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Geometric and Topological Combinatorics

Organised by
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Abstract. The 2007 Oberwolfach meeting "Geometric and Topological Combinatorics" presented a great variety of investigations where topological and algebraic methods are brought into play to solve combinatorial and geometric problems, but also where geometric and combinatorial ideas are applied to topological questions.

Mathematics Subject Classification (2000): 05-06, 54-06, 52-06.

Introduction by the Organisers

The 2007 Oberwolfach meeting "Geometric and Topological Combinatorics" was organized by Anders Björner (KTH and Mittag-Leffler Institute, Stockholm), Gil Kalai (Hebrew University, Jerusalem), and Günter M. Ziegler (TU Berlin). It consisted of six one-hour lectures by Isabella Novik, Herbert Edelsbrunner, Carsten Schultz, Igor Pak, Alexander Barvinok and Roy Meshulam, as well as twenty-seven half-hour talks, a problem session (led by Gil Kalai, also documented below), two software demonstrations (on `polymake` by Michael Joswig, and on `LattE` by Jesus De Loera), and many more informal sessions, group discussions, and a great variety of small group and pairwise discussions. It was a lively week!

As will become obvious from the sequence of extended abstracts presented below, the conference treated a broad spectrum of topics from Topological Combinatorics (such as poset topology, graph coloring, etc.), Discrete Geometry (polytopes, triangulations, arrangements, Coxeter groups, and polyhedral surfaces), as well as Geometric Topology (triangulated manifolds, persistent homology of geometric data, etc.).

The manifold connections between these themes, with refinements of well-established bridges as well as completely new links between seemingly distant themes, problems, methods, and theories show that the area is very much alive. Even more so this is demonstrated by substantial progress on older problems, which on this conference included Isabella Novik's opening lecture about centrally symmetric polytopes (joint works with Nati Linial and with Alexander Barvinok), or still on the first day Ed Swartz' report about the f-vectors of triangulated manifolds.

Of course there is no way to present the richness and depth of the work and presentations of the conference's program on a one page report, or a short collection of abstracts. All the following can offer is an overview of the official program of the conference. It does not cover all the additional smaller presentations, group discussions and blackboard meetings (for example, Tom Braden was coerced into an additional evening presentation of the topology of hypertoric varieties "by popular demand"), nor the lively interactions that occured during the week — for example, a surprising connection was made at the problem session between permutation patterns that appeared in Jonas Sjöstrand's lecture, and a conjecture posed by Alex Postnikov; in subsequent work, the problem was solved by Axel Hultman, Svante Linusson, John Shareshian, and Jonas Sjöstrand (paper in preparation).

We are extremely grateful to the Oberwolfach institute, to its director and to all of its staff for providing a perfect setting for an inspiring, intensive week of "Geometric and Topological Combinatorics."

Anders Björner, Gil Kalai, Günter M. Ziegler
Stockholm / Jerusalem / Berlin, March 2007

Workshop: Geometric and Topological Combinatorics

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Abstracts

Complexity of Delaunay triangulation for points on lower-dimensional polyhedra

Dominique Attali
(joint work with Nina Amenta and Olivier Devillers)

The Delaunay triangulation of a set of points is a data structure, which in low dimensions has applications in mesh generation, surface reconstruction, molecular modeling, geographic information systems, and many other areas of science and engineering. Like many spatial partitioning techniques, however, it suffers from the "curse of dimensionality": in higher dimensions, the complexity of the Delaunay triangulation increases exponentially. Its worst-case complexity is bounded precisely by the Upper Bound Theorem [13], which states that the number of simplices in the Delaunay triangulation of n points in dimension d is at most $O(n^{\lceil \frac{d}{2} \rceil})$. This bound is achieved exactly by the vertices of a cyclic polytope, which all lie on a one-dimensional curve known as the *moment curve.* Indeed all of the examples that we have of point sets which have Delaunay triangulations of complexity $O(n^{\lceil d/2 \rceil})$ are distributed on one-dimensional curves. At the opposite extreme, points distributed uniformly at random inside the sphere have Delaunay triangulations of complexity $O(n)$ in any fixed dimension, with a constant factor which is exponential in the dimension [9]. Our goal in this work is to begin to fill in the picture for distributions between the two extremes, in which the points lie on manifolds of dimension $2 \leq p \leq d-1$.

As an easy first case, we consider a fixed polyhedral set (not necessarily convex) $\mathbb{P}$ of dimension p in $d > p$ dimensional space. Our point set S is a *sparse ϵ-sample* from $\mathbb{P}$. Sparse ϵ-sampling is a model, sometimes used in computational geometry, in which the sampling can be neither too dense nor too sparse. Let n be the number of points in S. We consider how the complexity of the Delaunay triangulation of S grows, as $n \to \infty$, with $\mathbb{P}$ remaining fixed. Our main result is that the number of simplices of all dimensions is $O(n^{(d-1)/p})$. The hidden constant factor depends, among other things, on the geometry of $\mathbb{P}$, which is constant since $\mathbb{P}$ is fixed.

While our result is purely combinatorial, it has both potential and immediate algorithmic implications. The Delaunay triangulation can be computed in optimal worst-case time in dimension $d \geq 3$ by the standard randomized incremental algorithm [8, 16], or deterministically [7]. While our result does not immediately improve these running times for the special case of points distributed on lower-dimensional manifolds [3], it is of course a necessary step towards such an improvement. Our result shows that Seidel's giftwrapping algorithm [15] runs in time $O(n^2 + n^{(d-1)/p} \lg n)$ in our special cases, which can be somewhat improved using more sophisticated data structures [4].

Prior work. The complexity of the Delaunay triangulation of a set of points on a two-manifold in $\mathbb{R}^3$ has received considerable recent attention, since such point sets arise in practice, and their Delaunay triangulations are found nearly always to have linear size. Golin and Na [12] proved that the Delaunay triangulation of a large enough set of points distributed uniformly at random on the surface of a fixed convex polytope in $\mathbb{R}^3$ has expected size $O(n)$. They later [11] established an $O(n \lg^6 n)$ upper bound with high probability for the case in which the points are distributed uniformly at random on the surface of a non-convex polyhedron.

Attali and Boissonnat considered the problem using a sparse ϵ-sampling model similar to the one we use here, rather than a random distribution. For such a set of points distributed on a polygonal surface $\mathbb{P}$, they showed that the size of the Delaunay triangulation is $O(n)$ [1]. In a subsequent paper with Lieutier [2] they considered "generic" surfaces, and got an upper bound of $O(n \lg n)$. Specifically, a "generic" surface is one for which each medial ball touches the surface in at most a constant number of points.

The genericity assumption is important. Erickson considered more general point distributions, which he characterized by the *spread*: the ratio of the largest inter-point distance to the smallest. The spread of a sparse ϵ-sample of n points from a two-dimensional manifold is $O(\sqrt{n})$. Erickson proved that the Delaunay triangulation of a set of points in $\mathbb{R}^3$ with spread Δ is $O(\Delta^3)$. Perhaps even more interestingly, he showed that this bound is tight for $\Delta = \sqrt{n}$, by giving an example of a sparse ϵ-sample of points from a cylinder that has a Delaunay triangulation of size $\Omega(n^{3/2})$ [10]. Note that this surface is not generic and has a degenerate medial axis.

To the best of our knowledge, there are no prior results for $d > 3$.

Overview of the proof. Our proof uses two samples, the original sparse ϵ-sample S from the polyhedron $\mathbb{P}$, and a sparse ϵ-sample M of a bounded subset $\mathcal{M}^*$ of the medial axis of $\mathbb{P}$. We prove that any Delaunay ball circumscribing points of S intersects the polyhedron in a set of points that is contained in an enlarged medial ball centered at a medial sample point $z \in M$. We then prove that each sample z in M is assigned at most a constant number of Delaunay balls.

Since M is a sparse ϵ-sample from a fixed $(d-1)$-dimensional set of constant volume, its cardinality is $m = O(\epsilon^{-(d-1)})$. Similarly, S is a sparse ϵ-sample of $\mathbb{P}$ and we get $n = \Omega(\epsilon^{-p})$. Eliminating ϵ gives $m = O(n^{(d-1)/p})$, and since each sample of M is charged for a constant number of Delaunay balls, this bound applies to the size of the Delaunay triangulation as well. This is the key insight: as a function of ϵ, the number of Delaunay balls depends only on the dimension of the medial axis, which is always $d-1$. The number of samples, n, depends on the dimension p of $\mathbb{P}$. As p increases, n increases, but the complexity of the Delaunay triangulation stays about the same. If written as a function of n, the complexity of the Delaunay triangulation goes down.

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Moduli of marked representations of path algebras

Eric Babson

Associate to a finite directed graph a (complex) path algebra. The nodes of the graph are the primitive idempotents of the algebra. Fix a multiset of nodes and an associated module P which is a direct sum of cyclic modules with summands indexed by the elements of the multiset. For every dimension d the set of codimension d submodules of P which contain all length d paths is naturally a projective variety G whose structure we study.

- G is paved by affine spaces indexed by the submodules generated by paths.
- G decomposes into affine bundles over the connected subvarieties of (length) homogeneous submodules hG.
- Each hG is a tower of Grassmannian bundles.

Counting magic squares, contingency tables, integer flows, and more

Alexander Barvinok
(joint work with Alex Samorodnitsky and Alex Yong)

Let $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$ be positive integer vectors such that

$$r_1 + \ldots + r_m = c_1 + \ldots + c_n = N.$$

A *contingency table* with the margins (R, C) is a non-negative $m \times n$ integer matrix $D = (d_{ij})$ with the row sums $r_1, \ldots, r_m$ and the column sums $c_1, \ldots, c_n$. Let us choose an $m \times n$ non-negative matrix $W = (w_{ij})$ of weights. We define

$$T(W; R, C) = \sum_{D=(d_{ij})} \prod_{ij} w_{ij}^{d_{ij}},$$

where the sum is taken over all contingency tables with the margins (R, C). We agree that $w_{ij}^0 = 1$.

If $w_{ij} = 1$ for all i, j (we denote this matrix by $\mathbf{1}$) then $T(\mathbf{1}; R, C)$ is just the number of contingency tables with the margins (R, C). If $m = n$ and $R = C = (t, \ldots, t)$ (we denote these vectors by $\mathbf{t}$) then $T(\mathbf{1}; \mathbf{t}, \mathbf{t})$ is the number of *magic squares* with the line sums t. If $w_{ij} \in \{0, 1\}$ then $T(W; R, C)$ is the number of integer feasible flows in a bipartite graph defined by W with the "demands" $c_1, \ldots, c_n$ and "supplies" $r_1, \ldots, r_m$.

A simple combinatorial construction reduces counting integer feasible flows in any network (even a network with *capacities* on edges) to counting integer feasible flows in a bipartite network and hence to computing $T(W; R, C)$.

We define a non-negative quantity $T'(W; R, C)$ with the following properties:

- There is a randomized algorithm, which, given W, R, C and $\epsilon > 0$, computes the value of $T'(W; R, C)$ within relative error ϵ in time polynomial in N and ϵ^{-1};
- We have

  $$T'(W; R, C) \leq T(W; R, C) \leq \alpha(W; R, C) T'(W; R, C)$$

  for some $\alpha(W; R, C)$, such that, additionally,
- there is a randomized algorithm, which, given W, R, C and ϵ, computes the value of $T(W; R, C)$ within relative error ϵ in time polynomial in ϵ^{-1} and N and *linear* in $\alpha(W; R, C)$.

One can always choose

$$\alpha(W; R, C) = \frac{N^N}{N!} \min \left\{ \prod_{i=1}^{m} \frac{r_i!}{r_i^{r_i}}, \quad \prod_{j=1}^{n} \frac{c_j!}{c_j^{c_j}} \right\}.$$

Note that in this case $\alpha(W; R, C)$ does not depend on W. For example, if $R = C = (t, \ldots, t)$ then $\alpha(W; R, C)$ is of the order $t^{(n-1)/2}$ while the number of $n \times n$ non-negative integer matrices with the line sums t grows as $t^{(n-1)^2}$ for $t \longrightarrow +\infty$.

Alex Samorodnitsky proved that one can choose

$$\alpha(\mathbf{1}; \mathbf{t}, \mathbf{t}) \leq N^{\log N}.$$

Numerical experiments conducted by Alex Yong suggest that, possibly,

(1) the bound can be replaced by N^{β} for some absolute reasonably small constant β (say, $\beta = 1/2$) and
(2) the bound can be extended to $\alpha(\mathbf{1}; R, C)$ with sufficiently generic R and C.

but we are unable to prove that so far.

The functional $T'(W; R, C)$ is defined as follows. Let us consider an $N \times N$ random matrix A which consists of mn blocks $r_i \times c_j$ filled by $w_{ij} x_{ij}$, where x_{ij} are independent standard exponential variables. Then

$$T(W; R, C) = \frac{\mathbf{E} \text{ per } A}{r_1! \cdots r_m! c_1! \cdots c_n!},$$

where "per" stands for the usual permanent of a matrix. For an $N \times N$ positive matrix $A = (a_{ij})$ we define

$$\sigma(A) = \prod_{i=1}^{N} (\lambda_i \mu_i),$$

where λ_i and μ_i are positive numbers such that the matrix $B = (b_{ij})$ with

$$b_{ij} = a_{ij} \lambda_i \mu_j \quad \text{for all} \quad i, j$$

is *doubly stochastic*, that is, has the row and column sums equal to 1. The function $\sigma(A)$ is well-defined, and, moreover, *log-concave*, that is,

$$\sigma\left(\frac{1}{2}A_1 + \frac{1}{2}A_2\right) \geq \sigma^{1/2}(A_1)\, \sigma^{1/2}(A_2)$$

for any two positive matrices A_1, A_2. Hence we define

$$T'(W; R, C) = \frac{\mathbf{E}\ \sigma(A)}{r_1! \cdots r_m! c_1! \cdots c_n!} \frac{N!}{N^N}.$$

The algorithm for computing $T'(W; R, C)$ uses the two ingredients: a strongly polynomial time algorithm for computing $\sigma(A)$, due to N. Linial, A. Samorodnitsky, and A. Wigderson, and randomized polynomial time algorithms for integration of log-concave densities, due to D. Applegate, R. Kannan, M. Dyer, A. Frieze, N. Polson, L. Lovász, and S. Vempala. The algorithm for computing $T(W; R, C)$ uses, in addition, a randomized polynomial time approximation algorithm for computing the permanent of a non-negative matrix due to M. Jerrum, A. Sinclair, and E. Vigoda.

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Combinatorial complexity in O-minimal geometry

Saugata Basu

1. Introduction

Over the last twenty years there has been a lot of work on bounding the topological complexity (measured in terms of their Betti numbers) of several different classes of subsets of $\mathbb{R}^k$ – most notably semi-algebraic and semi-Pfaffian sets. The usual setting for proving these bounds is as follows. One considers a semi-algebraic (or semi-Pfaffian) set $S \subset \mathbb{R}^k$ defined by a Boolean formula whose atoms consists of $P > 0, P = 0, P < 0,\ P \in \mathcal{P}$, where $\mathcal{P}$ is a set of polynomials (resp. Pfaffian functions) of degrees bounded by a parameter (resp. whose Pfaffian complexity is bounded by certain parameters) and $\#\mathcal{P} = n$. It is possible to obtain bounds on the Betti numbers of S in terms of n, k and the parameters bounding the complexity of the functions in $\mathcal{P}$.

1.1. **Known Bounds in the Semi-algebraic and Semi-Pfaffian cases.** In the semi-algebraic case, if we assume that the degrees of the polynomials in $\mathcal{P}$ are bounded by d, and denoting by $b_i(S)$ the i-th Betti number of S, then it is shown in [6] that, $\sum_{i \geq 0} b_i(S) \leq n^{2k} O(d)^k$.

A similar bound is also shown for semi-Pfaffian sets [6].

In another direction, we also have reasonably tight bounds on the sum of the Betti numbers of the realizations of all realizable sign conditions of the family $\mathcal{P}$. A sign condition on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$, and the realization of a sign condition σ is the set, $\mathcal{R}(\sigma) = \{\mathbf{x} \in \mathrm{R}^k \mid \mathrm{sign}(P(\mathbf{x})) = \sigma(P), \forall P \in \mathcal{P}\}$. It is shown in [3] that, $\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathcal{R}(\sigma)) \leq \sum_{j=0}^{k-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = n^{k-i} O(d)^k$.

2. New Results

Notice that the above bounds are products of two quantities – one that depends only on n (and k), and another part which is independent of n, but depends on the parameters controlling the complexity of individual elements of $\mathcal{P}$ (such as degrees of polynomials in the semi-algebraic case, or the degrees and the length of the Pfaffian chain defining the functions in the Pfaffian case). It is customary to refer to the first part as the *combinatorial part* of the complexity, and the latter as the algebraic (or Pfaffian) part.

While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of primary interest (the algebraic part is assumed to be bounded by a constant). The motivation behind this point of view is the following. In problems in discrete and computational geometry, one typically encounters arrangements of a large number of objects in $\mathbb{R}^k$ (for some fixed k), where each object is of "constant description complexity" (for example, defined by a polynomial inequality of degree bounded by a constant). Thus, it is the number of objects that constitutes the important parameter, and the algebraic complexity of the individual objects are thought of as small constants. It is this second setting that is our primary interest in this paper.

The main results of this paper generalize (combinatorial parts of) the above mentioned bounds to sets which are definable in an arbitrary o-minimal structure over a real closed field R. The proofs of the theorems stated below, as well as several other results, can be found in the full paper [2].

2.1. **Admissible Sets.** We now define the sets that will play the role of objects of "constant description complexity".

Definition 1. *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure on a real closed field* R *and let $T \subset \mathrm{R}^{k+\ell}$ be a definable set. Let $\pi_1 : \mathrm{R}^{k+\ell} \to \mathrm{R}^k$ (resp. $\pi_2 : \mathrm{R}^{k+\ell} \to \mathrm{R}^\ell$), be the projections onto the first k (resp. last ℓ) co-ordinates.*

We will call a subset S of R^k to be a (T, π_1, π_2)-set if $S = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$ for some $\mathbf{y} \in \mathrm{R}^\ell$, and when the context is clear we will denote $T_{\mathbf{y}} = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$. In this paper, we will consider finite families of (T, π_1, π_2)-sets, where T is some fixed definable set for each such family, and we will call a family of (T, π_1, π_2)-sets to be a (T, π_1, π_2)-family. We refer to a finite (T, π_1, π_2)-family as an arrangement *of (T, π_1, π_2)-sets.*

Definition 2. *Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, such that each $S_i \subset \mathrm{R}^k$ is a (T, π_1, π_2)-set. For $I \subset \{1, \ldots, n\}$, we let $\mathcal{A}(I)$ denote the set $\bigcap_{i \in I \subset [1\ldots n]} S_i \cap \bigcap_{j \in [1\ldots n] \setminus I} \mathrm{R}^k \setminus S_j$, and we will call such a set to be a basic $\mathcal{A}$-set. We will denote by, $\mathcal{C}(\mathcal{A})$, the set of non-empty connected components of all basic $\mathcal{A}$-sets.*

For any definable set $X \subset \mathrm{R}^k$, we let $b_i(X)$ denote the i-th Betti number of X, and we let $b(X)$ denote $\sum_{i=0}^k b_i(X)$. We define the *topological complexity* of an arrangement $\mathcal{A}$ of (T, π_1, π_2)-sets to be the number $\sum_{D \in \mathcal{C}(\mathcal{A})} \sum_{i=0}^k b_i(D)$.

2.2. **Combinatorial and Topological Complexity of Arrangements.** We have the following theorems.

Theorem 3. *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field* R *and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $C = C(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family $\mathcal{A} = \{S_1, \ldots, S_n\}$ of subsets of R^k the following holds.*

(1) *For every $i, 0 \le i \le k$, $\sum_{D \in \mathcal{C}(\mathcal{A})} b_i(D) \le C \cdot n^{k-i}$. In particular, the combinatorial complexity of $\mathcal{A}$, which is equal to $\sum_{D \in \mathcal{C}(\mathcal{A})} b_0(D)$, is at most $C \cdot n^k$.*
(2) *The topological complexity of any m cells in the arrangement $\mathcal{A}$ is bounded by $m + C \cdot n^{k-1}$.*

Theorem 4 (Topological Complexity of Projections)**.** *Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure, and let $T \subset \mathbb{R}^{k+\ell}$ be a definable, closed and bounded set. Let $k = k_1 + k_2$ and let $\pi_3 : \mathbb{R}^k \to \mathbb{R}^{k_2}$ denote the projection map on the last k_2 co-ordinates. Then, there exists a constant $C = C(T) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$, with $|\mathcal{A}| = n$, and an $\mathcal{A}$-closed set $S \subset \mathbb{R}^k$, $\sum_{i=0}^{k_2} b_i(\pi_3(S)) \leq C \cdot n^{(k_1+1)k_2}$.*

2.3. **Cylindrical Definable Cell Decompositions.** The fact that given any finite family $\mathcal{A}$ of definable subsets of R^k, there exists a Cylindrical Definable Cell Decomposition (cdcd for short) of R^k adapted to $\mathcal{A}$ is classical (see [4, 5]). We prove a quantitative version of this result. Such quantitative versions are known in the semi-algebraic as well as semi-Pfaffian categories, but is missing in the general o-minimal setting.

Theorem 5 (Quantitative cylindrical definable cell decomposition)**.** *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exist constants $C_1, C_2 > 0$ depending only on T, and definable sets, $\{T_i\}_{i \in I}$, $T_i \subset \mathrm{R}^k \times \mathrm{R}^{2(2^k-1)\cdot\ell}$, depending only on T, with $|I| \leq C_1$, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$ with $S_i = T_{\mathbf{y}_i}, \mathbf{y}_i \in \mathrm{R}^\ell, 1 \leq i \leq n$, some sub-collection of the sets*

$$\pi^{\leq k}_{k+2(2^k-1)\cdot\ell}\left({\pi^{>k}_{k+2(2^k-1)\cdot\ell}}^{-1}(\mathbf{y}_{i_1}, \ldots, \mathbf{y}_{i_{2(2^k-1)}}) \cap T_i\right),$$
$$i \in I,\ 1 \leq i_1, \ldots, i_{2(2^k-1)} \leq n,$$

form a cdcd of R^k compatible with $\mathcal{A}$. Moreover, the cdcd has at most $C_2 \cdot n^{2(2^k-1)}$ cells.

2.4. **Application.** We end with an application which generalizes a Ramsey-type result due to Alon et al. [1] from the class of semi-algebraic sets of constant description complexity to (T, π_1, π_2)-families.

Theorem 6. *Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let F be a closed definable subset of $\mathrm{R}^\ell \times \mathrm{R}^\ell$. Then, there exists a constant $1 > \varepsilon = \varepsilon(F) > 0$, depending only on F, such that for any set of n points, $\mathcal{F} = \{\mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathrm{R}^\ell\}$ there exists two subfamilies $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, with $|\mathcal{F}_1|, |\mathcal{F}_2| \geq \varepsilon n$ and either, for all $\mathbf{y}_i \in \mathcal{F}_1$ and $\mathbf{y}_j \in \mathcal{F}_2$, $(\mathbf{y}_i, \mathbf{y}_j) \in F$, or for no $\mathbf{y}_i \in \mathcal{F}_1$ and $\mathbf{y}_j \in \mathcal{F}_2$, $(\mathbf{y}_i, \mathbf{y}_j) \in F$.*

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The complete cd-index of a Bruhat interval

Louis J. Billera
(joint work with Francesco Brenti)

For definitions involving Coxeter systems and Bruhat order, we refer to [3]. For any interval $[u, v]$ in the Bruhat order of a Coxeter system (W, S), we consider all directed u-v paths in the associated Coxeter graph. We assume the edges of this graph have been labeled by an arbitrary reflection order, as in [3, Figure 5.1].

The labels on the edges of each u-v path, considered as a sequence of positive integers, have an associated *descent set*, which we encode by a composition α. It follows that $\alpha \models k$, where $k \equiv l(v) - l(u) (\mathrm{mod}\ 2)$ and $l(w)$ is the length of a reduced expression for $w \in W$. For each such α, we let $b_\alpha = b_\alpha(u, v)$ denote the number of such u-v-paths having descent composition α. The numbers $b_\alpha(u, v)$ are independent of the choice of reflection order [3, Proposition 5.5.5].

Let $\mathbf{a}$ and $\mathbf{b}$ be noncommuting indeterminates (say, over the integers $\mathbf{Z}$), both of degree 1. For any $\alpha \models n$, we assign a corresponding degree $n - 1$ **ab**-word by

$$\alpha = (a_1, a_2, \ldots, a_k) \mapsto \mathbf{m}_\alpha := \mathbf{a}^{a_1 - 1}\mathbf{b} \cdots \mathbf{a}^{a_{k-1}-1}\mathbf{b}\mathbf{a}^{a_k - 1},$$

and define the polynomial

$$\widetilde{\Psi}_{u,v}(\mathbf{a}, \mathbf{b}) := \sum_\alpha b_\alpha\, \mathbf{m}_\alpha \in \mathbf{Z}[\mathbf{a}, \mathbf{b}].$$

Finally let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$; note that $\deg \mathbf{c} = 1$ and $\deg \mathbf{d} = 2$.

Theorem 1. *For any Bruhat interval* $[u, v]$, $\widetilde{\Psi}_{u,v}(\mathbf{a}, \mathbf{b}) \in \mathbf{Z}[\mathbf{c}, \mathbf{d}]$.

The resulting **cd**-polynomial will, by slight abuse of notation, also be denoted $\widetilde{\Psi}_{u,v} = \widetilde{\Psi}_{u,v}(\mathbf{c}, \mathbf{d})$ and will be called the *complete* **cd**-*index* of the Bruhat interval $[u, v]$. If we write

$$\widetilde{\Psi}_{u,v} = \sum_w [w]_{u,v}\, w,$$

where $[w]_{u,v}$ is the coefficient of the **cd**-word w, then it follows that $[w]_{u,v} = 0$ unless $\deg w \equiv l(v) - l(u) - 1 (\mathrm{mod}\ 2)$.

We denote by $\Psi_{u,v}$ the homogeneous component of highest degree $l(v) - l(u) - 1$ of $\widetilde{\Psi}_{u,v}$. $\Psi_{u,v}$ is the ordinary **cd**-index of the Eulerian poset $[u, v]$ (see, for example, [8]).

We denote by $P_{u,v}(q)$ the Kazhdan-Lusztig polynomial of the Bruhat interval $[u, v]$ (see, for example, [3, §5.1] for the definition of $P_{u,v}$). We can express $P_{u,v}$ in terms of the complete **cd**-index of $[u,v]$ and otherwise known quantities. Let

$$B_k(q) := \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k+1-2i}{k+1} \binom{k+1}{i} q^i$$

be the k-th *ballot polynomial*, and let $C_k = \frac{1}{2k+1}\binom{2k+1}{k} = \frac{1}{k+1}\binom{2k}{k}$ be the k^{th} Catalan number. We say a **cd**-word w is *even* if it is a monomial in $\mathbf{c}^2$ and $\mathbf{d}$; in this case $w = \mathbf{c}^{2k_1}\mathbf{d}\mathbf{c}^{2k_2}\mathbf{d}\cdots\mathbf{c}^{2k_{m-1}}\mathbf{d}\mathbf{c}^{2k_m}$, and we define $C_w = C_{k_1}C_{k_2}\cdots C_{k_m}$, $|w| = \deg w$ and $|w|_{\mathbf{d}} = m-1$, the number of $\mathbf{d}$'s in w.

Theorem 2. *Let $[u,v]$ be a Bruhat interval, $n = l(v) - l(u)$ and $\widetilde{\Psi} = \sum_w [w]_{u,v}\, w$ its complete* **cd***-index. Then the Kaszhdan-Lusztig polynomial of $[u,v]$ is*

$$P_{u,v}(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i \; q^i \, B_{n-2i}(-q),$$

where

$$a_i = a_i(u,v) = [\mathbf{c}^{n-2i}]_{u,v} + \sum_{w \text{ even}} (-1)^{\frac{|w|}{2}+|w|_{\mathbf{d}}} \, C_w \, [\mathbf{c}^{n-2i}\mathbf{d}w]_{u,v}.$$

Restricting the above formula to the terms of degree $n = l(v) - l(u) - 1$, we get the formula for the g-polynomial of the dual interval given in [1].

Corollary 3 (Bayer-Ehrenborg)**.** *If $\widehat{a}_i$ is a_i restricted to those $[w]$ with $\deg w = n$, then*

$$g([u,v]^*, q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \widehat{a}_i \; q^i \, B_{n-2i}(-q).$$

Thus $P_{u,v}(q) - g([u,v]^*, q)$ depends only on the lower degree terms $\widetilde{\Psi}_{u,v} - \Psi_{u,v}$ of the complete **cd**-index of $[u,v]$.

Example 4. *If $W = S_4$, all permutations of $\{1,2,3,4\}$, $u = 1234$, the identity, and $v = 4231$, then one can compute*

$$\widetilde{\Psi}_{u,v} = \mathbf{c}^4 + \mathbf{d}\mathbf{c}^2 + 2\mathbf{c}\mathbf{d}\mathbf{c} + 2\mathbf{c}^2\mathbf{d} + 2\mathbf{d}^2 + 2\mathbf{c}^2 + \mathbf{1},$$

and so

$$\begin{aligned} P_{u,v}(q) &= a_0\, q^0 B_4(-q) + a_1\, q^1 B_2(-q) + a_2\, q^2 B_0(-q) \\ &= [\mathbf{c}^4]\left(1 - 3q + 2q^2\right) + \left([\mathbf{c}^2\mathbf{d}] + [\mathbf{c}^2]\right) q(1-q) \\ &\qquad\qquad + \left([\mathbf{d}^2] - [\mathbf{d}\mathbf{c}^2] + [\mathbf{d}] + [\mathbf{1}]\right) q^2 \\ &= (1 - q + q^2) + (2q - q^2) = 1 + q \end{aligned}$$

with $g([u,v]^*, q) = 1 - q + q^2$.

We end with two conjectures and two questions.

Conjecture 5. *All the coefficients of $\widetilde{\Psi}_{u,v}$ are nonnegative. Since $[u,v]$ is a Gorenstein* poset, it follows from a theorem of Karu [7] that the coefficients of $\Psi_{u,v}$ are all nonnegative.*

Conjecture 6. *The coefficients $a_i(u,v)$ are all nonnegative for any Bruhat interval $[u,v]$. This, for example, is implied by the conjectured nonnegativity of the coefficients of $P_{u,v}$.*

Problem 7. *Do the* **cd***-coefficients $[w]_{u,v}$ depend only on the* poset $[u,v]$ *(and not on its structure as a Bruhat interval)? This property, called* combinatorial invariance*, is known to hold for all u and v when $\deg w = l(v) - l(u) - 1$ (the usual* **cd***-coefficients) and for all w when $u = e$. The latter follows from recent results of Brenti, et al.* [4] *and Delanoy* [5].

Problem 8. *What is the complexity of computing the $P_{u,v}$ or, equivalently, the coefficients $a_i(u,v)$, or even* all *$[w]_{u,v}$ in terms of the relevant input data, that is, the Coxeter matrix of the system (W,S), $l(v)$ and $l(u)$? See* [6].

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Combinatorics of arrangements and topology of hypertoric varieties

Tom Braden
(joint work with Nicholas Proudfoot)

Hypertoric varieties, which were first considered by Bielawski and Dancer [3], are hyperkähler or quaternionic analogues of toric varieties. Whereas the geometry and topology of a toric variety X_P is governed by a convex polyhedron

$$P = \bigcap_{i=1}^{n} \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{v}_i, \mathbf{x} \rangle \geq -\alpha_i\},$$

the corresponding hypertoric variety $Y_{\mathcal{H}}$ is governed by the hyperplane arrangement $\mathcal{H}$ with hyperplanes

$$H_i = \{\langle \mathbf{v}_i, \mathbf{x} \rangle = -\alpha_i\}.$$

Here $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Q}^d$ are the normal vectors to the hyperplanes and $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$. The toric variety can be obtained as a symplectic quotient of affine space $\mathbb{C}^n$ by the action of an $n-d$ dimensional subtorus of $(S^1)^n$ determined by the $\mathbf{v}_i$, and the hypertoric variety is obtained by a related hyperkähler quotient construction applied to the same torus acting on quaternionic affine space $\mathbb{H}^n$. The resulting space is $4d$-dimensional and contains the toric varieties corresponding to the chambers of the arrangement as half-dimensional (in fact, Lagrangian) subvarieties.

Just as the study of the topology of toric varieties has given rise to new invariants and interesting results about convex polytopes, topological invariants of hypertoric varieties lead to interesting invariants of arrangements (unlike the polytope case, though, these invariants were already known to combinatorists). For instance, if $\mathcal{H}$ is a simple arrangement, meaning that each codimension k flat is contained in exactly k hyperplanes, then the resulting hypertoric variety is rationally smooth, and the equivariant cohomology ring $H^*_T(Y_\mathcal{H};\mathbb{R})$ is canonically isomorphic to the face ring $\mathbb{R}[\Delta_\mathcal{H}]$ of the independence complex of $\mathcal{H}$, while the ordinary cohomology is the quotient of this ring by a set of linear forms [5, 6, 7]. In particular, the Betti numbers of $Y_\mathcal{H}$ are the h-numbers of $\Delta_\mathcal{H}$. At the other extreme, when the arrangement $\mathcal{H}$ is central, Proudfoot and Webster [8] showed that the intersection cohomology Betti numbers of $Y_\mathcal{H}$ are the h-numbers of the broken circuit complex $\Delta^{bc}_\mathcal{H}$ of the arrangment. This complex and its face ring depend on the choice of an ordering of the hyperplanes, although its h-numbers do not.

We present a canonical functorial computation of the T-equivariant intersection cohomology $IH^*_T(X_\mathcal{H})$, along the lines of the theory discovered by Barthel, Brasselet, Fieseler, and Kaup [1, 2] and Bressler and Lunts [4]. Let $L = L_\mathcal{H}$ be the lattice of flats of $\mathcal{H}$, whose elements are all possible intersections of hyperplanes, endowed with the order topology. We put a sheaf of rings $\mathcal{A}$ on it by declaring the stalk $\mathcal{A}_F$ at a flat F to be the symmetric algebra over the quotient of V by the linear span of F. A *minimal extension sheaf* $\mathcal{L}$ is a sheaf of graded modules over $\mathcal{A}$ satisfying four conditions:

(1) $\mathcal{L}_V \cong \mathbb{R}$
(2) $\mathcal{L}_F$ is a free $\mathcal{A}_F$-module for each flat F,
(3) $\mathcal{L}$ is flabby: $\mathcal{L}(L) \to \mathcal{L}(U)$ is surjective for any open set $U \subset L$, and
(4) $\mathcal{L}$ is minimal with respect to (1)-(3).

In fact, the construction for toric varieties in [1, 2, 4] is exactly the same, except with the lattice of flats replaced by the lattice of faces of the corresponding polyhedron.

Theorem. *A minimal extension sheaf $\mathcal{L}$ exists and is unique up to a scalar automorphism. Fixing a generator of $\mathcal{L}_V \cong \mathbb{R}$, there are canonical isomorphisms*

$$\mathcal{L}(L_\mathcal{H}) \cong IH^*_T(X_\mathcal{H})$$

and

$$\mathcal{L}(L_\mathcal{H})/V\mathcal{L}(L_\mathcal{H}) \cong IH^*(X_\mathcal{H}).$$

The fact that these isomorphisms are canonical is reflected in the fact that the minimal extension sheaf $\mathcal{L}$ has only scalar automorphisms as a sheaf of graded $\mathcal{A}$-modules.

We use this description to prove an unexpected result: the equivariant intersection cohomology group $IH_T(Y_{\mathcal{H}})$ carries a canonical ring structure. In general intersection cohomology does not have a multiplicative structure, and we do not yet have a geometric explanation of this fact. The ring in question is the "broken-circuit" ring $R_{\mathcal{H}}$ studied by Proudfoot and Speyer in [9]. For a central arrangement $\mathcal{H}$ it is the ring generated by the inverses of the linear forms defining the hyperplanes. It is a free $\mathrm{Sym}(V)$-algebra with graded rank given by $h(\Delta_{\mathcal{H}}^{bc})$. We show that the rings $R_{\mathcal{H}_F}$ assigned to the localizations of the arrangement at each flat carry a system of algebra homomorphisms making them into a sheaf of rings over $\mathcal{A}$, and that this sheaf satisfies the axioms (1)-(4).

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A formula for the volume of the polytope of doubly-stochastic matrices

J.A. De Loera

(joint work with F. Liu and R. Yoshida)

Let B_n denote the convex polytope of $n \times n$ doubly stochastic matrices; that is, the set of real non-negative matrices with all row and column sums equal to one. The polytope B_n is often called the *Birkhoff polytope* or the *assignment polytope.* Given a positive integer t the lattice points of the dilation $tB_n = \{X \in \mathbb{R}^{n^2} : \frac{X}{t} \in B_n\}$ consist of the $n \times n$ arrays of non-negative integers with the same row and column sums value t. We investigate the multivariate generating function

$$f(tB_n, \mathbf{z}) = \sum_{\alpha \in tB_n \cap \mathbb{Z}^{n^2}} \mathbf{z}^{\alpha}$$

of the lattice points of tB_n. Our first main result is an explicit formula for $f(tB_n, \mathbf{z})$ as a sum of multivariate rational functions, with each summand having a combinatorial meaning in terms of the directed trees (arborescences, cycles, and cuts of a directed complete graph. Our approach is based in the lattice point rational functions as developed in [6, 1] and the theory of Gröbner bases of toric ideals as outlined in [7].

To state the theorem we need some preliminaries: We call a directed spanning tree with all arcs pointing away from a root ℓ an *ℓ-arborescence*. The set of all ℓ-arborescences on the nodes $\{1, 2, \ldots, n\}$ will be denoted by $\mathbf{Arb}(\ell, n)$. It is well known that the cardinality of $\mathbf{Arb}(\ell, n)$ is n^{n-2}. As an example, note that there are only three trees for K_3, thus the three 3-arborescences A, B, C for K_3 are depicted in the upper part of Figure 1.

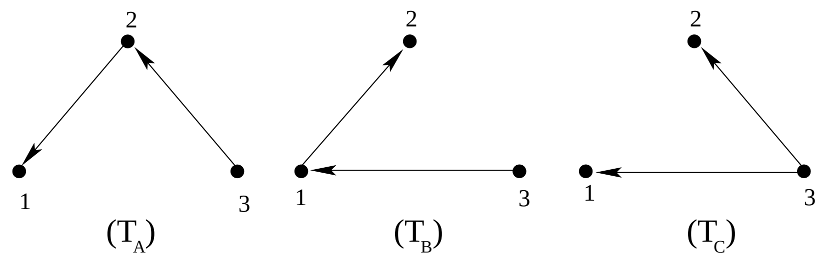

FIGURE 1. arborescences

For any arborescence $T \in \mathbf{Arb}(\ell, n)$ and directed edge e not in T we define the $n \times n$ matrix $W^{T,e}$ whose entries are

$$W^{T,e}(i,j) = \begin{cases} 1, & \text{if } i \neq j \text{ and } (i,j) \in cycle(T+e) \text{ has} \\ & \text{the same orientation as } e, \\ -1, & \text{if } i \neq j \text{ and } (i,j) \in cycle(T+e) \text{ has the} \\ & \text{opposite orientation as } e, \\ -1, & \text{if } i = j \text{ and } i \text{ is a vertex in two edges of } cycle(T+e) \text{ with} \\ & \text{both edges having same orientation as } e., \\ 1, & \text{if } i = j \text{ and } i \text{ is a vertex in two edges of } cycle(T+e), \text{ with} \\ & \text{both edges having opposite orientation of } e., \\ 0, & \text{in all other cases,} \end{cases}$$

where $cycle(T+e)$ denote the unique cycle created by adding e to T.

We also associate to a permutation σ its corresponding permutation matrix, i.e., the matrix whose $(i, \sigma(i))$ entry is 1 and zero otherwise. From now on $W^{T,e}\sigma$ denotes the usual matrix multiplication of $W^{T,e}$ and σ. This way, $z^{t\sigma}$ is it is naturally equal to $\prod z^t_{k,\sigma(k)}$

Theorem 1. *The multivariate generating function for the lattice points of the supporting cone $C_n(\sigma)$ at the vertex σ, for σ a permutation in S_n, is given by*

$$f(tC_n(\sigma),\mathbf{z}) = \sum_{T\in\mathbf{Arb}(\ell,n)} \mathbf{z}^{t\sigma} \prod_{e\notin E(T)} \frac{1}{(1-\prod z^{W^{T,e}\sigma})} \tag{1}$$

where $W^{T,e}(i,j)$ is the matrix specified above in in terms of the unique cycle in $T+e$. and $\mathbf{z}^{t\sigma} = \prod_{k=1}^{n} z^t_{k,\sigma(k)}$. Therefore

$$f(tB_n,\mathbf{z}) = \sum_{\sigma\in S_n} f(tC_n(\sigma),\mathbf{z}). \tag{2}$$

Besides its combinatorial content, our formula for $f(tB_n,\mathbf{z})$ is interesting because of its relation to the problem of computing the volume of B_n (see [2, 3, 5] and the references therein for information on prior work): For a d-dimensional polytope P with integer vertices, for any positive integer t, the number $e(P,t)$ of lattice points contained in the dilation tP is a polynomial function of degree $dim(P)$ in the variable t. This polynomial is called the *Ehrhart polynomial* of P. Note that $e(B_n,t)$ is exactly the number of $n\times n$ matrices with nonnegative integer entries and all row and column sums equal to t. Its leading coefficient is the *normalized volume* of P in units equal to the volume of the fundamental domain of the affine lattice spanned by P (see for example Chapter 4 of [6]). As the second main result of this article we extract, via residue calculations on our formula for $f(tB_n,\mathbf{z})$, the following explicit combinatorial formula for the leading coefficient of $e(B_n,t)$ and thus the volume of B_n:

Theorem 2. *For any choice of fix $\ell\in[n]$, the normalized volume of polytope of doubly stochastic $n\times n$ matrices is given by the formula*

$$vol(B_n) = \frac{1}{((n-1)^2)!} \sum_{\sigma\in S_n} \sum_{T\in\mathbf{Arb}(\ell,n)} \frac{\langle c,\sigma\rangle^{(n-1)^2}}{\prod_{e\notin E(T)}\langle c, W^{T,e}\sigma\rangle}. \tag{3}$$

In the formula T represents one of n^{n-2} ℓ-arborescences in the complete graph K_n and $W^{T,e}(i,j)$ is the $0,-1,1$ matrix associated to the unique oriented cycle $T+e$. Finally, the vector c is any vector such that $\langle c, W^{T,e}\rangle$ is non-zero for all pairs (T,e) of an arborescence T and an arc $e\notin E(T)$.

We also derive an explicit formula for other coefficients of the Ehrhart polynomial. As one nice final application of our Theorem 1 we also discuss similar formulas for semi-magic squares with structural zeros or forbidden entries (i.e. fixed entries are equal zero).

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An introduction to persistent homology

Herbert Edelsbrunner
(joint work with John Harer)

Persistent homology is an algebraic tool for measuring topological features of shapes and functions. It casts the multi-scale organization we frequently observe in nature into a mathematical formalism. This talk gives a short history of persistent homology [1, 2, 3] and present its basic concepts. Besides the mathematics we focus on algorithms and mention the various connections to applications, including to biomolecules, biological networks, data analysis, and geometric modeling.

To convey a feeling for the idea we consider the special case of a Morse function $f : \mathbb{M} \to \mathbb{R}$ on a compact manifold of finite dimension. It has only non-degenerate critical points all of which have distinct critical values. We choose regular values $t_0 < t_1 < \ldots < t_m$ bracketing the m critical values and let $\mathbb{M}_j = f^{-1}(-\infty, t_j]$ be the *sublevel set* containing the first j critical points. Morse theory tells us that $\mathbb{M}_j$ is homotopy equivalent to the result of attaching a p-dimensional cell along its boundary to $\mathbb{M}_{j-1}$, where p is the index of the j-th critical point. As we pass from $\mathbb{M}_{j-1}$ to $\mathbb{M}_j$ there are two possibilities for how homology might change. The first is that the dimension p homology group increases rank by one, that is, $\beta_p(\mathbb{M}_j) = \beta_p(\mathbb{M}_{j-1}) + 1$. The second is that $\beta_{p-1}(\mathbb{M}_j) = \beta_{p-1}(\mathbb{M}_{j-1}) - 1$. To distinguish the two cases, we call the critical point in the first case *positive* since the sum of Betti numbers increases and the critical point in the second case *negative* since the sum of Betti numbers decreases. Persistence gives a pairing between some of the positive critical points of index p and the negative critical points of index $p + 1$. The idea is that a homology class is born at a particular time, dies at a later time, and its persistence is the difference. To make this precise, we use the maps between homology groups induced by the inclusions $\mathbb{M}_i \subseteq \mathbb{M}_j$ whenever $i \leq j$. We say a homology class α is *born at* $\mathbb{M}_i$ if it does not come from a class in $\mathbb{M}_{i-1}$. Furthermore, if α is born at $\mathbb{M}_i$ we say it *dies entering* $\mathbb{M}_j$ if the image of the map induced by $\mathbb{M}_{i-1} \subseteq \mathbb{M}_{j-1}$ does not contain the image of α but the image of the map induced by $\mathbb{M}_{i-1} \subseteq \mathbb{M}_j$ does. If α is born at $\mathbb{M}_i$ and dies entering $\mathbb{M}_j$ then we pair the corresponding critical points, x and y, and say their *persistence* is $f(y) - f(x)$. Homology classes that are born at $\mathbb{M}_i$ and do not die are not paired by this method, but require an extension of the formulation. Persistence is coded in the *persistence diagrams*, $\mathrm{Dgm}_p(f)$, which includes the point $(f(x), f(y))$ whenever x is a positive critical point of index p that is paired with the negative critical point y of index $p+1$. All points live in the half-space above the diagonal and persistence is easily visible as the vertical to that line.

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The g-theorem matrices are totally nonnegative

Alexander Engström

(joint work with Michael Björklund)

We present a proof of a conjecture by Björner [1] on the relation between the f–vectors and g–vectors of simplicial polytopes.

The linear transformation between the f-vector and g-vector of a d–dimensional simplicial polytope is $f = gM_d$ where M_d is a $(\lfloor d/2 \rfloor + 1) \times d$ matrix given by

$$m_{ij} = \binom{d+1-i}{d-j} - \binom{i}{d-j} \qquad i = 0, \ldots, \lfloor d/2 \rfloor, \text{ and } j = 0, \ldots, d-1.$$

According to the g-theorem [2, 3, 7] this is a bijection between the f-vectors of d-dimensional simplicial polytopes and the M-sequences of length $\lfloor d/2 \rfloor + 1$.

Björner [1] used the g–theorem and some properties of the M_d matrices to prove a comparison theorem for f-vectors of simplicial polytopes. Let $S(n, d)$ and $C(n, d)$ denoted the stacked and the cyclic d–polytope on n vertices. The comparison theorem states that if

$$f_r(S(n_1, d)) \leq f_r(P) \leq f_r(S(n_2, d))$$

for a simplicial polytope P and some integers n_1, n_2, and r, then

$$f_s(S(n_1, d)) \leq f_s(P) \leq f_s(S(n_2, d))$$

for all $s > r$. The main technical property of the M_d matrices in the proof was that their 2×2 minors are nonnegative. Björner conjectured that all minors of M_d are nonnegative, and now we will outline the proof of that.

There is a beautiful correspondence by Lindström [6], and Gessel & Viennot [5], between totally nonnegative matrices and enumeration of weighted paths in directed graphs. For a complete description of the correspondence and modern developments, consult for example [4]. The weight of a path in a graph is the product of the weights of its edges. The weight matrix of a weighted acyclic directed graph G, with $\{a_1, a_2, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_l\} \subseteq V(G)$, is a $k \times l$ matrix

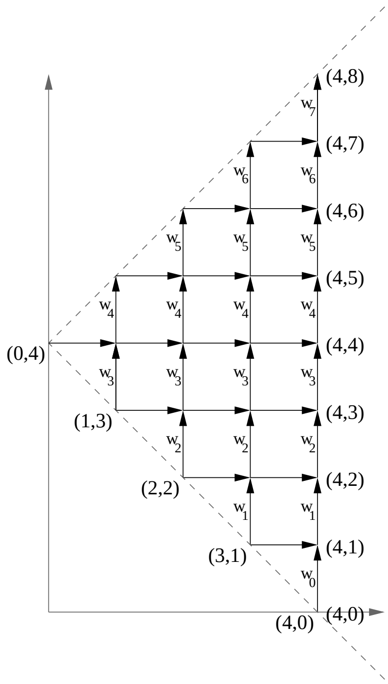

FIGURE 1. The weighted planar graph T_9

whose (i, j)–entry is the sum of the weights of the paths from a_i to b_j. If the graph is planar with nonnegative weights, and $a_k, \ldots, a_2, a_1, b_1, b_2, \ldots b_l$ is in that order on the boundary of a plane embedding of it, then the weight matrix is totally nonnegative.

From any subset V of $\mathbb{Z}^2$ one can construct a planar acyclic directed graph with vertex set V, and edges of types $(x, y) \to (x + 1, y)$ and $(x, y) \to (x, y + 1)$. For any $n \geq 2$ let T_n be the graph with vertex set

$$\{(x, y) \in \mathbb{Z}^2 \mid x \leq \lceil n/2 \rceil - 1, y - x \leq \lfloor n/2 \rfloor, \text{ and } x + y \geq \lceil n/2 \rceil - 1\}.$$

The weight of any horizontal edge of T_n is 1, and the weight of vertical edges $(x, y) \to (x, y + 1)$ is w_y. The graph T_9 is depicted in Figure 1. Define w_y as $(n-y)/(y+1)$. For all $0 \leq i \leq \lceil n/2 \rceil - 1$ and $0 \leq j < n$, let the sum of the weights of the directed paths from $(\lceil n/2 \rceil - 1 - i, i)$ to $(\lceil n/2 \rceil - 1, j)$ be the (i, j)–entry of the $\lceil n/2 \rceil \times n$ weight matrix $M(T_n)$.

Using classic path counting techniques one can show that M_d is a submatrix of $M(T_{d+1})$. Since $M(T_{d+1})$ is totally nonnegative, M_d is also that, and the conjecture is proved.

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On the complexity of Minkowski sums of polytopes

Komei Fukuda
(joint work with Christophe Weibel)

Consider the Minkowski sum of n convex polytopes in R^d $P = P_1 + \cdots + P_n$, and let $v_i = f_0(P_i)$ be the number of vertices of P_i. The fundamental problem is to determine the tight upper bounds for the face numbers $f_k(P)$ for $0 \leq k \geq d-1$ in terms of $v_1, \ldots, v_n$. There are two special cases where the tight bounds are known. When $n = 1$, McMullen's upper bound theorem [2] gives explicit upper bound formula. When $v_1 = v_2 = \cdots = v_n = 2$, tight bounds ("zonotope bounds") are attained by n line segments in general position, see, e.g. [4].

One can easily formulate trivial upper bounds:

$$f_0(P) \leq \Pi_{i=1}^n f_0(P_i) = \Pi_{i=1}^n v_i \tag{1}$$

$$f_k(P) \leq \sum_{\substack{1 \leq s_i \leq v_i \\ s_1 + \cdots + s_n = k+n}} \prod_{i=1}^n \binom{v_i}{s_i}, \forall 0 \leq k \leq d-1. \tag{2}$$

In a recent paper [1], it has been shown that the above upper bounds are tight for the following cases:

- The bound (1) is tight if $n \leq d-1$.
- The bound (2) is tight if $d \geq 4$, $n \leq \lfloor \frac{d}{2} \rfloor$, and $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - n$.

It has been shown by [3] that the first result above is best possible. More precisely, if $n \geq d$, the bound (1) is not tight.

Every nonempty face of a Minkowski sum is *decomposed* uniquely into a sum of faces of each summand. We will say this decomposition is *exact* when the dimension of the sum is equal to the sum of the dimensions of the summands. When all facets have an exact decomposition, we will say the summands are *relatively in* *general position.* The maximal number of k-faces in a sum can be attained when summands are relatively in general position for each k.

It appears extremely difficult to find tight upper bounds for the number of higher dimensional faces, even for the special case of $n = 2$. The only nontrivial tight bound for the number of facets (and edges) we know is for $d = 3$ [1],

$$f_2(P) \leq v_1v_2 + v_1 + v_2 - 6, \tag{3}$$

$$f_1(P) \leq 2v_1v_2 + v_1 + v_2 - 8. \tag{4}$$

The key lemma used to prove these bounds is a linear equation satisfied by the f-vectors of the Minkowski sum and the summands, when the summand polytopes are relatively in general position. This equation is best presented if we use a new vector δ which represents the increment of the number of faces in each dimension:

$$\delta = \delta(P_1, \ldots, P_n) = f(P) - (f(P_1) + \cdots + f(P_n)). \tag{5}$$

Then the equation for $d = 3$ is written as $2\delta_2 - \delta_1 = 0$. One can interpret this equation quite easily if one considers the Minkowski sum operation as the merging of two outer normal fans: whenever a new facet is created in the sum, two new edges are also created.

Our main result is the following extension of the equation to any dimension.

Theorem 1. *Let $P_1, \ldots, P_n$ be d-dimensional polytopes relatively in general position, and $P = P_1 + \cdots + P_n$ their Minkowski sum. Then*

$$\sum_{k=0}^{d-1} (-1)^k k\delta_k = 0.$$

Whether or not this leads to new tight upper bounds is to be investigated.

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Shelling Coxeter-like complexes and sorting on trees

Patricia Hersh

In [1], Babson and Reiner intoduced a common generalization of Coxeter complexes and chessboard complexes which they call *Coxeter-like complexes*, defined as follows. Given any finite group G and any minimal generating set S, define the *parabolic subgroups* of G to be the subgroups generated by subsets of S. Then the

Coxeter-like complex $\Delta(G,S)$ is the unique regular cell complex whose cells are (left) cosets of parabolics, with a cell σ contained in the closure of a cell τ if and only if the coset associated to τ is contained in the coset associated to σ. When G is the symmetric group and S is a generating set of $n-1$ transpositions, S may be described by a tree T on n nodes with an edge $e_{i,j}$ for each transposition $(i,j) \in S$. In this case, $\Delta(G,S)$ is a simplicial complex, denoted Δ_T. When T is a path, Δ_T is the Coxeter complex of type A; when T has a unique vertex v that is not a leaf, then Δ_T is isomorphic to the chessboard complex $M_{\deg(v),\deg(v)+1}$.

More generally, Babson and Reiner introduce type-selected versions as well, by assigning capacities to the vertices of T; letting $\mathbf{m}$ be a vector listing the vertex capacities, denote by $\Delta_{(T,\mathbf{m})}$ the simplicial complex whose facets are labellings of T with values $\{1,\ldots,\sum_{i=1}^{n} \mathrm{cap(v_i)}\}$ in which each vertex v_i receives exactly $\mathrm{cap(v_i)}$ labels, and each label is assigned to exactly one vertex. Lower dimensional faces are obtained by merging neighboring components, with each of the resulting edge eliminations causing the dimension of the face to drop by one.

Our starting point was to prove a conjecture of Babson and Reiner from [1], namely that Δ_T is at least $(n-b-1)$-connected, for T with n nodes, b of which are leaves. In the course of proving the conjecture we did several things:

(1) axiomatized notions of inversions and weak order for the labellings of any fixed tree, viewing permutations as the labellings of a linear array
(2) proved that whenever $(T,\mathbf{m})$ admits such an inversion function, $\Delta_{(T,\mathbf{m})}$ is shellable
(3) constructed inversion function for "distributable capacity trees", i.e., trees with each vertex v satisfying $\mathrm{cap(v)} \geq \mathrm{deg(v)} - 1$
(4) located for any tree T a pure, full-dimensional subcomplex of $\Delta_T^{(n-b)}$ which is isomorphic to the Coxeter-like complex of a distributable capacity tree, hence is shellable
(5) extended the shelling of this subcomplex to a shelling of the entire $(n-b)$-skeleton of Δ_T

This yielded the following result.

Theorem 1. *If T is any tree with n nodes, b of which are leaves, then $\Delta_T^{(n-b)}$ is shellable, implying that Δ_T is at least $(n-b-1)$-connected.*

Step 2 may also be used in the reverse direction to show that certain trees do not admit inversion functions by proving that their associated complexes are not shellable. For example, the construction of Shareshian and Wachs of nonvanishing homology in dimension $\nu_{m,n}$ for chessboard complexes immediately gives such a homological obstruction to shellability for any chessboard complex $M_{m,n}$ with $m \leq n < 2m-1$ (see [8]); this in turn implies that a tree T with exactly one non-leaf vertex v and all leaves having capacity one does not admit an inversion function unless $\mathrm{cap(v)} \geq \mathrm{deg(v)} - 1$. More generally, one may prove nonshellability of Δ_T for many trees T by locating faces whose links are nonshellable chessboard complexes or close relatives which for similar reasons also are not shellable.

It turns out that Δ_T is not shellable for most trees T with low vertex capacities relative to the vertex degrees, giving a new way of quantifying and trying to better understand via topology and combinatorics the fact that sorting is easier on linear arrays than other trees.

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Twisted identities in Coxeter groups

Axel Hultman

Given a Coxeter system (W,S) equipped with an involutive automorphism θ, the set of *twisted identities* is

$$\iota(\theta) = \{\theta(w^{-1})w \mid w \in W\}.$$

For $X \subseteq W$, let $\mathrm{Br}(X)$ denote the subposet of the Bruhat order on W induced by X. The set $\iota(\theta)$ and the poset $\mathrm{Br}(\iota(\theta))$ are important in various ways. Here are some examples:

Example 1. (cf. [6, Example 3.1] and [4, Example 3.2]). *Suppose θ is the involution $\theta : W \times W \to W \times W$ given by $(v,w) \mapsto (w,v)$, where W is any Coxeter group. Observe that $\iota(\theta) = \{(w,w^{-1}) \mid w \in W\}$. Hence, we have a bijection $W \leftrightarrow \iota(\theta)$ in this case. Furthermore, this bijection gives a poset isomorphism $\mathrm{Br}(\iota(\theta)) \cong \mathrm{Br}(W)$. Therefore, (Bruhat orders on) twisted identities generalise (Bruhat orders on) Coxeter groups.*

A *twisted involution* in W is an element which is sent to its inverse by θ. Denote by $\mathfrak{I}(\theta)$ the set of twisted involutions. Clearly,

$$\iota(\theta) \subseteq \mathfrak{I}(\theta) \subseteq W.$$

Example 2. *Consider a connected, reductive linear algebraic group G (over $\mathbb{C}$, say). Let B be a Borel subgroup and $T \subseteq B$ a maximal torus. Given an involutive automorphism $\Theta : G \to G$ which preserves T and B, let K be the fixed point subgroup. By means of left translations, B acts on the symmetric variety G/K. This gives rise to a finite number of orbits that may be ordered by containment of their Zariski closures. Richardson and Springer* [6, 7] *studied this poset V by defining an order preserving map $\varphi : V \to \mathrm{Br}(\mathfrak{I}(\theta))$. Here, the underlying Coxeter group W is the Weyl group N/T (where N is the normaliser of T in G) and θ is induced by Θ.*

In general, φ is neither injective nor surjective. However, $\iota(\theta)$ is always contained in the image. Moreover, φ produces an isomorphism $V \cong \mathrm{Br}(\iota(\theta))$ in certain interesting cases. For instance, with $G = \mathrm{SL}_{2n}$, we may define an automorphism so that $K \cong \mathrm{Sp}_{2n}$ as in [6, Example 10.4]*. The corresponding poset V, governing the cell decomposition of $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$, is then isomorphic to $\mathrm{Br}(\iota(\theta))$, where W is the symmetric group $\mathfrak{S}_{2n}$ and θ is given by conjugation with the longest element in W (i.e. the reverse permutation $i \mapsto 2n+1-i$).*

Example 3. *Let $\mathrm{Fix}(\theta)$ denote the fixed point set of θ. It is known (*[2, 5, 8]*) that $\mathrm{Fix}(\theta)$ is itself a Coxeter group.*

Observe that $\theta(w^{-1})w = \theta(v^{-1})v \Leftrightarrow vw^{-1} \in \mathrm{Fix}(\theta) \Leftrightarrow v \in \mathrm{Fix}(\theta)w$. In other words, there is a bijection between $\iota(\theta)$ and the set of cosets $\mathrm{Fix}(\theta)\backslash W$. Thus, $\iota(\theta)$ can be thought of as a quotient of Coxeter groups.

This work is chiefly devoted to the study of $\mathrm{Br}(\iota(\theta))$. The questions that we have strived to answer emerge from a context which we now briefly describe.

It is a fact that $\mathrm{Br}(W)$ is graded with rank function given by the Coxeter length ℓ. Furthermore, a fundamental result due to Björner and Wachs [1] asserts that (the order complex of) every (open) interval in $\mathrm{Br}(W)$ is a PL sphere. Recent results on $\mathrm{Br}(\mathfrak{I}(\theta))$ produce a similar picture. It is graded with a certain combinatorially explicit rank function ρ [3]. Moreover, every interval in $\mathrm{Br}(\mathfrak{I}(\theta))$ is a PL sphere [4].

In this spirit, it is natural to pose the following problems:

Problem 4. *For which (W, S) and θ is $\mathrm{Br}(\iota(\theta))$ graded?*

Problem 5. *Describe the topology of the intervals of $\mathrm{Br}(\iota(\theta))$.*

We do not know the complete solution to either of the problems. Our main results on $\mathrm{Br}(\iota(\theta))$ are these partial answers:

Theorem 6. *If $s\theta(s)$ is never of odd order for $s \in S$ unless s is a fixed point of θ, then $\mathrm{Br}(\iota(\theta))$ is graded with rank function ρ.*

For example, Theorem 6 applies whenever the Coxeter graph is a tree with a θ-fixed point. By way of contrast, there are examples of (W, S) and θ such that $\mathrm{Br}(\iota(\theta))$ is not graded[1].

Theorem 7. *Under the same conditions on $s\theta(s)$ as above, every interval in $\mathrm{Br}(\iota(\theta))$ is either a PL sphere or $\mathbb{Z}$-acyclic, the former case occurring precisely when the interval coincides with an interval in $\mathrm{Br}(\mathfrak{I}(\theta))$.*

If these conditions are not met, the homotopy type of an interval can be computed in certain special cases. We conjecture that "$\mathbb{Z}$-acyclic" can be replaced by "ball" in Theorem 7.

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Products of foldable triangulations and real roots of polynomial systems

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(joint work with Nikolaus Witte)

This is about how to obtain non-trivial (a priori) lower bounds for the number of real roots of certain sparse polynomial systems. Before we explain what our contribution is we start out by summing up results of Kushnirenko [4] and Soprunova and Sottile [5].

Let $P \subset \mathbb{R}^d_{\geq 0}$ be a d-dimensional lattice polytope contained in the positive orthant, that is, a convex polytope whose vertices have non-negative integer coordinates. Furthermore, let $\lambda : P \to \mathbb{R}$ be a convex lifting function which induces a (regular) triangulation of P (denoted as P^λ) which is *dense*, that is, its vertex set is $P \cap \mathbb{Z}^d$. Finally, we assume that P^λ is *foldable*, that is, its vertex-edge graph is

[1] This happens, for instance, when W is the affine group $\widetilde{A_2}$ and θ is the unique non-trivial automorphism.

$(d+1)$-colorable in the graph-theoretic sense. Since P^λ is a simplicial ball (and hence simply connected), P^λ is foldable if and only if the dual graph of P^λ is bipartite; see [2]. The coloring function $c : P \cap \mathbb{Z}^d \to \{0, 1, \ldots, d\}$ is unique up to renaming the colors.

Example 1. *The subdivision into unit intervals is a regular, dense, and foldable triangulation of the interval* $[0, g] \subset \mathbb{R}$ *for arbitrary* $g \geq 1$*.*

Each choice of a coefficient function $\alpha : \{0, 1, \ldots, d\} \to \mathbb{R}\setminus\{0\} : i \mapsto \alpha_i$ defines a P^λ-*Wronski polynomial*

$$\sum_{m \in P \cap \mathbb{Z}^d} \alpha_{c(m)} x^m \qquad \in \mathbb{R}[x_1, \ldots, x_d] \, .$$

A P^λ-Wronski polynomial has P as its Newton polytope. A *Wronski system* with respect to P^λ is a system of d such Wronski polynomials. The *normalized volume* $\nu(P)$ is the volume of P, multiplied by $d!$. This is always an integer.

Theorem 2 (Kushnirenko [4])**.** *A Wronski system with respect to* P^λ *has at most* $\nu(P)$ *complex roots.*

If the coefficients are, say, chosen at random then this upper bound is attained with probability one.

Soprunova and Sottile [5] define the *signature* $\sigma(P^\lambda)$ as follows: As the dual graph of P^λ is bipartite, we can call the maximal simplices of the triangulation either “black” or “white”. Now the signature is the size difference between the black and the white simplices, where, however, only those maximal simplices count whose normalized volume is an odd integer. The signature of the unit interval subdivision of $[0, g]$ from Example 1 is zero if g is even, and it is one if g is odd.

Theorem 3 (Soprunova and Sottile [5])**.** *A Wronski system with respect to* P^λ *which is* generic *(in the sense that it has exactly* $\nu(P)$ *many complex roots) and which satisfies additional geometric constraints (where also the lifting function* λ *plays its role) has at least* $\sigma(P^\lambda)$ *real roots.*

In the case $d = 1$ (where the additional conditions are always satisfied) of the Example 1 this reduces to the basic fact that a real polynomial of odd degree has at least one real zero.

The purpose of our paper [3] is to investigate Wronski systems with respect to products of lattice polytopes. Of course, there is no canonical choice for a product triangulation, and even the probably most natural candidate (which already appears in Eilenberg and Steenrod [1]), the *simplicial product* $\times_{\mathrm{stc}}$ (or *Cartesian product*), depends on orderings of vertices of the factors.

Theorem 4. *Let* $P \subset \mathbb{R}^m_{\geq 0}$ *and* $Q \subset \mathbb{R}^n_{\geq 0}$ *be non-negative full-dimensional lattice polytopes with regular, dense, and foldable triangulations* P^λ *and* Q^μ *which are nice for some value* $s_0 \in (0, 1]$*. Further choose any color consecutive vertex orderings for* P^λ *and* Q^μ*. Then there is a lifting function* $\omega : (P \times Q) \cap \mathbb{Z}^{m+n} \to \mathbb{R}$ *such that* $(P \times Q)^\omega = P^\lambda \times_{\mathrm{stc}} Q^\mu$ *is nice for* s_0*. Moreover, the number of real solutions of* *any Wronski polynomial system associated with* $(P \times Q)^\omega$ *is bounded from below by*

$$\sigma\left((P \times Q)^\omega\right) = \sigma_{m,n}\ \sigma(P^\lambda)\ \sigma(Q^\mu)\ .$$

In particular, it is true that $P^\lambda \times_{\mathrm{stc}} Q^\mu$ is again regular, dense, and foldable. The notion "nice" in our theorem is related to the geometric conditions in the Soprunova and Sottile result. The number $\sigma_{m,n}$, that is, the signature of the staircase triangulation of the product of an m-simplex with an n-simplex, is explictly known. See the paper [3] for the details.

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A combinatorial application of factorization norms

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(joint work with Adi Shraibman)

Let $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathbb{R}^d$ be vectors of norm ≥ 1. How many choices of sign vectors $(v_1, \ldots, v_n) \in \{\pm 1\}^n$ are there for which $\sum v_i \mathbf{w}_i$ has small norm ? (or falls into a small ball etc.). Such questions go under the names of Littlewood and Offord and have received considerable attention over the last 70 years. As it turns out, the optimal choice is for all the $\mathbf{w}_i$ to reside in a single one-dimensional subspace. It is natural to consider what happens if such configurations are ruled out and the $\mathbf{w}_i$ are assumed to be "spread out" in $\mathbb{R}^d$. Indeed questions of such nature have been studied over the years. We consider here a natural condition that formalizes the assumption that the given vectors are spread out and are not too short. We show that under this condition the answer to the Littlewood-Offord problem is substantially smaller than in general. We are able to establish tight bounds for the problems under consideration.

Let $r, a_1, \ldots, a_n$ be real numbers, and denote by $N_r(\{a_i\}_1^n)$ the set

$$N_r(\{a_i\}_1^n) = \{\mathbf{v} \in \{\pm 1\}^n \text{ such that } |\sum_i a_i v_i| \leq r\}.$$

A famous question of Littlewood and Offord asks for the largest possible size of a set $N_r(\{a_i\}_1^n)$ over all choices of real numbers $a_1, \ldots, a_n \geq 1$. It was proved by Erdős [1] that, for every r, this maximum equals the sum of the $\lfloor r \rfloor + 1$ largest binomial coefficients $\binom{n}{i}$, $i = 1, \dots, n$. A natural extension of the Littlewood-Offord question concerns the largest cardinality of the set

$$N_r(\{\mathbf{w}_i\}_1^n) = \{\mathbf{v} \in \{\pm 1\}^n\} | \sum_i \mathbf{w}_i v_i \in \mathcal{B}\},$$

where the vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{R}^d$ all have norm $\|\mathbf{w}_i\|_2 \geq 1$, and $\mathcal{B}$ is an open ball of radius r. Katona [4] and Kleitman [5] independently proved that, for $r < 1$, the answer remains $\binom{n}{\lfloor n/2 \rfloor}$ for $d = 2$, and Kleitman [6] later showed that the same holds for general d as well. The proofs of Kleitman and Katona initiated the study of M-part Sperner theorems in extremal set theory. Bounds for large d and $r \geq 1$ are also known, e.g. [2].

That the bound $\binom{n}{\lfloor n/2 \rfloor}$ is tight for general d can easily be seen by letting all $\mathbf{w}_i$ be the same vector. In this view, it becomes natural to consider what happens when we stay away from such a degenerate situation. Indeed such questions were investigated by Halász [3]. He considered the same problem under certain restrictions on the possible choices of vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{R}^d$. Further problems in the same general vein were studied more recently by Tao and Vu [9]. They were able to provide a structural result in the spirit of Freiman's theorem which characterizes those sets of integers $a_1, \dots, a_n$ for which $N(\{a_i\}_1^n)$ is large.

We take a different view at this set of problems. If W is the $d \times n$ matrix whose columns are the vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{R}^d$, then $N_r(\{\mathbf{w}_i\}_1^n)$ is the number of sign vectors $\mathbf{v}$ for which $W\mathbf{v} \in \mathcal{B}$. In other words, we are interested in the image of the n-dimensional discrete cube under the linear transformation W. Some normalization is clearly necessary. The original assumption is a lower bound on the ℓ_2-norm of W's columns. One can consider various ways to quantify the requirements that W's columns are not too short (in ℓ_2 norm) and are reasonably "spread out in space". A possible way to capture these requirements is to assume that *the rows in W's generalized inverse are not too long in ℓ_2 norm.* Surprisingly, perhaps, the conclusion under this assumption is much stronger than in previous Littlewood-Offord-type theorems.

Theorem 1. *Let W be a real $d \times n$ matrix with rank n. Assume that every row in $(W^tW)^{-1}W^t$ is a vector of ℓ_2 norm ≤ 1. Then for every $r > 0$*

$$|\{\mathbf{v} \in \{\pm 1\}^n \text{ such that } \|W\mathbf{v}\|_2 \leq r\}| \leq \sum_{i \leq r^2} \binom{n}{i}.$$

The bound is tight in the sense that there are matrices W as above for which the number of such sign vectors is $\sum_{i \leq cr^2} \binom{n}{i}$ for some absolute constant $c > 0$.

We point out that the bound in the theorem is independent of d and also is much smaller than in Littlewood-Offord-type theorems as long as $r \ll \sqrt{n}$.

Theorem 1 can be extended in several ways. Namely, it extends verbatim when the target set is *any* ball of radius r. Also, if we combine the vectors with $(0, 1)$ coefficients rather than with ± 1, a similar statement holds.

In my talk I presented the (very brief) proof. Its main ingredient are (i) the following notion from Banach spaces theory, see for example [10].

Definition 2. *Let B be a real matrix, $\gamma_2(B)$ is defined by*

$$\gamma_2(B) = \min_{XY=B} \|X\|_{2\to\infty}\|Y\|_{1\to 2}.$$

Here $\|Z\|_{p\to q}$ is the operator norm of the $m \times n$ real matrix Z viewed as a linear operator from l_p^n to l_q^m. In particular $\|X\|_{2\to\infty}$ (resp. $\|Y\|_{1\to 2}$) is the largest ℓ_2 norm of any row in X (resp. column in Y).

It is not hard to express γ_2 of a given matrix as the optimum of a certin simple semi-definite program. This in turn implies that $\gamma_2(\cdot)$ *is a norm*, a fact that we employ in the proof.

(ii) Sauer's lemma [8] on matrices with bounded VC dimension.

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Periodic foams and simplicial manifolds with small valence

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(joint work with Thom Sulanke and John M. Sullivan)

Chemist/metallurgists have found periodic foam structures in various materials, such as transition metal alloys, salt and gas hydrates [2, 3, 4]. Hereby, the Voronoi diagram for the positions of the occurring atoms forms a foam, which is a collection of CMC (constant mean curvature) surfaces that meet according to Plateau's rule (three cells around an edge, four cells meeting at a vertex). In turn, the Delaunay triangulation for the positions of the atoms in a periodic foam is built of nearly regular tetrahedra and is often called a tetrahedrally close-packed (TCP) structure.

All observed periodic foams in nature are built of only four types of cells, the dodecahedron with pentagons only and three further cells that have 2, 3, and 4 additional 6-gons as facets, respectively. Kusner and Sullivan [5] (see also [8, 9]) defined a TCP triangulation of a (closed) 3-manifold to be a triangulation s.t. each edge has valence 5 or 6 (i.e., each edge lies in 5 or 6 tetrahedra), and no triangle has two edges of valence 6.

According to Brady, McCammond, and Meier [1], every (closed, orientable) 3-manifold can be triangulated with edge valences at most 6. Sullivan asked in [10] whether every 3-manifold even has a TCP triangulation?

So far, TCP triangulations were known for the 3-torus (which give periodic foams) as well as for products of arbitrary surfaces with S^1 [9] and some spherical 3-manifolds (the boundary of the 600-cell and quotients thereof).

We add to this list a TCP triangulation of the Poincaré homology 3-sphere and of the octahedral space, respectively. Moreover, we determine all triangulated 3-manifolds with edge valences at most 5 (there are 4787 examples, $4761 \times S^3$, $22 \times \mathbb{R}\mathbf{P}^3$, $2 \times L(3,1)$, $1 \times L(4,1)$, and $1 \times S^3/Q$; cf. also [6]) as well as all proper 3-pseudomanifolds with edge valences at most 5 (there are 41 such examples, all suspensions of the projective plane $\mathbb{R}\mathbf{P}^2$).

We also show that (strongly connected) d-pseudomanifolds with codimension two faces of valence at most 4 are joins of boundaries of simplices.

(It was independently proven by Matveev and Shevchishin [7] that all triangulated 3-manifolds with edge degrees at most five are spherical and have at most 600 tetrahedra.)

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Large monochromatic components in two-colored grids

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(joint work with Aleš Přívětivý)

1. Introduction

For a graph G and an integer k, we define $\xi_k(G)$ as the smallest m such that there exists a coloring of the vertices of G by k colors with no monochromatic connected subgraph having more than m vertices. This graph parameter has been investigated in several papers (e.g., [7], [1], [6] and references therein).

Here we consider the ξ_2 for a geometrically defined class of graphs—cubic grids with diagonals. As a starting point, let us consider a planar graph T_n^2 obtained by adding a diagonal to every inner face of an $n \times n$ square grid. The well-known planar HEX lemma asserts that for any red-blue coloring of the vertices, there exists a red path connecting the top and bottom sides or a blue path connecting the left and right sides [5]. In particular, $\xi_2(T_n^2) \geq n$.

Linial asked for a d-dimensional analog of this statement: What can be said about ξ_2 for a d-dimensional grid of side n? Since the d-dimensional grid in itself is bipartite, we have to add some diagonals in order to get a meaningful question, similar to the planar case. There are at least two reasonable ways of adding diagonals.

1.1. **Triangulated grids.** First we consider a d-dimensional *triangulated grid* T_n^d, which is defined geometrically as follows. We begin with the d-dimensional solid cube $[1,n]^d$ and we subdivide it into the grid of unit cubes; each unit cube is of the form $[i_1, i_1+1] \times [i_2, i_2+1] \times \cdots \times [i_d, i_d+1]$, $1 \leq i_1, i_2, \ldots, i_d \leq n-1$. Then we triangulate each of the unit cubes in such a way that the simplices of all of these triangulations taken together form a triangulation of $[1,n]^d$ (here a triangulation is a simplicial complex in the sense of algebraic topology). Then T_n^d is the graph of such a triangulation; that is, the vertex set is $[n]^d$ (where $[n]$ is the shorthand for the set $\{1, 2, \ldots, n\}$), and the edges of T_n^d are the edges of the triangulation. Thus, for given d and n, T_n^d is not defined uniquely, but rather it stands for an arbitrary graph from a (finite) family.

The obvious "horizontal layer" coloring, by the parity of the first coordinate, shows that $\xi_2(T_n^d) \leq n^{d-1}$, and one might perhaps suspect this coloring to be optimal. However, for $d \geq 3$, there are better colorings at least for *some* of the possible triangulated grids. Namely, let us define the *ith diagonal layer* $L_n^d(i)$ by $L_n^d(i) = \{x \in [n]^d : \sum_{j=1}^d x_j = i\}$, and the *diagonal-layer coloring* by coloring all $L_n^d(i)$ with even i red and all $L_n^d(i)$ with odd i blue. It can be checked that there are triangulated grids T_n^d with no edges connecting $L_n^d(i)$ and $L_n^d(i')$ with $|i-i'| \geq 2$, and for these, the largest monochromatic connected component is the largest diagonal layer. For $d=3$, for example, $\max_i |L_n^d(i)|$ is approximately $\frac{3}{4}n2$.

We prove that the diagonal-layer coloring is not far from optimal in the worst case. To state the result precisely, let us define, for $\alpha \in [0,1]$ and given n and d, $i_\alpha = \min\{i : \sum_{j \leq i} |L_n^d(j)| > \alpha n^d\}$. It can be checked that $i_{1/2} = \left\lceil \frac{(n+1)d}{2} \right\rceil$ and that $L_n^d(i_{1/2})$ is either the single largest diagonal layer or one of the two largest diagonal layers. We prove the following.

Theorem 1. *For every $d \geq 3$ and $n \geq 3$, we have $\xi_2(T_n^d) \geq |L_n^d(i_{2/3})| - |L_n^{d-1}(i_{1/2})|$. The last quantity is of order $\frac{n^{d-1}}{\sqrt{d}}$.*

We conjecture that the right answer is $|L_n^d(i_{1/2})|$; that is, the diagonal-layer coloring is optimal in the worst case.

1.2. **Grid with all diagonals.** Next, we define the graph D_n^d, the d-dimensional *grid with diagonals*, as the graph with vertex set $[n]^d$ and edge set $\{\{u,v\} : u,v \in [n]^d, \|u-v\|_\infty \leq 1\}$, where $\|u-v\|_\infty = \max_i |u_i - v_i|$. In this case we show that the horizontal-layer coloring is almost optimal (the remaining lower-order term is probably an artifact of the proof method):

Theorem 2. *We have $\xi_2(D_n^d) \geq n^{d-1} - d^2 n^{d-2}$.*

We should remark that there is a different d-dimensional generalization of the planar HEX lemma appearing e.g. in Gale [5] (also see Linial and Saks [8] for a different proof and an application in computer science). It asserts that if the vertices of T_n^d are colored by d colors, then there is a monochromatic path connecting two opposite facets of the cube $[1,n]^d$. In particular, $\xi_d(T_n^d) \geq n$.

It seems natural to conjecture that $\xi_k(T_n^d)$ is of order n^{d-k+1} for d fixed and all $k = 3,4,\ldots,d-1$ (there are simple "column" colorings providing the upper bounds). This problem appears challenging and the techniques used in the present paper for the case $k=2$ do not seem applicable.

2. Outline of the proofs

By a *separator* in a graph G we mean a subset $S \subseteq V(G)$ whose removal disconnects G into components of size at most $\frac{1}{2}|V(G)|$. The main part of the proof of both Theorem 1 and Theorem 2 is the following proposition:

Proposition 3. *Let $G = G(K)$ be the graph of a finite simply connected simplicial complex K. For an arbitrary coloring of $V(G)$ by two colors there exists a monochromatic separator $S \subseteq V(G)$ such that the subgraph induced in G by S is connected.*

Here by a graph of a simplicial complex K we mean the 1-skeleton (edges of the graph correspond to 1-dimensional simplices of K), and we call K simply connected if the polyhedron of K is simply connected; that is, every closed loop in it can be contracted to a point (see, e.g., [10] for more background). In particular, the assumptions of Proposition 3 apply to the graphs T_n^d.

Theorems 1 and 2 are derived from it using suitable *isoperimetric inequalities.* For a graph G and a set $A \subseteq V(G)$ the *vertex boundary* of A in G is the set $\text{vert-bd}_G(A) = \{v \in V(G) : v \notin A, \{u,v\} \in E(G) \text{ for some } u \in A\}$, and the *edge boundary* of A in G is $\text{edge-bd}_G(A) = \{\{u,v\} \in E(G) : u \in A, v \notin A\}$. A vertex-isoperimetric inequality for G estimates the minimum possible size of $\text{vert-bd}_G(A)$ as a function of $|A|$, and similarly for an edge-isoperimetric inequality and $\text{edge-bd}_G(A)$.

Let G_n^d be the graph of the ordinary grid (not triangulated). Explicitly, $V(G_n^d) = [n]^d$ and $u, v \in V(G_n^d)$ are connected by an edge if $\|u - v\|_1 = \sum_{i=1}^d |u_i - v_i| = 1$. We have the following two vertex-isoperimetric inequalities:

Proposition 4. *(i) For any set* $A \subseteq [n]^d$ *with* $\frac{1}{4}n^d \leq |A| \leq \frac{3}{4}n^d$ *we have* $|\text{vert-bd}_{D_n^d}(A)| \geq n^{d-1} - d^2 n^{d-2}$.

(ii) For every $\alpha \in (0, \frac{1}{2})$ *and every* $A \subseteq [n]^d$ *with* $\alpha n^d \leq |A| \leq (1-\alpha)n^d$ *we have* $|\text{vert-bd}_{G_n^d}(A)| \geq |L_n^d(i_{1-\alpha})| - |L_n^{d-1}(i_{1/2})|$.

Part (i) is based on an edge-isoperimetric inequality from [2], while (ii) follows from vertex-isoperimetric inequality due to [3]. The derivations are not too difficult but rather lengthy and technical.

To prove Theorem 1, we consider a red-blue coloring of the vertices of T_n^d, and we choose a monochromatic connected separator S in it according to Proposition 3. We may assume $|S| \leq \frac{1}{3}n^d$, for otherwise, we would be done. Then by a standard argument, $[n]^d \setminus S$ can be partitioned into sets A and B so that no edge connects A to B and $\frac{1}{3}n^d \leq |A| \leq \frac{2}{3}n^d$. Since $S \supseteq \text{vert-bd}_{T_n^d}(A) \supseteq \text{vert-bd}_{G_n^d}(A)$, Proposition 4 (ii) shows that $|S| \geq |L_n^d(i_{2/3})| - |L_n^{d-1}(i_{1/2})|$ as claimed in Theorem 1. The asymptotic estimate for this quantity given in the theorem can be obtained from a quantitative version of the central limit theorem due to Feller [4] by a technical calculation. Theorem 2 follows from Proposition 4 (i) in almost the same way.

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Topological Helly type theorems

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(joint work with Gil Kalai)

Let X be a simplicial complex on the vertex set V, and let $\mathbb{K}$ be a field. X is *d-Leray* over $\mathbb{K}$, if $\tilde{\mathrm{H}}_i(Y;\mathbb{K}) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Let $L_{\mathbb{K}}(X)$ denote the minimal d such that X is d-Leray over $\mathbb{K}$. E.g. $\mathrm{L}_{\mathbb{K}}(X) = 0$ iff X is a simplex. $\mathrm{L}_{\mathbb{K}}(X) \leq 1$ iff X is the clique complex of a chordal graph.

The class $\mathfrak{L}^d_{\mathbb{K}}$ of d-Leray complexes over $\mathbb{K}$ arises naturally in the context of Helly type theorems [3]. The *Helly number* $\mathrm{h}(\mathfrak{F})$ of a family of sets $\mathfrak{F}$, is the minimal positive integer h such that if a finite subfamily $\mathfrak{K} \subset \mathfrak{F}$ satisfies $\bigcap \mathfrak{K}' \neq \emptyset$ for all $\mathfrak{K}' \subset \mathfrak{K}$ of cardinality $\leq h$, then $\bigcap \mathfrak{K} \neq \emptyset$. The *nerve* $N(\mathfrak{F})$ of a family of sets $\mathfrak{F}$, is the simplicial complex whose vertex set is $\mathfrak{F}$ and whose simplices are all $\mathfrak{F}' \subset \mathfrak{F}$ such that $\bigcap \mathfrak{F}' \neq \emptyset$. It is easy to see that for any field $\mathbb{K}$

$$\mathrm{h}(\mathfrak{F}) \leq 1 + \mathrm{L}_{\mathbb{K}}(N(\mathfrak{F})).$$

For example, if $\mathfrak{F}$ is a finite family of convex sets in $\mathbb{R}^d$, then by the Nerve Lemma $N(\mathfrak{F})$ is d-Leray over $\mathbb{K}$, hence follows Helly's Theorem: $\mathrm{h}(\mathfrak{F}) \leq d+1$. This argument actually proves the Topological Helly Theorem: If $\mathfrak{F}$ is a finite *good cover* in $\mathbb{R}^d$ (i.e. a finite family of closed sets in $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathfrak{F}$ is either empty or contractible), then $\mathrm{h}(\mathfrak{F}) \leq d+1$.

Nerves of families of convex sets however satisfy a much stronger combinatorial property called *d-collapsibility* [8], that leads to some of the deeper extensions of Helly's Theorem. It is of considerable interest to understand which combinatorial properties of nerves of families of convex sets in $\mathbb{R}^d$ extend to arbitrary d-Leray complexes. In the following we describe some results in this direction.

The Colorful Helly Theorem and its dual version, the colorful Carathéodory Theorem, are powerful results with numerous applications in discrete geometry.

Theorem 1 (Lovász, Bárány [2])**.** *Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_{d+1}$ be $d+1$ finite families of convex sets in $\mathbb{R}^d$. If $\bigcap_{i=1}^{d+1} F_i \neq \emptyset$ for all choices of $F_1 \in \mathfrak{F}_1, \ldots, F_{d+1} \in \mathfrak{F}_{d+1}$ then $\bigcap_{F \in \mathfrak{F}_i} F \neq \emptyset$ for some $1 \leq i \leq d+1$.*

In [4] we prove a matroidal topological extension of Theorem 1.

Theorem 2 ([4])**.** *Let X be a simplicial complex on V and let M be a matroidal complex on V such that $M \subset X$. Then there exists a simplex $\tau \in X$ such that $\rho_M(V - \tau) \leq \mathrm{L}_{\mathbb{K}}(X)$.*

Theorem 2 implies the following

Theorem 3 ([4])**.** *Let $\mathfrak{F} = \{F_1, \ldots, F_n\}$ be a good cover in $\mathbb{R}^d$ and suppose M is a matroid on $[n] = \{1, \ldots, n\}$ such that $\bigcap_{i \in I} F_i \neq \emptyset$ for all independent sets $I \in M$. Then there exists a $\tau \subset [n]$ such that $\bigcap_{i \in \tau} F_i \neq \emptyset$ and $\rho_M(V - \tau) \leq d$.*

Let $\mathfrak{F}$ be a family of sets and let r be a positive integer. A family of sets $\mathfrak{G}$ is an *$(\mathfrak{F}, r)$-family* if for any finite $\mathfrak{G}' \subset \mathfrak{G}$, the intersection $\bigcap \mathfrak{G}'$ is a union of at most r disjoint sets from $\mathfrak{F}$. The following result was conjectured by Grünbaum and Motzkin, and proved by Amenta [1].

Theorem 4 (Amenta)**.** *Let $\mathfrak{F}$ be a family of compact convex sets in $\mathbb{R}^d$. Then for any $(\mathfrak{F}, r)$-family $\mathfrak{G}$*

$$\mathrm{h}(\mathfrak{G}) \leq r(d+1) \ .$$

In [6] we study Leray numbers of projections. One consequence of our results is the following

Theorem 5 ([6])**.** *Let $\mathfrak{F}$ be a finite good cover in some topological space. Then for any $(\mathfrak{F}, r)$-family $\mathfrak{G}$*

$$\mathrm{L}_{\mathbb{Q}}(N(\mathfrak{G})) \leq r\mathrm{L}_{\mathbb{Q}}(N(\mathfrak{F})) + r - 1 \ .$$

Theorem 5 implies a topological extension of Amenta's theorem.

Theorem 6 ([6])**.** *If $\mathfrak{F}$ is a good cover in $\mathbb{R}^d$, and $\mathfrak{G}$ is an $(\mathfrak{F}, r)$-family then*

$$\mathrm{h}(\mathfrak{G}) \leq r(d+1) \ .$$

Estimates of Leray numbers of projections also lead to the following

Theorem 7 ([5])**.** *Let $X_1, \ldots, X_r$ be simplicial complexes on the same finite vertex set. Then*

$$\mathrm{L}_{\mathbb{K}}\Big(\bigcap_{i=1}^{r} X_i\Big) \leq \sum_{i=1}^{r} \mathrm{L}_{\mathbb{K}}(X_i) \tag{1}$$

$$\mathrm{L}_{\mathbb{K}}\Big(\bigcup_{i=1}^{r} X_i\Big) \leq \sum_{i=1}^{r} \mathrm{L}_{\mathbb{K}}(X_i) + r - 1 \ . \tag{2}$$

Theorem 7 was first conjectured in a different but equivalent form by Terai [7], in the context of monomial ideals . Let $S = \mathbb{K}[x_1, \ldots, x_n]$ and let M be a graded S-module. Let $\beta_{ij}(M) = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(\mathbb{K}, M)_j$ be the graded Betti numbers of M. The *Castelnuovo-Mumford regularity* of M is the minimal $\rho = \mathrm{reg}(M)$ such that $\beta_{ij}(M)$ vanish for $j > i + \rho$.

For a simplicial complex X on $[n] = \{1, \ldots, n\}$ let I_X denote the ideal of S generated by $\{\prod_{i \in A} x_i : A \notin X\}$. Fundamental results of Hochster on the Betti numbers of I_X, imply that $\mathrm{reg}(I_X) = \mathrm{L}_{\mathbb{K}}(X) + 1$. The case $r = 2$ of Theorem 7 is therefore equivalent to the following result conjectured by Terai [7].

Theorem 8 ([5])**.** *Let X and Y be simplicial complexes on the same vertex set. Then*

$$\mathrm{reg}(I_X + I_Y) = \mathrm{reg}(I_{X \cap Y}) \leq \mathrm{reg}(I_X) + \mathrm{reg}(I_Y) - 1$$
$$\mathrm{reg}(I_X \cap I_Y) = \mathrm{reg}(I_{X \cup Y}) \leq \mathrm{reg}(I_X) + \mathrm{reg}(I_Y) \ .$$

Theorem 8 can also be formulated in terms of projective dimension.

Theorem 9 ([5]).

$$\mathrm{pd}(I_X \cap I_Y) \leq \mathrm{pd}(I_X) + \mathrm{pd}(I_Y)$$
$$\mathrm{pd}(I_X + I_Y) \leq \mathrm{pd}(I_X) + \mathrm{pd}(I_Y) + 1\ .$$

Theorems 8 and 9 extend to arbitrary monomial ideals using standard polarization arguments.

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Cellular resolutions of multiplier ideals of sums

Ezra Miller

(joint work with Shin-Yao Jow)

Let X be a smooth complex algebraic variety and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. Applications in algebraic geometry of the multiplier ideal sheaves

$$\mathcal{J}(\mathfrak{a}^\alpha) = \mathcal{J}(X, \mathfrak{a}^\alpha) \subseteq \mathcal{O}_X$$

for real numbers $\alpha > 0$ have led to investigations of their behavior with respect to natural algebraic operations. For example, Demailly, Ein, and Lazarsfeld [2] proved that given two ideal sheaves $\mathfrak{a}_1$ and $\mathfrak{a}_2$, one has

$$\mathcal{J}((\mathfrak{a}_1\mathfrak{a}_2)^\alpha) \subseteq \mathcal{J}(\mathfrak{a}_1^\alpha)\mathcal{J}(\mathfrak{a}_2^\alpha).$$

For the subtler case of sums, on the other hand, Mustaţă [6] showed that

$$\mathcal{J}((\mathfrak{a}_1 + \mathfrak{a}_2)^\alpha) \subseteq \sum_{0 \leq t \leq \alpha} \mathcal{J}(\mathfrak{a}_1^{\alpha - t})\mathcal{J}(\mathfrak{a}_2^t),$$

and Takagi [7] later refined this to

$$\mathcal{J}((\mathfrak{a}_1 + \mathfrak{a}_2)^\alpha) = \sum_{0 \leq t \leq \alpha} \mathcal{J}(\mathfrak{a}_1^{\alpha - t}\mathfrak{a}_2^t), \tag{1}$$

where he proved it more generally when X is $\mathbb{Q}$-Gorenstein.

Takagi used characteristic p methods to deduce (1), which makes his work distinctly different from [2] and [6], where the arguments proceed by geometric techniques such as log resolutions and sheaf cohomology. Our purpose is to show that such geometric techniques, combined with combinatorial methods from topology and commutative algebra, can recover Takagi's equality (1) and generalize it. As a consequence, we derive a new proof of Howald's formula for multiplier ideals of monomial ideals [3], and demonstrate how it can be reformulated to hold for all ideals. Our main results center around the following.

Theorem 1. *Fix nonzero ideal sheaves* $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}$ *on a* $\mathbb{Q}$*-Gorenstein complex variety* X. *For any real* $\alpha, \beta > 0$, *there is a resolution* $0 \to \mathcal{J}_r \to \cdots \to \mathcal{J}_0 \to 0$ *of the multiplier ideal* $\mathcal{J}((\mathfrak{a}_1 + \cdots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta)$ *by sheaves* $\mathcal{J}_i$ *that are finite direct sums of multiplier ideals of the form* $\mathcal{J}(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$ *for various nonnegative* $\lambda \in \mathbb{R}^r$ *with* $\sum_{i=1}^r \lambda_i = \alpha$. *Every distinct ideal sheaf of that form appears as a summand of* $\mathcal{J}_0$.

Part of the final claim of Theorem 1 is that there are only finitely many distinct multiplier ideals of the form $\mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$ for $\lambda_1 + \cdots + \lambda_r = \alpha$. In particular, the surjection $\mathcal{J}_0 \twoheadrightarrow \mathcal{J}(X, (\mathfrak{a}_1 + \cdots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta)$ in our resolution implies the following (finite) summation formula.

Corollary 2. $\mathcal{J}(X, (\mathfrak{a}_1 + \cdots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta) = \sum_{\lambda_1 + \cdots + \lambda_r = \alpha} \mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$.

Corollary 2 reduces the calculation of the multiplier ideals of arbitrary polynomial ideals to those of principal ideals. In the special case of a monomial ideal $\mathfrak{a} = \langle \mathbf{x}^{\gamma_1}, \ldots, \mathbf{x}^{\gamma_r} \rangle$, generated by the monomials in the polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$ with exponent vectors $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^d$, the summation formula becomes particularly explicit. For a subset $\Gamma \subseteq \mathbb{R}^d$, let $\operatorname{conv} \Gamma$ denote its convex hull. By the *integer part* of a vector $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$, we mean the vector $(\lfloor \nu_1 \rfloor, \ldots, \lfloor \nu_d \rfloor) \in \mathbb{Z}^d$ whose entries are the greatest integers less than or equal to the coordinates of ν.

Corollary 3 (Howald)**.** *If* $\mathfrak{a} = \langle \mathbf{x}^{\gamma_1}, \ldots, \mathbf{x}^{\gamma_r} \rangle$ *is a monomial ideal in* $\mathbb{C}[x_1, \ldots, x_d]$, *then* $\mathcal{J}(\mathfrak{a}^\alpha)$ *is generated by the monomials in* $\mathbb{C}[x_1, \ldots, x_d]$ *whose exponent vectors are the integer parts of the vectors in* $\operatorname{conv}\{\alpha \cdot \gamma_1, \ldots, \alpha \cdot \gamma_r\} \subseteq \mathbb{R}^d$.

Proof. Using the log resolution definition of multiplier ideals (not reproduced here; see [4]) and Corollary 2 with $\mathfrak{a}_j = \langle \mathbf{x}^{\gamma_j} \rangle$, it suffices to note that the divisor of a monomial has simple normal crossings, so no log resolution is necessary. □

It is easy to check that for $\alpha = 1$, the vectors in the conclusion of Corollary 3 are precisely those from Howald's result [3], namely those $\gamma \in \mathbb{N}^d$ such that $\gamma + (1, \ldots, 1)$ lies interior to the convex hull of all exponents of monomials in $\mathfrak{a}$.

Our approach to Theorem 1 is to construct a specific resolution satisfying the hypotheses, including the part about $\mathcal{J}_0$. The resolution we construct is *cellular*, in a sense generalizing the manner in which resolutions of monomial ideals can be cellular [1]; see [5, Chapter 4] for an introduction. In general, a complex in any abelian category could be called *cellular* if each homological degree is a direct sum indexed by the faces of a CW-complex, and the boundary maps are determined in a natural way from those of the CW-complex. An elementary way to phrase this in the present context is as follows.

Theorem 4. *Resume the notation from Theorem 1. There is a triangulation* Δ *of the simplex* $\{\lambda \in \mathbb{R}^r \mid \sum_{i=1}^{r} \lambda_i = \alpha\}$ *such that we can take*

$$\mathcal{J}_i = \bigoplus_{\sigma \in \Delta_i} \mathcal{J}_\sigma$$

to be a direct sum indexed by the set Δ_i *of* i*-dimensional faces* $\sigma \in \Delta$*, and the differential of* $\mathcal{J}_\bullet$ *is induced by natural maps between ideal sheaves, using the signs from the boundary maps of* Δ*. If* $\lambda \in \Delta_0$ *is a vertex then* $\mathcal{J}_\lambda = \mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$*.*

Comparing the final sentences of Theorems 1 and 4, a key point is that every possible multiplier ideal of the form $\mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$ occurs at some vertex $\lambda \in \Delta_0$. Writing down what it means for two such multiplier ideals to coincide, this stipulation provides strong hints as to the choice of triangulation.

The proof of exactness for the cellular resolution in Theorem 4 proceeds by lifting the problem to an appropriate log resolution $X' \to X$. Over X', we resolve the lifted ideal sheaf by a complex (of locally principal ideal sheaves in $\mathcal{O}_{X'}$) that is, analytically locally at each point of X', a cellular free complex over a polynomial ring. This cellular free complex turns out to be a cellular free resolution of an appropriate monomial ideal. Having cellularly resolved the lifted sheaf over X', the desired cellular resolution over $\mathcal{O}_X$ is obtained by pushing forward to X, using local vanishing for multiplier ideals [4, Theorem 9.4.4]. Thus Theorems 1 and 4 constitute a certain global version of cellular free resolutions of monomial ideals.

The acyclicity of the monomial cellular free resolution reduces to a simplicial homology vanishing statement for simplicial complexes that are obtained by deleting boundary faces from certain contractible manifolds-with-boundary. We deduce this vanishing statement from the following more general result, which is of independent interest. Its hypothesis is satisfied by the barycentric subdivision of any polyhedral homology-manifold-with-boundary. (To *delete a simplex* σ from a simplicial complex M means to remove from M every simplex containing σ.)

Proposition 5. *Fix a simplicial complex* M *whose geometric realization* $|M|$ *is a homology-manifold with boundary* ∂M*. Assume* M *satisfies the following condition:*

if σ *is a face of* M*, then* $\sigma \cap |\partial M|$ *is a face of* M*.*

Then deleting any collection of boundary simplices from M *results in a simplicial subcomplex whose (reduced) homology is canonically isomorphic to that of* M*.*

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On embeddability of skeleta of spheres

Eran Nevo
(joint work with Uli Wagner)

A well known result by Van Kampen [6] and by Flores [2] asserts that

Theorem 1. *The d-skeleton of the $(2d+2)$-simplex does not embed in the $2d$-sphere.*

We consider two generalizations of this theorem:

Theorem 2. *Let K be the $\lceil \frac{d-1}{2} \rceil$-skeleton of a triangulated d-sphere. Then K does not embed in the $(d-1)$-sphere.*

Remark 3. *The case d odd and K is the $\frac{d-1}{2}$-skeleton of the boundary of the $(d+1)$-simplex is Theorem 1. Note that this generalization is obvious if the triangulated d-sphere is piecewise linear.*

Conjecture 4. *Let K be the d-skeleton of a triangulated $2d$-sphere and let T be a missing d-face of K, i.e. $T \notin K$ and the boundary of T is contained in K. Then $K \cup \{T\}$ does not embed in the $2d$-sphere.*

Remark 5. *The case where the triangulated sphere is the one obtained from the boundary of the $(2d+1)$-simplex by Stellarly subdividing at a d-face is Theorem 1.*

Theorem 2 follows from the following theorem, inspired by Lovász and Schrijver [4, Theorem 1], which deals with antipodal faces in convex polytopes. Its proof is based on Van Kampen's obstruction.

Theorem 6. *Let K be a triangulated d-sphere and let $f : K \longrightarrow \mathbf{R}^d$ be a generic continuous map. Then there exist two disjoint faces $F, F' \in K$ such that $\dim(\mathrm{F}) + \dim(\mathrm{F}') = \mathrm{d}$ and $|f(F) \cap f(F')|$ is odd.*

The following weakening of Conjecture 4 is true: $K \cup \{T\}$ is not the d-skeleton of a triangulation of the $2d$-sphere. We can prove a bit more, as indicated in the theorem below.

Theorem 7. *Let S be a triangulated $2d$-sphere and let L be an induced subcomplex of S which is homeomorphic to the $(d-1)$-sphere. Let B be a triangulated d-ball with boundary combinatorially isomorphic to L by $g : \partial(B) \longrightarrow L$. Then the glued complex $skel_d(S) \cup_g B$ is not the d-skeleton of a triangulated $2d$-sphere.*

This result is inspired by the proof of Dancis [1] that a triangulated compact $2d$-manifold without boundary and with vanishing d-homology is determined by its d-skeleton, which goes back to Perles who considered the polytope's boundary case.

For the special case where the triangulated $2d$-sphere is neighborly, Conjecture 4 would follow (by shiftedness) from the following strong conjecture of Kalai and Sarkaria [3, Conjecture 27]:

Conjecture 8. *Let K be a simplicial complex on n vertices which is embeddable in the $(d-1)$-sphere. Then its algebraic shifting (symmetric or exterior) satisfies $\Delta(K) \subseteq \Delta(d,n)$, where $\Delta(d,n)$ is the symmetric shifting of the cyclic d-polytope on n vertices (its combinatorics is known).*

We verify this conjecture for a special case:

Example 9. *Let T be a missing d-face in the cyclic $(2d+1)$-polytope on n vertices, denoted by $C(2d+1,n)$, and let $K = \mathrm{skel_d(C(2d+1,n))} \cup \{\mathrm{T}\}$. Then $\Delta(K) \nsubseteq \Delta(2d+1,n)$ and K is not embeddable in the $2d$-sphere.*

The proof of this case relies on the notion of *higher minors*, introduced in [5]. One shows that the d-skeleton of the $(2d+2)$-simplex is a minor of K, hence by [5], Corollary 1.2, K is not embeddable in the $2d$-sphere.

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Face numbers of centrally symmetric polytopes

Isabella Novik

To characterize the numbers that arise as the face numbers of simplicial complexes of various types is a problem that has intrigued many researchers over the last half century and has been solved for quite a few classes of complexes, among them the class of all simplicial complexes [10, 9] as well as the class of all simplicial polytopes [2, 18]. One of the precursors of the latter result was the Upper Bound Theorem (UBT, for short) [12] that provided sharp upper bounds on the face numbers of *all* d-dimensional polytopes with n vertices. While the UBT is a classic by now, the situation for centrally symmetric polytopes is wide open. For instance, the largest number of edges, $\text{fmax}\,(d, n; 1)$, that a d-dimensional centrally symmetric polytope on n vertices can have is unknown even for $d = 4$ and no even conjectural bounds on this number exist.

Related to the problem of determining or at least establishing non-trivial bounds on $\text{fmax}\,(d, n; j)$, the maximum number of j-dimensional faces of a centrally symmetric d-dimensional polytope with n vertices, is the question:

Problem 1. *How neighborly can a centrally symmetric polytope be as a function of its dimension and the number of vertices?*

Below we report on two recent papers [11] and [1] that provide partial answers to both of the above problems.

Recall that a polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if for every $x \in P$, $-x$ belongs to P as well. A cs polytope P is called *k-neighborly* if every set of k of its vertices, no two of which are antipodes, is the vertex set of a face of P.

It is well-know that a general (*non-cs*) d-dimensional polytope with at least $d + 2$ vertices can be at most $\lfloor d/2 \rfloor$-neighborly, and this bound is attained for instance by d-dimensional cyclic polytopes [19, Example 0.6]. In contrast to the general case, the neighborliness of cs polytopes appears to be quite restricted and not sufficiently understood.

In a joint work with Nati Linial [11] we study possible neighborliness of cs polytopes. In particular we establish the correct asymptotics of $k(d, n)$— the largest integer k such that there exists a k-neighborly cs d-polytope with $2(n + d)$ vertices. Our main results are

Theorem 2.
$$\frac{C_1 d}{1 + \log \frac{n+d}{d}} \leq k(d, n) \leq 1 + \frac{C_2 d}{1 + \log \frac{n+d}{d}},$$
where $C_1, C_2 > 0$ are absolute constants independent of d and n. In particular, there exists a cs d-polytope with $4d$ vertices that is at least $\frac{d}{400}$-neighborly.

Theorem 3. $k(d, 2^{d-1} + 1 - d) = 1$. *In other words, a 2-neighborly cs d-polytope has at most 2^d vertices.*

These theorems extend results by Grünbaum [7, p.116], McMullen and Shephard [13], Halsey [8], Schneider [15], Burton [3], as well as more recent results by Donoho [4], and Rudelson and Vershynin [14]. The proof of the lower bound in Theorem 2 is based on studying the cs transforms of cs polytopes introduced in [13] and on a theorem in geometry of Banach spaces due to Garnaev and Gluskin [6]. The proof of the upper bound relies on a certain volume trick.

The same volume trick serves as a starting point of a joint paper with Sasha Barvinok [1] where we analyze $\text{fmax}\,(d, n; j)$, the maximum number of j-dimensional faces of a centrally symmetric d-dimensional polytope with n vertices. The main object of this paper is a natural centrally symmetric analog of cyclic polytopes –

bicyclic polytopes. These polytopes are defined as the convex hulls of finitely many (and symmetrically chosen) points on the *symmetric moment curve* in $\mathbb{R}^{2k}$:

$$\mathrm{SM}_{2k}(t) = \Big(\cos t,\ \sin t,\ \cos 3t,\ \sin 3t,\ \ldots,\ \cos(2k-1)t,\ \sin(2k-1)t\Big) \quad \text{for } t \in \mathbb{R}.$$

By studying the facial structure of bicyclic polytopes on one hand and using a volume trick on the other we establish the following results.

Theorem 4. *If d is a fixed even number and $n \longrightarrow \infty$, then*

$$1 - \frac{1}{d-1} + o(1) \le \frac{\mathrm{fmax}\,(d,n;1)}{\binom{n}{2}} \le 1 - \frac{1}{2^d} + o(1).$$

Theorem 5. *If $d = 2k$ is a fixed even number, $j \le k-1$, and $n \longrightarrow \infty$, then*

$$c_j(d) + o(1) \le \frac{\mathrm{fmax}\,(d,n;j)}{\binom{n}{j+1}} \le 1 - \frac{1}{2^d} + o(1),$$

where $c_j(d)$ is a positive constant.

We also conjecture that our bicyclic polytopes provide asymptotically the largest number of faces in all dimensions among all centrally symmetric polytopes with n vertices. We remark that in dimension four bicyclic polytopes were introduced and studied by Smilansky [16, 17], but to the best of our knowledge the higher-dimensional bicyclic polytopes have not yet been investigated. It is also worth mentioning that analysis of the face numbers of *random* cs polytopes was recently done in [5].

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Inflating polyhedral surfaces

Igor Pak

We prove that all polyhedral surfaces in $\mathbb{R}^3$ have volume-increasing isometric deformations. This resolves the conjecture of Bleecker who proved it for convex simplicial surfaces (1996). A version of this result is proved for all convex surfaces in $\mathbb{R}^d$. We also discuss limits on the volume of such deformations, present a number of conjectures and special cases.

Computing certain invariants of topological spaces of dimension three

Alexander Postnikov

For $x_1, \ldots, x_n \in \mathbb{R}$, the *permutohedron* $P_n = P_n(x_1, \ldots, x_n)$ is the convex hull of the points $(x_{w(1)}, \ldots, x_{w(n)}) \in \mathbb{R}^n$, where w ranges over all permutations in S_n. The permutohedron P_n lies in a hyperplane $\{x_1 + \cdots + x_n = \mathit{Const}\} \subset \mathbb{R}^n$. Let *Vol* denotes the volume form on this hyperplane obtained by inducing the usual volume on $\mathbb{R}^{n-1}$ via the projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$. The volume of P_n has many interesting combinatorial properties. For example, is is well-known that $\mathit{Vol}\, P_n(1, 2, \ldots, n)$ equals $(n+1)^{n-1}$, which the number of trees on $n+1$ labelled vertices. In general, $\mathit{Vol}\, P_n(x_1, \ldots, x_n)$ is a multivariate polynomial in $x_1, \ldots, x_n$, if we assume that $x_1 \geq \cdots \geq x_n$. Using Brion's formula [1], we can derive the following expression for this polynomial.

Theorem 1 ([3, Theorem 3.1])**.** *Let us fix distinct numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then*

$$\mathit{Vol}\, P_n = \frac{1}{(n-1)!} \sum_{w \in S_n} \frac{\lambda_{w(1)} x_1 + \cdots + \lambda_{w(n)} x_n)^{n-1}}{\prod_{i=1}^{n-1} (\lambda_{w(i)} - \lambda_{w(i+1)})} .$$

We can give a combinatorial interpretation of the coefficients of the polynomial $Vol\, P_n$, as follows. For a nonnegative integer sequence $(c_1, \ldots, c_n)$ such that $c_1 + \cdots + c_n = n-1$, let $(\epsilon_1, \ldots, \epsilon_{2n-2}) \in \{1, -1\}^{2n-2}$ be the sequence obtained by replacing each entry c_i with $1, \ldots, 1, -1$ (c_1 1's followed by one -1). Define the subset $I_{c_1,\ldots,c_n} = \{i \mid \epsilon_1 + \cdots + \epsilon_{2i-1} < 0\} \subset [n-1]$. Let $D_n(I)$ denote the number of permutation $w \in S_n$ whose *descent set* $\{i \mid w_i > w_{i+1}\}$ equals I.

Theorem 2 ([3, Theorem 3.2])**.**

$$Vol\, P_n = \sum (-1)^{|I_{c_1,\ldots,c_n}|} D_n(I_{c_1,\ldots,c_n}) \frac{x_1^{c_1}}{c_1!} \cdots \frac{x_n^{c_n}}{c_n},$$

where the sum is over nonnegative integer sequences $(c_1, \ldots, c_n)$ *such that* $c_1 + \cdots + c_n = n-1$.

We also found two other combinatorial formulas for $Vol\, P_n$ and for the Ehrhart polynomial of P_n. These formulas can be extended to more general class of polytopes obtained from the permutohedron P_n by parallel translations of facets. We call these more general polytopes *generalized permutohedra.* They include many interesting polytopes: the associahedron, the cyclohedron, the Pitman-Stanley polytope, and various generalized associahedra related to wonderful compactifications of De Concini-Procesi, graph associahedra of Carr-Devadoss, matroid polytopes, etc. Generalized permutohedra can be also described as

- Polytopes whose normal fan refines the braid arrangement.
- Minkowski summands of the permutohedron P_n.
- Minkowski sums and Minkowski differences of various faces of the coordinate simplex.

In [4], we investigated the combinatorial structure of generalized permutohedra (and of their various subclasses), their face numbers and h-vectors, etc. In particular, we proved the following formula for graph associahedra. For each graph associahedron P_G associated with a graph G, there is a class of G-trees, which are certain rooted trees on $[n]$.

Theorem 3 ([4])**.** *The graph associahedron* P_G *is a simple polytope, whose* h*-polynomial* $h(t) = \sum h_i\, t^i$ *is given by* $h(t) = \sum q^{des(T)}$*, where the sum is over* G*-trees* T *and* $des(T)$ *denotes the number of descent edges in* T*.*

Each graph associahedra is a flag polytope, that is its dual simplicial complex is the clique complex of a graph. According to Gal's conjecture [2], for such polytopes, the γ*-vector* $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ defined as

$$h(t) = \sum_i \gamma_i\, t^i (1+t)^{d-2i}$$

should have nonnegative coefficients $\gamma_i \geq 0$. We proved this conjecture for tree-associahedra.

Theorem 4 ([4])**.** *Gal's conjecture is true for tree-associahedra.*

Moreover, we gave a combinatorial interpretation for the coefficients γ_i in this case.

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A quasisymmetric function for matroids

Vic Reiner

(joint work with L. J. Billera and N. Jia)

A new isomorphism invariant of matroids is introduced, in the form of a quasisymmetric function. This invariant

- defines a Hopf morphism from the Hopf algebra of matroids to the quasisymmetric functions, which is surjective if one uses rational coefficients,
- is a multivariate generating function for integer weight vectors that give minimum total weight to a unique base of the matroid,
- is equivalent, via the Hopf antipode, to a generating function for integer weigh t vectors which keeps track of how many bases minimize the total weight,
- behaves simply under matroid duality,
- has a simple expansion in terms of P-partition enumerators, and
- is a valuation on decompositions of matroid base polytopes.

This last property leads to an interesting application: it can sometimes be used to prove that a matroid base polytope has no decompositions into smaller matroid base polytopes. Existence of such decompositions is a subtle issue arising in work of Lafforgue, where lack of such a decomposition implies the matroid has only a finite number of realizations up to scalings of vectors and overall change-of-basis.

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Polyhedral surfaces in wedge products

Thilo Rörig

(joint work with Raman Sanyal and Günter M. Ziegler)

We introduce the wedge product of two polytopes which is dual to the wreath product of Joswig and Lutz [6]. The wedge product of a p-gon and a $(q-1)$-simplex contains many p-gon faces of which we select a subcomplex $S_{p,2q}$. This subcomplex is a regular surface of *type* $\{p, 2q\}$, that is, all faces are p-gons, all vertices have degree $2q$, and the combinatorial automorphism group acts transitively on the flags of the surface. We show that for certain choices of parameters p and q there exists a realization of the wedge product such that the surface "survives" the projection to $\mathbb{R}^4$. Using methods from topological combinatorics we show that for a different choice of parameters such a realization does not exist.

1. Wedge Product

Consider a d_1-dimensional polytope P_1 with m_1 facets and a d_2-dimensional polytope P_2 with m_2 facets. The *wedge product* $P_1 \lhd P_2$ of P_1 and P_2 is a $(m_1 d_2 + d_1)$-dimensional polytope with $m_1 m_2$ facets. The construction is dual to the wreath product of Joswig and Lutz, i.e. $P_1 \lhd P_2 = (P_2^{\triangle} \wr P_1^{\triangle})^{\triangle}$. In the following we consider the wedge product $W_{p,q-1} = C_p \lhd \Delta_{q-1}$ of a p-gon C_p and a $(q-1)$-simplex Δ_{q-1}. The polytope $W_{p,q-1}$ is a simple $(p(q-1)+2)$-polytope with pq facets. It contains q^p p-gon faces which may be identified with vectors $(i_1, \ldots, i_p) \in \{1, \ldots, q\}^p$.

2. Polyhedral Surfaces

Little is known about the realization of arbitrary polyhedral surfaces. Triangulated surfaces can always be realized as a subcomplex of the six dimensional cyclic polytope and thus in $\mathbb{R}^5$ via Schlegel projection. By a lemma of Perles (cf. Grünbaum [5, p. 204]) we know that every 2-dimensional polyhedral complex embedded in any $\mathbb{R}^d$ also has a realization in $\mathbb{R}^5$. There is also a negative result by Betke and Gritzmann [2] showing that if the number of vertices of odd degree and the number of faces containing odd vertices satisfy a certain inequality then this surface cannot be embedded in any $\mathbb{R}^d$.

The 2-skeleton of the wedge product $W_{p,q-1}$ contains many p-gon faces. Each p-gon edge is contained in exactly q p-gons. We consider the following subset of the p-gon faces:

$$S_{p,2q} = \Big\{(i_1, \ldots, i_p) \in \{1, \ldots, q\}^p \,:\, \sum_{j=1}^{p} i_j \equiv 0, 1 \mod q\Big\}.$$

We use $S_{p,2q}$ to denote the set of p-gons as well as the subcomplex obtained by taking the p-gons, their edges and their vertices.

Theorem 1. *The subcomplex $S_{p,2q}$ of the wedge product $W_{p,q-1}$ is a closed, orientable, regular polyhedral surface with f-vector $(pq^{p-2}, pq^{p-1}, 2q^{p-1})$ and genus $1 + \frac{1}{2}q^{p-2}(pq - 2q - p)$. In particular, all faces of the surface are p-gons and every vertex has degree $2q$.*

This family of surfaces is a generalization of well known surfaces:

- $S_{3,6}$ is the classical Dyck's Regular Map [4],
- $S_{3,2q}$ are triangulated surfaces which occur in Coxeter and Moser [3], and
- $S_{p,4}$ are surfaces of McMullen, Schulz, and Wills [7] of "unusually large genus."

In the following we consider projections of polytopes. We say that a subcomplex of the polytope *survives* the projection if it is a subcomplex of the projected polytope as well.

Theorem 2. *For $q = 2$ there exists a polytope combinatorially equivalent to the wedge product $W_{p,1}$ such that the surface $S_{p,4}$ survives the projection to $\mathbb{R}^4$. Thus $S_{p,4}$ has a realization in $\mathbb{R}^3$.*

We prove this theorem by starting with the standard wedge product of a p-gon and an interval which contains the surface $S_{p,4}$ as a subcomplex. Then we deform the polytope to obtain a realization such that its the projection to $\mathbb{R}^4$ preserves the surface. A simpler form of this deformation technique was introduced by Amenta and Ziegler [1] and was used by Ziegler [10] to construct four dimensional polytopes of large fatness. The most recent account on deformed products and their applications can be found in [8] and Sanyal [9]. Via Schlegel diagrams we obtain a new way to realize some of the surfaces of McMullen, Schulz, and Wills of "unusually large genus" in $\mathbb{R}^3$. The same scheme of projecting high dimensional polytopes containing a surface can also be applied to surfaces of high complexity contained in products of polygons or products of simplices. The following theorem shows that it is not always possible to obtain a realization of the surfaces via projection.

Theorem 3. *For $q \geq 3$ and $p \geq 4$ there exists* no *polytope combinatorially equivalent to the wedge product $W_{p,q-1}$ such that the surface $S_{p,2q}$ survives the projection to $\mathbb{R}^4$.*

The proof proceeds in two steps. First we use Gale duality to associate an embeddability problem to the projection problem, where the surface which is supposed to survive the projection corresponds to a subcomplex of an associated polytope. In a second step, we use the connectivity lower bound for the $\mathbb{Z}_2$-index and Borsuk-Ulam theorem to show that the subcomplex cannot be embedded in the associated polytope. Hence the surface cannot survive the projection to $\mathbb{R}^4$.

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Topological obstructions to vertex numbers of Minkowski sums

Raman Sanyal

For convex polytopes $P_1, \dots, P_r \subset \mathbb{R}^d$ a (trivial) upper bound on the number of vertices of their Minkowski sum is given by

$$f_0(P_1 + \cdots + P_r) \leq \prod_{i=1}^{r} f_0(P_i).$$

In a recent article Fukuda & Weibel [1] prove that for every $r \leq d-1$ there are d-dimensional polytopes $P_1, \dots, P_r \subset \mathbb{R}^d$ with arbitrarily many vertices such that $P_1 + \cdots + P_r$ attains the upper bound.

In this talk we show that their construction is best possible, namely

Theorem 1. *Let $P_1, \dots, P_r \subset \mathbb{R}^d$ be $r \geq d$ polytopes with $f_0(P_i) \geq d+1$. Then*

$$f_0(P_1 + \cdots + P_r) < \prod_{i=1}^{r} f_0(P_i).$$

We make a couple of useful observations reducing the proof to one single case per dimension: every sum of d full-dimensional simplices fails to attain the upper bound. A vital insight is that the Minkowski sum of two polytopes P and Q is the projection of $P \times Q$ under the map $(x, y) \mapsto x + y$. Thus Theorem 1 is a consequence of the following stronger result.

Theorem 2. *Let P be a polytope combinatorially equivalent to a d-fold product of d-dimensional simplices and let $\pi : P \to \mathbb{R}^d$ be an affine projection. Then*

$$f_0(\pi P) < f_0(P).$$

Arguing by contradiction, we can assume that there is a pair (P, π) refuting the claim of Theorem 2. Following ideas developed in [3] the existence of such a pair gives rise to

(1) a simplicial sphere S of a fixed dimension N, and
(2) a simplicial complex K which is embedded in S as a subcomplex.

Using Sarkaria's coloring/embedding theorem (cf. [2]) we show that K does not embed into a sphere of dimension less than $N+1$, thereby yielding the desired contradiction.

The mentioned theorem bounds the dimension of the sphere in terms of the chromatic number of a graph associated to K. In the situation at hand these graphs turn out to be instances of the famous *Kneser graphs* whose chromatic numbers are well-known; see [4].

This is partially joint work with Thilo Rörig and Günter M. Ziegler.

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Topological proofs of the existence and non-existence of graph homomorphisms

Carsten Schultz
(joint work with Eric Babson and Anton Dochtermann)

In the proof of Kneser's Conjecture, Lovász has shown that if the neighbourhood complex of a graph G is $(k-1)$-connected, its chromatic number is at least $k+2$. Later formulations of this theorem replace the neighbourhood complex by the complex $\operatorname{Hom}(K_2, G)$, which is homotopy equivalent to it. One variant of this theorem is the following.

Theorem 1 (Lovász [3, 1])**.** *Let G be a graph with at least one edge. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \leq \chi(G) - 2.$$

Here $\operatorname{Hom}(H, G)$ denotes, for graphs H and G, a cell complex introduced by Lovász and studied by Babson and Kozlov [1] whose 0-cells correspond to the graph homomorphisms from H to G. The non-trivial automorphism of K_2 makes $\operatorname{Hom}(K_2, G)$ into a $\mathbb{Z}_2$-space, which is free if G has no loops. For a free $\mathbb{Z}_2$-space X, the *co-index* and the *cohomological index* are integer-valued invariants, and we have $\operatorname{coind}_{\mathbb{Z}_2} X \leq \text{cohom-ind}_{\mathbb{Z}_2} X$.

The emphasis on the cohomological index of the $\mathbb{Z}_2$-action comes from the work of Babson and Kozlov on the following theorem which proves a conjecture by Lovász. In it, C_{2r+1} is a cyclic graph equipped with an involution that flips an edge.

Theorem 2 (Babson, Kozlov [2, 4])**.** *Let G be a graph with an odd cycle and $r \geq 1$. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) \leq \chi(G) - 3.$$

The connection between these two theorems is given by the following result which has been presented in [7].

Theorem 3 (S [5])**.** *Let G be a graph with an odd cycle and $r \geq 1$. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(C_{2r+1}, G) + 1 \leq \text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G).$$

We start with a more detailed description of the connection between the spaces $\operatorname{Hom}(C_{2r+1}, G)$ and $\operatorname{Hom}(K_2, G)$. Since there are $\mathbb{Z}_2$-equivariant graph homomorphisms $C_{2r+3} \to C_{2r+1}$, which induce inclusion maps $\operatorname{Hom}(C_{2r+1}, G) \to \operatorname{Hom}(C_{2r+3}, G)$, it is possible to consider the union (or more exactly, the colimit) of all the spaces $\operatorname{Hom}(C_{2r+1}, G)$. The following result states that the $\mathbb{Z}_2$-homotopy type of this union is determined by the $\mathbb{Z}_2$-homotopy type of $\operatorname{Hom}(K_2, G)$. Theorem 3 follows from it.

Theorem 4 (S [5])**.** *Let G be a graph. Then*

$$\operatorname{colim}_r \operatorname{Hom}(C_{2r+1}, G) \simeq_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_{\mathrm{b}}, \operatorname{Hom}(K_2, G)).$$

The space $\mathbb{S}^1_{\mathrm{b}}$ occuring in this theorem is a 1-sphere with a left and a right $\mathbb{Z}_2$-action, the left action given by the antipodal map and the right action by a reflection. Imitating this, we define C'_m to be a cyclic graph on $2m$ looped vertices and equip it with $\mathbb{Z}_2$-actions in the same way.

Theorem 5 (Babson, Dochtermann, S [6])**.** *Let G be a graph and T a graph with a right $\mathbb{Z}_2$-action. Then*

$$\operatorname{colim}_r \operatorname{Hom}(T \times_{\mathbb{Z}_2} C'_{2r+1}, G) \simeq_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_{\mathrm{b}}, \operatorname{Hom}(T, G)).$$

This motivates the definition of a 2-parameter family of graphs.

Definition 6. *For $k, r \geq 1$ we set*

$$T_{k,r} := C_{2r+1} \times_{\mathbb{Z}_2} \underbrace{C'_{2r+1} \times_{\mathbb{Z}_2} \cdots \times_{\mathbb{Z}_2} C'_{2r+1}}_{k-1 \text{ factors}}.$$

Properties of $T_{k,r}$ include uniform vertex degrees of $2 \cdot 3^k$ and an odd girth of $2r + 1$. Combining Theorems 4 and 5, we obtain the following.

Theorem 7. *Let G be a graph and $k \geq 1$. Then*

$$\operatorname{colim}_r \operatorname{Hom}(T_{k,r}, G) \simeq_{\mathbb{Z}_2} \operatorname{Map}_{\mathbb{Z}_2}(\underbrace{\mathbb{S}^1_{\mathrm{b}} \times_{\mathbb{Z}_2} \cdots \times_{\mathbb{Z}_2} \mathbb{S}^1_{\mathrm{b}}}_{k \text{ factors}}, \operatorname{Hom}(K_2, G)).$$

We can now generalize and complement Theorem 3.

Corollary 8. *Let G be a graph and $k \geq 1$. Then*

$$\text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(T_{k,r}, G) + k \leq \text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G),$$
$$\lim_{r \to \infty} \operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(T_{k,r}, G) + k \geq \operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G).$$

Just using that the left hand side in Theorem 7 is empty if and only if the right hand size is, we derive a connection between the Lovász bound of Theorem 1 and the existence of graph homomorphisms from $T_{k,r}$ to a graph.

Consequences.

- For every graph G we have
$$\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \leq \max\{k\colon \text{Ex. } r \geq 1 \text{ and } T_{k,r} \to G\}$$
$$\leq \text{cohom-ind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) \leq \chi(G) - 2.$$
- It follows that $\chi(T_{k,r}) \geq k+2$, and explicitly constructed colourings show that $\chi(T_{k,r}) = k+2$.
- Let K be a Kneser graph with $\chi(K) = k+2$. Lovász' proof of Kneser's conjecture establishes $\operatorname{coind}_{\mathbb{Z}_2} \operatorname{Hom}(K_2, G) = k$. Therefore there exist an $r \geq 1$ and a graph homomorphism $T_{k,r} \to K$. So not only do we know that $\chi(K) \geq k+2$, we also know that this is detected by a graph homomorphism from a $T_{k,r}$ to K.

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Colorful subgraphs in Kneser-like graphs

Gábor Simonyi

(joint work with Gábor Tardos)

Following Lovász's celebrated proof of Kneser's conjecture a large variety of simplicial complexes has been associated to a graph, the paper [9] by Matoušek and Ziegler discusses several of these. A common property of all these complexes is that topological parameters of their geometric realizations give lower bounds for the chromatic number of the graph that is sharp in several interesting cases including those of Kneser graphs.

Let $B(G)$ and $B_0(G)$ be two of the simplical complexes mentioned above, their definition can be found in [9]. They give rise to Z_2-spaces, that is, topological spaces with a Z_2-action (or involution) on them. For a Z_2-space T the Z_2-index and Z_2-coindex are defined (see [8]) as follows. (We consider the sphere S^h being equipped with an involution which is the usual antipodality.) The former parameter, $\mathrm{ind}(T)$, is the minimum dimension h for which there exists a Z_2-map $T \to S^h$. The latter parameter, $\mathrm{coind}(T)$, is the maximum dimension h for which a Z_2-map $S^h \to T$ exists. It follows from the Borsuk-Ulam theorem that $\mathrm{ind}(T) \geq \mathrm{coind}(T)$ for every Z_2-space T.

By earlier work of several researchers (for references see [10]) one has

$$\chi(G) \geq \mathrm{ind}(B(G)) + 2 \geq \mathrm{ind}(B_0(G)) + 1 \geq \mathrm{coind}(B_0(G)) + 1 \geq \mathrm{coind}(B(G)) + 2,$$

where the index and coindex of a simplicial complex are meant to be the index resp. coindex of its geometric realization.

Csorba, Lange, Schurr, and Waßmer [3] have recently proved what they called the $K_{\ell,m}$ theorem: If $\mathrm{ind}(B(G)) = \ell + m - 2$ then G must contain the complete bipartite graph $K_{\ell,m}$ as a subgraph.

For those graphs where the $\mathrm{coind}(B_0(G)) + 1$ lower bound on the chromatic number is sharp, we prove the following colorful version of the above.

Theorem 1. *Let G be a graph satisfying $\chi(G) = \mathrm{coind}(B_0(G))+1$ and $c : V(G) \to \{1, \ldots, \chi(G)\}$ an optimal proper coloring of G. Let L, M give a bipartition of the color set, i.e., $L \cap M = \emptyset$ and $L \cup M = \{1, \ldots, \chi(G)\}$. Then there exists a complete bipartite subgraph $K_{|L|,|M|}$ of G with the property that the colors appearing on one of its sides are exactly those in L, while the colors appearing on its other side are exactly those in M. In particular, all vertices of this subgraph are colored with a different color by c.*

The proof is based on a variant of the Borsuk-Ulam theorem due to Tucker [12] and Bacon [1].

There are several well-known and some less known families for which the above theorem applies. The better known families include those of Kneser graphs, Schrijver graphs, Borsuk graphs, and the so-called generalized Mycielskians of any graph on this list. For definitions of these families of graphs we refer to [8]. The less known classes include some of the homomorphism universal graphs for the invariant called local chromatic number (introduced in [4]) and homomorphism universal graphs for colorings called wide in [10]. (The latter are defined independently in [10] and [2].) The list also includes the odd chromatic (and not the even chromatic) rational (or circular) complete graphs $K_{p/q}$ (see e.g. [7]) that are also homomorphism universal graphs for a popular coloring invariant, the circular chromatic number. In the paper [11] we give the references which prove that all these graphs belong here.

The theorem above is in contrast to certain results we proved in [10] that imply that several of the above graphs can be properly colored with just one more color than its chromatic number in such a way that the only $K_{\ell,m}$ subgraphs with $\ell+m = \chi(G)$ which appear completely multicolored are those with $\ell = \lceil \chi(G)/2 \rceil, m = \lfloor \chi(G)/2 \rfloor$. Appearance of these latter are unavoidable in any proper coloring of these graphs (by an arbitrary number of colors) by results in [5, 6, 10].

This report is based on the paper [11] and is closely related to the work presented in [10].

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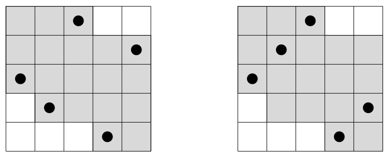
Bruhat intervals as rooks on skew Ferrers boards

Jonas Sjöstrand

We represent a permutation $\pi \in \mathfrak{S}_n$ of $\{1, 2, \ldots, n\}$ as an n by n rook diagram with a rook in row i and column j if $\pi(i) = j$. For a permutation π, define its *right hull*, denoted by $H_R(\pi)$, as the smallest (right-aligned) skew Ferrers board that covers all rooks of π. Figure 1(a) shows an example.

The *Bruhat order* on the symmetric group $\mathfrak{S}_n$ is the transitive closure of the directed Bruhat graph whose edges correspond to length-increasing transpositions (see e.g. [1]). As shown in Figure 1(b), applying a length-decreasing transposition to π results in a permutation whose rooks are covered by the right hull of π. Thus, the set $\mathfrak{S}(H_R(\pi))$ of all permutations whose rooks are covered by $H_R(\pi)$ is a lower order ideal in the Bruhat order on $\mathfrak{S}_n$. Now it is natural to ask when this order ideal is in fact a lower interval $[\mathrm{id}, \pi] = \{\rho \in \mathfrak{S}_n : \rho \leq \pi\}$. The following theorem answers this question. (We say that $\pi \in \mathfrak{S}_n$ avoids the pattern $\rho \in \mathfrak{S}_k$ if there do not exist indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_r) < \pi(i_s)$ if and only if $\rho(r) < \rho(s)$.)



(a) (b)

FIGURE 1. (a) The rook diagram of the permutation 35124, with the right hull $H_R(35124)$ shown as a shaded area. (b) The permutation 35124 becomes 32154 after exchanging rows 2 and 4. We do not leave the right hull of 35124 by this operation.

Theorem 1. $\mathfrak{S}(H_R(\pi))$ *equals the lower Bruhat interval* $[id, \pi]$ *in* $\mathfrak{S}_n$ *if and only if* π *avoids the patterns 4231, 35142, 42513, and 351624.*

Exactly the permutations avoiding the patterns in Theorem 1 have occured in a similar context twice.

(1) They are the permutations whose lower Bruhat intervals correspond to Schubert varieties "defined by inclusions" in the sense of Gasharov and Reiner [2].
(2) In the problem session of this workshop, Alexander Postnikov gave a conjecture involving our permutations. Let $\pi \in \mathfrak{S}_n$. For each inversion $i < j$, $\pi(i) > \pi(j)$ we define a hyperplane of $\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$ by the equation $x_i = x_j$. Let $r(\pi)$ be the number of regions of the hyperplane arrangement consisting of all such hyperplanes. Let $b(\pi)$ be the number of elements in the closed lower Bruhat interval $[\mathrm{id}, \pi]$.

Conjecture 2 (Postnikov, 2007)**.** *For any n-permutation* $\pi \in \mathfrak{S}_n$ *we have*

$$b(\pi) \geq r(\pi)$$

with equality if and only if π *avoids the patterns 4231, 35142, 42513, and 351624.*

Postnikov has shown that the conjecture holds for Grassmannian permutations (i.e. permutations with at most one descent), but otherwise it is still open.

Theorem 1 connects Bruhat intervals with the theory of rook numbers and rook polynomials developed by Riordan [4, secs. 7.2, 7.3, 7.4] and Goldman, Joichi, and White [3]. Using this connection we are able to compute the Poincaré polynomial of some particularly interesting intervals in the symmetric group $\mathfrak{S}_n \cong A_{n-1}$ and the hyperoctahedral group B_n.

For a Coxeter system (W, S) and a subset $J \subseteq S$ of the generators, let W_J denote the parabolic subgroup generated by J and let W^J denote the set of minimal representatives in cosets $wW_j \in W/W_j$; see [1] for detailed definitions. We label the generators according Figure 2.

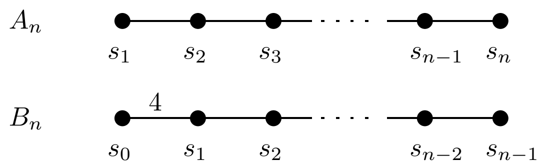


FIGURE 2. The Coxeter graphs of A_n and B_n.

To formulate our results we need to define the factorial q-analogue

$$[n]!_q = [n]_q[n-1]_q \cdots [1]_q,$$

where $[n]_q = 1 + q + q^2 + \cdots q^{n-1} = (1-q^n)/(1-q)$ is the q-analogue of the integer n. We also need the q-Stirling numbers of the second kind, defined by the recurrence

$$S_{n+1,k}(q) = q^{k-1}S_{n,k-1}(q) + [k]_q S_{n,k}(q), \quad \text{for } 0 \leq k \leq n$$

with the initial conditions $S_{0,0}(q) = 1$ and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.

Now, we are ready to present our results.

Theorem 3. *Let w be the maximal element of $A_{n-1}^{S\setminus\{s_k\}}$. Then the Poincaré polynomial of the Bruhat interval $[id, w]$ is*

$$\mathrm{Poin}_{[id,w]}(q) = q^{(n-k)k} \sum_{i=0}^{k} S_{k+1,i+1}(1/q)\, S_{n-k+1,i+1}(1/q)\, [i]!_q^2\, q^i.$$

Theorem 4. *Let w be the maximal element of $B_n^{S\setminus\{s_0\}}$. Then the Poincaré polynomial of the Bruhat interval $[id, w]$ is*

$$\mathrm{Poin}_{[id,w]}(q) = q^{\binom{n+1}{2}} \sum_{i=0}^{n} S_{n+1,i+1}(1/q)\, [i]!_q.$$

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f-Vectors of manifolds

Ed Swartz

The principal combinatorial invariant of a simplicial complex Δ is its f-vector, $(f_0, f_1, \dots, f_{d-1})$, where f_i is the number of i-dimensional faces. From here on, Δ is a triangulation of a $(d-1)$-dimensional homology manifold without boundary, $d \geq 3$. Frequently the h-vector, defined by,

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{n-i} f_{j-1}, \tag{1}$$

is easier to comprehend. For instance, Klee [2] proved that if Δ is a triangulation of a $(d-1)$-manifold without boundary (or more generally any semi-Eulerian complex), then

$$h_{d-i} - h_i = (-1)^d \binom{d}{i} [\chi(\Delta) - \chi(S^{d-1})]. \tag{2}$$

Another reason for introducing the h-vector is its close connection to the Hilbert function of the face ring of Δ. Let $V = \{v_1, \dots, v_n\}$ be the vertices of Δ. Define

$$\mathbb{C}[\Delta] = \mathbb{C}[x_1, \dots, x_n]/I_\Delta,$$

where I_Δ is the face ideal,

$$I_\Delta = \langle x_{i_1} \cdots x_{i_k} : \{v_{i_1}, \dots, v_{i_k}\} \notin \Delta \rangle.$$

A set $\Theta = \{\theta_1, \dots, \theta_d\}$ of one-forms in $\mathbb{C}[\Delta]$ is a *linear system of parameters (lsop)* if $\mathbb{C}[\Delta]/\langle\Theta\rangle$ is finite dimensional over $\mathbb{C}$.

Theorem 1. *(Schenzel)* [6]. *Let Θ be a lsop for $\mathbb{C}[\Delta]$. Then*

$$\dim_{\mathbb{C}}(\mathbb{C}[\Delta]/\langle\Theta\rangle)_i = h_i + \binom{d}{i} \sum_{j=2}^{i-1} (-1)^{i-j-1} \beta_{j-1}, \tag{3}$$

where the β_i are the rational Betti numbers of Δ.

Let h'_i be defined as the r.h.s. of (3). Then $(h'_0, \dots, h'_d)$ is an M-vector. By definition, an M-vector is a sequence $(a_0, \dots, a_d)$ such that the a_i are the Hilbert function of a homogeneous quotient of a polynomial ring. M-vectors are characterized by a nonlinear arithmetic condition due to Macaulay [3].

The g-vector of Δ is $(g_0, \dots, g_{\lfloor d/2 \rfloor})$, where g_i is defined by $g_i = h_i - h_{i-1}$. In view of (2) the g-vector of Δ encodes the same information as the entire f-vector. For a $(d-1)$-manifold X, let $\mathcal{G}(X)$ be the set of all possible g-vectors of trianglations of X.

Theorem 2. *For any simplicial complex $g_2 \leq \binom{g_1+1}{2}$.*
$\mathcal{G}(\mathbb{C}P^2) : g_1 \geq 3, g_2 \geq 6$ (S).
$\mathcal{G}(K3$ surfaces$) : g_1 \geq 10, g_2 \geq 55$ (S).
$\mathcal{G}((S^2 \times S^2)\#(S^2 \times S^2)) : g_1 \geq 6, g_2 \geq 18$ (S).

$\mathcal{G}(S^1 \times S^3) : g_1 \geq 5, g_2 \geq 15.$ *(S)*
$\mathcal{G}(S^1 \propto S^3) : g_1 \geq 6, g_2 \geq 15.$ *(J. Chestnut, J. Sapir, S)*

Other manifolds for which a complete description of their f-vectors are known are 2-manifolds (Ringel [5], Jungerman and Ringel [1]), S^3, $\mathbb{R}P^3$, $S^1 \times S^2$, $S^1 \propto S^2$, S^4 (Walkup [8]).

One of the main obstacles in dimensions five and above is the g-conjecture.

Conjecture 3. *If Δ is a sphere (PL-sphere, homology sphere), then the g-vector of Δ is an M-vector.*

The primary motivation for this conjecture is the g-theorem which states that the conjecture holds for boundaries of simplicial polytopes. Both Stanley's original proof [7], and McMullen's later proof [4] show the existence of a Lefschetz element $\omega \in (\mathbb{C}[\Delta]/\langle\Theta\rangle)_1$, such that multiplication

$$\omega^{2d-i} : (\mathbb{C}[\Delta]/\langle\Theta\rangle)_i \to (\mathbb{C}[\Delta]/\langle\Theta\rangle)_{d-i}$$

is an isomorphism.

Theorem 4. *(S) If at least $n - d$ of the links of vertices of Δ have Lefschetz elements, then there exists $\omega \in (\mathbb{C}[\Delta]/\langle\Theta\rangle)_1$ such that multiplication*

$$\omega : (\mathbb{C}[\Delta]/\langle\Theta\rangle)_i \to (\mathbb{C}[\Delta]/\langle\Theta\rangle)_{i+1}$$

is a surjection for all $i \geq \lceil d/2 \rceil$.

Corollary 5. *(S) If at least $n - d$ of the links of vertices of Δ have Lefschetz elements, then for all $i \geq \lceil d/2 \rceil$,*

$$h'_i \geq h'_{i+1} + \binom{d-1}{i}\beta_{i-1}.$$

In short, the existence of Lefschetz elements for spheres has strong consequences for f-vectors of manifolds. Conversely, if all manifolds satisfy the conclusion of the above corollary, then the Gorenstein property of spheres would allow a proof of the g-conjecture

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On the local chromatic number of odd quadrangulations

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(joint work with Bojan Mohar and Gábor Simonyi)

Our starting point is the following nice theorem of Youngs [4]; Archdeacon, Hutchinson, Nakamoto, Negami, Ota [1]; and Mohar, Seymour [3]:

Theorem 1. *An odd quadrangulation of a surface has no proper 3-coloring.*

Here the parity of the quadrangulation is defined by orienting the faces of the quadrangulation and counting the edges along which the orientation is inconsistent. This parity is clearly independent of the orientation. As any quadrangulation of an orientable surface is even the result is about quadrangulations of non-orientable surfaces. Youngs' result was about the first such surface, the projective plane, where the condition "odd" can equivalently be replaced by "non-bipartite". The other two mentioned papers generalize this result to arbitrary non-orientable surfaces.

Our goal was to generalize the above result to local chromatic number introduced in [2]. A proper coloring of a graph G is called a *local k-coloring* if at most $k-1$ different colors appear among the neighbors of any one vertex. The *local chromatic number* of G is the smallest k for which a local k-coloring exists. This parameter is clearly bounded by the chromatic number but can be arbitrarily smaller than that. To prove a lower bound of 4 for the local chromatic number means showing that any proper coloring has a vertex with at least three differently colored neighbors.

We found that the generalization we are after holds for some surfaces but not for all. Our main result is as follows:

Theorem 2. *An odd quadrangulation of a non-orientable surface of genus at most 4 has no local 3-coloring. All higher genus non-orientable surfaces admit odd quadrangulations of local chromatic number* 3.

Our example of an odd quadrangulation of the genus 5 non-orientable surface admits a local 3-coloring using 6 colors. As all graphs having a local 3-coloring with 4 colors are in fact 3-colorable the smallest number of colors we can hope for is 5. For 5 colors the threshold lies higher:

Theorem 3. *No odd quadrangulation of a non-orientable surface of genus 6 or less admits a local 3-coloring with 5 colors. All higher genus non-orientable surfaces have odd quadrangulations with such a coloring.*

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Topology of Rees products of posets and q-Eulerian polynomials

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(joint work with John Shareshian)

A poset operation called Rees product was recently introduced by Björner and Welker [1] in their study of connections between poset topology and commutative algebra. Through our study of Rees products of posets, we discovered a remarkable q-analog of a classical formula for the exponential generating function of the Eulerian polynomials. The Eulerian polynomials enumerate permutations according to the number of descents or the number of excedances. Our q-Eulerian polynomials are the enumerators for the joint distribution of the excedance statistic and the major index. There is a vast literature on q-Eulerian polynomials that involves other combinations of Eulerian and Mahonian permutation statistics, but this is the first result to address the combination of excedance number and major index. Although poset topology led us to conjecture our formula, symmetric function theory provided the proof. We prove a symmetric function generalization of our formula, which also yields other enumerative results.

The subject of permutation statistics originated in the early 20th century work of Major Percy MacMahon [3, Vol. I, pp. 135, 186; Vol. II, p. viii], [4] and has developed into an active and important area of enumerative combinatorics over the last four decades. MacMahon studied four fundamental permutation statistics, the inversion index (inv), the major index (maj), the descent number (des), and the excedance number (exc) (see [6] for the definitions).

MacMahon [3, Vol. I, p. 186] observed that the descent number and excedance number are equidistributed, that is, the number of permutations in the symmetric group $\mathfrak{S}_n$ with j descents equals the number of permutations with j excedances for all n and j. These numbers were first studied by Euler and have come to be known as the Eulerian numbers. They are the coefficients of the *Eulerian polynomials*[1]

$$A_n(t) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

[1] It is more common to define the Eulerian polynomials as $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)+1}$

There is a classical and well-known exponential generating function formula,

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)}-t}. \tag{1}$$

The permutation statistics des and exc are known as Eulerian statistics.

MacMahon [4] showed that the major index is equidistributed with the inversion statistic by establishing the first equality in

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q! = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)},$$

where

$$[n]_q := 1+q+\cdots+q^{n-1} \qquad \text{and} \qquad [n]_q! := [n]_q[n-1]_q\cdots[1]_q.$$

The permutations statistics maj and inv are known as Mahonian statistics.

One can look for nice q-analogs of the Eulerian polynomials by considering the joint distributions of the Mahonian and Eulerian statistics given above. Consider the four possibilities, $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}t^{\mathrm{des}(\sigma)}$, $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}t^{\mathrm{des}(\sigma)}$, $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}t^{\mathrm{exc}(\sigma)}$, and $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}t^{\mathrm{exc}(\sigma)}$. There are many interesting results on the first three q-Eulerian polynomials and on multivariate distributions of all sorts of combinations of Eulerian and Mahonian statistics. These include Stanley's q-analog of (1) involving the first q-Eulerian polynomial [7] .

We have found no mention of the fourth q-Eulerian polynomial $A_n^{\mathrm{maj,exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)}t^{\mathrm{exc}(\sigma)}$ anywhere in the literature prior to our paper [6], which announces the following q-analog of (1).

Theorem 1. *The q-exponential generating function for $A_n^{\mathrm{maj,exc}}(q,t)$ is given by*

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}\ , \tag{2}$$

where

$$\exp_q(z) := \sum_{n\geq 0}\frac{z^n}{[n]_q!}.$$

We first conjectured (2) by studying the Rees product $(B_n(q)\setminus\{(0)\}) * C_n$, where C_n is the chain with n elements, $B_n(q)$ is the lattice of subspaces of the vector space $\mathbb{F}_q^n$ and $\mathbb{F}_q$ is the field with q elements. The Rees product of two ranked posets is a subposet of the usual product poset (for the precise definition see [1] or [6]). The maximum elements of $(B_n(q)\setminus\{(0)\}) * C_n$ are of the form $(\mathbb{F}_q^n, j)$, where $j = 1,\ldots,n$. Let $I_{n,j}(q)$ be the induced subposet of elements of $(B_n(q)\setminus\{(0)\}) * C_n$ that are less than $(\mathbb{F}_q^n, j)$ and let $\beta_{n,j}(q)$ be the top Betti number of the order complex of $I_{n,j}(q)$. By computing the Betti number polynomials $\beta_n(q,t) := \sum_{j=1}^n \beta_{n,j}(q)t^j$ for small values of n we were led to conjecture that

$$\beta_n(q,qt) = A_n^{\mathrm{maj,exc}}(q,t). \tag{3}$$

By computing the Möbius invariant of a related poset we obtained

$$\sum_{n\geq 0}\beta_n(q,t)\frac{z^n}{[n]_q!}=\frac{(1-t)\exp_q(z)}{\exp_q(zt)-t\exp_q(z)}\ . \tag{4}$$

We had intended to prove (3) by combining (4) with a reference for (2) that we expected to find in the literature. Instead, we were surprised to discover that (2) was neither known nor easy to prove.

To prove Theorem 1 we introduce a family of quasisymmetric functions $Q_{n,j,k}$ for which the identity given in the following theorem specializes to a refinement of (2) involving the number fixed points of a permutation as well as maj and exc.

Theorem 2. *Let $H(z) := \sum_{n\geq 0} h_n z^n$, where h_n denotes the complete homogeneous symmetric function of degree n. Then*

$$\sum_{n,j,k\geq 0} Q_{n,j,k}t^j r^k z^n=\frac{(1-t)H(zr)}{H(zt)-tH(z)} \tag{5}$$

There are three main steps in the proof of (5). In the first step we modify a bijection that Gessel and Reutenauer [2] constructed to enumerate permutations with a fixed descent set and fixed cycle type. This yields an alternative characterization of the $Q_{n,j,k}$ involving bicolored necklaces. In the second step we use a bijection from multisets of bicolored necklaces to bicolered words, which involves Lyndon decompositions of words. This yields yet another characterization of $Q_{n,j,k}$. In the third step we generalize a bijection that Stembridge constructed to study the representation of the symmetric group on the cohomology of the toric variety associated with the type A Coxeter complex. This enables us to derive a recurrence relation for $Q_{n,j,k}$, which yields (5).

Stanley [8], using a formula of Procesi [5], proved that the generating function for the Frobenius characteristic of the representation of the symmetric group on the cohomology of the toric variety associated with the type A Coxeter complex equals the right hand side of (5) with $r = 1$. We prove that the same is true for the sign twisted representation of the symmetric group on homology of maximal open intervals in the Rees product $(B_n \setminus \{\emptyset\}) * C_n$, where B_n is the subset lattice. Hence the representations are isomorphic and have Frobenius characteristic equal to $Q_{n,j,1}$. We pose the problem of finding a direct isomorphism between these representations. In our proof of (5), we introduce a family of symmetric functions $Q_{\lambda,j}$, where λ is a partition of n. These symmetric functions refine the $Q_{n,j,k}$ and specialize to the maj enumerator of permutations of cycle type λ with j excedences. We conjecture that $Q_{\lambda,j}$ is the Frobenius characteristic of some representation.

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k-Sets and topological invariants

Uli Wagner

Let S be a set of $n \geq 3$ points in the affine plane $\mathbf{R}^2$, no three of them collinear. For an integer parameter k, a *k-edge* of S is a directed segment pq, spanned by two points of S, such that exactly k points of S lie to the right of the line through pq (and hence $n-2-k$ to the left of it). We denote the number of k-edges of S by $e_k(S)$.

The study of k-edges was initiated by Erdős, Lovász, Simmons, and Straus [6, 5] in the early 1970's. Since then, these objects, their higher-dimensional generalizations (*k-facets*), and a variety of closely related notions (*k-sets*, *levels in arrangements*, *higher order Voronoi diagrams*, etc.) have come to occupy a central place in discrete and computational geometry.

The most fundamental problem is this: What is the maximum number $e_k(n) := \max_{|S|=n} e_k(S)$ of k-edges that a set of n points in the plane can have? Of course, $e_0(n) = n$, since the 0-edges are just the edges of the convex hull of the point set; moreover, it is known [4] that $e_1(n) = \lfloor 3n/2 \rfloor$. Beyond that, no exact bounds are known, and even the asymptotic behaviour is far from understood.

In their original papers, Lovász et al. proved an upper bound of $O(n\sqrt{k+1})$ and a lower bound of $\Omega(n \log(k+1))$, and conjectured an upper bound of $O(n(k+1)^{\varepsilon})$ for every $\varepsilon > 0$ (with the implicit constants depending on ε). The currently best estimates are an upper bound of $O(n(k+1)^{1/3})$, due to Dey [3], and a lower bound construction of sets with $n \cdot e^{\Omega(\sqrt{\log k})}$ many k-edges, due to Tóth [8].

A crucial point in Dey's proof is the analysis of the number of crossings between k-edges. This analysis was later refined by Andrzejak, Aronov, Har-Peled, Seidel, and Welzl [1], who proved the identity

$$X_k + \sum_{p \in S} \overrightarrow{\deg}_k(p) = e_{<k}$$

for all sets of n points and $k < \frac{n-2}{2}$; here X_k is the number of crossings between k-edges, and $\overrightarrow{\deg}_k(p)$ is the out-degree of the point p, i.e., the number of directed k-edges that have p as their "tail"; and $e_{<k} = e_0 + \ldots + e_{k-1}$.

The set of k-edges of a point set can be partitioned so that the edges in each part form a locally convex polygonal curve: take a k-edge, start rotating the line it spans counterclockwise about the "head" of the edge until you hit the next k-edge, which will be the uniquely defined successor of the first, and continue. The left-hand-side of the above identity can be interpreted as the number of self-intersections of the resulting (multicomponent) *k-edge curve* Γ_k.

In order to fine further interesting structural properties of the collection of k-edges of a planar point set, we propose to study topogical invariants of the curve Γ_k. A first simple observation is that the *rotation number* (or *Whitney index*) of the curve, i.e., the total number of counterclockwise turns it makes, always equals $k+1$. (This is closely related to a basic lemma, already noted in [6, 5], that states that for any directed line ℓ, there are at most $k+1$ k-edges that cross ℓ from left to right).

Furthermore, we investigate two of the most basic finite-type invariants of plane curves, $J^{\pm}$, introduced by Arnold [2]. Using a formula of Viro [9] that expresses J^- in terms of local winding numbers of the curve, we show

Theorem 1. *For any set of n points in the plane and $0 \leq k < \frac{n-2}{2}$, $J^-(\Gamma_k) = 1 - (k+1)^2$.*

Polyak [7] introduces a variant of J^- for two curves, for which we obtain

Theorem 2. *For $0 \leq j < k < \frac{n-2}{2}$,*

$$J^-(\Gamma_j, \Gamma_k) = 1 - (j+1)(k+1).$$

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Splitting multidimensional necklaces

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(joint work with Mark de Longueville)

Our central new result (Theorem 2) is a consensus division theorem extending the well known splitting necklace theorem of Noga Alon to higher dimensions. The problem of consensus division arises when two or more competitive or cooperative parties, each guided by the same or different objective functions, divide an object according to some notion of fairness. There are many different mathematical reformulations of this problem depending on what kind of divisions are allowed, what kind of object is divided, whether the parties involved are cooperative or not, etc. Early examples of problems and results of this type are the "ham sandwich theorem" of Steinhaus and Banach, the envy-free "cake-division problem" of Steinhaus, the equipartition of measurable sets by hyperplanes of Grünbaum and Hadwiger, and more recently the Hobby-Rice theorem, the "splitting necklace theorem" of Alon etc.

Theorem 1 (Splitting necklace theorem, N. Alon [1]). *Let $\mu_1, \mu_2, \ldots, \mu_n$ be a collection of n continuous probability measures on $[0,1]$. Let $k \geq 2$ and $N := n(k-1)$. Then there exists a partition of $[0,1]$ by N cut points into $N+1$ intervals $I_0, I_1, \ldots, I_N$ and a function $f : \{0,1,\ldots,N\} \to \{1,\ldots,k\}$ such that for each μ_i and each $j \in \{1,2,\ldots,k\}$,*

$$\sum_{f(p)=j} \mu_i(I_p) = 1/k.$$

This result has been recognized as one of central examples of combinatorial results that require topological methods for their proof. For this aspect of the problem and surveys of related results the reader is referred to [2], [8] and [9].

The setting of the "multidimensional splitting necklace theorem" naturally extends the framework of the original one-dimensional question. A group of k "thieves" is interested in a fair division of a cube $[0,1]^d$ where n (continuous, probability) measures $\mu_1, \ldots, \mu_n$ determine the value of each (measurable) subset $A \subset [0,1]^d$. A fair division is a partition $[0,1]^d = A_1 \cup \ldots \cup A_k$ such that $\mu(A_i) = \frac{1}{k}\mu([0,1]^d)$ for each i, where $\mu = (\mu_1, \ldots, \mu_n)$ is the associated vector-valued measure. The dissection of $[0,1]^d$ is performed by hyperplanes parallel to the sides of $[0,1]^d$ such that m_i of them are parallel to the coordinate hyperplane

$x_i = 0$. The cube is divided into $m_1 \cdot \ldots \cdot m_d$ elementary parallelepipeds and each A_i is obtained as the union of some of these parallelepipeds.

Theorem 2 (Multidimensional splitting necklace theorem, [7]). *A fair division is always possible if and only if*

$$m_1 + \ldots + m_d \geq n(k-1).$$

For d = 1 this result reduces to Alon's "splitting necklace theorem" (Theorem 1). The proof of Theorem 2 uses equivariant topological methods and relies on the fact that an associated configuration space $\Omega(I^d, [k])$ admits a topological shelling. This motivated the introduction of more general complexes $\Omega(Q; [k])$ of "vertex-colored polytopes", associated to an arbitrary convex polytope Q. More precisely, a vertex-colored polytope (modelled by Q) is a pair $Q^\phi := (Q, \phi)$ where $\phi : Vert(Q) \to [m]$ is a coloring function. Two vertex-colored polytopes Q^ϕ and Q^ψ are glued over a common face $F < Q$ if and only if $\phi|Vert(F) = \psi|Vert(F)$. The regular cell-complex constructed this way is denoted by $\Omega(Q; [k])$.

In this generality $\Omega(Q; [k])$ can be seen as a relative of moment-angle complexes $\mathcal{Z}_K$, [4] [6], "inflated simplicial complexes" [3], homotopy colimits of canonical diagrams over posets (simplicial complexes) etc.

From a completely different point of view, the complex $\Omega(I^2; [k])$ can be also interpreted as a candidate for a "continuous space of images" (or "rectangular patches space") in relation to the qualitative analysis of (digital camera) images, see [5]. In this context the square $I^2 = [0,1]^2$ is interpreted as the (continuous) pixel space while $[k]$ enumerates the set of "standard patches". An element $x \in \Omega(I^2; [k])$ encodes both a division of the "pixel space" I^2 into $m_1 \times m_2$ elementary rectangles as well as the recognition (classification) of each of the elementary rectangles as one of standard elementary images (patches) enumerated by $[k]$.

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Open problems session
Account by Uli Wagner

Problem 1 (Jesús De Loera)**.** Let $P \subseteq \boldsymbol{R}^d$ be a *rational* polytope. Consider the number $i_P(k) := |nP \cap \boldsymbol{Z}^d|$ of integer points in the k^{th} dilate of P. The *Ehrhart series* of P is the generating function $\sum_{n=0}^{\infty} i_P(k)t^k$. It is well-known (see, for instance, [3, Chapter 4]) that this generating function can be expressed as a rational function

$$\sum_n i_P(n)t^n = \frac{h_0^* + h_1^* t + \ldots + h_d^* t^d}{D},$$

where the denominator D is a product of terms of the form $(1-t^\alpha)^m$. The vector $(h_0^*, h_1^*, \ldots, h_d^*)$ is called the *h^*-vector* of P.

Specifically, consider the polytope of true *magic squares*, i.e., of square matrices $A = [a_{ij}] \in \boldsymbol{R}^{n\times n}$ for which all row sums $\sum_j a_{ij}$ and column sums $\sum_i a_{ij}$ as well as the diagonal sums $\sum_i a_{ii}$ and $\sum_i a_{i,n-i}$ are equal to some integer constant t. This polytope is not a lattice polytope, but it is rational. Using standard methods one can prove that the h^*-vector of this polytope is symmetric.

Conjecture. *The h^*-vector of the polytope of magic $n \times n$-matrices is unimodal*

This conjecture has been verified computationally for $n < 7$ using `LattE`.

Problem 2 (Alex Engström)**.** Let $G = (V, E)$ be a finite graph. The *independence complex* of G is the simplicial complex with vertex set V and whose simplices are the independent sets of G,

$$\mathrm{Ind}(G) = \{I \subseteq V : I \text{ is an independent set of } G\}.$$

Conjecture. *If G is triangle-free then* $\mathrm{Ind}(G)$ *has no torsion in homology with integer coefficients.*

Problem 3 (Günter M. Ziegler)**.** (This problem goes back to a question raised by Vladlen Koltun.) Let $Z \subseteq \boldsymbol{R}^d$ be a zonotope with n zones. In other words, Z is the Minkowski sum of n line segments $[-v_i, v_i]$, $v_i \in \boldsymbol{R}^d$, $1 \leq i \leq n$.

Consider the intersection of Z with a 2-dimensional affine flat F. What is the maximum number of vertices of such an intersection? More precisely, what is the order of magnitude of this maximum if d is fixed and n is large? The vertices of the intersection correspond to ridges (i.e., $(d-2)$-dimensional faces) of Z that are intersected by F. It is well-known that the facets of Z are in one-to-one correspondence with the cells of the central arrangement of hyperplanes $\{x \in \boldsymbol{R}^d : \langle v_i, x\rangle = 0\}$, $1 \leq i \leq n$. Thus, Z has only $O(n^{d-1})$ many facets altogether, and the intersection $Z \cap F$ can have at most as many vertices.

In dimension $d = 3$, there are matching lower bound examples, i.e., zonotopes whose intersection with some 2-flat has $\Omega(n^2)$ vertices [2].

Challenge. *Find a lower bound construction that matches the trivial upper bound in any given dimension d, i.e., construct d-dimensional zonotopes with n zones whose intersection with some 2-flat has* $\Omega(n^{d-1})$ *vertices.*

Problem 4 (Rade Živaljević)**.** Consider the standard construction of the *mapping cylinder* $\mathrm{MC}(X \to Y)$ of a continuous map $f : X \to Y$. Recall that the mapping cylinder is formally defined as the quotient of the disjoint union $X \times [0,1] \uplus Y$ by the equivalence relation generated by $(x, 0) \sim f(x)$. This has a straightforward generalization $\mathrm{MC}(X \xrightarrow{f_1} Y_1, \ldots, X \xrightarrow{f_n} Y_n)$ to the case of several mappings $f_i : X \to Y_i$, $1 \le i \le n$, defined on the same space X. Formally, this is the quotient space of the disjoin union $X \times [0,1] \times [n] \uplus Y_1 \cup \ldots \uplus Y_n$ by the equivalence relation generated by $(x, 1, i) \sim (x, 1, j)$ and $(x, 0, i) \sim f_i(x)$, $1 \le i, j \le n$.

Consider now the special case where $X = T^n = S^1 \times \ldots \times S^1$ is the n-dimensional torus, Y_i is the $(n-1)$-dimensional torus T^{n-1} obtained by deleting the i^{th} factor S^1, and $f_i : X \to Y_i$ is the natural projection. It is known [1] that the resulting mapping cylinder is homotopy equivalent to a wedge of spheres,

$$\mathrm{MC}(T^n \xrightarrow{f_1} T^{n-1}, \ldots, T^n \xrightarrow{f_n} T^{n-1}) \simeq \bigvee_{k=2}^{n} (k-1)\binom{n}{k} \boldsymbol{S}^{k+1}.$$

Challenge. *Prove this by elementary methods (say by non-pure shelling or by discrete Morse theory).*

Problem 5 (Gil Kalai)**.** There is a result (which goes back to a question posed by James Joseph Sylvester) that says the following: If $X \subseteq \boldsymbol{R}^d$ is a finite set of points, not all contained in a hyperplane, then there exists a hyperplane H such that $H \cap X$ is "simple" (in the sense that all but one point in this intersection lie on a $(d-2)$-dimensional flat $F \subseteq H$) but "not too simple" (in the sense that the points in the intersection $H \cap X$ affinely span H).

We can view X as a zero-dimensional real algebraic variety. Can the above result be generalized to higher-dimensional algebraic varieties? If $X \subseteq \boldsymbol{R}^d$ is a k-dimensional real algebraic variety that spans $\boldsymbol{R}^d$, is it true that there is a hyperplane H such that $X \cap H$ is "simple" but "not too simple" in some reasonable sense? What are the conditions on the degree of (the polynomials defining) X, the dimension k of X and the ambient dimension d, if any?

Problem 6 (Gil Kalai)**.** Consider a *(unit) ball polyhedron* P that is the intersection of a finite number of (unit) balls in $\boldsymbol{R}^d$. What can be said about the facial structure of P?

Problem 7 (Imre Bárány)**.** Let $K \subseteq \boldsymbol{R}^2$ be a convex body of area 1, and let $0 < t < 1/4$. Do there exist two *orthogonal lines* that subdivide K into four regions of areas t, t, $1/2 - t$, and $1/2 - t$, respectively?

It is not hard to show that this is true in the limit cases $t = 0$ and $t = 1/4$. Moreover, for $t = 0$, one of the lines can be made tangent to K. (Note that if this were not the case, then the answer to the above question would be no for sufficiently small t.)

The same question can be asked with the uniform distribution on K replaced by any other (continuous Borel) probability measure on $\boldsymbol{R}^2$.

Conjecture. *The answer to the question is yes in the first case of a convex body K, but no for general probability measures.*

Problem 8 (Alexander Postnikov)**.** Consider the symmetric group S_n with the strong Bruhat order $<$ (for which u is covered by v if $\ell(v) = \ell(u) + 1$ and u and v differ by a transposition, and $\ell(w) = \#\{(i,j) : i < j, w_i > w_j\}$ is the number of *descents*). For $w \in S_n$, define two numbers

$$B(w) := \#\{u \in S_n : u < w\},$$

and

$$R(w) := \#\text{regions in the hyperplane arrangement } \{\{x_i - x_j = 0\}, 1 \le i < j \le n, w_i > w_j\}.$$

It is known that $B(w) = R(w)$ if w is a *Grassmannian permutation*, i.e., if $\ell(w) = 1$. This is also true if the *Schubert variety* V_w of w is smooth.

Conjecture. *For all $w \in S_n$,*

(1) *$R(w) \le B(w)$;*

(2) *moreover, "=" holds iff w avoids all of a certain short list of patterns.*

Problem 9 (Nati Linial)**.** It is known [4] that for a random ± 1-matrix $A \in \{-1,+1\}^{n\times n}$,

$$\Pr[A \text{ singular}] \le \left(\frac{3}{4}\right)^n.$$

Challenge. *Find nontrivial ways to generate random square ± 1-matrices of low rank (other than, for instance, simply repeating a row or a column). More specifically, find an efficient algorithm that on input $n > r > 1$ draws a matrix from the uniform distribution on $n \times n$ matrices of rank r. At present much less than this is still a challenge so any nontrivial step in this direction would be of interest.*

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