Mathematisches Forschungsinstitut Oberwolfach

Report No. 5/2007

Computational Electromagnetism and Acoustics

Organised by
Ralf Hiptmair (Zürich)
Ronald H. W. Hoppe (Augsburg)
Patrick Joly (Le Chesnay)
Ulrich Langer (Linz)

February 4th – February 10th, 2007

Abstract. It is a moot point to stress the significance of accurate and fast numerical methods for the simulation of electromagnetic fields and sound propagation for modern technology. This has triggered a surge of research in mathematical modeling and numerical analysis aimed to devise and improve methods for computational electromagnetism and acoustics.

Numerical techniques for solving the initial boundary value problems underlying both computational electromagnetics and acoustics comprise a wide array of different approaches ranging from integral equation methods to finite differences. Their development faces a few typical challenges: highly oscillatory solutions, control of numerical dispersion, infinite computational domains, ill-conditioned discrete operators, lack of strong ellipticity, hysteresis phenomena, to name only a few. Profound mathematical analysis is indispensable for tackling these issues.

Many outstanding contributions at this Oberwolfach conference on Computational Electromagnetism and Acoustics strikingly confirmed the immense recent progress made in the field. To name only a few highlights: there have been breakthroughs in the application and understanding of phase modulation and extraction approaches for the discretization of boundary integral equations at high frequencies. Much has been achieved in the development and analysis of discontinuous Galerkin methods. New insight have been gained into the construction and relationships of absorbing boundary conditions also for periodic media. Considerable progress has been made in the design of stable and space-time adaptive discretization techniques for wave propagation. New ideas have emerged for the fast and robust iterative solution for discrete quasi-static electromagnetic boundary value problems.

Mathematics Subject Classification (2000): 65Mxx, 65Nxx, 65Rxx, 78-04.

Introduction by the Organisers

The Oberwolfach Conference on Computational Electromagnetism and Acoustics, held Feb 5-9, 2007, at the Mathematisches Forschungsinstitut Oberwolfach, was already the second in series, following the 2004 Oberwolfach Conference on Computational Electromagnetism. This time the scope was slightly broader, because there has been substantial progress in numerical techniques for high-frequency wave propagation. In this regime, the mathematical and numerical challenges in acoustics are very similar to those arising in the simulation of electromagnetic fields. Usually new ideas and methods are first explored for acoustics and then adapted to electromagnetics. Moreover, many researchers are active in both fields. So it is natural to cover them in the context of a single event.

Like the previous meeting, the 2007 conference brought together more than fifty researchers active in the field, hailing from more than 10 countries. Among them were a large number of leading experts, but also quite a few students and postdoctoral fellows. At this point the organizers acknowledge the generous support of the European Union and NSF that covered expenses for young European and overseas participants. The majority of the participants were applied mathematicians, but a sizable number of people with a background in engineering also attended, as appropriate for a field with close ties to engineering and the sciences.

A total of 28 presentations was given at the conference, eight of which were meant to give a survey of particular numerical techniques. They covered the following topics:

- *High order integral equation methods* by O. Bruno, page 284,
- *Robust integral equation methods for high-frequency scattering* by I. Graham, page 308,
- *Plane wave basis functions* by P. Monk, page 341
- *Getting even with the Maxwell stress tensor* by A. Bossavit, page 283
- *Discontinuous Galerkin methods for the Maxwell eigenvalue problem* by A. Buffa, page 288.

  We must not forget to mention that Annalisa Buffa was awarded the first Oberwolfach Todd Fellowship during the workshop. She received the prize from the director M. Greuel for her many fundamental contributions to the numerical analysis of partial differential equations.
- *Discontinuous Galerkin for time-domain electromagnetics: Pros and cons* by T. Warburton, page 357
- *Non-spurious spectral-like elements for Maxwell's equations* by G. Cohen, page 297
- *Multiscale methods for plasma modeling* by P. Degond, page 300

As is typical of an event that is focused on an area of applications rather than a specific mathematical technique, talks varied a lot in style and scope, ranging from discussions of specific theoretical issues to general discussion of a computational method. Nevertheless, all of them managed to convey exciting new insights and developments. Thus the conference made it possible to catch a glimpse of the

future of computational electromagnetism and acoustics. The following extended abstracts will also enable the reader to keep abreast of many crucial trends in the field.

Workshop: Computational Electromagnetism and Acoustics

Table of Contents

Abstracts

Artificial boundary conditions for one-dimensional Schrödinger equations: from linear to nonlinear problems

Xavier Antoine
(joint work with C. Besse, Lille, and S. Descombes, ENS Lyon)

During the last two decades, an increasing attention has been directed towards the numerical simulation of linear (setting $V = 0$) or nonlinear Schrödinger equations

$$\begin{array}{l} i\partial_t u + \partial_2^2 u + V(u)u = 0, \quad \text{in } \mathbb{R}_x \times \mathbb{R}_t^{*+}, \\ u(x,0) = u_0(x), \quad \text{in } \mathbb{R}_x, \end{array} \tag{1}$$

with V a suitable nonlinear function. This is partially due to its central application in quantum mechanics but not only. Indeed, Schrödinger-type equations are often met in practice as some simplified models of real-world situations [7]. Among them, let us mention for example its application in fluid mechanics, in computational ocean acoustics, in optical fiber design, in nonlinear electromagnetism laser devices, in seismic migration problems,... As a result, a clear understanding of the way of computing the numerical solution to these classes of equations must be carefully investigated.

The temporal part of the equation is often discretized through the standard Crank-Nicolson scheme as introduced and studied by Delfour *et al.* [6]

$$i\frac{u^{n+1} - u^n}{\delta t} + \partial_x^2\left(\frac{u^{n+1} + u^n}{2}\right) = 0. \tag{2}$$

It is proved that the resulting schemes for the linear and nonlinear Schrödinger equations in an infinite medium are unconditionnally stable. However, in a realistic situation, a boundary condition must be introduced to get a finite computational domain Ω of boundary Σ. A classical boundary condition is the Dirichlet boundary condition. Quite obviously, this Boundary Condition (BC) is not fully satisfactory since it generates some unphysical reflections at the boundary. In this sense, this condition must be seen only as an approximation of a kind of expected *perfectly absorbing* boundary condition, also often called *transparent* (TBC).

In the one-dimensional linear case and for a free-space potential, the TBC can be straightforwardly built using the Laplace transform. The resulting BC is then defined by a fractional half-order derivative operator as follows

$$\partial_{\mathbf{n}} u + e^{-i\frac{\pi}{4}}\partial_t^{1/2} u = 0, \quad \text{for } (x,t) \in \Sigma \times \mathbb{R}^{*+}, \tag{3}$$

defining the half-order derivative operator by the Riemann-Liouville integral formula

$$\partial_t^{1/2}\psi(t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{\psi(s)}{\sqrt{t-s}}ds, \tag{4}$$

and $\mathbf{n}$ as the outwardly directed unit normal to Σ. Since this operator is of memory-type, the discretization of the linear Schrödinger equation coupled to this

TBC is not direct if one wants to preserve the unconditionnally stable criterion. Therefore, a special attention must be concentrated to the way of designing such a discretization. We will present both the construction and discretization of a suitable unconditionnally stable scheme. Next, a few numerical experiments in the 1D linear case for a reference solution are reported. We give a formal technique for deriving some boundary conditions for the 1D nonlinear Schrödinger (NLS) equation. Unlike the one-dimensional case, these conditions are no longer exact and are generally called Artificial Boundary Conditions (ABCs). An example of such a condition is given for a cubic nonlinearity by the expression

$$\partial_{\mathbf{n}} u + e^{-i\frac{\pi}{4}} e^{i\mathbb{U}} \partial_t^{1/2}(e^{-i\mathbb{U}} u) + iq\frac{\partial_x |u|^2}{4} e^{i\mathbb{U}} I_t^{1/2}(e^{-i\mathbb{U}} u) = 0, \tag{5}$$

which is valid on $\Sigma \times \mathbb{R}^{*+}$. The nonlinear integral term $\mathbb{U}$ is defined by

$$\mathbb{U}(x,t) = q\int_0^t |u(x,s)|^2 ds. \tag{6}$$

The numerical discretization is next discussed and some numerical simulations are presented for the propagation of a soliton for the nonlinear model. Finally, a conclusion and a short discussion about other related extensions and problems will be discussed. A survey paper summarizing all these results is in preparation [1] and will be soon available.

This paper summarizes some results [2, 3, 4, 5] obtained through collaborations over the past years with C. Besse (Université de Lille), S. Descombes (ENS Lyon) and V. Mouysset (ONERA Toulouse & MIP-Toulouse).

References

[1] X. Antoine, A. Arnold, C. Besse, M. Ehrhardt and A. Schädle, *A review of transparent and artificial boundary conditions techniques for linear and nonlinear Schrödinger equations*, work in progress.

[2] X. Antoine and C. Besse, *Construction, structure and asymptotic approximations of a microdifferential transparent boundary condition for the linear Schrödinger equation* J. Math. Pures Appl. IX. Sér. **80** (2001), 701–738.

[3] X. Antoine and C. Besse, *Unconditionally stable discretization schemes of non-reflecting boundary conditions for the one-dimensional Schrödinger equation*, J. Comp. Phys. **188** (2003), 157–175.

[4] X. Antoine, C. Besse and S. Descombes, *Artificial boundary conditions for nonlinear Schrödinger equations*, SIAM J. Numer. Anal. **43** (2006), 2272–2293.

[5] X. Antoine, C. Besse and V. Mouysset, *Numerical schemes for the simulation of the two-dimensional Schrödinger equation using non-reflecting boundary conditions*, Math. Comp. **73** (2004), pp.1779–1799.

[6] M. Delfour, M. Fortin and G. Payre, *Finite-difference solutions of a nonlinear Schrödinger equation*, J. Comput. Phys. **44** (1981), pp. 277-288.

[7] C. Sulem and P.L. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.

Getting even with the Maxwell stress tensor: The question of electromagnetic forces, in differential-geometric language.

Alain Bossavit

The geometric view of Maxwell's equations (fields as differential forms, separation between non-metric and metric-dependent parts of the theory, reinterpretation of "coefficients" ϵ and μ as Hodge-like operators), offers a natural, fast and direct approach to discretization, which results in a unified presentation of apparently diverse methods, such as Weiland's "Finite Integration Technique" [1], Tonti's "cell method" [5], finite volumes [4], finite elements [3]. In many coupled problems (for instance, the design of magnetostrictive actuators), the numerical treatment of forces is an issue, hence the need to integrate classical tools, such as the Maxwell stress tensor, to this geometrical framework. In the proposed reinterpretation, Maxwell's tensor is a "twisted covector-valued 2-form", that is to say, an object which can be integrated on a surface endowed with crossing direction, the outcome of integration being a covector, which can be interpreted, thanks to the virtual work approach, as momentum flux, and hence, in the case of a closed surface, as a force: the force exerted by the electromagnetic field on matter enclosed by this surface. The Lie derivative appears as an essential ingredient in analytic manipulations one must carry through to get there.

The talk was in three parts: A reminder of the geometric formulation of Maxwell's theory, then a derivation, from first principles, of the analytic form of the stress "tensor", followed by a survey of the virtual work method in electromagnetism.

To give the flavor of the approach in this summary, let's restrain to magnetostatics in the simple case of uniform permeability μ, i.e., to the system of equations

$$\mathrm{d}b = 0,\ h = \nu b,\ \mathrm{d}h = j,$$

in 3D affine space, with given (time invariant) current density j, and ν (the inverse of μ) translation-invariant. (Small-case letters are used for differential forms b, h, etc., whose proxy vectors are the traditional B, H, etc.) In this situation, the only forces are Laplace forces ($J \times B$, in vectorial notation) on conductors. Main ideas: (1) Force f, in accordance to the virtual work principle, is dual to virtual displacement, which is represented by a vector. Therefore, force "is" a covector. (2) Momentum p is the time integral of force, $f = \partial_t p$, and force is "momentum flux", as argued by many teachers (most notably, perhaps, by Herrmann [2]). (3) Momentum is conserved, just as electric charge, for instance, is conserved: Charge conservation being expressed by $\partial_t q + \mathrm{d}j = 0$, where charge q is a twisted 3-form and j a twisted 2-form, momentum conservation should appear as $\partial_t p + \mathrm{d}m = 0$, with p , momentum density, a covector-valued twisted 3-form, and m a twisted covector-valued 2-form, the "stress tensor". Our aim is to determine m in terms of the field $\{h, b\}$.

We start from the expression of Laplace force as the *map* $v \to j \wedge i_v b$, indeed a covector-valued twisted 3-form. The dummy parameter v that appears there will be, from now on, a uniform vector field. Integrating over a volume V, the total

(contained in V) momentum $\int_V p$ satisfies $\partial_t \int_V p = v \to \partial_t[\int_V j \wedge i_v b]$. One has $\mathrm{d}(h \wedge i_v b) = \mathrm{d}h \wedge i_v b - h \wedge \mathrm{d}i_v b)$. Therefore,

$$\int_V j \wedge i_v b = \int_V \mathrm{d}h \wedge i_v b = \int_V \mathrm{d}(h \wedge i_v b) + \int_V h \wedge \mathrm{d}i_v b.$$

Since $\mathrm{d}b = 0$ here, $i_v \mathrm{d}b = \mathrm{L}_v b$, actually. The flow of v preserves ν: Therefore L_v and ν commute. Using that, and the fact that $h \wedge b$, a 3-form, is closed, we have

$$\mathrm{L}_v(h \wedge b) = \mathrm{L}_v h \wedge b + h \wedge \mathrm{L}_v b = \mathrm{d}i_v(h \wedge b) = \mathrm{L}_v \nu b \wedge b + h \wedge \mathrm{L}_v b$$

$$= \nu \mathrm{L}_v b \wedge b + h \wedge \mathrm{L}_v b = \mathrm{L}_v b \wedge \nu b + h \wedge \mathrm{L}_v b = 2\mathrm{L}_v b \wedge h = 2h \wedge \mathrm{L}_v b.$$

Hence, $\int_V j \wedge i_v b = \int_V \mathrm{d}(h \wedge i_v b) + 1/2 \int_V \mathrm{d}i_v(h \wedge b) = \int_{\partial V}[h \wedge i_v b + 1/2\, i_v(h \wedge b)]$. Thus, the sought-for m is the map $v \to -[h \wedge i_v b + 1/2\, i_v(h \wedge b)]$. To get this closer to the familiar expression for the Maxwell tensor, remark that $i_v(h \wedge b) = i_v h \wedge b - h \wedge i_v b$, hence m is the map $v \to -[i_v h \wedge b - 1/2\, i_v(h \wedge b)]$, the proxy of which is indeed (in coordinates with Einstein's convention) the standard $H^i B^j - 1/2\, H^k B^k \delta^{ij}$. The full Maxwell case, with homogeneous ν and ϵ, is treated by similar techniques, by which a similar contribution of the electric field to m is found, while the presence of momentum *in the field*, with density $v \to i_v d \wedge b$, is recognized. In the non-homogeneous case, the keypoint is that $\mathrm{L}_v \nu - \nu \mathrm{L}_v \neq 0$, hence additional body forces, $(|B|^2/2)\, \mathrm{grad}\, \nu$ in terms of proxies (vector B and scalar reluctivity ν).

References

[1] M. Clemens, T. Weiland, *Finite elements in computational electromagnetism*, PIER **32** (2001), 65–87.

[2] F. Herrmann, *Energy density and stress: A new approach to teaching electromagnetism*, Am. J. Phys., **57** (1989), 707–714.

[3] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica **11** (2002), 237–340.

[4] M. Remaki, *A new finite volume scheme for solving Maxwell's system*, COMPEL **19** (2000), 913–31.

[5] M. Tonti, *Finite formulation of the electromagnetic field*, PIER **32** (2001), 1–44.

Computational electromagnetics and acoustics: high-order solvers, high-frequency configurations, high-order surface representations

Oscar P. Bruno

We survey a set of algorithms and methodologies developed recently for the numerical solution of problems of scattering by complex bodies in three-dimensional space, and we briefly mention ongoing work in these areas. These methods, which are based on integral equations, high-order integration, Fast Fourier Transforms and highly accurate high-frequency integrators, can be used in the solution of problems of electromagnetic and acoustic scattering by surfaces and penetrable scatterers—even in cases in which the scatterers contain geometric singularities such as corners and edges. All of the solvers presented here exhibit high-order convergence, they run on low memories and reduced operation counts, and they

result in solutions with a high degree of accuracy. In particular, our approach to direct solution of integral equations results in algorithms that can evaluate accurately in a personal computer scattering from hundred-wavelength-long objects—a goal, otherwise achievable today only by super-computing. The high-order high-frequency methods we present, in turn, are efficient where our direct methods become costly, thus leading to an overall computational methodology which is applicable and accurate throughout the electromagnetic spectrum. These algorithms produce high-order accuracy by exploiting the high-order convergence of both trapezoidal rule integration and approximation via Fourier series for smooth periodic functions. Fast numerics in our algorithms results from use of $\mathcal{O}(N \log(N))$ Fast Fourier Transforms, high-frequency integrators and resolution of singularities.

Some of the basic concepts in the algorithms mentioned above were introduced as part of the surface scattering method [2, 3]. Assuming a smooth scattering surface, the algorithm proceeds by transforming all integrands in an integral equation formulation into smooth and periodic functions—using smooth partitions-of-unity to partition the integration domain and polar-coordinate changes of variables to eliminate kernel singularities. Appropriate uses of the trapezoidal rule are then made to produce a high-order, super-algebraically convergent integration algorithm; see [1] for a discussion of the properties of the trapezoidal rule in these regards. An accelerator based on use of equivalent sources and Fast Fourier Transforms then enables fast evaluation of the integral operator while preserving high order accuracy. Using the GMRES iterative linear algebra solver in conjunction with the "forward map" algorithm just described yields a fast, high-order solver which runs in $\mathcal{O}\left(N^{6/5}\right)$ to $\mathcal{O}\left(N^{4/3}\right)$ operations—depending on the complexity of the underlying scattering surface. A number of comparisons with other solvers and a discussion of merits in terms of accuracies, speed and storage offered by the present approach are presented in [2, 1]. Solvers for problems of volumetric scattering, with properties similar to those mentioned above for surface scattering, were presented in [5, 6, 7, 8]. A corresponding high-order $\mathcal{O}(1)$ surface-scattering solver for high-frequency problems, i.e., a solver that provides fixed accuracy in fixed computational times for arbitrarily high frequencies, was presented in [9, 11, 12, 10, 13]; see also the related text by I. Graham in the present Oberwolfach report and references therein. An extension of these high-frequency solvers to volumetric problems is given in [14]; this *fully three-dimensional* approximate solver has been used to evaluate the propagation of GPS signals, of a wavelength of the order of 20cm, through hundreds of kilometers within the atmosphere. As mentioned above, the high-order surface scattering solvers require subdivision of a scattering surface into smooth patches—possibly meeting at edges, corners, etc. A methodology to obtain such smooth representations in terms of Fourier series and a certain continuation method for the resolution of the Gibbs phenomenon was introduced in [4]; this approach can produce high-order representations of complex engineering surfaces on the basis of regular low order finite-element or even point-cloud renderings.

The use of integral-equation iterative solvers for the full Maxwell problem often requires exceedingly large numbers of iterations; a recent methodology [15] addresses this issue by incorporating an integral parametrix of a form similar to that of the operators already used as part of the integral formulation. This approach gives rise to order-of-magnitude improvements in the number of iterations needed for convergence to a given residual and/or accuracy—at a incremental time requirement of the order of 40% per iteration; see also the related contribution by David P. Levadoux in the present Oberwolfach Report and additional references therein. A similar situation arises in connection with the problem of scattering by open surfaces; our approach [16] introduces a generalization of the classical Calderón relations that applies to open surface configurations and which, together with a high-order implementation, has been used to produce results of very high accuracy in very small numbers of GMRES iterations and, thus, in short computational times.

We have found that some of the elements implicit in the integral solvers discussed above can be made to be useful in applications to other PDE problems and solvers, including time dependent problems, finite-element/finite-difference algorithms, etc. We mention one example of such offshoot methodologies: An algorithm for fast evaluation of full, nonlocal computational boundary conditions was introduced [17] on the basis of the equivalent-source/FFT acceleration methods mentioned above. This method, which runs at a cost much smaller than the underlying FEM/FD algorithms, can be used as a substitute for absorbing boundary conditions (ABC), allows for use of very small computational domains and, unlike most ABC approaches, gives rise to error-controllable approximations. An additional offshoot is currently being developed [18]: a version of the Douglas-Peaceman-Rachford ADI Scheme that, instead of finite-differences, it is based upon the continuation method that was mentioned above in connection with the surface representation problem. Unlike the previous versions of the ADI, which are mostly restricted to square domains, the new version can be successfully applied to non-square domains, yielding fast numerics and high-order accuracy in addition to the classical unconditional stability of the ADI. Thus far our algorithm has been demonstrated for two-dimensional configurations and for the heat and wave equations; additional extensions are currently being pursued.

The efforts surveyed above have sought to advance the state of the art in numerical solution and analysis of Partial Differential Equations with three main goals in mind: generality, accuracy and speed. We believe these properties are essential to render computational science a predictive science beyond the present confines of the field. The use and development of sophisticated mathematical methods together with aggressive pursuit of practical and general algorithms should be, in our view, leading forces in forthcoming developments in computational science.

References

[1] O. P. Bruno, *Fast, High-Order, High-Frequency Integral Methods for Computational Acoustics and Electromagnetics*, in "Topics in Computational Wave Propagation Direct and Inverse Problems Series" Lecture Notes in Computational Science and Engineering , Vol. 31 Ainsworth, M.; Davies, P.; Duncan, D.; Martin, P.; Rynne, B. (Eds.) (2003).

[2] O. P. Bruno and L. A. Kunyansky, *A Fast, High-Order Algorithm for the Solution of Surface Scattering Problems: Basic Implementation, Tests, and Applications*, J. Comput. Phys. **169** (2001) 80–110.

[3] O. P. Bruno and L. A. Kunyansky, *Surface scattering in 3-D: an accelerated high order solver*, Proc. R. Soc. Lond. A **457** (2001), 2921–2934.

[4] O. P. Bruno, Y. Han and M. Pohlman, *Accurate, high-order representation of complex three-dimensional surfaces via Fourier-Continuation analysis*, Submitted.

[5] O. P. Bruno, and A. Sei, *A Fast High-Order Solver for Problems of Scattering by Heterogeneous Bodies*, IEEE Trans. Antenn. Propag. **51** (2003), 3142–3154.

[6] E. M. Hyde, and O. P. Bruno, *A fast, high-order method for scattering by penetrable bodies in three dimensions,* J. Comput. Phys. **202** (2005), 236–261.

[7] O. P. Bruno, and E. M. Hyde, *An efficient, preconditioned, high-order solver for scattering by two-dimensional, inhomogeneous media*, J. Comput. Phys. **200** (2004), 670–694.

[8] O. P. Bruno, and E. M. Hyde, *Higher-order Fourier approximation in scattering by two-dimensional, inhomogeneous media*, SIAM J. Numer. Anal. **42** (2005), 2298-2819.

[9] O. P. Bruno, C. Geuzaine, A. Monro J. and F. Reitich, *Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case*, Philosophical Transactions of the Royal Society of London A **362** (2004), 629-645.

[10] C. Geuzaine, O. P. Bruno and F. Reitich, *On the $\mathcal{O}(1)$ solution of multiple-scattering problems*, IEEE Trans. Magn. **41** (2005), 1488–1491.

[11] O. P. Bruno, C. Geuzaine and F. Reitich, *A new high-order high-frequency integral equation method for the solution of scattering problems I: Single-scattering configurations*, *The 20th Annual Review of Progress in Applied Computational Electromagnetics (ACES 2004), April 19-23, 2004 - Syracuse, NY*

[12] O. P. Bruno, C. Geuzaine, and F. Reitich, *A new high-order high-frequency integral equation method for the solution of scattering problems II: Multiple-scattering configurations*, *The 20th Annual Review of Progress in Applied Computational Electromagnetics (ACES 2004), April 19-23, 2004 - Syracuse, NY*

[13] O. P. Bruno, and C. Geuzaine, *An $\mathcal{O}(1)$ Integration Scheme for Three-Dimensional Surface Scattering Problems*, To appear in the Journal of Computational and Applied Mathematics.

[14] J. Chaubell, C. O. Ao and O. P. Bruno, *Evaluation of Propagation of GPS Signals through the Atmosphere via High-Frequency Localization and Rytov's Approximation*, In preparation.

[15] O. P. Bruno, T. Elling, R. Paffenroth and C. Turc, *Electromagnetic integral equations requiring small numbers of GMRES iterations*, In preparation.

[16] O. P. Bruno and S. K. Lintner, In preparation.

[17] O. P. Bruno and D. Hoch, In preparation.

[18] O. P. Bruno and M. Lyon, In preparation.

Discontinuous Galerkin Method for Maxwell eigenproblems

Annalisa Buffa

Let Ω be a domain in $\mathbb{R}^n$, $n = 2, 3$ with outward normal and counterclockwise tangent unit vectors $\mathbf{n}$ and $\mathbf{t}$, respectively, on $\partial\Omega$. Consider the eigenproblem: find $\mathbf{u} \neq 0$ and ω such that

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{u}) = \omega^2 \varepsilon \mathbf{u},$$

with $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$, where μ and ε are the magnetic permeability and the electric permittivity, respectively. We set $\mathbf{V} = H_0(\mathrm{curl}; \Omega)$, $\mathbf{V}^0 = H_0(\mathrm{curl}; \Omega) \cap \mathrm{Ker}\{\mathrm{curl}\}$, and $\mathbf{W} = \mathbf{V} \cap \mathrm{Ker}\{\mathrm{div}\}$.

Define the (hermitian) bilinear forms $a : \mathbf{V} \times \mathbf{V} \to \mathbb{C}$ as

$$a(\mathbf{u}, \mathbf{v}) = (\mu^{-1}\nabla \times \mathbf{u}, \nabla \times \mathbf{v}).$$

The variational formulation of the eigenproblem we are interested in is the following: find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{W} \times \mathbb{C}$ such that

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall \mathbf{v} \in \mathbf{W}.$$

A standard way to discretize this problem consists in neglecting the constraint $\mathbf{u} \in \mathbf{W}$ and adding a zero frequency eigenspace corresponding to the *infinite-dimensional* space $\mathbf{V}^0$, i.e. adding an essential part to the spectrum $\sigma^{ess} = \{0\}$. This leads to the following variational problem.

Problem 1. *Find* $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$*:*

$$a(\mathbf{u}, \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v})_\varepsilon \qquad \forall \mathbf{v} \in \mathbf{V}.$$

This is the problem we discretize with a Discontinuous Galerkin (DG) technique. Conforming approximations are treated in [2], [6], [3].

Let $\mathcal{T}_h$ be a conformal, shape-regular partition of Ω into tetrahedra $\{K\}$, where $\mathtt{h} = \max_{K \in \mathcal{T}_h} h_K$, with $h_K = \mathrm{diam}(K)$. We denote by $\mathcal{F}_h$ the set of all the faces of elements in $\mathcal{T}_h$. On such a mesh, we define the following space:

$$\mathbf{V}_h := \{\mathbf{v} \in L^2(\Omega)^n : \mathbf{v}|_K \in \mathcal{P}^\ell(K)^{2n-3} \ \forall K \in \mathcal{T}_h\}, \tag{1}$$

where $\mathcal{P}^\ell(K)$ is the space of complex polynomials of total degree at most ℓ on K. For piecewise smooth vector-valued functions $\mathbf{v}$, on interior faces, we denote by $[\![\mathbf{v}]\!]_T$ and $\{\!\{\mathbf{v}\}\!\}$ the tangential jump and mean value of $\mathbf{v}$, respectively. On boundary faces, we set $[\![\mathbf{v}]\!]_T = \mathbf{n} \times \mathbf{v}$ and $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$.

With the help of these definitions, we introduce the following bilinear form (∇_h stands for the gradient element by element):

$$\begin{aligned} a_h^{IP}(\mathbf{u}, \mathbf{v}) =& (\mu^{-1}\nabla_h \times \mathbf{u}, \nabla_h \times \mathbf{v}) - \int_{\mathcal{F}_h} [\![\overline{\mathbf{v}}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \mathbf{u}\}\!\}\, ds \\ & - \int_{\mathcal{F}_h} [\![\overline{\mathbf{u}}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \mathbf{v}\}\!\}\, ds + \int_{\mathcal{F}_h} \mathtt{a} [\![\mathbf{u}]\!]_T \cdot [\![\overline{\mathbf{v}}]\!]_T\, ds, \end{aligned}$$

with $\mathtt{a} := a_{\mathrm{IP}}\mathtt{h}^{-1}$, $a_{\mathrm{IP}} > 0$ being a parameter independent of the mesh size and the material coefficients ([1]). See [10], [7] for variants of this method. For Maxwell (coercive) source problem, the discontinuous Galerkin discretization has been studied in several papers: see [9], [8] and the references therein.

In [5], we study, among others, the eigenvalue problem:

Problem 2. *Find* $(\mathbf{0} \neq \mathbf{u}_h, \omega_h) \in \mathbf{V}_h \times \mathbb{C}$*:*

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \omega_h^2(\mathbf{u}_h, \mathbf{v}_h)_\varepsilon \qquad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

It is not difficult to see that we need to prove that "*all discrete eigenvalues approaching the kernel are far from the others*". This is an important feature because, if not, eigenvalues approaching the kernel are spread all over the spectrum and it is very hard, if not impossible, to distinguish them from the discrete eigenvalues approaching the point spectrum.

We refer the reader to [4] for numerical tests.

References

[1] D.N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.

[2] D. Boffi. Fortin operator and discrete compactness for edge elements. *Numer. Math.*, 87:229–246, 2000.

[3] Daniele Boffi, Martin Costabel, Monique Dauge, and Leszek Demkowicz. Discrete compactness for the *hp* version of rectangular edge finite elements. *SIAM J. Numer. Anal.*, 44(3):979–1004 (electronic), 2006.

[4] A. Buffa, P. Houston, and I. Perugia. Discontinuous galerkin computation of the maxwell eigenvalues on simplicial meshes. *J. Comput. Appl. Math.*, 2006. to appear.

[5] Annalisa Buffa and Ilaria Perugia. Discontinuous Galerkin approximation of the Maxwell eigenproblem. *SIAM J. Numer. Anal.*, 44(5):2198–2226 (electronic), 2006.

[6] S. Caorsi, P. Fernandes, and M. Raffetto. On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems. *SIAM J. Numer. Anal.*, 38:580–607, 2000.

[7] C. Dawson, S. Sun, and M.F. Wheeler. Compatible algorithms for coupled flow and transport. *Comput. Methods Appl. Mech. Engrg.*, 193:2565–2580, 2004.

[8] P. Houston, I. Perugia, A. Schneebeli, and D. Schötzau. Interior penalty method for the indefinite time-harmonic Maxwell equations. *Numer. Math.*, 100:485–518, 2005.

[9] P. Houston, I. Perugia, and D. Schötzau. Mixed discontinuous Galerkin approximation of the Maxwell operator: Non-stabilized formulation. *J. Sci. Comp.*, 22:325–356, 2005.

[10] B. Rivière, M.F. Wheeler, and V. Girault. Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems, Part I. *Computational Geosciences*, 3:337–360, 1999.

Stability and Convergence of the Time-Domain Perfectly Matched Layer Method for Acoustic Scattering Problems

Zhiming Chen

We consider the acoustic scattering problem with the sound-hard boundary condition on the obstacle

$$\frac{\partial u}{\partial t} = -\mathrm{div}\mathbf{p}, \quad \frac{\partial \mathbf{p}}{\partial t} = -\nabla u \quad \text{in } [\mathbb{R}^2 \backslash \bar{D}] \times (0,T), \tag{1}$$

$$\mathbf{p} \cdot \mathbf{n}_D = g \quad \text{on } \Gamma_D \times (0,T), \tag{2}$$

$$\sqrt{r}(u - \mathbf{p} \cdot \hat{x}) \to 0 \quad \text{as } r = |x| \to \infty, \ \text{ a.e. } t \in (0,T), \tag{3}$$

$$u|_{t=0} = u_0, \quad \mathbf{p}|_{t=0} = \mathbf{p}_0. \tag{4}$$

Here u is the pressure and $\mathbf{p}$ is the velocity field of the wave. $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, g is determined by the incoming wave, and $\mathbf{n}_D$ is the unit outer normal to Γ_D. $u_0, \mathbf{p}_0$ are assumed to be supported in some circle $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$ for some $R > 0$. (3) is the radiation condition which corresponds to the well-known Sommerfeld radiation condition in the frequency domain. We remark that the results in this paper can be easily extended to solve the scattering problems with other boundary conditions such as the sound-soft or the impedance boundary condition on Γ_D.

One of the fundamental problem in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. An interesting alternative to the method of absorbing boundary conditions is the method of perfectly matched layer (PML). Since the work of Berenger [5] which proposed a PML technique for solving the time-dependent Maxwell equations in the Cartesian coordinates, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [17], Teixeira and Chew [16] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain.

There are two classes of time-domain PML methods for the wave scattering problems. The first class, called "split-field PML method" in the literature, includes the original Bérénger PML method. It is shown in Abarbanel and Gottlieb [2] that the Bérénger PML method is only weakly well-posed and thus may suffer instability in practical applications. The second class, the so-called "unsplit-field PML formulations" in the literature, is however, strongly well-posed. One such successful method is the uniaxial PML method developed in Sacks *et al* [15] and Gedney [10] for the Maxwell equations in the Cartesian coordinates. The unsplit-field PML methods in the curvilinear coordinates are introduced in Petropoulos [14] and [16] for Maxwell equations.

Although the tremendous attention and success in the application of PML methods in the engineering literature, there are few mathematical results on the convergence of the PML methods. For the Helmholtz equation in the frequency domain, it is proved in Lassas and Somersalo [13], Hohage *et al* [12] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. In Chen and Wu [7], Chen and Liu [6], an adaptive PML technique is proposed and studied in which a posteriori error estimate is used to determine the PML parameters. In particular, it is shown that the exponential convergence can be achieved for fixed thickness of the PML layer by enlarging PML medium properties. For the time-domain PML method, not much mathematical convergence analysis is known except the work in Hagstrom [11] in which the planar PML method in one space direction is considered for the wave equation. In de Hoop *et al* [8], Diaz and Joly [9], the PML system with point source is analyzed based on the Cagniard - de Hoop method.

The long time stability of the PML methods is also a much studied topic in the literature (see e.g. Bécache and Joly [3], Bécache *et al* [4], Appelö *et al* [1]). For a PML method to be practically useful, it must be stable in time, that is, the solution should not grow exponentially in time. We remark that the well-posedness of the PML system which follows from the theory of symmetric hyperbolic systems allows the exponential growth of the solutions. In [3, 4, 1] the stability of the PML systems is considered under the assumption of constant PML medium property which, however, violates the property of perfect matchness of the associated PML system. Thus those studies do not fully explain the success of practical applications of the PML methods. Our strategy to prove the stability of the PML method is based on the combination of the stability of the original scattering problem and the convergence of the PML method.

In this talk we will show the exponential convergence and stability of the unsplit-field PML method for the acoustic wave scattering problem in the polar coordinates. Our analysis starts with the well-posedness and stability of the scattering problem (1)-(4). We then introduce the upsplit-field PML method for (1)-(4) in the polar coordinates by following the procedure in [14] for Maxwell equations. Also the well-posedness of the initial boundary value problem of the PML system is established. Next we prove the exponential convergence of the PML method. Finally we consider the stability of the PML system.

One of the key ingredients in our analysis is the following uniform exponential decay property of the modified Bessel function $K_n(z)$

$$\frac{|K_n(s\rho_1+\tau)|}{|K_n(s\rho_2)|} \le e^{-\tau\left(1-\frac{\rho_2^2}{\rho_1^2}\right)},$$

for any $n \in \mathbb{Z}$, $s \in \mathbb{C}$ with $\Re(s) > 0$, $\rho_1 > \rho_2 > 0$, and $\tau > 0$. The proof depends on the Macdonald formula for the integral representation of the product of modified Bessel functions and extends our earlier uniform estimate in [6] for the first Hankel function $H_\nu^1(z)$, $\nu \in \mathbb{R}$, in the upper-half complex plane.

References

[1] D. Appelö, T. Hagstrom and G. Kreiss. *Perfectly matched layers for hyperbolic systems: General formulation, well-posedness and stability.* Preprint, University of New Mexico, 2006.

[2] S. Abarbanel and D. Gottlieb. *A Mathematical Analysis of the PML Method.* J. Comput. Phys. **134** (1997), 357-363.

[3] E. Bécache and P. Joly. *On the analysis of Bérénger's perfectly matched layers for Maxwell equations.* Math. Modelling Numer. Anal. **36** (2002), 87-119.

[4] E. Bécache, P.G. Petropoulos, and D. Gedney. *On the long-time behavior of unsplit perfectly matched layers,* IEEE Trans. Antennas and Propagation **52** (2004), 1335-1342.

[5] J.-P. Bérénger. *A perfectly matched layer for the absorption of electromagnetic waves.* J. Comput. Physics **114** (1994), 185-200.

[6] Z. Chen and X.Z. Liu. *An adaptive perfectly matched layer technique for time-harmonic scattering problems.* SIAM J. Numer. Anal. **43** (2005), 645-671.

[7] Z. Chen and H. Wu. *An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures.* SIAM J. Numer. Anal. **41**, (2003), 799-826.

[8] A.T. de Hoop, P.M. van den Berg, and R.F. Remis. *Absorbing boundary conditions and perfectly matched layers – An analytic time-domain performance analysis,* IEEE Trans. Mag. **38** (2002), 657-660.

[9] J. Diaz and P. Joly. *A time domain analysis of PML models in acoustics,* Comput. Methods Appl. Mech. Engrg. **195** (2006), 3820-2853.

[10] S.D. Gedney. *An anisotropic perfectly matched layer absorbing media for the truncation of FDTD latices.* IEEE Trans. Antennas Propagat. **AP-44** (1995), 1630-1639.

[11] T. Hagstrom. *Radiation boundary conditions for the numerical simulation of waves.* Acta Numerica (1999) 47-106.

[12] T. Hohage, F. Schmidt and L. Zschiedrich. *Solving time-harmonic scattering problems based on the pole condition. II: Convergence of the PML method.* SIAM J. Math. Anal., **35** (2003), 547-560.

[13] M. Lassas and E. Somersalo. *On the existence and convergence of the solution of PML equations.* Computing **60** (1998), 229-241.

[14] P.G. Petropoulos. *Reflectionless sponge layers as absorbing boundary conditions for the numerical solution of Maxwell equations in Rectangular, cylindrical, and spherical coordinates.* SIAM J. Appl. Math. **60** (2000), 1037-1058.

[15] Z.S. Sacks, D.M. Kingsland, D.M. Lee and J.F. Lee. *A perfectly matched anisotropic absorber for use as an absorbing boundary condition.* IEEE Trans. Antennas Propagat. **AP-43** (1995), 1460-1463.

[16] F.L. Teixeira and W.C. Chew. *Advances in the theory of perfectly matched layers.* In: W.C. Chew et al, (eds.), Fast and Efficient Algorithms in Computational Electromagnetics, pp. 283-346, Artech House, Boston, 2001.

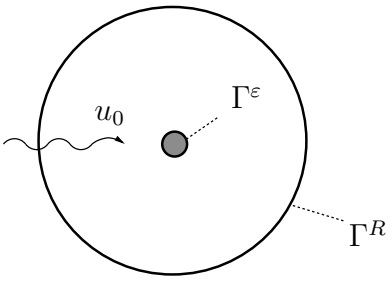
[17] E. Turkel and A. Yefet. *Absorbing PML boundary layers for wave-like equations.* Appl. Numer. Math. **27** (1998), 533-557.

A generalized Holland model for wave diffraction by thin wires

Xavier Claeys
(joint work with F.Collino and M.Duruflé)

1. General context and motivations

The general context of the problem we are interested in is the simulation of electromagnetic wave diffraction by thin wires. We make two assumptions on such obstacles: we suppose the wires to be perfectly conducting, and suppose their thickness is much smaller than the wavelength of the field. For example, antennas satisfy these hypothesis. We propose to study the model problem of acoustic diffraction by thin obstacles, and we wish to perform such computations using a volumic method with no mesh refinement.



To our knowledge there exists only one numerical method matching these previous requirements. It is called the Holland model and was proposed by Holland and Simpson in the engineer literature [1]. It is suitable for straigth wires aligned with the axis of a cartesian grid in a FDTD scheme for electromagnetics. This model assumes that the current is constant across any section of the wire and that the field has an electrostatic behavior close to the wire. It is also based on a averaging operator in a region as large as a cell close to the wire. This averaging operator involves a parameter called the lineic inductance that is to be chosen on a empirical basis. This model is widely used in volumic methods, it provides precise results (for a well chosen lineic inductance) and can be easily implemented. Unfortunately it lacks a solid theoretical basis. In particular, there exist only empirical formulas for the lineic inductance.

For the model problem of bidimensional acoustic diffraction, we present an augmented finite element method that matches the requirements stated before. From this alternative method, we are able to recover the Holland model. This method then generalizes the Holland model for arbitrary meshes, provides a rigorous setting for it and a theoretical expression of the lineic inductance.

2. Exact and simplified problem

We consider the 2D acoustic diffraction problem in a disc $\Omega_R = D(0,R)$ (with $\Gamma^R = \partial\Omega_R$) containing only one small obstacle $\Omega_\varepsilon = D(0,\varepsilon)$ (with $\Gamma^\varepsilon = \partial\Omega_\varepsilon$) and we suppose that $\varepsilon \to 0$. We impose a Dirichlet boundary condition on Γ^ε and an outgoing radiation condition on Γ^R. Then the problem we want to solve is the

following,

$$\begin{cases} \text{Find } (u^\varepsilon, p^\varepsilon) \in H^1(\Omega_R) \times H^{-1/2}(\Gamma^\varepsilon) \text{ such that} \\ \\ a(u^\varepsilon, v) + b^\varepsilon(p^\varepsilon, v) = \int_{\Gamma^R} (T_R u_0 + \frac{\partial u_0}{\partial n})\overline{v}\,,\ \forall v \in H^1(\Omega_R), \\ \\ b^\varepsilon(q, u^\varepsilon) = 0\,,\ \forall q \in H^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

Here a and b^ε are given by $a(u,v) = \int_{\Omega_R} \nabla u \cdot \nabla \overline{v} \ - k^2 \int_{\Omega^R} u\overline{v} + \int_{\Gamma^R} \overline{v} T_R u$ and $b^\varepsilon(q,v) = \int_{\Gamma^\varepsilon} q\overline{v}$, T_R is the usual Dirichlet-to-Neumann map, and $u_0(\mathbf{x}) = -e^{i\mathbf{k}\cdot\mathbf{x}}$ is the incident field we consider . In this formulation, we take the presence of the small obstacle into account using the bilinear form b^ε and Lagrange multipliers chosen in the space $H^{-1/2}(\Gamma^\varepsilon)$. The function p^ε is then the jump of the normal derivative of u^ε on Γ^ε, $p^\varepsilon = [\frac{\partial u^\varepsilon}{\partial n}]_{|\Gamma^\varepsilon}$. Using matched asymptotic expansions, we notice that p^ε is nearly a constant function on Γ^ε. We are then led to consider the simplified formulation

$$\begin{cases} \text{Find } (u^\varepsilon_{\rm ap}, p^\varepsilon_{\rm ap}) \in H^1(\Omega_R) \times P_0 \text{ such that} \\ \\ a(u^\varepsilon_{\rm ap}, v) + b^\varepsilon(p^\varepsilon_{\rm ap}, v) = \int_{\Gamma^R} (T_R u_0 + \frac{\partial u_0}{\partial n})\overline{v}\,,\ \forall v \in H^1(\Omega_R), \\ \\ b^\varepsilon(q, u^\varepsilon_{\rm ap}) = 0\,,\ \forall q \in P_0, \end{cases}$$

where we have replaced $H^{-1/2}(\Gamma^\varepsilon)$ by P_0 the space of constant functions on Γ^ε. This problem is well posed for ε sufficiently small (it satisfies inf-sup conditions uniform with respect to ε). Again using asymptotic analysis we can prove that the solution of this simplified problem is very close to the exact solution.

H	$=$	$H^1(\Omega_R)$	$H^1(\Omega_R \setminus \overline{\Omega_{R'}})$
$\|u^\varepsilon - u_0\|_H$	$\sim$	$\|\ln \varepsilon\|^{-1/2}$	$\|\ln \varepsilon\|^{-1}$
$\|u^\varepsilon - u^\varepsilon_{\rm ap}\|_H$	$\sim$	$\varepsilon^{1/2}$	ε

Here $\Omega_{R'} = D(0, R')$ with $0 < R' < R$ so that $\Omega_R \setminus \overline{\Omega_{R'}}$ is a region excluding a fixed neighborhood of the small obstacle. We see that $u^\varepsilon_{\rm ap}$ provides a better approximation of u^ε than the incident field u_0.

3. Locking free discrete formulation

Since the simplified formulation is validated by these error estimates, we decide to solve the corresponding discrete formulation. In this discrete problem we consider ε as a function of the mesh step h i.e $\varepsilon = \varepsilon(h)$. We make two assumptions on the mesh:

H1 The mesh is uniformly regular with no local refinement,

H2 There exists κ such that $\varepsilon(h) < \kappa\, h$.

The second hypothesis means that the mesh is coarse with respect to the size of the small obstacle. Then we can think of solving the simplified formulation with

standard finite elements (with $V^h = \mathbb{P}_q$ or $\mathbb{Q}_q$), unfortunately we are able to prove that such a discrete formulation is numericaly locking in the sense that there exists κ such that $\inf_{v_h \in V^h} \|u^\varepsilon_{\rm ap} - v_h\|_{H^1(\Omega_R)} > \kappa/|\ln \varepsilon|$. We are thus led to consider an augmented approximation space,

$$V^h_{\mathbf{e}} = V^h \oplus \text{span}\{\varphi^\varepsilon_{\rm ad}\} \quad \text{with} \quad \varphi^\varepsilon_{\rm ad}(\mathbf{x}) = \chi_0(\mathbf{x})\, \frac{\mathbf{1}_{|\mathbf{x}|>\varepsilon}}{2\pi} \ln \frac{|\mathbf{x}|}{\varepsilon}\,.$$

Here χ_0 is a smooth cut-off function that depends only on $|\mathbf{x}|$ ($\chi_0(\mathbf{x}) = 1$ for $|\mathbf{x}|$ small and $\chi_0(\mathbf{x}) = 0$ for $|\mathbf{x}|$ large). The corresponding augmented discrete formulation that we solve is then

$$\left\{ \begin{array}{l} \text{Find } (u^{\varepsilon,h}_{\rm ap}, p^{\varepsilon,h}_{\rm ap}) \in V^h_{\mathbf{e}} \times P_0 \text{ such that} \\ \\ a(u^{\varepsilon,h}_{\rm ap}, v^h) + b^\varepsilon(p^{\varepsilon,h}_{\rm ap}, v^h) = \int_{\Gamma^R} (T_R u_0 + \frac{\partial u_0}{\partial n}) \overline{v}^h\,,\ \forall v^h \in V^h_{\mathbf{e}}, \\ \\ b^\varepsilon(q, u^{\varepsilon,h}_{\rm ap}) = 0\,,\ \forall q \in P_0. \end{array} \right.$$

With this modification of the standard finite element method, the approximation becomes non-locking and consistant. Indeed on the figure 1, we ploted the relative error evaluated in a region excluding a fixed neighborhood of the obstacle versus h/λ, for four values of ε, using order 3 finite elements. According to the indicated slopes, the rate of convergence is optimal in regions that exclude a neigborhood of the small obstacle. Such an observation is valid with higher order finite elements. Moreover the quality of the approximation is not deteriorated when ε goes to 0.

ε/λ	10^{-2}	10^{-3}	10^{-4}	10^{-5}
slope	3.8883	4.1136	4.2392	4.3103

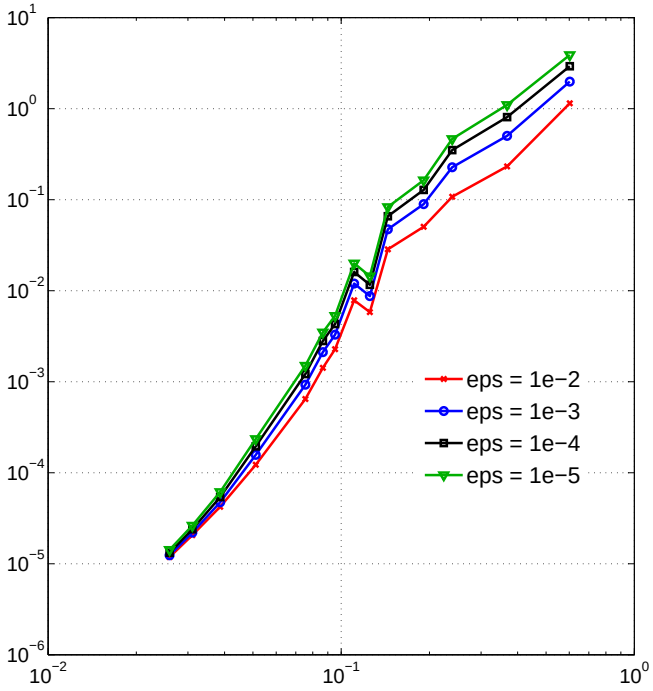


FIGURE 1. $\frac{\|u^\varepsilon_{\rm ap} - u^{\varepsilon,h}_{\rm ap}\|_{H^1}}{\|u^\varepsilon_{\rm ap}\|_{H^1}}$ versus h/λ

We are able to prove order 2 convergence rate (for sufficiently smooth data): $\|u^\varepsilon - u^{\varepsilon,h}_{\rm ap}\|_{H^1(\Omega_R\setminus\Omega_{R'})} < \kappa \ln 1/\varepsilon\ (\varepsilon + h^2)$, for $q \geqslant 2$ (q order of finite elements). How to prove optimal convergence remains for us an open question.

4. The Holland model

In [2], Collino and Millot proposed a continuous setting from which they can deduce the Holland model. On the basis of this continuous setting we can propose the following acoustic version of the Holland model on a finite element mesh,

$$\begin{cases} \text{Find } (u^{\varepsilon,h}_{\rm hol}, p^{\varepsilon,h}_{\rm hol}) \in V^h \times P_0 \text{ such that} \\ \\ a(u^{\varepsilon,h}_{\rm hol}, v^h) + b^\varepsilon(p^{\varepsilon,h}_{\rm hol}, v^h) = \int_{\Gamma^R} (T_R u_0 + \frac{\partial u_0}{\partial n})\overline{v}^h\ ,\ \forall v^h \in V^h_{\mathbf{e}}\,, \\ \\ b^\varepsilon(q, u^{\varepsilon,h}_{\rm hol}) - L^{\varepsilon,h}_{\rm hol}\, c(p^{\varepsilon,h}_{\rm hol}, q) = 0\,,\ \forall q \in P_0. \end{cases}$$

The parameter $L^{\varepsilon,h}_{\rm hol}$ is the lineic inductance; its value is not a priori known. The bilinear form c is defined by $c(p,q) = \overline{p}\, q$. We show how this discrete formulation

is related to our augmented discrete formulation. Consider $\mathcal{P}_h(\varphi^\varepsilon_{\rm ad}) \in V^h$ the unique element of the standard approximation space satisfying $a(\mathcal{P}_h(\varphi^\varepsilon_{\rm ad}), v^h) = a(\varphi^\varepsilon_{\rm ad}, v^h),\ \forall v^h \in V^h$ and define $\delta_h(\varphi^\varepsilon_{\rm ad}) = \varphi^\varepsilon_{\rm ad} - \mathcal{P}_h(\varphi^\varepsilon_{\rm ad})$. Then we can decompose the discrete solution using this new function, $u^{\varepsilon,h}_{\rm ap} = u^{\varepsilon,h}_{\rm reg} + p^{\varepsilon,h}_{\rm s}\,\delta_h(\varphi^\varepsilon_{\rm ad})$ with $u^{\varepsilon,h}_{\rm reg} \in V^h$. We also make a further assumption on the mesh,

H3 There exists $\kappa, \nu > 0$ such that $\kappa\, h^{1+\nu} < \varepsilon(h)$.

This hypothesis is reasonnable because ν can be chosen arbitrarily large. Then we prove that the regular part $u^{\varepsilon,h}_{\rm reg}$ is very close to the solution of the Holland problem. This result appears as a justification of the Holland model providing at the same time the correct value for the lineic inductance.

Theorem
Suppose that assumptions **H1**, **H2** *and* **H3** *are satisfied. Suppose also that we choose the value,*

$$L^{\varepsilon,h}_{\rm hol} = \frac{b^\varepsilon\big(\delta_h(\varphi^\varepsilon_{\rm ad})\big)^2}{a(\delta_h(\varphi^\varepsilon_{\rm ad}), \delta_h(\varphi^\varepsilon_{\rm ad}))}$$

for the lineic inductance. Then there exists $\kappa > 0$ *such that* $\|u^{\varepsilon,h}_{\rm reg} - u^{\varepsilon,h}_{\rm hol}\|_{H^1(\Omega_R)} \leqslant \kappa\, h^q$ *(q order of elements) .*

References

[1] R.Holland and L.Simpson, *Finite difference analysis of EMP coupling to thin struts and wires*, IEEE Trans. Electromagn. Compat. 1981.

[2] F.Collino and F.Millot, *Fils et méthodes d'éléments finis pour les équations de Maxwell. Le modèle de Holland revisité*, Reasearch Report 3472, INRIA (1998).

High-Order Numerical Methods for Maxwell's Equations on Unstructured Meshes

Gary Cohen

For a long time, Maxwell's equations were mainly solved in the time-harmonic domain. The evolution of radar techniques showed the limit of this formulation which can only treat monochromatic sources. Both engineers and researchers were then motivated to use equations in the time domain which can take into account large frequency sources in one resolution. The first and most popular approximation of Maxwell's equations in the time domain was provides by the Yee's scheme [25], commonly called FDTD (Finite Difference in the Time Domain) by enginees, which is basically a centered second order finite difference approximation of Maxwell's equation.

Although easy to implement, FDTD has some difficulties to treat complex geometries. In fact, the staircase approximation of curved boundary can produce spurious reflections which can substantially pollute the solution. On the other hand, finite element methods (FETD) have the major drawback of producing a

n-diagonal (n can grow up to several tens in 3D) mass matrix which must be inverted at each time-step, which is a serious handicap for FETD versus FDTD whose mass matrix is the identity matrix. This mass matrix does not present any difficulty to time harmonic problems, for which even the stiffness matrix must be inverted. For this reason, industry was reluctant to use FETD for a long time and FDTD remains the reference for Maxwell's equations in the time domain for 40 years!

The mass lumping technique is an efficient alternative to mass matrix inversion. However, this technique was well known for lower order continuous (or H^1) elements but not obvious for higher-order approximations. A first step towards a general mass lumping technique was made by Hennart et al. [17, 18] and independently by Young [26] which proposed to use Gauss-Lobatto quadrature formulas to get mass lumping for continuous quadrilateral or hexahedral finite elements. Besides mass lumping, these formulas ensure to keep the order induced by the finite element approximation [3]. First introduced for ODE or parabolic problems, this technique was extended to the wave equation by Cohen et al. [10] and renamed spectral element methods [20]. The non trivial extension of this technique to triangular and tetrahedral elements was later realized by Cohen et al [9] for triangles up to third order and Mulder et al. [21] for higher-order triangles and tetrahedra.

The problem of the mass matrix inversion was solved for the wave equation and remained a challenging problem for Maxwell's equations. A first try was done by Haugazeau et al. [16] but this approach remains restricted to first-order approximation. A second and natural step was to extend spectral element techniques to edge (or $H(curl)$) elements. This was done by Cohen et al. for orthogonal meshes for the first family [22, 11] and for any mesh for the second family of edge elements of any order [23, 12]. The extension to triangular and tetrahedral meshes was realized by Elmkies et al. [14, 15] but lead to efficient approximations up to the second-order elements.

Due to the storage of the stiffness matrix, even by using mass lumping techniques, FETD remained much more expensive than FDTD in terms of storage and, to a lesser extent, in computational time. This ultimate problem was solved by using a mixed $H(curl) - L^2$ formulation of Maxwell's equations based on $H(curl)$-conform definition of the *curl* operator in both spaces [12]. This technique provides a local definition of the stiffness matrices which induces a substantial gain of storage. It was later extended to acoustics [6] and linear elastodynamics [7]. A detailed presentation of all these techniques can be found in [4].

Unfortunately, although H^1 spectral elements and H^1 and $H(curl)$ triangular and tetrahedral elements behave quite well for any mesh, $H(curl)$ spectral elements present important parasitic waves for very distorted meshes, which are often used in industrial problems. For this reason, discontinuous Galerkin methods appeared as an efficient alternative for Maxwell's equations. First introduced by Hesthaven [19] for tetrahedra, this approach was adapted by Cohen et al. [8, 24] to the spectral element point of view, which provided a low-storage as well as fast method

to solve Maxwell's equations. This approach seemed to deal better with parasitic waves but eigenvalues considerations showed that such waves were however present in this method. All these remarks motivated us to discuss the numerical dissipation terms which can attenuate parasitic waves, as shown in [1, 2, 13]. This discussion is the new part of our survey.

An extended presentation of this work can be found in [5].

References

[1] D. Boffi, M. Costabel, M. Dauge, L. Demkowicz, *Discrete compactness for the hp version of rectangular edge finite elements*, SIAM Journal on Numerical Analysis, Vol. 44, No. 3, pp. 979-1004, 2006.

[2] A. Buffa, I. Perugia, *Discontinuous Galerkin approximation of the Maxwell eigenproblem*, SIAM Numer. Anal., to appear.

[3] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, exercises 4.1.4 and following, 1987.

[4] G. Cohen, *Higher Order Numerical Methods for Transient Wave Equations* , Springer, 2001.

[5] G. Cohen, M. Duruflé, *Non-spurious spectral like methods for Maxwell's equations*, J. Comp. Math. to appear, 2007.

[6] G. Cohen, S. Fauqueux, *Mixed finite elements with mass-lumping for the transient wave equation*, J. Comp. Acous. **8** (1), pp. 171-188, 2000.

[7] G. Cohen, S. Fauqueux, *Mixed spectral finite elements for the linear elasticity system in unbounded domains*, SIAM J. Sci. Comput., **26** (3), pp. 864-884, 2005.

[8] G. Cohen, X. Ferrieres, S. Pernet, *A Spatial High-Order Hexahedral Discontinuous Galerkin Methods to Solve Maxwell's Equations in Time Domain*, J. Comp. Phys., **217**, pp. 340-363 , 2006.

[9] G. Cohen, P. Joly, J. E. Roberts, N. Tordjman, *Higher order triangular finite elements with mass lumping for the wave equation.*, SIAM J. on Numer. Anal., **38** (6), pp. 2047-2078, 2001.

[10] G. Cohen, P. Joly, N. Tordjman, *Higher-order finite elements with mass lumping for the 1-D wave equation*, Finite elements in analysis and design, **17** (3-4), pp. 329-336, 1994.

[11] G. Cohen, P. Monk, *Gauss point mass lumping schemes for Maxwell's equations*, NMPDE Journal **14** (1), pp. 63-88, 1998.

[12] G. Cohen, P. Monk, *Mur-Nédélec finite element schemes for Maxwell's equations*, Comp. Meth. in Appl. Mech. Eng., **169** (3-4), pp. 197-217, 1999.

[13] M. Costabel, M. Dauge *Computation of resonance frequencies for Maxwell equations in non smooth domains*, Lecture notes in computational science and engineering, Vol. 31, Springer, 2003

[14] A. Elmkies, P. Joly, *Éléments finis d'arête et condensation de masse pour les équations de Maxwell : le cas 2D*, C.R.A.S., Math., **324**, série I, pp. 1287-1293, 1997.

[15] A. Elmkies, P. Joly, *Éléments finis d'arête et condensation de masse pour les équations de Maxwell : le cas de la dimension 3*, C.R.A.S., Math., **325**, série I, pp. 1217-1222, 1997.

[16] Y. Haugazeau, P. Lacoste,, *Condensation de la matrice masse pour les éléments finis mixtes de H(rot)*, C. R. Acad. Sci. Paris Sér. I Math., **316** (5), pp. 509–512, 1993.

[17] J.-P. Hennart, E. Sainz, M. Villegas, *On the efficient use of the finite element method in static neutron diffusion calculations*, Computational Methods in Nuclear Engineering, **1**, pp. 3–87, 1979.

[18] J.-P. Hennart, *Topics in finite element discretization of parabolic evolution problems*, Lecture Notes in Mathematics, Springer-Verlag, **909**, pp. 185–199, 1982.

[19] J. S. Hesthaven, T. Warburton, *High-order nodal methods on unstructured grids. I. Time-domain solution of Maxwell's equations*, J. Comput. Phys. **181**, pp. 1–34, 2002.

[20] Y. Maday, A. T. Patera, *Spectral element methods for the imcompressible Navier-Stokes equations*, State of the Art Survey in Computational Mechanics, ed. A. K. Noor, pp. 71-143, 1989.
[21] W. A. Mulder, *Higher-order mass-lumped finite elements for the wave equation*, J. Comput. Acoustics, **9** (2), pp. 671-680, 2001.
[22] J.-C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer.Math., **35** (3), pp. 315-341, 1980.
[23] J.-C. Nédélec, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer.Math., **50** (1), pp. 57-81, 1986.
[24] S. Pernet, *Étude de méthodes d'ordre élevé pour résoudre les équations de Maxwell dans le domaine temporel. Application à la détection et à la compatibilité électromagnétique*, PhD thesis, U. de Paris-Dauphine, 2004.
[25] K. Yee, *Numerical solutions of initial boundary value problems involving Maxwell's equations in isotropic media*, IEEE Trans. on Antennas and Propagation AP-16, pp. 302-307, 1966.
[26] L. C. Young, *An efficient finite element method for reservoir simulation*, Proc. of the 53rd Annual Fall Technical Conference and Exhibition of the Society of Petroleum Engineers of AIME, Houston, Texas Oct. 1-3, 1978.

High-frequency Plasma Wave Computations

Pierre Degond
(joint work with J.-G. Liu, M.H. Vignal)

In [8] and [10] (see also [6]), a new numerical discretization of the Euler-Poisson system has been proposed. The Euler-Poisson system under consideration consists of the isentropic Euler equations for the particle and momentum densities coupled with the Poisson equation through a source term modeling the electrostatic force. In dimensionless units, the coupling constant can be expressed in terms of a parameter ε which represents the scaled Debye length. When ε is small, the coupling is strong. In this situation, the particle density is constrained to be close to the background density of the oppositely charged particle, which we suppose to be uniform and equal to 1 in scaled units. The velocity then evolves according to the incompressible Euler equation. The limit $\varepsilon \to 0$ is called the quasineutral limit, since the charge density almost vanishes identically. When two or more particle species are considered, the limit $\varepsilon \to 0$ leads to a more complex model usually referred to in the physics literature as a quasineutral model.

The scheme which has been proposed in [8] and [6] is 'Asymptotic Preserving' in the quasineutral limit, which means that it becomes consistent with the limit model when $\varepsilon \to 0$. The goal of [10] is to analyze the stability properties of this scheme and to show that its stability domain is independent of ε. This stability analysis is performed on the Fourier transformed (with respect to the space variable) linearized system. We show that the stability property is more robust when a space-decentered scheme is used (which brings in some numerical dissipation) rather than with a space-centered one. The linearization is first performed about a zero mean velocity, and then about a non-zero mean velocity. At the various stages of the analysis, our scheme is compared with more classical schemes and its improved stability property is outlined. The analysis of a fully discrete (in space

and time) version of the scheme is also given. Finally, some considerations about a model nonlinear problem, the Burgers-Poisson problem, are also given.

The Euler-Poisson model is one of the most widely used fluid models in plasma and semiconductor physics (see e.g. [20], [1] for plasmas, and [23] for semiconductors). It can be derived from a moment expansion of kinetic models such as the Vlasov or Boltzmann Poisson equations supplemented with a convenient closure assumption (see references above).

There are two important physical length and time scales associated with this model (see e.g. [1], [20]): the Debye length and the electron plasma period. These two scales are related one to each other by the thermal speed which is an order one quantity. We are interested in the quasineutral regime where both parameters can be very small compared with typical macroscopic length and time scales. A standard explicit scheme must resolve these micro-scale phenomena in order to remain stable. The satisfaction of these constraints requires huge computational resources which make the use of explicit methods almost impracticable.

The search for schemes free of such constraints has been the subject of a vast literature. A number of works deal with Particle-in-Cell (PIC) methods which are specific discretizations of the kinetic model, the Vlasov-Poisson system (see e.g. the direct implicit method [3], [21] and the implicit moment method [24], [25]). These methods have proved extremely efficient in number of situations but there are still regions where short time steps must be used (see e.g. the recent discussion in [18]). So the improvement of classical time-stepping strategies, if possible, would offer attractive perspectives. More references can be found in [8]. We also mention that the strategy developed in the present paper has been applied to PIC codes in [9]

The present work deals with fluid models. For fluid models the literature is comparatively less abundant. We can refer to the pioneering work [16], and more recently to [2], [4], [26], [27]. When the fluid models are drift-diffusion models, implicit strategies have been proposed in [28], [29], [22].

To cancel the fast scales associated with electron plasma frequency, quasineutral models have been very frequently considered [15] (see also references in [8]). Recently, two-fluid quasineutral models have been studied [5], [7], [11], [12], [13], [14].

However, in situations where quasineutral and non quasineutral regions coexist, a specific treatment is needed to connect the quasineutral model with a non quasineutral model across the interface. Such situations arise in sheath problems (see e.g. [17], [19], [12] ; other references can be found in [8]). In such problems, one has often to deal with a dynamic interface the tracking of which gives rise to a complex numerical problem. Additionally, the interface dynamics is not a priori known, and must either be derived from an asymptotic analysis like in [12], [13], or [19], or must be inferred from physical considerations. In both cases, great care is required to ensure that the proper dynamics is implemented. Another problem is related to the fact that the quasineutral to non-quasineutral transition may not

be a sharp transition, but rather a fairly diffuse one, and its approximation into a sharp interface may actually lead to some unphysical behavior.

For these reasons, it is highly desirable to develop numerical methods which automatically shift from a quasineutral to a non-quasineutral model across the transition region when such a transition is encountered. The scheme proposed in [8] serves this purpose. Additionnally, it has been shown that the numerical cost of this scheme is the same as the standard strategy (we refer the reader to [8] for more detail). The work [10] provides elements towards a rigorous analysis of its stability properties.

References

[1] F. F. Chen, Introduction to plasma physics and controlled fusion, Volume 1, (Plenum Press 1974).

[2] H-H. Choe, N. S. Yoon, S. S. Kim and D-I. Choi, A New Unconditionally Stable Algorithm for Steady-State Fluid Simulation of High Density Plasma Discharge, J. Comput. Phys. 170, 550 (2001).

[3] B. I. Cohen, A. B. Langdon and A. Friedman, Implicit time integration for plasma simulation, J. Comput. Phys. 46,15 (1982).

[4] Ph. Colella, M. R. Dorr, D. D. Wake, A Conservative Finite Difference Method for the Numerical Solution of Plasma Fluid Equations, J. Comput. Phys. 149, 168 (1999).

[5] P. Crispel, P. Degond, C. Parzani and M-H. Vignal, Trois formulations d'un modèle de plasma quasi-neutre avec courant non-nul, C. R. Acad. Sci. Paris 338, 327 (2004).

[6] P. Crispel, P. Degond and M-H. Vignal, An asymptotically stable discretization for the Euler-Poisson system in the quasineutral limit, C. R. Acad. Sci. Paris, Ser. I, 341, 323 (2005).

[7] P. Crispel, P. Degond and M-H. Vignal,Quasi-neutral fluid models for current carrying plasmas, J. Comp. Phys, 205, 408 (2005).

[8] P. Crispel, P. Degond, M-H. Vignal, An asymptotically preserving scheme for the two-fluid Euler-Poisson model in the quasi-neutral limit, J. Comp. Phys. (2007).

[9] P. Degond, F. Deluzet, L. Navoret, An asymptotically stable Particle-in-Cell (PIC) scheme for collisionless plasma simulations near quasineutrality, C. R. Acad. Sci. Paris.

[10] P. Degond, J-G. Liu, M-H. Vignal, Analysis of an asymptotic preserving scheme for the Euler-Poisson system in the quasineutral limit, submitted.

[11] P. Degond, C. Parzani and M-H. Vignal, Un modèle d'expansion de plasma dans le vide, C. R. Acad. Sci. Paris Ser. I, 335, 399 (2002).

[12] P. Degond, C. Parzani and M-H. Vignal, A one-dimensional model of plasma expansion, Mathematical and Computer Modelling, 38, 1093 (2003).

[13] P. Degond, C. Parzani and M-H. Vignal, Plasma expansion in vacuum: modeling the breakdown of quasineutrality, SIAM Multiscale Modeling and Simulation 2, 158 (2003).

[14] P. Degond, C. Parzani and M-H. Vignal, On plasma expansion in vacuum, in: Free Boundary Problems: Theory and Applications, P. Colli, C. Verdi and A. Visintin, eds., International Series of Numerical Mathematics, vol 147 (Birkhäuser Verlag, Basel, 2004) p. 103.

[15] G. DiPeso, D. W. Hewett and G. F. Simonson, Extension of the Streamlined Darwin Model to Quasineutral Plasmas, J. Comp. Phys. 111, 237 (1994).

[16] S. Fabre, Stability analysis of the Euler-poisson equations, J. Comp. Phys. 101, 445 (1992).

[17] R. N. Franklin and J. R. Ockendon, Asymptotic matching of plasma and sheath in an active low pressure discharge, Journal of plasma physics, 4 (1970), pp. 3521–3528.

[18] Alex Friedman, Implicit multiscale PIC and related topics, Workshop on Multiscale Processes in Fusion Plasmas, IPAM, UCLA, January 10-14, 2005, http://www.ipam.ucla.edu/publications/fus2005/fus2005_5425.pdf

[19] S. Y. Ha, M. Slemrod, Global existence of plasma ion sheaths and their dynamics, Comm. Math. Phys. 238, 149 (2003).
[20] N. A. Krall and A. W. Trivelpiece, Principles of plasma physics (San Francisco Press, 1986).
[21] A. B. Langdon, B. I. Cohen and A. Friedman, Direct implicit large time-step particle simulation of plasmas, J. Comp. Phys. 51, 107 (1983).
[22] G. Lapenta, F. Iinoya and J. U. Brackbill, Particle-in-cell simulations of glow discharges in complex geometries, IEEE tran. Plasma Sci. 23, 769 (1995).
[23] P. A. Markowich, C. Ringhofer, C. Schmeiser, Semiconductor Equations, Springer, Wien, 1990.
[24] R. J. Mason, Implicit moment particle simulation of plasmas, J. Comp. Phys. 41, 233 (1981).
[25] R. J. Mason, Implicit moment PIC-hybrid simulation of collisional plasmas, J. Comp. Phys. 51,484 (1983).
[26] R. Schneider, C-D. Munz, The approximation of two-fluid plasma flow with explicit upwind schemes, Int. J. Numer. Model. 8, 399 (2005).
[27] U. Shumlak, and J. Loverich, Approximate Riemann solver for the two-fluid plasma model, J. Comp. Phys. 187, 620 (2003).
[28] P. L. G. Ventzek, T. J. Sommerer, R. J. Hoekstra and M. J. Kushner, Two-dimensional hybrid model of inductively coupled plasma sources for etching, Appl. Phys. Lett. 63, 605 (1993).
[29] P. L. G. Ventzek, R. J. Hoekstra and M. J. Kushner, Two-dimensional modeling of high plasma density inductively coupled sources for materials processing, J. Vac. Sci. Technol. B 12, 461 (1994).

Computation of harmonic wave propagation in infinite periodic media

Sonia Fliss
(joint work with Patrick Joly, Jing R. Li)

Periodic media play a major role in applications, in particular in optics for micro and nano-technology. From the point of view of applications, one of the main interesting features is the possibility offered by such media of selecting ranges of frequencies for which waves can or can not propagate. Mathematically, this property is linked to the gap structure of the spectrum of the underlying differential operator appearing in the model. For a complete, mathematically oriented presentation, we refer the reader to [7, 8]. Of course, there is a need for efficient numerical methods for computing the propagation of waves inside such structures. In real applications, the media are not really periodic but differ from periodic media only in bounded regions (small with respect to the total size of the propagation domain). In such a case, a natural idea is to reduce the pure numerical computations to these regions and to try to take advantage of the periodic structure of the problem outside: this is particularly of interest when the periodic regions contain a large number of periodicity cells.

My presentation was concerning a contribution of such methods in three particular situations, of increasing complexity. The solution of each case relies on the solution for the previous one.

1. Locally perturbed periodic waveguide

The first situation we have studied concerns the computation of the propagation of acoustic waves in a locally perturbed infinite periodic waveguide, namely an infinite structure which is periodic in one priviliged direction (the propagation direction) and bounded in the other transverse variables. We investigate the question of finding artificial (but exact) boundary conditions to reduce the numerical computation to a neighborhood of this perturbation.

Our goal was the generalization of the DtN approach to periodic waveguides, which is complicated by the fact that separation variables techniques (see [9, 4]) usually used in the case of homogeneous waveguides can no longer be used. However the notion of guided modes has a natural extension: the notion of Floquet modes. By revisiting the Floquet-Bloch theory [6], we propose a method for constructing a DtN operator, denoted Λ^W by solving local problems on a single periodicity cell. This is closely connected to operator-valued Ricatti equations (here, of stationary nature), a topic which is already present in many problems concerning artificial boundary conditions (see for instance [5]). It appears also that our method is similar to the transfer matrix approach developed for ordinary equations with periodic coefficients ([2]). However, except in a 1d-case, this theory cannot be applied directly to our problem due to the fact that the Cauchy problem for the Helmholtz equation is ill-posed in higher dimensions. For more details see [3].

2. Interface 2D Problem

The second situation presented concerns the transmission (and reflection) of an incident wave from a homogeneous half-space to a periodic half-space. This problem generalizes the classical problem of the diffraction by periodic gratings which is more widely treated in the literature (for example, see [1, 10]). We use the Floquet-Bloch Transform in the variable y parallel to the interface defined by:

$$\begin{aligned} \mathcal{F} \ : L^2(\mathbb{R}) &\rightarrow L^2(R_0 = [0,1] \times [0, 2\pi]) \\ \psi(y) &\rightarrow \mathcal{F}\psi(y,k) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \psi(y+n) e^{-\imath n k} \ , \end{aligned}$$

which basically transforms a function defined on the real line to a function defined in a rectangle. We can reduce ourselves to find exact boundary conditions for a family (depending on the continuous parameter, $k \in [0, 2\pi]$) of periodic half-waveguides problems with quasi-periodic conditions. As a consequence the DtN operator on the interface Σ for periodic half space problem, Λ^H, is described in terms of a family of "waveguide" DtN operators, $\Lambda^W(k)$, that we can determined using the method of the section 1:

$$\Lambda^H = \int^{\oplus} \Lambda^W(k) \, dk. \tag{1}$$

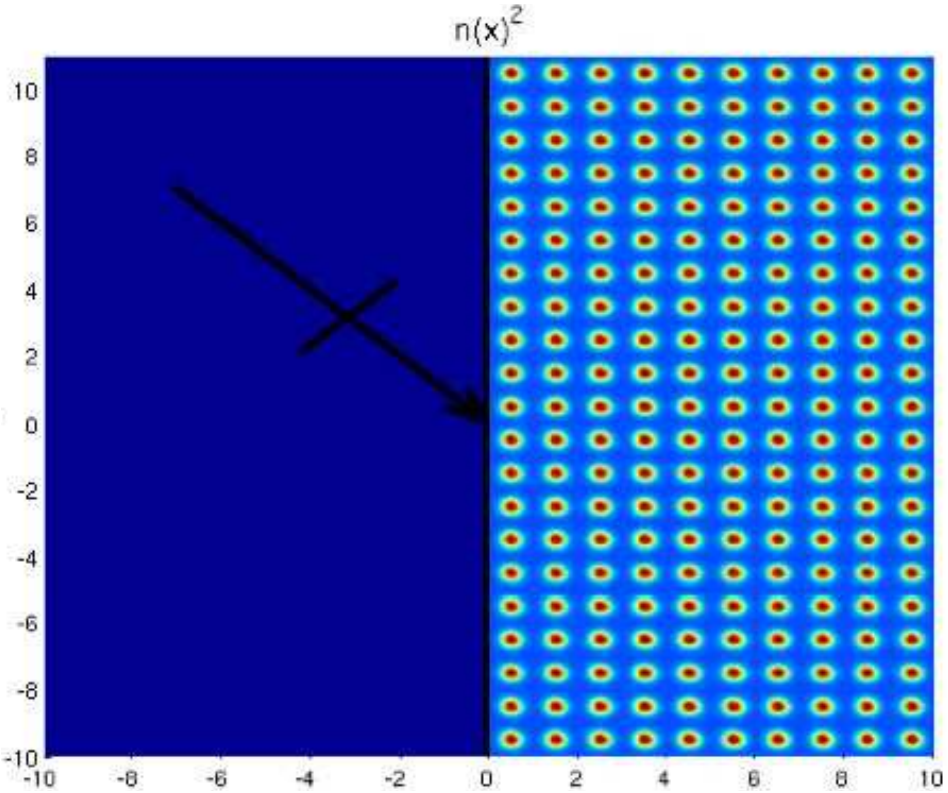


FIGURE 1. Isovalues of the refraction index $n(x)$ and direction of the incident plane wave ($\theta = \pi/4$)

We show in the following figures a computation that corresponds to the transmission-reflection of an incident plane wave in the homogeneous mediium, with angle of incidence $\pi/4$. We describe the propagation medium in the figure 1 where we represent the isovalues of the refraction index which varies from 1 to 2.

First, we choose a frequency so that the wavelength of the incident wave is about six times larger than the size of the periodicity cell. In this case, we are close to the application conditions of the homogenization theory (see figure 2). Next, we choose a five times larger frequency so that the wavelength is comparable to the size of the periodicity cell. The structure of the solution is more complicated in this case (see figure 3).

3. Locally perturbed periodic 2D plane

The third and last situation is the 2D-plane case: we are interested in propagation media which are a local perturbation of an infinite structure, periodic in two directions. Physically, the "defect" may be a geometrical defect, a perturbation of

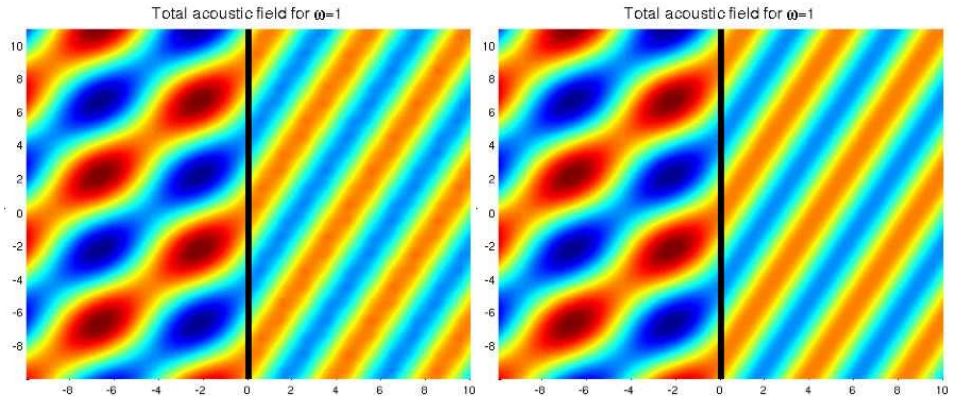


FIGURE 2. Low frequency case: comparison of the numerical solution of the initial problem to the one of the homogenized problem.

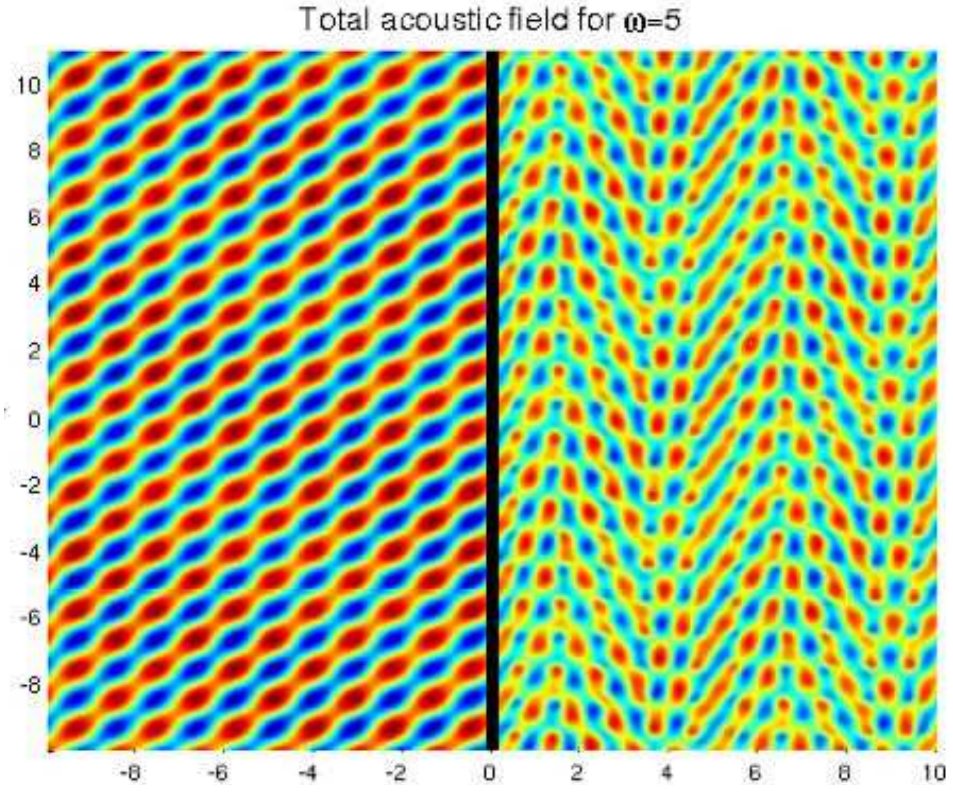


FIGURE 3. General Case: the structure of the solution is more complex.

a periodic refraction index or a local source. For simplicity, we will restrict ourselves to the case where the periodicity cell is a square and presents at least two symmetries, which is often the case in the applications. We want to restrict the computations to a square which contains the defect and whose side is a multiple of the size of the periodicity cell. The DtN operator we look for is defined on the boundary Γ of the square (which is an original situation, even in the constant coefficient case).
The key point of the method relies on the factorization of the DtN operator as:

$$\Lambda = \Lambda^H \circ D. \tag{2}$$

The first operator, Λ^H is the DtN operator corresponding to a periodic half space problem, that we can find using the method of section 2. The second operator, D, is an operator that maps functions defined on Γ, into functions defined on Σ, where Σ is an infinite line containing on of the sides of Γ. It is characterized as the unique solution of the operator valued linear equation :

$$D^H \circ D - D = 0. \tag{3}$$

In ()3),D^H is an operator acting on functions defined on Σ and can be computed using the techniques of the halfspace problem (see section 2). Moreover D is in the affine space of linear operators from $L^2(\Gamma)$ in $L^2(\Sigma)$ that leave invariant the restriction to the segment of Σ that coincides with one of the sides of Γ. The numerical resolution of this equation is in progress.

References

[1] T. Abboud, *Electromagnetic waves in periodic waves*, Second International Conference on Mathematical and Numerical Aspects of Wave Propagation(Newark, DE,1993), SIAM, Philadelphia,PA (1993), 1–9.

[2] A. Figotin and V. Gorentsveig, *Localized electromagnetic waves in periodic-layered dielectric medium with a defect*, Phy. Rev.B., vol. **58(1)**, (1998), 180–188.

[3] P. Joly, J.-R. Li and S. Fliss, *Exact boundary conditions for periodic waveguides containing a local perturbation*, Communications in Computational Physics, vol. **1(6)**, (2006), 945–973.

[4] J.B. Keller and D. Givoli, *Exact Non-reflecting Boundary Conditions*, J. Comp. Phys., vol. **82**, (1989), 172–192.

[5] J. Henry and A.M. Ramos, *Factorization of second-order elliptic boundary value problems by dynamic programming*, Nonlinear anal., vol. **59(5)**, (2004), 629–647.

[6] P. Kuchment, *Floquet theory for partial differential equations*, Operator theory: Advances and applications, vol. **60**. Birkhäuser Verlag, Basel (1993).

[7] P. Kuchment, *The mathematics of photonics crystals (chapter 7)*, Mathematical Modelling in optical science, vol. **22** of Frontiers in applied Mathematics. SIAM, Philadelphia (2001).

[8] P. Kuchment, *On some spectral problems of mathematical physics*, Partial differential equations and inverse problems, volume 362 of Contemp. Math., Amer.Math.Soc., Providence, RI (2004), 241–276.

[9] M. Lenoir and A. Tounsi, *The localized finite element method and its application to the two dimensional sea-keeping problem*, SIAM J.Numer.Anal., vol. **25(4)**, (1988), 729–752.

[10] T. Arens, S.N. Chandler-Wilde and J.A. DeSanto, *On integral equation and least squares methods for scattering by diffraction gratings*, Communications in Computational Physics,vol.**1**, (2006), 1010–1042.

Robust boundary integral methods in high frequency scattering

Ivan G. Graham

This report gives a brief survey of robust numerical methods for computing the scattering of a high-frequency incident plane wave $u_I(\mathbf{x}) = \exp(\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{d}})$ by a bounded obstacle $\Omega \subset \mathbb{R}^m$ with boundary Γ (where $\widehat{\mathbf{d}}$ is a unit vector and $m = 2$ or 3). The scattered wave u_S satisfies the Helmholtz equation:

$$\Delta u_S + k^2 u_S = 0 \quad \text{in} \quad \Omega' = \mathbb{R}^d \backslash \Omega \ ,$$

together with the Sommerfeld radiation condition. The total wave $u = u_I + u_S$ is required to satisfy a suitable boundary condition on Γ, for example $u = 0$ (the "sound-soft" case). We will be concerned with methods based on boundary integral formulations. In the sound-soft case, the complementary Cauchy data is the normal derivative of the total wave $v = \partial_\nu u$ which is found by solving the "combined potential equation":

$$\mathcal{R}_k v := \left(\frac{1}{2}I + \mathcal{D}_k' - i\eta\mathcal{S}_k\right) v = \partial_\nu u_I - i\eta u_I =: f_k \ , \tag{1}$$

where $\mathcal{S}_k$ is the single layer potential operator, $\mathcal{D}_k'$ is its normal derivative (i.e. the adjoint of the double layer potential). We shall follow normal practice and choose the *coupling parameter* as $\eta = k$.

There has been considerable recent interest in devising numerical methods for this problem which work well across a range of frequencies, including the high-frequency case $k \gg 1$. In this case due to the oscillatory solution, classical piecewise polynomial boundary element methods require at least $\mathcal{O}(k^{d-1})$ degrees of freedom, and have solution complexity of at least this order (for example using fast multipole or panel-clustering algorithms). Faster methods (motivated by asymptotic analysis), write the solution in terms of explicit oscillatory functions, and approximate only the (hopefully less-oscillatory) remainder.

Although high-frequency asymptotics for wave scattering has been well-established since the 1960's (e.g. [11, 5, 17, 10, 18]), they were not seriously used for numerical purposes until several papers of Abboud et. al. in the mid-90's (e.g. [1]). In the context of (1) the method of [1] amounts to writing

$$v(\mathbf{x}) = k\exp(ik\mathbf{x}.\widehat{\mathbf{d}})V(\mathbf{x}) \ , \tag{2}$$

which is the classical "Geometric Optics" approximation and V is known to be well behaved, at least away from shadow boundaries, when Γ is smooth and convex. A formal argument in [1] suggests that the approximation of V by standard piecewise polynomial boundary spaces of order m yields an approximation error of $\mathcal{O}(hk^{1/3})^m$, and (assuming k independent stability), an improved solution complexity (via the Galerkin method) of $\mathcal{O}(k^{(d-1)/3})$. Giladi and Keller [12] proposed a collocation-based method based on an extension of (2) which also took into account creeping waves arising from diffraction at shadow boundaries.

While both [1, 12] indicate the importance of efficient quadratures for the oscillatory integrals which arise in implementations, Bruno et. al. (e.g. [2, 3]) puts

quadrature at the forefront through devising an extension of the classical Nyström scheme for (1) in the high-frequency case. Introducing the ansatz (2) into (1) yields an integral equation for V, the integral terms in which are approximated by using a partition of unity to localise around singular and stationary points and then applying the trapezoidal rule on the component parts. Since the function V is not so well-behaved near shadow boundaries, coordinate stretching is employed there before discretisation. The resulting matrix-free discretisation was solved iteratively [2, 3] and errors independent of k were observed. The method described extends to the 3D smooth convex case provided the stationary points are identified.

Two recent developments have started the rigorous analysis of methods such as these. In [8] we have analysed the Galerkin solution of the integral equation for V which is obtained by substituting (2) into (1) in the case of smooth convex Γ in 2D. Algebraic polynomial approximations of V are employed in certain carefully chosen illuminated, shadow and shadow boundary zones of Γ. We undertake a complete error analysis of this type of approximation and prove its stability and superalgebraic convergence for fixed k. Most importantly, by a careful analysis of the k dependence of the constants appearing in the error analysis, we also prove that, to obtain bounded discretisation error as $k \to \infty$, the polynomial degrees need to grow only as $\mathcal{O}(k^{1/9+\epsilon})$, for any $\epsilon > 0$. Under an assumption on the coercivity of the integral operator $\mathcal{R}_k$ (which is proved by Fourier analysis to be satisfied by circular scatterers), we prove that the actual Galerkin error enjoys the same property. In experiments the error appears to be almost independent of k.

While [8] concerns smooth scatterers (and requires delicate estimates near shadow boundaries), parallel work by Chandler-Wilde and Langdon (e.g. [7]) considered the case of scattering from convex polygons. Here, diffraction at corner points, rather than at shadow boundaries, is the key feature. In [7] a novel direct analysis of the boundary integral equation representation obtained explicit decomposition of the solution as a sum of the geometric optics approximation and additional diffraction terms containing explicit oscillations and unknown slowly varying terms. The latter are approximated by piecewise polynomials and the discretisation error is shown to remain bounded provided the number of degrees of freedom grows logarithmically in k. The results in [7] are inspired by earlier results of the same group (e.g. [6]) which proposed an algorithm for scattering by an impedance plane and proved rigorously that uniformly accurate solutions are obtained for $k \to \infty$ with k-independent cost. This was the first such theoretical result for any scattering algorithm.

For other recent work on the important related field of oscillatory quadrature, we refer, for example, to [15, 14] and the references therein. We also mention that considerable progress has been made in the challenging area of multiple scattering and refer to [4, 9] for further details. Finally we mention that for scattering problem's with more general data than plane waves, one may use standard boundary element spaces enriched with plane waves in a range of directions on each element e.g. [13]. These are more efficient than standard piecewise polynomials in the

high-frequency case, in the sense that while the complexity remains empirically at $\mathcal{O}(k^{d-1})$, a smaller asymptotic constant is observed compared with classical methods.

References

[1] Abboud, T., Nédélec, J.-C., Zhou, B.: Improvement of the integral equation method for high-frequency problems, in Proceedings of 3rd International Conference on Mathematical Aspect's of Wave Propagation Problems, SIAM , p. 178 - 187 (1995).

[2] Bruno, O.P., Geuzaine, C. A., Monro, J.A., Reitich, F.: Prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency: the convex case. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **362**, 629–645 (2004).

[3] Bruno, O.P. and Geuzaine, C. A.: An $\mathcal{O}(1)$ integration scheme for three-dimensional surface scattering problems, Journal of Computational and Applied Mathematics, Published online, July 2006.

[4] Bruno, O.P., Geuzaine, C.A., Reitich, F.: On the $\mathcal{O}(1)$ solution of multiple-scattering problems. IEEE Trans. Magn. **41**, 1488-1491 (2005).

[5] Buslaev, V. S.: Short-wave asymptotic behaviour in the problem of diffraction by smooth convex contours (in Russian). Trudy Mat. Inst. Steklov. **73**, 14–117 (1964). Abbreviated English translation: On the shortwave asymptotic limit in the problem of diffraction by convex bodies. Soviet Physics Doklady **7**, 685–687 (1963)

[6] Chandler-Wilde, S. N., Langdon, S. and Ritter, L. : A high-wave number boundary-element method for an acoustic scattering problem, Phil. Trans. R. Soc. Lond. A, 362 (2004), pp. 647–671.

[7] Chandler-Wilde, S. N. and Langdon, S. : A Galerkin boundary element method for high frequency scattering by convex polygons, University of Reading preprint, 2006, to appear in SIAM J Numer Anal.

[8] Dominguez, V. , Graham, I.G. and Smyshlyaev, V.P.: A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering, Numer. Math., to appear.

[9] Ecevit, F., Reitich, F., Analysis of multiple scattering iterations for high-frequency scattering problems. I: The two dimensional case, report 137, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig (2006).

[10] Filippov, V.B.: Rigorous justification of the shortwave asymptotic theory of diffraction in the shadow zone. J. Sov. Math. **6**, 577-626 (1976)

[11] Fock, V.A.: Electromagnetic Diffraction and Propagation Problems. Pergamon Press, New York (1965)

[12] Giladi, E., Keller, J.B.: An asymptotically derived boundary element method for the Helmholtz equation. In 20th Annual Review of Progress in Applied Computational Electromagnetics, Syracuse, New York (2004)

[13] Honnor, M.E., Trevelyan, J. and Huybrechts, D.: Numerical evaluation of 2D partition boundary integrals for Helmholtz problems, Preprint, 2006.

[14] Huybrechts, D. and Vandewalle, S.: A sparse discretisation for integral equation formulations of high-frequency scattering problems, Report TW447, K.U. Leuven (2006).

[15] Iserles, A. and Nørsett, S.P.: On quadrature methods for highly oscillatory integrals and their implementation, BIT 44 (2004), 755-772.

[16] Melrose, R.B., Taylor, M.E.: Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. Adv. in Math. **55**, 242-315 (1985)

[17] Morawetz, C.S., Ludwig, D.: An inequality for the reduced wave equation and the justification of geometrical optics. Comm. Pure Appl. Math. **21**, 187-203 (1968)

[18] Ursell, F.: Creeping modes in a shadow. Proc. Camb. Phil. Soc. **68**, 171–191 (1968)

Energy Conserving Explicit Local Time Stepping For Second-order Wave Equations

Marcus J. Grote
(joint work with Julien Diaz)

Adaptivity and mesh refinement are certainly key for the efficient numerical solution of partial differential equations. However, locally refined meshes impose severe stability constraints on explicit time-stepping schemes, where the maximal time-step allowed by a CFL condition is dictated by the smallest elements in the mesh. When mesh refinement is restricted to a small region, the use of implicit methods, or a very small time step in the entire computational domain, are very high a price to pay. To overcome that stability restriction, we propose local time-stepping schemes, which allow arbitrarily small time steps where small elements in the mesh are located. When combined with a symmetric finite element discretization in space with an essentially diagonal mass matrix, the resulting fully discrete scheme is explicit and exactly conserves a discrete energy. Starting from the standard second order "leap-frog" scheme, time integrators of arbitrary order of convergence are derived. Numerical experiments illustrate the efficiency of these methods and validate the theory.

We consider the scalar wave equation

$$u_{tt} - \nabla \cdot \left(c^2 \, \nabla u \right) = f \qquad \text{in } (0,T) \times \Omega, \tag{1}$$

with appropriate initial and boundary conditions. Here Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. The source term f lies in $L^2(0,T;L^2(\Omega))$ and we assume that the speed of propagation, $c(x)$, is piecewise smooth, bounded, and strictly positive.

Starting from the weak formulation of (1), we then consider either standard conforming finite elements with mass lumping [2] or a recent symmetric interior penalty discontinuous Galerkin (DG) formulation [1] for the spatial discretization of (1). Both eventually lead to a second-order system of ordinary differential equations of the form:

$$M \frac{d^2 y}{dt^2} + Ky = F, \tag{2}$$

where both M, K are symmetric positive definite (SPD); moreover, $M^{1/2}$ can be obtained at low cost, since M is (block-)diagonal. For $F = 0$ the energy

$$E[y](t) = \dot{y}^\top M \dot{y} + y^\top K y$$

is conserved in time. Hence, we now seek time discretizations of (2), which also conserve (a discrete version of) the energy.

We let $F = 0$ and multiply (2) by $M^{-1/2}$ to obtain

$$\frac{d^2 z}{dt^2} + M^{-1/2} K M^{-1/2} z = 0, \tag{3}$$

with $z = M^{1/2}y$. Next, we set $A = M^{-1/2}KM^{-1/2}$, which is also symmetric positive definite, and thus rewrite (3) as

$$\frac{d^2z}{dz^2} + Az = 0 \ . \tag{4}$$

The (exact) solution $z(t)$ of (4) satisfies

$$z(t+\Delta t) - 2z(t) + z(t-\Delta t) = -\Delta t^2 \int_{-1}^{1} (1-|\theta|) Az(t+\theta\Delta t)\, d\theta. \tag{5}$$

The integral on the right-hand side of (5) represents a weighted average of $Az(s)$ over the interval $(t-\Delta t, t+\Delta t)$, which needs to be approximated in any numerical algorithm. For instance, if we simply replace $Az(t+\theta\Delta t)$ by $Az(t)$ in (5) and evaluate the remaining θ-dependent integral, we obtain the well-known second order leap-frog scheme with time step Δt,

$$\frac{z_{m+1} - 2z_m + z_{m-1}}{\Delta t^2} = -Az_m, \qquad z_m \simeq z(t_m), \tag{6}$$

which, however, would require Δt to be comparable in size to the smallest elements in the mesh.

Instead, following [3], we now split the vector $z(t)$ in two parts:

$$z(t) = (I-P)z(t) + Pz(t) = z^{[\text{coarse}]}(t) + z^{[\text{fine}]}(t) \ .$$

The projection matrix P is diagonal: its diagonal entries, equal to zero or one, identify the unknowns associated with the locally refined region, that is where smaller time steps are needed. To circumvent the severe CFL restriction on Δt in (6), we shall treat $z^{[\text{fine}]}(t)$ differently from $z^{[\text{coarse}]}(t)$ in

$$\begin{aligned} &z(t+\Delta t) - 2z(t) + z(t-\Delta t) \\ &\qquad = -\Delta t^2 \int_{-1}^{1} (1-|\theta|) A \left[z^{[\text{coarse}]}(t+\theta\Delta t) + z^{[\text{fine}]}(t+\theta\Delta t) \right] d\theta. \end{aligned}$$

Thus, we replace

$$A \left[z^{[\text{coarse}]}(t+\theta\Delta t) + z^{[\text{fine}]}(t+\theta\Delta t) \right] \simeq Az^{[\text{coarse}]}(t) + APq(\theta\Delta t),$$

where $q(\tau)$ solves the differential equation

$$\frac{d^2q}{d\tau^2}(\tau) = -A(I-P)z(t) - APq(\tau), \tag{7}$$

$$q(0) = z(t), \qquad q'(0) = 0. \tag{8}$$

We can then show that

$$z(t+\Delta t) - 2z(t) + z(t-\Delta t) = 2(q(\Delta t) - q(0)) + O(\Delta t^4). \tag{9}$$

Hence for each time step Δt, the algorithm to advance $z(t)$ from t to $t+\Delta t$ proceeds as follows: first, we solve (7) with the leap-frog scheme using p time steps each of size $\Delta\tau = \Delta t/p$, $p \geq 1$, which yields $q(\Delta t)$. Then, we update $z(t+\Delta t)$ by using (9). Here, a key observation in (7) is that the term $A(I-P)z(t)$ is independent of τ. For time step of size Δt, this algorithm therefore requires only one multiplication

by $(I - P)A$ and p multiplications by PA. Those p multiplications, however, only affect the unknowns in the highly refined region where P is nonzero, and hence the additional work will be small if the refined region only occupies a fraction of the entire mesh.

We have proved that the resulting overall algorithm is second-order accurate in time and exactly conserves a discrete energy. The CFL condition of the new scheme is usually comparable to that of the standard leap-frog scheme without local refinement. If a small overlap across at most three elements is allowed into the coarse region immediately next to the refined region, the CFL condition is optimal, in the sense that the same time step can be used.

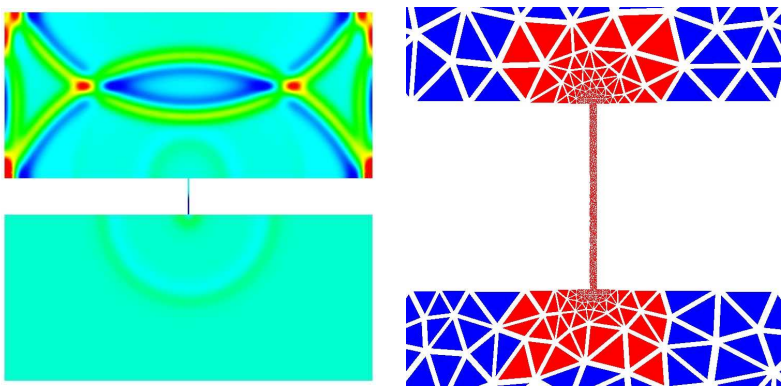


FIGURE 1. Left: The solution is shown at time $t = 0.2$. Right: the highly refined mesh inside the narrow channel.

To illustrate the versatility of the method, we consider a computational domain that consists of two rectangles connected by a very narrow channel. We use the IP-DG formulation [1] with P^3 elements on a triangular mesh, which is highly refined inside the narrow region, as shown in the right frame of Fig. 1. Since the mesh size inside the refined region is about 17 times smaller than in the surrounding coarse region, we take $p = 17$ local time steps for every time step Δt.

As shown in Fig. 1, the wave is initiated by a pulse in the upper region, which propagates outward until it impinges on the boundaries. A fraction of the wave then penetrates the channel and generates a circular outgoing wave as it reaches the opposite lower region. Further reflections occur as the wave moves back and forth inside the channel, subsequently generating multiple circular waves in the upper and lower domains.

REFERENCES

[1] M.J. Grote, A. Schneebeli, and D. Schötzau, *Discontinuous Galerkin finite element method for the wave equation*, SIAM Numer. Analysis **44** (2006), 2408–2431.

[2] G. Cohen, P. Joly, J.E. Roberts, and N. Tordjman, *Higher order triangular finite elements with mass lumping for the wave equation*, SIAM Numer. Analysis **38** (2001), 2047–2078.

[3] E. Hairer, C. Lubich, G. Wanner, *Geometric Numerical Integration*, 2nd edition, Springer-Verlag Berlin Heidelberg, 2006

Optimal local radiation boundary condition sequences, optimal perfectly matched layers, and why they are the same

Thomas Hagstrom

A few years back efficient temporally local but spatially nonlocal compressions of exact radiation boundary conditions for the scalar wave equation and Maxwell's equations were constructed [1, 2]. To achieve an accuracy, ϵ, up to a time, T, for a wavelength, λ, it was shown to require a boundary operator described by $O(\ln \frac{1}{\epsilon} \cdot \ln \frac{cT}{\lambda})$ functions per harmonic, an estimate we suspect is optimal. Although the associated computational complexity is small compared with the cost of the interior solve, these conditions suffer from the need to compute spatial transforms and from a lack of geometric flexibility. We present here a new sequence of local radiation boundary conditions which, experimentally at least, satisfy the same complexity estimate as the spatially nonlocal conditions, without the need for harmonic transforms and with the possibility of implementations in rectangular domains.

Consider a planar artificial boundary, $x = 0$. Our starting point is a representation of the solution as a superposition of propagating and evanescent plane waves, easily derived via Fourier transforms under the assumption that all inhomogeneities lie in the left half plane $x \leq -\delta$ for some $\delta > 0$.

$$\begin{aligned} u(x,y,z,t) &= \int_0^{\frac{\pi}{2}} \Phi\left(ct - x\cos\phi, y, z, \phi\right) d\phi \\ &\quad + \int_0^{\infty} e^{-\sigma x} \Upsilon(y,z,t,\sigma) d\sigma, \end{aligned} \tag{1}$$

where u satisfies the wave equation. Following the treatment of analogous expressions in deriving the translation formulas for the fast multipole method [3], approximate (1) by some quadrature rule:

$$u(x,y,z,t) \approx \sum_{j=0}^{n_p-1} w_j \Phi_j(ct - x\cos\theta_j, y, z) + \sum_{j=1}^{n_e} d_j e^{-\sigma_j x} \Upsilon_j(t,y,z). \tag{2}$$

Local boundary conditions with n_p+n_e auxiliary functions can now be constructed which are exact on this approximate representation independent of the unknown functions Φ_j and Υ_j. Following [4, 5, 6] recursively define for $j = 0, \ldots, n_p - 1$:

$$\left(\cos\theta_j \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\psi_j = \left(\cos\theta_j \frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\psi_{j+1} \tag{3}$$

with $\psi_0 = u$ and for $j = 1, \ldots, n_e$:

$$\left(\sigma_j + \frac{\partial}{\partial x}\right)\psi_{n_p+j-1} = \left(\sigma_j - \frac{\partial}{\partial x}\right)\psi_{n_p+j} \tag{4}$$

$$\psi_{n_p+n_e} = 0. \tag{5}$$

Upon proving by induction that each of these auxiliary fields satisfies the governing equations, one derives evolution equations along the boundary $x = 0$ by eliminating the normal derivatives. This leads to the local boundary condition sequences. Alternatively, one can interpret the recursions as a finite volume discretization of a PML (with nonstandard parameters). (See [7, 8] for related ideas). In particular, if we perform a Laplace transform in time the equations formally resemble a discrete PML with frequency-dependent grid spacings: (One could also rewrite a standard PML as a local radiation boundary condition in this way.)

To use these conditions on polygonal domains, corner and edge compatibility conditions must be derived to provide boundary conditions for the auxiliary hyperbolic systems. Here the PML interpretation is useful, allowing the construction of corner compatibility conditions in analogy with standard corner layers.

Error estimates follow from estimates of the complex reflection coefficients. So far, we have simply evaluated these numerically. Based on the nuemrical results (see the review article [10] for more details) we can compare the complexity of various boundary treatments as functions of the dimensionless parameter $\frac{cT}{\lambda}$. In particular we consider:
Padé Parameters:

$$\cos\theta_j = 1, \quad j = 0, \ldots, n_p - 1, \quad n_e = 0, \tag{6}$$

Gauss-Radau-Rokhlin-Yarvin (GRRY) Parameters:

$$\theta_j = \frac{\pi(c_j + 1)}{4}, \quad j = 0, \ldots, n_p - 1, \tag{7}$$

where c_j are the left endpoint Gauss-Radau nodes on $[-1, 1]$ and

$$\sigma_j = \beta\lambda d_j, \quad j = 0, \ldots, n_e, \tag{8}$$

where d_j are the Yarvin-Rokhlin nodes [9]. For the experiments we chose $n_p = n_e$ and $\beta = 5$.
PML: we choose an average absorption coefficient of $\bar{sigma} = 1$ and a complex frequency shift parameter of $\alpha = 0.1$. We vary the layer width L and assume that the number of points in the layer is proportional to $\lambda^{-1}L$.

The results of this study can be approximately fit with the following exponential convergence models:

$$\begin{aligned} \mathrm{err}_{\mathrm{PML}} &\propto \exp\left(-1.3\sqrt{\frac{\lambda}{cT}}n_l\right), \\ \mathrm{err}_{\mathrm{Pade}} &\propto \exp\left(-2.0\left(\frac{\lambda}{cT}\right)n_p\right), \\ \mathrm{err}_{\mathrm{GRRY}} &\propto \exp\left(-0.29\frac{(n_p+n_e)}{\ln\left(\frac{cT}{\lambda}\right)}\right). \end{aligned} \tag{9}$$

Numerical experiments are in agreement with the predictions of the reflection coefficients, though with actual errors an order of magnitude or two below the

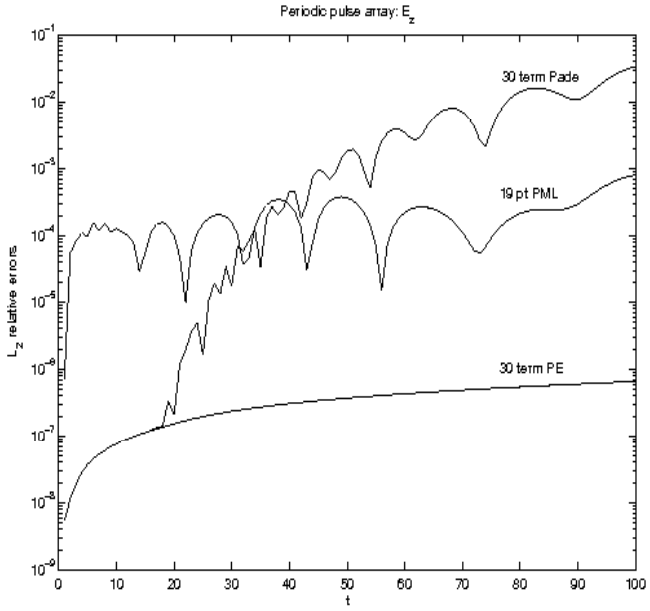


FIGURE 1. Errors for various boundary truncations with 30 degrees of freedom - Maxwell's equations with a transversely periodic transient line source.

worst case levels determined by the reflection coefficients. For example, the following results are obtained if one solves Maxwell's equations with a transversely periodic transient line source. The advantages of the new boundary conditions are apparent.

Note that the error estimates for PML and Padé parameters are in agreement with the exact error formulas of Diaz and Joly [11, 12] and de Hoop et al [13]. Publications detailing the new boundary conditions are in preparation [14, 15, 16]. For recent reviews which discuss other approaches see [17, 10].

The author acknowledges support by ARO Grant DAAD19-03-1-0146 and NSF Grant DMS-0610067. Any conclusions or recommendations expressed in this talk are those of the author and do not necessarily reflect the views of ARO or NSF.

REFERENCES

[1] B. Alpert, L. Greengard, and T. Hagstrom, *Rapid evaluation of nonreflecting boundary kernels for time-domain wave propagation*, SIAM J. Numer. Anal. **37** (2000), 1138–1164.

[2] B. Alpert, L. Greengard, and T. Hagstrom, *Nonreflecting boundary conditions for the time-dependent wave equation*, J. Comput. Phys., **180** (2002), 270–296.

[3] L. Greengard, J. Huang, V. Rokhlin, and S. Wandzura, *Accelerating fast multipole methods for low frequency scattering*, IEEE Comput. Sci. Engng. **5** (1998), 32–38.

[4] T. Hagstrom and T. Warburton, *High-order local radiation boundary conditions: Corner compatibility conditions and extensions to first order systems*, Wave Motion **39** (2004), 327–338.
[5] D. Givoli, T. Hagstrom, and I. Patlashenko, *Finite element formulation with high order absorbing boundary conditions for time-dependent waves*, Comput. Meth. Appl. Mech. **195** (2006), 3666–3690.
[6] T. Hagstrom, D. Givoli, M. de Castro, and D. Tzemach, *Local high-order ABCs for time-dependent waves in guides*, J. Comput. Acoust. (2007), to appear.
[7] S. Asvadurov, V. Druskin, M. Guddati, and L. Knizherman, *On optimal finite difference approximation of PML*, SIAM J. Numer. Anal. **41** (2003), 287–305.
[8] M. Guddati and K.-W. Lim, *Continued fraction absorbing boundary conditions for convex polygonal domains*, Int. J. Num. Meth. Engng.**66** (2006), 949–977.
[9] N. Yarvin and V. Rokhlin, *Generalized Gaussian Quadratures and Singular Value Decompositions of Integral Operators*, SIAM J. Sci. Comp., **20** (1998), 699–718.
[10] T. Hagstrom and S. Lau, *Radiation boundary conditions for Maxwell's equations: a review of accurate time-domain formulations*, J. Comput. Math. (2007), to appear.
[11] J. Diaz and P. Joly, *An analysis of higher-order boundary conditions for the wave equation*, SIAM J. Appl. Math. **65** (2005), 1547–1575.
[12] J. Diaz and P. Joly, *A time-domain analysis of PML models in acoustics*, Computer Meth. Appl. Mech. Engrg. **195** (2006), 3820–3853.
[13] A. de Hoop, P. van den Berg, and R. Remis, *Absorbing boundary conditions and perfectly matched layers - an analytic time-domain performance analysis*, IEEE Trans. on Magnetics **38** (2002), 657–660.
[14] T. Hagstrom and T. Warburton, *Complete radiation boundary conditions: minimizing the long time error growth of local methods*, in preparation.
[15] T. Warburton and T. Hagstrom, *Complete radiation boundary conditions for electromagnetic scattering*, in preparation.
[16] T. Hagstrom, A. Mar-Or, D. Tzemach, and D. Givoli, *High-order local radiation boundary conditions for the wave equation: extensions and improvements*, in preparation.
[17] T. Hagstrom, *New results on absorbing layers and radiation boundary conditions*, in *Topics in Computational Wave Propagation*, M. Ainsworth, P. Davies, D. Duncan, P. Martin, and B. Rynne, eds., Springer-Verlag, (2003), 1–42.

Computational Aeroacoustics using Nonmatching Grids and PML

Manfred Kaltenbacher
(joint work with Max Escobar, Bernd Flemisch, Barbara Wohlmuth)

A large amount of the total noise in our daily life is generated by turbulent flows (e.g. airplanes, cars, air conditioning systems, etc.). The physics behind the generation process is quite complicated and still not fully understood. Numerical simulation is one important way to analyze the generation of flow induced sound.

In our contribution, we propose a hybrid approach for the computation of flow induced noise. The flow is computed by a finite volume method using a large eddy simulation to solve the incompressible Navier-Stokes equations [9]. Based on Lighthill's acoustic analogy [16, 17], we solve the inhomogeneous wave equation

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho_0 v_i v_j),$$

with ρ' the acoustic density, c the speed of sound, ρ_0 the ambient density and $\boldsymbol{v}$ the flow velocity by applying the finite element method. The inhomogeneous term of the wave equation is calculated from the fluid flow data within the fluid region, and to obtain reliable results, the discretization of the wave equation within this region has to be very fine. However, outside the flow region the computational grid can be much coarser. In order to efficiently resolve this mismatch of the grid sizes in the two regions, we use independently generated grids and deal with the situation of nonconforming grids within the framework of the mortar finite element method, [4, 5, 2, 6, 15, 19]. Therewith, in each subdomain we have to solve the wave equation for the acoustic pressure $p_i' : \Omega_i \times (0,T) \to \mathbb{R}$,

$$\frac{1}{c^2}\ddot{p_i'} - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} p_i' = f_i \quad \text{in } \Omega_i \times (0,T)\,,\ i = 1,2,$$

with appropriate initial conditions at $t = 0$ and boundary conditions on the global boundary Γ. In a strong setting, one would impose continuity in the trace and flux of the pressure along the interface Γ_{I}, i.e.,

$$p_1' = p_2' \quad \text{and} \quad \boldsymbol{\nabla} p_1' \cdot \boldsymbol{n} = \boldsymbol{\nabla} p_2' \cdot \boldsymbol{n} \quad \text{on } \Gamma_{\mathrm{I}}.$$

Instead, we enforce just the flux coupling in a strong sense by introducing the Lagrange multiplier

$$\lambda = -\boldsymbol{\nabla} p_1' \cdot \boldsymbol{n} = -\boldsymbol{\nabla} p_2' \cdot \boldsymbol{n}\,,$$

and set the continuity in the trace in a weak sense, i.e.,

$$b(p', \mu) = < p_1' - p_2', \mu >_{\Gamma_{\mathrm{I}}} = 0,$$

for all test functions μ out of the Lagrange multiplier space $M = H^{-1/2}(\Gamma_{\mathrm{I}})$, [11]. This will lead to a symmetric evolutionary saddle point problem, [7], for which it can be shown that the approximation error is bounded by assuming enough smoothness for the acoustic pressure p' as well as for the Lagrange multiplier λ, [1, 10]. For the fast solution of such systems we refer to [20].

A second crucial point for an efficient computation of flow induced sound is the ability of the numerical scheme to fully take into account free radiation. Thus, reflections at the computational boundaries have to be avoided. To fulfill this requirement, we have developed an enhanced PML (Perfectly Matched Layer) method, which allows for computational domains to be a fraction of the acoustic wavelength. There has been much research work on the PML-technique since the first introduction of this technique in [3], see, e.g., [8, 12, 13, 18]. The idea of splitting the physical quantities into an x-, y- and z-component as presented in [3] can be fully adopted for the wave equation. Moreover, in the frequency domain, a rearrangement of the resulting equations allows to obtain a modified Helmholtz equation just for the acoustic pressure

$$\eta_y\eta_z \frac{\partial}{\partial x}\left(\frac{1}{\eta_x}\frac{\partial \hat{p}}{\partial x}\right) + \eta_x\eta_z \frac{\partial}{\partial y}\left(\frac{1}{\eta_y}\frac{\partial \hat{p}}{\partial y}\right) + \eta_x\eta_y \frac{\partial}{\partial z}\left(\frac{1}{\eta_z}\frac{\partial \hat{p}}{\partial z}\right) + \eta_x\eta_y\eta_z\, k^2\, \hat{p} = f\,.$$

In the above equation $k = \omega/c$ denotes the acoustic wave number and the coefficients η_i depend on the damping functions σ_i

$$\eta_x = 1 - j\frac{\sigma_x}{\omega} \qquad \eta_y = 1 - j\frac{\sigma_y}{\omega} \qquad \eta_z = 1 - j\frac{\sigma_z}{\omega}\,.$$

The choice of these damping functions σ_i is of great importance in order to obtain a stable method [14].

References

[1] Alain Bamberger, Roland Glowinski, and Quang Huy Tran, *A domain decomposition method for the acoustic wave equation with discontinuous coefficients and grid change*, SIAM J. Numer. Anal. **34** (1997), no. 2, 603–639.

[2] Faker Ben Belgacem, *The mortar finite element method with Lagrange multipliers*, Numer. Math. **84** (1999), no. 2, 173–197.

[3] J.P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comp. Phys. **114** (1994).

[4] C. Bernardi, Y. Maday, and A. T. Patera, *Domain decomposition by the mortar element method*, Asymptotic and numerical methods for partial differential equations with critical parameters (Beaune, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 384, Kluwer Acad. Publ., Dordrecht, 1993, pp. 269–286.

[5] ———, *A new nonconforming approach to domain decomposition: the mortar element method*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), Pitman Res. Notes Math. Ser., vol. 299, Longman Sci. Tech., Harlow, 1994, pp. 13–51.

[6] D. Braess and W. Dahmen, *Stability estimates of the mortar finite element method for 3-dimensional problems*, East-West J. Numer. Math. **6** (1998), no. 4, 249–263.

[7] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Verlag, 1991.

[8] F. Collino and P. Monk, *The perfectly matched layer in curvilinear coordinates*, SIAM J. Sci. Comp. **19** (1998), 2061–2090.

[9] F. Durst and M. Schäfer, *A Parallel Block-Structured Multigrid Method for the Prediction of Incompressible Flows*, Int. J. Num. Methods Fluids **22** (1996), 549–565.

[10] B. Flemisch, *Non-matching triangulations of curvilinear interfaces applied to electr-mechanics and elasto-mechanics*, Ph.D. thesis, University of Stuttgart, 2006.

[11] B. Flemisch, M. Kaltenbacher, and B.I. Wohlmuth, *Elasto-Acoustic and Acoustic-Acoustic Coupling on Nonmatching Grids*, Int. J. Numer. Meth. Engng **67** (2006), no. 13, 1791–1810.

[12] I. Harari, M. Slavutin, and E. Turkel, *Analytical and numerical studies of a finite element pml for the helmholtz equation*, Int. J. Comp. Acoustics **8** (2000), 121–127.

[13] Fang Q. Hu, *Absorbing boundary conditions*, Int. J. Comp. Fluid Dyn. **18** (2004), 513–522.

[14] M. Kaltenbacher, *Perfectly matched layer technique for computational acoustics in the time-harmonic case*, Journal of Computational Acoustics (2006), (submitted).

[15] Chisup Kim, Raytcho D. Lazarov, Joseph E. Pasciak, and Panayot S. Vassilevski, *Multiplier spaces for the mortar finite element method in three dimensions*, SIAM J. Numer. Anal. **39** (2001), no. 2, 519–538.

[16] M.J. Lighthill, *On sound generated aerodynamically I. general theory*, Proc. Roy. Soc. Lond. (1952), no. A 211, 564–587.

[17] ———, *On sound generated aerodynamically II. turbulence as a source of sound*, Proc. Roy. Soc. Lond. (1954), no. A 222, 1–22.

[18] I. Singer and E. Turkel, *A perfectly matched layer for helmholz equation in a semi-infinite strip*, Int. J. Comp. Phys. **201** (2004), 439–465.

[19] Barbara I. Wohlmuth, *Discretization methods and iterative solvers based on domain decomposition*, Lecture Notes in Computational Science and Engineering, vol. 17, Springer, 2001.

[20] ———, *A v-cycle multigrid approach for mortar finite elements*, SIAM J. Numer. Anal. **42** (2004), no. 6, 2476–2495.

Parallel auxiliary space AMG for definite Maxwell problems

Tzanio V. Kolev
(joint work with Panayot S. Vassilevski)

Motivated by the needs of large mullti-physics simulation codes, we are interested in algebraic solvers for the linear systems arising in time-domain electromagnetic simulations. Our focus is on finite element discretizations, and we are developing scalable parallel preconditioners which employ only fine-grid information, similar to algebraic multigrid (AMG) for diffusion problems. In the last few years, the search for efficient algebraic preconditioners for $\boldsymbol{H}(\mathrm{curl})$ bilinear forms has intensified. The attempts to directly construct AMG methods had some success, see [12, 1, 7]. Exploiting available multilevel methods on auxiliary mesh for the same bilinear form led to efficient auxiliary mesh preconditioners to unstructured problems as shown in [4, 8]. A computationally more attractive approach was recently proposed by Hiptmair and Xu [5]. In contrast to the auxiliary mesh idea, the method in [5] uses a nodal $\boldsymbol{H}^1$-conforming auxiliary space on the same mesh. This significantly simplifies the computation of the corresponding interpolation operator.

In the present talk, we consider several options for constructing unstructured mesh AMG preconditioners for $\boldsymbol{H}(\mathrm{curl})$ problems and report a summary of computational results from [10, 9]. Our approach is slightly different than the one from [5], since we apply AMG directly to variationally constructed coarse-grid operators, and therefore no additional Poisson matrices are needed on input. We also consider variable coefficient problems, including some that lead to a singular matrix. Both type of problems are of great practical importance and are not covered by the theory of [5].

We are interested in solving the following variational problem stemming from the definite Maxwell equations:

$$\text{(1)} \qquad \text{Find } \mathbf{u} \in \boldsymbol{V}_h \ : \quad (\alpha \ \mathrm{curl}\, \mathbf{u}, \mathrm{curl}\, \mathbf{v}) + (\beta\, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})\,, \quad \text{for all } \mathbf{v} \in \boldsymbol{V}_h\,.$$

Here we consider $\alpha > 0$ and $\beta \geq 0$ which are scalar coefficients, but extensions to (semi)definite tensors are possible. We allow β to be zero in part or the whole domain (in which case the resulting matrix is only semidefinite and for solvability, the right-hand side should be chosen to satisfy compatibility conditions). Let $\boldsymbol{A}_h$ be the stiffness matrix corresponding to (1), where $\boldsymbol{V}_h$ is the (lowest order) Nédélec space associated with a triangulation $\mathcal{T}_h$.

Let S_h be the space of continuous piecewise linear finite elements associated with the same mesh $\mathcal{T}_h$ as $\boldsymbol{V}_h$, and $\boldsymbol{S}_h$ be its vector counterpart. Let G_h and $\boldsymbol{\Pi}_h$ be the matrix representations of the mapping $\varphi \in S_h \mapsto \nabla\varphi \in \boldsymbol{V}_h$ and the nodal interpolation from $\boldsymbol{S}_h$ to $\boldsymbol{V}_h$, respectively. Note that G_h has as many rows as the number of edges in the mesh, with each row having two nonzero entries: $+1$ and

-1 in the columns corresponding to the edge vertices. The sign depends on the orientation of the edge. Furthermore, $\boldsymbol{\Pi}_h$ can be computed based only on G_h and on the coordinates of the vertices of the mesh.

The auxiliary space AMG preconditioner for $\boldsymbol{A}_h$ is a subspace correction method utilizing the subspaces $\boldsymbol{V}_h$, $G_h S_h$, and $\boldsymbol{\Pi}_h \boldsymbol{S}_h$. Its additive form reads (cf. [13])

$$\Lambda_h^{-1} + G_h B_h^{-1} G_h^T + \boldsymbol{\Pi}_h \boldsymbol{B}_h^{-1} \boldsymbol{\Pi}_h^T \,, \tag{2}$$

where Λ_h is a smoother for $\boldsymbol{A}_h$, while B_h and $\boldsymbol{B}_h$ are efficient preconditioners for $G_h^T \boldsymbol{A}_h G_h$ and $\boldsymbol{\Pi}_h^T \boldsymbol{A}_h \boldsymbol{\Pi}_h$ respectively. Since these matrices come from elliptic forms, the preconditioner of choice is AMG (especially for unstructured meshes).

If β is identically zero, one can skip the subspace correction associated with G_h, in which case we get a two-level method.

The motivation for (2) is that any finite element function $\mathbf{u}_h \in \boldsymbol{V}_h$ allows for decomposition of the form (cf., [5]) $\mathbf{u}_h = \mathbf{v}_h + \boldsymbol{\Pi}_h \mathbf{z}_h + \nabla \varphi_h$ with $\mathbf{v}_h \in \boldsymbol{V}_h$, $\mathbf{z}_h \in \boldsymbol{S}_h$ and $\varphi_h \in S_h$ such that the following stability estimates hold,

$$h^{-1} \|\mathbf{v}_h\|_0 + \|\mathbf{z}_h\|_1 \le C \, \|\operatorname{curl} \mathbf{u}_h\|_0 \quad \text{and} \quad \|\nabla \varphi_h\|_0 \le C \, \|\mathbf{u}_h\|_0. \tag{3}$$

A parallel solution algorithm based on (2) was implemented in the *hypre* library [6], under the name AMS (Auxiliary space Maxwell Solver). The internal AMG V-cycles employ *hypre*'s algebraic multigrid solver BoomerAMG [2]. Our results on unstructured meshes in two and three dimensions, including problems with variable coefficients and zero conductivity, clearly demonstrate the scalability of this preconditioner on hundreds of processors. A sample of the numerical experiments to be presented is shown in Table 1, using the following notation: np is the number of processors in the run, N is the global size of the problem, and t denotes the average time to solution (in seconds) on a machine with 2.4GHz Xeon processors.

References

[1] P. Bochev, C. Garasi, J. Hu, A. Robinson and R. Tuminaro, *An improved algebraic multigrid method for solving Maxwell's equations*, SIAM J. Sci. Comp. **25** (2003), 623–642.

[2] V. Henson and U. Yang, *BoomerAMG: A parallel algebraic multigrid solver and preconditioner*, Appl. Numer. Math. **41** (2002), 155–177.

[3] R. Hiptmair, *Multigrid method for Maxwell's equations*, SIAM J. Numer. Anal. **36** (1999), 204–225.

[4] R. Hiptmair, G. Widmer, and J. Zou, *Auxiliary space preconditioning in H(curl)*, Numerische Mathematik **103** (2006), 435–459.

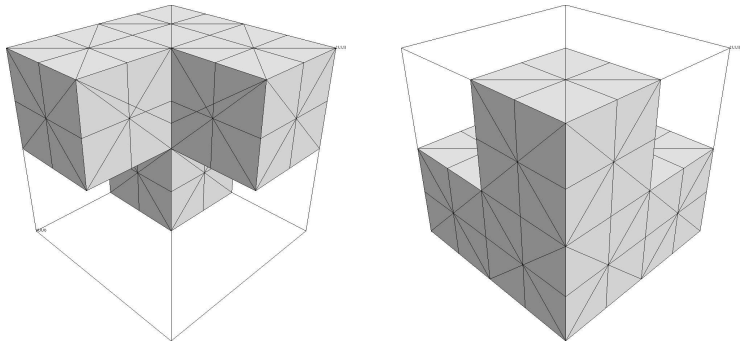
[5] R. Hiptmair and J. Xu, *Nodal auxiliary space preconditioning in H(curl) and H(div) spaces*, Research Report No. **2006-09** (May 2006). Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule, CH-8092 Zürich, Switzerland.

[6] *hypre*: High performance preconditioners, `http://www.llnl.gov/CASC/hypre/`.

[7] J. Jones and B. Lee, *A multigrid method for variable coefficient Maxwell's equations*, SIAM J. Sci. Comp. **27** (2006), 1689–1708.

[8] Tz. Kolev, J. Pasciak and P. Vassilevski, *H(curl) auxiliary mesh preconditioning*, Num. Lin. Alg. Appl. (2007), to appear.

[9] Tz. Kolev and P. Vassilevski, *Some experience with a H1-based auxiliary space AMG for H(curl) problems*. LLNL Technical Report **221841** (June, 2006).



np	N	p									t
		-8	-4	-2	-1	0	1	2	4	8	
1	83,278	9	9	9	9	9	9	10	11	11	5s
2	161,056	10	10	10	10	10	10	10	11	11	9s
4	296,032	11	12	12	12	11	11	12	13	13	9s
8	622,030	13	13	13	12	12	12	13	15	14	12s
16	1,249,272	13	13	13	13	13	13	13	15	14	13s
32	2,330,816	15	15	15	15	15	15	15	16	15	14s
64	4,810,140	16	16	16	16	16	15	16	18	17	17s
128	9,710,856	16	16	16	16	16	16	16	17	17	23s
256	18,497,920	19	19	19	19	19	19	19	21	20	27s
512	37,864,880	21	20	20	20	20	20	20	23	22	32s
1024	76,343,920	20	20	20	20	20	20	20	21	21	56s

TABLE 1. Number of AMS-PCG iterations with tolerance 10^{-6}, for the problem (1) on the unit cube with $\alpha = 1$, and $\beta \in \{1, 10^p\}$ having different values in the shown regions (cf. [3]).

[10] Tz. Kolev and P. Vassilevski, *Parallel H1-based auxiliary space AMG solver for H(curl) problems*. LLNL Technical Report **222763** (July, 2006).

[11] J. Pasciak and J. Zhao, *Overlapping Schwarz methods in H(curl) on polyhedral domain*, East West J. Numer. Anal. **10** (2002), 221–234.

[12] S. Reitzinger and J. Schöberl, *An algebraic multigrid method for finite element discretization with edge elements*, Num. Lin. Alg. Appl. **9** (2002), 215–235.

[13] J. Xu, *The auxiliary space method and optimal preconditioning techniques for unstructured grids*, Computing **56** (1996), 215–235.

A framework for (3+1)-decomposition of 4D electrodynamics based on differential forms

Stefan Kurz
(joint work with B. Flemisch)

In recent years, numerical methods based on discrete exterior calculus have led to significant progress in the discretization of electrodynamics [2]. The starting point for such methods is a formulation of Maxwell's equations and the constitutive laws in terms of differential forms that are defined on a three-dimensional differentiable manifold. This approach does not depend on the choice of specific charts or coordinates. Moreover, the application of metric concepts is confined to the formulation of constitutive laws. In such formulations, time is regarded as a universal one-dimensional parameter. The three-dimensional manifold plus the one-dimensional time parameter constitute a (3+1)-dimensional formulation of electrodynamics. It is obvious that this formulation depends on an implicit choice of an observer, which is dubbed laboratory observer and considered inertial.

In many engineering applications the interaction between the electromagnetic field and moving bodies is of great interest. Frequently, a Lagrangian description is employed, where the unknowns are defined on a mesh which moves and deforms together with the considered objects. What is the correct form of Maxwell's equations and the constitutive laws under such circumstances? To answer this question, it is appealing to formulate the equations of electrodynamics independently of any observer or reference frame [3]. The choice of a specific observer with certain properties should become explicit in the course of deriving the (3+1)-dimensional decomposition.

The aim of our work is to tackle this question by using the language of differential forms. We start from Maxwell's equations and constitutive relations in four-dimensional flat Minkowskian space-time in terms of the 2-forms of electromagnetic field and excitation, the 1-form of electromagnetic four-current density and the 1-vector field of material four-velocity [4, 5, 7].

The next step is to introduce a pair of a vector field and a 1-form, describing an observer in space-time. The observer gives rise to a fibration of space-time, called relative space. In case of an holonomic observer, there exists as well a foliation of space-time, called relative time. This definition comprises more specific classes of observers, like locally or globally inertial observers [6]. Once an observer is defined, a (3+1)-decomposition mechanism can be established, which does primarily not depend on an underlying metric or on the choice of a spatial coordinate system [7, 8, 9]. With the help of this mechanism, all fundamental operators like exterior derivative, Hodge, and contraction can easily be decomposed.

An application to four-dimensional electrodynamics yields the common notions of three-velocity and three-dimensional fields, and provides general (3+1)-Maxwell's equations and constitutive laws. It becomes obvious that if the observer responsible for the decomposition is holonomic or locally inertial, then Maxwell's equations or constitutive laws adopt their simple form, respectively.

While this essentially constitutes an Eulerian point of view, we proceed to a four-dimensional Lagrangian description by transformation to the canonical reference space. The previous observation becomes utilized by introducing two observers to the Lagrangian setting, one being holonomic and the other one locally inertial. In order to benefit from their individual advantages, a transformation law connecting the two observers is established, again simply by plugging in the decomposition mechanism. This concludes a convenient description of (3+1)-dimensional electrodynamics. As an application of this approach, we consider the classical paradoxon by Schiff, where the fields of a spherical capacitor are described in rotating coordinates [10, 11, 12].

We are also concerned with relating our approach to standard results. In particular, the notion of a Lorentz boost is introduced and extended to p-forms by means of the exterior p-compound. Lorentz boosts in connection with globally inertial observers allow the introduction of equivalence classes of physically equivalent fields. The (3+1)-decomposition of the equivalence relation identifies physically equivalent three-dimensional subspaces and fields inherited from different (3+1)-decompositions. This way, we arrive at transformation laws for differential forms and vector fields coinciding with the well-known relations from textbooks.

Another important topic to be addressed is the connection of the constitutive relations with the choice of the metric. In particular, we reconsider the conditions under which it is possible to construct a four-dimensional metric out of the knowledge of the electric permittivity and the magnetic permeability tensors [4]. In this context, it is especially interesting to find out if a linear mapping between 2-forms can be factorized in the sense that it can be written as the exterior 2-compound of a linear mapping between 1-forms.

The decomposition framework can be applied to low frequency electromagnetism as well, where it turns out that all rigid observers are equivalent. This goes beyond the standard principle of Galilean relativity, where only globally inertial observers are regarded as equivalent. We emphasize that this property is restricted to the low frequency limit of Maxwell's equations, it justifies the usual textbook analysis of induction machines from the rotor's point of view.

References

[1] B. Flemisch, S. Kurz, and B. Wohlmuth "A Framework for Maxwell's Equations in Non-Inertial Frames Based on Differential Forms," IANS Preprint 2006/011, Universität Stuttgart, `http://preprints.ians.uni-stuttgart.de/`, 2006.

[2] R. Hiptmair, "Finite elements in computational electromagnetism," Acta Numerica, pp. 237–339, 2002.

[3] T. Matolcsi, *Spacetime Without Reference Frames*. Akadémiai Kiáo (Publishing House of the Hungarian Academy of Sciences), Budapest, 1993.

[4] D. van Dantzig, "The fundamental equations of electromagnetism, independent of metrical geometry," Proc. Cambridge Phil. Soc., vol. 30, pp. 421 ff, 1934.

[5] E. Jan Post, *Formal Structure of Electromagnetics*. North-Holland, 1962.

[6] B. Mashhoon, "The hypothesis of locality in relativistic physics," Physics Letters A, vol. 145, pp. 147–153, 1990.

[7] F. Hehl and Y. Obukhov, *Foundations of Classical Electrodynamics*. Boston: Birkhäuser, 2003.
[8] M. Fecko, "On 3+1 decompositions with respect to an observer field via differential forms," J. Math. Phys., vol. 38, no. 9, pp. 4542–4560, 1997.
[9] J. Kocik, "Relativistic observer and Maxwell's equations: an example of a non-principal Ehresmann connection," Preprint P-98-10-029, Department of Physics, UIUC, Urbana, IL61801, `http://www.math.siu.edu/Kocik/pracki/obs-gph.pdf`.
[10] L. Schiff, "A question in general relativity," Proc. Nat. Acad. Sci. USA, vol. 25, pp. 391–395, 1939.
[11] T. Mo, "Theory of electrodynamics in media in non-inertial frames and applications," J. Math. Phys., vol. 11, no. 8, pp. 2589–2610, Aug. 1970.
[12] J. van Bladel, *Relativity and Engineering*. Berlin: Springer-Verlag, 1984.

Non-Conformal Domain Decomposition Methods for Maxwell Equations - Frequency Domain Approach

Jin-Fa Lee
(joint work with Kezhong Zhao, Seung Cheol Lee, Vineet Rawat)

1. Introduction

The recently developed dual primal-finite element tearing interconnecting (DP-FETI) domain decomposition method (DDM) [7, 4, 1, 2] has been shown to be effective in solving a number of electrically large electromagnetic problems. In this approach, DDM decomposes the entire problem domain into an arbitrary number of smaller domains, taking advantage of repetitions in the structure. The FETI process translates information of a sub-domain's volume into information only on the volume's bounding surface, through the computation of a "transfer function" or a numerical Green's function in the form of a matrix. The numerical Green's function, once obtained, can be readily combined with other domains as a matrix-vector multiplication in the DDM solution process.

In some situations such as the analysis of an antenna array in the presence of frequency selective surfaces with different periodicity, geometrically conforming DDM may not be easily applied; difficulties arise due to the nature, varying size, and relative position of the sub-structures. In this abstract, we implement a fast geometrically non-conforming DP-FETI DDM to help alleviate these problems. The present method is largely composed of three major ingredients; 1) Decoupling of the geometrically conforming and non-conforming domains by the use of the FETI algorithm and the 1^{st} order Robin transmission condition [5]; 2) Solution of the resulting finite element matrix equation with multiple right-hand-sides using the pMUS preconditioner [9]; 3) Compression of the numerical Green's function via an efficient rank-revealing SVD algorithm [10][11].

2. Cement DP-FETI DDM

To apply the domain decomposition method, the original problem domain Ω is first partitioned into N non-overlapping and non-conforming sub-domains:

$$\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \cdots \bigcup \Omega_N \tag{1}$$

Subsequently in i^{th} domain, the boundary value problem (BVP) is written as:

$$\begin{aligned} \nabla \times \frac{1}{\mu_{ri}} \nabla \times \vec{E}_i^{(n)} - k^2 \epsilon_{ri} \vec{E}_i^{(n)} &= -jk\eta \vec{J}_i \quad \text{in} \quad \Omega_i \\ \vec{j}_i^{(n)} - jk\vec{e}_i^{(n)} &= -\vec{g}_i^{(n-1)} \quad \text{on} \quad \Gamma_i \end{aligned} \tag{2}$$

Note that an additional unknown carrying the physical meaning of electric current on the boundary, namely $\vec{j}_i = \hat{n}_i \times \nabla \times \vec{E}_i \mid_{\Gamma_i} \in \mathcal{H}_{div}^{-1/2}(\Gamma_i)$, is introduced on the domain interface Γ_i. Also note that $\vec{E}_i$ is the electric field inside domain Ω_i, $\vec{e}_i = \hat{n}_i \times \vec{E}_i \times \hat{n}_i$ is the electric field on Γ_i, and $\hat{n}_i$ is the outward normal of the i^{th} domain. Superscript $^{(n)}$ stands for n^{th} iteration of alternating Schwarz algorithm. k, η, ϵ_{ri} and μ_{ri} are the free-space wave number, impedance and relative permittivity and permeability of the medium in the i^{th} domain, respectively. Equation 2 is the Robin transmission condition. The function space $\mathcal{H}_{div}^{-1/2}(\Gamma_i)$ is where the well-known RWG (Rao-Wilton-Glisson) basis functions reside [8]. Moreover, the right hand side of 2 is defined as $\vec{g}_i^{(n-1)} = \vec{j}_{neig\{i\}}^{(n-1)} + jk\vec{e}_{neig\{i\}}^{(n-1)}$, where $neig\{i\}$ indicates the neighboring domains of domain i. The above system corresponds to a Schwarz-type iteration scheme, where the complex parameter jk, acts as a convergence acceleration parameter. We would like to point out here that the only difference between geometrically conforming and non-conforming DDM is the neighboring domain information.

The system of equations is appropriately tested through a Galerkin testing procedure, resulting in a matrix form

$$\mathbf{K}_i u_i^{(n)} = y_i + g_i^{(n-1)}, \qquad \forall i = 1, 2, \cdots N \tag{3}$$

where

$$\mathbf{K}_i = \begin{pmatrix} \mathbf{A}_i^{II} & \mathbf{A}_i^{IB} & 0 \\ \mathbf{A}_i^{BI} & \mathbf{A}_i^{BB} & \mathbf{D}_i \\ 0 & \mathbf{D}_i^T & \mathbf{T}_i \end{pmatrix}, u_i^{(n)} = \begin{pmatrix} E_i^{(n)} \\ e_i^{(n)} \\ j_i^{(n)} \end{pmatrix}, y_i = \begin{pmatrix} b_i \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

and,

$$g_i^{(n-1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbf{D}_{ij}^T & \mathbf{T}_{ij} \end{pmatrix} \begin{pmatrix} 0 \\ e_j^{(n-1)} \\ j_j^{(n-1)} \end{pmatrix} \tag{5}$$

The matrix sub-block entries are given in ref. [2].

The solution process of the problem 3 would require a large number of sub-problem solutions on every outer-loop iteration n. To alleviate this deficiency, we apply the DP-FETI algorithm. The algorithm is derived based on the observation

that at each iteration, the right hand side of 3 (c.f. 5) is updated using only the information of surface unknowns. The algorithm starts by eliminating the internal primal unknowns, via

$$\mathbf{R}_{DP}u_i^{(n)} = \mathbf{R}_{DP}\mathbf{K}_i^{-1}y_i + \mathbf{R}_{DP}\mathbf{K}_i^{-1}\mathbf{R}_D^T\mathbf{R}_D g_i^{(n-1)} \tag{6}$$

where $\mathbf{R}_{DP}$ and $\mathbf{R}_D$ are the surface and dual restriction operators, respectively [1]. Namely $v = [e \;\; j]^T = \mathbf{R}_{DP}u$ and $j = \mathbf{R}_D u$. Then, the iteration for the surface unknowns can be rewritten as

$$v_i^{(n)} = x_i^0 + \mathbf{Z}_i\tilde{g}_i^{(n-1)}, \forall i = 1, 2, \cdots N \tag{7}$$

Once convergence of 7 is achieved, the interior unknowns can be recovered via a single additional iteration of 3.

3. Compression of the numerical Green's function

The DP-FETI algorithm achieves speed-up by reducing the matrix solution of 3 to a single matrix-vector multiplication of 6 using the pre-computed iteration matrix $\mathbf{Z}_i$. Each column of the iteration matrix $\mathbf{Z}_i$ corresponds to the responses of the surface electric field and current when excited by a unit electric current source on the boundary surface. Thus the iteration matrix $\mathbf{Z}_i$ can be otherwise viewed as the "numerical Green's function" of the domain i. The matrix $\mathbf{Z}_i$ therefore possesses properties very similar to those of the impedance matrix in the method of moments. Namely, the matrix consists of many numerically rank deficient sub-blocks. Consequently, the matrix $\mathbf{Z}_i$ can be represented in data-sparse fashion via multi-level partitioning of the boundary surface. The coupling representing two well-separated group interactions termed $\mathbf{A}$ can be accurately approximated via the rank-revealing QR factorization [3] or the fully pivoted adaptive cross approximation (ACA) algorithm [10].

References

[1] S.C. Lee, M. Vouvakis, and J.F. Lee, *A Non-overlapping Domain Decomposition Method with Non-Matching Grids for Modeling Large Finite Antenna Arrays*, J. Comput. Phys., **203** (2005), 1–21.

[2] M. Vouvakis, Z. Cendes, and J.F. Lee, *A FEM Domain Decomposition Method for Photonic and Electromagnetic Band Gap Structures*, IEEE Trans. Ant. Prop., **54, no. 2**, (2006), 721–733.

[3] S.C. Lee, M. Vouvakis, K. Zhao, and J.F. Lee, *Analyzing Microwave Devices using an IE-QR Accelerated Symmetric FEBI*, Internal Journal of Numerical Methods in Engineering, **203, No. 64**, (2005), 528–546.

[4] B. Smith, P. Bjørstad, and W. Gropp, *Domain Decomposition Parallel Multilevel Methods for Elliptic Partial Differential Equation*, Cambridge University Press, New York, (1996).

[5] B. Després, P. Joly, and J.E. Roberts, *A domain decomposition method for the harmonic Maxwell equations*, Iterative Methods in Linear Algebra, North-Holland, Amsterdam, (1992), 245–252.

[6] R. Hoppe, *Mortar edge element method in R3*, East West J. Num An. **7, no. 3**, (1999), 159–222.

[7] C. Farhat, and F.-X. Raux, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Internat. J. Numer. Methods Engrg., **32**, (1991), 1205–1227.

[8] S.M. Rao, D.R. Wilton, and A.W. Glisson, *Electromagnetic scattering by surfaces of arbitrary shape*, IEEE Trans. Antennas Propagat., **30, no. 3**, (1982), 409–418.
[9] J.F. Lee and D.K. Sun, *p-type multiplicative Schwarz (pMUS) method with vector finite elements for modeling three-dimensional waveguide discontinuities*,IEEE Trans. Micro. Theory Tech., **52**, (2004), 864–870.
[10] K. Zhao, M.N. Vouvakis, and J.F. Lee, *The Adaptive Cross Approximation Algorithm for Accelerated Method of Moments Computations of EMC Problems*, IEEE Trans. Electromagn. Compat., **47, no. 4**, (2005), 763–773.
[11] M. Bebendorf and S. Rjasanow, *Adaptive Low-Rank Approximation of Collocation Matrices*, Computing, **70**, (2003), 1–24.
[12] M.N. Vouvakis, K. Zhao and J.F. Lee, *A Domain Decomposition Approach for Non-Conformal Couplings Between Finite and Boundary Elements for Unbounded Electromagnetic Problems in R3*, J. Comput. Phys., to appear.

Recent advances in the pre-conditioning of integral equations of electromagnetism

David P. Levadoux

The pertinence of integral methods for solving Maxwell's equations in harmonic regime requires no further proof. Using them in combination with a fast multipole algorithm and an iterative solver makes them efficient and precise methods that can solve problems involving hundreds of thousands of unknowns. But the efficiency of iterative methods depends on the conditioning of the linear systems, so it is absolutely crucial to have either high-performance pre-conditioners or intrinsically well-conditioned integral equation formulations. It is in this field, which is a strategic one because it is necessary for solving very large calculation configurations, that the french aerospace lab ONERA has been conducting fundamental research for the past few years.

The main research theme, begun in 1998 [11], is based on a generalized combined source integral equation (GCSIE) [12] [13], also named generalized Brakhage-Werner integral equation by different authors [5] [8]. The principal idea is to find a way to incorporate in the integral equation formalism, information on the scattering phenomenon that can be extracted beforehand using the means offered by asymptotic theories (physical and/or mathematical). Very close in spirit, but in the area of direct integral equation methods is the generalized combined field integral equation (GCFIE) of Contopanagos *et al.* [7] and Adams [2]. Actually, both GCSIE and GCFIE equations depend on the choice of an operator whose purpose is to approximate the admittance of the diffracting body as best as possible. In the limit case where this approximation is exact, the integral operator to be inverted becomes the identity. It is worth to insist that this crucial property is not only verified for the GCSIE (which was the main motivation of this equation), though unnoticed by their authors, also for the GCFIE. We thus try to construct approximations of the admittance such that the operator underlying the equation appears as a mathematically controllable perturbation of the identity. When the GCSIE or GCFIE are discretized, we arrive at a linear system that is by its very nature well-conditioned.

A quite natural way to build approximations of the admittance is to pull back onto the boundary Γ of the scatterer, the well known admittance of the tangent plane. Viewing the admittance of the plane as the trace of a potential, say $\mathbf{n} \times \mathcal{V}_k$ (k being the wave number, $\mathbf{n}$ the unit normal vector to Γ), we are obviously conveyed to import this formula onto the boundary and to read a first approximation as $\widetilde{Y} = \mathbf{n} \times \mathcal{V}_k$. The resulting GCSIE/GCFIE equations are indeed compact perturbations of the identity, but suffer in return from irregular frequencies. However, according to some high frequency considerations, and because it is fruitful from a numerical point of view to deal with operators as local as possible, a solution to remove these spurious modes is to localize this first approximation. While Contopanagos *et al.* [7] and Adams [2] propose to introduce a damping coefficient in the wave number, we suggested to use a quadratic partition of unity (U_p, χ_p)[1] on the boundary, leading to the following pseudo-local approximation

$$\widetilde{Y} = -2 \sum_p \chi_p \, \mathbf{n} \times \mathcal{V}_k \, \chi_p \quad . \tag{1}$$

Another technique to pull back the admittance operator of the plane is first to read it in the Fourier domain and then import on Γ the formula, replacing formally the Fourier parameter by a cotangent vector. One obtains the symbol of a new candidate operator able to feed any generalized combined formalism [8].

From a theoretical point of view, the GCSIE formalism with the admittance model (1) is now well understood. One of its foremost features is that it has no internal resonance frequency problem, and shows a condition number independent of the mesh refinement as well as (almost) of the frequency [4]. As to the experimental results, we point out that, for example, the Channel cavity, which models an aircraft air intake was processed at 7 GHz (300 000 unknowns) with the GCSIE in half the computation time needed with a classical equation (Fig. 1).

A second research theme concerns the construction of non-standard pre-conditioners. Contrary to the pre-conditioners used in industry, which are often algebraic and consequently leave out the mathematical properties of the equation from which the linear system derives, we construct analytical pre-conditioners from the parametrix (pseudo-inverse) of the equation. Actually, we follow the way opened first by McLean and Tran [15] and independently by Steinbach [14], to construct a new class of preconditioners of analytical nature. Roughly speaking, one can say these authors observed that the product of two Galerkin matrices, one coming from a single layer potential, the other from a hyper-singular operator, produces a linear system whose condition number is bounded independently of h, the mesh-size.

In electromagnetism, first applications of this principle were given by Adams [3] [1] and Christiansen-Nédélec [6] to build new preconditionners of the electric field integral equation (EFIE) with the help of a well-known Calderón integral relation.

[1] $(U_p)_p$ is a set of patches recovering Γ, and $(\chi_p)_p$ a family of truncation functions with support in U_p and such that $\sum_p \chi_p^2 = 1$.

Indeed, one aim of this second axis of research we pursued at ONERA is to show that this latter kind of preconditioners also exists for the CFIE. But the main problem we have to face is that, contrary to the EFIE, there does not exist a formula so convenient as the Calderón one, allowing to easily find a parametrix of the CFIE. To find it, we have to perform a more accurate symbolic analysis of this equation which is a crucial step. A parametrix we have found is presently being tested. The first results are encouraging, even though it seems to be only well-suited for low-frequency applications (Fig. 2). An effort in the future will be to give a more accurate parametrix for the high-frequency regime. A few solutions have been investigated in [10]. Furthermore, as a theoretical consequence of this work, an original proof has been established for the numerical convergence of the CFIE equation [9] which was until recently an open problem.

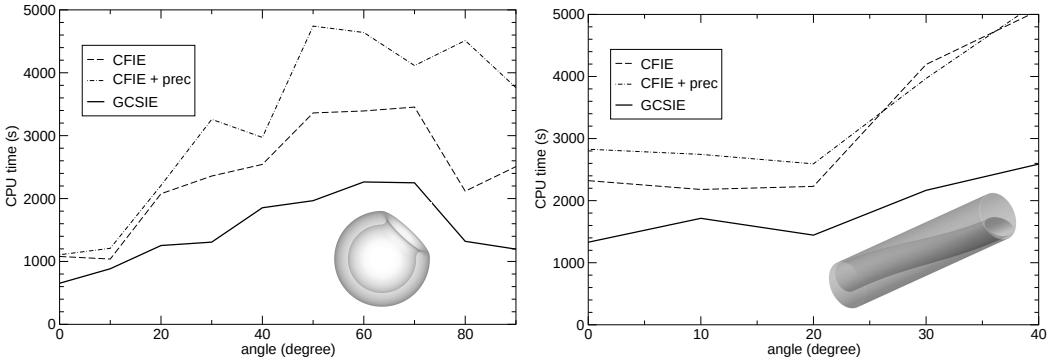


Figure 1. *Resolution time in function of the incidence for the hollow sphere* [6] *at* 2.8 *GHz (left) and the Channel cavity* [4] *at* 7 *GHz (right) with around* 300 000 *DoF.*

References

[1] R. J. Adams. Stabilisation procedure for electric field integral equation. *Electronics Letters*, 35(23):2015–2016, November 1999.

[2] R. J. Adams. Combined field integral equation formulations for electromagnetic scattering from convex objects. *IEEE Trans. Antennas Propag.*, 52(5):1294–1303, May 2004.

[3] R. J. Adams. Physical and analytical properties of a stabilized electric field integral equation. *IEEE Trans. Antennas Propag.*, 52(2):362–372, February 2004.

[4] F. Alouges, S. Borel, and D. P. Levadoux. A stable well-conditioned integral equation for electromagnetism scattering. *J. Comp. Appl. Math*, 2006, accepted.

[5] X. Antoine and M. Darbas. Alternative integral equations for the iterative solution of acoustic scattering problems. *Quart. J. Mech. Appl. Math.*, 58(1):107–128, 2005.

[6] S. H. Christiansen and J.-C. Nédélec. A preconditioner for the electric field integral equation based on Calderón formulas. *SIAM J. Numer. Anal.*, 40(3):1100–1135, 2002.

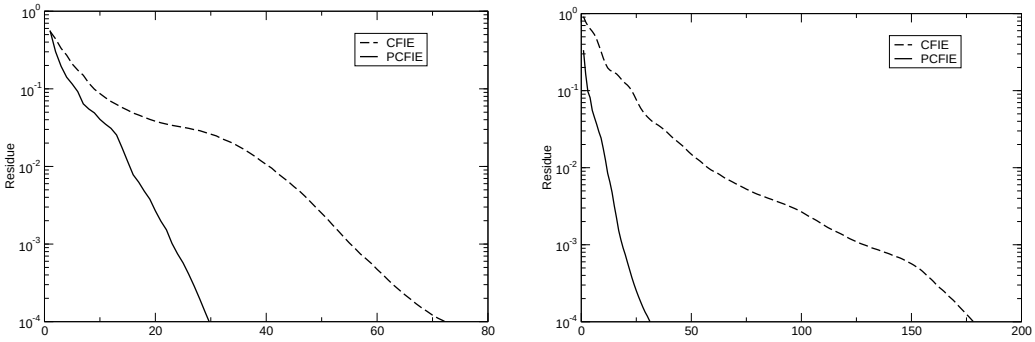


FIGURE 2. *Convergence curves for the hollow sphere* [6] *(left) and the Channel cavity* [4] *(right) at* 150 *MHz with around* 4 000 *DoF.*

[7] H. Contopanagos, D. Dembart, M. Epton, J. J. Ottusch, V. Rokhlin, J. Visher, and S. Vandzura. Well-conditioned boundary integral equations for three-dimensional electromagnetic scattering. Technical Report YALEU/DCS/RR-1198, Yale University, June 2000.

[8] M. Darbas. Generalized combined field integral equations for the iterative solution of the three-dimensional maxwell equations. *Applied Mathematics Letters*, 19(8):834–839, August 2006.

[9] D. P. Levadoux. Proposition de préconditionneurs pseudo-différentiels pour l'équation CFIE de l'électromagnétisme. *M2AN*, 39(1):147–155, 2005.

[10] D. P. Levadoux. New preconditioners for the CFIE equation of electromagnetism. In *IABEM 2006 conference*, pages 55–58, Graz, July 2006.

[11] D. P. Levadoux and B. L. Michielsen. Analysis of a boundary integral equation for high frequency Helmholtz problems. In *Waves Conference*, pages 765–767, Golden, Co, 1-5 June 1998.

[12] D. P. Levadoux and B. L. Michielsen. An efficient high-frequency boundary integral equation. In *Antennas and Propagation Society International Symposium*, volume 2, pages 765–767, Orlando, FL, 11-16 July 1999.

[13] D. P. Levadoux and B. L. Michielsen. Nouvelles formuations intégrales pour les problèmes de diffraction d'ondes. *M2AN*, 38(1):157–175, January/February 2004.

[14] O. Steinbach. *Gebietzerlegungsmethoden mit Randintegralgleichungen und effiziente numerische Lösungsverfahren für gemischte Randwertprobleme*. PhD thesis, University of Stuttgard, 1996.

[15] W. McLean, T. Tran. A preconditioning strategy for boundary element galerkin methods. *Numer. Methods Partial Differential Equations*, 13:283–301, 1997.

Inverse Obstacle Scattering: Some Theory and Numerics

HONGYU LIU
(joint work with Jun Zou)

In this short note we would like to report some recent advances we have made in the theory and numerics on inverse obstacle scattering.

In inverse obstacle scattering theory, one of the most fundamental problems is the uniqueness issue. It is on how many measurement data one can use to uniquely determine the underlying scatterer. There is a widespread belief that a

sound-soft scatterer can be uniquely determined by a single incident plane wave at an arbitrarily fixed wave number and incident direction, whereas a sound-hard scatterer can be uniquely determined by a single incident plane wave at some fixed (probably small) wave number and incident direction. However, this remains to be a challenging open problem (see, e.g., [3] and [5]).

Extensive study has been focused on establishing the uniqueness for polyhedral-type scatterers in the past few years. In [2], it is proved that a single (sound-soft or sound-hard) polygon in $\mathbb{R}^2$ can be uniquely determined by at most two incident plane waves but under the assumption of some 'non-trapping' condition. The condition was then lifted in [4] and the result was also improved to use one incoming wave, but still for a single polygonal obstacle. In [1], the very general sound-soft polyhedral scatterer is considered and the uniqueness is established by a single incoming wave.

On the contrary, no much progress has been made for general polyhedral-type sound-hard scatterers in higher dimension $\mathbb{R}^N (N \geq 3)$ until very recently. In our recent work [9], we considered much more general polyhedral scatterer in $\mathbb{R}^N$, possibly consisting of finitely many solid polyhedra and subsets of $(N-1)$-dimensional hyperplanes. It is then proved that a general sound-hard polyhedral scatterer in $\mathbb{R}^N (N \geq 2)$ is uniquely determined by N far-field patterns corresponding to N incident plane waves given by a fixed wave number and N linearly independent incident directions. Whereas for a general sound-soft polyhedral scatterer, the global uniqueness is established by a single incident plane wave. Both uniqueness results are now known to be optimal by the recent results in our work [11]. As one of the highlights in this work, we would like to mention that the methodology utilized in the paper for its proof is applicable for both sound-hard and sound-soft scatterers. This is remarkable considering the essential difference between the Dirichlet problem (corresponding to sound-soft scatterers) and Neumann problem (corresponding to sound-hard scatterers).

In our recent work [10], we proved that a solid polygonal scatterer consisting of finitely many solid polygons, which may be sound-soft or sound-hard, or even two types mixed together, is uniquely determined by a single incident plane wave. The novelty of this uniqueness result lies in two aspects: first, it is shown that if the scatterer does not contain any crack-type components, then one can uniquely determine it by a single incident wave; second, the uniqueness is established without knowing a priori the physical properties of the underlying scatterers, but must be either sound-hard or sound-soft (It is noted that uniqueness issue in such setting has only ever been considered in [6], where the global uniqueness is established by infinitely many incident plane waves). However, the uniqueness result in this paper for solid type scatterers by a single incident wave is incomplete for the $\mathbb{R}^3$ case. But we do prove another uniqueness result for scatterers in $\mathbb{R}^N (N \geq 2)$ which contain both solid- and crack-type components. It is shown that for a general polyhedral scatterer in $\mathbb{R}^N (N \geq 2)$ similar to that considered in [9], with

unknown a priori physical properties (but must be either sound-hard or sound-soft), the global uniqueness can be established by N different incident waves. This result is also known to be optimal by the recent results in our work [11].

In the work [12], a systematic and rather complete study of the interlacing character of the zeros for Bessel and spherical Bessel functions and their respective derivatives is presented. The results are then used to derive the global uniqueness in inverse obstacle scattering problems with balls or discs. It is shown that in the resonance region, the shape of a sound-soft/sound-hard ball in $\mathbb{R}^3$ or a sound-hard/sound-soft disc in $\mathbb{R}^2$ is uniquely determined by a single far-field datum measured at some fixed spot corresponding to a single incident plane wave. This seems to be an important result in the field as it is the first to establish the unique determination by a single far-field datum measured at one fixed spot. While all the other existing uniqueness results require far-field data observed at least in one open subset on the unit sphere with non-zero measure.

In the work [8], we extend the uniqueness study with polyhedral scatterers from acoustic scattering setting to the electromagnetic scattering setting. Some novel reflection principle for the time-harmonic Maxwell equations is developed. Then it is used to prove that a general perfect polyhedral conductor in $\mathbb{R}^3$, possibly consisting of finitely many solid polyhedra and subsets of planes, can be uniquely determined by two incident electromagnetic plane waves at fixed wave number and incident direction, and two linearly independent polarizations.

As one of the most important reconstruction algorithms in inverse scattering field, the linear sampling method has been intensively and extensively studied (see [3] and [13] for a comprehensive review). The method has been proven to be numerically quite successful and shown to possess several remarkable merits. However, it is computationally very expensive. In our recent work [7], we proposed a novel multilevel linear sampling method. The new method resembles the popular multi-level techniques in scientific computing and is shown to possess the asymptotically optimal computational complexity. For an $n \times n$ sampling mesh-grid in $\mathbb{R}^2$ or an $n \times n \times n$ sampling mesh-grid in $\mathbb{R}^3$, the proposed algorithm only requires to solve $\mathcal{O}(n^{N-1})(N = 2, 3)$ far-field equations for a $\mathbb{R}^N$ problem, and this is in sharp contrast to the original version of the linear sampling method which needs to solve n^N far-field equations instead. Numerical experiments have illustrated the promising feature of the new algorithm in significantly reducing the computational costs.

References

[1] G. Alessandrini and L. Rondi, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proc. Amer. Math. Soc. **6** (133), (2005), 1685-1691. Corrigendum: http://arxiv.org/abs/math.AP/0601406

[2] J. Cheng and M. Yamamoto, *Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves*, Inverse Problems **19** (2003), 1361-1384.

[3] D. Colton and R. Kress, *Using fundamental solutions in inverse scattering*, Inverse Problems **22** (2006), R49-R66.

[4] J. Elschner and M. Yamamoto, *Uniqueness in determining polygonal sound-hard obstacles with a single incoming wave*, Inverse Problems **22** (2006), 355-364.
[5] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, New York, 1998.
[6] A. Kirsch and R. Kress, *Uniqueness in inverse acoustic obstacle scattering*, Inverse Problems **9** (1993), 285–299.
[7] J. Z. Li, H. Y. Liu and J. Zou, *Multileve linear sampling method for inverse scattering problems*, submitted to SIAM J. Sci. Comput., 2006.
[8] H. Y. Liu, M. Yamamoto and J. Zou, *Reflection Principle for Maxwell's Equations and An Application to Inverse Electromagnetic Scattering Problem*, preprint, 2007.
[9] H. Y. Liu and J. Zou, *Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers*, Inverse Problems **22** (2006), no. 2, 515–524.
[10] H. Y. Liu and J. Zou, *Uniqueness in Determining Multiple Polygonal or Polyhedral Scatterers of Mixed Type*, submitted to SIAM J. Math. Anal., 2006. Also in Technical Report, CUHK-2006-03 (337), The Chinese University of Hong Kong.
[11] H. Y. Liu and J. Zou, *On Unique Determination of Partially Coated Polyhedral Scatterers with Far Field Measurements*, Inverse Problems **23** (2007), 297–308.
[12] H. Y. Liu and J. Zou, *Zeros of Bessel and Spherical Bessel Functions and Their Applications for Uniqueness in Inverse Acoustic Obstacle Scattering Problems*, IMA J. Appl. Math., in revision, 2006.
[13] R. Potthast, *A survey on sampling and probe methods for inverse problems*, Inverse Problems **22** (2006), no. 2, R1–R47.

Stable FEM-BEM Coupling for Helmholtz Transmission Problems

Patrick Meury
(joint work with Ralf Hiptmair)

Introduction

We consider acoustic scattering at a non-smooth penetrable object and coupled boundary element finite element schemes for its numerical simulation. Straightforward coupling approaches are haunted by instabilities at wave numbers related to interior resonances, the so-called spurious modes.

A remedy is offered by adopting the idea underlying the widely used combined field integral equations. We apply it in the form of modified trace operators. These will also feature regularizing operators to offset the lack of compactness of the double-layer integral operator on non-smooth surfaces.

Helmholtz Transmission Problem

The mathematical model for the acoustic scattering problem boils down to the following transmission problem for the Helmholtz equation

$$
\begin{aligned}
-\Delta U - \kappa^2 n(\mathbf{x}) U &= f(\mathbf{x}) \quad \text{in } \Omega, & -\Delta U^s - \kappa^2 U^s &= 0 \quad \text{in } \Omega^+, \\
\gamma_D^+ U^s - \gamma_D^- U &= g_D \quad \text{on } \Gamma, & \gamma_N^+ U^s - \gamma_N^- U &= g_N \quad \text{on } \Gamma, \\
\frac{\partial U^s}{\partial r} - i\kappa U^s &= o(r^{-1}) \quad \text{uniformly for } r := |\mathbf{x}| \to \infty,
\end{aligned} \tag{1}
$$

with the refractive index $n \in L^\infty(\Omega)$, the source term $f \in H^{-1}(\Omega)$ and the wave number $\kappa > 0$. In the case of excitation by an incident field U^i the generic jump data $g_D \in H^{1/2}(\Gamma)$ and $g_N \in H^{-1/2}(\Gamma)$ evaluate to the Dirichlet and Neumann data of U^i on the boundary Γ:

$$g_D := -\gamma_D U^i\,, \qquad\qquad g_N := -\gamma_N U^i\,.$$

It is well known that the transmission problem (1) has a unique solution $U \in H_{\rm loc}(\mathbb{R}^3,\,\Delta)$.

Symmetric FEM-BEM coupling

The classical approach to the coupling of boundary integral equations and variational formulations in Ω goes back to Costabel [3]. First, integration by parts shows that a solution U of problem (1) will fulfill

$$\mathsf{a}\,(U,V) - (\gamma_N^- U, \gamma_D^- V)_\Gamma = (f,V)_\Omega \quad \forall V \in H^1(\Omega)\,, \tag{2}$$

where we have used the abbreviation

$$\mathsf{a}\,(U,V) := \int_\Omega \mathbf{grad}\; U \cdot \mathbf{grad}\; \overline{V} - \kappa^2 n(\mathbf{x})\, U\, \overline{V}\mathrm{d}\mathbf{x}\,, \qquad U,V \in H^1(\Omega)\,.$$

Using the transmission conditions of (1) together with (2) and the the following symmetric Dirichlet-to-Neumann map

$$\mathrm{DtN}_\kappa^+ \;:=\; -\mathsf{W}_k + \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right) \circ \mathsf{V}_\kappa^{-1} \circ \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)\,, \tag{3}$$

we finally end up with the following equations.

Find $U \in H^1(\Omega)$, $\vartheta \in H^{-1/2}(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\varphi \in H^{-1/2}(\Gamma)$ there holds

$$\begin{aligned} \mathsf{a}(U,V) + (\mathsf{W}_\kappa(\gamma_D^- U), \gamma_D^- V)_\Gamma - ((\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa)(\vartheta), \gamma_D^- V)_\Gamma &= \mathsf{f}(V)\,, \\ (\varphi, (\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa)(\gamma_D^- U))_\Gamma + (\varphi, \mathsf{V}_\kappa(\vartheta))_\Gamma &= \mathsf{g}(\varphi)\,, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \mathsf{f}(V) &:= (f,V)_\Omega - (g_N, \gamma_D^- V)_\Gamma - (\mathsf{W}_\kappa(g_D), \gamma_D^- V)_\Gamma\,, \\ \mathsf{g}(\varphi) &:= (\varphi, (\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id})(g_D))_\Gamma\,. \end{aligned}$$

Although the bilinear form associated with (4) satisfies a Gårding inequality the variational formulation still suffers from resonant frequencies. The numerical experiment displayed on figure 1 clearly indicates the presence of resonant freuquencies in the ϑ-component of the discrete solutions at certain frequencies.

Regularized FEM-BEM coupling

In order to get rid of the resonant frequencies we replace the standard Dirichlet trace by a Robin-type (mixed) boundary condition, which will ensure uniqueness of solutions for the coupled problem. Thus we introduce the following generic trace transformation operator

$$\mathcal{T} := \begin{bmatrix} \mathsf{Id} & i\eta\,\mathsf{M} \\ 0 & \mathsf{Id} \end{bmatrix} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\,, \tag{5}$$

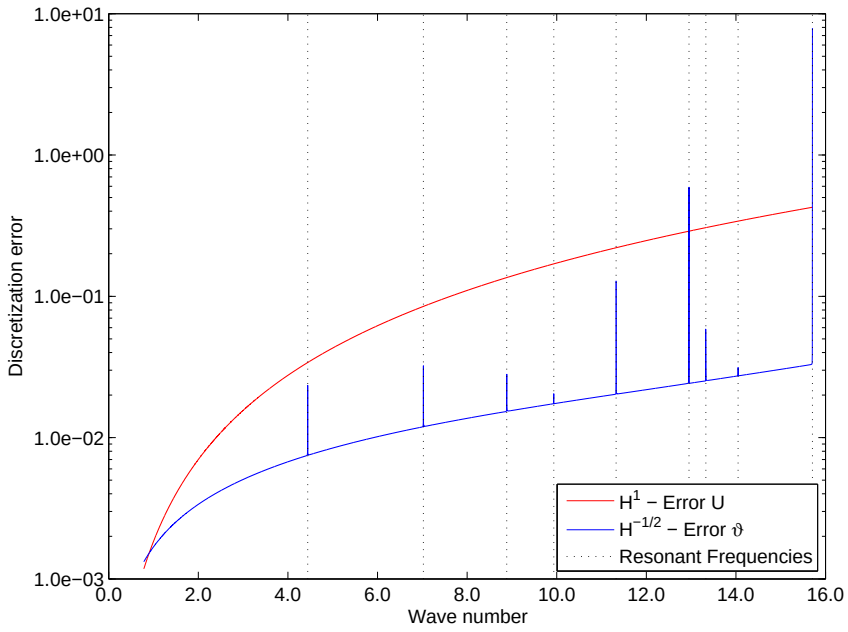


FIGURE 1. Discretization errors for symmetric FEM-BEM coupling on the unit square.

where the regularization operator $\mathsf{M} : H^{-1}(\Gamma) \mapsto H^1(\Gamma)$ is defined by

$$\mathsf{M} := (\mathsf{Id} - \Delta_\Gamma)^{-1} : H^{-1}(\Gamma) \mapsto H^1(\Gamma) \tag{6}$$

and will ensure coercivity of the coupled problem. It is easy to verify that $\mathsf{M} : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$ is compact and that $(\vartheta, \mathsf{M}(\vartheta))_\Gamma > 0$ holds for all $\vartheta \in H^{-1/2}(\Gamma) \setminus \{0\}$, see [2]. For later purpose we also define the sesquilinear form $\mathsf{b} : H^1(\Gamma) \times H^1(\Gamma) \mapsto \mathbb{C}$ by

$$\mathsf{b}(p,q) := (\mathbf{grad}_\Gamma\, p, \mathbf{grad}_\Gamma\, q)_\Gamma + (p,q)_\Gamma\,, \qquad p,q \in H^1(\Gamma)\,. \tag{7}$$

The trace transformation operator (5) can be used to define a generalized version of the Calderón projector, from which the following Dirichlet-to-Neumann map can be derived

$$\begin{aligned} \mathrm{DtN}_\kappa^+ := -\mathsf{W}_\kappa + (\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa) \\ \circ (\mathsf{V}_\kappa + i\eta\mathsf{M} \circ (\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa))^{-1} \circ (\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} + i\eta\mathsf{M} \circ \mathsf{W}_\kappa)\,. \end{aligned} \tag{8}$$

Thus combining (2) with the transmission conditions from (1) and the Dirichet-to-Neumann map (8), we finally end up with the following system of equations.

Find $U \in H^1(\Omega)$, $\vartheta \in H^{-1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\varphi \in H^{-1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$
\begin{aligned}
\mathsf{a}(U,V) + (\mathsf{W}_\kappa(\gamma_D^- U), \gamma_D^- V)_\Gamma + ((\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa)(\vartheta), \gamma_D^- V)_\Gamma &= \mathsf{f}(V)\,, \\
(\varphi, (\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id})(\gamma_D^- U))_\Gamma + (\varphi, \mathsf{V}_\kappa(\vartheta))_\Gamma - i\eta(\varphi, p)_\Gamma &= \mathsf{g}(V)\,, \\
-(\mathsf{W}_\kappa(\gamma_D^- U), q)_\Gamma - ((\mathsf{K}'_\kappa + \tfrac{1}{2}\mathsf{Id})(\vartheta), q)_\Gamma + \mathsf{b}(p,q) &= \mathsf{h}(q)\,,
\end{aligned} \tag{9}
$$

with right hand sides

$$
\begin{aligned}
\mathsf{f}(V) &:= (f,V)_\Omega - (g_N, \gamma_D^- V)_\Gamma - (\mathsf{W}_\kappa(g_D), \gamma_D^- V)_\Gamma\,, \\
\mathsf{g}(\varphi) &:= (\varphi, (\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa)(g_D))_\Gamma\,, \\
\mathsf{h}(q) &:= -(\mathsf{W}_\kappa(g_D), q)_\Gamma\,.
\end{aligned}
$$

It can be shown that the underlying sesquilinear form satisfies a Gårding inequality and that the variational formulation only admits unique solutiones for all frequencies $\kappa > 0$ and all right hand side load data, for details see [4]. Thus we have obtained a variational formulation, which is amenable to a standard Ritz-Galerkin discretization using mixed finite elements. Furthermore, it can be shown that the auxiliary solution component p vanishes everywhere on the interface boundary Γ, hence we conclude that regularization has no impact on the overall convergence rate. The same numerical experiment carried out for the regularized FEM-BEM

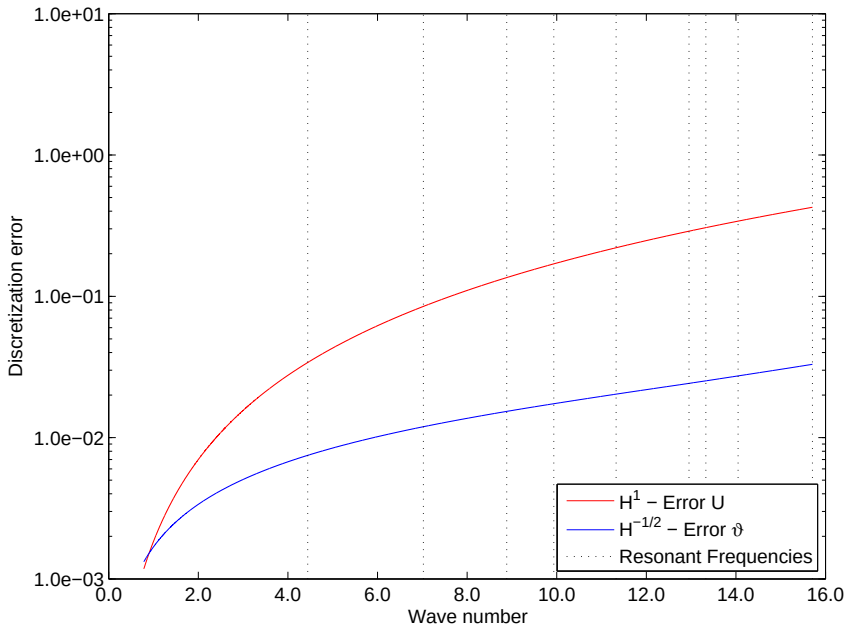


FIGURE 2. Discretization errors for regularized FEM-BEM coupling on the unit square.

coupling shows no resonant frequencies in the ϑ-component, see figure 2.

References

[1] A. Buffa, S. Sauter, *On the acoustic single-layer potential: Stabilization and Fourier analysis*, SIAM J. Sci. Comp., to appear.
[2] A. Buffa, R. Hiptmair, *Regularized combined field integral eqautions*, Numer. Math., **100** (2005), pp. 1-19.
[3] C. Costabel, *Symmetric methods for the coupling of finite elements and boundary elements*, in boundary Elements IX, C. Brebbia, W. Wendland, and G. Kuhn, eds., Springer-Verlag, Berlin, 1987, pp. 411-420.
[4] R. Hiptmair, P. Meury, *Stable FEM-BEM coupling for Helmholtz Transmission Problems*, SIAM J. Numer. Anal., **44** (2006), pp. 2106-2130.
[5] R. Hiptmair, P. Meury *Stable FEM-BEM Coupling for Maxwell Transmission Problems*, Tech. Rep., SAM, ETH Zürich.

Hierarchically and Calderón preconditioned Time Domain Integral Equation Solvers

Eric Michielssen

(joint work with Francesco Andriulli, Hakan Bağcı, Kristof Cools, Femke Olyslager, Giuseppe Vecchi, Francesca Vipiana)

Marching on in time (MOT)-based time domain integral equation (TDIE) solvers represent an increasingly appealing avenue for analyzing transient electromagnetic interactions with large and complex structures. Compared to their differential equation counterparts, MOT-TDIE solvers automatically impose radiation conditions, do not require unknown fields to be discretized throughout homogeneous volumes, and are highly immune to numerical dispersion. In addition, long-standing questions regarding their stability (when applied to the analysis of high-frequency scattering from relatively smooth structures that permit modeling by nondense meshes) have been addressed effectively by the introduction of averaging/filtering techniques [10], implicit time stepping methods [2], smooth and carefully tailored temporal basis functions [6], and space-time Galerkin schemes [5]. Finally, the computational efficiency of MOT-TDIE solvers has soared following the development of plane wave time domain (PWTD) [8] and time domain adaptive integral methods (TD-AIM) [9] capable of rapidly evaluating transient fields due to wideband source constellations.

MOT-TDIE solvers for analyzing electromagnetic scattering from perfect electrically conducting objects, obtained by enforcing electric field boundary conditions, implicitly time advance electric surface current densities by iteratively solving sparse (MOT) systems of equations at all time steps. They do not suffer from a CFL time step constraint and operate on unstructured meshes [4]. In other words, their time step sizes δ_t are dictated solely by excitation bandwidths, never by the spatial resolution δ_r of the meshes on which they operate. Meshes used by MOT-TDIE solvers only need to abide by two constraints: they must resolve (i)

the shortest wavelength in the excitation and (ii) the structure's geometric features. When constraint (i) drives mesh construction – this situation is the norm when analyzing electromagnetic scattering from relatively smooth surfaces subject to high-frequency excitations - the MOT-TDIE system matrix tends to be well-conditioned and the number of iterations per time step small. In contrast, when constraint (ii) dictates surface mesh resolution – this often happens when analyzing electromagnetic radiation from, and guidance on, geometrically intricate and mixed-scale surfaces subject to low- to medium-frequency excitations – the MOT-TDIE system matrix tends to be ill-conditioned and the number of iterations per time step large. In this scenario, the MOT-TDIE system matrix condition number grows without bound as the time step size $\delta_t \to \infty$ (temporal or low frequency breakdown) and/or the mesh resolution $\delta_r \to 0$ (spatial or dense mesh breakdown).

Temporal breakdown phenomena can be mitigated in part by using loop-star (quasi-Helmholtz decomposed) bases, which have been shown to effectively cap the MOT-TDIE system matrix condition number as $\delta_t \to \infty$; unfortunately, in practice, when dense meshes are used, this cap remains prohibitively high. We have shown that the use of properly constructed hierarchical bases in MOT-TDIE solvers yields MOT system matrices that are better-conditioned than those constructed by using loop-star bases; the same hierarchical bases previously were shown to adequately address low-frequency breakdown problems in frequency domain integral equation solvers [1]. Iterative solvers applied to MOT system matrices constructed using these hierarchical bases therefore converge rapidly, irrespective of the time step size. A preliminarily implementation of this approach has been tested on a model of a space shuttle. The condition numbers of the MOT system matrices obtained with the hierarchical basis are plotted in Figure 1 as a function of the time step size δ_t; they are compared with the condition numbers of the MOT matrices obtained using standard Raviart-Thomas basis functions. The advantages of the hierarchical basis are evident.

To date, no cure for spatial breakdown phenomena has been developed. Recently, to remedy this situation, we developed a comprehensive family of semi-analytical regularizers/preconditioners that derive from (new time domain) Calderon identities [3]. The new Calderón-preconditioned MOT-TDIE solvers give rise to well-conditioned MOT systems, irrespective of the mesh resolution. We tested a preliminary implementation of this approach on a sphere of unit electric radius. In Figure 2 the condition numbers of the Calderón preconditioned matrices are plotted versus $1/\delta_r$ and compared with the condition numbers of the non preconditioned MOT-TDIE solver. The Calderón preconditioned formulation compares favorably to the standard MOT-TDIE approach. Furthermore, we showed that the spectral properties of the Calderón regularizer ensure the non-oscillatory (low-frequency) stability of MOT-TDIE solvers. This follows from the fact that the preconditioned operator resolves solenoidal functions of any time signature, including "DC". Finally the use of the Calderón regularizer in a combined field integral equation was studied. The resulting equation is immune to temporal and

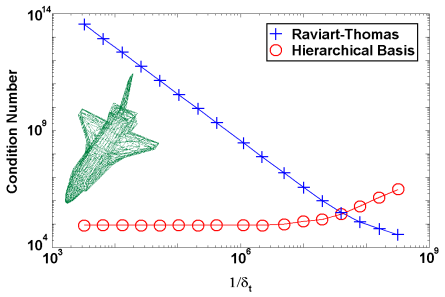


FIGURE 1. Condition numbers of the matrices obtained with the hierarchical basis versus $1/\delta_t$ and comparison with the condition numbers obtained with the standard Raviart-Thomas basis.

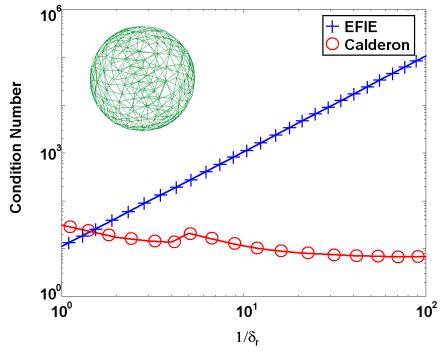


FIGURE 2. Condition numbers of the matrices obtained with the Calderón preconditioning versus $1/\delta_r$ and comparison with the condition numbers obtained with the standard EFIE.

spatial breakdown, and immune to the resonance problems affecting the solutions of standard MOT-TDIE solvers [7].

The new formulations permit the transient analysis of low- to medium-frequency electromagnetic field interactions with geometrically intricate and mixed-scale structures, and can be applied to the analysis of microwave circuits and system-on-a-chip technology.

REFERENCES

[1] F. ANDRIULLI, A. TABACCO, AND G.VECCHI, *A Multiresolution Approach to the Electric Field Integral Equation in Antenna Problems*, SIAM J. Sci. Comput., 29 (2007), pp. 1–21.

[2] R. CICCHETTI AND A. FARAONE, *Transient emission from microstrip interconnects: Theoretical formulation and CAD modeling*, IEEE Tran. on Electr. Comp., 38 (1996), pp. 367–375.

[3] K. COOLS, F. P. ANDRIULLI, AND E. MICHIELSSEN, *Time-Domain Calderón Identities and Preconditioning of the Time-Domain EFIE*, IEEE Antennas and Propagation Society International Symposium, (2006), pp. 2975–2978.

[4] M. Costabel, *Time-dependent problems with the boundary integral equation method*, Chapter 22 of Encyclopedia of Computational Mechanics, Wiley and Sons, (2006).
[5] V. Jandhyala, E. Michielssen, and R. Mittra, *A memory efficient, adaptive algorithm for multipole accelerated capacitance computation in a stratified dielectric medium*, Int. J. Microwave and Millimeter-Wave Computer Aided Engg., 6 (1996), pp. 381–390.
[6] E. Michielssen and A. Boag, *A multilevel matrix decomposition algorithm for analyzing scattering from large structures*, IEEE Trans. Antennas Propagation, 44 (1996), pp. 1086–1093.
[7] B. Shanker, A. A. Ergin, K. Aygun, and E. Michielssen, *Analysis of transient electromagnetic scattering from closed surfaces using a combined field integral equation*, IEEE Trans. Antennas Propagation, 48 (2000), pp. 1064–1074.
[8] C. F. Wang, F. Ling, and J. Jin, *A fast full-wave analysis of scattering and radiation from large finite arrays of microstrip antennas*, IEEE Trans. Antennas Propagation, 46 (1998), pp. 1467–1474.
[9] A. E. Yilmaz, J.-M. Jin, and E. Michielssen, *Time domain adaptive integral method for surface integral equations*, IEEE Trans. Antennas Propagation, 52 (2004), pp. 2692–2708.
[10] J. Zhao and Z. F. Li, *A time-domain full-wave extraction method of frequency-dependent equivalent circuit parameters of multiconductor interconnection lines*, IEEE Trans. Microwave Theory Tech., 45 (1997), pp. 23–31.

A survey of methods using plane-waves to approximate wave propagation problems

Peter Monk

The approximation of Laplace's equation on a reasonably well behaved bounded domain can be accomplished rather efficiently because of the availability of convenient approximating subspaces (e.g. *hp*-finite elements) and essentially optimal matrix solvers (e.g. multigrid methods). If, instead of Laplace's equation, we consider the Helmholtz equation $\Delta u + \kappa^2 u = 0$, the situation is much less rosy. Solutions are typically oscillatory with a wavelength $2\pi/\kappa$ where κ is the wavenumber. Even high order finite elements require at least π degrees of freedom per wave length to capture the waves [1]. Since many applications, such as acoustic noise propagation, require the solution on domains spanning many wavelengths this can result in very large computational problems. For lower order finite elements, even more degrees of freedom per wavelength are needed and "pollution error" increases the computational burden still further [8]. To make matters worse, solvers such as multigrid require a global coarse grid solve on a mesh that resolves the waves in order to retain a mesh independent convergence rate.

In view of the difficulties mentioned above, it is natural to consider other methods for discretizing the Helmholtz equation. Since the mid-1920's (cf. [5]) there have been many attempts to solve partial differential equations using "complete families" of solutions of the underlying equation A typical method is to expand the unknown acoustic field u using a sum of plane-waves $\{\exp(\mathrm{i}\kappa d_j \cdot x)\}_{j=1}^{N}$ where the directions d_j are typically chosen uniformly distributed on the unit disk assuming we are approximating a problem in 2D (alternatively Bessel functions could be used, but plane-waves are easier to manipulate). A solution is then found by choosing the coefficients in the expansion to provide a good fit to the appropriate

boundary condition in a convenient norm. For example we may wish to impose the impedance boundary condition $\frac{\partial u}{\partial n} - \mathrm{i}\kappa u = g$ where $g \in L^2(\Gamma)$ and n is the unit outward normal to the domain. Elementary examples show that the resulting linear system is often ill-conditioned when larger numbers of basis functions are used and that, depending on the exact solution, the convergence rate can be rather slow. To some extent these problems can be counteracted by subdividing the domain Ω and using different local expansions on each sub-domain.

Probably the best know method of this type is the Partition of Unity Finite Element Method (PUFEM) originating in Melenk's thesis [11, 12] (a rich source of information on plane-wave based methods). This uses a standard weak formulation of the boundary value problem but instead of using a general finite element subspace of $H^1(\Omega)$, the PUFEM uses finite element "hat" basis functions multiplied by plane-waves (a different expansion for each node in the mesh). Melenk shows that the interpolation error depends on the local approximation properties of the plane-waves, and proves convergence of the method as the mesh is refined (h-convergence). Another common and convenient refinement path is to fix the mesh and increase the number of plane-waves per node, but a proof of convergence in this case is an open problem. The method has been tested in a number of studies (see e.g. [9]) that show the need for a fast method to compute the integrals appearing in matrix assembly (see the contribution of J. Trevelyan on page 354in this volume). In addition, if Ω contains materials of differing refractive index, the method must be modified. One possible modification is to move to a discontinuous basis and Lagrange multipliers to enforce continuity where the refractive index jumps [10].

For complex materials in which the refractive index changes at many interfaces it is reasonable to abandon the PUFEM continuous element philosophy altogether. Tezaur and Farhat [17] have taken this route using plane-wave basis functions piecewise on each element in the mesh, and weakly enforcing continuity by Lagrange multipliers that are also plane waves. Their interesting numerical results show that increasing the number of plane waves per element results in a method of increasing order. An analysis of this method is not currently available.

An alternative to Lagrange multipliers is to use least squares to minimize the jump in the numerical solution and it's normal derivative across inter-element boundaries. This method can be quite successful in practice [15] and an initial error analysis is available [16, 13]. An advantage is that the method results in a Hermitian positive definite matrix problem. However, matrix conditioning remains a problem, so that if conjugate gradients are used to solve the linear system, the iteration must often be terminated early to avoid poor results.

A related method, called the Ultra Weak Variational Formulation (UWVF) of the Helmholtz equation, has been proposed and analyzed in [3]. This method enforces approximate continuity between elements using a weak formulation that can be understood as a discontinuous Galerkin method [4, 2]. Exploiting the link between UWVF and upwind discontinuous Galerkin methods, a proof of global convergence is given in [2]. Several numerical studies have extended the original

UWVF [7, 6] and I have found it to be a very useful technique for academic scattering problems provided due care is taken to control the ill-conditioning still present in the plane wave basis. A particular advantage is that simple BiCGstab can be applied to solve the linear system (although the iteration count is mesh dependent).

Underlying all the convergence studies mentioned above is the approximation properties of plane waves. Results for convergence under mesh refinement can be found in [3] (but using rather severe norms on the data) and other results are in [12, 13]. Approximation of plane-waves by planes-waves is considered in [14]. It would be very useful to have theoretical results on convergence rates and inverse estimates more akin to those in polynomial based finite element theory.

The methods discussed here all extend to Maxwell's equations and linear elasticity. This talk was restricted to methods using a mesh covering Ω, but there has been considerable success in applying plane-waves in the discretization of boundary integral equations (see the abstracts by I. Graham and J. Trevelyan on pages 308, 354 of this volume).

At present it is by no means clear that the use of plane-waves or other local solutions of the Helmholtz equation will provide a clear advantage over traditional finite elements. Numerical evidence suggests that at least $N = O(h\kappa)$ directions for the plane-waves must be chosen on each element of radius h to provide any accuracy. While this is better than for finite elements (since, in 2D, N degrees of freedom are used per element rather than $O(N^2)$ for a finite element method) it is unlikely to allow these methods to be used for very high frequency problems unless a better (perhaps adaptive) choice of plane-wave directions can be made. However there are several interesting mathematical problems to be studied, and, at least in the case of the UWVF and least-squares methods, the numerical linear algebra problem is simplified.

References

[1] M. Ainsworth, *Discrete dispersion relation for hp-version finite element approximation at high wave number*, SIAM J. Numer. Anal., 42 (2004), pp. 553–575.

[2] A. Buffa and P. Monk, *Error estimates for the ultra weak variational formulation of the Helmholtz equation*. In preparation.

[3] O. Cessenat and B. Després, *Application of the ultra-weak variational formulation of elliptic PDEs to the 2-dimensional Helmholtz problem*, SIAM J. Numer. Anal., 35 (1998), pp. 255–99.

[4] G. Gabard, *Discontinuous Galerkin methods with plane waves for time-harmonic problems*. to appear.

[5] I. Herrera, *Boundary Methods: an Algebraic Theory*, Pitman, 1984.

[6] T. Huttunen, J. Kaipio, and P. Monk, *The perfectly matched layer for the ultra weak variational formulation of the 3D Helmholtz equation*, Int. J. Numer. Meth. Eng., 61 (2004), pp. 1072–92.

[7] T. Huttunen, P. Monk, and J. Kaipio, *Computational aspects of the Ultra Weak Variational Formulation*, J. Comput. Phys., 182 (2002), pp. 27–46.

[8] F. Ihlenburg, *Finite Element Analysis of Acoustic Scattering*, vol. 132 of Applied Mathematical Sciences, Springer, Berlin, 1998.

[9] O. Laghrouche, P. Bettess, and R. Astley, *Modelling of short wave diffraction problems using approximating systems of plane waves*, Int. J. Numer. Meth. Eng., 54 (2002), pp. 1501–33.
[10] O. Laghrouche, P. Bettess, E. Perrey-Debain, and J. Trevelyan, *Wave interpolation finite elements for Helmholtz problems with jumps in the wave speed*, Comput. Meth. Appl. Mech. Eng., 194 (2005), pp. 367–81.
[11] J. Melenk, *On generalized finite element methods*, PhD thesis, University of Maryland, College Park, MD, 1995.
[12] J. Melenk and I. Babuška, *The partition of unity finite element method: Basic theory and applications*, Comput. Meth. Appl. Mech. Eng., 139 (1996), pp. 289–314.
[13] P. Monk and D. Wang, *A least squares method for the Helmholtz equation*, Comput. Meth. Appl. Mech. Eng., 175 (1999), pp. 121–36.
[14] E. Perrey-Debain, *Plane wave decomposition in the unit disc: Convergence estimates and computational aspects*, J. Comput. Appl. Math., 193 (2006), pp. 140–56.
[15] B. Pluymers, W.Desmet, D. Vandepitte, and P. Sas, *On the use of a wave based prediction technique for steady-state structural-acoustic radiation analysis*, CMES, 7 (2005), pp. 173–83.
[16] M. Stojek, *Least-squares Trefftz-type elements for the Helmholtz equation*, Int. J. Numer. Meth. Eng., 41 (1998), pp. 831–49.
[17] R. Tezaur and C. Farhat, *Three-dimensional discontinuous Galerkin elements with plane waves and Lagrange multipliers for the solution of mid-frequency Helmholtz problems*, Int. J. Numer. Meth. Eng., 66 (2006), pp. 796–815.

A Discontinuous Galerkin Method Based on a Space-time Expansion

Claus-Dieter Munz
(joint work with Gregor Gassner, Frieder Lörcher, Michael Dumbser)

Overview. Our research objectives are the construction and application of accurate and efficient schemes for linear and nonlinear wave propagation. We presented the recently developed space-time expansion discontinuous Galerkin (STE-DG) approach for a general system of linear wave equations and its extension to the unsteady compressible Navier-Stokes equations. The basis of the scheme is a variational formulation of the equations under consideration. The spatial polynomial of the DG approach is expanded in time using the so called Cauchy-Kovalevskaya (CK) procedure. With a polynomial of order N in space the CK procedure generates an approximation of order N in time as well yielding a scheme being accurate of order N+1 in space and time. The locality and the space-time nature of the presented method give the interesting feature that the time steps may be different in each grid cell. Hence, we drop the common global time step and propose for a time-dependent problem that any grid cell runs with its own time step determined by the local stability restriction. In spite of the local time steps the scheme is fully explicit and as in the DG approach the polynomial order could be chosen arbitrarily, the scheme is theoretically of arbitrary order of accuracy in space and time for transient calculations. The fields of applications are computational fluid dynamics (CFD) in combination with computational aeroacoustics (CAA) as well as computational electromagnetics.

A Discontinuous Galerkin Scheme Based on a Space-Time Expansion. The class of DG schemes is quite interesting for practical calculations in complex geometries, because the DG schemes really reproduce the order of accuracy even on distorted unstructured grids. The approximate solution in the discontinuous Galerkin schemes may be discontinuous at grid cell interfaces which allows the approximation of strong gradients on coarse grids or even discontinuities. Usually, high order time discretization of DG schemes for unsteady computations is done separately from space discretization by the method of lines approach using a Runge Kutta scheme (RK-DG).

Our actual topic of research are explicit discontinuous Galerkin (DG) schemes which provide the time approximation in one single step. Ideas of the so called ADER approach (Arbitrary order using DERivatives) were extended to the discontinuous Galerkin framework in [3]. Based on these ideas we propose the space-time expansion discontinuous Galerkin scheme which are based on a space-time Taylor expansion about the barycenter $\vec{x}_i$ of a grid cell Q_i and about the old time level t_n:

$$U(\vec{x},t) = U(\vec{x}_i,t_n) + \sum_{j=1}^{N_i} \frac{1}{j!}((t-t_n)\frac{\partial}{\partial t} + (\vec{x}-\vec{x}_i)\cdot\vec{\nabla})^j\, U(\vec{x}_i,t_n). \tag{1}$$

With this space time Taylor series it is possible to approximate U at all space time points $(\vec{x},t) \in Q_i \times [t_n;t_{n+1}]$, which is needed for the evaluation of the volume and surface integrals. While the space derivatives are known from the approximate solution at time t_n, the time and mixed space-time derivatives are the problem. These are replaced by pure space derivatives using the differential equation several times. This is called the Cauchy-Kovalevskaya or Lax-Wendroff procedure, see, e.g. [4] for more details. This expansion gives the values for the Gaussian quadrature in the case of nonlinear problems by which the surface and volume space-time integrals in the variational formulation are approximated. In the linear case the polynomials may be integrated analytically resulting in quadrature-free DG schemes.

The so-called space-time expansion discontinuous Galerkin scheme (STE-DG) scheme may be locally adapted to the behavior of the solution. This includes h-adaptivity, which is especially efficient due to the possibility of using nonconforming mesh nodes, and p-adaptivity allowing the degree of the local approximation space to vary from grid cell to grid cell. The STE-DG scheme is also locally adapted with respect to the time approximation, see [1], [2]. Each grid cell may evolve in time with a local time step corresponding to the local stability restrictions. This technique strongly increases the performance for multi-scale problems or distorted grids where the local time steps strongly vary in the computational domain.

Applications The STE-DG scheme was developed for a general system of linear wave equations as the Maxwell equations. The Cauchy-Kovalevskaya procedure may be written for linear equations in a very general form. It is possible to implement the method to a computer code such that the Jacobian matrices

of the system of evolution equations are input values. The same programm is then able to solve any linear system of wave equations. Numerical simulations of several applications are performed including noise propagation in turbo engines, electromagnetic and seismic wave propagation. The local time stepping guarantees efficiency of the simulation on the entire domain. Large grid cells in the far field allow larger time steps there.

References

[1] F. Lörcher and G. Gassner and C.-D. Munz, *A discontinuous Galerkin scheme based on a space-time expansion. I. Inviscid compressible flow in one space dimension*, J. of Sci. Comp., doi:0.1007/s10915-007-9128-x.

[2] G. Gassner and F. Lörcher and C.-D. Munz, *A discontinuous Galerkin scheme based on a space-time expansion. II. Viscous flow equaqtions in multi dimensions*, submitted to J. of Sci. Comp.

[3] M. Dumbser and C.-D. Munz, *Building blocks for arbitrary high order discontinuous Galerkin schemes*, J. of Sci. Comp. **27** (2006), 215–230.

[4] F. Lörcher and C.-D. Munz, *Lax-Wendroff-type schemes of arbitrary order in several space dimensions*, IMA J. Numer. Anal., doi:10.1093/imanum/drl031.

Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems

Serge Piperno

The accurate modeling of systems involving electromagnetic waves, in particular through the resolution of the time-domain Maxwell equations on space grids, is of strategic interest. Although the FDTD method is still prominent, the use of structured grids is a huge constraint when a complex geometry is to be analyzed. Many different types of methods have been proposed in order to handle complex geometries and heterogeneous configurations by dealing with unstructured tetrahedral meshes (Finite Element Time-Domain methods [1], Finite Volume Time-Domain methods [2]), which all partially fail in being at the same time efficient, easily extendible to high orders of accuracy via *hp*-refinement, stable, and energy-conserving. The Discontinuous Galerkin (DG) methods enjoy an impressive favor nowadays and are now used in various applications, taking advantage of their ability to achieve a high order of accuracy by simply choosing suitable basis functions [3, 19] on unstructured meshes. Nevertheless, all the most popular explicit time-domain methods cannot deal easily with configurations involving small details in the geometry: explicit methods based on finite element meshes (FVTD, FETD, DGTD) would be able to handle locally refined unstructured grids, but the time-integration would remain a concern. Implicit time-schemes are expensive, while explicit time-schemes have a stability constraint on the time-step directly related to the smallest elements in the mesh.

We propose an original strategy to overcome these difficulties in the particular context where an energy conservation property is verified [5]. The idea is to introduce locally implicit time-integration or explicit local time-stepping. Algorithmic

solutions have already been proposed in the geometrical integration literature for Hamiltonian systems [6, 9] and applied in time-domain wave propagation problems [7, 8]. The Maxwell's equations can be written as an infinite-dimensional Hamiltonian system of PDEs. In order to fully exploit the properties of symplectic schemes, the most commonly used technique to design 'symplectic' numerical methods consists in (1) discretizing the Maxwell's equations with a numerical method leading to a finite-dimensional Hamiltonian system, (2) considering this system has an input for symplectic time-integrators. It can be proved that a DGTD space discretization based on centered non-dissipative numerical fluxes leads to such a discrete finite-dimensional Hamiltonian system [11]. It takes the form $\mathbb{M}^\epsilon \partial_t \mathbb{E} = \mathbb{S}\mathbb{H}$, $\mathbb{M}^\mu \partial_t \mathbb{H} = -{}^t\mathbb{S}\mathbb{E}$, where $\mathbb{M}^\epsilon$, $\mathbb{M}^\mu$ are symmetric positive definite matrices (block diagonal matrices with blocks related to each element given by integrals of the form $\int_K {}^t\vec{\varphi}_{K,i}\bar{\bar{\varepsilon}}_K\vec{\varphi}_{K,j}$ and $\int_K {}^t\vec{\varphi}_{K,i}\bar{\bar{\mu}}_K\vec{\varphi}_{K,j}$). If the spatial domain is decomposed into two subdomains, then the DGTD equations read

$$\left\{\begin{array}{l} \mathbb{M}_1^\epsilon \partial_t \mathbb{E}_1 = \mathbb{S}_1\mathbb{H}_1 - \mathbb{B}\mathbb{H}_2, \\ \mathbb{M}_1^\mu \partial_t \mathbb{H}_1 = -{}^t\mathbb{S}_1\mathbb{E}_1 + \mathbb{B}\mathbb{E}_2, \end{array}\right. \qquad \left\{\begin{array}{l} \mathbb{M}_2^\epsilon \partial_t \mathbb{E}_2 = \mathbb{S}_2\mathbb{H}_2 - {}^t\mathbb{B}\mathbb{H}_1, \\ \mathbb{M}_2^\mu \partial_t \mathbb{H}_2 = -{}^t\mathbb{S}_2\mathbb{E}_2 + {}^t\mathbb{B}\mathbb{E}_1, \end{array}\right.$$

in function of $\mathbb{E} \equiv (\mathbb{E}_1,\ \mathbb{E}_2)$ and $\mathbb{H} \equiv (\mathbb{H}_1,\ \mathbb{H}_2)$, for some matrix $\mathbb{B}$,

A locally-implicit symplectic scheme We assume the set of elements has been partitioned into two classes: one made of particularly small elements (implicitly time-advanced), the other one gathering all other elements (explicitly time-advanced). We propose the following implicit-explicit algorithm: starting from unknowns at time $t^n = n\Delta t$, we perform the 3 following sub-steps:

$$(1) \qquad \left\{\begin{array}{l} \mathbb{M}_1^\mu \mathbb{H}_1^{n+\frac{1}{2}} = \mathbb{M}_1^\mu \mathbb{H}_1^n + \frac{\Delta t}{2}(-{}^t\mathbb{S}_1\mathbb{E}_1^n + \mathbb{B}\mathbb{E}_2^n), \\ \mathbb{M}_1^\epsilon \mathbb{E}_1^{n+\frac{1}{2}} = \mathbb{M}_1^\epsilon \mathbb{E}_1^n + \frac{\Delta t}{2}(\mathbb{S}_1\mathbb{H}_1^{n+\frac{1}{2}} - \mathbb{B}\mathbb{H}_2^n), \\ \mathbb{M}_2^\epsilon \mathbb{E}_2^{n+1} = \mathbb{M}_2^\epsilon \mathbb{E}_2^n + \Delta t(\mathbb{S}_2\frac{\mathbb{H}_2^n + \mathbb{H}_2^{n+1}}{2} - {}^t\mathbb{B}\mathbb{H}_1^{n+\frac{1}{2}}), \\ \mathbb{M}_2^\mu \mathbb{H}_2^{n+1} = \mathbb{M}_2^\mu \mathbb{H}_2^n + \Delta t(-{}^t\mathbb{S}_2\frac{\mathbb{E}_2^n + \mathbb{E}_2^{n+1}}{2} + {}^t\mathbb{B}\mathbb{E}_1^{n+\frac{1}{2}}), \\ \mathbb{M}_1^\epsilon \mathbb{E}_1^{n+1} = \mathbb{M}_1^\epsilon \mathbb{E}_1^{n+\frac{1}{2}} + \frac{\Delta t}{2}(\mathbb{S}_1\mathbb{H}_1^{n+\frac{1}{2}} - \mathbb{B}\mathbb{H}_2^{n+1}), \\ \mathbb{M}_1^\mu \mathbb{H}_1^{n+1} = \mathbb{M}_1^\mu \mathbb{H}_1^{n+\frac{1}{2}} + \frac{\Delta t}{2}(-{}^t\mathbb{S}_1\mathbb{E}_1^{n+1} + \mathbb{B}\mathbb{E}_2^{n+1}). \end{array}\right.$$

This algorithm is obviously reversible. It can be sequentially read as made of five operations, the central one corresponding to the implicit midpoint-rule for the "implicit" subdomain. If $\mathbb{B} = \mathbb{O}_d$, this algorithm reduces to the juxtaposition of the explicit Verlet-method and the implicit midpoint-rule. The stability of the algorithm (1) can be shown using an energy approach. We have the following

Lemma 1. *The following quadratic form $\mathcal{E}^n$ of numerical unknowns $\mathbb{E}_e^n$, $\mathbb{E}_i^n$, $\mathbb{H}_e^n$, and $\mathbb{H}_i^n$ is exactly conserved* [11] *(i.e. $\mathcal{E}^{n+1} = \mathcal{E}^n$) through Algorithm (1):*

$$(2) \qquad \mathcal{E}^n = \mathcal{E}_e^n + \mathcal{E}_i^n + \mathcal{E}_c^n \ \textit{with} \ \left\{\begin{array}{l} \mathcal{E}_e^n = {}^t\mathbb{E}_e^n\mathbb{M}_e^\epsilon\mathbb{E}_e^n + {}^t\mathbb{H}_e^{n+\frac{1}{2}}\mathbb{M}_e^\mu\mathbb{H}_e^{n-\frac{1}{2}}, \\ \mathcal{E}_i^n = {}^t\mathbb{E}_i^n\mathbb{M}_i^\epsilon\mathbb{E}_i^n + {}^t\mathbb{H}_i^n\mathbb{M}_i^\mu\mathbb{H}_i^n, \\ \mathcal{E}_c^n = -\frac{\Delta t^2}{4}{}^t\mathbb{H}_i^n{}^t\mathbb{B}\,(\mathbb{M}_e^\epsilon)^{-1}\,\mathbb{B}\mathbb{H}_i^n. \end{array}\right.$$

Remark. Thus Algorithm (1) is clearly stable for Δt small enough. A more closer investigation is required to determine a sufficient condition on Δt for having

a stable coupled scheme. It is apparently the one of the explicit subdomain (for standard 2D triangular meshes on acoustics).

A multi-scale fully-explicit symplectic scheme. A fully explicit algorithm inspired from [10] can be proposed, where different classes of space elements are time-advanced using different time steps. The definition of the algorithms with N classes using time steps of the form $\Delta t/2^{N-k}$ can be given in a recursive way [11]. Let us denote by $R^N(\tau)$ the algorithm for advancing in time N classes over the time interval $\tau > 0$. We decide that $R^1(\tau)$ is the Verlet method with $\Delta t = \tau$. Then $R^{N+1}(\tau)$ is recursively constructed as:

(1) advance all elements with class $k \leq N$ with $R^N(\Delta t/2)$; if required, use values at time t^n for unknowns in elements of class $N+1$;
(2) advance all elements with class $k = N+1$ with the Verlet method (i.e. $R^1(\Delta t)$); if required, use values at time $t^n + \Delta t/2$ for unknowns in elements of class $k \leq N$;
(3) advance all elements with class $k \leq N$ with $R^N(\Delta t/2)$; if required, use values at time t^{n+1} for unknowns in elements of class $N+1$;

The reader can check that the scheme is reversible. The proof of stability for such a scheme is built using energy conservation. In the simple case of $R^2(\Delta t)$, it can be proved that an energy is exactly conserved [11]. We have the following

Theorem 1. *The algorithm $R^2(\Delta t)$ conserves an energy and is stable under the following condition (fulfilled for Δt small enough):*

$$\left\{ \begin{array}{l} M_X \equiv \mathbb{M}_1^\epsilon - \dfrac{\Delta t^2}{16}\mathbb{S}_1 \left(\mathbb{M}_1^\mu\right)^{-1} {}^t\mathbb{S}_1 \text{ is positive definite} \\ M_Y \equiv \mathbb{M}_2^\epsilon - \dfrac{\Delta t^2}{4}\mathbb{S}_2 \left(\mathbb{M}_2^\mu\right)^{-1} {}^t\mathbb{S}_2 \text{ is positive definite} \\ \rho\begin{pmatrix} {}^t\mathbb{P}_1 M_X^{-1}\mathbb{P}_1 & -2\mathbb{P}_1 M_X^{-1}\mathbb{Q}_1 \\ -2{}^t\mathbb{Q}_1 M_X^{-1\,t}\mathbb{P}_1 & 4{}^t B M_X^{-1}\mathbb{Q}_1 \end{pmatrix} \Delta t^2 \;<\; \dfrac{16}{\max(\rho(M_X^{-1}), \rho(M_Y^{-1}))}, \end{array} \right.$$

where $\mathbb{P}_1 = \begin{pmatrix} 0 & S_1 \\ -{}^t S_1 & 0 \end{pmatrix}$, $\mathbb{Q}_1 = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}$ *and $\rho(M)$ denotes the spectral radius of M.*

It seems very difficult to prove a sufficient stability condition on Δt for $R^2(\Delta t)$. Although numerical results on standard 2D triangular meshes show there is no additional constraint on Δt than those derived from local CFL-like stability conditions, it has been shown on 1D cases that stability can breakdown when the time step and the space size are refined in the same place. This might mean that the stability condition on Δt is no longer local. Investigations are under way to obtain an explicit sufficient stability condition on Δt.

References

[1] A. Elmkies and P. Joly, *Éléments finis d'arête et condensation de masse pour les équations de Maxwell: le cas de dimension 3*, C. R. Acad. Sci. Paris Sér. I Math. **325** 11 (1997), 1217–1222.
[2] M. Remaki, *A new finite volume scheme for solving Maxwell's system*, COMPEL **19** (2000), 913–931.

[3] J. Hesthaven and C. Teng, *Stable spectral methods on tetrahedral elements*, SIAM J. Sci. Comput. **21** (2000), 2352–2380.
[4] J. Hesthaven and T. Warburton, *Nodal high-order methods on unstructured grids. I: Time-domain solution of Maxwell's equations*, J. Comput. Phys. **181** (2002), 186–221.
[5] L. Fezoui, S. Lanteri, S. Lohrengel and S. Piperno, *Convergence and Stability of a Discontinuous Galerkin Time-Domain method for the 3D heterogeneous Maxwell equations on unstructured meshes*, ESAIM M2AN, **39** (2005), 1149–1176.
[6] J. M. Sanz-Serna and M. P. Calvo, *Numerical Hamiltonian Problems*, Chapman and Hall, London, U.K. (1994).
[7] T. Hirono, W. W. Lui, and K. Yokoyama, *Time-domain simulation of electromagnetic field using a symplectic integrator*, IEEE Microwave Guided Wave Lett., **7** (1997), 279–281.
[8] R. Rieben, D. White, and G. Rodrigue, *High-Order Symplectic Integration Methods for Finite Element Solutions to Time Dependent Maxwell Equations*, IEEE Trans. Antennas and Propagation, **52** (2004), 2190–2195.
[9] W. Huang and B. Leimkuhler, *The adaptive Verlet method*, SIAM J. Sci. Comput., **18** (1997), 239–256.
[10] D. J. Hardy, D. I. Okunbor and R. D. Skeel, *Symplectic variable step size integration for N-body problems*, Appl. Numer. Math., **29** (1999), 19–30.
[11] S. Piperno, *Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems*, ESAIM M2AN, **40** (2006), 815–841.

Equilibrated Error Estimator for Maxwell's Equations

Joachim Schöberl
(joint work with Dietrich Braess)

Recently, a posteriori error estimates without constants have attracted much interest; see Ainsworth and Oden [1], Neittaanmäki and Repin [17], Vejchodský [20], Luce and Wohlmuth [14] and also Ladeveze and Leguillon [13]. At first glance they look like estimators which use local Neumann problems as introduced by Bank and Weiser [5], but they are based on a comparison of primal and dual forms of the variational problems. Following Prager and Synge (1949) in the special case of the Poisson equation $-\Delta u = f$ in Ω, one compares a finite element approximation $u_h \in H^1(\Omega)$ and a function $\sigma \in H(\mathrm{div})$ which satisfies the equilibrium condition $\mathrm{div}\,\sigma + f = 0$. In principle, the latter can be obtained via mixed methods, but in practical computations a feasible function σ is constructed by an equilibration of ∇u_h.

The equilibration can be done by solving local problems; see [1, Chapter 6.4]. The solution of local problems by polynomials of sufficiently high order is avoided in [20] by the combination with a variant of the hypercircle method and in [14] by the introduction of a dual mesh. We will go a different way. A small portion of the error that results from the data oscillation is estimated in the classical manner. So the local problems become transparent, and a generalization to other types of elliptic problems is now natural (although not trivial). In the 2D case there is even a simple geometrical interpretation of the resulting equilibration procedure.

In the presented work [6] we also establish a posteriori error estimators for Maxwell's equations with similar properties. There is an analogue to the result

of Prager and Synge although we have to deal with different Sobolev spaces and edge elements. The equilibration in $H(\mathrm{curl})$, however, is more involved since the splitting of the residual currents into local divergence-free currents has to be done with more constraints. Moreover, the constraints refer to currents on geometrical objects with different dimensions.

To overcome these obstacles we proceed as we have shown for the Poisson equation. We extend the Raviart–Thomas elements and Nédélec elements to finite element spaces such that the differential operators curl and div act on distributions. We show that the differential operators and the extended spaces still form exact sequences. The sequences generalize the de Rham sequences and the discrete analoga that were frequently used in the last years for constructing and understanding new finite element spaces [2, 3, 12].

References

[1] Ainsworth, M. and Oden, T.J. (2000): A Posteriori Error Estimation in Finite Element Analysis. Wiley, Chichester.

[2] Arnold, D.N., Falk, R.S. and Winther, R. (2000): Multigrid in $H(div)$ and $H(curl)$. Numer. Math. 85, 197–217

[3] Arnold, D.N., Falk, R.S. and Winther, R. (2006): Differential complexes and stability of finite element methods. I. The de Rham complex. In "Proceedings of the IMA workshop on Compatible Spatial Discretizations for PDE" pp. 23–46

[4] Arnold, D.N., Falk, R.S. and Winther, R. (2006): Finite element exterior calculus, homological techniques, and applications. Acta Numerica (to appear)

[5] Bank, R.E. and Weiser, A. (1985): Some a posteriori error estimators for elliptic partial differential equations. Math. Comp. 44, 283–301

[6] D. Braess and J. Schöberl. Equilibrated Residual Error Estimator for Maxwell Equations (submitted), available as Report 2006-19, Johann Radon Institute for Computational and Applied Mathematics (RICAM)

[7] Beck, R., Hiptmair, R., Hoppe, R., and Wohlmuth, B. (2000): Residual based a posteriori error estimators for eddy current computations. M^2AN, 34(1), 159–182

[8] Bossavit, A. (2005): Discretization of electromagnetic problems: The "General finite differences", In "Handbook of Numerical Analysis. Volume XIII (Ciarlet, P.G. ed.), pp. 105–197. Elsevier, Amsterdam

[9] Braess, D. (2001): Finite Elements: Theory, Fast Solvers and Applications in Solid Mechanics. Cambridge University Press.

[10] Brezzi, F. and Fortin, M. (1991): Mixed and Hybrid Finite Element Methods. Springer-Verlag, Berlin – Heidelberg – New York

[11] Demkowicz, L. (2006): Computing with *hp* Finite Elements. I. One- and Two-Dimensional Elliptic and Maxwell Problems. CRC Press, Taylor and Francis.

[12] Hiptmair, R. (2002): Finite elements in computational electromagnetism Acta Numerica 2002, 237–339

[13] Ladeveze, P. and Leguillon, D. (1983): Error estimate procedure in the finite element method and applications. SIAM J. Numer. Anal. 20, 485–509

[14] Luce, R. and Wohlmuth, B. (2004): A local a posteriori error estimator based on equilibrated fluxes. SIAM J. Numer. Anal. 42, 1394–1414

[15] Monk, P. (2003): Finite Element Methods for Maxwell's Equations. Clarendon Press, Oxford

[16] Nédélec, J.C. (1986): A new family of mixed finite elements in $\mathbb{R}^3$. Numer. Math. 50, 57–81.

[17] Neittaanmäki, P. and Repin, S. (2004): Reliable Methods for Computer Simulation. Error control and a posteriori estimates. Elsevier. Amsterdam

[18] Prager, W. and Synge, J.L. (1949): Approximations in elasticity based on the concept of function spaces. Quart. Appl. Math. 5, 241–269
[19] Schöberl, J. (2006): A posteriori error estimates for Maxwell equations. Math. Comp. (to appear),
[20] Vejchodský, T. (2004): Local a posteriori error estimator based on the hypercircle method. In " Proc. of the European Congress on Computational Methods in Applied Sciences and Engineering" (ECCOMAS 2004), Jyvaskylä, Finland.
[21] Verfürth, R. (1996): A Review of A Posteriori Error Esimation and Adaptive Mesh-Refinement Techniques. Wiley-Teubner, Chichester – New York – Stuttgart

High Order Finite Element Maxwell Solvers for Particle In Cell Codes

Eric Sonnendrücker

Particle-In-Cell (PIC) codes have become an essential tool for the simulation of plasmas and charged particle beams. However, because of their robustness most available codes are still based today on the Yee scheme for Maxwell's equations on a structured uniform grid and the Cloud In Cell formulation of Birdsall for the coupling between particles and mesh [1]. We propose here a general mathematical framework for electromagnetic PIC codes that can be used on structured as well as unstructured or hybrid grids. It can accomodate high order Maxwell solvers and arbitrary smoothing functions for the particles. Moreover, it includes an exact discrete continuity equation. We notice that the lowest order version of our formulation corresponds on a rectangular grid to the traditional Cloud-In-Cell (CIC) version of the PIC method coupled to a Yee Maxwell solver and the Villasenor-Buneman charge deposition scheme [5].

We consider the Maxwell equations on a two dimensional domain Ω for which the electromagnetic field obeys to two sets of decoupled equations, the first of which involving the (E_x, E_y, B_z) components (TE mode) and the second involving the (E_z, B_x, B_y) components (TM mode). We present only the first system, the other can be dealt with in a similar manner. This system reads

$$\frac{\partial \mathbf{E}}{\partial t} - c^2 \mathbf{curl}\, B = -\frac{1}{\varepsilon_0}\mathbf{J}, \tag{1}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl} \mathbf{E} = 0, \tag{2}$$

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{3}$$

where $\mathbf{E} = (E_x, E_y)^T$, $B = B_z$, $\mathbf{curl}\, B_z = (\partial_y B_z, -\partial_x B_z)^T$, $\operatorname{curl} \mathbf{E} = \partial_x E_y - \partial_y E_x$, and $\operatorname{div} \mathbf{E} = \partial_x E_x + \partial_y E_y$.

For simplicity, we assume that the whole boundary is perfectly conducting so that $\mathbf{E} \cdot \tau = 0$, where τ is the unit vector tangential to the boundary. Moreover we assume that there are no particles at the vicinity of the boundary so that $\mathbf{J} \cdot \mathbf{n} = 0$ and $\mathbf{E} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal to the boundary.

Gauss' law (3) which links the divergence of the electric field to the charge density is a consequence of the two evolution equations called Ampere's law (1)

and Faraday's law (2) provided it is satisfied at the initial time and the continuity equation $\partial_t \rho + \operatorname{div} \mathbf{J} = 0$ is verified.

There are two ways of choosing the appropriate function spaces where we are going to defined our mixed variational formulation. We choose to use $H(\operatorname{curl}, \Omega)$ for $\mathbf{E}$ and $L^2(\Omega)$ for B. Then the mixed variational formulation reads: Find $(\mathbf{E}, B) \in H(\operatorname{curl}, \Omega) \times L^2(\Omega)$ such that

$$\frac{d}{dt} \int_\Omega \mathbf{E} \cdot \mathbf{F} + c^2 \int_\Omega B_z \operatorname{curl} \mathbf{F} = -\frac{1}{\varepsilon_0} \int_\Omega \mathbf{J} \cdot \mathbf{F} \quad \forall \mathbf{F} \in H(\operatorname{curl}, \Omega). \tag{4}$$

$$\frac{d}{dt} \int_\Omega B_z \, C + \int_\Omega \operatorname{curl} \mathbf{E} \, C = 0 \quad \forall C \in L^2(\Omega). \tag{5}$$

$$- \int_\Omega \mathbf{E} \cdot \nabla q = \frac{1}{\varepsilon_0} \int_\Omega \rho \, q \quad \forall q \in H^1(\Omega). \tag{6}$$

If the continuity equation is satisfied, (6) is a consequence of (4).

Let us now introduce conforming discretizations of our function spaces. We shall denote by $W_k^0 \subset H^1(\Omega)$, $W_k^1 \subset H(\operatorname{curl}, \Omega)$, and $W_k^2 \subset L^2(\Omega)$, the finite dimensional spaces used for the k-th order ($k \geq 1$)) discretization. The three discrete spaces need in addition to verify that

$$W_k^0 \stackrel{grad}{\longrightarrow} W_k^1 \stackrel{\operatorname{curl}}{\longrightarrow} W_k^2$$

is an exact sequence. This is obtained by defining on quads

$$W_k^0 = \{q_h \in C^0(\omega) \mid q_{h|K} \in \mathbb{Q}_k(K)\},$$

$$W_k^1 = \{(F_{x,h}, F_{y,h}) \in H(\operatorname{curl}, \Omega)) \mid F_{x,h|K} \in \mathbb{Q}_{k-1,k}(K), F_{y,h|K} \mathbb{Q}_{k,k-1}(K)\},$$

$$W_k^2 = \{q_h \in L^2(\Omega) \mid q_{h|K} \in \mathbb{Q}_{k-1}(K)\},$$

and in the same manner on triangles W_k^0 is the space based on Lagrange $\mathbb{P}_k$ finite elements, W_k^1 is the space edge elements of degree k [4] and W_k^2 is a space of discontinuous finite elements locally $\mathbb{P}_{k-1}$. Triangles and quads can be organized as desired so as to ensure H^1 conformity for W_k^0 and $H(curl)$ conformity for W_k^1. As our finite dimensional subspaces are included in the continuous spaces, we simply get the discrete variational formulations by replacing the continuous spaces by their finite dimensional finite element analogs. Let us denote by $\mathbb{E}$, $\mathbb{B}$ the column vectors containing the degrees of freedom. And define the matrices

$$M_1 = ((\int_\Omega \varphi_i^1 \varphi_j^1))_{1 \leq i \leq N_1, 1 \leq j \leq N_1}, \; M_2 = ((\int_\Omega \varphi_i^2 \varphi_j^2))_{1 \leq i \leq N_2, 1 \leq j \leq N_2},$$

$$R = ((\int_\Omega \operatorname{curl} (\varphi_i^1) \varphi_j^2))_{1 \leq i \leq N_1, 1 \leq j \leq N_2},$$

Then the matrix formulations of Ampere's and Faraday's laws read

$$M_1 \frac{d}{dt} \mathbb{E} - c^2 R^T \mathbb{B} = -\frac{1}{\varepsilon_0} \mathcal{J}, \quad M_2 \frac{d}{dt} \mathbb{B} + R \mathbb{E} = 0.$$

In a PIC method the particle distribution function is approximated by

$$f_h(\mathbf{x},\mathbf{p},t) = \sum_k w_k \delta(\mathbf{x}-\mathbf{x}_k(t))\delta(\mathbf{p}-\mathbf{p}_k(t)).$$

Then the charge and current densities are defined by

$$\rho_h(\mathbf{x},t) = \int f_h(\mathbf{x},\mathbf{p},t)\,d\mathbf{p} = \sum_k w_k \delta(\mathbf{x}-\mathbf{x}_k(t)),$$

$$\mathbf{J}_h(\mathbf{x},t) = \int f_h(\mathbf{x},\mathbf{p},t)\mathbf{v}(\mathbf{p})\,d\mathbf{p} = \sum_k w_k \mathbf{v}_k(t)\delta(\mathbf{x}-\mathbf{x}_k(t)),$$

where the velocity $\mathbf{v}(\mathbf{p})$ is linked to the momentum by $\mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}}{\gamma m}$, $\gamma = \sqrt{1+\frac{\mathbf{p}^2}{c^2}}$ is the Lorentz factor, and c the velocity of light.

These expressions of the charge and current densities if used directly in the discrete variational formulation do not verify the continuity equation. Hence, Gauss' law will not be satisfied automatically. Such formulations are yet used in practice, but they need a correction scheme to yield accurate simulations. Such schemes have been discussed by several authors.

Let us now propose a charge conserving formulation for the current density keeping the formulation for the charge density

$$\int_\Omega \mathbf{J}_h^{n+\frac{1}{2}} \cdot \varphi_j^1\,d\mathbf{x} = \sum_k \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{v}_k^{n+\frac{1}{2}} \cdot \varphi_j^1(\mathbf{x}_k(t))\,dt. \tag{7}$$

Indeed as $\nabla\varphi^0 \in W_1^k$, we can prove that

$$\int_\Omega \mathbf{J}_h^{n+\frac{1}{2}} \cdot \nabla\varphi_j^0\,d\mathbf{x} = \frac{1}{\Delta t}\Big(\int_\Omega \rho_h^{n+1}\varphi_j^0\,d\mathbf{x} - \int_\Omega \rho_h^n\varphi_j^0\,d\mathbf{x}\Big),$$

which is the discrete formulation of the continuity equation in our framework.

A straightforward computation shows that the basis functions of our first order space W_1^1 and W_1^2 on squares of side h are of the following form

$$\varphi^1_{i+\frac{1}{2},j} = \begin{pmatrix} S_h^0(x-x_i)S_h^1(y-y_j) \\ 0 \end{pmatrix}, \qquad \varphi^1_{i,j+\frac{1}{2}} = \begin{pmatrix} 0 \\ S_h^1(x-x_i)S_h^0(y-y_j) \end{pmatrix},$$

$$\varphi^2_{i+\frac{1}{2},j+\frac{1}{2}} = S_h^0(x-x_i)S_h^0(y-y_j).$$

In this case our finite element solver with mass lumping, i.e. based on the Cohen-Monk elements [2], is algebraically identical to Yee's scheme, and the charge conserving expression of the current density reads

$$J_{x,i+\frac{1}{2},j} = \frac{v_x^{n+\frac{1}{2}}}{\Delta t}\int_{t_n}^{t_{n+1}} S_h^0(x_i-x_k(t))S_h^1(y_j-y_k(t))\,dt,$$

$$J_{y,i,j+\frac{1}{2}} = \frac{v_y^{n+\frac{1}{2}}}{\Delta t}\int_{t_n}^{t_{n+1}} S_h^1(x_i-x_k(t))S_h^0(y_j-y_k(t))\,dt,$$

which is exactly the charge conserving expression given by Villasenor and Buneman.

References

[1] C. K. Birdsall, A. B. Langdon, *Plasma physics via computer simulation*, Institute of Physics, Bristol (1991) p. 359.
[2] G. Cohen, P. Monk, Gauss point mass lumping schemes for Maxwell's equations. Numer. Methods Partial Differential Equations 14 (1998), no. 1, 63–88.
[3] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numerica (2002), pp. 237-339.
[4] J.-C. Nédélec, A new family of mixed finite elements in R^3. Numer. Math. 50 (1986), no. 1, 57–81.
[5] J. Villasenor, O. Buneman, Rigorous charge conservation for local electromagnetic field solvers, Computer Physics Communications 69 (1992) 306-316.

Numerical steepest descent evaluation of 2D partition of unity boundary integrals for Helmholtz problems

Jon Trevelyan

(joint work with Mick Honnor, Daan Huybrechs)

We address frequency domain problems of linear wave scattering by a sound hard, non-convex obstacle of boundary Γ, governed by the Helmholtz equation

$$\Delta\phi + k^2\phi = 0 \tag{1}$$

$$\frac{\partial\phi(x)}{\partial\nu} = 0, x \in \Gamma \tag{2}$$

in which we seek the potential $\phi(x) \in \mathbb{C}, x \in \mathbb{R}^2$ and k is the wavenumber. We consider the solution of (1) in the unbounded 2D region Ω resulting from the scattering of an incident plane wave $\phi^i = Ae^{ika\cdot x}$.

It is well known that in order to capture accurately a wave field, the nodal spacing in a BEM or FEM discretisation is limited to approximately $\lambda/10$, where λ is the wavelength under consideration. Methods for circumventing these restrictions have been the subject of considerable interest in recent years. Bettess (2004) provides a useful review. One popular approach is the Fast Multipole Method (FMM) (e.g Darve (2000)). Other notable methods include the use of FFT and stationary phase methods (e.g. Bruno (2001)).

A further class of methods is based on the inclusion of the wave nature of the solution in some way into the formulation. One example is the Ultraweak Variational Formulation (Cessenat & Després (1998), Huttunen *et al.* (2002), Gamallo & Astley (2006)). Abboud *et al.* (1995) included the incident wave in the shape function using an integral equation approach. This can be highly effective at higher frequencies, but is unsuitable for non-convex scatterers.

Melenk & Babuska (1996) presented the Partition of Unity Method (PUM) as a general technique in which the approximation space is enriched by the inclusion of *sets of* functions known to populate the solution. Laghrouche & Bettess (2000),

Farhat *et al.* (2000), Ortiz & Sanchez (2000) and Mayer & Mandel (1997) have thus enriched finite elements using sets of plane waves. Boundary integral PUM implementations (PUBEM) have been presented by de la Bourdonnaye (1994), under the name Microlocal Discretisation, Perrey-Debain *et al.* (2002, 2003, 2004), Chandler-Wilde & Langdon and Arden *et al.* (2005). The PUBEM is characterised by a plane wave expansion of the potential so that the boundary integral equation yields a system of linear equations in the unknown amplitudes. However, the integrals in the 2D PUBEM, in which we integrate over an element of boundary Γ_j in the conventional isoparametric mapping $\xi(x)(\xi \in (-1,1), x \in \Gamma_j)$, are typically of the highly oscillatory form

$$I(x', \Gamma_j, m) = \int_{-1}^{1} \frac{\partial G(x', \xi)}{\partial \nu} e^{ikd_{jm} \cdot x(\xi)} N_j(\xi) J(\xi) d\xi \tag{3}$$

where $J(\xi)$ is the Jacobian of the mapping. Unfortunately, the present authors have found that the asymptotic methods generate extremely complicated expressions for the coefficients when applied to integrals of the form (3). Huybrechs & Vandewalle (2006) have presented an integration scheme based on the classical numerical steepest descent method (Bender & Orszag (1978)). This involves changing the path of integration to one *in the complex plane* and over which the integral becomes *non-oscillatory* and *exponentially decaying*. In the current work we apply this to integrals of the form (3).

We denote with h the complex plane whose real axis coincides with ξ, and find a suitable path in h to simplify the evaluation of the integral. For reasons of generality we assume the conventional notation:

$$I = \int_{-1}^{1} f(\xi) e^{ikg(\xi)} d\xi \tag{4}$$

For the PUBEM integrals we find

$$f(\xi) = \frac{k}{4} \frac{\partial r}{\partial \nu} \sqrt{\frac{a_1}{kr}} N_j J \left(\frac{8}{kr} q_1 e^{i(-b_1)} - p_1 e^{i\left(-b_1 + \frac{\pi}{2}\right)} \right) \tag{5}$$

$$g(\xi) = r + d_{jm} \cdot x \tag{6}$$

where p_1, q_1, a_1, b_1 come from the series approximations to Bessel functions provided by Press *et al.* (2003). This allows the integrals to be treated using numerical steepest descent. The integration paths for these functions are the loci satisfying

$$g(h) = g(\xi) + ip \tag{7}$$

and are determined using a linear Taylor Series approximation followed by Newton-Raphson iteration. Substitution of (7) into (4) yields an exponentially decaying integral with respect to p. Numerical tests show that for the majority of PUBEM integrals, the integration may be performed over each element using

only two steepest descent paths, using only 8 Gauss-Laguerre points on each path. Care needs to be taken with regard to stationary points at which $g'(h) = 0$ on the real interval, for which a further coordinate transformation in the spirit of Telles (1987) is used. A similar treatment is used for cases in which $g'(h)$ is small at the ends of the element.

Numerical tests on scattering by a cylinder of radius a, with wavenumber k such that $ka = 240\pi$, show that the scheme presented herein uses $365\text{x}10^6$ integration points, comparing favourably with the $8.8\text{x}10^9$ points used for comparable accuracy in previous works with conventional Gauss-Legendre on the real interval.

References

[1] T. Abboud, J.C. Nédélec & B. Zhou, *Improvement of the integral equation method for high frequency problems.* In Third Int. Conf. on Mathematical Aspects of Wave Propagation Phenomena, SIAM, Philadelphia, U.S., (1995), 178-187.

[2] S. Arden, S.N. Chandler-Wilde and S. Langdon, *A collocation method for high frequency scattering by convex polygons*, Reading University Numerical Analysis Report 7/05, (2005).

[3] C.M. Bender & S.A. Orszag , *Advanced Mathematical Methods for Scientists and Engineers*, (1978), McGraw-Hill, New York.

[4] P. Bettess, *Short-wave scattering: problems and techniques*, Phil. Trans. Royal Soc. London A, **362**, (2004), 421-443.

[5] O.P. Bruno and L.A. Kunyansky, *Surface Scattering in Three Dimensions: an Accelerated High-Order Solver*, Phil. Trans Royal Soc. London, Ser. A., **457** (2001), 1-14.

[6] O. Cessenat and B. Després, *Application of the ultra-weak variational formulation of elliptic PDE's to the two-dimensional Helmholtz problem*, SIAM J. Num. An., **35** (1998), 255-299.

[7] E. Darve, *The fast multipole method: numerical implementation*, J. Comput. Phys. **160** (2000), 195-240.

[8] A. de la Bourdonnaye (1994), *Convergence of the approximation of wave functions by oscillatory functions in the high frequency limit*, C.R. de l'Acad. des Sc., Série I, **318**, 765-768.

[9] C. Farhat, I. Harari & L. Franca, *The discontinuous enrichment method*, Report No. CU-CAS-00-20, University of Colorado (Boulder), College of Engineering (2000).

[10] P. Gamallo and R.J. Astley, *The partition of unity finite element method for short wave acoustic propagation on non-uniform potential flows*, International Journal for Numerical Methods in Engineering, **65** (2006), 425-444.

[11] T. Huttunen, P. Monk & J.P. Kaipio, *Computational Aspects of the Ultra Weak Variational Formulation*, J. Comput. Phys., **182** (2002), 27-46.

[12] D. Huybrechs & S. Vandewalle, *On the evaluation of highly oscillatory integrals by analytic continuation*, SIAM J. Numer. Anal., **44** (2006), 1026-1048.

[13] O. Laghrouche and P. Bettess, *Short wave modelling using special finite elements*, Journal of Computational Acoustics, **8** (2000), 189-210.

[14] S. Langdon and S.N. Chandler-Wilde, *Implementation of a Boundary Element Method for High Frequency Scattering by Convex Polygons, in Advances in Boundary Integral Methods*, K. Chen (ed.), University of Liverpool (2005).

[15] P. Mayer and J. Mandel, *The finite ray element method for the Helmholtz equation of scattering: first numerical experiments*, UCD/CCM Report 111 (1997).

[16] J.M. Melenk & I. Babuška, *The Partition of Unity Finite Element Method. Basic Theory and Applications*, Comput Meth Appl Mech Eg., **139** (1996), 289-314.

[17] P. Ortiz & E. Sanchez, *An improved partition of unity finite element model for diffraction problems*, Int. J. Num. Meth. in Engg, **50** (2000), 2727-2740.

[18] E. Perrey-Debain, J. Trevelyan & P. Bettess, *New special wave boundary elements for short wave problems*, Communications in Numerical Methods in Engineering, **18** (2002), 259-268.

[19] E. Perrey-Debain, J. Trevelyan & P. Bettess, *Plane wave interpolation in direct collocation boundary element method for radiation and wave scattering: numerical aspects and applications*, Journal of Sound and Vibration, **261** (2003), 839-858.
[20] E. Perrey-Debain, O. Laghrouche, P. Bettess & J. Trevelyan, *Plane-wave basis finite elements and boundary elements for three-dimensional wave scattering*, Phil. Trans. Royal Soc. London A, **362** (2004a), 561-577.
[21] W.H. Press, S.A. Teukolsky, W.T. Vetterling and B.P. Flannery, *Numerical Recipes in Fortran 77*, Second Edition, Cambridge University Press, Cambridge (2003).
[22] J.C.F. Telles, *A self-adaptive co-ordinate transformation for efficient numerical evaluation of general boundary element integrals*, Int. J. Num. Meth. in Engg, **24** (1987), 959-973.

A Survey of Discontinuous Galerkin Methods for Time-Domain Electromagnetics

Tim Warburton

The time-domain Maxwell's equations of electromagnetics present a number of impressive challenges for available numerical methods. Typical simulations require propagation of high frequency travelling wave solutions over distances that are long compared to the length scales (say wavelengths) intrinsic to the solution. Furthermore, the travelling wavefronts may encounter complicated scattering surfaces or pass through abrupt material changes in the course of propagation. The solution may also exhibit singular behaviour at rough features of these scatterers. Furthermore, some existing numerical methods, most notably the standard finite element method, are known to admit spurious solutions for internal cavity resonance simulations. Finally, for external scattering problems special care must be taken as the solutions propagates through the outer boundary of the domain.

We first introduce the first order non-dimensionalized Maxwell's equations

$$\frac{\partial \vec{H}}{\partial t} + \nabla \times \vec{E} = 0 \, , \; \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H} = 0 \, , \tag{1}$$

and suitable boundary conditions, where $\vec{H}$ is the magnetic field and $\vec{E}$ the electric field. A natural choice of finite element spaces are compatible subspaces of the natural energy space for these fields is $H\,(\mathrm{curl}) \times H\,(\mathrm{div})$ discretized by edge and face elements. But a major drawback of this approach is that the mass matrices induced by these matrices have globally coupled inverses, and this means that explicit time integration requires the inversion of these matrices at every step. This is a particular obstacle to performing efficient simulations on large scale distributed computing platforms.

Seeking an alternative to the mixed finite elements methods, a pseudo-spectral multi-domain method that uses quadrilateral elements applied to electromagnetic wave scattering was proposed by Gottlieb and Hesthaven [9], and extended to three-dimensional computations using hexahedral elements in Yang [22] and Yang & Hesthaven [23]. Inter-element continuity was imposed strongly through a characteristic treatment at boundary degrees of freedom and filters were employed to maintain stability. This method also required a careful treatment at element

corners to maintain stability, and the restriction to quadrilateral elements limited the geometric flexibility of the method. However, the ability to use high-order elements without incurring significant computational linear algebra overheads was very attractive. The basic philosophy of this approach was sound but the lack of a variational framework was responsible for the filtering and delicate inter-element treatment to maintain stability.

It was thus natural to consider the discontinuous Galerkin time-domain (DGTD) method since it has similar locality properties. The strong characteristic treatment of the pseudo-spectral method is replaced by a flux treatment to weakly enforce continuity of the solution fields between elements and at the domain boundaries. Originally developed for neutron transport, the discontinuous Galerkin methods have found numerous applications [15] because of their suitability for solving hyperbolic partial differential equations. We will briefly introduce DG methods for Maxwell's equations for the first order PDE's, and also for the second order curl-curl form. For the first order form of the equations most discretized DGTD methods can be recovered from the following variational statement:

Assuming a partition of non-overlapping elements $\Omega_h = \bigcup D^k$ we find an approximation for the magnetic and electric fields, $\left(\vec{H}, \vec{E}\right) \in V^h \times W^h$ such that

$$\left(\vec{\phi}, \frac{\partial \vec{H}}{\partial t} + \nabla \times \vec{E}\right)_{D^k} = \left(\vec{\phi}, -\vec{n} \times \left(\vec{E}^* - \vec{E}^-\right)\right)_{\partial D^k} , \tag{2}$$

$$\left(\vec{\psi}, \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H}\right)_{D^k} = \left(\vec{\psi}, \vec{n} \times \left(\vec{H}^* - \vec{H}^-\right)\right)_{\partial D^k} , \tag{3}$$

for all $\left(\vec{\phi}, \vec{\psi}\right) \in V^h \times W^h$, where $\vec{H}^-$ and $\vec{E}^-$ are to be understood as the trace of the numerical magnetic and electromagnetic fields from inside the k'th element D^k, $\vec{n}$ is the outwards facing normal on the faces of D^k, and the space $V^h \times W^h$ is a product of broken spaces typically consisting of polynomial spaces on each element with no assumption of continuity.

To close this elemental description of the Maxwell's equations we weakly encode boundary conditions on the element boundary through the star variables $\vec{H}^*$ and $\vec{E}^*$. There is a certain amount of freedom in how to specify the star variables. The simplest choice that yields a stable method is to use a central average of the negative and positive traces of the fields to form the star variables [17, 1, 8] , i.e.

$$\vec{H}^* := \frac{\vec{H}^+ + \vec{H}^-}{2} , \ \vec{E}^* := \frac{\vec{E}^+ + \vec{E}^-}{2} . \tag{4}$$

A trivial L^2 stability analysis yields

$$\frac{d\epsilon_h}{dt} := \frac{d}{dt} \sum_k \left\{ \left(\vec{H}, \vec{H}\right)_{D^k} + \left(\vec{E}, \vec{E}\right)_{D^k} \right\} = 0 , \tag{5}$$

and thus for this choice of boundary states, a numerical measure of total energy of the DGTD solution is conserved. However, this method yields sub-optimal convergence rates and unfortunately provides no control over the inherently unphysical solutions inherent in the choice of broken polynomial spaces.

An alternative choice for the boundary state was proposed independently by Warburton [21], Woodruff, Kopriva, & Hussaini, [16], and Hesthaven & Warburton [11, 12, 13]. Based on the upwind flux treatment used in the finite volume solver for Maxwell's equations by Shankar, Hall, & Mohammadian [20] the star variables are obtained by solving the one-dimensional Riemann problem at the interface based on the two trace states

$$\vec{H}^* := \frac{\vec{H}^+ + \vec{H}^-}{2} + \vec{n} \times \frac{\vec{E}^+ - \vec{E}^-}{2}, \tag{6}$$

$$\vec{E}^* := \frac{\vec{E}^+ + \vec{E}^-}{2} - \vec{n} \times \frac{\vec{H}^+ - \vec{H}^-}{2}, \tag{7}$$

and a simple energy analysis demonstrates that the total numerical energy for this method decays in time in proportion to the square of the element boundary seminorm of the tangential jumps in the solution. Thus the method is dissipative, but fortunately it only dissipates a solution if it has unphysical jumps between elements in contrast to the straight forward averaging approach which does not dissipate these unphysical solutions.

The variational framework of the DGTD method allows for a relatively general choice of elements to discretize the domain. Hesthaven & Warburton [11, 12, 13] used high order nodal triangular and tetrahedral elements and Woodruff, Kopriva, & Hussaini [16] demonstrated how quadrilateral elements with hanging nodes can be used. At this workshop Cohen demonstrated DGTD combined with meshes formed from subdividing tetrahedral elements into hexahedral elements (see Cohen, Ferrieres, and Pernet [7] for further details) thus maintaining high geometric flexibility and the tensor-product efficiency of the hexahedral elements.

No special time stepping methods needed to be developed for DGTD, and this is reflected in the number of papers exploring alternative schemes. Hesthaven & Warburton [11] combined the upwind DGTD with the five stage, fourth order, low storage RK of Carpenter & Kennedy [2] because of its advantageous stepping region. Recently, Chen, Cockburn, & Reitich [3] discussed the positive aspects of using the strong stability preserving Runge-Kutta methods balancing arbitrary order in time with extra computational work (because of extra stages) and extra storage requirements together with a more involved handling of time dependent forcing to avoid stage order reduction. Piperno proposes the use of symplectic time integrators to maintain conservation of a discrete energy in time for the central flux DGTD [18, 19].

The variational framework in Equation 2 was extended by Hesthaven & Warburton [14] with the addition of extra fields on the trace of each face of each

element. The DGTD variational statement was modified to

$$\left(\vec{\phi}, \frac{\partial \vec{H}}{\partial t} + \nabla \times \vec{E}\right)_{D^k} = \left(\vec{\phi}, -\vec{n} \times \left(\vec{E}^* - \vec{E}^-\right)\right)_{\partial D^k} , \tag{8}$$

$$\left(\vec{\psi}, \frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{H}\right)_{D^k} = \left(\vec{\psi}, \vec{n} \times \left(\vec{Q}^- + \vec{H}^* - \vec{H}^-\right)\right)_{\partial D^k} , \tag{9}$$

$$\left(\vec{\chi}, \frac{\partial \vec{Q}}{\partial t}\right)_{\partial D^{k,f}} = \left(\vec{\chi}, \frac{\tau}{2}\vec{n} \times \left(\vec{E}^+ - \vec{E}^-\right)\right)_{\partial D^{k,f}} , \tag{10}$$

where an additional vector field $\vec{Q} \in X^h$ is sought such that this is true for all $\vec{\chi} \in X^h$. Here $D^{k,f}$ denotes face f of the element k, and X^h is the union of the element boundary traces of the broken polynomial spaces in W^h. It was shown how it is possible to use central fluxes for $\vec{E}$ and $\vec{H}$ and instead of upwind damping the penalty τ penalizes the energy available for spurious modes inherent in the central flux DGTD. The drawback to this approach is the possibility of having to choose smaller time steps, dt, to satisfy a modified CFL condition. This modified DGTD scheme is very similar to the interior penalty discontinuous Galerkin method for the second order Maxwell's equations proposed by Grote, Schneebeli, & Schötzau [10]. Interestingly their analysis demonstrates optimal order accuracy in L^2 and this result may extend to the modified DGTD of Hesthaven & Warburton [14].

Thus far all the DGTD formulations have been based on conventional local polynomial spaces on each element. Cockburn, Li, & Shu [6] used the flexibility of the choice of basis available to DGTD by using a locally divergence free polynomial basis for the magnetic field. With this choice it was found to be advisable to use a Lax-Friedrichs flux formulation, and this extra penalization term on jumps in the normal component of the magnetic fields mean this method falls out of the general variational statement of Equation 2. Recently, an additional alternative method was introduced by Chung & Engquist [4, 5] using local basis functions for the electric and magnetic fields that are defined on their own staggered meshes. Their a priori analysis shows an optimal order accuracy.

In conclusion, their is a healthy diversity of DGTD methods and extensions for solution of the time-domain Maxwell equations in first order and second order forms.

References

[1] N. Canouet, L. Fezoui, and S. Piperno, *Discontinuous Galerkin Time-Domain solution of Maxwell's equations on locally-refined nonconforming Cartesian grids*, COMPEL, **24**, (2005), 1381–1401.

[2] M. H. Carpenter and C. A. Kennedy, *Fourth order 2N-storage Runge-Kutta scheme*, NASA-TM-109112, NASA Langley Research Center, VA, (1994).

[3] M-H. Chen, B. Cockburn, F. Reitich, *High-order RKDG Methods for Computational Electromagnetics*, J. of Sci. Comp., **22-23**, (2005), 205–226.

[4] E. T. Chung and B. Engquist. *Staggered grid discontinuous galerkin methods for Maxwell's equations*, 7th International conference on mathematical and numerical aspects of waves, (2005).
[5] E. T. Chung and B. Engquist. *A new discontinuous galerkin method for Maxwell's equations*, The 8th US national congress on computational mechanics, (2005).
[6] B. Cockburn, F. Li, and C.-W. Shu, *Locally Divergence-Free Discontinuous Galerkin Methods for the Maxwell Equations*, J. Comp. Phys., **194**, (2004), 588-610.
[7] G. Cohen, X. Ferrieres, and S. Pernet, *A spatial high-order hexahedral discontinuous Galerkin method to solve Maxwell's equations in time domain* J. Comp. Phys., **217**, (2006), 340–363.
[8] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno, *Convergence and Stability of a Discontinuous Galerkin Time-Domain method for the 3D heterogeneous Maxwell equations on unstructured meshes*, M2AN, **39** (2005), 1149–1176.
[9] D. Gottlieb and J. S. Hesthaven, *Spectral Simulations of Electromagnetic Wave Scattering*, J. Comp. Phys., **135** (1990), 216–230.
[10] M. Grote, A. Schneebeli, and D. Schötzau, *Interior Penalty Discontinuous Galerkin Method for MaxwellÕs Equations: Optimal L^2 Norm Error Estimates*, IMA J. Numer. Analysis, In press.
[11] J. S. Hesthaven and T. Warburton, *Nodal High-Order Methods on Unstructured Grids: I. Time-Domain Solution of Maxwell's Equations*, J. Comp. Phys., **181** (2002), 186–221.
[12] J. S. Hesthaven and T. Warburton, *Discontinuous Galerkin Methods for the Time-Domain MaxwellÕs Equations*, Applied Computational Electromagnetics Society Newsletter, **19** (2004).
[13] J. S. Hesthaven and T. Warburton, *High-Order Accurate Methods for Time-domain Electromagnetics*, Comp. Mod. Engin. Sci. **5** (2004), 395–408.
[14] J. S. Hesthaven and T. Warburton, *High-order nodal discontinuous Galerkin methods for the Maxwell eigenvalue problem*, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, **362** (2004), 493–524.
[15] J. S. Hesthaven and T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*. Springer Texts in Applied Mathematics, Springer Verlag, New York (in preparation 2007).
[16] D. A. Kopriva, S. L. Woodruff, and M. Hussaini, *Discontinuous spectral element approximation of Maxwell's equations* Discontinuous Galerkin Methods, B. Cockburn, G. E. Karniadakis, and C.-W. Shu (Eds.), Lecture Notes in Computational Science and Engineering, **11** (1999), Springer.
[17] S. Piperno, M. Remaki, and L. Fezoui, *A non-diffusive finite volume scheme for the 3D Maxwell equations on unstructured meshes*, SIAM J. Num. Anal., **39**, (2002), 2089–2108.
[18] S. Piperno. *Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems* , ECCOMAS CFD 2006 Conference.
[19] S. Piperno, *Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems*, Oberwolfach Conference on Computational Electromagnetics and Acoustics, Mathematisches Forschungsinstitut Oberwolfach, Germany, (2007)
[20] V. Shankar, W. F. Hall, and A. H. Mohammadian, *A CFD-Based Finite-Volume Procedure for Computalional Electromagnetics*, Interdisciplinary Applications of CFD Methods, AIAA Paper **89** (1987).
[21] T. Warburton, *Application of the discontinuous galerkin method to Maxwell's equations using unstructured polymorphic hp-finite elements* Discontinuous Galerkin Methods, B. Cockburn, G. E. Karniadakis, and C.-W. Shu (Eds.), Lecture Notes in Computational Science and Engineering, **11** (1999), Springer.
[22] B. Yang, *Spectral Methods and Absorbing Boundary Conditions for Maxwell's Equations*, Ph.D. Dissertation, Division of Applied Mathematics, Brown University, (1997).
[23] B. Yang and J. S. Hesthaven, *Multidomain Pseudospectral Computation of Maxwell's Equations in 3-D General Curvilinear Coordinates*, Appl. Numer. Math. **33** (2000), 281–289.

Poisson Preconditioning for Discrete H(curl)-elliptic Boundary Value Problems

J. Xu
(joint work with R. Hiptmair)

On $\Omega \subset \mathbb{R}^3$ we consider the $\mathbf{H}(\mathbf{curl}, \Omega)$-elliptic variational model problem

$$(1) \quad \mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + \tau\,(\mathbf{u}, \mathbf{v})_0 = (\mathbf{f}, \mathbf{v})_0\,,\ \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)\,,$$

with source term $\mathbf{f} \in (L^2(\Omega))^3$ and $\tau > 0$. Problems of this type arise in the modeling of magnetic diffusion phenomena (eddy current model). The Galerkin finite element discretization of (1) by means of low order edge elements, see [5, Ch. 3], leads to large sparse linear systems of equations. These are ill-conditioned for fine meshes, which entails using preconditioners to speed up CG-type iterative solvers. Their performance should be independent of mesh resolution. Geometric multigrid preconditioners are known to achieve this [1, 4, 8]. However, they require hierarchies of nested meshes. Algebraic multigrid (AMG) methods that dispense with this requirement have been proposed in [11, 3], but they noticeably deteriorate on fine meshes, let alone permit a comprehensive theoretical analysis.

We propose a preconditioner for the discretized version of (1) that is very easy to implement and allows a rigorous proof of its asymptotic mesh independence also on highly non-uniform meshes. We merely take for granted that good preconditioners for discrete 2nd-order elliptic boundary value problems are available. Our approach builds on ideas first outlined in [2].

I. Auxiliary space preconditioning. Let $a(\cdot,\cdot)$ denote the inner product of the Hilbert space V. An auxiliary space preconditioner for the variational problem $u \in V :\ a(u,v) = f(v)\ \forall v \in V$ requires the following ingredients: an auxiliary space W equipped with an inner product $d(\cdot,\cdot)$, a continuous transfer operator $\Pi : W \mapsto V$, and a smoother $\mathsf{S} : V \mapsto V$ associated with another inner product $s(\cdot,\cdot)$ on V.

Theorem 1 ([12])**.** *If*

$$\exists c_s > 0 : \quad a(v,v) \le c_s s(v,v) \quad \forall v \in V\,,$$

and for every $v \in V$ there are $v_0 \in V$ and $w \in W$ such that $v = v_0 + \Pi w$ and

$$(2) \qquad s(v_0, v_0) + a(\Pi w, \Pi w) \le c_0^2 a(v,v)\,,$$

for some $c_0 > 0$ independent of v, then we find the following bound for the spectral condition number

$$\kappa(\mathsf{BA}) \le c_0^2 (c_s + \|\Pi\|^2)\,,$$

where the auxiliary space preconditioner *is defined as*

$$\mathsf{B} := \mathsf{S}^{-1} + \Pi \circ \mathsf{D}^{-1} \circ \Pi^*\,,$$

and $\mathsf{D} : W \mapsto W'$ is the operator induced by $d(\cdot,\cdot)$.

II. Regular decompositions. As can be seen from (2), uniformly stable splittings are at the heart of auxiliary space preconditioning. For the space $\mathbf{H}_0(\mathbf{curl},\Omega)$ the so-called regular *Helmholtz-type decomposition* have become a key tool in mathematical and numerical analysis, see [5, Sect. 2] for details:

Theorem 2. *There are continuous linear operators* $\mathsf{R} : \mathbf{H}_0(\mathbf{curl},\Omega) \mapsto (H_0^1(\Omega))^3$, $\mathsf{Z} : \mathbf{H}_0(\mathbf{curl},\Omega) \mapsto H_0^1(\Omega)$ *such that* $\mathsf{R} + \mathbf{grad}\,\mathsf{Z} = \mathsf{Id}$ *on* $\mathbf{H}_0(\mathbf{curl},\Omega)$ *and* $\mathsf{R}u = 0 \Leftrightarrow \mathbf{curl}\,u = 0$.

This confirms that the splitting

$$\mathbf{H}_0(\mathbf{curl},\Omega) = (H_0^1(\Omega))^3 + \mathbf{grad}\,H_0^1(\Omega) \tag{3}$$

can be done in a stable way. Thus, in light of Thm. 1 the statement of Thm. 2 can be read as follows:

$$\mathsf{B} := \mathsf{I} \circ (\mathsf{Id} - \boldsymbol{\Delta})^{-1} \circ \mathsf{I}^* + \mathbf{grad} \circ (\mathsf{Id} + (\mathbf{grad})^*\,\mathbf{grad})^{-1} \circ (\mathbf{grad})^* \tag{4}$$

will supply a "preconditioner" for the operator A associated with $a(\cdot,\cdot)$. Here, I designates the trivial injection $\mathsf{I} : (H_0^1(\Omega))^3 \mapsto \mathbf{H}_0(\mathbf{curl},\Omega)$ arising from the continuous embedding $(H_0^1(\Omega))^3 \subset \mathbf{H}_0(\mathbf{curl},\Omega)$. For the case $\tau = 1$ we conclude

$$\kappa(\mathsf{BA}) \le \|\mathsf{R}\|^2 + \|\mathsf{Z}\|^2 \ . \tag{5}$$

Analogous results hold for $\mathbf{H}(\mathbf{curl},\Omega)$, $\mathbf{H}(\mathrm{div},\Omega)$, and $\mathbf{H}_0(\mathrm{div},\Omega)$.

III. Discrete regular decompositions. Alas, "preconditioning" of a variational problem set in the infinite dimensional function space $\mathbf{H}_0(\mathbf{curl},\Omega)$ is of no practical interest. What is needed is a discrete version of Thm. 2. To state it, let us denote by $\boldsymbol{E}_h \subset \mathbf{H}_0(\mathbf{curl},\Omega)$ the space of lowest order edge element functions, and by $S_h \subset H_0^1(\Omega)$ the standard piecewise linear continuous finite element space. Further, we write $\boldsymbol{V}_h := (S_h)^3 \subset (H_0^1(\Omega))^3$ and let $\Pi_h : \boldsymbol{V}_h \mapsto \boldsymbol{E}_h$ stand for the edge based finite element interpolation operator.

Theorem 3. *For any* $\mathbf{v}_h$ *there is* $\boldsymbol{\Psi}_h \in \boldsymbol{V}_h$, $p_h \in S_h$, *and* $\widetilde{\mathbf{v}}_h \in \boldsymbol{E}_h$ *such that*

$$\mathbf{v}_h = \widetilde{\mathbf{v}}_h + \Pi_h \boldsymbol{\Psi}_h + \mathbf{grad}\,p_h \ , \tag{6}$$

and

$$\left\|h^{-1}\widetilde{\mathbf{v}}_h\right\|_{L^2(\Omega)}^2 + \|\boldsymbol{\Psi}_h\|_{H^1(\Omega)}^2 \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{L^2(\Omega)}^2 \ , \ \|p_h\|_{\mathbf{H}_0(\mathbf{curl},\Omega)} \lesssim \|\mathbf{v}_h\|_{\mathbf{H}_0(\mathbf{curl},\Omega)} \ .$$

The constants are allowed to depend on Ω *and the shape regularity of the mesh.*

This result first appeared in [6], see also [7].

IV. Auxiliary space preconditioner for edge elements. Interpreting Thm. 3 against the backdrop of Thm. 1 suggests the following components for an auxiliary spaced preconditioner for (1) on $\boldsymbol{E}_h$:

- auxiliary space $W := \boldsymbol{V}_h \times S_h$ (standard continuous linear FE spaces),
- transfer operator $\Pi := \Pi_h \times \mathbf{grad}$ (a local operator),
- and a simple Gauss-Seidel relaxation as smoother.

The norm on the auxiliary space W is the conventional H^1-norm, so that the operator D from Thm. 1 boils down to discrete (vector) Laplacians. So application of the preconditioner involves solving a total of four Dirichlet problems for $-\Delta$ discretized by means of linear Lagrangian finite elements.

From Thm. 3 one readily concludes that the condition number of the preconditioned system will merely depend on Ω, the shape regularity of the meshes, and a bound for τ. Robustness with respect to $\tau \to 0$ can also be inferred.

V. Further remarks.

- The lack of $L^2(\Omega)$-stability of regular decompositions does not permit us to establish robustness of the preconditioner for $\tau \to \infty$. Under additional regularity assumptions this can be achieved using the genuine $L^2(\Omega)$-orthogonal Helmholtz decomposition, see [7].
- Along the same lines auxiliary space preconditioning in $\mathbf{H}_0(\mathrm{div}, \Omega)$ can be done. Also natural boundary conditions can be accommodated.
- The case of variable coefficients,

$$a(\mathbf{u}, \mathbf{v}) := (\alpha\, \mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v})_0 + (\beta \mathbf{u}, \mathbf{v})_0 \ , \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \ ,$$

  $\alpha, \beta \in L^\infty(\Omega)$, cannot yet be adequately dealt with by the theory. The heuristics is that the coefficient α should enter the inner product in $\boldsymbol{V}_h$, whereas β shows up in the inner product for S_h.
- Numerical experiments studying the performance of the proposed auxiliary space preconditioner and generalizations of it are reported in [7, 10, 9]. See also the contribution of T. Kolev at the Oberwolfach conference covered in this report.

References

[1] D. Arnold, R. Falk, and R. Winther, *Multigrid in H(div) and H(**curl**)*, Numer. Math., 85 (2000), pp. 175–195.

[2] R. Beck, *Algebraic multigrid by component splitting for edge elements on simplicial triangulations*, Tech. Rep. SC 99-40, ZIB, Berlin, Germany, December 1999.

[3] P. Bochev, C. Garasi, J. Hu, A. Robinson, and R. Tuminaro, *An improved algebraic multigrid method for solving Maxwell's equations*, SIAM J. Sci. Comp., 25 (2003), pp. 623–642.

[4] R. Hiptmair, *Multigrid method for Maxwell's equations*, SIAM J. Numer. Anal., 36 (1999), pp. 204–225.

[5] ———, *Finite elements in computational electromagnetism*, Acta Numerica, 11 (2002), pp. 237–339.

[6] R. Hiptmair, G. Widmer, and J. Zou, *Auxiliary space preconditioning in $\mathbf{H}_0$(**curl**, Ω)*, Numer. Math., 103 (2006), pp. 435–459.

[7] R. Hiptmair and J. Xu, *Nodal auxiliary space preconditioning in H(curl) and H(div) spaces*, Report 2006-09, SAM, ETH Zürich, Zürich, Switzerland, 2006.

[8] R. Hiptmair and W.-Y. Zheng, *Local multigrid in $\mathbf{H}$(**curl**)*, Tech. Rep. 2007-03, SAM, ETH Zürich, Zürich, Switzerland, March 2007.

[9] T. Kolev and P. Vassilevski, *Parallel H^1-based auxiliary space AMG solver for H(**curl**) problems*, Report UCRL-TR-2222763, LLNL, Livermore, CA, 2006.

[10] ———, *Some experience with a H^1-based auxiliary space AMG for H(**curl**) problems*, Report UCRL-TR-221841, LLNL, Livermore, CA, 2006.

[11] S. Reitzinger and J. Schöberl, *Algebraic multigrid for edge elements*, Numerical Linear Algebra with Applications, 9 (2002), pp. 223–238.
[12] J. Xu, *The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids*, Computing, 56 (1996), pp. 215–235.

Reporter: Ralf Hiptmair

Participants

Prof. Dr. Mark Ainsworth
Department of Mathematics
University of Strathclyde
Livingstone Tower
26, Richmond Street
GB-Glasgow , G1 1XH

Prof. Dr. Xavier L. Antoine
Institut Elie Cartan
-Mathematiques-
Universite Henri Poincare, Nancy I
Boite Postale 239
F-54506 Vandoeuvre les Nancy Cedex

Prof. Dr. Alfredo Bermudez
Departemento de Matematica Aplicada
Facultad de Matematicas
Universidad de Santiago
E-15706 Santiago de Compostela

Prof. Dr. Oszkar Biro
IGTE
TU Graz
Kopernikusgasse 24
A-8010 Graz

Prof. Dr. Pavel B. Bochev
Sandia National Laboratories
Computational Mathematics and
Algorithms Department
MS 1320, P.O.Box 5800
Albuquerque , NM 87185-1320
USA

Prof. Dr. Daniele Boffi
Dipartimento di Matematica
Universita di Pavia
Via Ferrata 1
I-27100 Pavia

Prof. Dr. Anne-Sophie Bonnet Ben-Dhia
ENSTA/UMA
32, Boulevard Victor
F-75739 Paris Cedex 15

Prof. Dr. Alain Bossavit
Laboratoire de Genie Electrique
de Paris (LGEP), Supelec
Universites Paris VI et Paris XI
11, Rue Joliot-Curie
F-91192 Gif-sur-Yvette -CEDEX

Prof. Dr. Oscar P. Bruno
Applied Mathematics 217-50
California Institute of Technology
Pasadena , CA 91125
USA

Dr. Annalisa Buffa
IMATI - CNR
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Simon N. Chandler-Wilde
Department of Mathematics
University of Reading
WhiteKnights
P.O.Box 220
GB-Reading , RG6 6AX

Prof. Dr. Zhiming Chen
Inst. of Computational Mathematics
Chinese Academy of Sciences
P.O.Box 2719
Beijing 100080
CHINA

Prof. Dr. Snorre Harald Christiansen
CMA
c/o Department of Mathematics
University of Oslo
P.O.Box 1053 - Blindern
N-0316 Oslo

Xavier Claeys
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Prof. Dr. Gary Cohen
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Dr. Francis Collino
CERFACS
42, Ave. Gustave Coriolis
F-31057 Toulouse Cedex

Prof. Dr. Martin Costabel
Departement de Mathematiques
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Monique Dauge
I.R.M.A.R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Dr. Pierre Degond
Institut de Mathematiques
Lab. MIP
Universite Paul Sabatier
F-31062 Toulouse Cedex 4

Prof. Dr. Leszek Demkowicz
Institute for Computational
Engineering and Sciences (ICES)
University of Texas at Austin
ACES 6.332
Austin , TX 78712-1085
USA

Prof. Dr. Bruno Despres
CEA/DAM-DIF/DSSI
BP 12
F-91680 Bruyeres-le-Chatel

Sonia Fliss
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Prof. Dr. Ivan G. Graham
Department of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath Somerset BA2 7AY

Prof. Dr. Marcus Grote
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel

Dr. Houssem Haddar
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Prof. Dr. Thomas Hagstrom
Dept. of Mathematics and Statistics
University of New Mexico
Albuquerque , NM 87131-1141
USA

Holger Heumann
Seminar für Angewandte Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Ralf Hiptmair
Seminar für Angewandte Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Dr. Thorsten Hohage
Institut für Numerische
und Angewandte Mathematik
Universität Göttingen
Lotzestr. 16-18
37083 Göttingen

Prof. Dr. Ronald H.W. Hoppe
Lehrstuhl für Angewandte
Mathematik I
Universität Augsburg
Universitätsstr. 14
86159 Augsburg

Prof. Dr. Patrick Joly
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Prof. Dr. Manfred Kaltenbacher
Lehrstuhl für Sensorik
Friedrich-Alexander-Universität
Erlangen
Paul-Gordan-Str. 3/5
91052 Erlangen

Dr. Tzanio V. Kolev
Centre of Applied and Scientific
Computing
Lawrence Livermore National Lab.
P.O.Box 808, L-561
Livermore , CA 94551
USA

Prof. Dr. Robert Kotiuga
Department of ECE
Boston University
8 Saint Mary's Street
Boston MA 02215
USA

Prof. Dr. Stefan Kurz
ETAS GmbH
ETASIMK
Borsigstr. 14
70469 Stuttgart

Prof. Dr. Ulrich Langer
Institut für Numerische Mathematik
Johannes Kepler Universität Linz
Altenbergerstr.69
A-4040 Linz

Prof. Dr. Jin-Fa Lee
The ElectroScience Laboratory
The Ohio State University
1320 Kinnear Rd.
Columbus , OH 43212
USA

Prof. Dr. David Levadoux
ONERA
Centre de Palaiseau
Chemin de la Huniere
F-91761 Palaiseau

Prof. Dr. Jing-Rebecca Li
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Hongyu Liu
Department of Mathematics
The Chinese University of Hong Kong
Shatin
Hong Kong
P.R. CHINA

Patrick Meury
Seminar für Angewandte Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Eric Michielssen
Elc. Eng. & Comp. Science Dept.
The University of Michigan
Ann Arbor , MI 48109-2122
USA

Prof. Dr. Peter Monk
Department of Mathematical Sciences
University of Delaware
501 Ewing Hall
Newark , DE 19716-2553
USA

Prof. Dr. Claus-Dieter Munz
Institut f. Aerodynamik
und Gasdynamik
Universität Stuttgart
Pfaffenwaldring 21
70569 Stuttgart

Prof. Dr. Joseph E. Pasciak
Department of Mathematics
Texas A & M University
College Station , TX 77843-3368
USA

Prof. Dr. Ilaria Perugia
Seminar für Angewandte Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Serge Piperno
CERMICS - ENPC
Cite Descartes, Champs-sur-Marne
6 et 8 avenue Blaise Pascal
F-77455 Marne La Vallee Cedex 2

Dominique Savelief
Universite Paul Sabatier
U.F.R. M.I.G.
118, route de Narbonne
F-31062 Toulouse Cedex 4

Dr. Joachim Schöberl
Center for Computational
Engineering Science
RWTH Aachen
Pauwelstraße 19
52065 Aachen

Prof. Dr. Eric Sonnendrücker
Institut de Recherche
Mathematique Avancee
ULP et CNRS
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Dr. Manuel Torrilhon
Seminar für Angewandte Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Jon Trevelyan
School of Engineering
University of Durham
South Road
GB-Durham DH1 3LE

Prof. Dr. Tim C. Warburton
Computational and Applied Math.
Rice University
6100 Main Street
Houston , TX 77005-1892
USA

Prof. Dr. Jinchao Xu
Department of Mathematics
Pennsylvania State University
University Park , PA 16802
USA

Prof. Dr. Jun Zou
Department of Mathematics
The Chinese University of Hong Kong
Shatin
Hong Kong
P.R. CHINA