Suppose that general fibers $f^{-1}(Q)$ for $Q \in Y$ are isomorphic to $\mathbb{A}^2$. Then there exists a Zariski open subset U of Y such that $f^{-1}(U) \cong U \times \mathbb{A}^2$.*

Theorem 4. *Let f be a polynomial in $\mathbb{C}[x, y, z]$ such that the associated morphism $f : \mathbb{A}^3 \to \mathbb{A}^1$ defined by $P \mapsto f(P)$ has general fibers isomorphic to $\mathbb{A}^2$. Then $\mathbb{C}[x, y, z] = \mathbb{C}[f, g, h]$ with $g, h \in \mathbb{C}[x, y, z]$. In particular, all the fibers of f are isomorphic to $\mathbb{A}^2$.*

An open set U of X is called an $\mathbb{A}^n$-*cylinder* or a *cylinderlike open set* if $U \cong \mathbb{A}^n \times U_0$. In the case $F = \mathbb{A}^1$, an $\mathbb{A}^1$-fibration has a strong tie with a G_a-action.

Lemma 5. *Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} A_0$ and let $f : X \to Y$ be an $\mathbb{A}^1$-fibration. Then the generic fiber X_K is isomorphic to $\mathbb{A}^1_K$ iff there exists a G_a-action on X such that A_0 is contained in the ring of invariants in A.*

The following result of Suzuki-Zaidenberg [13, 14] allows us to compute the topological Euler number of an affine surface with an F-fibration. One can see the special roles that $\mathbb{A}^1$ or $\mathbb{A}^1_*$ will play.

Theorem 6. *Let X be a smooth affine surface with a morphism $f : X \to C$ with connected general fibers F, where C is a smooth curve. Let F_i $(1 \le s \le \ell)$ exhaust all singular fibers. Then we have the equality of topological Euler numbers*

$$e(X) = e(C) \cdot e(F) + \sum_{i=1}^{\ell} (e(F_i) - e(F)).$$

Furthermore, $e(F_i) \ge e(F)$ for all i. If the equality holds for some i, then F is $\mathbb{A}^1$ or $\mathbb{A}^1_*$, and F_i is isomorphic to F for all i if taken with reduced structures.

Singularity on an affine normal variety with $\mathbb{A}^n$-fibration is an interesting object to study.

Theorem 7. *Let X be a normal affine surface and let $f : X \to C$ be either an $\mathbb{A}^1$-fibration or $\mathbb{A}^1_*$-fibration, where C is a smooth curve. Then X has only cyclic quotient singularity.*

If V is a normal projective surface and $f : V \to B$ is a $\mathbb{P}^1$-fibration, V has only rational singularity. A more general result is given by Flenner-Zaidenberg [3].

Theorem 8. *Let V be an algebraic variety and let $P \in V$ be an isolated Cohen-Macaulay singularity. If there exists a Zariski open set U of V such that U is covered by closed rational curves not passing through the point P. Then P is a rational singularity.*

There are some recent trials by Kishimoto [9] on singularities of affine 3-folds which contain $\mathbb{A}^1$-cylinders. If there exists an $\mathbb{A}^1$-fibration (resp. $\mathbb{A}^1_*$-fibration) on a smooth algebraic surface X then $\overline{\kappa}(X) = -\infty$ (resp. $\overline{\kappa}(X) \le 1$). Degenerate fibers of an $\mathbb{A}^1$ (or $\mathbb{A}^1_*$) fibration are very easy to describe.

Lemma 9. *Let $f : X \to C$ be an $\mathbb{A}^1$-fibration or an $\mathbb{A}^1_*$-fibration on a smooth affine surface X and let $f^*(Q)$ be a singular fiber of f. In the case of $\mathbb{A}^1$-fibration, $f^*(Q)_{\mathrm{red}}$ is a disjoint sum of the $\mathbb{A}^1$. In the case of $\mathbb{A}^1_*$-fibration, $f^*(Q) = \Gamma + \Delta$, Γ is $0, mC$ or $m_1C_1 + m_2C_2$, where $C \cong \mathbb{A}^1_*$, $C_1 \cong C_2 \cong \mathbb{A}^1$ with $(C_1 \cdot C_2) = 1$, $\Delta \cap \Gamma = \emptyset$, and Δ_{red} is a disjoint sum of the $\mathbb{A}^1$ if $\Delta \ne 0$.*

The following results are due to Asanuma [1] and Asanuma-Bhatwadekar [2].

Theorem 10. *If R is a regular local ring and let A be a finitely generated flat R-algebra such that $A \otimes \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})^{[n]}$ for every $\mathfrak{p} \in \operatorname{Spec} R$, where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then A is stably isomorphic to $R^{[n]}$, i.e., $A \otimes_R R^{m]} \cong R^{[n+m]}$.*

Theorem 11. *Let A be an affine domain over a 1-dimensional noetherian domain R containing the field $\mathbb{Q}$. Suppose that A is flat over R and $A \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})^{[2]}$ for every $\mathfrak{p} \in \operatorname{Spec} R$. Then there exists $\xi \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[\xi]$.*

There are many results where the fibrations are used in crucial ways. To wit, we can list the following results.
(1) In the Abhyankar-Moh-Suzuki theorem, an embedded line $V(f)$ in $\mathbb{A}^2$ defines an $\mathbb{A}^1$-fibration $f : \mathbb{A}^2 \to \mathbb{A}^1$ whose closed fibers are all geometrically integral, and hence defines an $\mathbb{A}^1$-bundle. Similarly, any embedded line in an ML_0-surface with the Picard number 0 is a fiber-component of an $\mathbb{A}^1$-fibration whose closed fibers are all geometrically irreducible.
(2) In the proof by Gurjar-Miyanishi [4] of Lin-Zaidenberg theorem which asserts that *any irreducible contractible curve C is conjugate to a curve defined by $x^m = y^n$ with* $\gcd(m, n) = 1$, an $\mathbb{A}^1_*$-fibration $\phi : \mathbb{A}^2 - \{P\} \to \mathbb{P}^1$ is used effectively to determine the equation of C, when C has a singular point P and $\overline{\kappa}(\mathbb{A}^2 - C) = 1$.
(3) An affine surface X is isomorphic to $\mathbb{A}^2$ iff X is factorial, $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ and X has an $\mathbb{A}^1$-fibration $f : X \to C$.
(4) A smooth affine surface X has $\overline{\kappa}(X) = -\infty$ iff X has an $\mathbb{A}^1$-fibration $f : X \to C$.
(5) In order to determine the equation of a generically rational polynomial (or a field generator by Russell) $f \in \mathbb{C}[x, y]$ up to an automorphism of $\mathbb{C}[x, y]$, one effective way is to consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^1$ and look into degenerate fibers. This method worked in the cases of polynomials of simple type or of quasi-simple type.
(6) By Kaliman [5], we have the following characterization of $\mathbb{A}^3$.

Theorem 12. *Let $X = \operatorname{Spec} A$ be an affine 3-fold. Then $X \cong \mathbb{A}^3$ iff $A^* = \mathbb{C}^*$, A is factorial, $H_3(X; \mathbb{Z}) = 0$ and X contains a cylinderlike open set $U \times \mathbb{A}^2$ such that every irreducible component of $X \setminus U \times \mathbb{A}^2$ has at most isolated singularities.*

(7) We often use Kawamata's addition formula of log Kodaira dimensions.

Theorem 13. *Let $f : X \to Y$ be an F-fibration of smooth algebraic varieties with $\dim F \leq 1$. Then $\overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(Y)$.*

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The special automorphism group of $R[t]/(t^m)[x_1,\dots,x_n]$ and coordinates of a subring of $R[t][x_1,\dots,x_n]$

Stéphane Vénéreau
(joint work with Arno van den Essen and Stefan Maubach)

Let R be a commutative with unity ring and $R^{[d]} = R[x_1,\cdots,x_d]$ a polynomial ring in d indeterminates over R. A polynomial $f = f(x_1,\cdots,x_d) \in R[x_1,\cdots,x_d]$ is called a *hyperplane* (or a *plane* resp. *line* if $d-1 = 2$ resp. 1) if $R[x_1,\cdots,x_d]/(f) \simeq R^{[d-1]}$ and it is called a *variable* if there exists an automorphism of R-algebra $\alpha : R[x_1,\cdots,x_d] \to R[x_1,\cdots,x_d]$ such that $\alpha(x_1) = f$. One sees easily that variables are hyperplanes but the converse is one of the major problems in affine algebraic geometry, it is often referred to as the Abhyankar-Sathaye Problem or Conjecture. For $d = 2$ the problem is completely solved thanks to the Abhyankar-Moh-Suzuki Theorem and its generalizations for some coefficients rings on one hand and thanks to the construction of *bad lines* (lines that are not variables) for the remaining rings on the other hand (see [2]). The idea is to introduce another notion : residual variables. We say that a polynomial $f \in R^{[d]}$ is a *residual variable* if its canonical image in $k(\wp)^{[d]}$ is a variable for every prime ideal $\wp \in \operatorname{Spec} R$, where $k(\wp)$ denotes the residue class field $R_\wp/\wp R_\wp$. Variables are easily seen to be residual variables and, again, the converse has to be studied. In case $d = 2$ it is (almost) completely known for which rings residual variables are variables and for which there are, say, *bad residual variables* (see [2]). From $d = 3$ and when the ring R is of characteristic 0 (as well as all its residue class fields) both problems remain unsolved. The question we ask here is: can one generalize the construction in [2] of bad lines and bad residual variables (in the char. 0 case) to get bad hyperplanes and bad residual variables? Unfortunately(?) the answer is no but this answer has nevertheless some "positive" consequences.

The starting point of the construction of bad objects in [2] is the following example: to simplify take $R = \mathbb{F}_2[t^2, t^3] \subset \mathbb{F}_2[t]$, where $\mathbb{F}_2$ denotes the field with two elements. We define the following automorphism γ of $\mathbb{F}_2[t][x, y]$:

$$\begin{cases} \gamma(x) & = x + (y + tx)^2 = x + y^2 + t^2x^2 \\ \gamma(y) & = y + tx. \end{cases}$$

Remark that $\gamma(x) \in R[x, y]$, however an easy argument shows that it is not a variable there but only over the larger ring $\mathbb{F}_2[t]$. One can also see that it is a residual variable, whence a bad residual variable (note here that this is not a line, examples of bad lines being more complicated).

We will study a slight modification of this example, which is nicer in some way and has the same properties:

$$\begin{cases} \gamma(x) & = x - y^2 + (y + t(x - y^2))^2 & \equiv x & \mod t^2\mathbb{F}_2[t][x, y] \\ \gamma(y) & = y + t(x - y^2) - t\gamma(x) & \equiv y + ty^2 & \mod t^2\mathbb{F}_2[t][x, y]. \end{cases}$$

To ease notations consider the canonical epimorphism

$$\begin{array}{ccc} \mathbb{F}_2[t] & \rightarrow & \mathbb{F}_2[t]/(t^2) \\ a & \mapsto & \bar{a} \end{array}$$

as well as its extensions

$$\mathbb{F}_2[t][x, y] \rightarrow \mathbb{F}_2[t]/(t^2)[x, y],\ \mathrm{Aut}\mathbb{F}_2[t][x, y] \rightarrow \mathrm{Aut}\mathbb{F}_2[t]/(t^2)[x, y]$$

for which we use the same notation $(\bar{f}(x, y),\ \bar{\gamma}, \dots)$.
Then the fact that $\gamma(x)$ is a bad residual variable of $R[x, y]$ comes from the following:

$$\begin{array}{rcl} \bar{\gamma} & \in & \mathrm{Aut}_x\mathbb{F}_2[t]/(t^2)[x, y] \cap \overline{\mathrm{Aut}\mathbb{F}_2[t][x, y]} \\ \text{BUT } \bar{\gamma} & \notin & \overline{\mathrm{Aut}_x\mathbb{F}_2[t][x, y]} \end{array}$$

where the index x in Aut_x stands for " automorphisms preserving x". Indeed, every automorphism of $\mathbb{F}_2[t][x, y]$ preserving x has degree one in y, none can therefore give $\bar{\gamma}(y) = y + ty^2$.

In view of generalizing such examples in higher dimensions one asks: for $d \geq 3$ can one do the same with, say, $\mathbb{C}$ instead of $\mathbb{F}_2$ or more generally with R a reduced ring containing $\mathbb{Q}$ instead of $\mathbb{F}_2$? Replacing y by $d - 1$ indeterminates $y_1, \cdots, y_{d-1} =: Y$ we ask then:
Is there an automorphism $\bar{\gamma}$ such that

$$\begin{array}{rcl} \bar{\gamma} & \in & \mathrm{Aut}_x R[t]/(t^2)[x, Y] \cap \overline{\mathrm{Aut}R[t][x, Y]} \\ \text{BUT } \bar{\gamma} & \notin & \overline{\mathrm{Aut}_x R[t][x, Y]}\ ? \end{array}$$

For this, there has to be some

$$\alpha \in \mathrm{Aut}_x R[t]/(t^2)[x, Y] \setminus \overline{\mathrm{Aut}_x R[t][x, Y]}.$$

This alone is easy to find: define $\alpha \in \mathrm{Aut}_x R[t]/(t^2)[x,Y]$ by $\alpha(y_1) = y_1 + ty_1^2$, $\alpha(y_{2,3,\dots}) = y_{2,3,\dots}$. Then the jacobian determinant of α, $j(\alpha)$, is $1+2ty_1$ and

$$j(\alpha) = 1 + 2ty_1 \notin \overline{R[t][x,Y]^*} = \overline{R^*} = R^* \subset R[t]/(t^2)[x,Y]$$

where * stands for the set of units. It follows that

$$\alpha \notin \overline{\mathrm{Aut}_x R[t][x,Y]}.$$

But if $\alpha \in \overline{\mathrm{Aut}R[t][x,Y]}$, as required in the question, then $j(\alpha) \in \overline{R[t][x,Y]^*} = R^* \subset R[t]/(t^2)[x,Y]$. So one may assume that $j(\alpha) = 1$. The eventual generalization of the construction of counter-examples depends now on the following

Question 1.

$$\text{Is } \mathrm{SAut}R[t][Y] \longrightarrow \mathrm{SAut}R[t]/(t^m)[Y] \text{ surjective ?}$$

Where SAut denote the group of automorphisms with jacobian determinant equal to one.

Answer 1 (van den Essen, Maubach). YES

There is no more x since it can be considered as an element of R now.
We want to sketch the proof for $m = 2$ and $Y = y_1, y_2$:
take $\alpha \in \mathrm{SAut}R[t]/(t^2)[y_1,y_2]$. Up to multiplying it by an automorphism of $R[y_1,y_2]$ which is clearly in the image, one may assume $\alpha \equiv \mathrm{Id} \mod t$. The condition $j(\alpha) = 1$ implies that

$$\begin{cases} \alpha(y_1) &= y_1 + t\frac{\partial P}{\partial y_2} \\ \alpha(y_2) &= y_2 - t\frac{\partial P}{\partial y_1} \end{cases} \text{ for some } P \in R[y_1,y_2]$$

we will denote $\alpha = \alpha_P$. Remark that $\alpha_P\alpha_Q = \alpha_{P+Q}$. One may therefore assume that $P = ry_1^{i_1}y_2^{i_2}$, for some $r \in R$. But such a monomial decomposes as well:

$$y_1^{i_1}y_2^{i_2} = \sum_q r_q(y_1 + qy_2)^{i_1+i_2} \text{ for some } q, r_q \in \mathbb{Q}.$$

Whence one may assume $P = r(y_1+qy_2)^d$. Now we define a Locally Nilpotent Derivation of $R[t][y_1,y_2]$: $Dy_1 = q$, $Dy_2 = -1$. One has $D(rtd(y_1+qy_2)^{d-1}) = 0$ hence $rtd(y_1+qy_2)^{d-1}D$ is a LND too and $e^{rtd(y_1+qy_2)^{d-1}D}$ is the desired automorphism of $R[t][y_1,y_2]$ since

$$\begin{cases} e^{rtd(y_1+qy_2)^{d-1}D}(y_1) &\equiv y_1 + rtd(y_1+qy_2)^{d-1}q \mod (t^2) \\ e^{rtd(y_1+qy_2)^{d-1}D}(y_2) &\equiv y_2 - rtd(y_1+qy_2)^{d-1} \mod (t^2) \end{cases}$$

and

$$\begin{cases} \overline{e^{rtd(y_1+qy_2)^{d-1}D}(y_1)} &= y_1 + t\frac{\partial P}{\partial y_2} &= \alpha(y_1) \\ \overline{e^{rtd(y_1+qy_2)^{d-1}D}(y_2)} &= y_2 - t\frac{\partial P}{\partial y_1} &= \alpha(y_2) \end{cases}$$

□

Consequently there is no possible generalization of the construction of bad lines and bad residual variables. We derive however from this result (Question and Answer 1 above) some positive ones such as:

Theorem 2. *Let R be a ring containing $\mathbb{Q}$ and m a positive integer. If $f \in R[t^m, t^{m+1}, \cdots, t^{2m-1}][x_1, \cdots, x_n]$ is a variable of $R[t][x_1, \cdots, x_n]$ then it is a variable of $R[t^m, t^{m+1}, \cdots, t^{2m-1}][x_1, \cdots, x_n]$.*

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Algebraic vector bundles over smooth real affine varieties

S. M. Bhatwadekar

Let $X = \operatorname{Spec} A$ be a smooth affine variety of dimension $n \geq 2$ over a field k. It is a well known fact that an algebraic vector bundle E over X of rank d corresponds to a projective A-module P of rank d and E has a nowhere vanishing section if and only if P splits off a free summand of rank 1 (i.e. $P \simeq A \oplus Q$). A classical result of Serre ([S]) says that a projective A-module P splits off a free summand of rank one if the rank of $P > n$.

Now we suppose that P is a projective A-module of rank $n = \dim A$. Let $CH_0(X)$ denote the group of zero cycles of X modulo rational equivalence. We can associate to P an element of $CH_0(X)$ viz. the top Chern class $C_n(P)$ of P. A result of Mumford can be used to construct a smooth affine variety $X = \operatorname{Spec} A$ of dimension $n \geq 2$ over the field $\mathbb{C}$ of complex numbers and a projective A-module P of rank n such that $C_n(P) \neq 0$. Since $C_n(P) = 0$ is a necessary condition for P to split off a free summand of rank 1, P can not have a free summand of rank 1. In view of this, it is natural to ask :

Question 1. Does P have a free summand of rank 1 if $C_n(P) = 0$?

M.P. Murthy settled the above question affirmatively for smooth affine varieties over algebraically closed fields ([Mu], Theorem 3.8)) . However, as is shown by the example of the tangent bundle of an even dimensional real sphere, $C_n(P) = 0$ is not always a sufficient condition to conclude that $P \simeq A \oplus Q$, if the base field is not algebraically closed. Incidentally, all the known examples of projective modules exhibiting such a behaviour are algebraic vector bundles over **even** dimensional real varieties. In view of these examples, it is of interest to explore whether one can classify examples of projective modules over smooth real affine varieties which have the property that the top Chern class of the projective module vanishes, but the projective module does not split off a free summand of rank one. More precisely one can ask the following:

Question 2. Let $X = \operatorname{Spec} A$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbb{R}$ (the field of real numbers) and P a projective A-module of rank n. Under what further restrictions, does $C_n(P) = 0$ imply that $P \simeq A \oplus Q$?

Recently the above question has been settled in complete generality in my joint work with Mrinal Das and Satya Mandal ([BDM], Theorem 4.30). As a special case of this result one can show:

Theorem 3. *Let X be as above. Let $K_A = \wedge^n(\Omega_{A/\mathbb{R}})$ denote the canonical module of A. Assume that the manifold $X(\mathbb{R})$ of real points of X is connected. Let P be a projective A-module of rank n such that its top Chern class $C_n(P) \in CH_0(X)$ is zero. Then $P \simeq A \oplus Q$ in the following cases:*

(1) *n is odd.*
(2) *$X(\mathbb{R})$ is not compact.*
(3) *n is even, $X(\mathbb{R})$ is compact and $\wedge^n(P) \not\simeq K_A$.*

Moreover, if n is even and $X(\mathbb{R})$ is compact then there exists a projective A-module P of rank n such that $P \oplus A \simeq A^n \oplus K_A$ (hence $C_n(P) = 0$ and $\wedge^n(P) \simeq K_A$), but P does not have a free summand of rank 1.

Note that the above theorem says that if $\dim(X)$ is odd then the only obstruction for an algebraic vector bundle of top rank over X to split off a trivial subbundle of rank 1 is algebraic, namely the possible nonvanishing of its top Chern class. However if $\dim X$ is even and $X(\mathbb{R})$ is compact and connected, then, apart from the possible nonvanishing of its top Chern class, the only other obstruction for an algebraic vector bundle of top rank over X to split off a trivial subbundle of rank 1 is purely topological viz. the associated topological vector bundle and the manifold $X(\mathbb{R})$ have the same orientation. Moreover, in this case, the theorem assures us that indeed these obstructions genuinely exist. The above result leads us to ask the following:

Question 4. *Let $X = \operatorname{Spec} A$ be a smooth affine variety of* **odd** *dimension $n \geq 2$ over a field k of characteristic 0. Let P be a projective A-module of rank n such that $C_n(P) = 0$ in $CH_0(X)$. Then, does there exist a projective A-module Q of rank $n-1$ such that $P \simeq A \oplus Q$?*

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Two topics on the geometry of singular plane curves of type (d, ν)

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(joint work with Masahito Ohkouchi, Mohammad Saleem)

Let C be a plane curve of degree d. We call C a *plane curve of type* (d, ν) if the maximal multiplicity of singular points on C is equal to ν. We observe two topics on singular plane curves of type (d, ν).

1. Classification

A unibranched singularity is called a *cusp*. Rational cuspidal plane curves of type $(d, d-2)$ and $(d, d-3)$ were classified around 1996–2000 by Flenner-Zaidenberg [3, 4] (See also Fenske [2] and Sakai-Tono [12] for some cases). A cusp P can be described by its multiplicity sequence $\underline{m}_P = (m_0, m_1, m_2, \ldots)$. In order to describe a multibranched singular point P on C, we introduce the notion of "the system of the multiplicity sequences" of P.

Definition 1. Let $P \in C$ be a multibranched singular point, having r local branches $\gamma_1, \ldots, \gamma_r$. Let $\underline{m}(\gamma_i) = (m_{i0}, m_{i1}, m_{i2}, \ldots)$ denote the multiplicity sequences of the branches γ_i, respectively. We define the *system of the multiplicity sequences*, which will be denoted by the same symbol $\underline{m}_P(C)$, to be the combination of $\underline{m}(\gamma_i)$ with brackets indicating the coincidence of the centers of the infinitely near points of the branches γ_i. For instance, for the case in which $r = 3$, we write it in the following form:

$$\left\{ \begin{pmatrix} m_{1,0} \\ m_{2,0} \\ m_{3,0} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho} \\ m_{2,\rho} \\ m_{3,\rho} \end{pmatrix} \begin{matrix} \begin{pmatrix} m_{1,\rho+1} \\ m_{2,\rho+1} \end{pmatrix} \cdots \begin{pmatrix} m_{1,\rho'} \\ m_{2,\rho'} \end{pmatrix} \begin{matrix} m_{1,\rho'+1}, \ldots, m_{1,s_1} \\ m_{2,\rho'+1}, \ldots, m_{2,s_2} \end{matrix} \\ m_{3,\rho+1}, \ldots, m_{3,s_3} \end{matrix} \right\}.$$

We also use some simplifications.

Example 2. We examine our notations for ADE singularities.

P	A_{2n-1}	D_{2n-1}	D_{2n}	E_7
$\underline{m}_P(C)$	$\binom{1}{1}_n$	$\left\{ \binom{2}{1} \; {}_{2n-3} \right\}$	$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \binom{1}{1}_{n-2} \right\}$	$\binom{2}{1}\binom{1}{1}$

Example 3. The hyperelliptic curve $y^2 = \prod_{i=1}^{2g+2}(x - a_i)$ has one singularity P on the line at infinity with $\underline{m}_P = \binom{g}{g}\binom{1}{1}_g$.

Using these notations, we can describe any kind of singularities of plane curves. As for irreducible plane curves of type $(d, d-2)$, we have the following

Theorem 4 ([10, 11]). *We can classify the singularities of irreducible plane curves C of type $(d, d-2)$ with genus $g \geq 0$. Furthermore, we can give the algorithm to find a Cremona transformation which transforms C into a line (if $g = 0$), a smooth cubic (if $g = 1$), or a hyperelliptic curve $y^2 = \prod_{i=1}^{2g+2}(x - a_i)$ (if $g \geq 2$).*

2. Gonality

In case the curve C has genus $g \geq 2$, the *gonality* of C, denoted by G, is an important birational invariant. By definition, $G = \min\{\deg(\varphi)\}$, among all non-constant morphisms $\varphi : \tilde{C} \to \mathbf{P}^2$, where $\tilde{C}$ is the non-singular model of C. If C is of type (d, ν), then via the projection from the singular point with multiplicity ν to a line, one has the bound: $G \leq d - \nu$. Let us recall some known results.

(1) Brill-Noether upper bound: $G \leq (g+3)/2$,
(2) 1979 Namba [5]: $G = d - 1$, if C is non-singular,
(3) 1990 Coppens-Kato [1]: $G = d-2$, if C has only nodes and ordinary cusps and if $g \geq d(d-4)/4 - \{1$ (if d is even), $3/4$ (if d is odd)$\}$.

In [6], we obtained two criteria for the equality $G = d-\nu$. But, it is not difficult to construct examples with $G < d - \nu$ (See [6]). Recently, we also obtained two criteria for the inequality $G \geq d - \nu - q$ ([8]).

Definition 5. We define a quadratic function $Q(x) = x(x-d) + d + \delta - \nu$, where the δ is the well known invariant with which the genus g of C is given by the formula: $g = (d-1)(d-2)/2 - \delta$. We set $q = Q([d/\nu])$.

Theorem 6. *Let C be an irreducible plane curve of type (d, ν) with $d/\nu \geq 2$.*

(i) *If $q \leq 0$, then $G = d - \nu$.*
(ii) *If $q \geq 1$, then $G \geq d - \nu - q$.*

Definition 7. Let $m_1, \ldots, m_n$ denote the multiplicities of all singular points of C. We here include infinitely near singular points. We set $\eta = \sum_{i=1}^{n} (m_i/\nu)^2$. Let ν_i denote the i-th largest multiplicity of points on C. We also define the secondary invariant $\sigma = (\nu_2/\nu) + (\nu_3/\nu) + (\nu_4/\nu)$. Clearly, we have $3 \geq \sigma \geq 3/\nu$.

Definition 8. We define the following functions.

$$\begin{aligned}
h(\eta,\nu,q) &= \frac{\eta}{2(1+q/\nu)} + \frac{1+q/\nu}{2}, \\
g(\eta,\nu,q) &= \frac{\sqrt{\eta}+\sqrt{\eta-4/\nu}+2}{2} - \frac{2(q/\nu)}{\sqrt{\eta}+\sqrt{\eta-4/\nu}-2}, \\
f_3(\eta,\nu,q) &= \frac{3\sqrt{\eta}-(1+1/\nu+q/\nu)}{2}, \\
f_2(\eta,\nu,q) &= 2\sqrt{\eta}-(1+1/\nu+q/\nu).
\end{aligned}$$

For $k = 2, 3$, we set $\chi_k(\eta,\nu,q) = \max\Big\{h(\eta,\nu,q), \min\big\{f_k(\eta,\nu,q), g(\eta,\nu,q)\big\}\Big\}$.

Theorem 9. *Let C be an irreducible singular plane curve of type (d, ν) such that $\eta \geq 4/\nu$. Let q be a non-negative integer. We have $G \geq d - \nu - q$, either if*

(i) $d/\nu > \chi_3(\eta,\nu,q)$ *and* $d/\nu \geq \sigma - q/\nu$, *or if*
(ii) $d/\nu > \chi_2(\eta,\nu,q)$.

Theorems 6, 9, are complementary. Combining them, we obtain positive values $B(d,\nu)$, so that if $g \geq B(d,\nu)$, then we have the equality $G = d - \nu$. For small d, ν, the values of $B(d,\nu)$ are given in the following table:

d	7	8	9	10	11	12	13	14	15	16	17	18
$B(d,2)$	8	11	15	19	24	29	35	41	48	55	63	71
$B(d,3)$	7	11	16	22	29	32	40	49	53	63	74	79
$B(d,4)$	4	8	13	19	26	34	43	53	64	69	81	94

Note that $B(d,2)$ is optimal (See examples in [1]). For $(d,\nu) = (7,3)$, the possible pairs (g,G) (derived from Theorems 6, 9) are illustrated as follows. The black circle means the existence of examples.

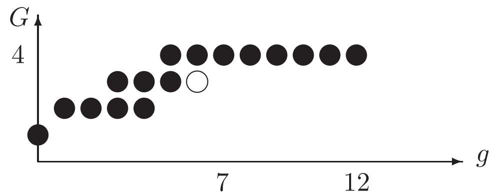

FIGURE 1. $(d,\nu) = (7,3)$

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Singularities appearing on generic fibers

Stefan Schröer

Let S be a regular scheme over an algebraically closed field of characteristic $p > 0$, endowed with a dominant morphism $S \to B$. Then the generic fiber S_η is a regular, but the geometric generic fiber $S_{\bar{\eta}}$ might be singular. This happens, for example, in quasielliptic surfaces, or for the addition map $\mathrm{Hilb}^2(A) \to A$ if A is an abelian surface in characteristic two.

A natural question: What kind of singularities may appear on geometric generic fibers? The following terminology is useful: Let X be a scheme of finite type over some field F of characteristic $p > 0$. We say that X *descends to a regular scheme* if $X \simeq Y \otimes_E F$ for some regular E-scheme Y, for some subfield $E \subset F$ so that the field extension is purely inseparable.

Throughout, we assume that X descends to a regular scheme. Then X is locally of complete intersection. If $x \in X$ is a point of codimension ≥ 2, the Zariski–Nagata Theorem leads to the following result on local fundamental groups:

Theorem 1. $\pi_1^{\mathrm{loc}}(\mathcal{O}_{X,x}^{\mathrm{sh}}) = 0$.

The subscheme $X' \subset X$ defined by the jacobian ideal $\mathcal{J} \subset \mathcal{O}_X$ satisfies some strong conditions as well. Suppose that X' is discrete, and let $d = \dim(X)$.

Theorem 2. *The Tjurina number $h^0(\mathcal{O}_{X'})$ is divisible by p, the sheaf $\mathcal{O}_{X'}$ has finite projective dimension, and the length formula $l(\mathcal{O}/\mathcal{J}^{[p]}) = p^d l(\mathcal{O}/\mathcal{J})$ holds.*

Finally, suppose that $x \in X$ is a point of codimension one or two.

Theorem 3. *The stalk of the tangent sheaf $\Theta_{X,x}$ is free, and $\Omega^1_{X,x} = \mathcal{O}_{X,x} \oplus M$ contains an invertible direct summand.*

Using all these criteria, we are able to determine which rational double points descend to regular schemes. Hirokado obtained this already for odd primes. For $p = 2$, the rational double points are A_n with $n + 1 = 2^e$, and D_n^0 with $n \geq 4$, and E_7^0, E_8^0. Here we use notation of Artin, who gave the classification of rational double points in positive characteristics.

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Survey of the Jacobian Conjecture

David Wright

1. The General Assertion

The renowned Jacobian Conjecture is stated below. It listed by Smale [20] as Problem 16 in his list of 18 great unsolved mathematical problems for the 21st century.

Conjecture 1.1 (Jacobian Conjecture (JC)). For any integer $n \geq 1$ and polynomials $F_1, \ldots, F_n \in \mathbb{C}[X_1, \ldots, X_n]$, the polynomial map $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism if the determinant $|JF|$ of the Jacobian matrix $JF = (D_i F_j)$ is a nonzero constant.

Here and throughout this survey we write D_i for $\partial/\partial X_i$. We will continue to write JF for the Jacobian matrix of a polynomial map F, and the determinant of this matrix will be denoted by $|JF|$. A polynomial map satisfying the hypotheses of the above conjecture will be called a *Jacobian map.* The Jacobian Conjecture in a fixed dimension n will be denoted by JC_n. Following standard notational practice we write $\mathbb{C}^{[n]}$ for $\mathbb{C}[X_1, \ldots, X_n]$.

2. Formulation and Early History

The conjecture was stated by O. H. Keller in 1939 [Ke]. Early proofs, all eventually shown to be incorrect, were given by W. Engel [9], B. Segre [16], [17], [18], and Gröbner [10]. Shafarevich asserted the conjecture as fact [19], mistakenly believing it to be obvious. The 2-dimensional problem JC_2 was discussed in detail by Abhyankar in [2], where he proves that it suffices that for all Jacobian maps (F_1, F_2) the curves $F_i = 0$ have only one (common) point at infinity; he then show that in fact they have at most two such points. Moh [14] showed there are no counterexamples to JC_2 where the degrees of F_1 and F_2 are ≤ 100. The author [23] showed that $F = (F_1, F_2)$ is an automorphism if and only if $JF \in \mathrm{GE}_2(\mathbb{C}^{[2]})$, GE_2 being the subgroup of GL_2 generated by elementary and diagonal matrices.

There are many other partial results for general n, some of which are: A Jacobian map F is an automorphism if $\mathbb{C}[X]/\mathbb{C}[F]$ is a birational extension, or an integral extension (see [3]), or if $\mathbb{C}(X)/\mathbb{C}(F)$ is a Galois extension of fields [5], or if F is injective, or if the total degree of each F_i is ≤ 2 [22] (see [3] for a short proof of this due to S. Oda).

3. Stability Reductions

These are well-known reductions that can be made if we allow n to increase.

Theorem 1 (Cubic Homogeneous Reduction)**.** *The Jacobian Conjecture is true if it holds for all polynomial maps F having the form $F = X - H$ with H homogeneous of degree* 3.

Remark 3.1. In the case where $F = X - H$ with H homogeneous of degree $d \geq 2$ we have $|JF| \in \mathbb{C}^* \Leftrightarrow |JF| = 1 \Leftrightarrow$ JH is a nilpotent matrix $\Leftrightarrow (JH)^n = 0$.

The reduction to the cubic homogeneous case was proved in [3] and by Jagžev in [11]. The following reduction was proved by Drużkowsky in [8]:

Theorem 2 (Cubic Linear Reduction)**.** *The Jacobian Conjecture is true if it holds for all polynomial maps F having the form $F = X - H$ with $H = L^3$ where $L = L_1, \ldots, L_n$ is a linear homogeneous map.*

The following reduction is due to de Bondt and van den Essen [6]:

Theorem 3 (Symmetric Homogeneous Reduction)**.** *The Jacobian Conjecture is true if it holds for all polynomial maps F having the form $F = X - H$ with $H = \nabla P$ homogeneous of degree 3.*

The condition $H = \nabla P$ is equivalent to the condition JH is symmetric matrix. In this situation P is called the *potential function* for H. This situation occurs precisely when the Jacobian matrix of H is the Hessian matrix of P: $JH = \operatorname{Hess} P = (D_i D_j P)$. If H is homogeneous of degree d, P can, of course, be taken to be homogeneous of degree $d + 1$.

4. Formal Inverse

If F has the form $X - H$ with H having terms of degree ≥ 2, then F has a "formal inverse" $F^{-1} = G = (G_1, \ldots, G_n)$ where $G_i \in \mathbb{C}[[X_1, \ldots, X_n]]$. Then JC is just the assertion that if $|JF| = 1$ then G is a polynomial map. It is known that the degree of the inverse of a polynomial map of degree d is bounded by d^{n-1} (Gabber (see [3]) and Rusek-Winiarski ([15]). This gives finitude to the process of determining if F^{-1} is a polynomial. One might hope to do this using one of the known formulas for the formal inverse.

Formulas for F^{-1} were given by Abhyankar-Gurjar in 1974 [1], Bass-Connell-Wright in 1883 [3], and Zhao in 2004 and 2006 [25], [26]. We mention two resulting formulas which apply to the symmetric situation. In this case it can easily be shown that $F^{-1} = X + \nabla Q$ where $Q \in \mathbb{C}[[X_1, \ldots, X_n]]$, and thus our goal is to show Q is a polynomial. The first is due to Zhao [26].

Theorem 4 (Laplacian Formula)**.** *With $F = X - \nabla P$ and $F^{-1} = X + \nabla Q$ as above, we have $Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \cdots$ where*

$$Q^{(m)} = \frac{1}{2^m m!(m+1)!} \triangle^m (P^{m+1})$$

Here $\triangle$ is the familiar Laplacian operator $\sum_{i=1}^n D_i^2 P$.

The following combinatorial is due to the author [24]. (A similar formula was announced in [13] without proof.)

Theorem 5 (Combinatorial Formula)**.** *With $F = X - \nabla P$ and $F^{-1} = X + \nabla Q$ as above, we have $Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \cdots$ where*

$$Q^{(m)} = \sum_{T \in \mathbb{T}_m} \frac{1}{|\operatorname{Aut} T|} \mathcal{Q}_{T,P}$$

with $\mathcal{Q}_{T,P}$ being defined by

$$\mathcal{Q}_{T,P} = \sum_{\ell:E(T)\to\{1,\dots,n\}} \prod_{v\in V(T)} D_{adj(v)}P$$

The notation of the above is as follows: $\mathbb{T}_m$ is the set of isomorphism classes of free (i.e., unrooted) trees with m vertices; for T such a tree, $E(T)$ and $V(T)$ denote the sets of edges and vertices of T, respectively; for $v \in V(T)$, $\mathrm{adj}(v)$ denotes the set of edges $\{e_1, \dots, e_s\}$ in T which contain v, and $D_{\mathrm{adj}(v)} = D_{\ell(e_1)} \cdots D_{\ell(e_s)}$.

In [24] the author derives a number of consequences from the above formula, including the solution for the homogeneous case (any degree) where $(JH)^3 = 0$ and the cubic homogeneous case where $(JH)^4 = 0$. Sharp bounds on the degree of the inverse are obtained in these cases.

5. The Vanishing Conjecture

W. Zhao established the following criterion ([27]):

Theorem 6. *These conditions are equivalent, for $P \in \mathbb{C}[X_1, \dots, X_n]$:*
1. $\mathrm{Hess}\, P$ *is nilpotent.*
2. $\triangle^m(P^m) = 0$ *for* $m = 1, \dots, n$.
3. $\triangle^m(P^m) = 0$ *for all* $m \geq 1$.

Then Zhao formulated:

Conjecture 5.1 (Vanishing Conjecture (VC)). For every homogeneous polynomial $P \in \mathbb{C}[X_1, \dots, X_n]$ of degree 4, $\triangle^m(P^m) = 0 \ \forall m \geq 1 \implies \triangle^m(P^{m+1}) = 0$ $\forall m >> 0$.

The following remarkable assertion, due to Zhao, follows from Theorems 3 and 4.

Theorem 7. *VC $\Leftrightarrow$ JC.*

6. The Dixmier Conjecture

Suppose F is a Jacobian map. Define derivations Δ_i, $i = 1, \dots, n$, on $\mathbb{C}^{[n]}$ by

$$\Delta_i(U) = |J(F_1, \dots, F_{i-1}, U, F_{i+1}, \dots, F_n)|$$

for $U \in \mathbb{C}^{[n]}$. Then the Δ_is commute and $\Delta_i(F_j) = \delta_{ij}$. Moreover, they are locally nilpotent on $\mathbb{C}[F_1, \dots, F_n]$. It is not difficult to see that if each Δ_i is locally nilpotent on
$C^{[n]}$ then F is an automorphism.

This relates the JC to a problem about the Weyl algebra

$$\mathbb{A}_n = \mathbb{C}[X_1, \dots, X_n, D_1, \dots, D_n]$$

of differential operators on $\mathbb{C}^{[n]}$. It is defined by the relations $[X_i, X_j] = [D_i, D_j] = 0$, $[D_i, X_j] = \delta_{ij}$ (where $[A, B] = AB - BA$). The conjecture below, for $n = 1$, was made by Dixmier [7].

Conjecture 6.1 (Dixmier Conjecture (DC)). Every $\mathbb{C}$-endomorphism of $\mathbb{A}_n$ is an automorphism.

Since $\mathbb{A}_n$ is know to be a simple ring, DC is equivalent to showing the surjectivity of all $\mathbb{C}$-endomorphisms.

A Jacobian map F induces an endomorphism φ of $\mathbb{A}_n$ sending X_i to F_i, D_i to Δ_i by virtue of the fact that the defining relations for $\mathbb{A}_n$ are satisfied. If this endomorphism is surjective, it is easy to conclude that F is an automorphism. The proof is given in [3], where it is attributed to Kac and Vaserstein. Thus we have DC $\implies$ JC; in fact, $\mathrm{DC}_n \implies \mathrm{JC}_n$, where DC_n is the Dixmier Conjecture for a fixed n. A recent breakthrough proved by Tsuchimoto [21] and Belov-Kanel and Kontsevich [4] asserts that:

Theorem 8. *DC $\Leftrightarrow$ JC.*

More specifically, the proof establishes that $\mathrm{JC}_{2n} \implies \mathrm{DC}_n$.

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Cancellation with 2-dimensional UFDs

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(joint work with Leonid Makar-Limanov)

Let $\mathbf{k}$ denote a field of arbitrary characteristic. For a ring A, let $A^{[n]}$ denote the polynomial ring in n indeterminates over A. We are motivated by the following

Cancellation Theorem. If A is a finitely generated domain over an algebraically closed field $\mathbf{k}$ such that $A^{[1]} \cong \mathbf{k}^{[3]}$, then $A \cong \mathbf{k}^{[2]}$.

This result is a special case of a theorem due to the combined work of Takao Fujita, Masayoshi Miyanishi, and Tohru Sugie in zero characteristic and Peter Russell in prime characteristic [8, 11, 14]. In their work, $A^{[1]}$ and $\mathbf{k}^{[3]}$ are replaced by $A^{[n]}$ and $\mathbf{k}^{[n+2]}$, respectively, where $n > 0$.

Even for the special case we are considering, the only known proofs are the original one and a recent proof of Rajendra Gurjar [9] which relies on the topological methods of Mumford and Ramanujam [12, 13]. Our intention is to present purely algebraic techniques which yield a self-contained proof of the Cancellation Theorem as well as other similar results. We also hope that the algebraic approach will be easier to use in the case of higher dimensions (in which the problem is open). To obtain these cancellation results we first prove a statement on the AK invariant.

The AK invariant, defined below, has already helped to recover and generalize other similar cancellation results. In [4] we generalize the following cancellation theorem of Shreeram Abhyankar, Paul Eakin, and William Heinzer [1]: If A and B are finitely generated domains with transcendence degree 1 over an algebraically closed field $\mathbf{k}$ such that $A^{[n]} \cong B^{[n]}$, then $A \cong B$. Connections to the Cancellation Problem not withstanding, the AK invariant seems to be a useful tool in its own right for studying rings.

Suppose $\delta : A \to A^{[1]}$ is a homomorphism, where A is a commutative ring with identity. We write $\delta = \delta_t : A \to A[t]$ if we wish to emphasize an indeterminate t. We call δ an *exponential map on* A if

(i) $\varepsilon_0 \delta_t$ is the identity on A, where $\varepsilon_0 : A[t] \to A$ is evaluation at $t = 0$, and
(ii) $\delta_s \delta_t = \delta_{s+t}$, where δ_s is extended to a homomorphism $A[t] \to A[s,t]$ by $\delta_s(t) = t$.

Define $A^\delta = \{a \in A \,|\, \delta(a) = a\}$, a subring of A called the *ring of* δ*-invariants.* Let $\mathrm{EXP}(A)$ denote the set of all exponential maps on A. We define the *AK invariant*, or *ring of absolute constants of* A as

$$\mathrm{AK}(A) = \bigcap_{\delta \in \mathrm{EXP}(A)} A^\delta.$$

Any isomorphism $\varphi : A \to B$ of rings restricts to an isomorphism $\varphi : \mathrm{AK}(A) \to \mathrm{AK}(B)$. Indeed, if $\delta \in \mathrm{EXP}(A)$ then $\varphi\delta\varphi^{-1} \in \mathrm{EXP}(B)$. Remark that $\mathrm{AK}(A) = A$ if and only if the only exponential map on A is the standard inclusion $\delta(a) = a$ for all $a \in A$.

The main result is

Theorem 1. *Let A be a domain which is either finitely generated as a ring or finitely generated over a field* $\mathbf{k}$*. If* $\mathrm{AK}(A) = A$ *then* $\mathrm{AK}(A[x]) = A$.

The following two corollaries follow quickly from the main result.

Corollary 2. *Let A be a domain which is finitely generated over an algebraically closed field* $\mathbf{k}$*. If* $A^{[1]} \cong \mathbf{k}^{[3]}$ *then* $A \cong \mathbf{k}^{[2]}$.

Corollary 3. *Let A and B be finitely generated 2-dimensional UFDs over an algebraically closed field* $\mathbf{k}$*. Suppose* $A^* = \mathbf{k}^*$ *where* * *denotes the set of units. If* $A^{[1]} \cong B^{[1]}$ *then* $A \cong B$.

The second corollary is false without the hypothesis of unique factorization. The first counterexample (over the complex numbers) is due to Wlodzimierz Danielewski [6]. In [3] the AK invariant is used to demonstrate that Danielewski's surfaces provide a counterexample over any field of any characteristic, not necessarily algebraically closed. We are hopeful that the assumption on the units of A can be removed in the second corollary. However, we cannot increase the dimension. Counterexamples of 3-dimensional UFDs were recently discovered by David Finston and Stefan Maubach [7]. For a proof of the main theorem and first corollary, see [5].

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Algebraic curves on $\mathbb{C}^2$ with $\chi = 0$

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(joint work with Henryk Żołądek)

The problem, we deal with, consists of classifying all plane curves in $\mathbb{C}^2$ of Euler characteristic zero. By purely topological reasons, any such curve C has to be rational and may have at most two places at infinity. Actually, there are two possibilities

(A) C is a rational curve with one place at infinity (i.e. a polynomial curve) with one finite self–intersection. The index of intersection of the two branches at the self–intersection point can be larger than 1. This case is dealt with in [BZ1]
(B) C is an embedding of $\mathbb{C}^*$ into $\mathbb{C}^2$ (an annulus) with one or two places at infinity. We refer to [BZ2] for more details.

In this report we describe mostly curves from the case B. Curves we are dealing with are allowed to have more singular points but cuspidal. The classification problem is a generalisation of Zaidenberg–Lin theorem (if $\chi(C) = 1$, C is topologically a line).

The way to attack this problem is to find several complementary conditions that a rational curve of degree d has to satisfy so that it does not have any finite self–intersection; a typical rational curve has $\frac{1}{2}(d-1)(d-2)$ double points. Special curves have all these double points *hidden* either at infinity or at finite singular points. The number of double points hidden at the given singularity is the δ-invariant of the singular point. We have a well–known formula

$$\sum 2\delta_i = (d-1)(d-2), \tag{0.1}$$

where the sum is taken over all singular points of a rational curve, and δ_i is the number of double points at the ith singular point.

We introduce another invariant, called the external codimension $\operatorname{ext}\nu$ of a singular point, which can be calculated in terms of the blow–up diagram of the singularity. For a cuspidal singularity of multiplicity m we can prove that $2\delta \leq m(\operatorname{ext}\nu - m+2)$, whereas for a singular point with two branches $2\delta \leq m(\operatorname{ext}\nu - m+3)$. The invariant $\operatorname{ext}\nu$ and the above inequalities resemble the $\overline{M}$ invariant of Orevkov [Or] and the Zaidenberg–Orevkov inequality [ZO], however our inequality is slightly stronger.

In order to have a control over the external codimensions of all singular points of a given curve, we consider a space $Curv = Curv_{a,b,c,d}$ of curves of the form

$$\begin{cases} x(t) = t^a + \alpha_1 t^{a-1} + \cdots + \alpha_{a+b} t^{-b} \\ y(t) = t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}. \end{cases} \tag{0.2}$$

The space $Curv$ is of course $\mathbb{C}^{a+b+c+d}$. There is a group of transformations of $\mathbb{C}^2$ preserving the space $Curv_{a,b,c,d}$. The dimension g of this group depends heavily on actual values of a, b, c and d. The dimension counting argument suggests that the following inequality should be true

$$\sum \operatorname{ext}\nu_i \leq a+b+c+d-g, \tag{0.3}$$

where the sum is taken over all singular points and the contribution from singularities at infinity should be suitably adjusted. The inequality (0.3), altough very natural, is a conjecture. We were able to prove only a weakened version of that inequality. Therefore our result is still incomplete in a sense. We have classified all curves of type (A) and (B) satisfying the inequality (0.3). All cases known to us satisfy this inequality.

The last but not least condition comes from bounding the sum of multiplicities of finite singular points. If a singular point has a multiplicity m_i, then $\dot{x}$ has the order at least $m_i - 1$. Hence

$$\sum (m_i - 1) \leq a+b. \tag{0.4}$$

Putting together (0.1),(0.3) and (0.4) we obtain the set of conditions for a curve. Now straightforward, but complicated calculations allow to exhibit all curves that satisfy these inequalities. In fact we obtain a list of 21 cases of curves of type (A) (including 16 series and 5 exceptional cases) and 23 cases of curves of type (B) (including one continuous family, 18 discrete series and 4 exceptional cases).

Curves of type (B) contain 7 series and 2 exceptional cases of smooth embeddings of $\mathbb{C}^*$ into $\mathbb{C}^2$.

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Embeddings of a family of Danielewski hypersurfaces

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(joint work with Lucy Moser-Jauslin)

In this talk, we are interested by general questions about equivalence and stable equivalence of polynomials in $\mathbb{C}[x, y, z]$. We explaine how new kinds of phenomena follow from the study of the embeddings of some hypersurfaces, called Danielewski hypersurfaces, in $\mathbb{C}^3$. Our main results can be resumed in the following examples.

Theorem. Let Q_k be, for all integer $k \geq 1$, the polynomial of $\mathbb{C}[x, y, z]$ defined by $Q_k = x^2y - z^2 - x(1 - z^2)^k$. Then, the following statements hold:

(1) For all $c \in \mathbb{C}$, $V(Q_k + c) \cong V(Q_{k'} + c)$ and they are smooth and irreducible surfaces;
(2) $Q_k \not\sim Q_{k'}$ if $k \neq k'$ (see definition below);
(3) Q_k and $Q_{k'}$ are stably (and analytically) equivalent for all $k, k' \geq 1$.

1. Context

We start by given a few definitions and well-known results.

Definition. Two polynomials P and Q of $\mathbb{C}[x_1, \ldots, x_n]$ are said to be *equivalent* if there exists a polynomial automorphism ϕ of $\mathbb{C}^n$ such that $\phi^*(P) = Q$. We denote $P \sim Q$.

We should remark that if two polynomials P and Q are equivalent, then the zero sets of $P - c$ and $Q - c$ are isomorphic as algebraic varieties.
This remark is very useful when we want to prove that two given polynomials are not equivalent. It was, in fact, until now the only method one can find in the literature to distinguish equivalence classes of polynomials. Therefore, we can ask if the converse is true. When P is a coordinate, there are some cases in which the converse is true. By coordinate we mean a polynomial of $\mathbb{C}[x_1, \ldots, x_n]$ equivalent to x_1.

For $n = 2$, the famous theorem of Abhyankar, Moh and Suzuki asserts that if a polynomial $P \in \mathbb{C}[x_1, x_2]$ has at least one of these fibers isomorphic to $\mathbb{C}$, then it is a coordinate.

For $n = 3$, S. Kaliman proved that if the general fibers of a polynomial in 3 variables are isomorphic to $\mathbb{C}^2$, then this polynomial is a coordinate.

The case $n > 3$ is still open. S. Vénéreau gave an explicit example of a polynomial $v = y + x^2z + xy(yu + z^2) \in \mathbb{C}[x, y, z, u]$ such that $V(v - c) \cong \mathbb{C}^3$ for all $c \in \mathbb{C}$ but for which it is unknown if v is or not a coordinate. Nevertheless G. Freudenburg showed that, when viewed as a polynomial in 5 variables, this polynomial v becomes equivalent to x. We say that such a v is a *stable coordinate.* This leads to the following natural questions which were posed in [2].

Definition. Two polynomials $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$ are stably equivalent if $\exists$ $m \in \mathbf{N}$ with $P \sim Q$ in $\mathbb{C}[x_1, \ldots, x_{n+m}]$.

Stable coordinate conjecture. If a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ is stably equivalent to x_1, then it is a coordinate.

Stable equivalence problem. If two polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ are stably equivalent, are they equivalent?

The conjecture is true for $n \geq 3$ and is open for $n > 3$. The problem was solved affirmatively in the case $n = 2$ by L. Makar-Limanov, P. van Rossum, V. Shpilrain and Y.-T. Yu. The result given at the first part of the talk give counter-examples for $n = 3$.

2. Danielewski hypersurfaces

The polynomials Q_k defined previously came from the study of so called Danielewski hypersurfaces. That means hypersurfaces of $\mathbb{C}^3$ defined by an equation of the form $x^n y = Q(x, z)$ with $deg(Q(0, z)) \geq 2$. Surfaces of this kind were first studied by Danielewski when he used it to construct counter-examples to the *cancellation problem.*

In this talk, we give the complete classification of equivalence classes of polynomials of the form $P_r = x^2y - z^2 - xr(z^2)$, where r is a polynomial in one variable.

Theorem 1. $P_r \sim P_s \Leftrightarrow \exists\, a \in \mathbb{C}^*$ with $s = ar$.

Theorem 2. The three following conditions are equivalent;

(1) P_r and P_s are stably (algebraically) equivalent;
(2) either $r(0) = s(0) = 0$ or $r(0)s(0) \neq 0$;
(3) P_r and P_s are equivalent by a holomorphic automorphism of $\mathbb{C}^3$.

We proved the theorem 1 by using Makar-Limanov's techniques. Indeed, since the Makar-Limanov invariant of Danielewski surfaces with $n = 2$ is non trivial and is equal to $\mathbb{C}[x]$, we can show that an automorphism which sends P_r to P_s must have a certain form. Then, it suffices to check all the automorphisms of this particular form. We also prove theorem 2 on an example.

Finally, we give a result obtained in the same way in collaboration with A. Dubouloz.

Theorem. Let $\theta : \mathbb{C}^* \times V \longrightarrow V$ be defined by

$$\theta(a; x, y, z) = (ax, a^{-2}(1 - ax)((1 + x)y + z^2 - 1), z).$$

θ is a $\mathbb{C}^*$-action on the smooth surface $V = V(x^2y - (1-x)(z^2-1))$ which can be extended holomorphically but not algebraically to a $\mathbb{C}^*$-action on $\mathbb{C}^3$. Moreover, for every $a \neq 1$, $\theta(a; .)$ does not extend to an (algebraic) automorphism of $\mathbb{C}^3$.

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Actions of algebraic groups on affine algebraic varieties

Shulim Kaliman

Suppose that G is an algebraic group and X is a smooth complex affine algebraic variety. Two algebraic G-actions Φ_1 and Φ_2 on X are equivalent if $\Phi_2 = \alpha \circ \Phi_1 \circ \alpha^{-1}$ for some $\alpha \in \mathrm{Aut}X$. The topic of this talk is recent developments in the following classification problem: for G-actions on X with given properties describe all equivalence classes. Starting with results of Gutwirth, Rentchler, and others in 1960's on classification of $\mathbb{C}^*$-actions and $\mathbb{C}_+$-actions on $\mathbb{C}^2$ we go to the Koras-Russell Linearization Theorem that states that every $\mathbb{C}^*$-action on $\mathbb{C}^3$ is equivalent to a linear one. We discuss the idea of the proof including the introduction of the Makar-Limanov invariant. One of the central steps was strengthened recently by Koras and Russell as follows

Let S be a normal contractible surface of $\bar{\kappa}(S) = -\infty$ with quotient singularities only. Then $\bar{\kappa}(S_{\mathrm{reg}}) = -\infty$.

In turn this fact is crucial in the coming Gurjar's result:

For every reductive group G acting on a contractible smooth affine algebraic variety X that admits a dominant morphism from a Euclidean space $\mathbb{C}^n$ so that $X//G$ is two-dimensional this quotient is isomorphic to the quotient of $\mathbb{C}^2$ with respect to a linear action of a finite group.

Another consequence of the Linearization theorem is Popov's paper (based also on his previous result with Kraft) that proves that

Every action of a connected reductive group on $\mathbb{C}^3$ is linearizable, i.e. it is equivalent to a representation.

For higher-dimensional tori we have the following generalization of the old Bialynicki-Birula theorem (Demushkin, Berchtold, and Hausen)

Let X be a toric variety of dimension n with a canonical action Φ of torus $T = (\mathbb{C}^*)^n$. Then any other effective action of T on X is equivalent to Φ and, furthermore, any effective action of $(\mathbb{C}^*)^{n-1}$ on X is equivalent to the action of an $(n-1)$-dimensional subtorus of T generated by Φ.

Contrary to the situation on $\mathbb{C}^2$ there are non-triangular $\mathbb{C}_+$-actions on $\mathbb{C}^3$ (Bass) and not all free $\mathbb{C}_+$-actions on $\mathbb{C}^4$ are translations (Winkelmann). However we have the following (Kaliman)

Let Φ be a $\mathbb{C}_+$-action on a smooth factorial three-dimensional X with $H_2(X) = H_3(X) = 0$. Suppose that the action is free and $S = X//\Phi$ is smooth. Then Φ is a translation, i.e. X is isomorphic to $S \times \mathbb{C}$ and the action is generated by a translation on the second factor.

Since $\mathbb{C}^3//\mathbb{C}_+ \simeq \mathbb{C}^2$ for any nontrivial $\mathbb{C}_+$-action (Miyanishi) we have

A free $\mathbb{C}_+$-action on $\mathbb{C}^3$ is equaivalent to a translation.

In fact the smoothness assumption can be dropped in some cases (Kaliman, Saveliev):

Let Φ be a $\mathbb{C}_+$-action on a smooth three-dimensional contractible X. Then the quotient $X//\Phi$ is a smooth contractible surface.

Since smooth contractible surfaces are rational (Gurjar, Shastri) we have

A smooth contractible threefold X with a nontrivial $\mathbb{C}_+$-action is rational.

This is a partial answer to the Van de Ven conjecture in dimension 3 which states that smooth contractible affine algebraic varieties are rational. (For smooth contractible affine threefolds with a nontrivial $\mathbb{C}^*$-action rationality is proven by Gurjar, Shastri, and Pradeep).

On a normal affine surface S Flenner and Zaidenberg studied Dolgachev-Pinkhman-Demazure (DPD) presentation and the question about its uniqueness which is equivalent to the following fact:

Given an effective $\mathbb{C}^*$-action Φ on S any other effective $\mathbb{C}^*$-action is equivalent either to Φ or to Φ^{-1}. In particular, there are at most two equivalence classes.

They established that except for Gizatullin (i.e. quasi-homogeneous) surfaces this fact holds. Then Russell noticed that besides $\mathbb{C}^2$ (that has infinite number of such equivalence classes) there are other smooth affine surfaces with more than two equivalence classes. Namely they are Danilov-Gizatullin surfaces (i.e. the complements to ample sections in Hirzebruch surfaces). However in the coming paper of Flenner, Kaliman, and Zaidenberg the uniqness of the DPD presentation will be shown for the rest of Gizatullin surfaces except for those whose standard zigzag has exactly one vertex of weight less than -2 (in which case the answer is still under investigation).

There are non-linearizable actions of connected reductive groups different from tori on $\mathbb{C}^4$ (Schwarz). Actually for any such a group there exists such an action on some $\mathbb{C}^n$, $n \geq 4$ (Knop). Non-linearizable actions of finite groups on $\mathbb{C}^4$ were found by Jauslin-Moser, Masuda, and Petrie. Later new ideas were brought by Asanuma who showed that there is a non-linearizable $\mathbb{R}^*$-action on $\mathbb{R}^5$. Applying his method Derksen and Kutzschebauch discovered a non-linearizable analytic

$\mathbb{C}^*$-action on $\mathbb{C}^4$. In conclusion we discuss Asanuma's fourfolds that provide potential counterexamples to the Linearization and Cancellation Conjectures, and also a potential example of an exotic algebraic structure that is not exotic analytic structure.

Open algebraic surfaces with logarithmic Kodaira dimension zero

Hideo Kojima

Open algebraic surfaces with logarithmic Kodaira dimension zero have been studied by several authors. Irrational open algebraic surfaces with $\overline{\kappa} = 0$ were classified by Iitaka and Sakai. In [6], Zhang classified the Iitaka surfaces which are almost minimal open rational surfaces with $\overline{\kappa} = 0$ and $\overline{p}_g > 0$ (for the definitions, see [5, Chapter 2]). Log Enriques surfaces (normal projective rational surfaces with only quotient singular points and with numerically trivial canonical divisors) were studied by Blache, Kudryavtsev, Oguiso and Zhang. The author [4] established a classification theory of smooth affine surfaces with $\overline{\kappa} = 0$ in any characteristic and gave a classification of the strongly minimal smooth affine surfaces with $\overline{\kappa} = 0$.

In this article, we consider smooth open rational surfaces with $\overline{\kappa} = \overline{p}_g = 0$ and with non-contractible boundaries at infinity.

Let S be a smooth open rational surface with $\overline{\kappa}(S) = \overline{p}_g(S) = 0$ and (X, B) an SNC-completion of S (i.e., X is a smooth projective surface and B is a simple normal crossing divisor on X such that $S \cong X - B$). Let $C + K_W = (C + K_W)^+ + (C + K_W)^-$ be the Zariski decomposition of $C + K_W$, where $(C + K_W)^+$ is the nef part of $C + K_W$. Since $\overline{\kappa}(S) = 0$ and the pair (W, C) is almost minimal, $(C + K_W)^+$ is numerically equivalent to zero and $C^\# := C - (C + K_W)^-$ is an effective $\mathbb{Q}$-divisor. Let I be the smallest positive integer such that $IC^\#$ is an integral divisor. Since W is a rational surface, we have

$$\overline{P}_n(W - C) = \begin{cases} 1, & \text{if } I|n \\ 0, & \text{if } otherwise. \end{cases}$$

By using [1, Theorem C], we obtain the following lemma.

Lemma 1. *With the same notation and assumptions as above, assume further that $\lfloor C^\# \rfloor \neq 0$, where $\lfloor C^\# \rfloor$ is the integral part of $C^\#$. Then $I \in \{2, 3, 4, 6\}$.*

Now, set $C' := C - \lfloor C^\# \rfloor$. Then there exists a birational morphism $\mu : W \to V$ such that (V, D') $(D' := \mu_*(C'))$ is an almost minimal model of the pair (W, C'). Set $D := \mu_*(C)$.

Lemma 2. *With the same notation as above, the following assertions* (1) $\sim$ (4) *hold true.*

(1) $D := \mu_*(C)$ *is a simple normal crossing divisor.*
(2) $\overline{P}_n(V-D) = \overline{P}_n(X-B)$ *for any integer* $n \geq 1$. *In particular,* $\overline{\kappa}(V-D) = \overline{\kappa}(X-B) = 0$.
(3) $D^{\#} = \mu_*(C^{\#})$. *Moreover,* $\lfloor C^{\#} \rfloor \neq 0 \Longleftrightarrow \lfloor D^{\#} \rfloor \neq 0$.
(4) (V, D) *and* $(V, D - \lfloor D^{\#} \rfloor)$ *are almost minimal.*

We call the pair (V, D) a strongly minimal model of the pair (X, B) (or the surface S).

Recently, the author classified the strongly minimal models (V, D) in the case where $\lfloor C^{\#} \rfloor \neq 0$ ($\Longleftrightarrow \lfloor D^{\#} \rfloor \neq 0$), though we cannot reproduce the classification here for lack of space. In the following theorem, we give the classification in the case $I = 6$.

Theorem 1. *With the same notation and assumptions as above, assume further that* $I = 6$ *and* $\lfloor D^{\#} \rfloor \neq 0$. *Then the pair* (V, D) *is Fujita's* $Y\{2,3,6\}$.

Here we recall Fujita's $Y\{2,3,6\}$ (cf. [2, §8]). Let $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let ℓ_1, ℓ_2 and ℓ_3 be three distinct irreducible curves with $\ell_i \sim \ell$, where ℓ is a fiber of a fixed ruling on V_0, and let $\overline{\ell}_1$, $\overline{\ell}_2$, and $\overline{\ell}_3$ be three distinct curves with $\overline{\ell}_i \sim M_0$, where M_0 is a minimal section. Put $P_1 := \ell_1 \cap \overline{\ell}_2$, $P_2 := \ell_2 \cap \overline{\ell}_3$, $P_3 := \ell_3 \cap \overline{\ell}_1$ and $P_4 := \ell_1 \cap \overline{\ell}_3$. Let $\mu_0 : V_1 \to V_0$ be the blowin-up with centers $P_1, \ldots, P_4$. Put $E_i := \mu_0^{-1}(P_i)$, $1 \leq i \leq 3$. Let $\mu_1 : V_2 \to V_1$ be the blowing-up with centers $Q_1 := E_1 \cap \mu_0'(\overline{\ell_2})$, $Q_2 := E_2 \cap \mu_0'(\ell_2)$ and $Q_3 := E_3 \cap \mu_0'(\ell_3)$. Put $V := V_2$ and

$$D := \mu_1'(E_1 + E_2 + E_3 + \mu_0'(\sum_{i=1}^{3}(\ell_i + \overline{\ell}_i))).$$

The pair (V, D) (or $V - D$) is called Fujita's $Y\{2,3,6\}$.

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Exotic embeddings of smooth affine varieties

Zbigniew Jelonek

We find examples of exotic embeddings of smooth affine varieties into $\mathbb{C}^n$ in large codimensions. We show also examples of affine smooth, rational algebraic varieties X, for which there are algebraically exotic embeddings $\Phi : X \to X \times \mathbb{C}^l$, which are holomorphically trivial. Using this we construct an infinite family $\{C_{2p+3}\}$ (p is a prime number) of complex manifolds, such that every C_{2p+3} has at least two different algebraic (quasi-affine) structures. We show also that there is a natural connection between Abhyankar-Sathaye Conjecture and the famous Quillen-Suslin Theorem.

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Infinite automorphisms groups of algebraic manifolds

De-Qi Zhang

Consider the question of Gizatullin-Harbourne-McMullen (see [8] page 409 and [10] §12), where $\mathrm{Aut}^*(X) := \mathrm{Im}(\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Pic}(X)))$:

Question 1. Let X be a smooth projective (complex) rational surface. If $\mathrm{Aut}^*(X)$ is infinite, is there then a birational morphism φ of X to a surface Y having an anti-pluricanonical curve and an infinite subgroup $G \subset \mathrm{Aut}^*(Y)$ such that G lifts via φ to X?

The result below answers Question 1 in the case of null entropy.

Theorem 2. *(see* [15]*). Let X be a smooth projective rational surface and $G \leq \mathrm{Aut}(X)$ an infinite subgroup of null entropy. Then we have:*

(1) *There is a G-equivariant smooth blowdown $X \to Y$ such that $K_Y^2 \geq 0$ and hence Y has an anti-pluricanonical curve.*
(2) *Suppose further that $\mathrm{Im}(G \to \mathrm{Aut}(\mathrm{Pic}(X)))$ is also an infinite group. Then the Y in (1) can be so chosen that $-K_Y$ is nef of self intersection zero and Y has an anti 1-canonical curve.*

The result below is applicable when $G \geq \mathbb{Z} \times \mathbb{Z}$.

Theorem 3. *(see* [15]*). Let X be a smooth projective surface and $G \leq \mathrm{Aut}(X)$ a subgroup. Assume that there is a sequence of groups*

$$H \trianglelefteq A \trianglelefteq G$$

satisfying the following three conditions: (1) $\mathrm{Im}(H \to \mathrm{Aut}(\mathrm{NS}(X)))$ is finite; (2) A/H is infinite and abelian; and (3) $|G/A| = \infty$.

Then G contains a subgroup S of null entropy and infinite order.
In particular, when X is rational, there is an S-equivariant smooth blowdown $X \to Y$ such that Y has an anti-pluricanonical curve.

Remark. Conditions like the ones in Theorems 2 and 3 are probably necessary. See [1, Theorem 3.2]. The blowdown process $X \to Y$ to the minimal pair (Y, S) in Theorems 2 and 3 is necessary, as observed by [8].

Let $\sigma : \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{NS}_{\mathbb{Q}}(X))$ be the natural homomorphism.
Note that the T itself in (2) below is also finite when X is a rational or $K3$ surface (see Sterk [13] Lemma 2.1, and Torelli theorem).

Theorem 4. *(see [15]). Let X be a smooth projective surface and $G \leq \mathrm{Aut}(X)$ a subgroup of positive entropy. Then G satisfies either:*

1. *G contains the non-abelian free group $\mathbb{Z} * \mathbb{Z}$; or*
2. *There is a $B \trianglelefteq G$ such that $|G/B| \leq 2$ and $B = \langle h_m \rangle \ltimes T$ (semi-direct product) with h_m positive entropy and $\sigma(T)$ finite.*

See [12] Theorems 2.1 or 1.3 for groups of null entropy and [11] Theorem 1.1 for $K3$ groups (and more generally for hyperkähler manifolds), where the cyclic-ness of B/T in our result here is replaced by abelian-ness.

Next we show that the dynamics of automorphisms on all projective complex manifolds (of dimension 3, or of any dimension but assuming the Good Minimal Model Program or Mori's Program) are canonically built up from the dynamics on just three types of manifolds: Complex Tori, weak Calabi-Yau Manifolds X (i.e., $\kappa(X) = 0 = q(X)$), and Rationally Connected Manifolds. For the update on dynamics, see the survey [6], and [2], [5], [10].
A pair (X, g) is *rigidly parabolic* if $g|X$ is parabolic (i.e., $g|X$ is of null entropy and $\mathrm{ord}(g|X) = \infty$) and if every pair (Y, g) (dominated by (X, g)) is with $g|Y$ parabolic. A pair (X, g) is of *primitively positive entropy* if $g|X$ is of positive entropy and if every pair (Y, g) (of lower dimension and dominated by (X, g)) is with $g|Y$ parabolic.

Theorem 5. *(see [16]). Let X be a smooth projective complex manifold of $\dim X \geq 2$, and with $g \in \mathrm{Aut}(X)$. Then we have:*

1. *Suppose that (X, g) is either rigidly parabolic or of primitively positive entropy. Then the Kodaira dimension $\kappa(X) \leq 0$.*
2. *Suppose that $\dim X = 3$ and g is of positive entropy. Then $\kappa(X) \leq 0$, unless $d_1(g^{-1}) = d_1(g) = d_2(g) = e^{h(g)}$ and it is a Salem number. Here $d_i(g)$ are dynamical degrees and $h(g)$ is the entropy.*

Theorem 6. *(see* [16]*). Let X be a smooth projective complex manifold of* $\dim X \geq 1$*, and $g \in \mathrm{Aut}(X)$. Suppose that the pair (X, g) is either rigidly parabolic or of primitively positive entropy. Then we have:*

(1) *The albanese map* $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ *is a g-equivariant surjective morphism with connected fibres.*
(2) *The irregularity $q(X)$ satisfies $q(X) \leq \dim X$.*
(3) *$q(X) = \dim X$ holds if and only if X is g-equivariantly birational to an abelian variety* (*i.e., a projective torus*)*.*
(4) $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ *is a smooth morphism if $q(X) < \dim X$.*

In view of Theorem 5 we have only to treat the dynamics on those X with $\kappa(X) = 0$ or $-\infty$. This is done in [16]. As an illustrations, we have the simple 3-dimensional formulations of them as in (7) $\sim$ (9) below.

Corollary 7. *(see* [16]*). Let X' be a smooth projective complex threefold, with $g \in \mathrm{Aut}(X')$. Assume that the Kodaira dimension $\kappa(X') = 0$, irregularity $q(X') > 0$, and the pair (X', g) is either rigidly parabolic or of primitively positive entropy. Then there are a g-equivariant birational morphism $X' \to X$, a pair $(\tilde{X}, g)$ of a torus $\tilde{X}$ and $g \in \mathrm{Aut}(\tilde{X})$, and a g-equivariant etale Galois cover $\tilde{X} \to X$. In particular, X is a* **Q***-torus.*

Theorem 8. *(see* [16]*). Let X' be a smooth projective complex threefold, with $g \in \mathrm{Aut}(X')$. Assume $\kappa(X') = -\infty$, and (X', g) is either rigidly parabolic or of primitively positive entropy. Then there is a g-equivariant birational morphism $X \to X'$ with X smooth projective, such that:*

(1) *If $q(X) = 0$ then X is rationally connected.*
(2) *Suppose that $q(X) \geq 1$ and the pair (X, g) is of primitively positive entropy. Then $q(X) = 1$ and the albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a smooth morphism with every fibre F a smooth projective rational surface of Picard number* $\mathrm{rank}\ \mathrm{Pic}(F) \geq 11$.

Corollary 9. *(see* [16]*). Let X' be a smooth projective complex threefold. Suppose that $g \in \mathrm{Aut}(X')$ is of positive entropy. Then there is a pair (X, g) birationally equivariant to (X', g), such that one of the cases below occurs.*

(1) *There are a* 3*-torus $\tilde{X}$ and a g-equivariant etale Galois cover $\tilde{X} \to X$.*
(2) *X is a weak Calabi-Yau threefold.*
(3) *X is a rationally connected threefold.*
(4) $d_1(g^{-1}|X) = d_1(g|X) = d_2(g|X) = e^{h(g|X)}$ *and it is a Salem number.*

The following confirms the conjecture of Guedj [7] page 7 for automorphisms on 3-dimensional projective manifolds.

Theorem 10. *Let X be a smooth projective complex threefold admitting a cohomologically hyperbolic automorphism g in the sense of* [7] *p.3. Then either X is a weak Calabi-Yau threefold, or X is rationally connected, or there is a g-equivariant birational morphism $X \to T$ onto a* **Q***-torus. In particular, the Kodaira dimension $\kappa(X) \leq 0$.*

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Algorithmic problems in polynomial algebras

Vladimir Shpilrain

In this talk, we show how two ideas from combinatorial group theory, namely, the "peak reduction" method and the "small cancellation" technique can be used in the context of polynomial algebras.

The "peak reduction" method is a powerful combinatorial technique with applications in many different areas of mathematics as well as theoretical computer science. It was introduced by Whitehead (see [6] or [1]), a famous topologist and group theorist, who used it to solve an important algorithmic problem concerning automorphisms of a free group. Since then, this method was used to solve numerous problems in group theory, topology, combinatorics, and probably in some other areas as well.

In general, this method is used to find some kind of canonical form of a given object P under the action of a given group (or a semigroup) T of transformations. The principal idea behind this method is rather simple: one chooses the *complexity* of an object P one way or another, and declares a canonical form of P an object P' whose complexity is minimal among all objects $t(P),\ t \in T$. To actually find a "canonical model" P' of a given object P, one tries to arrange a sequence of sufficiently simple transformations so that the complexity of an object decreases *at* *every step*. To prove that such an arrangement is possible, one uses "peak reduction"; that means, if in some sequence of simple transformations the complexity goes up (or remains unchanged) before eventually going down, then there must be a pair of *subsequent* simple transformations in the sequence (a "peak") such that one of them increases the maximum degree (or leaves it unchanged), and then the other one decreases it. Then one tries to prove that such a peak can always be reduced.

In the context of polynomial algebras, objects are polynomials; their complexity is their degree; the group of transformations is the group of polynomial automorphisms; simple transformations are elementary and linear automorphisms. (An elementary automorphism is one that changes just one variable.)

Below are two sample results obtained by using the "peak reduction" method. Here $K[x, y]$ is the algebra of two-variable polynomials over a field K.

Theorem 1. [7] *Let K be a field of characteristic 0, and $p = p(x, y) \in K[x, y]$. If $\deg(p)$ cannot be decreased by a single elementary or linear automorphism of $K[x, y]$, then it cannot be decreased by any automorphism of $K[x, y]$.*

Theorem 2. [3] *Let $p = p(x, y) \in K[x, y]$. If the maximum of $\deg_x(p)$ and $\deg_y(p)$ cannot be decreased by a single elementary or linear automorphism of $K[x, y]$, then it cannot be decreased by any automorphism of $K[x, y]$.*

Comparing these two results shows that the "complexity" of an input polynomial may be chosen in different ways. In this particular situation, both choices are viable, but choosing the maximum of $\deg_x(p)$ and $\deg_y(p)$, as opposed to just $\deg(p)$, yields an easier proof. The lesson here therefore is that choices of both the set T of "basic" transformations and the "complexity" of inputs may have significant impact on the feasibility of a relevant "peak reduction".

Now we are going to touch upon another ideas from combinatorial group theory, the "small cancellation" technique. These ideas were introduced by Nielsen early in the 20th century, and taken further by Greendlinger in the 1960s. For an exposition of the foundations of this theory as well as for numerous applications in group theory, we refer to [1]. Here we give a brief exposition of the principal idea of the small cancellation theory. Suppose we have a "reduced" set S of elements of a free group (the precise meaning of "reduced" is not important to us here). Then the length of any nontrivial element in the subgroup generated by S cannot be smaller than $1/2$ of the minimum length of elements in S. This was proved by Nielsen. Later, Greendlinger established a similar fact for elements from the *normal closure* of S, under more stringent conditions on S.

To apply a similar idea in the polynomial algebra situation, we use the following standard "glossary": a free group corresponds to a polynomial algebra, elements of a free group correspond to polynomials, the length of a free group element corresponds to the degree of a polynomial, subgroups of a free group correspond to subalgebras of a polynomial algebra.

The following analog of Nielsen's result in the polynomial algebra situation is a consequence of a recent result of Shestakov and Umirbaev [4]. Let $p = p(x, y)$ and

$q = q(x, y)$ be a pair of polynomials in $K[x, y]$, reduced in the following sense: no transformation of the form $(p, q) \to (p, q+c\cdot p^k)$, $c \in K$, or $(p, q) \to (p+c\cdot q^k, q)$, can reduce the maximum degree of the pair (p, q). Let furthermore the polynomials p and q be algebraically independent. Then any nontrivial polynomial in the subalgebra $K[p, q]$ of the algebra $K[x, y]$ has the degree higher than 1/2 of the minimum of the degrees of p and q.

Shestakov and Umirbaev themselves used this degree estimate to solve a well-known problem due to Nagata by showing that a particular automorphism of $K[x, y, z]$ (suggested by Nagata) is not tame [5]. It has turned out to be useful in other situations as well; in particular, this degree estimate was used in [2] in proving that two *stably equivalent* polynomials in $K[x, y]$ are necessarily equivalent. It appears, however, that there is no reasonable way to generalize Shestakov-Umirbaev's degree estimate to subalgebras generated by more than two polynomials; this is explained in [2].

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Locally nilpotent derivations and Hilbert's 14th Problem

Daniel Daigle

In Hilbert's famous list of 23 mathematical problems, the 14th item is:

H14. *Let $\mathbf{k}$ be a field and $R = \mathbf{k}^{[n]}$. If K is a field such that $\mathbf{k} \subseteq K \subseteq \operatorname{Frac}(R)$, is $K \cap R$ finitely generated as a $\mathbf{k}$-algebra?*

Here, $\mathbf{k}^{[n]}$ denotes a polynomial ring in n variables over $\mathbf{k}$ and $\operatorname{Frac}(R)$ is the field of fractions of R. The talk is a survey of this problem, with a particular viewpoint. We begin by stating five special cases of H14 and explaining the relations between them.

The first three special cases are concerned with group actions and invariants. Suppose that $\mathbf{k}$ is an algebraically closed field and $R = \mathbf{k}^{[n]}$.

H14-GA. *If G is an algebraic group acting algebraically on $\mathbb{A}^n = \operatorname{Spec} R$, is the ring of invariants R^G finitely generated as a $\mathbf{k}$-algebra?*

Here it should be noted that $\mathrm{Frac}(R^G) \cap R = R^G$, so H14-GA is indeed a special case of H14. We state two special cases of H14-GA. The first one is:

H14-LinGA. *If G is an algebraic group acting algebraically on $\mathbb{A}^n = \operatorname{Spec} R$* **by linear automorphisms**, *is R^G finitely generated as a $\mathbf{k}$-algebra?*

This problem was Hilbert's motivation for proposing H14 and, for that reason, is sometimes referred to as the *Original 14th Problem.* The second special case of H14-GA which we consider is:

H14-ConnGA. *If G is a* **connected** *algebraic group acting algebraically on $\mathbb{A}^n = \operatorname{Spec} R$, is R^G finitely generated as a $\mathbf{k}$-algebra?*

However we note that H14-ConnGA and H14-GA are equivalent: if G_0 is the connected component of G containing the identity element then one can show that

$$R^G \text{ is finitely generated over } \mathbf{k} \Leftrightarrow R^{G_0} \text{ is finitely generated over } \mathbf{k}.$$

We consider two other special cases of H14, related to derivations and their rings of constants (or kernels). Let $R = \mathbf{k}^{[n]}$ where $\mathbf{k}$ is a field of characteristic zero.

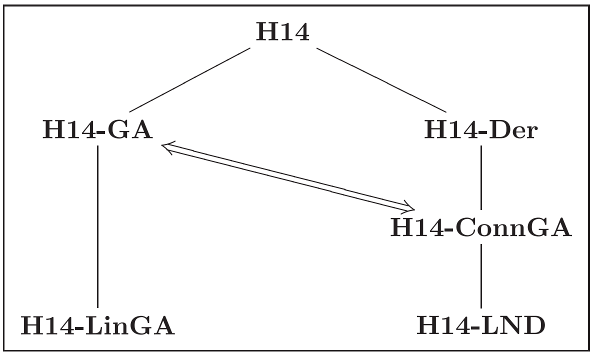
H14-Der. *If $D : R \to R$ is a $\mathbf{k}$-derivation, is $\ker D$ finitely generated as a $\mathbf{k}$-algebra?*

Here, $\ker D = \{f \in R \mid D(f) = 0\}$. As $\mathrm{Frac}(\ker D) \cap R = \ker D$, H14-Der is a special case of H14. An even more special case is:

H14-LND. *If $D : R \to R$ is a* **locally nilpotent** *derivation, is $\ker D$ finitely generated as a $\mathbf{k}$-algebra?*

Here, one should recall that a derivation $D : R \to R$ is *locally nilpotent* if for each $f \in R$ there exists $N > 0$ such that $D^N(f) = 0$. It is well-known that each locally nilpotent derivation $D : R \to R$ determines an algebraic action of $G_a = (\mathbf{k}, +)$ on $\mathbb{A}^n$ satisfying $R^{G_a} = \ker D$, and that all G_a-actions on $\mathbb{A}^n$ are obtained in this way. Thus H14-LND can also be viewed as the special case $G = G_a$ of H14-ConnGA.

Moreover, H14-ConnGA is a special case of H14-Der. Indeed, it follows from Nowicki [7] that if G is a connected algebraic group acting algebraically on $\mathbb{A}^n = \operatorname{Spec} R$, then R^G is the kernel of some $\mathbf{k}$-derivation of R. So we have the following hierarchy of special cases of H14:



Next we describe the current status of H14, H14-Der and H14-LND. Unfortunately there is no time to discuss the cases related to group actions.

Zariski's Theorem (cf. [8]). *Let A be a normal affine domain over a field $\mathbf{k}$ and let K be a field such that $\mathbf{k} \subseteq K \subseteq \operatorname{Frac} A$. If $\operatorname{trdeg}(K/\mathbf{k}) \leq 2$, then $K \cap A$ is finitely generated as a $\mathbf{k}$-algebra.*

The case $\operatorname{trdeg}(K/\mathbf{k}) = 3$ of H14 remained completely open until Kuroda found several counterexamples in 2005–2006. In particular [6] implies: *Let $\mathbf{k}$ be a field of characteristic zero and $e \geq 3$ an integer. Then there exists a field K such that $\mathbf{k} \subset K \subset \mathbf{k}(X, Y, Z)$, $[\mathbf{k}(X, Y, Z) : K] = e$ and $K \cap \mathbf{k}[X, Y, Z]$ is not finitely generated.* Then it easily follows:

Corollary. *Let $\mathbf{k}$ be a field of characteristic zero, $3 \leq d \leq n$ integers, $R = \mathbf{k}^{[n]}$. Then there exists a field K such that $\mathbf{k} \subset K \subset \operatorname{Frac}(R)$, $\operatorname{trdeg}(K/\mathbf{k}) = d$ and $K \cap R$ is not finitely generated as a $\mathbf{k}$-algebra.*

This and Zariski's Theorem settle H14, in the sense that for each pair (n, d) of integers we know whether or not there exists a counterexample to H14 with $R = \mathbf{k}^{[n]}$ and $\operatorname{trdeg}(K/\mathbf{k}) = d$.

Next, consider the status of H14-Der. Let $R = \mathbf{k}^{[n]}$ where $\operatorname{char} \mathbf{k} = 0$. ¿From Zariski's Theorem and the elementary fact that the kernel of a derivation is algebraically closed, one immediately obtains:

Corollary. *If $R = \mathbf{k}^{[n]}$ with $n \leq 3$, then the kernel of any $\mathbf{k}$-derivation $D : R \to R$ is a finitely generated $\mathbf{k}$-algebra.*

In 2005, Kuroda [5] showed that there exists a $\mathbf{k}$-derivation of $\mathbf{k}^{[4]}$ whose kernel is not finitely generated. So H14-Der is settled. (Note however that Kuroda's derivation is not locally nilpotent.)

The status of H14-LND can be summarized by saying that the problem

(i) has a positive answer when $n \leq 3$,
(ii) has a negative answer when $n \geq 5$,
(iii) is open when $n = 4$, but there are partial results.

For (i), see the last corollary, above. To explain (ii) and (iii), we need the following notions. Let $R = \mathbf{k}[X_1, \dots, X_n] = \mathbf{k}^{[n]}$ and recall that a $\mathbf{k}$-derivation $D : R \to R$ is *triangular* if $DX_i \in \mathbf{k}[X_1, \dots, X_{i-1}]$ holds for all i. It is easy to see that

> *If $D : R \to R$ is triangular then it is locally nilpotent and moreover* $\ker D$ *contains a variable of R,*

where by a *variable* of R we mean an element $f \in R$ for which there exist $f_2, \dots, f_n$ satisfying $R = \mathbf{k}[f, f_2, \dots, f_n]$. Freudenburg and this author gave the following example in [2]: *Let $R = \mathbf{k}[a, b, x, y, z] = \mathbf{k}^{[5]}$ and define a $\mathbf{k}$-derivation $D : R \to R$ by*

$$D = a^2 \frac{\partial}{\partial x} + (ax + b)\frac{\partial}{\partial y} + y \frac{\partial}{\partial z}.$$

Then $\ker D$ *is not finitely generated as a* $\mathbf{k}$*-algebra.* In this example D is triangular, hence locally nilpotent, which explains (ii). Regarding (iii), Freudenburg and this author gave the following results in [4] and [3]:

(1) *The kernel of any triangular derivation of* $\mathbf{k}^{[4]}$ *is finitely generated.*
(2) *Given $m \in \mathbb{N}$, there exists a triangular derivation of* $\mathbf{k}^{[4]}$ *whose kernel cannot be generated by fewer than m elements.*

In unpublished work, Bhatwadekar improved statement (1) as follows: *Let D be a locally nilpotent derivation of* $\mathbf{k}^{[4]}$ *whose kernel contains a variable. Then* $\ker D$ *is finitely generated.* It is still not known whether there exists a locally nilpotent derivation of $\mathbf{k}^{[4]}$ whose kernel is not finitely generated.

The talk concluded with a result from [1] pertaining to the problem of classifying the locally nilpotent derivations of $\mathbf{k}^{[3]}$.

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Questions on Plane Polynomial Curves

Avinash Sathaye

Assume k to be a ground field of characteristic zero. By a plane polynomial curve we mean a plane curve parametrized by polynomials $P(t), Q(t)$ in one variable t over k. They will be assumed monic for convenience. The resulting coordinate ring $k[P(t), Q(t)]$ will be denoted by A. The set of t-degrees of non zero members of A will be denoted by $\Gamma(A)$ and called the degree semigroup of the curve $C: x = P(t), Y = Q(t)$.

Most of the results below are either published in my joint work with Jon Stenerson [1] or sketched here. References not explicitely mentioned here are located in the same paper. It is well known that $\Gamma(A)$ is planar, which means it is generated by a sequence of positive integers $\delta_0, \cdots, \delta_h$ satisfying three defining properties:

(1) Set $d_i = \mathrm{GCD}(\delta_0, \cdots, \delta_{i-1})$ for $i = 1, \cdots, h+1$. Set $n_i = d_i/d_{i-1}$ for $i = 1, \cdots, h$. Then $d_{h+1} = 1$ and $n_i > 1$ for $2 \leq i \leq h$.
(2) $\delta_i n_i$ is in the semigroup generated by $\delta_0, \cdots, \delta_{i-1}$.
(3) Set $q_i = \delta_{i-1} n_{i-1} - \delta_i$. Then $q_i > 0$ for all $i \geq 2$.

Abhyankar-Moh theory developed for the Epimorphism Theorem established these properties for the degree semigroups of any plane curve with one place at infinity. (The degree of a non zero element in the coordinate ring of such a "one place curve" is defined as the negative of its order at the place at infinity.) My old proof of the converse is given in [1].

Let us call a planar semigroup "polynomial" if it arises from a plane polynomial curve. I had conjectured that the semigroup with δ generator sequence $6, 22, 17$ is not polynomial. This was recently established by three Japanese mathematicians M. Fujimoto, M. Suzuki and K. Yokoyama.[2], using extensive computer calculations. They also found a smaller example that I had missed, namely, $6, 21, 4$ and established it also as non polynomial.

While this is satisfactory, we need a more conceptual proof (rather than this "Simon says" proof by computer). I outlined a procedure for this new example $6, 21, 4$ by establishing two reductions:

Let P, Q, R denote the polynomials in $k[t]$ such that $\deg_t(P) = 6, \deg_t(Q) = 21$ and $\deg_t(R) = 4$. What we need to arrange is that for some polynomial $\phi(X, Y)$ we get $\phi(P, Q) = R$. Moreover, we may assume that $\phi(P, Q)$ is simply of the form $Q^2 - (P^7 +$ lower degree monomials in P. First, we show that P, R are both polynomial expressions in a quadratic polynomial, which we may assume to be t^2 without loss of generality. Thus $P = P_1(t^2), R = R_1(t^2)$, where $P_1(X), Q_1(X)$ have degrees $3, 2$. Then it can be further argued that the polynomial Q must be of the form $tH(t^2)$, for some $H(X) \in k[X]$ of degree 10. We simply how to arrange that $t^2 H(t^2)^2 - G(P(t^2))$ has degree at most 4. This can be rewritten as $XQ_1(X)^2 = G(P_1(X)) + aX^2 + bX + c$ where $0 \neq a, b, c \in k$. As a consequence, the calculations become much simpler and humanly doable. This reduction process due to the composite nature of the ring generated by the polynomials of degree

4, 6 might prove to be a useful ingradient for further determination of polynomial planar semigroups.
A basic question which naturally arises in the semigroup constructions is to estimate the minimum possible degree of $P^m - Q^n$ when the expression is not zero. Naturally, this is of interest when at least the top degrees cancel, i.e. the numbers n, m are proportional to the degrees of monic polynomials P, Q respectively. In our paper [1] we had formulated and established the result that the drop in the degree is at most $k - 1$ where k is the total number of distinct roots of P, Q in the algebraic closure of k. We had also established the optimality of this lower bound when the degrees of P, Q are mutually coprime.
Indeed, this question has been of intense interest in Number Theory, especially for the case $m = 3, n = 2$. Our main theorem was already proved by Davenport (1965) and Mason (1984). The question of establishing the optimality of the estimate is recently carried out for the same case $m = 2, n = 3$ by Stothers (1991) and Zannier (1995-96). These already involve deep results in Algebraic Geometry. However, the active work only appears in journals in Number Theory (See Acta Arithmatica for Zannier's work). Perhaps, the algebraists need to find a simple algebraic proof of such a simple algebraic statement!

I presented one recent application of this estimate to a questions raised by Peter Russell. He was interested in determining pairs of embedded lines in the plane which intersect in exactly two points. It is easy to presume that one line is simply given by $y = 0$ and the other is given by parametrization $x = P(t), y = Q(t)$ where P, Q are monic of degrees n, m respectively and $n < m$. Thus, by the Embedded Line Theorem of Abhyankar-Moh-Suzuki, there exists a polynomial $H(X)$ of degree $d = m/n$ such that $Q - H(P)$ becomes a polynomial whose t-degree divides n. Using this fact, we can even assume that P is an approximate d-th root of Q. Then the estimate gives $\deg_t(Q-P^d) \geq m-(n+2)+1 = m-n-1$. It is also less than $m - n$ by the approximate root assumption. Thus, we deduce that it is exactly $m-n-1$. Since this is not in the semigroup generated by m, n, we deduce that it must divide n. Simple calculations lead to the conclusion that either we have the trivial case of $n = 1$ or we have an embedding with $n = 2, m = 4$.

In general, we can deduce a result like $\deg_t(P) \leq r$ where r is the number of distinct roots of Q in the algebraic closure.

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Commuting derivations on UFDs

Stefan Maubach
(joint work with H. Derksen and A. van den Essen)

Consider the following conjecture:

Commuting derivations conjecture (CD(n)): If $D_1, \ldots, D_{n-1}$ are $n-1$ commuting derivations on $\mathbb{C}^{[n]}$ which are linearly independent over $\mathbb{C}^{[n]}$, then the intersection of their kernel $(\mathbb{C}^{[n]})^{D_1,\ldots,D_n} = \mathbb{C}[f]$ where f is a coordinate.

CD(3) is true (see [2]), and has some nice consequences, but there is no recent hope of any progress for proving CD(n) if $n \geq 4$.

In this talk, I will prove the following theorem:

Theorem: Let A be a UFD over $\mathbb{C}$ with $\operatorname{trdeg}_{\mathbb{C}} Q(A) = n+1(\geq 1)$, $A^* = \mathbb{C}^*$, and let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations (linearly independent over A). Now $A^{D_1,\ldots,D_n} = \mathbb{C}[f]$ for some $f \in A \backslash \mathbb{C}$, and

(1) If $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ are independent over $A/(f-\alpha)$, then $A/(f-\alpha) \cong \mathbb{C}^{[n]}$. There are only finitely many $\alpha \in \mathbb{C}$ for which $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ are dependent over $A/(f-\alpha)$.

(2) In the case that $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ are independent over $A/(f-\alpha)$ for each $\alpha \in \mathbb{C}$, then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n+1$ variables.

I will elaborate on some of the lemma's which one uses for this theorem (which are interesting in their own right) and discuss some questions arising from these lemma's and the theorem. As a remark, it is possible to generalize the theorem, which essentially is a theorem about $(G_a)^n$-actions, to free unipotent actions, but the techniques are different.

The work is joint with H. Derksen and A. van den Essen , and the preprint [1] can be found on my website.

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Andersén-Lempert-theory for automorphisms of affine space - generalizations and applications

Frank Kutzschebauch
(joint work with Shulim Kaliman)

1. Introduction

The main theorem of Andersén-Lempert-theory (due to Forstneric and Rosay) allows to construct holomorphic automorphisms of $\mathbb{C}^n$ with prescribed behavior on compact polynomially convex subsets in the following sense:

Theorem 1. *Let Ω be an open set in $\mathbb{C}^n$ $(n \geq 2)$ and let $\Phi : [0,1] \times \Omega$ be a map of class $\mathcal{C}^2$ such that for every $t \in [0,1]$ the map $\Phi_t : \Omega \to \mathbb{C}^n$ is injective and holomorphic. Assume that each domain $\Phi_t(\Omega)$ is Runge in $\mathbb{C}^n$ and does not intersect X. If Φ_0 can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing X, then for every $t \in [0,1]$ the map Φ_t can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing X.*

Striking applications of this theorem are:

(A) For all $0 < k < n$ there are proper holomorphic embeddings of $\varphi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ so that for no holomorphic automorphisms α of $\mathbb{C}^n$ the embedding $\alpha \circ \varphi$ is standard. (Forstneric, Globevnik, Rosay, Rudin) Observe the difference to the algebraic situation, where we have

Conjecture 2 (Abhyankar-Sathaye)**.** *Every polynomial embedding $\phi : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ is rectifiable.*

(B) These embeddings lead to non-linearizable holomorphic group actions on affine spaces for all compact Lie groups. (Derksen, Kutzschebauch)

(C) Recall that every Stein manifold of dimension $n > 1$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N = \left[\frac{3n}{2}\right] + 1$, and this N is the smallest possible by the examples of Forster. The corresponding embedding theorem with N replaced by $N' = \left[\frac{3n+1}{2}\right] + 1$ was proved by Eliashberg and Gromov. For even values of $n \in \mathbb{N}$ we have $N = N'$ and hence their result is the best possible; for odd values of n the optimal result was obtained by Schürmann, also for Stein spaces with bounded embedding dimension. Using A-L theory one can prove the embedding theorem with interpolation on discrete sequences (Forstneric, Prezelj, Ivarsson, Kutzschebauch)

Theorem 3. *Let X be a Stein manifold of dimension $n > 1$, and let $\{a_j\}_{j\in\mathbb{N}} \subset X$ and $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^m$ be discrete sequences without repetitions. If $m \geq N = \left[\frac{3n}{2}\right] + 1$ then there exists a proper holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^m$ satisfying*

$$f(a_j) = b_j \qquad (j = 1, 2, \ldots). \tag{1.1}$$

(D) The embedding problem for open Riemann surfaces into $\mathbb{C}^2$ got recent progress by work of Fornæss Wold:

Theorem 4. *Any finitely connected subset of $\mathbb{C}$ and any finitely connected subset of any torus (no punctures) can be properly holomorphically embedded into $\mathbb{C}^2$*

(E) the same holds with interpolation on a discrete subset (Fornæss Wold, Løw, Kutzschebauch)

(F) Some open problems in complex dynamics were solved by Fornæss Wold and Peters, the most striking is:

There is a Fatou-Bieberbach domain in $\mathbb{C}^2$ such that the Hausdorff measure of its boundary near every point is equal to 4.

2. Generalizations

The main algebraic result behind the main theorem of Andersén-Lempert-theory is the following observation:

Each polynomial vector field on $\mathbb{C}^n$ $(n \geq 2)$ is a finite sum of completely integrable polynomial vector fields

where a holomorphic vector field on a complex manifold is completely (or globally) integrable if its phase flow generates a holomorphic $\mathbb{C}_+$-action on this manifold.

The first generalization of the Andersén-Lempert theory was made by Varolin who extended it from Euclidean spaces to a wider class of algebraic complex manifolds. He realized also that instead of presenting algebraic vector fields as a finite sum of integrable algebraic fields one can use Lie combinations of those fields. This leads to the following.

Definition 5. A complex manifold X has the density property if in the compact-open topology the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by globally integrable holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X. An affine algebraic manifold has the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by globally integrable algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it (clearly the algebraic density property implies the density property).

Varolin and Toth established the density property for some manifolds including semisimple complex Lie groups and some homogenous spaces of semisimple Lie groups. They made a conjecture related to our **main motivation How to detect $\mathbb{C}^n$ among affine algebraic varieties / Stein manifolds)?**

Conjecture 6 (Varolin-Toth). *If X is a Stein manifold with the density property and X is diffeomorphic to $\mathbb{R}^{2n}$, then X is biholomorphic to $\mathbb{C}^n$.*

In this connection we would also like to mention the following

Question. (Zaidenberg) *Is there a complex affine algebraic variety biholomorphic to $\mathbb{C}^n$ but not isomorphic to $\mathbb{C}^n$?*

Important consequences of the density property are

(1) [Varolin] (Fatou-Bieberbach maps of the first kind) For each point $x \in X_p$ there is an injective but not surjective holomorphic map $f : \mathbb{C}^{n+1} \to X_p$ with $f(0) = x$. In particular all Eisenman measures on X_p vanish identically.

(2) [Varolin] (Fatou-Bieberbach maps of the second kind) For each point $x \in X_p$ there is an injective but not surjective holomorphic map $f : X_p \to X_p$ with $f(x) = x$.

(3) [Varolin] The holomorphic automorphism group of X acts k-transitively on X for all $k \in \mathbf{N}$.

(4) [Kaliman, Kutzschebauch] Density implies Gromov's spray

Theorem 7. *Any Stein manifold X with the density property admits a spray.*

3. New Examples

The authors gave two classes of new examples of manifolds with the density property described in the following two theorems

Theorem 8. *Let G be a linear algebraic group whose connected component is different from a torus or $\mathbb{C}_+$. Then G has the algebraic density property.*

Theorem 9. *Let $p \in \mathbb{C}[x_1, x_2, ..., x_n]$ be a polynomial (a holomorphic function) with smooth reduced zero fibre, i.e., the partial derivatives $p_i = \partial p/\partial x_i$ of p have no common zeros on the zero fiber of p. Then the hypersurface*

$$X_p := \{(\bar{x}, u, v) \in \mathbb{C}^{n+2} : uv = p(\bar{x}), \quad \bar{x} = (x_1, x_2, \ldots, x_n)\}$$

has the algebraic density property (density property).

Extremely interesting are the examples:

$$H_1 = \{P_1(x, y, z, u, v) = uv - [(xz + 1)^3 - (yz + 1)^2 - z]/z = 0\} \subset \mathbb{C}^5$$

and

$$H_2 = \{P_2(x, y, z, t, u, v) = uv - (x + x^2 y + z^2 + t^3) = 0\} \subset \mathbb{C}^6.$$

Remark. The hypersurfaces H_1 and H_2 are either counterexamples to one of the conjectures of Abhyankar-Sathaye resp. Varolin-Toth or they give a positive answer to Zaidenbergs question.

4. Holomorphic automorphisms with control on non-compact sets

The following theorem of the authors solves a problem posed by Forstneric:

Theorem 10. *Let X be a closed algebraic subset of $\mathbb{C}^n$ of codimension at least 2 such that the Zariski tangent space $T_x X$ has dimension at most $n - 1$ for any point $x \in X$. Then the geometric structure of vector fields vanishing on X has the algebraic density property.*

It implies the following generalization of the main theorem of Andersén-Lempert theory, giving control on compact sets and at the same time on algebraic subvarieties of codimension at least 2.

Theorem 11. *Let X be an algebraic subvariety of $\mathbb{C}^n$ of codimension at least 2 and Ω be an open set in $\mathbb{C}^n$ $(n \geq 2)$. Let $\Phi : [0,1] \times \Omega$ be a map of class $\mathcal{C}^2$ such that for every $t \in [0,1]$ the map $\Phi_t : \Omega \to \mathbb{C}^n$ is injective and holomorphic. Assume that each domain $\Phi_t(\Omega)$ is Runge in $\mathbb{C}^n$ and does not intersect X. If Φ_0 can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing X, then for every $t \in [0,1]$ the map Φ_t can be approximated on Ω by holomorphic automorphisms of $\mathbb{C}^n$ fixing X.*

As an application we have a stronger version of a result of Buzzard and Hubbard answering a question by Siu.

Corollary 12. *Any point z in the complement of an algebraic subset X of $\mathbb{C}^n$ of codimension at least 2 has a neighborhood U in $\mathbb{C}^n \setminus X$ which is biholomorphic to $\mathbb{C}^n$ (such U is called a Fatou-Bieberbach domain).*

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Finite subgroups of the plane Cremona group

Igor V. Dolgachev
(joint work with Vassily A. Iskovskikh)

The Cremona group $\mathrm{Cr}_k(n)$ over a field k is the group of birational automorphisms of the projective space $\mathbb{P}^n_k$, or, equivalently, the group of k-automorphisms of the field $k(x_1, x_2, \ldots, x_n)$ of rational functions in n independent variables. The group $\mathrm{Cr}_k(1)$ is the group of automorphisms of the projective line, and hence it is isomorphic to the projective linear group $\mathrm{PGL}_k(2)$. Already in the case $n = 2$ the group $\mathrm{Cr}_k(2)$ is not well understood in spite of extensive classical literature (e.g. [2], [3]) on the subject as well as some modern research and expositions of classical results (e.g. [1]). Very little is known about the Cremona groups in higher-dimensional spaces.

In the talk I discuss the classical problem of classification of finite subgroups of the plane Cremona group $\mathrm{Cr}(2)$ over the field of complex numbers. The classification of finite subgroups of $\mathrm{PGL}_{\mathbb{C}}(2)$ is well-known and goes back to F. Klein. It consists of cyclic dihedral, tetrahedral, octahedral and icosahedral groups. Groups of the same type and order constitute a unique conjugacy class in $\mathrm{PGL}_{\mathbb{C}}(2)$. The goal is to find a similar classification in the two-dimensional case.

The story of this problem begins in 1894. G. Castelnuovo, as an application of his theory of adjoint linear systems, proved that any element of finite order in $\mathrm{Cr}(2)$ leaves invariant either a net of lines, or a pencil of lines, or a linear system of cubic curves with $n \leq 8$ base points. A similar result was claimed earlier by S. Kantor in his memoir which was awarded a prize by the Accademia delle Scienze di Napoli in 1883. However his arguments, as was pointed out by Castelnuovo, required justifications. Kantor went much further and announced a similar theorem for arbitrary finite subgroups of $\mathrm{Cr}(2)$ [5]. He proceeded to classify possible groups in each case (projective groups, groups of de Jonquiérs type, and groups of type M_n). A much clearer exposition of his results can be found in a paper of A. Wiman [6]. Unfortunately, Kantor's classification, even with some correction made by Wiman, is incomplete for two main reasons. First, only maximal groups were considered and even some of them were missed. Second, although Kantor was aware of the problem of conjugacy of subgroups, he did not attempt to fully investigate this problem.

The goal of our work with V. Iskovskikh work is to complete Kantor's classification. We use a modern approach to the problem initiated in the works of Yu. Manin and V. Iskovskikh, a survey of their results can be found in [4]. It makes a clear understanding of the conjugacy problem via the concept of a rational G-surface. This is meant to be a pair (S, G) which consists of a nonsingular projective rational surface S over a field k and a finite group G acting on it by biregular automorphisms (the *geometric case*) or, acting on $S \otimes_k K \cong \mathbb{P}^2_K$ by the Galois action (the *arithmetic case*). A birational G-equivariant k-map $S - \to \mathbb{P}^2_k$ realizes G as a finite subgroup of $\mathrm{Cr}_k(2)$ (geometric case) or a finite subgroup of $\mathrm{Cr}_K(2)$ (arithmetic case). In the geometric case, two birational isomorphic G-surfaces define conjugate subgroups of $\mathrm{Cr}_k(2)$, and conversely a conjugacy class of a finite subgroup G of $\mathrm{Cr}_k(2)$ can be realized as a birational isomorphism class of G-surfaces. In this way classification of conjugacy classes of subgroups of $\mathrm{Cr}_k(2)$ becomes equivalent to the birational classification of G-surfaces. A G-equivariant analog of the theory of minimal models of surfaces allows one to concentrate on the study of minimal G-surfaces, i.e. surfaces which cannot be G-equivariantly birationally and regularly mapped to another G-surface. Minimal G-surfaces turn out to be G-isomorphic either to $\mathbb{P}^2_k$, or a conic bundle, or Del Pezzo surface of degree $d = 9 - n \leq 6$. This leads to groups of projective transformations, or groups of de Jonquiéres type, or groups of type M_n, respectively. To complete the classification one requires

- to classify all finite groups G which may occur in a minimal G-pair (S, G);
- to determine when two minimal G-surfaces are birationally isomorphic.

To solve the first part of the problem one has to compute the full automorphism group of a conic bundle surface or a Del Pezzo surface (in the latter case this was essentially accomplished by Kantor and Wiman), then make a list of all finite subgroups which act minimally on the surface (this did not come up in the works of Kantor and Wiman). The second part is less straightforward. For this we use the ideas from Mori's theory to decompose a birational map of rational G-surfaces into elementary links. This theory was successfully applied in the arithmetic case (see [4]) and we borrow these results with obvious modifications adjusted to the geometric case. Here we use the analogy between k-rational points in the arithmetic case (fixed points of the Galois action) and fixed points of the G-action. As an important implication of the classification of elementary G-links is the rigidity property of groups of type M_n with $n \geq 6$: birationally isomorphic minimal Del Pezzo surface (S, G) of degree $d \leq 3$ are biregularly isomorphic. This saves much of the painful analysis of possible conjugacy for a lot of groups.

The large amount of group-theoretical computations needed for the classification of finite subgroups of groups of automorphisms of conic bundles and Del Pezzo surfaces makes us to be aware of some possible gaps in our final classification. This seems to be a destiny of enormous classification problems. We hope that our hard work will be useful for the future faultless classification of conjugacy classes of Cr(2).

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PROBLEMSESSION

Shreeram S. Abhyankar

Let $f = X + X^2Y$ and k be a field of char $k = 0$.

Show that there does not exist g, h in $k[X, Y, Z]$ such that the Jacobian of f, g, h relative to X, Y, Z equals a non zero constant.

Tatiana Bandmann

Two polynomials $p(x,y), q(x,y) \in \mathbb{Z}[x,y]$ define the map $f : \mathbb{A}^2(\mathbb{F}) \to \mathbb{A}^2(\mathbb{F})$, where

$$f(x,y) = (p(x,y), q(x,y))$$

and the field $\mathbb{F}$ may be $\mathbb{C}$, $\mathbb{Q}$ or $\mathbb{F}_q$ for prime q. We assume that $p(0,0) = q(0,0) = 0$. We are interested in the following

Property $P(\mathbb{F})$: There exists a periodic for f point $a_F \in \mathbb{A}^2(\mathbb{F})$ such that $a_F \neq (0,0)$.

Question 1
Assume that f is dominant and has **Property** $P(\mathbb{F}_q)$ for all q. Does it have a periodic point rational over $\mathbb{Q}$?

Question 2
Find all the pairs of polynomials $p, q \in \mathbb{Z}[x,y]$ such that the corresponding map has **Property** $P(\mathbb{F}_q)$ for all q.

Example

$$p(x,y) = -x(y-2)(x^2 + y^2 - 2y)$$

$$q(x,y) = x^2(y-2)(x^2 + y^2 - 2y)$$

Properties:
1. $f(0,0) = (0,0)$.
2. The orbit $\{f^{(n)}(x,y)\}n \in \mathbb{N}$ is Zariski dense for a generic point (x,y).
3. For any prime q the map has periodic point rational over $\mathbb{F}_q$.

Alexei Belov-Kanel

Question 1. Is the authomorphism group of the Weil-algebra W_n over $\mathbb{C}$ isomorphic to the group of polynomial symplectomorphisms of $\mathbb{C}^{2n}$?

Since the situation is unclear in view to the Jacobian Conjecture, one can ask a similar question for semigroups of endomorphisms and polynomial symplectomorphisms. (The problem is due to M.Kontsevich).

Tame authomorphism groups are isomorphical. The isomorphism of the authomorphism groups in the case $n = 1$ was established by L.Makar-Limanov.

The previous problem can be deduced from the following (hypothetical) fact: Let M be a holonomic W_n module. One can define its reduction modulo suffitiently large prime, in particular, infinitely large.

After reduction we shall have the annihilator in the center. This annihilator defines an algebraic variety (a subvariety of the spectrum $Z(W_n)$).

Question 2. a) Is it Lagrangian?
b) Can any Lagrangian variety be obtained in such a way?

Note that the graph of the symplectomorphism is Lagrangian and the corresponding holonomical module gives us a corresponding endomorphism of the Weil algebra. We can prove that only for W_n if $n = 1$. (This problem was also formulated by M.Kontsevich.)

The construction of homomorphism between authomorphisms groups is based on infinitely large primes. If we go back and forward using different ultrafilters, we get elements of $Aut(Aut(W_n))$ and $Aut(Sumpl)$.

Hence it is interesting to describe these groups. Let $n > 1$.

Question 3. Is it true that these groups are generated by conjugations and authomorphisms of the ground field?

A similar question can be asked for subgroups $Aut(Aut(W_n))$ and $Aut(Sumpl)$ – authomorphisms of $Aut(W_n)$ and $Sumpl(C^n)$ as *ind*-schemes.

Also it is interesting to consider $Aut(Aut(k[x_1, \dots, x_n])$ in set theoretical sense and as *ind*-schemes.

Daniel Daigle

Define a sequence $\{H_n\}_{n=0}^{\infty}$ of homogeneous polynomials in $\mathbb{C}[X, Y, Z]$ by $H_0 = Y$, $H_1 = X$ and for $n \geq 1$:

$$H_{n+1} = \frac{r^{a_n} + H_n^3}{H_{n-1}} \quad \text{where } r = X^3 + XYZ - Y^3 \text{ and } a_n = \deg H_n.$$

This sequence appeared in Section 4.2 of Freudenburg's paper [3] with the same notation, and the proof that the H_n are indeed polynomials can be found there. Note that $H_2 = XZ - Y^2$, $H_3 = X^5 + 2X^3YZ - 2X^2Y^3 + X^2Z^3 - 2XY^2Z^2 + Y^4Z$ and $\{\deg H_n\}_{n=1}^{\infty} = \{1, 2, 5, 13, 34, 89, \dots\}$ (every other term in the Fibonacci sequence).

For each $n \in \mathbf{N}$, let U_n (resp. V_n) be the hypersurface in $\mathbb{A}^3$ defined by the equation $H_n = 1$ (resp. $XY = Z^{a_n} + 1$). I claim that (a) for each $n \in \mathbf{N}$ there is an isomorphism $U_n \cong V_n$ of algebraic surfaces, and that (b) if $n \geq 3$ then no algebraic automorphism of $\mathbb{A}^3$ maps one hypersurface onto the other; in other words, U_n is (when $n \geq 3$) a **nonstandard** embedding of the Danielewski surface $XY = Z^{a_n} + 1$. Because I tend to believe that these embeddings are nonstandard in a very strong sense, I find it interesting to ask:

> **Question:** *When $n \geq 3$, does there exist a holomorphic automorphism of $\mathbb{C}^3$ which maps U_n onto V_n? What about real diffeomorphisms, or even homeomorphisms?*

Proof of claims (a) *and* (b). One can prove that the $\mathbb{C}$-algebra $\mathbb{C}[X,Y,Z]/(H_n - 1)$ is generated by the three elements $\alpha = \pi(H_{n-1})$, $\beta = \pi(H_{n+1})$ and $\gamma = \pi(r)$, where

$$\pi : \mathbb{C}[X,Y,Z] \to \mathbb{C}[X,Y,Z]/(H_n - 1)$$

is the canonical epimorphism. As $\alpha\beta = \gamma^{a_n} + 1$, this proves the claim (a) that the surfaces U_n and V_n are isomorphic. As to claim (b), one argues as follows: suppose that there exists an algebraic automorphism of $\mathbb{A}^3$ which maps U_n onto V_n. Then it is not hard to see that for every $\lambda \in \mathbb{C}$ (including $\lambda = 0$), the surface $H_n = \lambda$ is isomorphic to the surface $XY = Z^{a_n} + \mu$ for some $\mu \in \mathbb{C}$; in particular, the surface $H_n = \lambda$ is normal for every λ; since $H_n = 0$ is not normal when $n \geq 3$, (b) is proved. □

We now make some remarks which help to understand the context.

Ubiquity of $\{H_n\}_{n=1}^{\infty}$. This sequence was discovered and rediscovered by several authors. We have already mentioned Freudenburg's paper [3]; there it is shown that for each n, $\mathbb{C}[H_n, H_{n+1}]$ is the kernel of a locally nilpotent derivation of $\mathbb{C}[X,Y,Z]$. The sequence $\{H_n\}_{n=1}^{\infty}$ also appeared in unpublished work of Gizatullin, in relation with automorphisms of $\mathbb{C}[X,Y,Z]$. Now consider the curve $V(H_n) \subset \mathbb{P}^2$ whose equation is $H_n = 0$. Then $V(H_3)$ is Yoshihara's rational quintic [6]. More generally, all $V(H_n)$ are "Kashiwara curves": in [4], they correspond to the case where the divisor Γ is a linear chain. The $V(H_n)$ are also "Orevkov curves": in [5] Orevkov defines curves $C_j \subset \mathbb{P}^2$ where $j > 0$ is either odd or a multiple of 4, and uses these to show that a certain inequality (involving degree of a rational curve and highest multiplicity of a singular point) is best possible; the $V(H_n)$ correspond exactly to the C_j with j odd, i.e., to the Orevkov curves whose complements have logarithmic Kodaira dimension $-\infty$. In my joint work [2] with Peter Russell, the curves $V(H_n)$ appear in the *basic* affine rulings of $\mathbb{P}^2$.

One can also show that, up to automorphism of $\mathbb{P}^2$, the $V(H_n)$ are precisely the curves $C \subset \mathbb{P}^2$ whose complement $\mathbb{P}^2 \setminus C$ is completable by a rational zigzag, or equivalently, whose complement $\mathbb{P}^2 \setminus C$ has trivial Makar-Limanov invariant.

All this shows that $\{H_n\}_{n=1}^{\infty}$ is indeed a remarkable sequence of polynomials!

Finally I want to point out that the fact that the surface "$H_n = 1$" is isomorphic to a surface "$XY =$ nonconstant polynomial in Z" is a special case of a general phenomenon: whenever $f \in \mathbb{C}[X,Y,Z]$ is an irreducible polynomial which is annihilated by two locally nilpotent derivations $D_1 \neq 0 \neq D_2$ such that $\ker D_1 \neq \ker D_2$, the last theorem in my talk—which is the main theorem of [1]—implies that, for general $\lambda \in \mathbb{C}$, the surface $f = \lambda$ is isomorphic to a surface "$XY =$ nonconstant polynomial in Z".

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R. V. Gurjar

Let X be a smooth projective, rational surface/$\mathbf{C}$ and $C \subset X$ a smooth irreducible curve. Show that $\pi_1(X - C)$ is finite.

D.-Q. Zhang and I have verified this (Math. Annalen, **306** (1996), p. 15-30) when the log Kodaira dimension $\overline{\kappa}(X - C) \leq 1$.

If this question has an affirmative answer then we get a striking consequence.

Corollary. *Let X be a smooth projective, rational surface/$\mathbf{C}$ and let $f : X \to \mathbf{P^1}$ be a genus g fibration. Then f has atmost one multiple fiber.*

Recall that a scheme-theoretic fiber $\Sigma a_i C_i$ of f is called a multiple fiber if *gcd* $(a_1, a_2, \ldots) > 1$.

This Corollary has been directly verified by D.-Q. Zhang and myself when $g \leq 12$, except for $g = 7, 11$. It is also true for infinitely many higher values $g = 14, 17, 18, 20, \cdots$.

Shulim Kaliman

Let X be a smooth affine contractible threefold (over complex numbers) equipped with a nontrivial $\mathbb{C}_+$-action on it. Suppose that $\pi : X \to S = X//\mathbb{C}_+$ is the quotient morphism. It is known that S is a smooth contractible surface that contains a closed curve Γ for which $\pi^{-1}(S \setminus \Gamma)$ is isomorphic to $(S \setminus \Gamma) \times \mathbb{C}$ over $S \setminus \Gamma$. Furthermore, if Γ is chosen minimal possible then each of its irreducible components is a polynomial curve.

The problem is to classify all such curves Γ which may be a crucial step in classification of nontrivial $\mathbb{C}_+$-actions on $\mathbb{C}^3$.

More specific questions are: (1) is each irreducible component of Γ *contractible? smooth? Is* Γ *itself contractible (provided it is connected)?*

The motivation for such questions is the following: for $S = \mathbb{C}^2$ the only known examples of Γ are (a) the finite union of parallel lines (in a suitable coordinate system) or (b) a pair of lines meeting transversally at one point. Case (b) is quite nontrivial; it was first found by Freudenburg and then it was studied extensively by Daigle and Russell.

In the case when the Kodaira logarithmic dimension of S is 1 the answer to the problem is positive [1] and furthermore Γ coincides with the only line in S (discovered by Gurjar and Miyanishi) which is also the only polynomial curve in S.

For S of general type the existence polynomial curves in S is unknown but a theorem of Zaidenberg forbids the existence of contractible curves in S.

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Hanspeter Kraft and Peter Russell

In the joint paper [1] which is still "under construction" we prove the following result.

Proposition 1. *Let* k *be an algebraically closed field of infinite transcendence degree over the prime field. Let* $p\colon S \to X$ *and* $q\colon T \to X$ *be two morphisms where* S*,* T *and* X *are* k*-varieties. Assume that for all* $x \in X$ *the two (schematic) fibers* $S_x := p^{-1}(x)$ *and* $T_x := q^{-1}(x)$ *are isomorphic. Then there is a dominant étale morphism* $\phi\colon U \to X$ *and an isomorphism* $S \times_X U \cong T \times_X U$ *over* U*:*

$$\begin{array}{ccccccc} S & \longrightarrow & S \times_X U & \longrightarrow & T \times_X U & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{\ \phi\ } & U & = & U & \xrightarrow{\ \phi\ } & X \end{array}$$

Remark. Under the assumptions of the proposition assume in addition that an algebraic group G acts on S and T such that p and q are both equivariant with respect to the trivial action on X and that the isomorphisms proposition holds G-equivariantly, i.e., there is an étale morphism $U \to X$ and a G-equivariant isomorphism $S \times_X U \simeq T \times_X U$ over U.

In the proof we use in an essential way the assumption about the ground field, namely that k has infinite transcendence degree over its prime field. So our problem is the following:

Problem 2. Show that the proposition above holds over any algebraically closed field k, or give counter examples over $\overline{\mathbb{F}_p}$ or $\overline{\mathbb{Q}}$.

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Ignacio Luengo

In the theory of normal surface singularities and their topological/analytical invariants, one of the fundamental questions is what kind of analytic invariants of an analytic complex normal surface singularity can be determined from the topology (i.e. from the link) of the singularity. To have a chance to answer these type of questions, one has to assume two types of restrictions: a topological one – e.g. that the link is a rational homology sphere – and an analytic one – e.g. that the singularity is $\mathbb{P}$–Gorenstein. For such class of singularities several conjectures were presented and discussed in Oberwolfach 2003 Singularities workshop: the so called "Seiberg-Witten invariant conjecture"(of Nicolaescu and Némethi), the "Universal abelian cover conjecture" (of Neumann and Wahl) and the "Geometric genus conjecture" of Némethi.

In [6] we found counter-examples to these conjectures using hypersurface superisolated singularities. This class of singularities "contains" in a canonical way the theory of complex projective plane curves. They were introduced in by Luengo in order to show that the μ-constant stratum, in general, is not smooth.

The Seiberg-Witten Conjecture of Nicolaescu and Némethi is a generalization of the "Casson invariant conjecture" of Neumann and Wahl [10].

If the link of a normal surface singularity $(X, 0)$ is a rational homology sphere then the geometric genus p_g of $(X, 0)$ has an "optimal" topological upper bound. Namely,

$$p_g \leq \mathbf{sw}(M) - (K^2 + s)/8. \qquad \text{(SWC)}$$

Moreover, if $(X, 0)$ is a $\mathbb{Q}$-Gorenstein singularity then in (SWC) the equality holds.

Here, $\mathbf{sw}(M)$ is the Seiberg-Witten invariant of the link M of $(X, 0)$ associated with its canonical *spin*^{c} structure, K is the canonical cycle associated with a fixed resolution graph G of $(X, 0)$, and s is the number of vertices of G (see [9] for more details).

A hypersurface singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \cdots$ (where f_j is homogeneous of degree j) is called *superisolated* if the projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with isolated singularities $\{p_i\}_{i=1}^{\nu}$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$. The link of f is a rational homology sphere if the curve C is rational and cuspidal (i.e. if all the germs (C, p_i) are locally irreducible).

In [6] some superisolated singularities with $\nu = \#Sing(C) \geq 2$ which do not satisfy (SWC) were showed. Moreover, in all the counterexamples $p_g > \mathbf{sw}(M) - (K^2 + s)/8$ (contrary to the inequality predicted by the general conjecture !). On the other hand, even after an intense search of the existing cases, the authors were not able to find any counterexample with $\nu = 1$.

To understand the relationship between (SWC) and the pair $(\mathbb{P}^2, C)$, (C being the rational cuspidal curve which is the tangent cone of the corresponding superisolated surface singularity) leads the authors of [3, 4, 5] to the classification problem of the rational cuspidal projective plane curves. That is, to determine, for a given d, whether there exists a projective plane curve of degree d having a fixed number of unibranch singularities of given topological type. One of the integers which help in the classification problem is the logarithmic Kodaira dimension $\bar{\kappa}$ of open surface $\mathbb{P}^2 \setminus C$. The classification of curves with $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ has been recently finished by Miyanishi and Sugie [7], Tsunoda [14]. and Tono [12].

This remarkable problem of classification is not only important for its own sake, but it is also connected with crucial properties, problems and conjectures in the theory of open surfaces, and in the classical algebraic geometry:

• **Coolidge and Nagata problem**, see [1, 8]. It predicts that every rational cuspidal curve can be transformed by a Cremona transformation into a line, (it is verified in all known cases).

• **Orevkov's conjecture** [11] which formulates an inequality involving the degree d and numerical invariants of local singularities. In a different formulation, this is equivalent with the positivity of the virtual dimension of the space of curves with fixed degree and certain local type of singularities which can be geometrically realized.

• **Rigidity conjecture** of Flenner and Zaidenberg, [2]. Fix one of 'minimal logarithmic compactifications' (V, D) of $\mathbb{P}^2 \setminus C$, that is V is a smooth projective surface with a normal crossing divisor D, such that $\mathbb{P}^2 \setminus C = V \setminus D$, and (V, D) is minimal with these properties. The sheaf of the logarithmic tangent vectors $\Theta_V\langle D\rangle$ controls the deformation theory of the pair (V, D), The *rigidity conjecture* asserts that every $\mathbb{Q}$-acyclic affine surfaces $\mathbb{P}^2 \setminus C$ with logarithmic Kodaira dimension $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$ is rigid and has unobstructed deformations. That is,

$$h^1(\Theta_V\langle D\rangle) = 0 \qquad \text{and} \qquad h^2(\Theta_V\langle D\rangle) = 0.$$

In fact, the Euler characteristic $\chi(\Theta_V\langle D\rangle) = h^2(\Theta_V\langle D\rangle) - h^1(\Theta_V\langle D\rangle)$ must vanish because $h^0(\Theta_V\langle D\rangle) = 0$. Note that the open surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic if and only if C is a rational cuspidal curve.

The aim of the propose talk is to present some of these conjectures and related problems, and to complete them with some results and new conjectures from the recent work of the authors in [3].

1. Using results by Tono is proved that $\chi(\Theta_V\langle D\rangle) \geq 0$ and

Theorem 1 *Let C be an irreducible, cuspidal, rational projective plane curve with $\bar{\kappa}(\mathbb{P}^2 - C) = 2$. The following conditions are equivalent:*

(i) $\chi(\Theta_V\langle D\rangle) = 0$,

(ii) *Orevkov's conjecture is true.*
(iii) $\chi(\Theta_V\langle D\rangle) \leq 0$.

In such a case, the curve C can be transformed by a Cremona transformation of $\mathbb{P}^2$ into a straight line (i.e., the Coolidge-Nagata problem has a positive answer).

2. Author's 'compatibility property' is a sequence of inequalities, conjecturally satisfied by the degree and local invariants of the singularities of a rational cuspidal curve.

Consider a collection $(C,p_i)_{i=1}^{\nu}$ of locally irreducible plane curve singularities, let $\Delta_i(t)$ be the characteristic polynomial of the monodromy action associated with (C,p_i), and $\Delta(t) := \prod_i \Delta_i(t)$, with $\deg\Delta(t) = 2\sum\delta(C,p_i)$. Then $\Delta(t)$ can be written as $1+(t-1)\delta+(t-1)^2Q(t)$ for some polynomial $Q(t)$. Let c_l be the coefficient of $t^{(d-3-l)d}$ in $Q(t)$ for any $l=0,\ldots,d-3$.

Conjecture CP *Let $(C,p_i)_{i=1}^{\nu}$ be a collection of local plane curve singularities, all of them locally irreducible, such that $2\delta=(d-1)(d-2)$ for some integer d. If $(C,p_i)_{i=1}^{\nu}$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve then*

$$c_l \leq (l+1)(l+2)/2 \quad \textit{for all} \quad l=0,\ldots,d-3.$$

The main result of [3] is:

Theorem 2 *If $\bar\kappa(\mathbb{P}^2\setminus C)$ is ≤ 1, then Conjecture CP is true (in fact with $n_l=0$). Moreover, (SWC) holds for the corresponding superisolated singularity.*

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L. Makar-Limanov

Consider the subgroup of automorphisms of $\mathbb{C}[x, y, z]$ which is generated by tame automorphisms and exponential automorphisms (automorphisms which are exponents of locally nilpotent derivations). Call this subgroup the subgroup of generalized tame automorphisms. A. Joseph conjectured many years ago that any automorphism of $\mathbb{C}[x, y, z]$ is tame in this sense. E. g. the Nagata automorphism is in this group.

Potential counterexample: Let $\Delta = x^2y + z^2$. Consider the following automorphism of $\mathbb{C}[x, y, z]$.

$\gamma(x) = x$, $\gamma(y) = y - 2xyz - \Delta^2 + 2(3z^2\Delta + 3xz\Delta^2 + x^2\Delta^3)$, $\gamma(z) = z + x\Delta$.

Is this automorphism generalized tame? The automorphism γ is a composition of two exponential automorphisms of $\mathbb{C}(x, y, z)$. So it is similar to the Nagata automorphism which is a composition of two tame automorphisms of $\mathbb{C}(x, y, z)$.

Stefan Maubach

Commuting derivations conjecture (CD(n)): Let $D_1, \ldots, D_{n-1}$ be pairwise commuting locally nilpotent derivations on $\mathbb{C}^{[n]}$, which are linearly independent over $\mathbb{C}^{[n]}$. Then the common kernel, $(\mathbb{C}^{[n]})^{D_1,\ldots,D_{n-1}}$, equals $\mathbb{C}[f]$ where f is a coordinate.

The common kernel is always of the form $\mathbb{C}[f]$, the point is that it should be a coordinate. The conjecture is open for $n \geq 4$.

Generalized coordinates: The idea is to generalize the concept of “stable coordinate” to more general rings.

Let $g \in \mathbb{C}^{[n]}/I$ where $I = (f_1, f_2, \ldots, f_m)$ is an ideal. Take some $G \in \mathbb{C}^{[n]}$ such that $G + I = g$. We define g as a *generalized coordinate* if

$$f_1Y_1 + f_2Y_2 + \ldots + f_mY_m + G \in \mathbb{C}^{[n]}[Y_1, \ldots, Y_m]$$

is a *stable* coordinate.

Conjecture: The definition of “generalized coordinate” coincides with “stable coordinate” in case $\mathbb{C}^{[n]}/I \cong \mathbb{C}^{[p]}$ for some p (i.e. on a polynomial ring, generalized coordinates are stable coordinates).

Masayoshi Miyanishi

The *Equivariant Jacobian Conjecture* is stated as follows:

Let X be a normal affine variety defined over $\mathbb{C}$ with an effective algebraic action of an algebraic group G and let $\varphi : X \to X$ be an unramified endomorphism which commutes with the G-action. Then φ is a finite morphism.

If the topological Euler number of X is 1, we may ask if φ is an automorphism. The Equivariant Jacobian Conjecture is reduced to the following two conjectures.

(1) *Let $Y = X//G$ be the algebraic quotient and let $\psi : Y \to Y$ be the induced endomorphism. Then ψ is a finite morphism.*

(2) *Suppose ψ is a finite morphism. Then φ is a finite morphism.*

The second conjecture is related to the following problem.

Equivariant Ax Problem. Suppose ψ is an automorphism without assuming that φ is unramified. Is φ an automorphism ?

To consider these conjectures, it is necessary to answer the following question.

Let φ and ψ be as above. Is ψ unramified?

The answer is yes if G is a connected reductive algebraic group or if $G = G_a$, X is factorial and $\dim X = 2$. The case $G = G_a$ follows if the answer to the following question is affirmative.

Let B be a factorial affine domain defined over $\mathbb{C}$, δ a locally nilpotent derivation on B and $A = \operatorname{Ker}\delta$. Let M be a maximal ideal of B and let $m = M \cap A$. Then δ extends to (not necessarily locally nilpotent) derivations δ_M and $\hat{\delta}_M$ of B_M and $\hat{B}_M$ respectively. It is known that $\operatorname{Ker}\delta_M = A_m$. Is $\operatorname{Ker}\hat{\delta}_M$ equal to $\hat{A}_m$?

The answer is yes if $\delta(B) \cap A \not\subset m$.

Vladimir Popov

Below all algebraic varieties are taken over an algebraically closed field k of characteristic zero.

1. Birationally nonequivalent linear actions

Let G be a reductive algebraic group. In 1992 P. Katsylo published the following

Conjecture 1. ([Ka]) *Let V and W be finite dimensional algebraic G-modules with trivial stabilizers of points in general position. Then the following properties are equivalent:*

(i) $\dim V = \dim W$*;*

(ii) *there exists a G-equivariant birational map* $V \overset{\simeq}{\dashrightarrow} W$.

In [Ka] Conjecture 1 was proved for $G = \mathbf{SL}_2$, $\mathbf{PSL}_2$, and the symmetric groups S_n, $n \leqslant 4$. However E. Tevelev observed (unpublished) that Conjecture 1 fails for one-dimensional spaces and $G = \mathbf{Z}/n$, $n \neq 2, 3, 4, 6$; the same observation was independently made in [RY]. In 2000 new counterexamples to Conjecture 1 have been found in [RY], where a birational classification of finite dimensional G-modules for diagonalizable G has been obtained. Being sceptical about Conjecture 1, in 1993 I suggested to consider $W = V^*$, the dual module of V:

Problem 2. Are there a connected semisimple group G and a finite dimensional algebraic G-module V with trivial stabilizers of points in general position such that V and V^* are not birationally G-isomorphic?

This problem was communicated to some people, see, e.g., [RV]. So far it is still open.

It is well known that, if G is connected, then V^* is V "twisted" by an automorphism of G. This naturally leads to the following generalization. Let H be an algebraic group acting on an algebraic variety X,

$$H \times X \to X, \quad (h, x) \mapsto h \cdot x.$$

Let $\sigma\colon H \to H$, $h \mapsto {}^{\sigma}h$, be an automorphism of H. Consider the following new action of H on X:

$$H \times X \to X, \quad (h, x) \mapsto {}^{\sigma}h \cdot x.$$

Then the new H-variety appearing in this way is denoted by ${}^{\sigma}X$ and called X "twisted" by σ. Problem 2 can be now generalized as follows:

Problem 3. Are there a connected semisimple group G, a finite dimensional algebraic G-module V with trivial stabilizers of points in general position, and an automorphism σ of G such that V and ${}^{\sigma}V$ are not birationally G-isomorphic?

Note that if in Problem 3 one replaces V by a G-variety X, then the answer is positive (see [RV] where this is proved for $G = \mathbf{PGL}_n$).

It is clear that one should consider only outer automorphisms σ in Problem 3. Also, it is easily seen that if H is special in the sense of J.-P. Serre, [S], then X and ${}^{\sigma}X$ are always birationally G-isomorphic, cf. [RV]. In particular, answering Problems 2 and 3 for simple G, one should consider only

$$\mathbf{SL}_n/\mu_d \text{ where } d \neq 1, \text{ and the groups of types } \mathsf{D}_l \ (l \text{ is odd in Problem 2}),\ \mathsf{E}_6.$$

In particular, we have the following special

Problem 4. Let $G = \mathbf{SL}_d/\mu_d$. Let V be the d-th symmetric power of k^d endowed with the natural action of G. Let σ be the automorphism of G induced by the automorphism $g \mapsto (g^{\mathsf{T}})^{-1}$ of $\mathbf{SL}_d$. Are V and ${}^{\sigma}V$ birationally G-isomorphic?

Note that for $d > 3$ in Problem 4, stabilizers of points in general position in V are trivial, [P].

2. Cayley degrees of simple algebraic groups

Let G be a connected reductive algebraic group and let $\operatorname{Lie} G$ be its Lie algebra. Consider the action of G on $\operatorname{Lie} G$ via the adjoint representation and on G by conjugation.

Definition 5. ([LPR_1]) G is called *Cayley group* if G and $\operatorname{Lie} G$ are birationally G-isomorphic.

All simple Cayley groups have been classified in [LPR_1, Theorem 1.31]: they are precisely the groups from the list

$$\mathbf{SL}_n, n \leqslant 3; \quad \mathbf{SO}_n, n \neq 2, 4; \quad \mathbf{Sp}_{2n}; \quad \mathbf{PGL}_n.$$

For every G, by [LPR_1, Prop. 10.5] there always exists a dominant G-equivariant rational map $G \dashrightarrow \operatorname{Lie} G$. So the following number is well defined:

Definition 6. ([LPR_1]) The Cayley degree $\operatorname{Cay}(G)$ of G is the minimum of degrees of dominant rational G-equivariant maps $G \dashrightarrow \operatorname{Lie} G$.

So G is Cayley if and only if $\operatorname{Cay}(G) = 1$. In general, $\operatorname{Cay}(G)$ "measures" how far G is from being Cayley.

Problem 7. ([LPR_1]) Find the Cayley degrees of connected simple algebraic groups.

In [LPR_1], [LPR_2]) it is proved that

$$\operatorname{Cay}(\mathbf{SL}_n) \leqslant n-2, \text{ for } n \geqslant 3; \quad \operatorname{Cay}(\mathbf{SL}_n/\mu_d) \leqslant n/d;$$

$$\operatorname{Cay}(\mathbf{Spin}_n) = \begin{cases} 2 & \text{for } n \geqslant 6, \\ 1 & \text{for } n \leqslant 5; \end{cases}$$

$$\operatorname{Cay}(\mathbf{G}_2) = 2; \quad \operatorname{Cay}(\mathbf{G}_2 \times \mathbf{G}_m^2) = 1.$$

In particular, this implies that $\mathbf{Cay}(\mathbf{SL}_4) = 2$ and $2 \leqslant \mathbf{Cay}(\mathbf{SL}_5) \leqslant 3$.

Problem 8. Find $\operatorname{Cay}(\mathbf{SL}_5)$.

At the moment no examples of groups whose Cayley degree is bigger than 2 are known.

Problem 9. Is there G such that $\mathrm{Cay}(G) > 2$? Is there a simple such G?

More generally,

Problem 10. Given a $d \in \mathbf{N}$, is there G such that $\mathrm{Cay}(G) > d$? Is there a simple such G?

3. Singularities of two-dimensional quotients

Using a result of [KR], it was recently proved in [G_2] that if a complex reductive algebraic group G acts algebraically on $\mathbf{C}^n$ and the categorical quotient $\mathbf{C}^n/\!\!/G$ is two-dimensional, then $\mathbf{C}^n/\!\!/G$ is isomorphic to $\mathbf{C}^2/\Gamma$, where Γ is a finite group acting algebraically on $\mathbf{C}^2$. This theorem can be considered as a generalization of C. T. C. Wall's conjecture for the linear action of G on $\mathbf{C}^n$ proved in [G_1].

This result, discussed in Koras' talk at this Workshop, prompted the following question.

Problem 11. (M. Miyanishi) What are the groups Γ occuring in the above situation?

I conjecture that the following holds.

Conjecture 12. *If G is connected, then Γ is cyclic.*

Note that it was conjectured in [P_1] and proved in [Ke] that if in the above situation the group G is connected semisimple and the action of G on $\mathbf{C}^n$ is linear, then Γ is trivial.

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David Wright

Conjecture: Let R be a UFD(even a principle ideal domain or discrete valuation ring) and let $\mathrm{GA}_2(R) = \mathrm{Aut}_R(R^{[n]})$. Show that all elements of $\mathrm{GA}_2(R)$ are stably tame.

Describe the group $\mathrm{GA}_2(R)$ (generators, relations) for R a principle ideal domain (e.g. $R = k[t]$, k field).

In regard to the second problem, it was shown in [1] that, when R is a polynomial group over a field k, $\mathrm{GA}_2(R)$ has an amalgamated free product structure $\mathrm{Af}_2(k) *_B H$. However the group H is not well-understood.

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Stéphane Vénéreau

A Problem of M. Zaidenberg

Shortly expressed, the question raised by M. Zaidenberg is the following:

$$p(x_1, \cdots, x_m) + q(y_1, \cdots, y_n) \text{ is a variable} \Longrightarrow p \text{ or } q \text{ is a variable?}$$

Where $p \in R[x_1, \cdots, x_m]$ and $q \in R[y_1, \cdots, y_n]$, R is a commutative with unity ring and $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ are disjoint sets of indeterminates. The sum $p + q$ is seen as a polynomial in $R[x_1, \cdots, x_m, y_1, \cdots, y_n]$ and one says that a polynomial $f \in R[z_1, \cdots, z_l]$ is a *variable* if there exists an automorphism of R-algebra $\alpha : R[z_1, \cdots, z_l] \to R[z_1, \cdots, z_l]$ such that $\alpha(z_1) = f$.

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