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Normal Families and Complex Dynamics

Organised by
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Abstract. The schedule comprised more than 25 ordinary and problem session talks from a broad range of areas in function theory, including but not limited to: Nevanlinna theory, iteration of rational functions, dynamics of transcendental entire and meromorphic functions, function algebras, Riemann surfaces, each of them in close connection to the main topic Normal Families and Complex Dynamics.

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Introduction by the Organisers

The workshop *Normal Families and Complex Dynamics*, organised by Phil Rippon (Milton Keynes), Norbert Steinmetz (Dortmund) and Lawrence Zalcman (Ramat-Gan) was held February 18th–February 24th, 2007.

Normal families and Nevanlinna theory. Almost half a century after it first appeared, Hayman's seminal work[1] continues to exert a pervasive influence on the study of the relation between properties enjoyed by meromorphic functions on the complex plane and the normality of families of meromorphic functions on plane domains.

If $a, b \in \mathbb{C}$, $a \neq 0$, and f is a transcendental meromorphic function, then $f + a(f')^n - b$ vanishes infinitely often for $n \geq 3$. In his talk *Mingliang Fang* proved that this remains true for $n = 2$, thus answering an old question of Ye. He also showed that if f is a meromorphic function on $\mathbb{C}$, which has all but finitely many poles multiple and shares the distinct values $0, a, b$ (counting multiplicity) with f', then $f \equiv f'$ unless a and b are related in a very specific way. *Jürgen Grahl* considers the condition $\psi(z) \equiv f^n(z) + af^{(k)}(xz) - b \neq 0$, where $a, b \in \mathbb{C}$,

[1] *Picard values of meromorphic functions and their derivatives*, Ann. Math. **70** (1959), 9–42.

$a \neq 0$, and $0 < |x| \leq 1$. It turns out that for a family $\mathcal{F}$ of holomorphic functions on the unit disc $\mathbb{D}$, this condition implies that $\mathcal{F}$ is normal at 0 if $n \geq 3$ and $k \geq 1$. For meromorphic functions on $\mathbb{C}$, this same condition implies that f is constant for $n \geq k+5$. *Shahar Nevo*'s contribution focuses on results just beyond the range of application of normal family techniques. For instance, the family $\mathcal{F}$ of meromorphic functions on a plane domain D, all of whose zeros have multiplicity at least $k+1$ and such that $f^{(k)}(z) \neq 1$ for each $f \in \mathcal{F}$ and $z \in D$, is not normal; but it is quasinormal of order 1. As a consequence, one can show that the derivative of a transcendental meromorphic function which has at most finitely many simple zeros takes on every nonzero value infinitely often.

Escaping sets. The complex dynamics talks focused on two main topics, one being properties of the escaping set. The early work on the escaping set of a general transcendental entire function was done by Eremenko, who established the close connection between the escaping set and the Julia set. He asked two questions that have provoked great interest: *are all components of the escaping set unbounded and, more strongly, are these components path connected to infinity?*

Walter Bergweiler described a generalisation of Wiman-Valiron theory to a meromorphic function with a direct tract. Using this, one can show that many properties of the escaping set that are true for an entire function but false for a general meromorphic function (for example, that this set has at least one unbounded component) remain true for a meromorphic function with a direct tract. *Dierk Schleicher* described an example of an entire function of bounded type (that is, one in the Eremenko-Lyubich class) for which every path component of the escaping set is bounded, thus showing that the answer to the strong form of Eremenko's question is 'no'. However, for an entire function of bounded type and finite order the escaping set is a disjoint union of curves, so in this case the answer is 'yes'. *Lasse Rempe* discussed an analog of Böttcher's theorem (that two polynomials of the same degree are conformally conjugate near infinity) for bounded type entire functions. If two such functions are quasiconformally equivalent near infinity, then they are quasiconformally conjugate on a set related to the escaping set, and this explains why their Julia sets look similar near infinity. *Boguslawa Karpińska* showed how coding trees can be used to describe the dynamics on the Julia set of an entire function whose inverse function singularities are compactly contained in an immediate basin of attraction, in particular discussing which points of the Julia set are accessible from within the basin. *Gwyneth Stallard* spoke about sufficient conditions for the escaping set of an entire function to be connected, giving a number of topological conditions which guarantee that this is true. From these conditions, and earlier work proving the absence of unbounded Fatou components, it can be deduced that the answer to Eremenko's question is 'yes' for many functions of small growth.

Connectivity of Fatou components. The second main topic in the complex dynamics part focused on connectivity properties of Fatou components.

Phil Rippon gave conditions for a meromorphic function with a finite number of poles to have a Baker wandering domain, and deduced that the eventual connectivity of a wandering domain of such a function is either two or infinity, in the case of a Baker wandering domain, and one otherwise. *Patricia Domínguez* showed how to use quasi-conformal surgery to construct examples of various types of functions (e.g. meromorphic functions and functions meromorphic outside small sets) with doubly-connected wandering domains in various configurations. *Marcus Stiemer* considered a class of generalized Blaschke products that can have zeros outside the unit disc as well as inside, and showed how to construct examples of functions in this class which have Fatou components of any given connectivity number. *Richard Stankewitz* discussed the dynamics of polynomial semigroups with bounded postcritical set and described some of their properties; for example, if the Fatou set has two doubly connected components, then the Julia set contains a Cantor family of quasicircles. *Nuria Fagella* discussed work on a generalization of a result of Shishikura, showing that if a meromorphic function has a multiply connected Fatou component which is of a certain type, then the function must have a weakly repelling fixed point.

Hausdorff dimension and measure. *Ludwik Jaksztas* described work related to the limiting behaviour of the Hausdorff dimension of certain quadratic Julia sets near the parameter values $1/4$ and $-3/4$. *Jörn Peter* extended a result of McMullen by showing that for certain exponential functions with an attracting fixed point there is a gauge function, related to the Schröder function of the fixed point, with respect to which the Hausdorff measure of the Julia set is infinity.

Miscellanea. *Dzmitry Dudko* described a proof, due to Adam Epstein, of the Fatou-Shishikura inequality. *Lukas Geyer* presented a survey on the type problem, which consists in deciding whether a Riemann surface (X, f) spread over the plane is hyperbolic or parabolic. He showed that (X, f) is type-stable if it has uniformly separated singularities, but showed also that there exists a parabolic type-stable surface (X, f) whose singularities are not uniformly separated. *Aimo Hinkkanen*'s talk focussed on majorants of analytic functions, i.e. the question whether or not conditions imposed on analytic functions on the boundary of their domain of definition remain valid in the interior. A typical question is to ask whether $|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$ with $z_1 \in \partial G$ fixed and $z_2 \in \partial G$ implies $|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)$ for $z_2 \in G$, C some constant. Under reasonable circumstances this was proved with C independent of μ. *James Langley* talked about value distribution properties of functions in the classes S and B (functions meromorphic in $\mathbb{C}$ having finitely many resp. a bounded set of singularities of f^{-1}). In particular he proved the conjectures of Mues for functions in B having finite lower order, and Gol'dberg's conjecture for functions in S, without any further restriction. *Oliver Roth* proved existence of a Blaschke product with prescribed sequence (z_ν) of critical points (satisfying the Blaschke condition); existence was known before only in the case of finitely many points. Roth's proof is based on solvability of the elliptic boundary value problem $\Delta u = 4|h(z)|^2 e^{2u}$ on $\mathbb{D}$, $u \to \infty$ as $z \to$ ß∂D, with h the Blaschke procuct with zeros z_ν. *Nikita Selinger* gave an interesting description of Douady's and Hubbard's proof of Thurston's theorem, which characterizes when a postcritically finite branched cover of the Riemann sphere is equivalent to a rational map.

Workshop: Normal Families and Complex Dynamics

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Abstracts

Dynamics of meromorphic functions with direct singularities

Walter Bergweiler
(joint work with P. J. Rippon and G. M. Stallard)

For a function f meromorphic in the plane the Fatou set $F(f)$ is defined as the set where the iterates f^n of f are defined and form a normal family, and the Julia set $J(f)$ is its complement. The escaping set $I(f)$ is defined as the set of all $z \in \mathbb{C}$ for which $f^n(z) \to \infty$ as $n \to \infty$.

Eremenko [3] proved that if f is an entire transcendental function, then $I(f) \neq \emptyset$, $\partial I(f) = J(f)$, $I(f) \cap J(f) \neq \emptyset$ and $\overline{I(f)}$ has no bounded components. Eremenko conjectured that in fact all components of $I(f)$ are unbounded. This conjecture is still open, but it is known that $I(f)$ has at least one unbounded component [4]. Domínguez [2] proved that $I(f) \neq \emptyset$, $\partial I(f) = J(f)$ and $I(f) \cap J(f) \neq \emptyset$ also holds for meromorphic functions f. However, $I(f)$ need not have unbounded components. In fact, for $f(z) = \frac{1}{2}\tan z$ the sets $I(f)$ and $J(f)$ are totally disconnected.

Here we show that some of the results for entire functions remain true for meromorphic functions which have a direct singularity over infinity. These are, by definition, the meromorphic functions f for which there exists $R > 0$ such that $f^{-1}(\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\})$ contains a component D with the property that $f(z) \neq \infty$ for all $z \in D$. We call such a component D a *direct tract* of f. (Actually our results do not require that f is meromorphic outside the direct tract, but for simplicity we restrict to functions meromorphic in the plane.)

Theorem 1. *Let f be a meromorphic function with a direct singularity over infinity. Then $I(f)$ has an unbounded component.*

Theorem 2. *Let f be a meromorphic function with a direct singularity over infinity. Then $J(f) \cap I(f)$ contains continua.*

The proofs of the above results for entire functions use the Wiman-Valiron theory or the maximum principle and thus do not carry over to meromorphic functions with poles. Our main tools are results whose statements are similar to those of the Wiman-Valiron theory, or of Macintyre's theory of flat regions.

To describe these results, we note that for a non-constant subharmonic function $v : \mathbb{C} \to [0, \infty)$ the function

$$B(r, v) := \max_{|z|=r} v(z)$$

is increasing, convex in $\log r$ and tends to ∞ as r tends to ∞. Hence

$$a(r, v) := \frac{dB(r, v)}{d\log r} = rB'(r, v)$$

exists except perhaps for a countable set of r-values, and $a(r, v)$ is non-decreasing.

Note that if f, D, R are as above, then the function $v : \mathbb{C} \to [0, \infty)$ defined by

$$v(z) = \begin{cases} \log \dfrac{|f(z)|}{R} & \text{if } z \in D, \\ 0 & \text{if } z \notin D, \end{cases} \tag{1}$$

is subharmonic. While for a nonconstant function v subharmonic in the plane we only know that

$$\lim_{r\to\infty} \frac{B(r, v)}{\log r} > 0,$$

functions of the form (1) have faster growth.

Theorem 3. *Let D be a direct tract of f and let v be defined by* (1)*. Then*

$$\lim_{r\to\infty} \frac{B(r, v)}{\log r} = \infty \tag{2}$$

and

$$\lim_{r\to\infty} a(r, v) = \infty. \tag{3}$$

Our main tool to study functions with a direct tract is the following result. Here we denote the open disk of radius r around a point $a \in \mathbb{C}$ by $D(a, r)$.

Theorem 4. *Let D be a direct tract of f and let $\tau > \frac{1}{2}$. Let v be defined by* (1) *and let z_r be a point satisfying $|z_r| = r$ and $v(z_r) = B(r, v)$. Then there exists a set $F \subset [1, \infty)$ of finite logarithmic measure such that if $r \in [1, \infty) \setminus F$, then $D(z_r, r/a(r, v)^\tau) \subset D$. Moreover,*

$$f(z) \sim \left(\frac{z}{z_r}\right)^{a(r,v)} f(z_r) \quad \textit{for} \quad z \in D\left(z_r, \frac{r}{a(r, v)^\tau}\right) \tag{4}$$

as $r \to \infty$, $r \notin F$.

Since $a(r, v) \to \infty$ as $r \to \infty$ by Theorem 3 we see that (4) can also be written in the form

$$f(z_r e^h) \sim e^{a(r,v)h} f(z_r) \quad \text{for} \quad |h| \le a(r, v)^{-\tau}, \tag{5}$$

again as $r \to \infty$, $r \notin F$.

The asymptotic relations (4) and (5) are very similar to the main result of Wiman-Valiron theory, except that the central index is replaced by $a(r, v)$. However, the methods of Wiman and Valiron based on the Taylor series of f do not apply. The main tools we use are a lower bound of the maximum modulus of a subharmonic function due to Tsuji, Jensen's formula for subharmonic functions, and certain growth lemmas for real functions.

As in Wiman-Valiron theory, an important consequence of (4) or (5) is the following result, which can be deduced from them for example by Rouché's theorem.

Theorem 5. *For each* $\beta > 1$ *there exists* $\alpha > 0$ *such that if* f, D, v, z_r *and* F *are as in Theorem 4 and if* $r \notin F$ *is sufficiently large, then*

$$\left\{ z \in \mathbb{C} : \frac{|f(z_r)|}{\beta} \leq |z| \leq \beta |f(z_r)| \right\} \subset f\left(D\left(z_r, \frac{\alpha r}{a(r,v)} \right) \right).$$

More precisely, $\log f$ *is univalent in* $D(z_r, \alpha r/a(r,v))$ *and for* $\gamma > \pi$ *the constant* α *can be chosen such that* $\log f\,(D(z_r, \alpha r/a(r,v)))$ *contains the square*

$$\{ z \in \mathbb{C} : |\text{Re}\, z - \log |f(z_r)|| \leq \log \beta, |\text{Im}\, z - \arg f(z_r)| \leq \gamma \}$$

if $r \notin F$ *is sufficiently large, where the branches of* log *and* arg *are chosen such that* $\text{Im}(\log f(z_r)) = \arg f(z_r)$.

The Wiman-Valiron theory was the main tool in Eremenko's proof [3] that $I(f) \neq \emptyset$ if f is a transcendental entire function. Using Theorem 5 instead of the Wiman-Valiron method in his argument we see that if f has a direct tract D, then there exists $z \in D$ such that $f^n(z) \to \infty$ as $n \to \infty$, with $f^n(z) \in D$ for all n. The result that z can be chosen such that all $f^n(z)$ belong to the same tract seems to be new even for entire functions.

With

$$M(r) := \max_{|z|=r, z \in D} |f(z)| = \exp B(r,v)$$

Eremenko's argument yields that there exists z_0 such that

$$|f^{n+1}(z_0)| \sim M(|f^n(z_0)|) \tag{6}$$

as $n \to \infty$. It follows from (2) that $M(\rho) > \rho$ for large ρ, say $\rho > \rho_0 > R$. Hence $M^n(\rho) \to \infty$ as $n \to \infty$ for $\rho > \rho_0$. For such ρ we define

$$A(f,D,\rho) := \{ z \in D : f^n(z) \in D \text{ and } |f^n(z)| \geq M^n(\rho) \text{ for all } n \in \mathbb{N} \}.$$

In particular it follows that $|f^n(z)| \to \infty$ as $n \to \infty$ for $z \in A(f,D,\rho)$ so that $A(f,D,\rho) \subset I(f)$.

As in [1] the existence of points z_0 satisfying (6) yields the following result.

Theorem 6. *Let* D *be a direct tract of* f. *Then* $A(f,D,\rho) \neq \emptyset$.

Combining the above reasoning with the methods of [4] we obtain the following result which clearly implies Theorem 1.

Theorem 7. *Let* D *be a direct tract of* f. *Then all components of* $A(f,D,\rho)$ *are unbounded.*

References

[1] W. Bergweiler and A. Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Cambridge Philos. Soc. **126** (1999), 565-574.

[2] P. Domínguez, *Dynamics of transcendental meromorphic functions*, Ann. Acad. Sci. Fenn. (Ser. A, I. Math.) **23** (1998), 225-250.

[3] A. E. Eremenko, *On the iteration of entire functions*, in Dynamical systems and ergodic theory, Banach Center Publications **23**, Polish Scientific Publishers, Warsaw 1989, 339-345.

[4] P. J. Rippon and G. M. Stallard, *On questions of Fatou and Eremenko*, Proc. Amer. Math. Soc. **133** (2005), 1119-1126.

A problem concerning Newton's method (problem session)

Walter Bergweiler

Let f be an entire transcendental function and let $N_f(z) = z - f(z)/f'(z)$ be the associated Newton function. Suppose that N has an invariant Baker domain; that is, there exists a domain U with $N(U) \subset U$ where the iterates of N tend to ∞.

Question 1. Does there exists a positive constant K and a curve γ tending to ∞ such that $|f(z)| \leq |z|^K$ for $z \in \gamma$?

Question 2. Is the order of f at least $\frac{1}{2}$?

The above questions are related to a question of Douady who had asked whether f must have the asymptotic value 0 under the above hypotheses. This was shown to be true under mild additional hypotheses [1] but turned out to be false in general [2]. However, for the counterexample in [2] there exists a curve γ tending to ∞ where $|f(z)| = O\left(|z|^{1/3}\right)$ as $z \to \infty$, $z \in \gamma$. It seems possible to replace $\frac{1}{3}$ by another constant here, but the question is, whether there is always a growth restriction like this on a curve.

An affirmative answer to the first question would imply an affirmative answer to the second one because of the classical $\cos \pi\rho$-theorem.

References

[1] X. Buff and J. Rückert, *Virtual immediate basins of Newton maps and asymptotic values*, Int. Math. Res. Not. 2006, 65498, 1–18.

[2] W. Bergweiler, D. Drasin and J. K. Langley, *Baker domains for Newton's method*, to appear in Ann. Inst. Fourier.

Doubly connected wandering domains for meromorphic functions

Patricia Domínguez

(joint work with Guillermo Sienra)

Let $f : X \to X$ denote a meromorphic function, where $X = \mathbb{C}$ or $\widehat{\mathbb{C}}$, and f^n the n-th iterate of f. We define the *Fatou set* $F(f)$ as the set of those points $z \in X$ such that $\{f^n\}_{n\in\mathbb{N}}$ is meromorphic and forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$ of f. If U is a component of $F(f)$, $f^k(U)$ is contained in a unique component U_k of $F(f)$, for each $k \neq n$. When $U_k \neq U_n$ the component U is called *wandering component*. Otherwise U is either pre-periodic or periodic.

We will deal in this talk with the following classes of maps.

$\mathcal{R} = \{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} |$ f is rational of degree at least two $\}$.

$\mathcal{E} = \{f : \mathbb{C} \to \mathbb{C} |$ f is transcendental entire $\}$.

$\mathcal{M} = \{f : \mathbb{C} \to \widehat{\mathbb{C}} |$ f is transcendental meromorphic with at least one not omitted pole $\}$.

$\mathcal{K} = \{f : \widehat{\mathbb{C}} \setminus B \to \widehat{\mathbb{C}} |$ $B \neq \emptyset$, B is a compact countable set and f non-constant meromorphic $\}$.

The set B is the closure of the isolated essential singularities of f, observe that $\mathcal{M} \subset \mathcal{K}$. The class $\mathcal{K}$ has been introduced by Bolsch in [6, 7]. Independently, broader generalizations have been given by Baker, Domínguez and Herring in [4], [12]. The Fatou and the Julia set can be defined as above for functions in class $\mathcal{K}$. General properties of $J(f)$ and $F(f)$ are the same as for the other classes of functions, but different proofs are needed.

Sullivan in [15] proved that wandering domains do not arise for functions in class $\mathcal{R}$ (rational functions). The analogue of Sullivan's result holds for the class $\mathcal{E}$ of transcendental entire functions f such that the set of singularities of f^{-1} is finite, we shall denote this set by $S(f)$. Transcendental entire functions outside the set $S(f)$ can have wandering domains.

One of the problems of wandering domains concerns their connectivity. For the class $\mathcal{E}$ wandering domains may be either simply connected or multiply connected with infinite connectivity. Examples of these facts have been constructed by Baker [2], Herman [11], Eremenko and Lyubich [9] and Devaney [8]. It has been an open question whether the connectivity of a wandering domain for $f \in \mathcal{E}$ could be finite but greater than one. Recently Kisaka and Shishikura in [13] gave the first example of such a wandering domain. They used quasi-conformal surgery to construct a function in class $\mathcal{E}$ with a wandering domain with connectivity two.

Baker, Kotus and Yinian in [3] solved the connectivity problem for functions in class $\mathcal{M}$. They constructed examples of bounded and unbounded wandering domains either with finite or infinite connectivity by using results on complex approximation.

The main task in this work is to give examples of wandering domains for functions in class $\mathcal{M}$ without using results of complex approximation. The construction is based on quasi-conformal surgery. In addition our construction gives examples of wandering domains for functions in class $\mathcal{K}$. The following theorem give us the construction of a doubly connected wandering domain in class $\mathcal{M}$ and in class $\mathcal{K}$.

Theorem A.

(1) *There exists a function $h \in \mathcal{M}$ with a doubly connected wandering domain in the Fatou set.*

(2) *There exists a function h in class $\mathcal{M}$ such that the Fatou set $F(h)$ has a doubly connected wandering domain U. Moreover, there is a simply connected wandering domain $V \subset F(h)$ which is surrounded by the inner boundary of U.*

(3) *For every natural k there exists a function h in class $\mathcal{M}$ such that the Fatou set $F(h)$ has m-nested doubly connected wandering domains W_i, and a simply connected wandering domain $V \subset F(h)$ such that V is surrounded by the inner boundary of W_i.*

Observe that in Theorem A the case (3) contains the case (2) which contains case (1). However, we display it in three cases in order to understand the surgery construction. For the class $\mathcal{K}$ we have the following results.

Theorem B. *There exists a function h in class $\mathcal{K}$ such that the Fatou set $F(h)$ has a doubly connected wandering domain.*

Theorem C. *There exists a function h in class $\mathcal{K}$ such that the Fatou set $F(h)$ has a doubly connected wandering domain which is unbounded on one side.*

References

[1] L. Ahlfors, *Lectures on Quasi-conformal mappings.* Van Nostrand 1966.
[2] I.N. Baker, *Wandering domains in the iteration of the entire functions,* Proc. London Math. Soc. **49** (1984), 563-576.
[3] I.N Baker, J. Kotus and Lü Yinian, *Iterates of Meromorphic Fucntions II: Examples of Wandering Domains,* J. London. Math. Soc. **42** (1990), 267-278.
[4] I.N Baker, P. Domínguez and M. E. Herring, *Dynamics of functions meromorphic outside a small set,* Ergod. Th. Dynam. Sys. **21** (2001), 647-672.
[5] W. Bergweiler, *On the Julia set of analytic self-maps of the punctured plane,* Analysis **15** (1995), no 3, 251-256.
[6] A. Bolsch, *Repulsive periodic points of meromorphic functions,* Complex Variables Theory and Application **31** (1996), 75-79.
[7] A. Bolsch, *Iteration of meromorphic functions with countably many singularities,* PhD Thesis TU Berlin (1997).
[8] R. L. Devaney, *Dynamics of entire maps, Dynamics System and Ergodic Theory,* Banach Center (23) Polish Scientific Publishers Warsaw, (1989), 221-228.
[9] A. Eremenko and M. Yu. Lyubich, *Examples of entire functions with pathological dynamics,* J. London Math. Soc. (2) **36** (1978), 458-468.
[10] P. Domínguez and N. Fagella, *Existence of Herman rings for meromorphic functions,* Complex Variables Theory and Applications **49** (2004), 851-870.
[11] M. Herman, *Examples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann,* Bull. Soc. Math. France **112** (1984), 93-142.
[12] M. E. Herring, *An extension of the Julia-Fatou theory of iteration,* PhD. Thesis Imperial College, UK (1994).
[13] M. Kisaka and M. Shishikura, *On multiply connected wandering domains of entire functions,* to appear in *Transcendental dynamics and complex analysis*, LMS Lecture Notes, Cambridge University Press 2007.
[14] M. Shishikura, *On the quasi-conformal surgery of rational functions.,* Ann. Scient. Ec. Norm. Sup. **20** (1987), 1-29.
[15] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I: Solutions of the Fatou-Julia problem on wandering domains,* Ann. of Math **122** (3) (1985), 401-418.

Refined Fatou-Shishikura Inequality

Dzmitry Dudko

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map with degree $D > 1$. The famous Fatou-Shishikura inequality says that the number of non-repelling periodic points is less or equal $2D - 2$, the number of critical points. In this inequality any parabolic periodic point is counted as ν times where ν is a multiplicity of parabolic point. In 1999 A. Epstein presented a new independent proof of Fatou-Shishikura inequality which gives a slightly better result.

For any parabolic periodic point z_0 the first return to z_0 can be expressed as

$$z \to p(z + z^{N+1} + \alpha z^{2N+1}) + O(z^{2N+2}) \tag{1}$$

where $N = \nu n$, p is primitive n-root of unity and ν is a number of cycles of petals. If $\operatorname{Re}\beta = \operatorname{Re}(\frac{N+1}{2} - \alpha) \leq 0$ then in the new variant of the inequality the parabolic periodic point z_0 can be counted as $\nu + 1$ instead of ν.

The idea of the proof is to consider the space of forms on Riemann sphere and the linear map $\nabla_f = I - f_*$ on this space, where f_* is pushforward. Assume that f is not a Lattès example. Let $M(\mathbb{P}^1)$ be the space of all meromorphic quadratic differentials q on the Riemann sphere $\mathbb{P}^1$, and $Q(\mathbb{P}^1)$ be subspace of all $q \in M(\mathbb{P}^1)$ with at worst simple poles. Denote by $D(\mathbb{P}^1)$ the quotient $M(\mathbb{P}^1)/Q(\mathbb{P}^1)$. Each class $[q] \in D(\mathbb{P}^1)$ depends only on non-simple poles of q. The pullback f^* and pushforward f_* are well defined linear operators on these spaces, so $\nabla_f = I - f_*$ is also well defined.

Let $D(f) = ker\nabla_f|_{D(\mathbb{P}^1)} \in D(\mathbb{P}^1)$ then it is possible to show that

$$D(f) \cong \bigoplus_{\langle z\rangle \subset \mathbb{P}^1} D_{\langle z\rangle}(f),$$

where the subspace $D_{\langle z\rangle}(f)$ consists of all $[q] \in D(f)$ where all non-simple poles of q are on periodic cycle $\langle z\rangle$. The dimension of $D_{\langle z\rangle}(f)$ is one for attracting, repelling or irrational indifferent periodic cycles, $\nu + 1$ for parabolic, and 0 for superattracting.

Let class $[q]$ be from the subspace $D(f)$. According to the Thurston Rigidity principle $\nabla_f q \neq 0$ if q has only simple poles. And this fact follows from the inequality

$$\|q\| = \int_{\mathbb{P}^1} |q| = \int_{\mathbb{P}^1} f_*|q| < \int_{\mathbb{P}^1} |f_* q| = \|f_* q\|\,.$$

But if q is able to have poles with order more than one then it can happen that $\nabla_f q = 0$. But then the following inequality has to hold:

$$0 < \int_{\mathbb{P}^1} (f_*|q| - |f_* q|) = 2\pi Res(f : q),$$

where $Res(f : q) = \frac{1}{2\pi} \int_{\mathbb{P}^1} (f_*|q| - |q|)$ is the dynamical residue. From this inequality follows that if $[q] \in D(f)$ and $Res(f : q) \leq 0$ then $\nabla_f q \neq 0$.

Let $D^b_{\langle z\rangle}(f) = \{[q] \in D(f)|\ Res(f : q) \leq 0\}$ and $\gamma_{\langle z\rangle}$ be the dimension of $D^b_{\langle z\rangle}(f)$. Then

$$\gamma_{\langle z\rangle} = \begin{cases} 0 & \text{if } \langle z\rangle \text{ is repelling or superattracting,} \\ 1 & \text{if } \langle z\rangle \text{ is attracting or irrational indifferent,} \\ \nu & \text{if } \langle z\rangle \text{ is parabolic and } \operatorname{Re}\beta > 0, \\ \nu + 1 & \text{if } \langle z\rangle \text{ is parabolic and } \operatorname{Re}\beta \leq 0 \end{cases}$$

So it is possible to construct the space $D(f)$ with dimension $\gamma_f = \sum_{\langle z\rangle} \gamma_{\langle z\rangle}$, in such way that $\nabla_f q \neq 0$ if $[q] \in D(f)$.

Finally we can construct the injective linear map from $D(f)$ to the space $Q(\mathbb{P}^1, A^+)/Q(\mathbb{P}^1, A)$ where dimension $Q(\mathbb{P}^1, A^+)/Q(\mathbb{P}^1, A)$ is equal to the number of infinite tails of critical orbits of f. From the last statement follows the refined Fatou-Shishikura inequality.

References

[1] A. Epstein, *Infinitesimal Thurston Rigidity and the Fatou-Shishikura Inequality. http://arxiv.org/PS_cache/math/pdf/9902/9902158.pdf*
[2] J. Milnor, *Dynamics in one complex variable.*

Connectivity of Julia sets of transcendental meromorphic functions

Núria Fagella
(joint work with Xavier Jarque and and Jordi Taixés)

Shishikura in 1990 proved the following:

Theorem. *If R is a rational map with disconnected Julia set, then R has at least two weakly repelling fixed points* (repelling or derivative = 1).

A corollary is that the Julia set of Newton's method of any polynomial must be connected. We partially generalize this theorem to transcendental meromorphic functions by showing:

Theorem (F., Jarque, Taíxes). *If f is a transcendental meromorphic function and U is a multiply connected component of $\mathcal{F}(f)$ which is either an immediate attracting or parabolic basin, or a Herman ring or a preperiodic component, then f has at least one weakly repelling fixed point.*

Put together with Bergweiler's result on wandering domains, we have the following

Corollary. *If f is an entire transcendental map and N_f is its Newton function, then all connected components of $\mathcal{F}(N_f)$ are simply connected, except maybe Baker domains.*

The tools for the proof are surgery, Fatou's theorem on existence of weakly repelling fixed points and a theorem of Xavier Buff on virtually repelling fixed points.

Normal families and their applications to the uniqueness of entire functions

Mingliang Fang
(joint work with Jianming Chang and Lawrence Zalcman)

1. Bloch's Principle and value distribution

In [4], Hayman proved

Theorem A. *Let f be a transcendental meromorphic function and a a nonzero complex number. Then $f' + af^n$ assumes every complex value infinitely often for each positive integer $n \geq 5$.*

The proof of Theorem A also shows that if f is meromorphic on the plane and $f' + af^n$ $(n \geq 5)$ fails to assume some complex value b, then f must be constant. On the other hand, Mues [5] showed that for $n = 3$ and 4 and each $b \neq 0$, there exists a transcendental function f such that $f' + af^n \neq b$.

Considering a similar problem, Ye [6] proved

Theorem B. *Let f be a transcendental meromorphic function and a a nonzero complex number. Then $f + a(f')^n$ assumes every complex value infinitely often for each positive integer $n \geq 3$.*

Ye asked whether Theorem B remains valid for $n = 2$. (It clearly fails for $n = 1$, as one sees by taking $f(z) = e^{-z/a}$.) We give an affirmative answer to this question.

Theorem 1. *Let f be a transcendental meromorphic function and a a nonzero complex number. Then $f + a(f')^2$ assumes every complex value infinitely often.*

Note that the condition $f + a(f')^2 \neq b$ for some $b \in \mathbb{C}$ does *not* imply that a meromorphic function on $\mathbb{C}$ is constant. Indeed, for $f(z) = z^2$ and $a = -1/4$, we have $f + a(f')^2 \equiv 0$.

According to Bloch's Principle [1], for each theorem of Liouville-Picard type, there is a corresponding criterion for the normality of families of meromorphic functions on plane domains. The normality criterion corresponding to Theorem A was established independently by J.K. Langley, S.Y. Li, and X.J. Li. In fact, as follows from an argument due to X.C. Pang, the condition that $f' + af^n$ omit some fixed value implies normality even in the cases $n = 3$ and 4, where there is no corresponding Picard-type theorem. (Thus, the *converse* of Bloch's Principle fails in these cases.) See [7] for complete references and a discussion.

It is therefore natural to ask whether there exists a normality criterion corresponding to Theorem B and Theorem 1. It is easy to see that the answer is negative. However, if we require an additional condition, we obtain a positive result.

Theorem 2. *Let $\mathcal{F}$ be a family of meromorphic functions on the plane domain D, let $n \geq 2$ be a positive integer, and let $a \neq 0$ and b be complex numbers. If, for each $f \in \mathcal{F}$, all zeros of f are multiple and $f + a(f')^n \neq b$ on D, then $\mathcal{F}$ is normal on D.*

2. Normal families and uniqueness theorems for entire functions

Let f and g be two meromorphic functions in a domain D in $\mathbb{C}$, and let a be a complex number. If $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicity, then we say that f and g share the value a **CM** in D; if $f(z) = a$ if and only if $g(z) = a$, then we say that f and g share the value a **IM** in D.
Let S be a set of complex numbers. Set

$$E(S, f) = \bigcup_{a \in S} \{z : \ f(z) - a = 0\},$$

where a zero of multiplicity m is counted m times in the set.

Nevanlinna proved the following theorem.

Theorem C. *Let a_j $(j = 1, 2, 3, 4)$ be four distinct finite complex numbers. Suppose that f and g are two non-constant entire functions. If $E(a_j, f) = E(a_j, g)$ for $j = 1, 2, 3, 4$, then $f \equiv g$.*

In 1976, Gross posed the following question.

Question A. Can one find a finite set S such that any two non-constant entire functions f and g satisfying $E(S, f) = E(S, g)$ must be identical? If such a set exists, how large must it be?

In 1994, Yi proved that such a set exists.

Theorem D. *There exists a finite set S containing 7 elements such that if f and g are two non-constant entire functions and $E(S, f) = E(S, g)$, then $f \equiv g$.*

In 1977, Rubel and Yang proved

Theorem E. *Let a, b be two distinct finite numbers, and let f be a non-constant entire function. If $E(a, f) = E(a, f')$, and $E(b, f) = E(b, f')$, then $f \equiv f'$.*

We pose the following question.

Question B. Let S be such that if a non-constant entire function f and its derivative f' satisfy $E(S, f) = E(S, f')$, then $f \equiv f'$. How large must S be?

In [3], using the theory of normal families, we study the uniqueness of entire functions and obtain the following result.

Theorem F. *There exists a set S with 3 elements such that if a non-constant entire function f and its derivative f' satisfy $E(S, f) = E(S, f')$, then $f \equiv f'$. The number 3 is best possible.*

In 1986, Jank-Mues-Volkmann proved

Theorem G. *Let f be a non-constant entire function, and let a be a nonzero constant. If $E(a, f) = E(a, f')$ and $f''(z) = a$ whenever $f'(z) = a$, then $f \equiv f'$.*

We pose the following question.

Question C. Let $b(\neq a)$ be a nonzero finite complex number. If $f''(z) = a$ whenever $f'(z) = a$ is replaced by $f''(z) = b$ whenever $f'(z) = b$ in Theorem G, is it still true that $f \equiv f'$?

In [3], we also proved

Theorem H. *Let f be a non-constant entire function, $k \geq 2$ a positive integer, and let a, b be two constants such that $b \neq 0$. If f and f' share a **CM**, and $f^{(k)}(z) = b$ whenever $f'(z) = b$, then $f(z) = de^{cz} + \frac{c-1}{c}a$, where c, d are two nonzero constants and $c^{k-1} = 1$.*

In particular, $f \equiv f'$ for $k = 2$.

Naturally, we ask that whether Theorem H is valid or not if f and f' share a **CM** is replaced by f and f' share a **IM** ?

In [2], we have proved the following more general result.

Theorem 3. *Let f be a nonconstant entire function; let a be a finite nonzero complex number; and let $k \geq 2$ be a positive integer. Suppose that $f(z) = a \Longrightarrow f'(z) = a$, and $f'(z) = a \Longrightarrow f^{(k)}(z) = a$. Then either $f(z) = Ce^{\lambda z} + a$ or $f(z) = Ce^{\lambda z} + \frac{a(\lambda-1)}{\lambda}$, where C, λ are nonzero constants with $\lambda^{k-1} = 1$.*

Most recently, continuing our study of the uniqueness of entire functions by using the theory of normal families, we have proved the following results.

Theorem 4. *Let f be a nonconstant entire function and let $S = \{a, b, c\}$, where a, b and c are distinct complex numbers. If f satisfies $E(S, f) = E(S, f')$, then one of the following cases must occur:*
(i) $f = Ce^z$;
(ii) $f(z) = Ce^{-z} + \frac{2}{3}(a + b + c)$ and $(2a - b - c)(2b - c - a)(2c - a - b) = 0$;
(iii) $f(z) = Ce^{\frac{-1\pm\sqrt{3}i}{2}z} + \frac{3\pm\sqrt{3}i}{6}(a + b + c)$ and $a^2 + b^2 + c^2 - ab - bc - ca = 0$,
where C is a nonzero constant.

Theorem 5. *There exists a set S of three distinct complex values such that if a nonconstant meromorphic function f, all but finitely many of whose poles are multiple, satisfies $E(S, f) = E(S, f')$, then $f \equiv f'$.*

References

[1] W. Bergweiler, *Bloch's Principle,* Comput. Methods Funct. Theory **6** (2006), 77-108.

[2] J. M. Chang and M. L. Fang, *Normal families and uniqueness of entire functions and their derivatives,* Acta Math. Sinica, English Series (2007), to appear.

[3] M. L. Fang and L. Zalcman, *Normal families and uniqueness theorems for entire functions,* J. Math. Anal. Appl. **280** (2003), 273-283.

[4] W. K. Hayman, *Picard values of meromorphic functions and their derivatives,* Ann. of Math. **70** (1959), 9-42.

[5] E. Mues, *Über ein Problem von Hayman,* Math. Z. **164** (1979), 239-259.

[6] Y. S. Ye, *A Picard Type Theorem and Bloch Law,* Chinese Ann. Math. Ser. B **15** (1994), 75-80.

[7] L. Zalcman, *New light on normal families,* Proceedings of the Ashkelon Workshop on Complex Function Theory (May, 1996), ed. L. Zalcman, Bar-Ilan University, 1997, pp. 237-245.

Type Problem for Riemann Surfaces

Lukas Geyer
(joint work with Sergei Merenkov)

A *surface spread over the sphere* is a pair (X, f), where X is a topological surface and $f \to X \to \widehat{\mathbb{C}}$ a continuous, open and discrete map into the Riemann sphere. According to a theorem of Stoïlow, there exists a unique conformal structure on X which makes f holomorphic. We will denote this Riemann surface by X_f.

From now on we will assume that X is open and simply connected. By the Uniformization Theorem, X_f is conformally equivalent to either $\mathbb{C}$ or $\mathbb{D}$. We say that (X, f) is of *parabolic* or *hyperbolic type*, respectively. A natural question is how topological and metric properties of f determine the type of X_f and the value distribution properties of f. This is a very hard problem with no satisfactory general theory to approach it. For a recent survey of results and open problems see [1].

A continuous, open and discrete map f has a discrete set C_f of *critical points*, i.e. points where the mapping is not locally injective. The image of a critical point under f is a *critical value*. A point $a \in \widehat{\mathbb{C}}$ is called an *asymptotic value* of f, if there exists a curve $\gamma\colon [0, t_0) \to X$ such that

$$\gamma(t) \to \infty \text{ and } \psi(\gamma(t)) \to a \text{ as } t \to t_0.$$

A point $a \in \widehat{\mathbb{C}}$ is a *singular value* of f if a is a critical or an asymptotic value of ψ.

A surface spread over the sphere (X, f) is *type-stable* if the conformal type of $X_{\phi\circ f}$ is the same as that of X_f for any homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Teichmüller showed that (X, f) is type-stable if f has a finite set of singular values [2], using the fact that the type is invariant under quasiconformal mappings.

Eremenko suggested in [1] that Teichmüller's result extends to the wider class of functions with uniformly separated singularities, in the following sense. The *Mazurkiewicz metric* on X is defined as $d(x, y) = \inf \operatorname{diam} f(K)$, where diam denotes the spherical diameter, and the infimum is taken over all continua in X containing x and y. If $\overline{X}$ denotes the completion of X with respect to the Mazurkiewicz metric, then f extends continuously to $\overline{X}$, and the points in $Z_f = \overline{X} \setminus X$ are the transcendental singularities of f. We say that f has *uniformly separated singularities* if there exists $r > 0$ such that $d(x, y) \geq r$ for $x, y \in C_f \cup Z_f$, $x \neq y$.

Theorem 1. *If (X, f) has uniformly separated singularities, then (X, f) is type-stable.*

We prove this theorem by quasiconformally modifying f and $\phi \circ f$ to have finitely many singular values and the same topological behavior, thus reducing it to Teichmüller's theorem.

Eremenko also asked whether the converse of Theorem 1 holds for the case of parabolic surfaces (X, f). We show that this is not the case.

Theorem 2. *There exists a parabolic type-stable surface (X, f) whose singularities are not uniformly separated.*

The example, a modification of a function with finitely many singular values, is a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$, symmetric with respect to the real line, with real critical points, and no asymptotic values. This function has a sequence of critical points c_n such that $d(c_{2k}, c_{2k+1}) \to 0$, i.e. singularities are not uniformly separated. Type-stability follows from the existence of an infinite sequence of nested annuli $A_k \to \infty$, each of which is mapped conformally by f to a fixed annulus $A \subset \widehat{\mathbb{C}}$. Now if ϕ is any homeomorphism of the sphere, then $\phi(A)$ has a fixed positive modulus, so the A_k form a nested sequence of annuli in $X_{\phi \circ f}$ with $A_k \to \infty$ and constant moduli. By a theorem of Grötzsch this implies parabolicity of the surface.

References

[1] A. Eremenko, *Geometric theory of meromorphic functions*, in: In the Tradition of Ahlfors-Bers, III, (Contemp. math., 355) AMS, Providence, RI, 2004, pp. 221-230.
[2] O. Teichmüller, *Eine Anwendung quasikonformer Abbildungen auf das Typenproblem*, Deutsche Math. **2** (1937) 321-327.

Two problems concerning the differential polynomial $P[f] := f^n + af^{(k)}$

Jürgen Grahl

The differential polynomial

$$P[f] := f^n + af^{(k)} \qquad (*)$$

plays an important role in several well-known Picard and Montel type theorems by W. K. Hayman [2], D. Drasin [1], J. Langley [3], X. Pang [4] and L. Zalcman [5]. We discuss two further problems concerning this differential polynomial: We show that at least in the case of *analytic* functions results similar to those mentioned above also hold if in $(*)$ we introduce a dilatation in the argument of $f^{(k)}$. More precisely, we have:

Theorem 1. *Let $\mathcal{F}$ be a family of functions analytic in the unit disk $\mathbb{D}$, $n \geq 3$, $k \geq 1$, $0 < |x| \leq 1$ and $a, b \in \mathbb{C}$, $a \neq 0$. Assume that*

$$f^n(z) + af^{(k)}(xz) + b \neq 0$$

for all $f \in \mathcal{F}$ and all $z \in \mathbb{D}$. Then $\mathcal{F}$ is normal in $z = 0$.

The questions whether (1) $\mathcal{F}$ is normal in the whole of $\mathbb{D}$ and whether (2) there are similar results for families of *meromorphic* functions remain open. The main tool in the proof of Theorem 1 is an extension of Nevanlinna theory which provides a method to control the so-called initial value terms (like $\log |f(\alpha)|$) which appear in the First and Second Fundamental Theorem. The same method can be used to show the following result:

Theorem 2. *Let $\mathcal{F}$ be a family of functions analytic in $\mathbb{D}$, $n \geq 3$, $k \geq 1$ and $0 < |x| \leq 1$. Assume that for each $f \in \mathcal{F}$ the zeros of f are of multiplicity $\geq k$ and*

$$f^n(z)f^{(k)}(xz) \neq 1$$

for all $z \in \mathbb{D}$. Then $\mathcal{F}$ is normal in $z = 0$.

The Picard type theorem corresponding to Theorem 1 can be proved by "classical" Nevanlinna theory, even for meromorphic functions. In fact, it is a special case of the following generalization of Hayman's result ([2], Theorem 8) which we state in the terms of Nevanlinna theory.

Theorem 3. *Let f and g be meromorphic functions in $\mathbb{C}$ such that f is non-constant, $g \not\equiv 0$ and*

$$T(r, g) + \overline{N}(r, g) \leq c \cdot T(r, f) + S(r, f)$$

for some $c > 0$. Let $n > c + 2$ be an integer and $\psi := f^n + g$. Then $\psi \not\equiv 0$ and ψ has a zero in $\mathbb{C}$. If f is transcendental, then ψ has infinitely many zeros in $\mathbb{C}$. In the case of entire f, these results also hold for $n > c + 1$.

Applying this result to $g(z) := af^{(k)}(xz) - b$, we obtain

Corollary 1. *Let f be a function meromorphic in $\mathbb{C}$, k, n be positive integers and $a, b, x \in \mathbb{C}$ such that $n \geq k + 5$, $0 < |x| \leq 1$ and $a \neq 0$. If*

$$\psi(z) := f^n(z) + af^{(k)}(xz) - b$$

has no zeros in $\mathbb{C}$, then f is constant. If f is transcendental, then ψ has infinitely many zeros in $\mathbb{C}$. In the case of entire f, these results hold for all $n \geq 3$ and $k \geq 1$.

In recent years, there has been a growing interest in problems concerning shared values of differential polynomials and uniqueness of entire or meromorphic functions. In this context, we study the question whether there hold any uniqueness results for the case of two entire functions f and g such that $P[f]$ and $P[g]$ share a value $b \in \mathbb{C}$ CM (counting multiplicities). Our main result is the following.

Theorem 4. *Let f and g be non-constant entire functions, $a, b \in \mathbb{C} \setminus \{0\}$ and let n and k be positive integers satisfying $n \geq 11$ and $n \geq k + 2$. Assume that the functions*

$$P[f] = f^n + af^{(k)} \qquad \text{and} \qquad P[g] = g^n + ag^{(k)}$$

share the value b CM. Then

$$\frac{P[f] - b}{P[g] - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b}$$

or $f = g$ and $f^{(k)} = g^{(k)} \equiv b$. If $g = f'$, then $f \equiv f'$ holds.

We do not know whether in the general case we can deduce that $f \equiv g$, not even for $k = 1$. Anyhow, the following theorem shows that there does not exist a "simple" counterexample (in the sense of $\frac{f}{g}$ having order 1), at least if $k = 1$.

Theorem 5. *Let f and g be non-constant entire functions, $a, b \in \mathbb{C} \setminus \{0\}$ and let $n \geq 2$ be an integer. Assume that*

$$\frac{P[f]-b}{P[g]-b} = \frac{f^n}{g^n} = \frac{af'-b}{ag'-b}$$

and that f and g share the value 0 CM. Then $\frac{f}{g}$ is either constant or has order at least 2.

References

[1] D. Drasin, *Normal families and the Nevanlinna theory,* Acta Math. **122** (1969), 231-263

[2] W. K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. Math. (2) **70** (1959), 9-42

[3] J. Langley, *On normal families and a result of Drasin*, Proc. Roy. Soc. Edin., Sect. A **98** (1984), 385-393

[4] X. Pang, *On normal criterion of meromorphic functions*, Sci. Sinica, Ser. A **33** (1990), 521-527

[5] L. Zalcman, *New light on normal families*, in: Proceedings of the Ashkelon Workshop on Complex Function Theory (May 1996), Isr. Math. Conf. Proc. **11** (1997), 237-245

Majorisation of analytic functions

Aimo Hinkkanen

The earliest majorisation result for analytic functions is the maximum modulus principle. In the simplest setting, for a function f analytic in G and continuous in $\overline{G}$ for a bounded domain G in the complex plane $\mathbb{C}$, it states that if $|f(z)| \leq M$ for all $z \in \partial G$, where M is a positive constant, then $|f(z)| \leq M$ for all $z \in \overline{G}$. Note that under these assumptions, it is clear that $|f|$ is bounded in $\overline{G}$, and we are merely asking for the best upper bound. Many generalisations of the maximum modulus principle have been obtained. Some are based on relaxing the assumptions by allowing $|f|$ to be at least hypothetically unbounded, perhaps in an unbounded domain G, or in a bounded domain G with a set of potential singularities on the boundary. In those cases one assumes that $|f(z)|$ does not grow faster than at a prescribed rate, possibly depending on the shape of the domain, as z approaches infinity or one of the finite exceptional boundary points. This leads to Phragmén–Lindelöf type of theorems.

Another type of generalisation is based on replacing the constant M by a function of distance, and considering more carefully how f can vary from one point to another. This leads one to consider distances between function values at different points. Such problems are more delicate. We wish to avoid the extra problems arising from possible singularities on the boundary, so we consider this question in the following setting.

Let G is a bounded domain, let f be analytic in G and continuous in $\overline{G}$, and let $\mu(t)$ be a non-negative non-decreasing function defined for $t \geq 0$. Suppose that $z_1 \in \partial G$ and that

$$|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|) \tag{1}$$

for all $z_2 \in \partial G$. One can ask whether we then have

$$|f(z_1) - f(z_2)| \le C\mu(|z_1 - z_2|) \tag{2}$$

for all $z_2 \in G$ for some absolute constant C.

One may note that by a fairly simple application of the the maximum modulus principle, the above result, if valid, implies that if (1) holds for all $z_1, z_2 \in \partial G$, then (2) holds for all $z_1, z_2 \in \overline{G}$.

Examples due to Smith and Stegenga [8] show that when we consider this situation for only one fixed $z_1 \in \partial G$, then the growth of the non-negative non-decreasing function

$$\mu_1(t) = \sup\{|f(z_1) - f(z_2)| : z_2 \in \partial G, |z_1 - z_2| \le t\}$$

can be sufficiently irregular to prevent the validity of such a result in general. Indeed, for any large $a > 0$ there exists a conformal mapping f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto a bounded domain, which can be taken to be the union of three rectangles, such that for some $t_0 \in (0, 2)$ (close to 2 if a is large), we have $|f(z)-f(-1)| \le 1$ whenever $z \in \partial\mathbb{D}$ and $|z+1| \le t_0$, while $|f(t_0-1)-f(-1)| \ge a$.

The situation changes if we majorize the function μ_1, effectively a kind of modulus of continuity of f, by a sufficiently regular function. In the particular case of the unit disk, the situation also changes if we allow both z_1 and z_2 vary on $\partial\mathbb{D}$, for then the best μ_1 becomes a subadditive function of t.

Thus we say that a non-negative non-decreasing function $\mu(t)$ defined for $t \ge 0$ is a **majorant** if $\mu(2t) \le 2\mu(t)$ for all $t \ge 0$. We ask whether (1) for all $z_2 \in \partial G$ implies (2) for all $z_2 \in G$ for an absolute constant C when μ is a majorant.

When considering Hölder continuity, we take $\mu(t) = t^\alpha$, where $0 < \alpha \le 1$. In this case, Hardy and Littlewood [2] obtained the desired conclusion for some absolute constant C when $G = \mathbb{D}$, and Walsh and Sewell [7] proved the same with $C = 1$ when G is a Jordan domain. Gehring, Hayman and the speaker [1] extended this result, with $C = 1$, to arbitrary domains. The speaker [3] generalised this, with $C = 1$, to arbitrary domains when $\log \mu(e^t)$ is a concave function of t for real t.

For general majorants, it was proved that (1) implies (2) for some absolute constant C by Tamrazov [9] and by Rubel, Shields, and Taylor [6] when G is simply connected. In fact, Tamrazov [9] proved a more general result for domains whose boundary is uniformly thick in a suitable sense, with C depending on this thickness, and his result for simply connected domains is a special case of this.

A good representative of a doubly connected domain is the annulus $G = \{z \in \mathbb{C} : 1/R < |z| < R\}$ for some large $R > 1$. If we take $z_1 \in \partial G$ to be on the inner component of ∂G, we may assume, for all practical purposes (when R is very large), that $z_1 = 0$ and $f(0) = 0$. If we now take $\mu(t) = \max\{1, t\}$, our assumption (1) can be taken to be $|f(z)| \le \max\{1, |z|\}$ for all $z \in \partial G$. As usual, write $M(r, f) = \max\{|f(z)| : |z| = r\}$. Then $M(1/R, f) \le 1$ and $M(R, f) \le R$. If we choose $z_2 = 1 \in G$ in (2), we see that we would like to show that $|f(1)| \le C$ for an absolute constant C. Hadamard's three-circles theorem states that $\log M(r, f)$

is a convex function of $\log r$, which implies that

$$\log|f(1)| \le \log M(1,f) \le \frac{1}{2}(\log M(1/R,f) + \log M(R,f)) \le \frac{1}{2}\log R,$$

which does not remain bounded by an absolute constant. This shows that there is still something to be proved here. Note that here the inner boundary component of ∂G is "thin" when viewed from $z_2 = 1 \in G$. This also exposes the fact that potential theoretic methods, based on using properties of $|f|$ alone (via the subharmonic function $\log|f|$), cannot yield the best result in all cases; the proofs in all earlier papers including [9], [6], [1], [3] were based on such methods. It will be necessary to find an effective way of using the argument of f, which has the special property that if G is bounded by finitely many disjoint Jordan curves, as we may assume by approximation (compare [4]), then $\arg f(z)$ changes by an integral multiple of 2π when z traverses any component of ∂G.

The first person to exploit the fact that one can do better for analytic functions than subharmonic functions by using the argument of f was Teichmüller, who in 1939 ([10], [11]) proved a sharp version of the three-circles theorem for annuli. In 1987, the speaker [4] extended Teichmüller's framework to multiply connected domains, but was able to prove a definite result only for doubly connected domains, obtaining $C = 1.63 \cdot 10^7$ for them.

The speaker has recently proved the following result [5], which shows that (1) implies (2) for an absolute constant C under all circumstances.

Theorem 1. *If G is a bounded domain, if f is analytic in G and continuous in $\overline{G}$, if μ is a majorant, and if (1) holds for all $z_1, z_2 \in \partial G$, then (2) holds for all $z_1, z_2 \in \overline{G}$ with $C = 3456$.*

If (1) holds for a fixed $z_1 \in \partial G$ and for all $z_2 \in \partial G$, then (2) holds for this z_1 and for all $z_2 \in \overline{G}$ with $C = 3456$.

The proof of Theorem 1 is based on noting that by approximation, we may assume that G is bounded by finitely many disjoint analytic Jordan curves, and, by Runge's theorem, that f is rational with only one pole (ignoring multiplicities) at a preassigned point in each component of $\overline{\mathbb{C}}\setminus\overline{G}$. Then one combines local results based on the use of logarithmic capacity, already articulated in [9] and [4], with consideration of the level sets of suitable rational functions.

References

[1] F.W. Gehring, W.K. Hayman, and A.Hinkkanen, *Analytic functions satisfying Hölder conditions on the boundary*, J. Approx. Theory **35** (1982), 243–249.

[2] G.H. Hardy and J.E. Littlewood, *Some properties of fractional integrals II*, Math. Z. **34** (1931), 403–439.

[3] A. Hinkkanen, *On the modulus of continuity of analytic functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 247–253.

[4] A. Hinkkanen, *On the majorization of analytic functions*, Indiana Univ. Math. J. **36** (1987), 307–331.

[5] A. Hinkkanen, *Majorization of the modulus of continuity of analytic functions*, preprint, 2007.

[6] L.A. Rubel, A.L. Shields, and B.A. Taylor, *Mergelyan sets and the modulus of continuity of analytic functions*, J. Approximation Theory **15** (1975), 23–40.
[7] W.E. Sewell, *Degree of Approximation by Polynomials in the Complex Domain*, Annals of Mathematical Studies, no. **9**, Princeton University Press, Princeton, N.J., 1942.
[8] W. Smith and D. Stegenga, *The local modulus of continuity of an analytic function*, Holomorphic functions and moduli, Vol. I, pp. 133–142, Math. Sci. Res. Inst. Publ., 10, Springer, New York, 1988.
[9] P.M. Tamrazov, *Contour and solid structural properties of holomorphic functions of a complex variable* (Russian), Uspehi Mat. Nauk **28** (1973), 131–161. English translation: Russian Math. Surveys **2**8 (1973), 141–173.
[10] O. Teichmüller, *Eine Verschärfung des Dreikreisesatzes*, Deutsche Math. **4** (1939), 16–22. Also pp. 276–282 in "Collected Papers".
[11] O. Teichmüller, *Collected Papers*, ed. by L.V. Ahlfors and F.W. Gehring, Springer, Berlin, 1982.

Estimation of derivative of the Hausdorff dimension of Julia set for quadratic family

Ludwik Jaksztas

We consider quadratic family of polynomials $f_c(z) = z^2 + c$, and its Julia set $J(f_c)$. Let $d(c)$ denotes Hausdorff dimension of $J(f_c)$. We discuss behaviour of $d(c)$ as a function of the parameter. In fact we are interested in real parameter $c \in (-3/4, 1/4)$.

David Ruelle in [4] proved, that $d(c)$ is real analytic in each hyperbolic component of the Mandelbrot set. So, $d(c)$ is also analytic on the interval $(-3/4, 1/4)$ which is included in the main cardioid.

We investigate behaviour $d(c)$ close to the ends of $(-3/4, 1/4)$. In [1] Bodart and Zinsmeister proved, that $d(c)$ is continuous from the left side at $1/4$. $d(c)$ is also continuous from the right at $-3/4$. We present two Theorems:

Theorem 1. [3] *There exist* $c_0 < 1/4$ *and* $K > 1$ *such that for all* $c \in (c_0, 1/4)$*,*

$$\frac{1}{K}\Big(\frac{1}{4} - c\Big)^{d(\frac{1}{4})-\frac{3}{2}} \le d'(c) \le K\Big(\frac{1}{4} - c\Big)^{d(\frac{1}{4})-\frac{3}{2}}.$$

Because $d(1/4) < 1,295$ see [2], exponent is negative, so $d'(c)$ tends to infinity when c tends to $1/4$ from the left side.

Theorem 2. *If* $d(-3/4) < 4/3$*, then there exist* c_0 *and* $K > 1$ *such that for all* $c \in (-3/4, c_0)$*,*

$$-K\Big(\frac{3}{4} + c\Big)^{\frac{3}{2}d(-\frac{3}{4})-2} \le d'(c) \le -\frac{1}{K}\Big(\frac{3}{4} + c\Big)^{\frac{3}{2}d(-\frac{3}{4})-2}.$$

Assumption $d(-3/4) < 4/3$ seems to be reasonable, because numerical experiments give us $d(-3/4) \approx 1,23$. Under this assumption, exponent also is negative in this case, so $d'(c)$ tends to minus infinity.

In the proofs of both Theorems is used formula, which follows from theory of thermodynamical formalism. Let $K(f_c)$ denotes the filled Julia set, and $\Phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_c$ be the Riemann map, which conjugates the action of z^2 to f_c.

Φ_c has homeomorphic extension to $\partial\mathbb{D}$, so we get conjugation z^2 on $\partial\mathbb{D}$ to f_c on $J(f_c)$. Let μ_c denote f_c-invariant measure on $J(f_c)$ of maximal dimension, and $\tilde{\mu_c} = (\Phi_c)_*(\mu_c)$. We have (see [3])

$$d'(c) = -\frac{d(c)}{\int_{\partial\mathbb{D}} \log|2\Phi_c| d\tilde{\mu_c}} \int_{\partial\mathbb{D}} \frac{\partial}{\partial c}(\log|2\Phi_c|) d\tilde{\mu_c}.$$

When the measure is normalized, integral in denominator is equal to the Lyapunov exponent. The main problem is how to estimate the second integral, and it's necessary to show that the main contribution comes from set, which corresponds to the neighbourhood of attracting fixed point (which becomes parabolic point for $c = 1/4$ or $c = -3/4$).

References

[1] O. Bodart, M. Zinsmeister, *Quelques resultats sur la dimension de Hausdorff des ensembles de Julia des polynomes quadratiques*, Fund. Math. (1996), 121–137.

[2] G. Havard, M. Zinsmeister *Le chou-fleur a une dimension de Hausdorff inferieure a 1,295*, Preprint, (1994).

[3] G. Havard, M. Zinsmeister, *Thermodynamic formalism and variations of the Hausdorff dimension of quadratic Julia sets*, Commun. Math. Phys. **210** (2000), 225–247.

[4] D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory and Dynamical Systems, **2** (1982), 99–108.

Coding trees and boundaries of attracting basins for some entire maps

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(joint work with Krzysztof Barański)

Let f be an entire transcendental map such that $\mathrm{Sing}(f^{-1})$ is contained in a compact subset of the immediate basin $B(z_0)$ of an attracting fixed point $z_0 \in \mathbb{C}$ (this assumption implies that $f \in \mathcal{B}$). It follows from [3] that $B(z_0)$ is simply connected. Moreover $B(z_0)$ is the only Fatou component and the Julia set $J(f)$ is equal to the boundary of $B(z_0)$.

We study the topological and combinatorial structure of $J(f)$ and properties of the Riemann mapping onto $B(z_0)$. In this area much work was done for the exponential family (see e.g. [2], [1]). We use the technique of geometric coding trees of preimages of points from $B(z_0)$. A geometric coding tree in an invariant domain U is obtained by connecting a point from U to its first preimages by some curves in U and taking their pull-backs by the branches of f^{-n}, $n > 0$.

For every point in $J(f)$ one can define its *itinerary* (similarily as for the exponentials in [2]) in the following way. Let $T^1, T^2, ..$ be connected components of $f^{-1}(\mathbb{C} \setminus D)$, where D is a topological disc containing $\mathrm{Sing}(f^{-1})$, and let T^r_s ($s \in \mathbb{Z}$) be fundamental domains in a tract T^r. Then the itinerary of a point $z \in J(f)$ is a pair of sequences $(s_0, s_1, ..)$, $(r_0, r_1, ..)$ such that $f^n(z) \in T^{r_n}_{s_n}$ for every $n \geq 0$. We show that a given itinerary is actually realizable by the trajectory of some point if and only if the corresponding branch of a coding tree does not converge to infinity.

As the next result we prove that if the codes of the tracts r_n are bounded and codes of the fundamental domains s_n grow slightly slower than the iterates of $E_\lambda(z) = \lambda \exp(z)$ (for some λ and x) then the corresponding branch converges to a point from $J(f)$. In this case the Julia set contains a point with this itinerary, accessible from $B(z_0)$. Moreover if the sequence r_n is bounded and s_n grows not faster than the iterates of an exponential function then the corresponding branch does not tend to infinity. On the other hand, if the codes are growing sufficiently fast then the branch tends to infinity and the itinerary is not realizable.

We prove also that the pull-back of a coding tree under the Riemann map has all branches convergent. This implies that there exists at most one point in the Julia set of a given itinerary which is accessible from $B(z_0)$.

References

[1] R. L. Devaney and L. R. Goldberg, *Uniformization of attracting basins for exponential maps*, Duke Math. J. **55** (1987), 253-266.

[2] R. L. Devaney and M. Krych, *Dynamics of exp(z)*, Ergodic Theory Dynam. Systems **4** (1984), 35-52.

[3] A. E. Eremenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989-1020.

Some value distribution properties of the class S

James Langley

1. Introduction

Let the function f be transcendental and meromorphic in the plane. A key role in complex dynamics [1] is played by the singular values of the inverse function f^{-1}, which are the critical and asymptotic values of f. In particular the class S, consisting of those f for which f^{-1} has finitely many singular values, has been studied extensively in iteration theory [1, 5]. Also important is the class B, for which infinity is not a limit point of singular values.

Moreover, the class S has played an important role in value distribution theory [8]. Collingwood [4] showed that every Nevanlinna deficient value a of f must be in the closure of the singular values: in particular if $f \in S$ then a must itself be a singular value. This result was subsequently extended by Teichmüller [18], who proved that the main inequality of Nevanlinna theory becomes an asymptotic equality for f in class S . More recently, an important result of Bergweiler and Eremenko [3] shows that if a meromorphic function f has finite order and finitely many critical values then f has finitely many asymptotic values and so is in class S. This was used in [3] to prove a long-standing conjecture of Hayman [8], and subsequently has found extensive applications such as [11].

Sharp lower bounds for the growth of a transcendental meromorphic function $f \in S$ follow at once from results in [10]. Define

$$L(f) = \liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^2},$$

where $T(r,f)$ is the Nevanlinna characteristic [8], and let $q(z,w)$ denote the chordal metric on the Riemann sphere.

Theorem 1. *There exists a positive absolute constant c_0 with the following property. Let $\{a_1,\dots,a_N\}$ be a finite set of distinct elements of the extended complex plane and set*

$$\alpha = \sup\{t > 0 : \exists a \in \mathbb{C}\cup\{\infty\} \text{ such that } q(a,a_j) > t \text{ for } j = 1,\dots,N\}.$$

Let f be a function transcendental and meromorphic in the plane such that all but finitely many critical points of f lie over $\{a_1,\dots,a_N\}$. Then

$$\exp\left(\frac{-1}{4L(f)}\right) \geq c_0\alpha \min\{q(a_j,a_{j'}) : j \neq j'\}. \tag{1}$$

If the function f has only three singular values then a Möbius map may be applied sending these to $0, 1, \infty$, and by Nevanlinna's first fundamental theorem this does not change $L(f)$. Theorem 1 then shows that $L(f) \geq d_0 > 0$ in this case, for an absolute constant d_0, and the sharp value $d_0 = \sqrt{3}/2\pi$ has been proved by Eremenko [6].

Examples constructed in [10] using modified elliptic functions show that (1) is sharp. It should be noted further that functions in class S may grow arbitrarily fast [14]. On the other hand functions in class B may grow arbitrarily slowly subject to being transcendental: to see this take $f = 1/g$, where g is an entire function of small growth with strongly separated zeros.

2. The conjectures of Mues and Gol'dberg

The next results consider deficiencies of derivatives of functions in the classes S and B. It was conjectured by Mues [15] that every transcendental meromorphic function f satisfies, for every positive integer n,

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(n)}) \leq 1, \tag{2}$$

in contrast to the usual upper bound 2 for the sum of deficiencies of f itself over all values [8]. This conjecture is known to be true when all poles of f are simple [7], and in the general case the best upper bound known for the left-hand side of (2) appears to be $(2n+2)/(2n+1)$ [9, 19].

The following two theorems were proved in [12], and show that the Mues conjecture is true for functions in B of finite lower order, and that for $f \in S$ of arbitrary growth the conjecture holds at least for the first derivative.

Theorem 2 ([12])**.** *Let f be transcendental and meromorphic of finite lower order in the plane such that $f \in B$. Let n be a positive integer. Then $\delta(b, f^{(n)}) = 0$ for every $b \in \mathbb{C}\setminus\{0\}$.*

Theorem 3 ([12])**.** *Let f be transcendental and meromorphic in the plane such that $f \in S$. Then $\delta(b, f') = 0$ for every $b \in \mathbb{C} \setminus \{0\}$.*

Finally we turn to the zeros of the second derivative. Pólya showed [8, p.63] that if f is meromorphic with at least two distinct poles then the kth derivative of f has at least one zero, for all sufficiently large k. In the same spirit it was conjectured by Gol'dberg that the frequency of distinct poles of f is already controlled by the frequency of zeros of f''. Again this is known to be true if all poles of f are simple [7]. Moreover, if f has finite order and f'' has finitely many zeros then f has finitely many poles [11], a sharp result which makes essential use of the connection between critical and asymptotic values established in [3]. The following result was proved in [13].

Theorem 4. *Let $f \in S$ be transcendental and meromorphic in the plane such that f''/f' is non-constant and has finite order. Then*

$$\log^+ M(r, f'/f'') = \log^+ (\max\{|f'(z)/f''(z)| : |z| = r\}) = o(T(r, f''/f'))$$

as $r \to \infty$ in a set of logarithmic density 1.

Of course if f is transcendental and f''/f' is constant then f has the simple form $f(z) = e^{Az+B} + C$ with A, B, C constants and in particular has no poles. It follows from Theorem 4 that if $f \in S$ has finite order then

$$\overline{N}(r, f) \leq N(r, 1/f'') + o(T(r, f))$$

holds as $r \to \infty$ in a set of logarithmic density 1, which is the inequality conjectured by Gol'dberg [11].

A key role in the proof of these results is played by the following lemma, which has extensive applications in complex dynamics and value distribution [1, 2, 17].

Lemma 1 ([5, 17])**.** *Let f be transcendental and meromorphic in the plane such that $f \in B$. Then there exist $L > 0$ and $M > 0$ such that*

$$\left| \frac{z_0 f'(z_0)}{f(z_0)} \right| \geq C \log^+ \left| \frac{f(z_0)}{M} \right| \quad \text{for} \quad |z_0| > L,$$

where C is a positive absolute constant, in particular independent of f, L and M.

References

[1] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), 151-188.

[2] W. Bergweiler, *On the zeros of certain homogeneous differential polynomials*, Arch. Math. (Basel) **64** (1995), 199-202.

[3] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), 355-373.

[4] E.F. Collingwood, *Sur les valeurs exceptionelles des fonctions entières d'ordre fini*, C.R. Acad. Sci. Paris **179** (1924), 1125-1127.

[5] A. Eremenko and M.Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier Grenoble **42** (1992), 989-1020.

[6] A. Eremenko, *Transcendental meromorphic functions with three singular values*, Illinois J. Math. **48** (2004), 701-709.

[7] G. Frank and G. Weissenborn, *Rational deficient functions of meromorphic functions*, Bull. London Math. Soc. **18** (1986), 29-33.
[8] W.K. Hayman, *Meromorphic functions*, Oxford at the Clarendon Press, 1964.
[9] K. Ishizaki, *Some remarks on results of Mues about deficiency sums of derivatives*, Arch. Math. (Basel) **55** (1990), 374-379.
[10] J.K. Langley, *Critical values of slowly growing meromorphic functions*, Comput. Methods Funct. Theory **2** (2002), 539-547.
[11] J.K. Langley, *The second derivative of a meromorphic function of finite order*, Bulletin London Math. Soc. **35** (2003), 97-108.
[12] J.K. Langley, *Deficient values of derivatives of meromorphic functions in the class S*, Comput. Methods Funct. Theory **4** (2004), 237-247.
[13] J.K. Langley, *Meromorphic functions in the class S and the zeros of the second derivative*, to appear, Comput. Methods Funct. Theory.
[14] S. Merenkov, *Rapidly growing entire functions with three singular values*, preprint 2005.
[15] E. Mues, *Über eine Defekt- und Verzweigungsrelation für die Ableitung meromorpher Funktionen*, Manuscripta Math. **5** (1971), 275-297.
[16] R. Nevanlinna, *Eindeutige analytische Funktionen*, 2. Auflage, Springer, Berlin, 1953.
[17] P.J. Rippon and G.M. Stallard, *Iteration of a class of hyperbolic meromorphic functions*, Proc. Amer. Math. Soc. **127** (1999), 3251-3258.
[18] O. Teichmüller, *Eine Umkehrung des zweiten Hauptsatzes der Wertverteilungslehre*, Deutsche Mathematik **2** (1937), 96-107.
[19] L. Yang, *Precise estimate of total deficiency of meromorphic derivatives*, J. d'Analyse Math. **55** (1990), 287-296.

X-universal functions (problem session)

Raymond Mortini

Problem 1. Let Ω be a domain in $\mathbb{C}$ and let X denote either the Fréchet space $H(\Omega)$ of all holomorphic functions on Ω or the unit ball $\mathsf{B} = \{f \in H(\Omega) : \sup_{z\in\Omega} |f(z)| \leq 1\}$. A function $f \in X$ is called X-universal for the sequence (ϕ_n) of holomorphic self-maps of Ω if the "orbit" $\mathcal{O}(f) = \{f \circ \phi_n : n \in \mathbb{N}\}$ is locally uniformly dense in X.
In the case where (ϕ_n) is a sequence of automorphisms, necessary and sufficient conditions are known for the existence of X-universal functions (see [2, 4, 3]). In the case of the unit disk, it is shown in [1] that if (ϕ_n) is a sequence of selfmaps of $\mathbb{D}$ with $|\phi_n(0)| \to 1$, then a necessary and sufficient for the existence of a B-universal function for the sequence (ϕ_n) is that $\limsup \frac{|\phi_n'(0)|}{1-|\phi_n(0)|^2} = 1$. For a detailed exposition see [5]. Our question is: *Give necessary and sufficient conditions on a sequence of selfmaps (ϕ_n) of an arbitrary domain for the existence of X-universal functions for (ϕ_n). In particular, does an annulus admit X-universal functions?*

Problem 2. In order to switch from B-universal functions to $H(\Omega)$-universal functions, an answer to the following question, interesting in its own, would be useful: *Let $\Omega \subseteq \mathbb{C}$ be a domain. Under which conditions $H^\infty(\Omega)$ is locally uniformly dense in $H(\Omega)$?*

References

[1] F. Bayart, P. Gorkin, S. Grivaux, R. Mortini, *Bounded universal functions for sequences of holomorphic self-maps of the disk*, preprint 2006.
[2] L. Bernal-González, A. Montes-Rodriguez, *Universal functions for composition operators*, Complex Variables Theory Appl. **27** (1995), 47-56.
[3] P. Gorkin, F. Leon-Saavedra, R. Mortini, *Bounded universal functions in one and several complex variables*, preprint 2006
[4] A. Montes-Rodriguez, *A note on Birkhoff open sets*, Complex Variables Theory Appl. **30** (1996), 193-198.
[5] R. Mortini, *Bounded universal functions in one and several complex variables.* Abstract in Oberwolfach Reports vol. 3 (3) (2006), 36-38; Mini-Workshop 0633b, Hypercyclicity and Linear Chaos, 13.08-19.08.2006.

Picard-Hayman Behavior of Derivatives of Functions with Multiple Zeros

S. Nevo
(joint work with X.C. Pang and L. Zalcman)

Quasinormality (P. Montel, 1922). A family $\mathcal{F}$ of functions meromorphic on a domain D is called quasinormal on D if every sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \overset{\chi}{\Rightarrow} f$ on $D\backslash E$, where $E \subset D$ satisfies $E_D^{(1)} = \emptyset$. If E can always be taken to satisfy $|E| \leq \nu$, then $\mathcal{F}$ is said to be quasinormal of order ν. For example, the family $\{nz : n \in \mathbb{N}\}$ is quasinormal of order 1 in $\mathbb{C}$.

The following Theorem 1 and Theorem 2 are the main results.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions on the plane domain D, all of whose zeros have multiplicity at least $k+1$. If there exists a holomorphic function φ univalent on D such that $f^{(k)}(z) \neq \varphi'(z)$ for all $f \in \mathcal{F}$ and $z \in D$, then $\mathcal{F}$ is quasinormal of order 1 on D.*

Theorem 1 is sharp. The family

$$\left\{ f_\alpha(z) = \frac{(z-\alpha/(k+1))^{k+1}}{k!(z-\alpha)} : \alpha \neq 0 \right\}$$

is quasinormal of order 1 but not normal. Also the order of multiplicity $k+1$ cannot be reduced. For example, in the case $k=2$, the family $\{n\cos^2 z : n \in \mathbb{N}\}$ in the domain $D = \{z : -1 < \operatorname{Im} z < 1\} \setminus \overline{\bigcup_{m\in\mathbb{Z}} B\left(\frac{\pi}{4} + m\frac{\pi}{2}, \frac{\pi}{8}\right)}$ is quasinormal of order ∞.

Theorem 1 improves a theorem from [3], which is the same but with the additional condition that the family $\mathcal{F}$ is quasinormal.

A consequence of Theorem 1 is

Theorem 2. *The derivative of a transcendental meromorphic function on $\mathbb{C}$ all but at most finitely many of whose zeros are multiple takes on every nonzero complex value infinitely often.*

History. In 1959, Hayman proved the following seminal result, which has come to be known as Hayman's Alternative.

Theorem A. [Hayman's Alternative] *Let f be a transcendental meromorphic function on the complex plane $\mathbb{C}$. Then either*

(i) *f assumes each value $a \in \mathbb{C}$ infinitely often, or*
(ii) *$f^{(k)}$ assumes each value $b \in \mathbb{C} \setminus \{0\}$ infinitely often for $k = 1, 2, \ldots$.*

Theorem B. *Let f be a meromorphic function on $\mathbb{C}$. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer k and all $z \in \mathbb{C}$, then f is constant.*

Our point of departure was

Theorem C. [2] *Let $\mathcal{F}$ be a family of meromorphic functions on a plane domain D. Suppose that for each $f \in \mathcal{F}$, $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer k and all $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

Observation. In many instances, the condition that a function have no zeros can be replaced by a weaker condition that the zeros of the function be of large enough multiplicity.

In the case $k = 1$, Theorem 2 extends a result of Wang and Fang, [4], which is the same as Theorem 2, but the multiplicity of the zeros needs to be at least 3. It also extends a result of Bergweiler and Eremenko, [1], which is the same as Theorem 2 but for functions of finite order. (We proved Theorem 2 for meromorphic functions of order $0 < \rho \leq \infty$.) Bergweiler and Eremenko also gave an example that their result is not true for functions of infinite order.

An Intermediate Result. The following weaker version of Theorem 2 follows almost immediately from Theorem 1.

Theorem 2'. *Let f be a transcendental meromorphic function on $\mathbb{C}$, all of whose zeros are multiple. Then f' assumes every nonzero complex value.*

Proof. Suppose not, so that (say) $f'(z) \neq 1$ for $z \in \mathbb{C}$. By Theorem A and the above result of Bergweiler and Eremenko, f has infinite order and hence has unbounded spherical derivative. Choose $z_n \to \infty$ so that $f^{\#}(z_n) \to \infty$, $\frac{z_n}{z_{n-1}} \to \infty$, and consider the family $\mathcal{F} = \{f_n\}$ on $\mathbb{C}$, where $f_n(z) = f(z_n z)/z_n$. Then $f_n'(z) = f'(z_n z) \neq 1$, so by Theorem 1, $\mathcal{F}$ is quasinormal of order 1 on $\mathbb{C}$. On the other hand,

$$f_n^{\#}(1) = \frac{|f_n'(1)|}{1 + |f_n(1)|^2} = \frac{|f'(z_n)|}{1 + |f(z_n)/z_n|^2}$$
$$\geq f^{\#}(z_n)$$

for $|z_n| \geq 1$. Thus $f_n^\#(1) \to \infty$; so by Marty's Theorem, no subsequence of $\{f_n\}$ can be normal at $z = 1$. We also have

$$f_n^\#\left(\frac{z_{n-1}}{z_1}\right) = \frac{|f'(z_{n-1})|^2}{1 + \frac{|f(z_{n-1})|^2}{|z_n|^2}} \geq f^\#(z_{n-1}) \to \infty.$$

Thus $\sup_{|z|\leq\varepsilon} f_n^\#(z) \to \infty$ for each $\varepsilon > 0$, so that no subsequence of $\{f_n\}$ can be normal at $z = 0$. The existence of *two* points of non-normality for any subsequence of $\{f_n\}$ contradicts the assertion that $\mathcal{F}$ is quasinormal of order 1 and establishes the theorem. □

Structure of the Proof of Theorem 1. Let $\{f_n\}_1^\infty$ be a sequence of $\mathcal{F}$.
Case I: Each $a \in D$ has $\delta_a > 0$, such that f_n has at most a single (multiple) zero for large enough n.

In this case $\{f_n\}$ is quasinormal in D, and if a_1 is a point of of nonnormality, then the limit function f of some subsequence of $\{f_n\}$ satisfies $f^{(k-1)} = \varphi(z) - \varphi(a_1)$. So by the univalence of φ, a_1 is the unique point of nonnormality.

It is shown that the nature of $\{f_n\}$ near a_1 is, roughly speaking, that of $\{f_\alpha\}$, $f_\alpha(z) = \frac{(z-\frac{\alpha}{k+1})^{k+1}}{k!(z-\alpha)}$, $\alpha \to 0$, near $z = 0$.

Case II: For some $a \in D$ no such δ_a exists. This case cannot occur!

References

[1] W. Bergweiler and Alexandre Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), 355–373.

[2] Ku Yongxing, *Un critère de normalité des familles de fonctions méromorphes*, Sci. Sinica Special Issue **1** (1979), 267–274 (Chinese).

[3] Xuecheng Pang, Shahar Nevo, and Lawrence Zalcman, *Quasinormal families of meromorphic functions*, Rev. Mat. Iberoamericana **21** (2005), 249–262.

[4] Wang Yuefei and Fang Mingliang, *Picard values and normal families of meromorphic functions with multiple zeros*, Acta Math. Sinica (N.S.) **14** (1998), 17–26.

Hausdorff measure of Julia sets of exponential functions

Jörn Peter

We consider the exponential family $\{f_\lambda\}_{\lambda\in\mathbb{C}\setminus\{0\}}$, consisting of all functions $f_\lambda(z) = \lambda e^z$. For a continuous and nondecreasing 'gauge function' $h : [0, a) \to \mathbb{R}_{\geq 0}$ (where a is any positive real number) with $h(0) = 0$ and a subset A of the complex plane, let

$$H^h(A) := \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam } A_i) : \bigcup_{i=1}^{\infty} A_i \supset A, \text{diam } A_i < \delta \right\}$$

denote the Hausdorff measure of the set A with respect to the gauge function h. Let $J(f_\lambda)$ denote the Julia set of the function f_λ, i.e.

$$J(f_\lambda) = \left\{ z \in \mathbb{C} : \{f_\lambda^n\}_{n\in\mathbb{N}} \begin{array}{l} \text{does not form a normal family} \\ \text{in any neighborhood of } z \end{array} \right\}.$$

Here (and in the following), f^n is the n-th iterate of a function f, i.e.

$$f^0(z) = z \text{ and } f^n(z) = f(f^{n-1}(z)) \text{ for all } n \in \mathbb{N}.$$

McMullen [1] showed that $\dim_H(\mathrm{J(f_\lambda)})$=2 for every λ, where $\dim_H$ denotes the Hausdorff dimension. He also remarked that the Hausdorff measure with respect to the gauge function

$$h(t) = t^2 \log^k \left(\frac{1}{t}\right)$$

is ∞, for any $k \in \mathbb{N}$. However, if the function f_λ is hyperbolic (i.e. f_λ has an attracting periodic cycle), then the Julia set of f_λ has zero Lebesgue measure.

We have found other functions for which this statement is still true, at least in the classical case where $0 < \lambda < \frac{1}{e}$. For such a λ, let β_λ be the unique real repelling fixed point of f_λ. Then there exists a unique analytic function $S_\lambda : \mathbb{C} \to \mathbb{C}$ such that

$$f_\lambda \circ S_\lambda = S_\lambda \circ \beta_\lambda, S_\lambda(0) = \beta_\lambda, S'_\lambda(0) = 1.$$

Define $\Phi_\lambda := (S_\lambda|_\mathbb{R})^{-1}$. After these preparations, we can now state our theorem:

Theorem 1. *Let $\lambda, \beta_\lambda, \Phi_\lambda$ be as above and let γ_λ be such that $\beta_\lambda^{\gamma_\lambda} > 2$. Then the Hausdorff measure of $J(f_\lambda)$ with respect to the gauge function $h_\lambda(t) = t^2\Phi_\lambda(\frac{1}{t})^{\gamma_\lambda}$ is ∞.*

It is easy to see that McMullen's remark follows from this theorem in the case that $0 < \lambda < 1/e$. The proof mainly follows the ideas of McMullen. He uses Frostman's Lemma for the proof, which we slightly generalize for our purposes:

Lemma 1. *Let μ be a mass distribution on $\mathbb{C}$, $0 < a, c < \infty$, $F \subset \mathbb{C}$ μ-measurable, $h : [0, a) \to \mathbb{R}_{\geq 0}$ with $h(0) = 0$. If*

$$\limsup_{r\to 0} \frac{\mu(B_r(x))}{h(r)} < c \text{ for all } x \in F, \tag{1}$$

then $H^h(F) \geq \frac{\mu(f)}{c}$.

The construction of a suitable mass distribution μ and a suitable subset F of $J(f_\lambda)$ such that (1) is true for all positive c and $h(t) = t^2\Phi_\lambda(\frac{1}{t})^{\gamma_\lambda}$ is the same as the one used by McMullen.

The mass distribution μ is defined by using the method of nested intersections: For $k \in \mathbb{N}$, let $E_k = \{F_k^1, ..., F_k^{n_k}\}$ be a collection of disjoint, compact, connected subsets of $\mathbb{C}$ with positive Lebesgue measure such that the following conditions are satisfied, where 'meas' denotes two-dimensional Lebesgue measure:

(1) There exists a decreasing sequence $(d_k) \underset{k\to\infty}{\longrightarrow} 0$ with $\max_i \operatorname{diam} F_k^i \leq d_k$

(2) For every $k \in \mathbb{N}$, there exists $\Delta_k > 0$ such that

$$\mathrm{dens}(\overline{E_{k+1}}, F) = \frac{\mathrm{meas}(E_{k+1} \cap F)}{\mathrm{meas}(F)} \geq \Delta_k$$

(where $F \in E_k$ and $\overline{E_{k+1}} = \bigcup_{i=1}^{n_{k+1}} F_{k+1}^i$)

(3) Every $F \in E_{k+1}$ is contained in some $F' \in E_k$.

Then μ is defined on E_k first by

$$\mu(F) = \frac{\mathrm{meas}(F)}{\sum_{i=1}^{n_1} \mathrm{meas}(F_1^i)} \text{ for } F \in E_1$$

and

$$\mu(F) = \frac{\mathrm{meas}(F)}{\sum_{\substack{i=1 \\ F_i \cap F_{k-1}^l \neq \emptyset}}^{n_k} \mathrm{meas}(F_i)} \cdot \mu(F_{k-1}^l) \text{ for } k \geq 2, F \in E_k,\ F \subset F_{k-1}^l$$

and is then extended by a well-known limit process. If we define $\overline{E} := \bigcap_{i=1}^{\infty} \overline{E_i}$, then μ is supported on $\overline{E}$ and $\mu(\overline{E}) = 1$.

With the above terminology, McMullen showed that the Hausdorff dimension of $J(f_\lambda)$ is 2 by using the following proposition:

$$\limsup_{k\to\infty} \frac{\sum_{i=1}^{k} |\log \Delta_i|}{|\log d_k|} \geq 2 - \dim_H \overline{E}.$$

We modify the proposition so that it fits for our purposes:

Lemma 2. *Let $a > 0, g : (0, a) \to \mathbb{R}_{\geq 0}$ be a decreasing, continuous function with*

$$\lim_{k\to\infty} g(d_k) \cdot \prod_{i=1}^{k} \Delta_i = \infty.$$

Let $h(r) = r^2 \cdot g(r)$. Then $H^h(\overline{E}) = \infty$.

For the proof of our theorem, we work with the same sets E_k as McMullen did, but we slightly improve his estimates for Δ_k and d_k.

References

[1] C. McMullen, *Julia sets of entire functions,* Trans. amer. math. Soc. **300** (1987), 329-342

Rigidity of escaping dynamics of entire functions

Lasse Rempe

1. Introduction

In the following, $f : \mathbb{C} \to \mathbb{C}$ will be a nonconstant, nonlinear entire function (that is, a holomorphic self-map of the complex plane). We are interested in studying the *escaping set*,

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

If f is a polynomial, then $I(f)$ is a completely invariant component of the Fatou set of f (the *basin of infinity*). A classical theorem of Böttcher implies that any two polynomials of the same degree are conformally conjugate near ∞. This conjugacy gives rise to "external rays" (also called "dynamic rays"), first introduced by Douady and Hubbard in their celebrated study of the Mandelbrot set [1]. These rays have formed the backbone of the successful dynamical study of polynomials; in particularly they are the basis of the famous "Yoccoz puzzle" (compare [6]).

We are interested in the case where f is a transcendental entire function. In this situation, the escaping set (which made its first appearance, albeit implicitly, in Fatou's original study of transcendental dynamics [4]) is never open, and always intersects the Julia set [2]. (In fact, in the cases we will be considering, we always have $I(f) \subset J(f)$.)

We ask whether there is an analog of Böttcher's theorem for the behavior of (certain) entire functions near infinity. At first glance, this may seem like a strange question: we might e.g. be comparing a function whose Julia set is the whole sphere with one that has an attracting cycle, and hence has points arbitrarily close to infinity which will be attracted to this cycle under iteration. However, under the right formulation of our question, there is a (rather general) positive answer.

An analog of Böttcher's theorem. To formulate our theorem, let us introduce the *Eremenko-Lyubich class*

$$\mathfrak{B} := \{f \text{ transcendental, entire} : \operatorname{sing}(f^{-1}) \text{ is bounded}\}$$

(where $\operatorname{sing}(f^{-1})$ is the set of all critical and asymptotic values of f).

Let us also say that two functions $f, g \in \mathfrak{B}$ are *quasiconformally equivalent near infinity* if there are quasiconformal maps $\phi, \psi : \mathbb{C} \to \mathbb{C}$ such that

$$\psi(f(z)) = g(\phi(z)) \tag{1}$$

whenever $|f(z)|$ and $|g(z)|$ are large enough. When (1) holds on all of $\mathbb{C}$, the maps are simply called *quasiconformally equivalent*. (Quasiconformal equivalence classes are the natural complex parameter spaces of entire functions [3].)

Theorem 1 ([7], 2006)**.** *Let $f, g \in \mathfrak{B}$ be quasiconformally equivalent near infinity. Then there exists a quasiconformal map $\vartheta : \mathbb{C} \to \mathbb{C}$ such that $\vartheta \circ f = g \circ \vartheta$ on*

$$A_R := \{z : |f^n(z)| \geq R \textit{ for all } n \geq 1\}$$

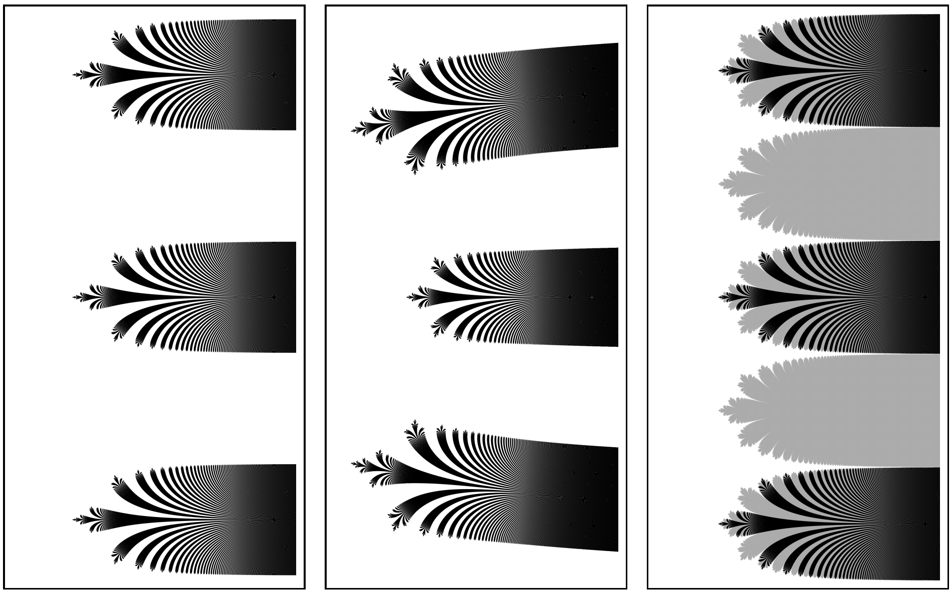

FIGURE 1. Three Julia sets of different entire functions: the first is $z \mapsto 2(\exp(z) - 1)$, the second one is $z \mapsto (z+1)\exp(z) - 1$; our results imply that these are quasiconformally conjugate on their Julia sets. The third picture shows a *subset* of the Julia set of a function of the form $z \mapsto \lambda \sinh(z)$ on which this function is quasiconformally conjugate to the previous two.

Furthermore, ϑ has zero dilatation on $\{z \in A_R : |f^n(z)| \to \infty\}$.

Thus, even though the structure of the escaping set can change dramatically within class $\mathfrak{B}$ (see [9] or Schleicher's abstract in this report), it will stay constant within any given parameter space. This is all the more surprising as other properties — the Hausdorff dimension of the escaping set, and the order of growth, for example — *do* change within such parameter spaces.

The theorem also explains why some function-theoretically very diverse functions turn out to have very similar Julia sets; compare Figure 1.

We should also note that the above theorem does not hold for general entire functions (outside class $\mathfrak{B}$). Indeed, the map $z \mapsto z - 1 + \exp(z)$ has a Baker domain (i.e., an invariant Fatou component contained in $I(f)$) containing a left half-plane, while $z \mapsto z + 1 + \exp(z)$ has no Baker domains at all.

Conjugacy on the escaping set. In the polynomial case, the conjugacy provided by Böttcher's theorem can be continued analytically to a conjugacy between between the entire basins of infinity, provided that no critical value of either polynomial escapes itself. This is not true in the transcendental setting: e.g. two elements of the exponential family $z \mapsto \exp(z) + \kappa$ will, in general, *not* be conjugate on their escaping sets, even if the singular values do not escape (compare [8]). However, if both functions are *hyperbolic* — that is, the postsingular set

$$\mathbb{P}(f) = \overline{\bigcup_{j \geq 0} f^n(\operatorname{sing}(f^{-1}))}$$

is compact and contained in the Fatou set — the situation is different.

Theorem 2 ([7], 2006)**.** *Suppose that $f, g \in \mathfrak{B}$ are hyperbolic and quasiconformally conjugate near infinity. Then $f|_{I(f)}$ is topologically conjugate to $g|_{I(f)}$.*

Rigidity and density of hyperbolicity. It turns out that the conjugacy constructed in Theorem 1 is "essentially unique". This has several important consequences, which have applications in current joint work with Sebastian van Strien. This work aims to transfer recent results of Kozlovski, Shen and van Strien [5] on density of hyperbolicity for real polynomials to the case of real transcendental maps.

Let us denote by $\mathbb{S}^{\mathbb{R}}$ the set of all real entire transcendental functions for which $\operatorname{sing}(f^{-1})$ is finite and contained in $\mathbb{R}$. For functions of this type, we have the following rigidity result.

Theorem 3 (R., van Strien, 2005)**.** *Suppose that $f, g \in \mathbb{S}^{\mathbb{R}}$ are topologically conjugate. Then f and g are quasiconformally conjugate.*

Results of this type can usually be used to establish density of hyperbolicity in real parameter spaces when coupled with theorems on the absence of *invariant line fields* on the Julia set. Work to establish the absence of such line fields for functions in $\mathbb{S}^{\mathbb{R}}$ is still ongoing, but some important progress has already been made. For example, we can prove the following.

Theorem 4 (R., van Strien, 2006)**.** *Suppose that $f \in \mathbb{S}^{\mathbb{R}}$ is bounded on the real axis. Then there exists a hyperbolic function $g \in \mathbb{S}^{R}$ arbitrarily close to f and quasiconformally equivalent to f.*

This theorem applies, in particular, to the real cosine family of maps $z \mapsto a\cos(z) + b\sin(z)$ with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

References

[1] A. Douady and J. Hubbard, *Etude dynamique des polynômes complexes*, Prépublications mathémathiques d'Orsay 2/4, 1984 / 1985.

[2] A. Eremenko, *On the iteration of entire functions*, in "Dynamical systems and ergodic theory", volume 23 of *Banach Center Publ.*, PWN, Warsaw (1989), 339–345.

[3] A. Eremenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989–1020.

[4] P. Fatou, *Sur l'itération des fonctions transcendantes entières*, Acta Math. **47** (1926), 337–370.

[5] O. Kozlovski, W. Shen, and S. van Strien. *Rigidity for real polynomials*, to appear in Ann. Math.

[6] J. Milnor, *Local connectivity of Julia sets: expository lectures*, in "The Mandelbrot set, theme and variations", volume 274 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge (2000), 67–116.
[7] L. Rempe. *Rigidity of escaping dynamics for transcendental entire functions*, Preprint (2006), submitted for publication.
[8] L. Rempe. *Topological dynamics of exponential maps on their escaping sets*, Ergodic Theory Dynam. Systems, **26** (2006), 1939–1975.
[9] G. Rottenfußer, J. Rückert, L. Rempe, and D. Schleicher, *Dynamic rays of bounded-type entire functions*, Manuscript (2006).

Multiply connected wandering domains of meromorphic functions

Phil Rippon
(joint work with Gwyneth Stallard)

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function and denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ is well-defined, meromorphic and forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$.

The set $F(f)$ is completely invariant, so for any component U of $F(f)$ there exists, for each $n = 0, 1, 2, \ldots$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of *period* p, assuming p to be minimal. There are then five possible types of periodic components; see [5]. If U_n is not eventually periodic, then we say that U is a *wandering component* of $F(f)$, or a *wandering domain*.

We use the name *Baker wandering domain* to denote a wandering component U of $F(f)$ such that, for n large enough, U_n is a bounded multiply connected component of $F(f)$ which surrounds 0, and $U_n \to \infty$ as $n \to \infty$. An example of this phenomenon with f an entire function was first given by Baker in [1].

If f is a transcendental entire function and U is a multiply connected component of $F(f)$, then U is a Baker wandering domain; see [2]. This need not be the case for meromorphic functions, even those with finitely many poles; see [7]. There are also examples of meromorphic functions with multiply connected wandering domains that are not Baker wandering domains. For example, in [4] Baker, Kotus and Lü used techniques from approximation theory to construct a meromorphic function with a k connected bounded wandering domain, where $k \in \{2, 3, \ldots\}$, which is not a Baker wandering domain. This example has infinitely many poles.

For any meromorphic function f we let $\operatorname{sing}(f^{-1})$ denote the set of inverse function singularities of f, which consists of the critical values and finite asymptotic values of f. Using a sufficient condition for Baker wandering domains given in [11] we obtain the following result.

Theorem 1. *Let f be meromorphic and have a finite number of poles, and let U be a multiply connected wandering domain of f.*

(a) *The component U is a Baker wandering domain if and only if infinitely many of the components U_n, $n = 0, 1, 2, \ldots$, are multiply connected.*

(b) *If*

$$sing(f^{-1}) \cap \bigcup_{n\geq 1} U_n = \emptyset,$$

then U_n is multiply connected for $n = 0, 1, 2, \ldots$, so U is a Baker wandering domain.

Theorem 1(a) is related to a result of Qiu and Wu [10], which gives a sufficient condition for a meromorphic function to have infinitely many weakly repelling fixed points. Note that Theorem 1(a) is false without the hypothesis that f has finitely many poles; see the finitely connected example of Baker, Kotus and Lü [4] mentioned earlier. Here we give an infinitely connected example of this type, based on an earlier example of an entire function with a bounded wandering domain due to Bergweiler.

Example 1. *If $\epsilon > 0$ is small enough, then the function*

$$f(z) = 2 + 2z - 2e^z + \frac{\epsilon}{e^z - e^a},$$

has a wandering domain U such that each component U_n, $n = 0, 1, 2, \ldots$, is bounded and infinitely connected, but U is not a Baker wandering domain.

Our second example shows that there exists a meromorphic function f, with just one pole, which has a multiply connected wandering domain U such that, for $n \geq 1$, the components U_n are simply connected. This is based on an earlier example of an entire function with a bounded wandering domain due to Devaney. Thus in Theorem 1(a) it is not sufficient to assume that U alone is a multiply connected wandering domain in order to deduce that U is a Baker wandering domain.

Example 2. *If λ and a are chosen so that $\lambda \sin a = 2\pi$ and $1 + \lambda \cos a = 0$, and $\epsilon > 0$ is small enough, then the function*

$$f(z) = z + \frac{\epsilon}{z} + \lambda \sin(z + a),$$

has a bounded doubly connected wandering domain U such that each component U_n, $n = 1, 2, \ldots$, is bounded and simply connected.

Next we discuss some general connectivity properties of Fatou components of transcendental meromorphic functions. For a domain U in $\mathbb{C}$, the *connectivity* $c(U)$ is the number of components of ∂U in the extended complex plane $\hat{\mathbb{C}}$; note that if U is unbounded, then we include ∞ in ∂U. Following Kisaka and Shishikura [9], we define the *eventual connectivity* of a component U of $F(f)$ to be c provided that $c(U_n) = c$ for all large values of n. Kisaka and Shishikura [9] showed that if f is entire and U is a multiply connected component of $F(f)$, and hence a Baker wandering domain, then the eventual connectivity of U exists and is either 2 or ∞. Moreover, they constructed the first example of an entire function f with a Baker wandering domain with eventual connectivity 2, thus answering an old question; see [5]. Earlier, Baker [3] constructed an example with infinite eventual connectivity.

Using Theorem 1(a), the Riemann-Hurwitz formula and results of Bolsch [6] and Herring [8], we obtain the following generalisation of the above result of Kisaka and Shishikura to meromorphic functions with a finite number of poles.

Theorem 2. *Let f be a meromorphic function with a finite number of poles, and let U be a wandering domain of f.*

(a) *If U is not a Baker wandering domain, then the eventual connectivity of U is 1.*
(b) *If U is a Baker wandering domain, then the eventual connectivity of U is either 2 or ∞.*

In the example of Baker, Kotus and Lü mentioned earlier (which has infinitely many poles) it can be shown that the wandering domains constructed have eventual connectivity k, where $k \in \{2, 3, \dots\}$. Thus Theorem 2(a) is false without the assumption that f has a finite number of poles. By modifying the construction of Baker, Kotus and Lü, we can obtain a meromorphic function f with a Baker wandering domain whose eventual connectivity is k. Thus Theorem 2(b) is also false without the assumption that f has a finite number of poles.

Finally we give examples to show that if U is a wandering domain of a meromorphic function f with a finite number of poles, then $c(U_n)$ is not necessarily monotonic decreasing (as would be the case if all the components U_n are bounded). These examples are obtained by starting with Example 2 and modifying it in a suitable left half-plane using techniques from approximation theory.

References

[1] I.N. Baker, *An entire function which has wandering domains,* J. Austral. Math. Soc. Ser. A, **22** (1976), 173–176.
[2] I.N. Baker, *Wandering domains in the iteration of entire functions,* Proc. London Math. Soc., **49** (1984), 563–576.
[3] I.N. Baker, *Some entire functions with multiply connected wandering domains,* Ergodic Theory Dynam. Systems, **5** (1985), 163–169.
[4] I.N. Baker, J. Kotus and Lü Yinian, *Iterates of meromorphic functions II: Examples of wanderings,* J. London Math. Soc., **42** (1990), 267–278.
[5] W. Bergweiler, *Iteration of meromorphic functions,* Bull. Amer. Math. Soc., **29** (1993), 151–188.
[6] A. Bolsch, *Periodic Fatou components of meromorphic functions,* Bull. London. Math. Soc., **31** (1999), 543–555.
[7] P. Domínguez, *Dynamics of transcendental meromorphic functions,* Ann. Acad. Sci. Fenn. Math. Ser. A (1), **23** (1998), 225–250.
[8] M.E. Herring, *Mapping properties of Fatou components,* Ann. Acad. Sci. Fenn. Math., **23** (1998), 263–274.

[9] M. Kisaka and M. Shishikura, *On multiply connected wandering domains of entire functions,* To appear in Transcendental dynamics and complex analysis, LMS Lecture Notes, Cambridge University Press.

[10] L. Qiu and S. Wu, *Weakly repelling fixed points and multiply connected wandering domains of meromorphic functions,* Sci. China Ser. A **49** (2006), 267–276.

[11] P.J. Rippon and G.M. Stallard, *Escaping points of meromorphic functions with a finite number of poles,* Journal d'Analyse Math., **96** (2005), 225–245.

Critical points of inner functions, nonlinear partial differential equations, and an extension of Liouville's theorem

Oliver Roth
(joint work with Daniela Kraus)

Theorem 1. *Let $\{z_j\} \subseteq \mathbb{D}$ be a Blaschke sequence. Then there exists a Blaschke product with critical points $\{z_j\}$ (counted with multiplicity) and no others.*

In case of finitely many critical points $z_1, \ldots, z_n \in \mathbb{D}$ the Blaschke product constructed in Theorem 1 is finite and of degree $n+1$. This follows from the proof of Theorem 1 and [4, Corollary 1.10]. The finite case in Theorem 1 has earlier been obtained by Heins [3, Theorem 29.1], Wang & Peng [11], Bousch [2] and Zakeri [12] using topological methods, and Stephenson [10, Theorem 21.1] using Circle Packing. Theorem 1 appears to be the first result of its kind for infinitely many critical points. A partial converse of Theorem 1 has been obtained by Heins [3], who showed that for any nonconstant holomorphic function $f : \mathbb{D} \to \mathbb{D}$ the critical points of f contained in any horocycle satisfy the Blaschke condition.

Sketch of proof. In a first step we define the Blaschke product

$$h(z) := \prod_{j=1}^{\infty} \frac{-\overline{z}_j}{|z_j|} \frac{z - z_j}{1 - \overline{z_j}\, z}$$

and show that the nonlinear elliptic Gaussian curvature equation

$$\Delta u = 4\,|h(z)|^2\, e^{2u} \text{ in } \mathbb{D}\,, \qquad \lim_{z \to \xi} u(z) = +\infty \text{ for every } \xi \in \partial\mathbb{D} \tag{1}$$

has a solution $u \in C^2(\mathbb{D}; \mathbb{R})$. It follows from the results of [4] that $e^{u(z)}\,|dz|$ is then a *complete* conformal Riemannian metric in $\mathbb{D}$ with curvature $-4\,|h(z)|^2$, so this step is closely related to the *Berger-Nirenberg* problem in differential geometry. In a second step we establish an extension of Liouville's theorem [5] to show that

$$u(z) = \log\left(\frac{|f'(z)|}{1 - |f(z)|^2} \frac{1}{|h(z)|}\right)$$

for some holomorphic function $f : \mathbb{D} \to \mathbb{D}$. Since $u(z)$ is finite everywhere, the function f has the critical points $\{z_j\}$ (counted with multiplicity) and no others. Now classical theorems from complex analysis (Fatou's theorem and Privalov's theorem) guarantee that f is an inner function, so a certain Frostman shift of f is a Blaschke product with critical points $\{z_j\}$ and no others. □

Building holomorphic maps with the help of the Liouville equation (i.e. (1) for $h \equiv 1$) is an old idea and can be traced back at least to the work of Schwarz [9], Poincaré [8], Picard [6, 7] and Bieberbach [1]. The main new aspect of the present work is to show that the same method can also be applied in situations when branch points occur even though branching complicates the treatment considerably. Roughly speaking, this is accomplished by replacing Liouville's equation by the Gaussian curvature equation (1) with the critical points encoded as the zeros of the holomorphic function $h(z)$.

References

[1] L. Bieberbach, *$\Delta u = e^u$ und die automorphen Funktionen,* Math. Ann. **77** (1916), 173-212.

[2] T. Bousch, *Sur quelques problèmes de dynamique holomorphe*, Thesis, Université Paris 11, Orsay, 1992.

[3] M. Heins, *On a class of conformal metrics,* Nagoya Math. J. **21** (1962), 1-60.

[4] D. Kraus, O. Roth, and St. Ruscheweyh, *A boundary version of Ahlfors' Lemma, locally complete conformal metrics and conformally invariant reflection principles for analytic maps,* Journal d'Analyse Math. to appear.

[5] J. Liouville, *Sur l'équation aux différences partielles $\frac{d^2 \log \lambda}{du dv} \pm \frac{\lambda}{2a^2} = 0$,* J. de Math. **16** (1853), 71-72.

[6] E. Picard, *De l'équation $\Delta u = e^u$ sur une surface de Riemann fermée,* J. de Math. **9** (1893), 273-291.

[7] E. Picard, *De l'integration de l'équation differentielles $\Delta u = e^u$ sur une surface de Riemann fermée,* J. Reine Angew. Math. **130** (1905), 243-258.

[8] H. Poincaré, H., *Les fonctions fuchsiennes et l'équation $\Delta u = e^u$,* J. de Math. **4** (1898), 137-230.

[9] H. A. Schwarz, *Preisaufgabe der Math.-Phys. Klasse der Königl. Ges. der Wissenschaften zu Göttingen für das Jahr 1891*, Nachr. Akad. Wiss. Göttingen (1890), 216.

[10] K. Stephenson, *Introduction to circle packing: the theory of discrete analytic functions*, Camb. Univ. Press, New York, 2005.

[11] Q. Wang and J. Peng, *On critical points of finite Blaschke products and the equation $\Delta u = \exp 2u$,* Kexue Tongbao **24** (1979), 583-586 (Chinese).

[12] S. Zakeri, *On critical points of proper holomorphic maps on the unit disk,* Bull. London Math. Soc. **30** (1996), 62-66.

Local boundary behaviour of analytic maps and the hyperbolic metric (problem session)

Oliver Roth

Theorem 1 (see [3])**.** *Let Γ be an open subarc of the unit circle $\partial\mathbb{D}$ and let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then the following conditions are equivalent.*

(a) *For every $\xi \in \Gamma$:* $\displaystyle \lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = +\infty\,.$

(b) *For every $\xi \in \Gamma$:* $\displaystyle \liminf_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} > 0\,.$

(c) *For every $\xi \in \Gamma$:* $\displaystyle \lim_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} = 1\,.$

(d) *For every* $\xi \in \Gamma$: $\quad \lim\limits_{z \to \xi} |f(z)| = 1$.

(e) f *has a holomorphic extension across the arc* Γ *with* $f(\Gamma) \subset \partial\mathbb{D}$.

The statement (c) $\Longleftrightarrow$ (d) for the global case $\Gamma = \partial\mathbb{D}$ has been obtained earlier by Heins [2] using a different method. Thus Theorem 1 is a (sharper) localised version of Heins' result. One is inclined to ask whether some of the implications of Theorem 1 are still valid when the arc Γ shrinks to a point.

Problem 1. *Suppose*

$$\lim_{z \to 1} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} = 1 \,.$$

What can be said about the boundary behaviour of f *at* $z = 1$ *?*

Going back to Theorem 1 we note that the implication (a) $\Longrightarrow$ (c) is contained in

Lemma (Boundary Ahlfors Lemma [3]). *Let* Γ *be an open subarc of the unit circle* $\partial\mathbb{D}$ *and let* $\lambda(z)\,|dz|$ *be a regular conformal pseudo-metric on the unit disk* $\mathbb{D}$ *with curvature bounded below by* -4. *If*

(A) $$\lim_{z \to \xi} \lambda(z) = +\infty$$

for every point $\xi \in \Gamma$, *then*

(B) $$\liminf_{z \to \xi} \frac{\lambda(z)}{\lambda_{\mathbb{D}}(z)} \geq 1$$

for every $\xi \in \Gamma$. *Here* $\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2)$ *denotes the hyperbolic metric in* $\mathbb{D}$. *In particular,*

(C) $$\lim_{z \to \xi} d_\lambda(0, z) = +\infty$$

for every point $\xi \in \Gamma$, *where* d_λ *denotes the distance function associated to* $\lambda(z)\,|dz|$.

The Lemma contains the Ahlfors Lemma [1] as a special case ($\Gamma = \partial\mathbb{D}$). The implication (C) $\Longrightarrow$ (B) (for conformal *metrics*) is contained in the work of Yau [5] and Bland [4]. In fact (see [3]), the conditions (A), (B) and (C) are equivalent in much more generality, i.e.

Corollary 1. *Let* $\Omega \subseteq \mathbb{C}$ *be a domain, let* Γ *be an open free* C^2*-subarc of* $\partial\Omega$, *let* $\lambda(z)\,|dz|$ *be a regular conformal metric on* Ω *with* $\kappa_\lambda \geq -4$ *and let* d_λ *be the associated distance function. Then the following are equivalent:*

(a) $\lambda(z)\,|dz|$ *is locally complete near* Γ, *i.e.* $d_\lambda(z, z_0) \to \infty$ *as* $z \to \xi$ *for every* $\xi \in \Gamma$ *for some (and then every)* $z_0 \in \Omega$.

(b) $\lim\limits_{z \to \xi} \lambda(z) = +\infty$ *for every* $\xi \in \Gamma$.

Problem 2. *What are the minimal regularity conditions on the boundary set* Γ *such that the two conditions (a) and (b) in Corollary 1 are still equivalent ?*

References

[1] L. Ahlfors, *An extension of Schwarz's lemma,* Trans. Amer. Math. Soc. **42** (1938), 359-364.
[2] M. Heins, *Some characterizations of finite Blaschke products of positive degree,* J. Anal. Math. **46** (1986), 162-166.
[3] D. Kraus, O. Roth, and St. Ruscheweyh, *A boundary version of Ahlfors' Lemma, locally complete conformal metrics and conformally invariant reflection principles for analytic maps,* Journal d'Analyse Math. to appear.
[4] J. S. Bland, *Local boundary regularity of the canonical Einstein-Kähler metric on pseudoconvex domains,* Math. Ann. **263** (1983), 289-301.
[5] S. T. Yau, *A general Schwarz Lemma for Kähler Manifolds,* Amer. J. Math. **100** (1978), 197-203.

Dynamic Rays for Entire Functions of Bounded Type

Dierk Schleicher

(joint work with Günter Rottenfußer, Johannes Rückert, and Lasse Rempe)

The study of the escaping set for polynomials has been one of the key ingredients in the successful study of polynomial dynamics. Polynomials (of degree $d \geq 2$) have superattracting fixed points at ∞, and it is well known that I is conformally equivalent to $\mathbb{C} \setminus \overline{\mathbb{D}}$ (if the Julia set is connected; if not, the necessary modifications are also well-understood). The set I consists naturally of dynamic rays. Many of the deepest results about the structure of polynomial Julia sets rely on the landing properties of dynamic rays at points of the Julia set. Since rays and their landing properties have been so successful in the study of polynomial dynamics, it has been a major motivation for our recent work to extend these ideas to transcendental entire functions. The study is very different, though, because the superattracting fixed point at ∞ is replaced by an essential singularity.

There has been substantial interest in the set $I(f)$ for quite a while. Fatou [7] observed that for certain entire functions, $I(f)$ contains curves to infinity and asked whether this property was true in greater generality. Eremenko [5] showed that for every entire function f, the set $I(f)$ is non-empty and unbounded, and he conjectured that every connected component (or even every path component) of $I(f)$ was unbounded; he also proved that the Julia set of f has the property that $J(f) = \partial I(f)$ and, if f was of bounded type, that $J(f) = \overline{I(f)}$. Devaney and coauthors showed that $I(f)$ contains curves for all maps in the exponential family $z \mapsto \lambda \exp(z)$ [2] and for a number of entire transcendental functions [4]; for exponential maps with the special dynamical property that the singular value converges to an attracting fixed point, they even proved that every path component of $I(f)$ is a curve to ∞. The same was shown for all exponential maps, even when the Julia set is the entire complex plane, in [14, 8]; these results provide a classification of all escaping points in terms of dynamic rays. This was extended to the cosine family $z \mapsto ae^z + be^{-z}$ in [12], and to entire functions of bounded type and finite order, subject to the dynamical condition that all singular and critical values converge to a single attracting fixed point, by Baranski [1]. Rempe [9] has shown some kind of stability of the set $I(f)$ within every parameter space of transcendental functions (see also his report in this volume). The first transcendental map where the Julia set is $\mathbb{C}$, every dynamcic ray lands and every point in the Julia set is the landing point of some dynamic rays, is the case of cosine maps $z \mapsto ae^z + be^{-z}$ with both critical orbits strictly preperiodic [13].

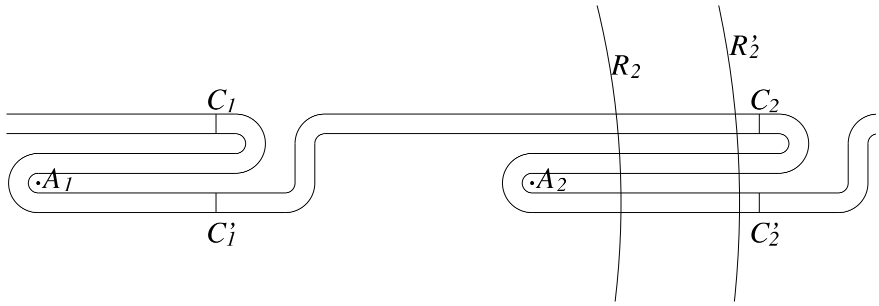

FIGURE 1. The tract of a bounded type entire function for which every path component of the escaping set is bounded, as constructed in [11]. The curves C_i and C_i' are geodesics of T; their images are semi-circles R_i, R_i' in $\mathbb{H}$.

Our first main result is the following.

Theorem 1 (Bounded Path Components). *There is an entire function f of bounded type where every path component of $I(f)$ is bounded.*

This result disproves the strong version of Eremenko's conjecture (that every path component of $I(f)$ is unbounded) even for bounded type entire functions. The proof consists of several steps: following Eremenko and Lyubich [6], we employ logarithmic coordinates. In logarithmic coordinates, every bounded type entire function has countably many tracts $T_i \subset \mathbb{C}$ with conformal isomorphisms $F_i : T_i \to \mathbb{H}$ (where $\mathbb{H}$ denotes the right half plane), and every escaping point of f gives rise to an orbit of escaping points of the F_i within $\bigcup_i T_i$.

We first give sufficient conditions on the tracts T_i so that the set of orbits that stay within $\bigcup T_i$ have all their path components bounded; then we show that such tracts actually exist within $\mathbb{H}$. Finally we construct bounded type entire functions which give rise to such tracts. We also sketch how to construct a hyperbolic entire function of bounded type where all path components of $I(f)$ are single points.

Our second main result is positive, proving Eremenko's conjecture in many cases.

Theorem 2 (Dynamic Rays Exist). *Every entire function f of bounded type and finite order has the property that $I(f)$ is the disjoint union of curves to ∞ (dynamic rays). The same is true for finite compositions of entire functions of bounded type and finite order.*

In fact, we give more general conditions on the shape of the tracts ("bounded wiggling") which imply the conclusion of the theorem, and we show that entire functions of bounded type and finite order satisfy these conditions.

References

[1] K. Barański, *Trees and hairs for entire maps of finite order*, Preprint (2005). http://www.mimuw.edu.pl/ baranski/hairs.pdf

[2] C. Bodelón, R. Devaney, M. Hayes, G. Roberts, L. Goldberg and J. Hubbard: *Dynamical convergence of polynomials to the exponential.* J. Diff. Eq. Appl. **6** (2000), 275-307.

[3] R. Devaney and M. Krych, *Dynamics of* $\exp(z)$, Erg. Theory Dynam. Sys. **4** (1984), 35-52.

[4] R. Devaney and F. Tangerman, *Dynamics of entire functions near the essential singularity*, Ergodic Theory Dynam. Systems **6** (1986), 489-503.

[5] A. Eremenko, *On the iteration of entire functions*, Dynamical systems and ergodic theory (Warsaw, 1986), Banach Center Publ., vol. 23, PWN, Warsaw, 1989, pp. 339-345.

[6] A. Eremenko and M. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989-1020.

[7] P. Fatou, *Sur l'itération des fonctions transcendantes entières*, Acta Math. **47** (1926), 337-370.

[8] M. Förster, L. Rempe and D. Schleicher, *Classification of Escaping Exponential Maps*, to appear in Proc. AMS.

[9] L. Rempe, *Rigidity of escaping dynamics for transcendental entire functions*, Preprint (2006). http://www.arXiv.org/abs/math.DS/0605058, arXiv:math.DS/0605058.

[10] G. Rottenfußer, *On the dynamical fine structure of entire transcendental functions*, doctoral thesis, International University Bremen (2005).

[11] G. Rottenfußer, J. Rückert, L. Rempe and D. Schleicher, *Dynamic rays of bounded-type entire functions.* Manuscript.

[12] G. Rottenfußer and D. Schleicher, *Escaping points of the cosine family*, to appear in: *Transcendental dynamics and complex analysis* (P. Rippon and G. Stallard, eds). http://www.arXiv.org/abs/math.DS/0403012, arXiv:math.DS/0403012.

[13] D. Schleicher, *The dynamical fine structure of iterated cosine maps and a dimension paradox.* Duke Math. Journal **136** 2 (2007), 343-356. http://www.arXiv.org/abs/math.DS/0406255, arXiv:math.DS/0406255.

[14] D. Schleicher and J. Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. (2) **67** (2003),380-400.

On Thurston's Characterization Theorem

Nikita Selinger

Thurston's characterization theorem reduces the question of whether a postcritically finite branched self-cover of $\mathbb{S}^2$ is equivalent to a rational map to a purely topological criterion. We follow the work of Douady and Hubbard [1] to see the connection between this question and holomorphic dynamics.

Let us call an orientation preserving branched covering $f \to \mathbb{S}^2 \to \mathbb{S}^2$ to be postcritically finite if all critical points have finite orbits, i.e. they are periodic or preperiodic. Denote by Ω_f the set of all critical points of f. Then the postcritical set is defined as: $P_f = \bigcup_{n>0} f^n(\Omega_f)$. Two postcritically finite maps $f \to \mathbb{S}^2 \to \mathbb{S}^2$ and $g \to \mathbb{S}^2 \to \mathbb{S}^2$ are Thurston-equivalent if there are two homeomorphisms $h_1, h_2 \to \mathbb{S}^2 \to \mathbb{S}^2$ with $g = h_1 \circ f \circ h_2^{-1}$, $h_1(P_f) = h_2(P_f) = P_g$, $h_1|_{P_f} = h_2|_{P_f}$, and h_1 and h_2 are isotopic relative to P_f. We omit the precise definition for "hyperbolic" orbifolds (see [1]): all postcritically finite branched covers have hyperbolic orbifolds, except for a few explicitly understood cases.

Thurston's theorem gives a purely topological way to determine whether a branched cover is equivalent to a rational function on the Riemann sphere. To state it we need to give a definition of a Thurston obstruction. A simple closed curve in $\mathbb{S}^2 \setminus P_f$ is *essential* if both connected components of its complement contain at least two points of P_f. The map f is a covering on $\mathbb{S}^2 \setminus P_f$ hence the preimage of any simple closed curve in $\mathbb{S}^2 \setminus P_f$ is a number of simple closed curves. A *multicurve* is a set of disjoint essential curves; a multicurve Γ is *f-stable* if every essential connected component of the preimage of Γ is homotopic to a curve in Γ. For every f-stable multicurve $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ we can construct the corresponding Thurston linear transformation (or the Thurston matrix) $f_\Gamma \to \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ as follows: let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\mathbb{S}^2 \setminus P_f$. Then set

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i,$$

where $d_{i,j,\alpha} = \deg f_{\gamma_{i,j,\alpha}} \to \gamma_{i,j,\alpha} \to \gamma_j$. Denote by $\lambda(\Gamma, f)$ the largest real eigenvalue of f_Γ. A *Thurston obstruction* is a multicurve Γ such that $\lambda(\Gamma, f) \geq 1$.

Now we are ready to formulate

Theorem 1 (Thurston's characterization theorem [1])**.** *Every postcritically finite orientation preserving branched cover $f \to \mathbb{S}^2 \to \mathbb{S}^2$ with hyperbolic orbifold is either Thurston-equivalent to a rational map g (which is necessarily unique up to conjugation by a Möbius transform), or f has a Thurston obstruction.*

The key idea of the proof is to define an analytic self-mapping σ of the Teichmüller space $\mathcal{T}_f$ modelled on $(\mathbb{S}^2, P_f)$. A point $\tau \in T_f$ can be represented as a homeomorphism $h \to \mathbb{S}^2 \to \widehat{\mathbb{C}}$ where two such homeomorphisms h_1, h_2 represent the same point in $\mathcal{T}_f$ if there exists a Möbius transformation m such that $h_1 \circ m$ and h_2 are isotopic relative P_f. Having this representation of $\mathcal{T}_f$ in mind we can define the mapping σ as follows: let τ be a point in $\mathcal{T}_f$ represented by h. By pulling back the standard complex structure σ_0 from $\widehat{\mathbb{C}}$ first by h and then by f we get an almost complex structure $\sigma_1 = f^* h^*(\sigma_0)$ on $\mathbb{S}^2$. By the Measurable Riemann Mapping theorem there exists a homeomorphism h_1 such that $\sigma_1 = h_1^*(\sigma_0)$. Set $\sigma(\tau) = \tau_1$ where τ_1 is represented by h_1.

One can see now the correspondence between rational functions equivalent to f and the fixed points of σ. Define $g = h_1^{-1} \circ f \circ h$. Then g preserves the standard complex structure on $\widehat{\mathbb{C}}$ hence g is rational. Comparing this with the definition of Thurston equivalence we see that if h and h_1 correspond to the same point in $\mathcal{T}_f$ then f is equivalent to g. On the other hand, if f is equivalent to some rational function g then the point which corresponds to both h_1 and h_2 from the definition of Thurston equivalence is fixed under the action of σ.

We know that the cotangent space to $\mathcal{T}_f$ at a point τ represented by h is isomorphic to the space $Q(h(P_f))$ of meromorphic quadratic differentials on $\widehat{\mathbb{C}}$ which are holomorphic in $\widehat{\mathbb{C}} \setminus h(P_f)$ and have at most simple poles in P_f. The norm $|q| = \int |q(x+iy)|dxdy$ on $Q(h(P_f)$ can be used to define a metric on T_f. The coderivative of σ equals $f_* \to Q(h_1(P_f)) \to Q(h(P_f))$ where f_* denotes push-forward by f:

$$f_* q(z) = \sum_{w \in f^{-1}(z)} (f^{-1})^*(w).$$

It is immediate to see that $||f_*|| \leq 1$. For f with hyperbolic orbifold it also follows that $||(f_*)^2|| < 1$, i.e. $d(\tau_1, \tau_2) < d(\sigma^2(\tau_1), \sigma^2(\tau_2))$ because the Teichmüller space is path connected. That already yields the uniqueness part of the theorem. Unfortunately, this is not enough to conclude that there exists a fixed point of σ because $\mathcal{T}_f$ is not compact.

The further investigation is based on the study of conformal annuli in $\widehat{\mathbb{C}} \setminus h(P_f)$. Every sufficiently short curve on a hyperbolic Riemann surface is surrounded by an annulus of big modulus. We can also denote by $m(\tau)$ the maximal modulus of a conformal annulus in $\widehat{\mathbb{C}} \setminus h(P_f)$ which is homotopic to an essential curve. This will clearly define a continuous function on $\mathcal{T}_f$. Here is the place where the definition of Thurston obstruction comes in. If we have a number of annuli $(A_1, A_2, \ldots, A_s)$ homotopic to the respective components of an f-stable multicurve $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ then the vector of moduli $M(h) = (\mod h(A_1), \mod h(A_2), \ldots, \mod h(A_s))$ changes by the action of σ approximately as predicted by f_Γ $M(h_1) = f_\Gamma(M(h))$ provided the moduli are large enough.

Suppose there exists a fixed point τ of σ. If Γ is a Thurston obstruction then starting from a point τ_1 in T_f with large enough $M(h)$ we will always have anuli with large moduli on $\widehat{\mathbb{C}} \setminus h(P_f)$, in particular larger than $m(\tau)$. Hence $\sigma^n(\tau_1)$ can not tend to τ. This contradicts the fact that σ is weakly contracting. On the other hand, if $\lambda(\Gamma, f) < 1$ for all Γ then $m(\sigma^n(\tau))$ is bounded for any τ which means that the sequence rests in a part of $\mathcal{T}_f$ where we can uniformly estimate the convergence rate. In this case we get a fixed point of σ.

In [2] the analogous statement for topological exponential maps was proven.

Theorem 2 (Postsingularly Finite Exponential Maps [2])**.** *A postsingularly finite topological exponential map f is either Thurston equivalent to a holomorphic exponential map (necessarily unique up to conformal conjugation), or f has a Thurston obstruction.*

Here, a topological exponential map is simply an orientation preserving universal cover $f \to \mathbb{S}^2 \setminus \{\infty\} \to \mathbb{S}^2 \setminus \{\infty, 0\}$. It has a unique singular value, the omitted value 0, and the latter is never periodic. A postsingularly finite exponential maps is thus one where the singular orbit is strictly preperiodic. It is easy to show that for entire maps, rational or transcendental, a Thurston obstruction would have to be of a particularly simple kind known as "Levy cycle".

The approach used in [2] is different from the original one of [1]. The theorems on decomposition and limiting models of quadratic differentials proven there give hope to get a new proof of the original Thurston's theorem which would be easier to generalise.

Our principal goal is to generalise the statement of Thurston's theorem to larger classes of postsingularly finite entire and meromorphic functions.

References

[1] A. Douady and J. Hubbard, *A Proof of Thurston's Topological Characterization of Rational Functions.* Acta Math. **171**, 263-297 (1993).

[2] J. Hubbard, M. Shishikura and D. Schleicher, *A Topological Characterization of Postsingularly Finite Exponential Maps and Limits of Quadratic Differentials*, submitted.

Escaping points of entire functions of small growth

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(joint work with Phil Rippon)

Let f be a transcendental entire function and denote by $f^n, n \in \mathbf{N}$, the nth iterate of f. The *Fatou set*, $F(f)$, is defined to be the set of points, $z \in \mathbf{C}$, such that $(f^n)_{n\in\mathbf{N}}$ forms a normal family in some neighbourhood of z. The complement, $J(f)$, of $F(f)$ is called the *Julia set* of f.

The escaping set

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

was first studied for a general transcendental entire function f by Eremenko [6]. He proved that $I(f) \neq \emptyset$, $J(f) = \partial I(f)$, $I(f) \cap J(f) \neq \emptyset$ and $\overline{I(f)}$ has no bounded components.

Eremenko conjectured that it may be possible to show that $I(f)$ has no bounded components, a problem that still remains open. In [8] we proved the following result which shows that Eremenko's conjecture is true whenever $F(f)$ has a multiply connected component.

Theorem 1. *Let f be a transcendental entire function and suppose that $F(f)$ has a multiply connected component U. Then $\overline{U} \subset I(f)$ and $I(f)$ is connected and unbounded.*

Here we prove the following result which generalises Theorem 1. We show that $I(f)$ is connected whenever there is a 'hole' in $I(f)$.

Theorem 2. *Let f be a transcendental entire function and suppose that there exists a bounded domain U with $\partial U \subset I(f)$ and $U \cap J(f) \neq \emptyset$. Then*
(a) $\gamma_n = \partial f^n(U)$ is in $I(f)$, $\gamma_n \to \infty$ and γ_n surrounds 0 for sufficiently large n;
(b) $I(f)$ is connected and unbounded.

We will show later that there are functions that satisfy the hypotheses of Theorem 2 but do not have any multiply connected Fatou components.

We proved Theorem 1 by considering the following subset of $I(f)$, which was introduced by Bergweiler and Hinkkanen in [4]:

$$A(f) = \{z : \text{ there exists } L \in \mathbf{N} \text{ such that } |f^n(z)| > M(R, f^{n-L}), \text{ for } n > L\}.$$

Here,

$$M(r, f) = \max_{|z|=r} |f(z)|$$

and R can be taken to be any value such that $R > \min_{z \in J(f)} |z|$. A key part of the proof was to show that $A(f)$ is equivalent to the set

$$B(f) = \{z : \text{ there exists } L \in \mathbf{N} \text{ such that } f^{n+L}(z) \notin \widetilde{f^n(D)}, \text{ for } n \in \mathbf{N}\},$$

where D is any open disc meeting $J(f)$ and $\widetilde{U}$ denotes the union of U and its bounded complementary components.

In practice, identifying which points belong to $B(f)$ will involve studying sets of the form

$$B_D(f) = \{z : f^n(z) \notin \widetilde{f^n(D)}, \text{ for } n \in \mathbf{N}\},$$

where D is an open disc meeting $J(f)$. We prove several results concerning the structure of such sets. Our main result is the following.

Theorem 3. *Let f be a transcendental entire function, let D be an open disc meeting $J(f)$ and suppose that there exists a bounded domain U with $\partial U \subset B_D(f)$ and $U \cap J(f) \neq \emptyset$. Then*
(a) $\gamma_n = f^n(\partial U) \to \infty$ and, for sufficiently large n, γ_n is in $B_D(f)$ and γ_n surrounds 0;
(b) $B_D(f)$, $B(f)$ and $I(f)$ are all connected.

We show that there are several conditions that are equivalent to the hypotheses of Theorem 3. This enables us to prove the following result.

Theorem 4. *Let f be a transcendental entire function and let D be an open disc meeting $J(f)$. Suppose that there exist simple closed Jordan curves γ_n such that*

$$\gamma_n \text{ surrounds } f^n(D)$$

and

$$f(\gamma_n) \text{ surrounds the bounded component of } \gamma_{n+1}^c.$$

Then there exists a bounded domain U with $\partial U \subset B_D(f)$ and $U \cap J(f) \neq \emptyset$. Thus both $B(f)$ and $I(f)$ are connected.

There are two different classes of functions that we are aware of that satisfy the hypotheses of Theorem 4. Firstly, any entire function with a multiply connected component of the Fatou set will satisfy these hypotheses. Secondly, many functions of order less than 1/2 have been shown to satisfy conditions which are stronger than the hypotheses of Theorem 4. These functions were originally studied in connection with a different conjecture (first raised by Baker [2] and described in detail in the survey article [7]) — namely that if f is a function of order at most 1/2, minimal type, then $F(f)$ has no unbounded component. In particular, it has been shown that the hypotheses of Theorem 4 are satisfied by functions of order less than 1/2 whose growth is sufficiently regular and by functions of order zero for which there exists $\epsilon \in (0,1)$ such that

$$\log\log M(r,f) < \frac{(\log r)^{1/2}}{(\log\log r)^{\epsilon}},$$

for large values of r.

There are many functions which satisfy the above growth condition and which do not have multiply connected components of the Fatou set. For example, Bergweiler and Eremenko showed in [3] that there are transcendental entire functions of arbitrarily small growth for which the Julia set is the whole plane. Also Baker [1] and Boyd [5] independently proved that there are transcendental entire functions of arbitrarily small growth for which every component of the Fatou set is simply connected. Further, every point in the Fatou set tends to zero under iteration.

We end with the following result. This gives a criterion related to the escaping set which is sufficient to ensure that a function has no unbounded Fatou components.

Theorem 5. *Let f be a transcendental entire function, let D be an open disc meeting $J(f)$ and suppose that there exists a bounded domain U with $\partial U \subset B_D(f)$ and $U \cap J(f) \neq \emptyset$. Then $F(f)$ has no unbounded component.*

Together with Theorem 4, this gives the following result.

Theorem 6. *Let f be a transcendental entire function and let D be an open disc meeting $J(f)$. Suppose that there exist simple closed Jordan curves γ_n such that*

$$\gamma_n \textit{ surrounds } f^n(D)$$

and

$$f(\gamma_n) \textit{ surrounds the bounded component of } \gamma_{n+1}^c.$$

Then $F(f)$ has no unbounded component.

References

[1] I. N. Baker. *Dynamics of slowly growing entire functions* Bull. Austral. Math. Soc. **63** (2001), 367-377.

[2] I. N. Baker, *The iteration of polynomials and transcendental entire functions* J. Austral. Math. Soc. (Series A) **30** (1981), 483-495.

[3] W. Bergweiler and A. E. Eremenko. *Entire functions of slow growth whose Julia set coincides with the plane* Ergod. Th. and Dynam. Sys. **20** (2000), 1577-1582.

[4] W. Bergweiler and A. Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Cambridge Philos. Soc. **126** (1999), 565–574.

[5] D. A. Boyd. *An entire function with slow growth and simple dynamics* Ergod. Th. and Dynam. Sys. **22** (2002), 317-322.

[6] A. E. Eremenko, *On the iteration of entire functions*, in Dynamical systems and ergodic theory, Banach Center Publications **23**, Polish Scientific Publishers, Warsaw 1989, 339–345.

[7] A. Hinkkanen, *Entire functions with bounded Fatou components*, To appear in Transcendental dynamics and complex analysis. LMS Lecture Note Series, CUP 2007.

[8] P. J. Rippon and G. M. Stallard, *On questions of Fatou and Eremenko*, Proc. Amer. Math. Soc. **133** (2005), 1119–1126.

Complex Dynamics of Rational Semigroups

Rich L. Stankewitz
(joint work with Hiroki Sumi)

We discuss the dynamics of rational semigroups as an extension of rational iteration theory. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G.J. Martin in [1]. A **rational semigroup** is a semigroup generated by non-constant rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ with the semigroup operation being the composition of maps. We denote by $\langle h_\lambda : \lambda \in \Lambda \rangle$ the rational semigroup generated by the family of maps $\{h_\lambda : \lambda \in \Lambda\}$. In this paper we assume each map of a rational semigroup is of degree two or more.

Definition 1. *Let G be a rational semigroup. We set*

$$F(G) = \{z \in \widehat{\mathbb{C}} | G \text{ is normal in a neighborhood of } z\} \text{ and } J(G) = \widehat{\mathbb{C}} \setminus F(G).$$

We call $F(G)$ the **Fatou set** *of G and $J(G)$ the* **Julia set** *of G. We also employ the familiar iteration notation $F(\langle g \rangle) = F(g) = F_g$ and $J(\langle g \rangle) = J(g) = J_g$.*

In contrast to iteration dynamics, $F(G)$ is only guaranteed to be forward invariant under each element of G (and thus $J(G)$ is backward invariant under each element of G). In fact, $J(G)$ is the smallest closed **backward invariant** set containing 3 or more points. However, if we wish to require that the Julia set of the semigroup G be completely invariant under each element of G, we would use the following.

1. Completely invariant Julia sets

Definition 2. *For a rational semigroup G we define the* ***completely invariant Julia set*** *of G, called $E(G)$, to be the smallest closed subset of $\widehat{\mathbb{C}}$ containing three or more points which is completely invariant under each element of G. We then define the* ***completely invariant set of normality*** *by $W(G) = \widehat{\mathbb{C}} \setminus E(G)$.*

It follows readily that $J(G) \subset E(G)$, but what else can be said about these two sets?

Theorem 1. [3] *For a rational semigroup G the set $W(G)$ can have only 0, 1, 2, or infinitely many components.*

Here are the few known types of $E(G)$ along with a conjecture.

Example 3. *(trivial case) If $J_f = J_g$, then $E(\langle f, g \rangle) = J_f = J_g = J(\langle f, g \rangle)$.*

Example 4. *Let $a \in \mathbb{C}, |a| > 1$ and $G = \langle z^2, z^2/a \rangle$. One can easily show $J(G) = \{z : 1 \leq |z| \leq |a|\}$ while $E(G) = \widehat{\mathbb{C}}$. Note that $Int(J(G)) \neq \emptyset$, yet $J(G) \neq \widehat{\mathbb{C}}$. However, $Int(E(G)) \neq \emptyset$ implies $E(G) = \widehat{\mathbb{C}}$ (see* [2]*).*

Theorem 2. [2] *For a rational semigroup G which contains two* ***polynomials*** *f and g, $J_f \neq J_g$ implies $E(G) = \widehat{\mathbb{C}}$.*

Example 5. *Set $f(z) = 2z - \dfrac{1}{z}$ and $g(z) = 2z - \dfrac{4}{z}$. Then J_f and J_g are Cantor subsets of $[-1,1]$ and $[-2,2]$ respectively and $E(\langle f,g\rangle) = \overline{R}$.*

Example 6. *For $f(z) = \dfrac{3z+5z^2}{1+3z+4z^2}$ and $g(z) = 2z^2-1$, it follows that $J_f \neq [-1,1] = J_g$, but $E(\langle f,g\rangle) = [-1,1]$.*

Conjecture 1. [4] *If G is a rational semigroup which contains two maps f and g such that $J_f \neq J_g$ and $E(G)$ is not the whole Riemann sphere, then $E(G)$ is Möbius equivalent to a line segment or a circle.*

2. Nearly Abelian polynomial semigroups.

Theorem 3. [1] *Let $\mathcal{F}$ be a family of* ***polynomials*** *of degree at least 2, and suppose that there is a set J such that $J_g = J$ for all $g \in \mathcal{F}$. Then $G = \langle \mathcal{F} \rangle$ is a nearly abelian semigroup.*

Note that under the hypotheses of Theorem 3 we have $J_h = J(G)$ for each generator $h \in \mathcal{F}$. It is, however, not the case that $J_h = J(G)$ for just one $h \in G$ necessarily implies that G is nearly abelian.

Example 7. *Let $f(z) = z^2-2$, $g(z) = 4z^2-2$ and $G = \langle f,g\rangle$. It follows that $J(G) = [-2,2] = J_f$, yet $J_f \neq J_g$.*

Conjecture 2. [4] *Let G be a* ***polynomial*** *semigroup such that $J_h = J(G)$ for some $h \in G$ where $J(G)$ is not a line segment. Then $J_f = J_g$ for all $f,g \in G$ (and hence G is nearly abelian by Theorem 3).*

3. Dynamics of postcritically bounded polynomial semigroups.

The **planar postcritical set** of a polynomial semigroup G is defined by $P^*(G) = \overline{\bigcup_{g\in G}\{\text{all critical values of } g\}} \setminus \{\infty\}$. Let $\mathcal{G}$ be the set of all **polynomial** semigroups G such that $P^*(G)$ is bounded in $\mathbb{C}$. Unlike in iteration, we may have $G \in \mathcal{G}$, yet still have $J(G)$ disconnected. We now state a few results regarding such semigroups $G \in \mathcal{G}$.

Theorem 4. [5] *Let $G \in \mathcal{G}$. Suppose that A is a doubly connected component of $F(G)$ and B satisfies one of the following conditions:*

- *B is a doubly connected component of $F(G)$ other than A,*
- *B is the connected component of $F(G)$ with $\infty \in B$,*
- *$B = \{z \in \mathbb{C} | \{g(z)\}_{g\in G}$ is bded in $\mathbb{C}\}$.*

Then, $\partial A \cap \partial B = \emptyset$ and $\overline{A}$ and $\overline{B}$ are separated by a Cantor family of quasicircles (with uniform dilatation) all which lie in $J(G)$.

In the following theorems $J_{\min}(G)$ denotes the unique component of $J(G)$ which meets $\{z \in \mathbb{C} | \{g(z)\}_{g\in G}$ is bded in $\mathbb{C}\}$.

Theorem 5. [5] *Let $H \in \mathcal{G}$ and let $G = \langle H, h_1, \dots, h_n \rangle$ be a polynomial semigroup generated by H and $h_1, \dots, h_n$. Suppose*
(1) $G \in \mathcal{G}$ and $J(G)$ is disconnected,
(2) $J(h_j) \cap J_{\min}(G) = \emptyset$ for each $j = 1, \dots, m$, and
(3) H is semi-hyperbolic.
Then, G is semi-hyperbolic.

This theorem would not hold if we were to replace both instances of the word *semi-hyperbolic* with the word *hyperbolic*, but we do have the following.

Theorem 6. [5] *Let $H \in \mathcal{G}$ and let $G = \langle H, h_1, \dots, h_n \rangle$ be a polynomial semigroup generated by H and $h_1, \dots, h_n$. Suppose*
(1) $G \in \mathcal{G}$ and $J(G)$ is disconnected,
(2) $J(h_j) \cap J_{\min}(G) = \emptyset$ for each $j = 1, \dots, m$,
(3) H is hyperbolic, and
(4) For each $j = 1, \dots, m$, the critical values of h_j do not meet $J_{\min}(G)$.
Then, G is hyperbolic.

For the proof of Theorems 4, 5, and 6, we use and develop many ideas from [6].

References

[1] A. Hinkkanen and G. J. Martin, *The dynamics of semigroups of rational functions I,* Proc. London Math. Soc. **73** (1996), 358-384.
[2] R. Stankewitz, *Completely invariant Julia sets of polynomial semigroups,* Proc. Amer. Math. Soc. **127** (1999), 2889-2898.
[3] R. Stankewitz, *Completely invariant sets of normality for rational semigroups,* Complex Variables Theory Appl. **40** (2000), 199-210.
[4] R. Stankewitz, T. Sugawa, and H. Sumi, *Some counterexamples in dynamics of rational semigroups,* Ann. Acad. Sci. Fenn. Math. **29** (2004), 357-366.
[5] R. Stankewitz and H. Sumi, *Structure of Julia sets of postcritically bounded polynomial semigroups*, preprint.
[6] H. Sumi, *Dynamics of postcritically bounded polynomial semigroups,* preprint.

Remark on Nevanlinna's Four-Point-Theorem (problem session)

Norbert Steinmetz

This problem session talk was given to remind the present audience-and possibly a wider one-of an old problem dating back to R. Nevanlinna's famous

Four-Point-Theorem ([6]). *Suppose that meromorphic functions f and g assume each of the values a_1, a_2, a_3, a_4 at the same points and with the same multiplicities* (phrased "f and g share a_1, a_2, a_3, a_4 CM"). *Then f and g have two common Picard values, $a_4 = \infty$, say, and $a_2 = \frac{1}{2}(a_1 + a_3)$, and satisfy $(f - a_2)(g - a_2) = (a_2 - a_1)(a_3 - a_2)$.* The most simple example is given by $a_1 = -1$, $a_2 = 0$, $a_3 = 1$, $a_4 = \infty$, $f(z) = e^z$, $g(z) = e^{-z}$.

The novelty and importance of this 1926 paper stems from the new and powerful methods rather than the fact that meromorphic instead of entire functions of finite order were considered (as was done by Pólya a few years before). Nevanlinna probably believed that the additional hypothesis "CM" was redundant. He wrote:

> *Es wäre nun interessant zu wissen, ob dieses Ergebnis auch dann besteht, wenn die* Multiplizitäten *der betreffenden Stellen nicht berücksichtigt werden. Einige [...] Ergebnisse sprechen vielleicht für die Vermutung [...]*

In 1979 G. Gundersen [1] gave an explicit counterexample to this conjecture; quite different counterexamples were found (and characterised) by M. Reinders [7] and the author [10] (the shared values are $(0, 1, \infty, -\frac{1}{8})$, $(0, 1, \infty, -1)$ and $(0, 1, \infty, \sqrt[3]{-1})$ with $\sqrt[3]{-1} \neq -1$, respectively). On the other hand Gundersen proved in two steps [1, 2] that the additional hypothesis "CM" may be dropped for two of the values. So it remains the

Question. *Do there exist different meromorphic functions f and g sharing four values, exactly one of them by "CM"? And if so, for which values does this happen?*

Actually there has been no real progress since Gundersen's 1983 paper (if we ignore those papers which make additional and sometimes artificial assumptions). Any serious result in this direction obtained in recent years, see [3, 4, 9, 11, 12, 13], may be found in E. Rudolph's thesis [8]-unfortunately unpublished. These results rely on the auxiliary function

$$\psi = \frac{f'g'(f-g)^2}{\prod_{\nu=1}^{3}(f-a_\nu)(g-a_\nu)}$$

which carries the whole information and was introduced by E. Mues in the early 1980's, but published only in 1989, see [5]. The properties of ψ and its factors

$$\psi_1 = \frac{f'(f-g)}{\prod_{\nu=1}^{3}(f-a_\nu)} \text{ and } \psi_2 = \frac{g'(f-g)}{\prod_{\nu=1}^{3}(g-a_\nu)}$$

may be stated as follows:

Theorem. *If meromorphic functions f and g share the values a_1, a_2, a_3 and $a_4 = \infty$, then ψ is entire, "small" and "almost" zero-free* (to be understood in the sense of Nevanlinna theory). *Moreover, if $a_4 = \infty$ is shared CM, then also ψ_1 and ψ_2 are entire and "almost" zero-free* (but not necessarily "small"). *Finally, if the spherical derivatives $f^\#$ and $g^\#$ are bounded, then ψ is a non-zero constant, and* (assuming $\psi \equiv 1$) $\psi_1 = e^Q$, $\psi_2 = e^{-Q}$ *with* $\deg Q \leq 2$ *hold.*

The hypothesis $f^\# + g^\# \leq C$ may always be achieved by applying the re-scaling technique, see L. Zalcman [14], *simultaneously* to both functions. Since all known counterexamples to Nevanlinna's conjecture have bounded spherical derivatives (at least in their most simple form), we may pose two problems-and give presumptive answers.

Problem 1. Determine all solutions (a_1, a_2, a_3, f, g) of the equation $\psi = 1$. *They are presumably known, see* [1, 6, 7, 10].

Problem 2. Determine all solutions of the systems $\{\psi_1 = e^z, \psi_2 = e^{-z}\}$ and $\{\psi_1 = e^{z^2}, \psi_2 = e^{-z^2}\}$, respectively. *Up to Möbius transforms-replace f and g with $T \circ f(az+b)$ and $T \circ g(az+b)$, respectively-, the solution of the first system is presumably given by $(-1, 0, 1, e^{-z}, e^z)$, while it seems unlikely that the second system has any solution.*

References

[1] G. G. Gundersen, *Meromorphic functions that share three or four values,* J. London Math. Soc. **20** (1979), 457-466.

[2] G. G. Gundersen, *Meromorphic functions that share four values,* Trans. Amer. Math. Soc. **277** (1983), 545-567.

[3] G. G. Gundersen, *Meromorphic functions that share three values IM and a fourth value CM*, Complex Variables **20** (1992), 99-106.

[4] B. Huang, *On the unicity of meromorphic functions that share four values,* Indian J. pure appl. Math. **35** (2004), 359-372.

[5] E. Mues, *Meromorphic functions sharing four values,* Complex Variables **12** (1989), 169-179.

[6] R. Nevanlinna, *Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen*, Acta Math. **48** (1926), 367-391.

[7] M. Reinders, *Eindeutigkeitssätze für meromorphe Funktionen, die vier Werte teilen,* Mitt. Math. Sem. Giessen **200** (1991), 15-38.

[8] E. Rudolph, *Über meromorphe Funktionen, die vier Werte teilen,* Diploma Thesis, Karlsruhe (1988).

[9] G. D. Song and J. M. Chang, *Meromorphic functions sharing four values*, Southeast Asian Bull. Math. **26** (2002), 629-635

[10] N. Steinmetz, *A uniqueness theorem for three meromorphic functions*, Ann. Acad. Fenn. Sci. **13** (1988), 93-110.

[11] H. Ueda, *Some estimates for meromorphic functions sharing four values*, Kodai Math. J. **17** (1994), 329-340.

[12] S. P. Wang, *On meromorphic functions that share four values*, J. Math. Anal. Appl. **173** (1993), 359-369.

[13] J. P. Wang, *Meromorphic functions sharing four values*, Indian J. pure appl. Math. **32** (2001), 37-46.

[14] L. Zalcman, *A heuristic principle in function theory* Amer. Math. Monthly **82** (1975), 813-817.

Generalized Blaschke products with Fatou components of arbitrary connectivity number

Marcus Stiemer

In this presentation, rational maps of the form

$$f(z) = e^{i\alpha} \prod_{j=0}^{2p} \frac{z - a_j}{1 - \overline{a_j} z}, \qquad a_j \in \mathbb{C} \setminus \partial\mathbb{D},\ \alpha \in [0, 2\pi),\ p \in \mathbb{N} \tag{1}$$

are considered. In contrast to classical Blaschke products, they are allowed to have zeros outside the unit disk $\mathbb{D}$. All these maps fix the unit circle $\partial\mathbb{D}$. Having one more zero in $\mathbb{D}$ than poles, $\partial\mathbb{D}$ is covered once by $f(\partial\mathbb{D})$, as follows by the argument principle. If no critical point lies on $\partial\mathbb{D}$, f is an orientation preserving diffeomorphism of the unit circle. Otherwise, f is still a homeomorphism of $\partial\mathbb{D}$, if and only if all critical points on $\partial\mathbb{D}$ are of even multiplicity. In any case the total number of critical points on $\partial\mathbb{D}$ is even (counting multiplicities), since f is symmetric to the unit circle, i.e. $f(1/\bar{z}) = 1/\overline{f(z)}$ for z in $\mathbb{C}$. All rational maps whose restriction to $\partial\mathbb{D}$ is an orientation preserving homeomorphism of the unit circle are of the form described above.

The Fatou sets of rational homeomorphisms of the unit circle can be classified depending on the rotation number $\mu(f) = \lim_{n\to\infty}(2\pi n)^{-1}h^n(0) \pmod 1$, where h^n denotes the nth iterate of h, and h is an arbitrary lift of f to the real line, i.e. $f(e^{it}) = e^{ih(t)}$, $t \in \mathbb{R}$ (see [4]):

- If $\mu(f)$ is irrational, then either $\partial\mathbb{D}$ is entirely contained in the Julia set of f, or $\partial\mathbb{D}$ is contained in an Arnol'd-Herman ring symmetric about $\partial\mathbb{D}$.
- If $\mu(f)$ is rational (say $k\mu(f) \in \mathbb{N}$ with a minimal $k \in \mathbb{N}$), then there is a finite number $m \in \mathbb{N}$ of fixed points of f^k on $\partial\mathbb{D}$. These fixed points divide $\partial\mathbb{D}$ in m arcs, which are contained entirely in an invariant Böttcher, Schröder of Leau domain except of one or both of the end points.

For the subsequent construction of rational maps with multiply connected Fatou components of a desired connectivity number, the following lemma will be essential to control the critical points of the given mapping (see [5]):

Lemma 1. *If f is of the form (1), then there exist at least two (or one multiple) critical points $\zeta_1, \zeta_2 = 1/\bar{\zeta}_1$ such that both of them are contained in the Julia set of f, or $|f^n(\zeta_j)| \to 1$ for $n \to \infty$ ($j = 1, 2$).*

Now we turn to multiply connected Fatou components: A first example of a rational map possessing a three-fold or four-fold connected Fatou component was presented by A. Beardon [2, p. 263]. About the same time, the existence of rational maps possessing a Siegel disk which has a preimage of an arbitrary given finite connectivity was established implicitly in [1] by use of quasiconformal surgery methods.

Define for $c > 1$, $0 < \rho < c - 1$, $\nu \in \mathbb{N}$, $\eta \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ the mappings $f_{c,\rho,\nu,\eta,\alpha}(z) = e^{i\alpha} f_{c,\rho,\nu,\eta}(z)$ with

$$f_{c,\rho,\nu,\eta}(z) = z^{1+\eta}\left(\frac{z-c}{1-cz}\right)^{\eta}\left(\frac{(z-c)^{\nu}-\rho^{\nu}}{(1/z-c)^{\nu}-\rho^{\nu}}\right)^{\eta}\left(\frac{(1/z-c)^{\nu}+\rho^{\nu}}{(z-c)^{\nu}+\rho^{\nu}}\right). \tag{2}$$

In the sequel we will write f instead of $f_{c,\rho,\nu,\eta,\alpha}$. The Böttcher domain about $z = 0$ will be denoted $\mathcal{A}_0(f)$. Since the unit circle is invariant, $\partial\mathbb{D} \cap \mathcal{A}_0(f) = \emptyset$. This implies $\mathcal{A}_0(f) \subset \mathbb{D}$ for all parameter values. Since all considered maps are symmetric to $\partial\mathbb{D}$, $z = \infty$ is also a super-attracting fixed point with Böttcher domain $\mathcal{A}_\infty(f) = 1/\overline{\mathcal{A}_0(f)}$. The maps f possess $\nu + 1$ zeros of order η in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ at c and $c + \rho e^{2\pi ik/\nu}$, $k = 0, \ldots, \nu - 1$. These zeros do not lie in $\mathcal{A}_0(f) \subset \mathbb{D}$. Further, there are simple poles of f outside $\overline{\mathbb{D}}$ in $c + \rho e^{2\pi ik/\nu + \pi i/\nu}$, $k = 0, \ldots, \nu - 1$. The exponent $1 + \eta$ in the first factor in (2) is chosen such that f has one more zero in $\mathbb{D}$ than outside and hence is of the form (1). For such functions, the following theorem holds:

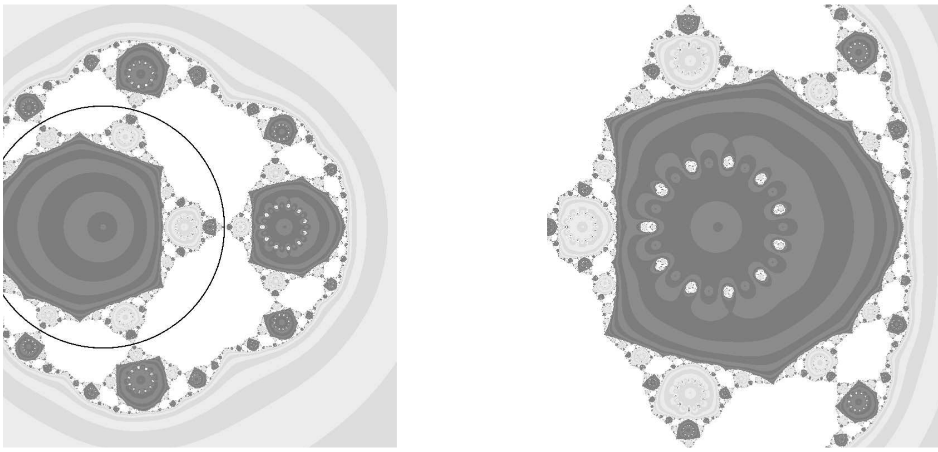

FIGURE 1. The Fatou sets of the mappings $f_{c,\rho,\nu,\eta,\alpha}$ contain components of connectivity $\nu+1$: On the left hand side, a Fatou set containing 12-fold connected components ($c = 1.5$, $\rho = 0.175$, $\nu = 11$, $\eta = 1$, $\alpha = 0$) is presented. On the right hand side a close-up view on the area [0.9,2.1]+i[-1.05,1.05] containing the multiply connected preimage of the immediate basin of attraction of 0 is given. The black circle on the left hand side represents the unit circle.

Theorem 1. *For all $c > 1$, $\nu \in \mathbb{N}$, $\eta \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $\rho > 0$ sufficiently small the mappings $f_{c,\rho,\nu,\eta,\alpha}$ possess Fatou components with connectivity number $\nu + 1$.*

One of the Fatou components described in Theorem 1 contains the $\nu+1$ (possibly multiple) zeros c and $c+\rho e^{2\pi ik/\nu}$, $k = 0,\dots,\nu-1$. The ν holes are situated about the poles $c + \rho e^{2\pi ik/\nu+\pi i/\nu}$, $k = 0,\dots,\nu-1$ (compare Fig. 1). The idea of the proof of Theorem 1 is as follows: We will show that for sufficiently small values of ρ the circle $\{z \in \mathbb{C} : |z - c| = 2\rho\}$ together with the *spokes* $\{z \in \mathbb{C} : z = c + te^{2\pi ik/\nu}, 0 \le t \le 2\rho\}$, $k = 0,\dots,\nu-1$, is mapped inside a neighborhood $U_\epsilon(0) = \{z \in \mathbb{C} : |z| < \epsilon\}$ of the origin that is invariant under $f = f_{c,\rho,\nu,\eta,\alpha}$.From the invariance of $U_\epsilon(0)$, it follows that the sequence of iterates is normal in $U_\epsilon(0)$, and hence, we have $U_\epsilon(0) \subset \mathcal{A}_0(f)$. From $\mathcal{A}_0(f) \subset \mathbb{D}$, we deduce that the zeros $c+\rho e^{2\pi ik/\nu}$, $k = 0,\dots,\nu-1$, do not lie in $\mathcal{A}_0(f)$ and are consequently all contained in the same preimage of $\mathcal{A}_0(f)$. This preimage possesses at least ν holes about the poles $c + \rho e^{2\pi ik/\nu+\pi i/\nu}$, $k = 0,\dots,\nu-1$. The final step is to show that $\mathcal{A}_0(f)$ is simply connected. This is done by counting critical points and by assuring that $\mathcal{A}_0(f)$ contains no other critical points than the origin itself.

In the examples presented above, multiply connected preimages of a simply connected Böttcher domain have been considered. Replacing in (2) z^2 (choose $\eta = 1$, for simplicity) by $z(z-d)/(1-dz)$, $0 < d < 1$, leads to multiply connected preimages of simply connected Schröder domains as long as the origin is an attracting fixed point and of a Siegel disk when 0 becomes indifferent and, e.g. $\alpha = (\sqrt{5}-1)\pi$. Further, replacing z^2 in (2) (again $\eta = 1$) by $(z^2 - d)/(1 - dz^2)$, $0 < d < 1$, and choosing $\alpha = 0$ leads to a family of rational mappings that contains functions with multiply connected preimages of simply connected Schröder or Leau domains.

References

[1] I. N. Baker, J. Kotus, Y. Lü, *Iterates of meromorphic functions III: Preperiodic domains*, Ergodic Theory Dyn. Syst. **11** (1991), 603-618.
[2] A. Beardon, *Iteration of rational functions*, Springer-Verlag (1990).
[3] N. Steinmetz, *Rational iteration*, Walter de Gruyter (1993).
[4] M. Stiemer, *Über rationale Homöomorphismen der Einheitskreislinie*, master's thesis, Universität Dortmund (1995), www.mathematik.uni-dortmund.de/lsix/diplom.php.
[5] M. Stiemer, *Rational maps with Fatou components of arbitrary connectivity number*, submitted (2007).

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