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Reelle Algebraische Geometrie

Organised by
Michel Coste (Rennes)
Claus Scheiderer (Konstanz)
Niels Schwartz (Passau)

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Abstract. This workshop was organized by Michel Coste (Rennes), Claus Scheiderer (Konstanz) and Niels Schwartz (Passau). The talks focussed on recent developments in real enumerative and tropical geometry, positivity and sums of squares, real aspects of classical algebraic geometry, semialgebraic and tame geometry, and topology and singularities of real varieties.

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Introduction by the Organisers

This workshop was the seventh of its kind held in Oberwolfach (after 1984, 1987, 1990, 1993, 1997 and 2002) and was attended by 52 participants from Europe and North America, among them 20 young post- docs and graduate students. The meeting comprised 25 talks of 50 minutes each, with a strong emphasis on recent progress and new directions. We try to give a brief overview of the topics involved:

1) Real enumerative geometry and tropical geometry

Enumerative geometry is concerned with counting the number of solutions to algebro-geometric problems. One very active topic is counting the number of rational curves passing through prescribed points (and variations thereof). The Welschinger invariants are the real analogue of Gromov-Witten invariants in complex geometry. Calculating both the complex invariants and their real counterparts was related to tropical geometry by Grigory Mikhalkin in his celebrated correspondence theorem. The talks of Erwan Brugallé, Ilia Itenberg and Jean-Yves Welschinger were devoted to these problems. Frédéric Bihan reported about improved fewnomial bounds, i.e. bounds on the number of solutions to sparse polynomial systems that depend only on the number of monomials involved. Alex

Degtyarev and Andrej Gabrielov gave talks on applications and generalisations of Grothendieck's dessins d'enfant.

2) Positivity and Sums of Squares

Representations of positive polynomials involving sums of squares of polynomials provide certificates for positivity that are interesting both theoretically and for numerical purposes. On the other hand, a classical result of Hilbert states that not every positive polynomial in more than one variable is a sum of squares. A constructive approach to this theorem was presented at the workshop by Bruce Reznick. Recent developments in this field were presented by Salma Kuhlmann, Vicky Powers, Marie-Françoise Roy, and Markus Schweighofer. The talk of Igor Klep was concerned with positivity and sums of squares in a non-commutative setting, relating such questions to a famous conjecture by Alain Connes about von Neumann algebras.

3) Topology and singularities of real varieties

Much research in this area is concerned with finding topological properties that characterise real algebraic sets or particular classes of varieties. Wojciech Kucharz spoke about the question which homotopy types of maps between real varieties are represented by algebraic morphisms. The talk of Benoît Bertrand was devoted to generalizations of Viro's patchworking method for constructing real varieties with a prescribed topology, and Nicolai Vorobjov reported on an effective version with explicit bounds of a theorem due to Coste and van den Dries that bounds the number of homotopy types of semialgebraic sets in terms of the complexity of the polynomial description. Riccardo Ghiloni spoke about semialgebraic deformations of real algebraic sets and presented a proof of the fact that every real algebraic set is semialgebraically homeomorphic to an irreducible one. Johannes Huisman addressed the problem of determining all rational real algebraic models of a topological surface, proving the uniqueness of such models for a large class of surfaces. Frédéric Mangolte spoke about real singular Del Pezzo surfaces and threefolds fibred by rational curves. Adam Parusiński reported on a complete classification of blow-analytic equivalence classes of function germs in two variables.

Other topics included a talk by Daniel Schaub on the Pierce-Birkhoff conjecture, which asks whether every piecewise polynomial function can be expressed as a surpremum of infima of polynomials. José Fernando spoke about semialgebraic analogues to classical results on rings of continuous functions and Andreas Fischer about analytic approximation results in o-minimal structures. Jean-Philippe Monnier presented a real version of the Riemann-Hurwitz inequality for the number of fixed points of an automorphism of an algebraic curve, allowing for more precise results in the real case that involve the number of connected components of the real locus. Vladimir Kostov's talk was concerned with the Schur-Szego composition for polynomials in one variable, especially those that have only real roots. Krzysztof Kurdyka spoke on trajectories of subriemmanian gradients of polynomials.

The schedule of the meeting left plenty of time for numerous discussions among the participants. There was no official programme beyond the talks (except the traditional hike on Wednesday afternoon) but many participants worked together in small groups in the seminar rooms until late in the evening. Several collaborations were continued and some new ones were started.

We cordially thank the Oberwolfach Institute and its staff for the splendid atmosphere they provided and gratefully acknowledge the financial support for young researchers.

Workshop: Reelle Algebraische Geometrie

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Abstracts

Algebraic and pseudoholomorphic patchworking
Benoît Bertrand

The results presented here were obtained jointly with Erwan Brugallé and are contained in [BB06] and [BB07] from which this note is adapted.

Viro's patchworking was applied in the proof of a lot of meaningful results in topology of real algebraic varieties. It is one of the most important method in this field. We consider here the case of curves in $\mathbb{C}P^2$ and in rational geometrically ruled surfaces Σ_n.

Viro Method allows to construct an algebraic curve A out of simpler curves A_i so that the topology of A can be deduced from the topology of the initial curves A_i. Namely one gets a curve with Newton polytope Δ out of curves whose Newton polytopes are the 2-simplices of a subdivision σ of Δ, and one can see the curve A as a gluing of the curves A_i. Moreover, if all the curves A_i are real, so is the curve A (see [Vir84] and [Vir89]).

One of the hypotheses of the Viro Method is that σ should be convex (i.e. the 2-simplices of σ are the domains of linearity of a piecewise linear convex function). In the original Patchworking Theorem, one also requires the curves A_i to be totally nondegenerate. In particular, one can only glue nonsingular curves. E. Shustin proved in [Shu98] that, under some numerical conditions depending on the types and number of the singularities, it is possible to patchwork singular curves keeping the singular points.

On the other hand, I. Itenberg and E. Shustin proved in [IS02] a pseudoholomorphic patchworking theorem: they showed that applying the Viro Patchworking with any subdivision (non necessarily convex) and with reduced curves A_i with arbitrary singularities, one can glue the A_i's, keeping singular points, to obtain a *real pseudoholomorphic curve*. More precisely, given some (maybe singular) curves A_i whose Newton polygons are the 2-simplices of a subdivision of the quadrangle with vertices $(0,0)$, $(k,0)$, (k,l) and $(l+nk,0)$, Itenberg and Shustin gave a way to construct a pseudoholomorphic curve C of bidegree (k,l) in the rational geometrically ruled surface Σ_n, whose position with respect to the pencil of lines can be deduced from the initial curves A_i. Isotopy types realizable by (algebraic or pseudoholomorphic) curves obtained via a patchworking procedure are called *patchworked curves.*

Pseudoholomorphic curves were introduced by M. Gromov in [Gro85] to study symplectic 4-manifolds. A real pseudoholomorphic curve C on $\mathbb{C}P^2$ or Σ_n is an immersed Riemann surface which is a J-holomorphic curve in some tame almost complex structure J such that the exceptional section (in Σ_n with $n \geq 1$) is J-holomorphic, $conj(C) = C$, and $conj_* \circ J_p = -J_p \circ conj_*$ (where $conj$ is the standard complex conjugation and p is any point of C). It as been realized since then that real pseudoholomorphic curves share a lot of properties with real algebraic ones (see for example [OS02], [OS] and [FLTO02]). It is still unknown if there exist nonsingular real pseudoholomorphic curves in $\mathbb{C}P^2$ which are equivariantly isotopic to no homologous real algebraic curves (this is the so called real symplectic isotopy problem). However, S. Orekov gave an example of a nonsingular real pseudoholomorphic curve in Σ_4 which is not istopic to any real algebraic curve of the same bidegree ([Ore]). Note that, not requiring the exceptional section in Σ_n to be J-holomorphic, J-Y. Welschinger constructed in [Wel02] examples of real pseudoholomorphic curves on Σ_n for $n \geq 2$ which are not isotopic to any real algebraic curve realizing the same homology class in $H_2(\Sigma_n; \mathbb{Z})$.

In the surfaces $\mathbb{R}\Sigma_n$, there is a natural pencil of lines $\mathcal{L}$, and one can study curves there up to fiberwise isotopy. Two curves C_1 and C_2 in $\mathbb{R}\Sigma_n$ are said to be *$\mathcal{L}$-isotopic* if there exists an isotopy $\phi(t, x)$ of $\mathbb{R}\Sigma_n$ mapping C_1 to C_2 such that for any $t \in [0; 1]$, for any $p \in C_1$ and for any fiber F of $\mathbb{R}\Sigma_n$, $\phi(t, F)$ is a fiber of $\mathbb{R}\Sigma_n$, and the intersection multiplicity of C_1 and F at p is the intersection multiplicity of $\phi(t, C_1)$ and $\phi(t, F)$ at $\phi(t, p)$.

There are several examples of nonsingular real pseudoholomorphic curves in $\mathbb{R}\Sigma_n$ which are $\mathcal{L}$-isotopic to no homologous real algebraic curves (see for example [OS02], [OS], [Bru]). However, as far as we know, none of those examples are constructed with the pseudoholomorphic patchworking of Itenberg and Shustin, and the question of the existence of a patchworked pseudoholomorphic curves with any kind of non-algebraic behaviour was open.

In [BB06] we proved that in the case of curves of bidegree $(3, 0)$ in Σ_n, the patchworked pseudoholomorphic curve is always isotopic to a real algebraic one in the same homology class.

On the other hand we constructed the first examples of patchworked real pseudoholomorphic curves in Σ_n whose position with respect to the pencil of lines cannot be realised by any homologous real algebraic curve.

Theorem 1. *For any $d \geq 3$ there exists a smooth real pseudoholomorphic patchworked curve of bidegree $(d, 0)$ in Σ_2 which is not $\mathcal{L}$-isotopic to any real algebraic curve in Σ_2 of the same bidegree.*

To prove this theorem we patwork, for any $d \geq 3$, a pseudoholomorphic curve in Σ_2 of bidegree $(d, 0)$ which has, in particular, four points of maximal tangency with respect to $\mathcal{L}$. The nonalgebraicity follows from the fact that a given subresultant would have too many roots if the curve were algebraic.

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New Fewnomial Bounds

Frédéric Bihan

(joint work with Frank Sottile)

In this talk, we present results obtained with Frank Sottile in [5]. Consider a polynomial system $f_1 = \cdots = f_n = 0$ defined by n Laurent real polynomials in n variables. We are interested in the number of complex or real solutions with non-zero coordinates. Let $\mathcal{W}$ denotes the support of this system, that is, the subset of $\mathbb{Z}^n$ consisting of all the exponent vectors $(a_1, \dots, a_n)$ corresponding to the monomials $x_1^{a_1} \cdots x_n^{a_n}$ of the system. Kouchnirenko [6] showed that the number of non-degenerate complex solutions to the system is at most the normalized volume of the convex hull of $\mathcal{W}$. This volume gives thus a bound on the number of real solutions. However, the resulting bound appears to be far from being sharp for systems having few monomials. For $n = 1$, this is a consequence of Descartes rule of signs. In general, this is due to Khovanskii [7] who showed that if the cardinality if $\mathcal{W}$ is $n+k+1$ (we can assume that $k \geq 0$), then the number of positive solutions to the system is at most

$$2^{\binom{n+k}{2}}(n+1)^{n+k}$$

(Multiplying by 2^n gives then a bound on the number of real solutions.) The first concrete result showing that Khovanskii's bound is likely overstated was due to Li, Rojas, and Wang [9] who showed that two trinomials in two variables have at most 5 positive solutions (and this is sharp). Such systems are particular cases of systems with $n = k = 2$ (we can assume that both trinomials have a common monomial), so that the Khovansky bound in this case is 5184. More recently, upper bounds have been obtained for systems with $k = 1$, that is, when $\mathcal{W}$ forms what is called a circuit (see [1] and [2]). The main idea was to reduce the study of such a system to that of particular univariate polynomial. In [5], we generalize this for any k and obtain the following new fewnomial bounds.

Theorem. *A system of n polynomials in n variables having a total of $n+k+1$ distinct monomials has fewer than*

$$\tfrac{e^2+3}{4}\, 2^{\binom{k}{2}} n^k$$

non-degenerate solutions in the positive orthant.

In fact, the paper [5] contains better bounds, but which are more difficult to state. Like the Khovansky bound, our bound also works for systems with real exponent vectors. In the case $n = k = 2$, we obtain the new upper bound of 15. A construction based on a sharpness result of [2] shows that our bound is nearly asymptotically optimal for k fixed and $n \to +\infty$, see [3]. The proof of our new fewnomial upper bound goes as follows. We generalize the reduction step which was done for $k = 1$ by reducing the study of the original system to that of a special k by k system called *Gale dual system.* Precisely, we show a bijection between the positive solutions to the original system to the solutions to the Gale system which are contained in a certain polyhedron. We use then what is called the Khovansky-Rolle theorem, see [8], toric tricks and a bit of combinatorics of polytopes. Other bounds have also been been obtained recently and which concern a fewnomial hypersurface of the positive orthant. Namely, bounds for the number of compact connected components [5], for the number of (possibly non compact) connected components [3], and finally for the total sum of the Betti numbers [4].

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Floor decompositions of tropical curves

Erwan Brugallé
(joint work with Grigory Mikhalkin)

1. Enumerative invariants of projective spaces

Let $n \geq 2$, $d \geq 1$ and $g \geq 0$ be some integers and $c = (c_0, \ldots, c_{n-2}) \in \big(\mathbb{N} \cup \{0\}\big)^n$ such that $\sum_{i=0}^{n-2}(n-1-i)c_i = (n+1)d + (n-3)(1-g)$. Moreover, **throughout the text we assume that** $g = 0$ **if** $n \geq 3$. Consider ω a generic configuration of c_i linear spaces of $\mathbb{C}P^n$ of dimension i with i varying from 0 to $n-2$, and consider the set $\mathcal{C}$ of algebraic irreducible curves of degree d and genus g in $\mathbb{C}P^n$ passing through all elements of ω The set $\mathcal{C}$ is finite, and if n, d, g and c are fixed, then the cardinal of $\mathcal{C}$ does not depend on ω. Let us define $N_c^{n,g}(d) = \sharp\mathcal{C}$. If $n = 2$ (resp. $n \geq 3$), then we will simply write $N^{2,g}(d)$ (resp. $N_c^n(d)$) instead of $N^{2,g}_{(3d-1+g)}(d)$ (resp. $N_c^{n,0}(d)$). The numbers $N_c^{n,0}(d)$ were first computed by Kontsevich, and Caporaso and Harris gave in an algorithm to compute the numbers $N^{2,g}(d)$.

When all the elements of ω are real, then one can consider the set

$$\mathbb{R}\mathcal{C} = \{\textit{real algebraic curves in } \mathcal{C}\}$$

Now the cardinal of $\mathbb{R}\mathcal{C}$ depends on ω. However, Welschinger proved that counting curves in $\mathbb{R}\mathcal{C}$ with respect to some sign, one obtain an invariant when $c_1 = \ldots = c_{n-2} = 0$. We denote by $W^n(d)$ this invariant. Welschinger invariants of $\mathbb{R}P^2$ were first computed by Itenberg, Kharlamov and Shustin.

Computations of Itenberg, Kharlamov and Shustin are based on a previous paper by Mikhalkin, where he gave, in particular, an algorithm to compute the numbers $\sharp\mathcal{C}$ and $\sharp\mathbb{R}\mathcal{C}$ for some special configurations of points in the plane.

In this talk, we present a generalisation to dimension 3 of this algorithm, and we exhibit new formulas for the numbers $N_c^3(d)$ and $W^3(d)$ in terms of *floor diagrams*. The main tool in the proof of our results is *tropical geometry*, thanks to Mikhalkin's Correspondence Theorems.

Notation. Given $\mathcal{D}$ a finite oriented graph, we denote the set of vertices (resp. edges) of $\mathcal{D}$ by $\overline{\mathcal{D}_0}$ (resp. $\mathcal{D}_1$). We also denote by $\mathcal{D}_0^\infty$ the set of sinks, and by $\mathcal{D}_1^\infty$ the set of edges adjacent to a sink. Finally we put $\mathcal{D}_0 = \overline{\mathcal{D}_0} \setminus \mathcal{D}_0^\infty$.

Given $v \in \mathcal{D}_0$, we denote by $\mathcal{A}(v)$ the set of edges adjacent to v, and we define the function $\epsilon_v : \mathcal{A}(v) \to \{\pm 1\}$ by $\epsilon_v(e) = 1$ if v is the origin of e, and $\epsilon_v(e) = -1$ otherwise.

By convention, we define $N^{1,0}(1) = W^1(1) = 1$ and $N^{1,0}(d) = W^1(d) = 0$ if $d \geq 2$.

2. Floor diagrams

Definition 1. A floor diagram of degree d and of genus g is a triple $(\mathcal{D}, w_0, w_1)$ where $\mathcal{D}$ is a connected finite oriented graph of genus g with no oriented cycle, and where $w_i : \mathcal{D}_i \to \mathbb{N}$ are two functions satisfying the following conditions

i. $\sum_{v\in\mathcal{D}_0} w_0(v) = d$,
ii. $\forall v \in \mathcal{D}_0,\ \sum_{e\in\mathcal{A}(v)} \epsilon(e)w_1(e) = w_0(v)$,
iii. $\forall e \in \mathcal{D}_1^\infty,\ w_1(e) = 1$.

Let $(\mathcal{D}, w_0, w_1)$ be a floor diagram of degree d and genus g, $n = 2$ or 3 and $c = (c_0, \ldots, c_{n-2}) \in \left(\mathbb{N} \cup \{0\}\right)^n$ such that $\sum_{i=0}^{n-2}(n-1-i)c_i = (n+1)d + (n-3)(1-g)$. Put $E_c = \{1, \ldots, \sum_{i=0}^{n-2} c_i\}$ and let $m : E_c \to \mathcal{D}$ a map injective on $m^{-1}(\mathcal{D}_1)$. We define **the dimension** of $l \in E_c$ as the integer $dim(l)$ such that $c_{dim(l)-1} < l \leq c_{dim(l)}$ (with the convention that $c_{-1} = 0$), **the rank** of $v \in \mathcal{D}_0$ as the integer $rk(v) = dim(max\{l \mid m(l) = v\})$, and **the freedom** of $e \in \mathcal{D}_1$ as the integer $fr(e) = n - 1 - \sum_{l\in m^{-1}(e)}(n-1-dim(l))$. For $e \in \mathcal{D}_1$, we denote by $\partial^0 e$ the set of its adjacent vertices in $\mathcal{D}_0$. Finally, we define $E_{c,m} = m^{-1}(\mathcal{D}_1) \cup_{v\in\mathcal{D}_0} max\{l\ |m(l) = v\}$.

Definition 2. A map Π which associate to any $e \in \mathcal{D}_1$ a subset of $\partial^0 e$ of cardinal $fr(e)$ is called a reconstruction map for m if for any $v \in \mathcal{D}_0$ one has

$$(n-2)\left(\sharp m^{-1}(v) - rk(v) + \sharp fr^{-1}(0) \cap \mathcal{A}(v) + \sum_{e\in fr^{-1}(1)\cap\mathcal{A}(v)} (1 - \sharp\Pi(E) \cap \{v\})\right) = nw_0(v) + n - 4$$

Definition 3. The quadruplet $(\mathcal{D}, w_0, w_1, m)$ is called an n-dimensional marked floor diagram of type c if the following conditions hold

i. $m|_{E_{c,m}} : E_{c,m} \to \mathcal{D}$ is an increasing map,
ii. if $dim(l) = 0$ and $m(l) \in \mathcal{D}_0$, then $m^{-1}(m(l)) = \{l\}$,
iii. there exists a reconstruction map for m.

Remark 4. For an n-dimensional marked floor diagram, the reconstruction map is unique.

Two n-marked floor diagrams of type c, $(\mathcal{D}, w_0, w_1, m)$ and $(\mathcal{D}', w_0', w_1', m')$, are said to be **isomorphic** if there exists a homeomorphism of oriented graphs $\phi : \mathcal{D} \to \mathcal{D}'$ such that $w_i = w_i' \circ \phi$ and $m = m' \circ \phi$. We call without distinction a marked floor diagram either an isomorphism class of marked floor diagrams, or one of its representative.

To each marked floor diagram $\mathcal{D}$ of type c, we associate now a complex multiplicity $\mu_c^{\mathbb{C}}(\mathcal{D})$ defined by

$$\mu_c^{\mathbb{C}}(\mathcal{D}) = \prod_{v\in\mathcal{D}_0}\left[N^{n-1,0}(w_0(v))w_0(v)^{rk(v)}\right]\prod_{e\in\mathcal{D}_1}\left[w_1(e)^{1+\sharp m^{-1}(e)}\prod_{v'\in\Pi(e)} w_0(v')\right].$$

If $g = 0$ and $c_i = 0$ for $i \geq 1$ then we also associate a real multiplicity $\mu^{\mathbb{R}}(\mathcal{D})$ to $\mathcal{D}$ defined by

$$\mu^{\mathbb{R}}(\mathcal{D}) = \prod_{v\in\mathcal{D}_0} W^{n-1}(w_0(v)) \quad if \quad w_1(\mathcal{D}_1) \cap 2\mathbb{N} = \emptyset \quad and \quad \mu^{\mathbb{R}}(\mathcal{D}) = 0\ otherwise.$$

Note that for $n = 2$, both real and complex multiplicities take a very simple form: one has $fr(e) = 0$ for any $e \in \mathcal{D}_1$ and $w_0(v) = 1$ for any $v \in \mathcal{D}_0$, so $\mu_c^{\mathbb{C}}(\mathcal{D})$ is the square of the product of the weight of all edges of $\mathcal{D}$, and $\mu^{\mathbb{R}}(\mathcal{D}) = \mu_c^{\mathbb{C}}(\mathcal{D})$ *mod* 2.

3. Main results

Theorem 5. *For $(n, g) = (2, g)$ or $(n, g) = (3, 0)$, the number $N_c^{n,g}(d)$ is equal to the sum of the complex multiplicity of all n-dimensional marked floor diagrams of degree d, genus g, and type c.*

Theorem 6. *For $n = 2$ or $n = 3$, the Welschinger invariant $W^n(d)$ is equal to $(-1)^{n\epsilon(d)}$ times the sum of the real multiplicity of all n-dimensional marked floor diagram of degree d, genus 0, and type $(c_0, 0, \ldots, 0)$.*

As said in the introduction, Theorems 5 and 6 are obtained via tropical geometry. Let us state the main ingredient in our tropical computations. Take n, d, g and c as in section 1, and take elements of ω to be linear tropical subspaces of $\mathbb{R}^n$.

Theorem 7. *Let HC be a hypercube containing all vertices of all elements of ω. Then HC contains all the vertices of any tropical curve of degree d and genus g in $\mathbb{R}^n$ passing through elements of ω.*

Zariski k-plets via dessins d'enfants

Alex Degtyarev

We apply the techniques of Grothendieck's *dessins d'enfants*, which has already proved useful in the study of real trigonal curves (joint work with I. Itenberg and V. Kharlamov), to the study of singular complex plane curves. As a first application, we construct asymptotically large collections of non-equivalent irreducible curves sharing the same set of singularities.

For the purpose of this paper, by a *type* of a singular point we mean its *PL*-homeomorphism class, and by an *equisingular deformation* (of plane curves) we mean a *PL*-equisingular deformation.

The definition below was suggested by E. Artal. For the sake of simplicity, we confine ourselves to the case of irreducible curves.

Definition 1. A collection $C_1, \ldots, C_k$ of irreducible (complex) plane curves is said to form a *Zariski k-plet* if all curves C_i have the same degree and the same combinatorial set of singularities but no two curves C_i, C_j, $i \neq j$, can be connected by an equisingular deformation.

Historically, the first example of Zariski pairs was found by O. Zariski, who constructed two distinct families of irreducible six cuspidal sextics. Later, more examples were found by E. Artal. A. Degtyarev, C. Eyrol, M. Oka, H. Tokunaga, M. Uludağ, and others. In most examples known, there are just a few (usually, two) families of curves sharing the same set of singularities, and the curves are distinguished by the fundamental group of the complement or even by the Alexander polynomial (which is an easily computable invariant of the fundamental group).

The principal result of this talk is the following theorem.

Theorem 2. *For each integer $m \geq 8$, there is a set of singularities shared by*

$$N(m) = \frac{1}{k}\binom{2k-2}{k-1}\binom{k}{[k/2]}\binom{[k/2]}{\epsilon}$$

pairwise not deformation equivalent irreducible plane curves $C_i \subset \mathbb{P}^2$ of degree m, where $k = [(m+2)/3]$ and $\epsilon = 3k - m \in \{0, 1, 2\}$. The fundamental groups of all curves C_i are abelian: one has $\pi_1(\mathbb{P}^2 \smallsetminus C_i) = \mathbb{Z}_m$.

Remark 3. It is easy to see that asymptotically the count $N(m)$ given by the theorem grows faster than a^m for any $a < 2$.

Remark 4. The set of singularities that is actually constructed below consists of two points, one point of type $\mathbf{A}_{5k-2}$ and one point of 'transversal' intersection of $(k-1)$ 'blocks': ϵ cusps $\mathbf{A}_2$, one block of type $\mathbf{E}_{12}$ (in Arnol′d's notation), $[k/2]-\epsilon$ blocks of type $\mathbf{J}_{2,1}$, and $[(k-3)/2]$ blocks of type $\mathbf{J}_{2,0}$.

Remark 5. In fact, the notion of Zariski pair/k-plet changes from paper to paper. Often, the condition that the curves C_i are not deformation equivalent is replaced by the stronger requirement that the pairs $(\mathbb{P}^2, C_i)$ (or even spaces $\mathbb{P}^2 \smallsetminus C_i$) should not be homeomorphic. At present, I do not know whether this stronger requirement holds for all/some of the curves given by the theorem.

The proof of the main theorem is based on the study of curves of degree m with a singular point O of multiplicity $(m-3)$. (Note that curves of degree m with a singular point of multiplicity $\geq (m-2)$ can easily be classified; they never form Zariski pairs.) Blowing O up converts such a curve C to a curve in the Hirzebruch surface Σ_1, and after a series of elementary transformations one arrives at a trigonal curve B in a certain Hirzebruch surface Σ_k; one can assume that B does not intersect the exceptional section of the surfaces and has no triple points. The deformation type of B (decorated with a number of distinguished points/fibers recording the converse transformation) determines that of C. Thus, essentially the problem reduces to the deformation classification of trigonal curves.

A trigonal curve B can be described by its functional j-invariant $j\colon \mathbb{P}^1 \to \mathbb{P}^1$. The j-invariant has three special values, 0, 1, and ∞; typically, the value $j = \infty$ corresponds to the singular fibers of B (in particular, its singular points), and the values $j = 0$ and 1 correspond to the fibers of B admitting complex multiplication. Generically, the pull-backs of 0 and 1 consist, respectively, of double and triple ramification points. Points of smaller multiplicity correspond to the singular fibers of B of types $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^{*}$, and $\tilde{\mathbf{A}}_2^{*}$.

In the special case when B is maximally singular, its j-invariant has no critical values other than 0, 1, or ∞ and its topology and, hence, analytic structure is uniquely described by its classical *dessin d'enfants*, which is defined as the plane graph $\Gamma_j = j^{-1}[0,1] \subset \mathbb{P}^1$ whose vertices are marked according to whether they project to 0 or 1. Assuming, further, that B has no singular fibers of type $\tilde{\mathbf{A}}_1^*$, one can disregard the vertices projecting to 0.

All vertices of the resulting graph Γ_j have valency at most 3, 1- and 2-valent vertices corresponding to the singular fibers of B of type $\tilde{\mathbf{A}}_0^{**}$ and $\tilde{\mathbf{A}}_2^*$, respectively. All other singular fibers of B are of type $\tilde{\mathbf{A}}_p$; they are in a one-to-one correspondence with the p-gonal faces of Γ_j. Conversely, any graph $\Gamma_j \subset \mathbb{P}^2$ with at most 3-valent vertices gives rise to a trigonal curve.

Proof of the theorem. All curves are constructed from trigonal curves in Σ_k having one singular fiber of type $\tilde{\mathbf{A}}_0^{**}$, one singular fiber of type $\tilde{\mathbf{A}}_{5k-2}$, and k singular fibers of type $\tilde{\mathbf{A}}_0^*$. Any such trigonal curve is obtained from a graph $\Gamma \subset \mathbb{P}^1$ constructed as follows: one starts with a binary tree with $(k-1)$ vertices, marks its root by adding an edge and a 1-valent vertex (in particular, this procedure rules out the possible symmetries of the graph), and completes the valency of each remaining vertex to three by attaching k 'leaves', each leaf consisting of a loop connected to the rest of the graph by an edge.

The number of such graphs (hence, the number of trigonal curves) is Catalan's number

$$C(k-1) = \frac{1}{k}\binom{2k-2}{k-1}.$$

Note that all singular fibers of the resulting curves are different, *i.e.*, no pair of singular fibers can be interchanged by a deformation. Now, in order to obtain a plane curve, one needs to perform $(k-1)$ elementary transformations, converting Σ_k to Σ_1, and blow down the exceptional section of Σ_1. The elementary transformations are chosen as follows:

(1) the only type $\tilde{\mathbf{A}}_0^{**}$ singular fiber contracts to a singular point of type $\mathbf{E}_6$;
(2) $[k/2]$ of the k type $\tilde{\mathbf{A}}_0^*$ singular fibers contract to singular points of type $\mathbf{A}_2$ or $\mathbf{D}_5$ (depending on whether the blow-up center is chosen on the curve);
(3) $[(k-3)/2]$ generic fibers contract to singular points of type $\mathbf{D}_4$.

The extra counts

$$\binom{k}{[k/2]} \quad \text{and} \quad \binom{[k/2]}{\epsilon}$$

in the statement of the theorem are due, respectively, to the choice of $[k/2]$ of the k singular fibers to be contracted and the choice of ϵ of them to be contracted to $\mathbf{A}_2$ instead of $\mathbf{D}_5$.

The fundamental groups of the resulting plane curves can easily be controlled. Essentially, they are all abelian due to the existence of type $\tilde{\mathbf{A}}_0^{**}$ singular fibers. □

Rings of semialgebraic functions

José F. Fernando
(joint work with J.M. Gamboa)

The study of rings of continuous functions is a difficult matter which has deserved a lot of attention from specialists in analysis, topology and algebra. The history of this theory is long and rich and its main development goes back to the 50's and 60's of XX century. This subject contributed in an important way to the appearance and evolution of well-known tools in Mathematics like the Stone–Cěch compactification, the theory of nets and filters, the spectrum and the maximal spectrum of a commutative ring, ... We refer the reader to [GH1], [GJ], [K1], [K2] and [M] for more details. See also the surveys [Ve1] and [Ve2].

Later in the 80's, after the birth and development of Real Algebraic Geometry, most results concerning rings of continuous functions were reviewed and studied from a different view-point which involves the use of some tools proper of this theory. We refer the reader to [ChD1], [ChD2], [S1], [S2], [Tr1] and [Tr2].

In this work, we study the main properties of rings of a smaller class of continuous functions, namely (continuous) semialgebraic functions over a semialgebraic subset of $\mathbb{R}^n$, continuing some previous work of the second author and Ruiz which goes to the 90's (see [GR] and [G], and also [CC]). Recall that a *semialgebraic subset* of $\mathbb{R}^n$ is a set which can be described as a finite boolean combination of polynomial equalities and inequalities. A *semialgebraic function* on a semialgebraic set M is a continuous function $f : M \to \mathbb{R}$ whose graph is a semialgebraic set. As it is well-known, the set $\mathcal{S}(M)$ of semialgebraic functions on M is an $\mathbb{R}$-algebra and the set $\mathcal{S}^*(M)$ of bounded semialgebraic functions on M is an $\mathbb{R}$-subalgebra of $\mathcal{S}(M)$. To refer to properties of both rings simultaneously, we will write $\mathcal{S}^\diamond(M)$ to allude either $\mathcal{S}(M)$ or $\mathcal{S}^*(M)$.

Our purpose is to study the topological and algebraic properties of the spectra and maximal spectra of such rings. Since the usual notations for these objects become cumbersome, we replace them by the following ones. Let $M \subset \mathbb{R}^n$ be a semialgebraic set. We denote

$$\operatorname{Spec}_{\mathrm{s}}(M) = \operatorname{Spec}(\mathcal{S}(M)) \qquad \operatorname{Spec}^*_{\mathrm{s}}(M) = \operatorname{Spec}(\mathcal{S}^*(M)),$$
$$\beta_{\mathrm{s}} M = \operatorname{Spec}_{\max}(\mathcal{S}(M)) \qquad \beta^*_{\mathrm{s}} M = \operatorname{Spec}_{\max}(\mathcal{S}^*(M)).$$

Again we may allude to the respective spectra of both rings simultaneously by setting $\operatorname{Spec}^\diamond_{\mathrm{s}}(M)$ equals $\operatorname{Spec}_{\mathrm{s}}(M)$ or $\operatorname{Spec}^*_{\mathrm{s}}(M)$ and $\beta^\diamond_{\mathrm{s}} M$ equals $\beta_{\mathrm{s}} M$ or $\beta^*_{\mathrm{s}} M$. As we will show in section 2.1 the real spectra and real maximal spectra of $\mathcal{S}^\diamond(M)$ coincide with its classical Zariski spectra and maximal spectra. Consequently we will not be concerned about real spectra.

We point out here the most remarkable results of this memoir. We are interested in analyzing until what extend the rings of semialgebraic functions on a semialgebraic set determines it up to homeomorphism, and how much information they provide about its semialgebraic compactifications. Namely,

(1) It is shown that the ring $\mathcal{S}(M)$ classifies the semialgebraic set M up to semialgebraic homeomorphism. On the other hand, the ring $\mathcal{S}^*(M)$ classifies the semialgebraic set M, up to semialgebraic homeomorphism and outside a finite subset $\eta(M) \subset M$ depending only on M. Moreover, the spectra $\operatorname{Spec}_{\mathrm{s}}(M)$ and $\operatorname{Spec}^*_{\mathrm{s}}(M)$ are homeomorphic if and only if the rings $\mathcal{S}(M)$ and $\mathcal{S}^*(M)$ are isomorphic, and this is so if and only if M is compact, or equivalently if both rings coincide. In contrast, the maximal spectra $\beta_{\mathrm{s}}M$ and $\beta^*_{\mathrm{s}}M$ are always homeomorphic; hence they will be denoted by $\beta_{\mathrm{s}}M$. Unfortunately, this last homeomorphism has no functorial character.

(2) Although these rings $\mathcal{S}^\diamond(M)$ do not enjoy some of the nice properties of rings of Nash or polynomial functions, namely, they are not noetherian, their ideals do not have in general primary decomposition,... however they are Gelfand rings and they have finite Krull dimension equal to the dimension of M. Recall that a ring is Gelfand if each prime ideal is contained just in one maximal ideal. As to the dimension, we also prove that the height of the maximal ideal consisting of all functions vanishing at a point $p \in M$ equals the local dimension of M at p.

(3) In the same vein as the classical Stone–Cěch compactification, we prove that $\beta_{\mathrm{s}}M$ is the smallest compactification of M such that each bounded semialgebraic function on M extends continuously to $\beta_{\mathrm{s}}M$. This is why we will call $\beta_{\mathrm{s}}M$ the *semialgebraic Stone–Cěch compactification of* M. Moreover, it can be characterized as the smallest compactification that dominates all the semialgebraic compactifications of M. These results suggest that the topology of $\beta_{\mathrm{s}}M$ can be rescued from the semialgebraic compactifications of M. In fact, we will be interested in determining the main topological properties of the residue $\partial M = \beta_{\mathrm{s}}M \setminus M$ and we prove first that it has finitely many connected components, whose number equals the number of connected components of the residue of a suitable semialgebraic compactification of M. Other remarkable properties of ∂M are that it is locally connected and that its local compactness can be characterized just in terms of the topology of M. We also describe and use in a crucial way some distinguished subsets of the residue ∂M, which have a deep geometric meaning.

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Smooth approximation of definable continuously differentiable functions in some in o-minimal structures

Andreas Fischer

Whitney's approximation theorem, cf. [4], states that every continuously differentiable function f can be approximated by an analytic function g, such that the difference between f and g and their derivatives is less than any given positive continuous function ε.

M. Shiota proved in [3] the semi-algebraic version of this theorem, that is, if f and ε are semi-algebraic, the statement remains valid and we can claim g to be semi-algebraic.

Recall that an o-minimal structure on $\mathbb{R}$ is a collection of boolean algebras S_n of definable sets, $n \in \mathbb{N}$, where S_n is a subset of the power set of $\mathbb{R}^n$ which contains all semi-algebraic sets, such that linear projections of definable sets are definable and all definable subsets of $\mathbb{R}$ are semi-algebraic. If the graph of a function f belongs to some S_n, then f is called definable.

Shiota's method does not apply to any other o-minimal structure.

Here we prove the following approximation theorem for a certain subclass of o-minimal structures in which the exponential function is definable.

Theorem 1. *Let $\mathcal{M}$ be an o-minimal expansion of the real exponential field with smooth cell decomposition. Let $U \subset \mathbb{R}^n$ be definable and open, and let $f : U \to \mathbb{R}$ be definable and continuously differentiable. Then, for every definable continuous function $\varepsilon : U \to (0, \infty)$, there is a definable smooth function $g : U \to \mathbb{R}$ such that*

$$|D_\alpha f(u) - D_\alpha g(u)| < \varepsilon(u),\ u \in U,\ \alpha_1 + ... + \alpha_n \leq 1.$$

Analogous to analytic approximation, we can conclude that definable open sets are definably $\mathcal{C}^\infty$ diffeomorph if and only if they are $\mathcal{C}^1$ diffeomorph, and also that we can separate definable disjoint closed sets by definable smooth functions.

Contrary to analytic approximation, our methods imply that the functions f and g coincide outside of any pregiven definable open neighbourhood of the closure of the set of non-smooth points. Therefore the separation argument can be strengthened as follows.

If A, B are definable closed disjoint subsets of $\mathbb{R}^n$, there is a definable smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $A \subset \{\varphi = 1\}$ and $B \subset \{\varphi = 0\}$.

This implies the following two consequences.

Firstly, if $U \subset \mathbb{R}^n$ is a definable open set, then the definable sheaf consisting of the smooth definable functions from U to $\mathbb{R}$ is fine.

Secondly, if A is a closed definable set, U a definable open neighbourhood of A and $f : U \to \mathbb{R}$ a definable smooth function, then there is a definable smooth function $F : \mathbb{R}^n \to \mathbb{R}$ which coincides with f in A.

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Non-algebraic dessins d'enfants and zeros of eigenfunctions of anharmonic oscillators

Andrej Gabrielov

Consider an eigenvalue problem

$$-y'' + P(z)y = \lambda y, \quad y(-\infty) = y(\infty) = 0. \tag{1}$$

Here $P = cz^d + \dots$ is a real polynomial of even degree d with $c > 0$. This is known as a (quantum) anharmonic oscillator. It is well known that the spectrum is discrete, the eigenvalues $\lambda_0 < \lambda_1 < \dots$ are real and simple, and $\lambda_n \to \infty$ as $n \to \infty$. Eigenfunctions are real entire functions of the order $(d+2)/2$, an eigenfunction ϕ_n with the eigenvalue λ_n has n real zeros.

If P is even, each eigenfunction ϕ is either even or odd, and can be normalized either as $\phi(0) = 1$ or as $\phi'(0) = 1$. Let ψ be another solution of $-y'' + Py = \lambda y$ (without boundary conditions) with the same eigenvalue λ as ϕ, normalized as $\psi(0) = 0$, $\psi'(0) = 1$ if $\phi(0) = 1$ and as $\psi(0) = 1$, $\psi'(0) = 0$ if $\phi(0) = 0$. Then $f = \phi/\psi$ is an odd meromorphic function of the order $(d+2)/2$ without critical points. At infinity, f has asymptotic values a_j in each of $d+2$ Stokes sectors S_j

separated by the Stokes directions $\theta = \pi(2j-1)/(d+2)$. Here j is considered a residue modulo $d+2$. These values satisfy $a_{j+1} \neq a_j$ and $a_0 = a_{d/2+1} = 0$. They are also symmetric with respect to the real and imaginary axes: $a_{-j} = \bar{a}_j,\ a_{j+d/2+1} = -a_j$.

Let $d = 4$. Then

$$a_2 = -\bar{a}_1,\ a_4 = a_1,\ a_5 = \bar{a}_1\ a_0 = a_3 = 0. \tag{2}$$

One can show that, a_1 in 2 is neither real or pure imaginary, so f has 5 distinct asymptotic values. Let us choose 4 disjoint loops $\Gamma_1, \dots, \Gamma_4$ with the ends at ∞ passing around the non-zero asymptotic values of f and such that their union is symmetric with respect to the real and imaginary axes. Then preimage $f^{-1}(\Gamma_1 \cup \dots \cup \Gamma_4$ defines a partition of $\mathbf{C}$ that has 6 unbounded domains (corresponding to the Stokes sectors) and each zero of f is inside one of its bounded domains. The set of edges of this partition is a planar graph which, after contraction of loops and some additional reduction, becomes a double-symmetric planar tree with 6 infinite ends. From the classification of such trees, we derive the following result

Theorem 1. *Any eigenfunction ϕ of the eigenvalue problem 1 for an even polynomial P of degree 4 has all its zeros either real or pure imaginary.*

A theorem by Nevanlinna allows one to recover f from a given planar graph and a set of asymptotic values. This allows us to derive analytic properties of eigenfunctions of anharmonic oscillators from the the combinatorial properties of planar trees, similar to representation of algebraic functions by dessins d'enfants.

Topology and irreducibility of real algebraic varieties — Deformations of real algebraic varieties in irreducible models

Riccardo Ghiloni

1. Introduction and main theorem

In complex algebraic and analytic geometry, the usual notion of irreducibility defines a topological invariant on varieties equipped with the euclidean topology. This follows easily from the equivalence between irreducibility and connectedness of the nonsingular locus, and from the fact that the topological codimension of the singular locus is ≥ 2.

The aim of this paper is to show that, in the real case, the situation is completely different. In fact, we prove that every real algebraic variety of positive dimension is irreducible up to a semialgebraic homeomorphism. More precisely, given a real algebraic variety X of positive dimension, we construct a semialgebraically trivial family $\{\xi_t : X_t \longrightarrow X\}_{t \in \mathbb{R}}$ of semialgebraic homeomorphisms between real algebraic varieties such that $X_0 = X$, ξ_0 is the identity map on X and, for each $t \in \mathbb{R} \setminus \{0\}$, X_t is irreducible. In the case X is nonsingular, the preceding family $\{\xi_t : X_t \longrightarrow X\}_{t \in \mathbb{R}}$ can be choosen in such a way that, for each $t \in \mathbb{R} \setminus \{0\}$, X_t is an irreducible nonsingular real algebraic variety and ξ_t is a Nash isomorphism.

In what follows, by a real algebraic set, we mean an algebraic subset of some $\mathbb{R}^n$. Unless otherwise indicated, all real algebraic sets are equipped with the euclidean topology. The notion of irreducibility is refered to the Zariski topology. Let X be a real algebraic set. We indicate by $\mathrm{Nonsing}(X)$ the set of nonsingular points of X of maximum dimension and by $\mathrm{Sing}(X)$ the set $X \setminus \mathrm{Nonsing}(X)$. If $X = \mathrm{Nonsing}(X)$, then X is called nonsingular. We define the Zariski dimension of X as the maximum dimension of the Zariski tangent spaces of X. Let Y be another real algebraic set and let $h : X \longrightarrow Y$ be a semialgebraic homeomorphism. We say that h is *good* if $h(\mathrm{Sing}(X)) = \mathrm{Sing}(Y)$, the restriction of h from $\mathrm{Sing}(X)$ to $\mathrm{Sing}(Y)$ is a biregular isomorphism and the restriction of h from $\mathrm{Nonsing}(X)$ to $\mathrm{Nonsing}(Y)$ is a Nash isomorphism.

Let us introduce the notion of semialgebraic deformation of a real algebraic set.

Definition. *Let X and X^* be real algebraic sets and let $\pi : X^* \longrightarrow \mathbb{R}$ be a regular function. A continuous semialgebraic map $\xi : X^* \longrightarrow X$ is called semialgebraic deformation of X parametrized by π if the restriction of ξ to $\pi^{-1}(0)$ is a biregular isomorphism and the map $h := (\xi, \pi) : X^* \longrightarrow X \times \mathbb{R}$ is a semialgebraic homeomorphism. Furthermore, if h is good (resp. h is a Nash isomorphism), then ξ is said to be a good semialgebraic deformation (resp. a Nash deformation) of X parametrized by π.*

Let X be a real algebraic set, let $\xi : X^* \longrightarrow X$ be a semialgebraic deformation of X parametrized by $\pi : X^* \longrightarrow \mathbb{R}$ and let $t \in \mathbb{R}$. It is immediate to verify that the restriction of ξ to $\pi^{-1}(t)$ is a semialgebraic homeomorphism and, if ξ is good, the following hold: the restriction of ξ from $\pi^{-1}(t) \cap \mathrm{Sing}(X^*)$ to $\mathrm{Sing}(X)$ is a biregular isomorphism, $\pi^{-1}(t) \setminus \mathrm{Sing}(X^*) \subset \mathrm{Nonsing}(\pi^{-1}(t))$ and the restriction of ξ from $\pi^{-1}(t) \setminus \mathrm{Sing}(X^*)$ to $\mathrm{Nonsing}(X)$ is a Nash isomorphism. The reader observes that, when X is nonsingular, the notions of good semialgebraic deformation and of Nash deformation of X coincide.

We are now in position to state our main theorem.

Theorem 1. *Let X be a real algebraic set of positive dimension r and let z be its Zariski dimension. Then there exist an integer $N = N(r, z)$ depending only on r and z, an algebraic subset X^* of $\mathbb{R}^N \times \mathbb{R}$ and, denoting by $\pi : X^* \longrightarrow \mathbb{R}$ the restriction to X^* of the natural projection of $\mathbb{R}^N \times \mathbb{R}$ onto the last factor $\mathbb{R}$, a good semialgebraic deformation $\xi : X^* \longrightarrow X$ of X parametrized by π such that, for each $t \in \mathbb{R} \setminus \{0\}$, $\pi^{-1}(t)$ is irreducible.*

As a consequence, we obtain:

Corollary 2. *Every real algebraic set of positive dimension is irreducible up to a good semialgebraic homeomorphism.*

In the nonsingular case, the preceding results can be restated as follows:

Theorem 3. *For each nonsingular real algebraic set X of positive dimension, there exists a Nash deformation $\xi : X^* \longrightarrow X$ of X parametrized by a regular submersion $\pi : X^* \longrightarrow \mathbb{R}$ such that, for each $t \in \mathbb{R} \setminus \{0\}$, $\pi^{-1}(t)$ is an irreducible nonsingular real algebraic set.*

Corollary 4. *Every nonsingular real algebraic set of positive dimension is Nash isomorphic to an irreducible nonsingular real algebraic set.*

The compact case of the latter result was originally proved by Tognoli in [5].
We have a conjecture:

Conjecture. *In Theorem 3, the adjective "nonsingular" can be omitted. In particular, every real algebraic set of positive dimension is Nash isomorphic to an irreducible real algebraic set.*

2. Sketch of the proof

Our proof of the main theorem is quite long and technical. Here, we sketch the proof of Corollary 2. The ideas presented below to prove this corollary are the same we use to prove the main theorem.

Sketch of the proof of Corollary 2. We organize the proof into six steps.

Step I: compactification. Let X be an algebraic subset of $\mathbb{R}^n$ of dimension $r > 0$ and let $S := \mathrm{Sing}(X)$. Since the Alexandrov compactification of X can be made algebraic, we may suppose that X is compact.

Step II: resolution of singularities. By Hironaka's resolution theorem, there exist a compact nonsingular real algebraic set $\widetilde{X}$, a finite union A of nonsingular hypersurfaces of $\widetilde{X}$ in general position and a regular map $\pi : \widetilde{X} \longrightarrow X$ such that $A = \pi^{-1}(S)$ and the restriction of π from $\widetilde{X} \setminus A$ to $X \setminus S$ is a biregular isomorphism.

Step III: Tognoli's approximation technique. By a classical approximation technique due to Tognoli [5], we may suppose that $\widetilde{X}$ is irreducible.

Step IV: adding a tail. Making use of a translation if needed, we may suppose that S is contained in the half space $\{\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 \leq -2\}$. Let q be a polynomial in $\mathbb{R}[\bar{x}]$ such that $S = q^{-1}(0)$. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. For each $t \in \mathbb{R}$, define $Q_t \in \mathbb{R}[\bar{x}, y]$ by the formula $(y^2 + q^4(\bar{x}))(y^2 + \sum_{i=1}^n x_i^2 - 1) + t(y^2 + 2q^4(\bar{x}))$. Observe that Q_0^{-1} is equal to the disjoint union of S and of the standard sphere S^n of $\mathbb{R}^{n+1}$. Moreover, for each $t \in \mathbb{R}$, S is contained in $Q_t^{-1}(0)$. It is easy to see that, for each $t \in \mathbb{R} \setminus \{0\}$, Q_t is irreducible as element of $\mathbb{C}[\bar{x}, y]$. Choosing a small positive real number t, the algebraic subset $T := Q_t^{-1}(0)$ of $\mathbb{R}^{n+1}$ is irreducible and equal to the disjoint union of S and of a Nash submanifold M of $\mathbb{R}^{n+1}$ arbitrarily close to S^n. Since S is algebraic and disjoint from M, there is a regular map $\rho : T' \longrightarrow T$ from a nonsingular real algebraic set T' to T such that $\rho(T') = M$ and the restriction of ρ from T' to M is a Nash isomorphism. Consider the product variety $\widetilde{X} \times \mathbb{R}^{n+1-r}$ (which is irreducible) and let $\psi : T' \longrightarrow \widetilde{X} \times \mathbb{R}^{n+1-r}$ be a Nash embedding whose image is contained in a contractible open subset of $(\widetilde{X} \setminus A) \times (\mathbb{R}^{n+1-r} \setminus \{0\})$. Such a ψ exists because $\dim(\widetilde{X} \times \mathbb{R}^{n+1-r}) = n+1$ and T' is Nash diffeomorphic to S^n. Improving Theorem 4 and Lemma 5 of [1], we may suppose that $T'' := \psi(T')$ is a nonsingular algebraic subset of $\widetilde{X} \times \mathbb{R}^{n+1-r}$ and $\psi : T' \longrightarrow T''$ is a biregular isomorphism. Indicate by $g : T'' \longrightarrow T$ the regular map $\rho \circ \psi^{-1}$. Applying an improved version of the Real Algebraic Blowing Down Lemma (see Proposition 2.6.1 of [2]) to the data $\widetilde{X} \times \mathbb{R}^{n+1-r} \supset T'' \xrightarrow{g} T$, we obtain an irreducible algebraic subset P of some $\mathbb{R}^N$ homeomorphic to $(\widetilde{X} \times \mathbb{R}^{n+1-r}) \cup_g T$, containing $(A \times \mathbb{R}^{n+1-r}) \cup (\widetilde{X} \times \{0\})$ and equal to the disjoint union of S and of a Nash submanifold P' of $\mathbb{R}^N$ arbitrarily close to $\widetilde{X} \times \mathbb{R}^{n+1-r}$.

Step V: Dubois–Efroymson cut. In 1974, Dubois and Efroymson [3] proved the following dimension theorem: "Let W be an irreducible real algebraic set and let Z be an algebraic subset of W such that $c := \dim(W) - \dim(Z) \geq 2$. Then there exists an irreducible algebraic subset of W of codimension 1 containing Z". In [4], we improve this result as follows: "Let $e \in \{1, \ldots, c-1\}$ and let $F : W \longrightarrow \mathbb{R}^e$ be a regular map vanishing on Z. Suppose there is a nonsingular point p of W such that $F(p) = 0$ and the rank of the differential of F at p is equal to e. Then there exists a regular map $G : W \longrightarrow \mathbb{R}^e$ such that, for each $t \in (-1, 1) \setminus \{0\}$, the set $\{w \in W \,|\, F(w) + tG(w) = 0\}$ is an irreducible algebraic subset of W of codimension e containing Z". Applying the latter result to the real algebraic sets $P \supset S$, we obtain an irreducible real algebraic set X^* equal to the disjoint union of S and of a Nash submanifold M^* of $\mathbb{R}^N$ arbitrarily close to $\widetilde{X} \times \{0\}$. Observe that the set $A^* := (A \times \mathbb{R}^{n+1-r}) \cap M^*$ is algebraic and the natural projection $\mu : A^* \longrightarrow A$ is a Nash isomorphism.

Step VI: real algebraic blowing down. Define the regular map $f : A^* \sqcup S \longrightarrow S$ by $f(x) := \pi(\mu(x))$ if $x \in A^*$ and $f(x) := x$ if $x \in S$. By the above–mentioned Real Algebraic Blowing Down Lemma, $X^* \cup_f S$ is homeomorphic to an irreducible real algebraic set X'. By construction of X^* and of X', it follows the existence of a good semialgebraic homeomorphism from X to X'. $\square$

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Rational real algebraic models of topological surfaces

Johannes Huisman
(joint work with I. Biswas)

Let X be a nonsingular rational real projective surface. Then the set $X(\mathbb{R})$ of real points of X is a compact connected topological surface. Comessatti showed that $X(\mathbb{R})$ is either nonorientable, or it is diffeomorphic to one of S^2 and $S^1 \times S^1$ [2]. Conversely, each of these topological surfaces admits a *rational real algebraic model*, or *rational model* for short. In other words, if S is a compact connected topological surface which is either nonorientable or diffeomorphic to one of S^2 and $S^1 \times S^1$, then there is a nonsingular rational real projective surface X such that $X(\mathbb{R})$ is diffeomorphic to S.

Two rational models X and Y of S are said to be *isomorphic* if there is a birational map

$$f\colon X \dashrightarrow Y$$

such that $X(\mathbb{R})$ is contained in the domain of definition of f, and

$$f_{|X(\mathbb{R})}\colon X(\mathbb{R}) \longrightarrow Y(\mathbb{R})$$

is a diffeomorphism. Equivalently, X and Y are isomorphic models if and only if the real algebraic varieties $X(\mathbb{R})$ and $Y(\mathbb{R})$ are biregularly isomorphic in the sense of [1] (see [5, p. 517–518]).

We address the following question. Given a compact connected topological surface S, what is the number of nonisomorphic rational models of S?

It is known that the topological surfaces S^2, $S^1 \times S^1$ and $\mathbb{P}^2(\mathbb{R})$ have exactly one rational model, up to isomorphism. Mangolte has shown that the same holds for the Klein bottle (the 2-fold connected sum of the real projective plane) [4, Theorem 1.3].

Mangolte asked how large n should be so that the n-fold connected sum of the real projective plane admits more than one rational model, up to isomorphism; see the comments following Theorem 1.3 in [4]. The following theorem shows that there is no such integer n.

Theorem 1. *Let S be a compact connected real two-manifold. Assume that S is either nonorientable, or it is diffeomorphic to one of S^2 and $S^1 \times S^1$. Then there is exactly one rational model of S, up to isomorphism. In other words, any two rational models of S are isomorphic.*

Our proof is based on the Minimal Model Program for real algebraic surfaces developed in [3], and some explicit constructions of birational maps between real algebraic surfaces whose restrictions to the real points are diffeomorphisms.

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Tropical Welschinger invariants

Ilia Itenberg

(joint work with Viatcheslav Kharlamov and Eugenii Shustin)

The talk is devoted to an enumeration of real rational curves interpolating fixed collections of real points in a real algebraic surface Σ, more precisely, to the following question: *given a real divisor* D *and a generic collection* $\boldsymbol{w}$ *of* $c_1(\Sigma)\cdot D-1$ *real points in* Σ*, how many of the complex rational curves belonging to the linear system* $|D|$ *and passing through the points of* $\boldsymbol{w}$ *are real* ? By rational curves we mean irreducible genus zero curves and their degenerations, so that they form in $|D|$ a projective subvariety $S(\Sigma, D)$; this subvariety is called the *Severi variety.* A curve on a real surface Σ is called real, if the curve is invariant under the involution $c:\Sigma\to\Sigma$ defining the real structure of Σ.

While, under mild conditions on Σ and D, the number of complex curves in question is the same for all generic collections $\boldsymbol{w}$ (it equals to the degree of $S(\Sigma, D)$), it is no more the case for real curves (except few very particular situations).

J.-Y. Welschinger [8, 9] discovered a way to attribute weights ± 1 to the real solutions in question so that the number of real solutions counted with weights becomes independent of the choice of a generic collection of real points. As an immediate consequence, the absolute value of the Welschinger invariant $W_{\Sigma,D}$ provides a lower bound on the number $R_{\Sigma,D}(\boldsymbol{w})$ of real solutions: $R_{\Sigma,D}(\boldsymbol{w}) \geq |W_{\Sigma,D}|$.

In some cases (for example, in the case of toric Del Pezzo surfaces; recall that there are five toric Del Pezzo surfaces: the projective plane $\mathbb{P}^2$, the product $\mathbb{P}^1\times\mathbb{P}^1$ of projective lines, and $\mathbb{P}^2$ with k blown up points in general position, where $k=1,2$ or 3) Welschinger invariants can be calculated using Mikhalkin's approach [5, 6] which deals with a corresponding count of tropical curves. In tropical geometry, complicated non-linear algebro-geometric objects are replaced by simpler piecewise-linear ones. For example, tropical plane curves are piecewise-linear graphs whose edges have rational slopes. Tropical curves can be seen as algebraic curves over the tropical semiring $(\max, +)$.

Using the tropical approach, we proved (see [1]) the logarithmic equivalence for the Welschinger and Gromov-Witten invariants of any toric Del Pezzo surface equipped with its tautological real structure, *i.e.*, the real structure which is provided by the toric structure.

Theorem 1 (see [1]). *Let Σ be a toric Del Pezzo surface equipped with its tautological real structure, and D an ample divisor on Σ. The sequences $\log W_{\Sigma,nD}$ and $\log GW_{\Sigma,nD}$, $n \in \mathbb{N}$, of the Welschinger invariants and the corresponding Gromov-Witten invariants are asymptotically equivalent. More precisely, $\log W_{\Sigma,nD} = \log GW_{\Sigma,nD} + O(n)$ and $\log GW_{\Sigma,nD} = (c_1(\Sigma) \cdot D) \cdot n \log n + O(n)$.*

We also defined (see [2]) a series of relative tropical Welschinger-type invariants of real toric surfaces. In the Del Pezzo case, these invariants can be seen as real tropical analogs of relative Gromov-Witten invariants, and are subject to recursive formulas of Caporaso-Harris type.

In the present talk, we consider generic collections of real points on the projective plane blown up at 4 real points in general position and prove that the logarithmic equivalence of the Welschinger and Gromov-Witten invariants holds in this situation as well.

Theorem 2. *Let Σ be the projective plane $\mathbb{P}^2$ blown up at 4 real points in general position, and D an ample divisor on Σ. The sequences $\log W_{\Sigma,nD}$ and $\log GW_{\Sigma,nD}$, $n \in \mathbb{N}$, of the Welschinger invariants and the corresponding Gromov-Witten invariants are asymptotically equivalent.*

The proof is based on a new version of the correspondence theorem, whose proof in turn uses an appropriate tropical Caporaso-Harris type formulas. In particular, we get recursive formulas that allow one to calculate Welschinger invariants of $\mathbb{P}^2$ blown up at 4 real points in general position.

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Sums of hermitian squares, Connes' embedding problem and the BMV Conjecture

Igor Klep
(joint work with Markus Schweighofer)

Throughout this note R will denote an associative ring with 1. If R is endowed with an involution $x \mapsto x^*$, we denote by $\mathrm{Sym}(R) := \{x \in R \mid x = x^*\}$ the set of *hermitian* elements and by $\Sigma^2(R) := \{\sum_{i=1}^m x_i^* x_i \mid m \in \mathbb{N},\, x_i \in R\} \subseteq \mathrm{Sym}(R)$ the set of *sums of hermitian squares*. We introduce an equivalence relation (*cyclic equivalence*) on R by declaring that $x \overset{\mathrm{cyc}}{\sim} y$ means that $x - y$ is a sum of commutators in R.

Let $\mathbb{k} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{k}^{s\times s}$ be the $\mathbb{k}$-algebra of square matrices over $\mathbb{k}$ of size $s \in \mathbb{N} := \{1, 2, \dots\}$ with the involution $A \mapsto A^*$, where A^* is the conjugate transpose of A. It is easy to show that for all $A \in \mathrm{Sym}(\mathbb{k}^{s\times s})$,

(a) A is positive semidefinite $\Leftrightarrow A \in \Sigma^2(\mathbb{k}^{s\times s})$;
(b) $\mathrm{tr}(A) = 0 \Leftrightarrow A \overset{\mathrm{cyc}}{\sim} 0$;
(c) $\mathrm{tr}(A) \geq 0 \Leftrightarrow \exists B \in \Sigma^2(\mathbb{k}^{s\times s}) : A \overset{\mathrm{cyc}}{\sim} B$,

where tr denotes the trace of a matrix. Let $n \in \mathbb{N}$, $\bar{X} := (X_1, \dots, X_n)$ be variables (or symbols) and let $\langle \bar{X} \rangle$ be the semigroup generated freely by $\bar{X}$ whose elements are words in the n letters $X_1, \dots, X_n$ (including the empty word 1). Let $\mathbb{k}\langle \bar{X} \rangle$ denote the corresponding semigroup algebra, i.e., the free associative $\mathbb{k}$-algebra over $\bar{X}$ consisting of polynomials in n noncommuting variables $\bar{X}$ with coefficients in $\mathbb{k}$. We endow this ring with the involution $p \mapsto p^*$ given by $X_i^* = X_i$ and $a^* = \overline{a}$ for $a \in \mathbb{k}$. For each word $w \in \langle \bar{X} \rangle$, w^* is thus the reverse word. Given $f \in \mathbb{k}\langle \bar{X} \rangle$, we say that:

(a),(a)' f is positive semidefinite everywhere (on the hypercube) if $f(A_1, \dots, A_n)$ is positive semidefinite for all $s \in \mathbb{N}$ and (contractions) $A_i \in \mathrm{Sym}(\mathbb{k}^{s\times s})$;
(b),(b)' the trace of f vanishes everywhere (on the hypercube) if $\mathrm{tr}(A_1, \dots, A_n) = 0$ for all $s \in \mathbb{N}$ and (contractions) $A_i \in \mathrm{Sym}(\mathbb{k}^{s\times s})$;
(c),(c)' the trace of f is nonnegative everywhere (on the hypercube) if $\mathrm{tr}(A_1, \dots, A_n) \geq 0$ for all $s \in \mathbb{N}$ and (contractions) $A_i \in \mathrm{Sym}(\mathbb{k}^{s\times s})$.

A *contraction* is a matrix A with $\|A\| \leq 1$ and the *hypercube* is the set

$$\{(A_1, \dots, A_n) \in \mathrm{Sym}(\mathbb{R}^{s\times s})^n \mid s \in \mathbb{N},\, \|A_i\| \leq 1\}.$$

One can ask if these geometric conditions can be translated into algebraic identities in the algebra $\mathbb{k}\langle \bar{X} \rangle$ which are analogous to the case of the algebra $\mathbb{k}^{s\times s}$. Despite the huge activity in the commutative case, these "noncommutative questions" have been hardly examined before the breakthrough work of Helton [Hel] in 2002 (cf. McCullough and Putinar [MP]):

(a) $f \in \mathrm{Sym}(\Bbbk\langle\bar{X}\rangle)$ is positive semidefinite everywhere $\Leftrightarrow f \in \Sigma^2(\Bbbk\langle\bar{X}\rangle)$.

Later Helton and McCullough [HM] studied (a)', proving:

(a)' $f \in \mathrm{Sym}(\Bbbk\langle\bar{X}\rangle)$ is positive semidefinite on the hypercube $\Leftrightarrow$
$\forall\varepsilon > 0 : f + \varepsilon \in M_{\Bbbk\langle\bar{X}\rangle}$.

Here $M_{\Bbbk\langle\bar{X}\rangle}$ denotes the quadratic module generated by $\{1 - X_1^2, \dots, 1 - X_n^2\}$ in $\Bbbk\langle\bar{X}\rangle$, i.e., $M_{\Bbbk\langle\bar{X}\rangle} = \{\sigma + \sum_{i,j} g_{ij}^*(1 - X_i^2)g_{ij} \mid \sigma \in \Sigma^2,\, g_{ij} \in \Bbbk\langle\bar{X}\rangle\}$.

In our joint work [KS1, KS2] we study (b), (b)', (c) and (c)'. We prove that for all $f \in \mathrm{Sym}(\Bbbk\langle\bar{X}\rangle)$,

(b), (b)' the trace of f vanishes everywhere (on the hypercube) $\Leftrightarrow f \sim 0$.

We also prove the following variant of (c):

Theorem. *For $f \in \mathbb{C}\langle\bar{X}\rangle$, $\tau(f(A_1, \dots, A_n)) \geq 0$ for every separable II_1-factor $\mathcal{F}$ with trace τ and all contractions $A_1, \dots, A_n \in \mathrm{Sym}(\mathcal{F}) \Leftrightarrow$ for every $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon \overset{\mathrm{cyc}}{\sim} g \in M_{\mathbb{C}\langle\bar{X}\rangle}$ for some $g \in \mathbb{C}\langle\bar{X}\rangle$.*

We were able to show that the matrix variant of (c) is equivalent to a problem of Alain Connes [Con] concerning von Neumann algebras and dating back to 1976.

Connes' Conjecture. *Let $\mathcal{R}$ denote the hyperfinite and let $\mathcal{F}$ be a separable type II_1 factor. If $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$, then $\mathcal{F}$ can be embedded (as a ring with involution) in the ultraproduct $\mathcal{R}^{\mathscr{U}}$.*

Here $\mathcal{R}^{\mathscr{U}}$ is an ultraproduct in the sense of von Neumann algebras with trace and is itself a factor of type II_1. This embedding problem has recently gained a lot of attention: Kirchberg [Kir] has shown in 1993 that it is equivalent to some interesting problems in the theory of operator algebras and Banach spaces.

Algebraic reformulation of Connes' Conjecture. *If $f \in \mathrm{Sym}(\mathbb{R}\langle\bar{X}\rangle)$ and the trace of f is nonnegative on the hypercube, then for every $\varepsilon > 0$ there is a $g \in M_{\mathbb{R}\langle\bar{X}\rangle}$ such that $f + \varepsilon \overset{\mathrm{cyc}}{\sim} g$.*

This algebraic formulation entails entirely new approaches to Connes' problem. Under the hypothesis that the conjecture holds, we were for example able to prove a version with bounds on the degree of the representation using the theory of real closed fields, valuation theory and model theory. Moreover, it is possible to systematically consider polynomials in few variables or of small degree to obtain ideas. We could already show that the conjecture holds for a certain class of polynomials in two variables. Finally, one can do numerical experiments to test special classes of polynomials. It is in fact possible to maximize $\lambda \in \mathbb{R}$ such that $f - \lambda$ has the desired representation by solving a sequence of SDPs.

For the remainder of this note let $n = 2$ and write (X, Y) instead of (X_1, X_2). In [KS2] we give examples of polynomials $f \in \mathrm{Sym}(\Bbbk\langle X, Y\rangle)$ such that the trace of f is nonnegative everywhere, but there are no $g \in \Sigma^2(\Bbbk\langle X, Y\rangle)$ with $f \overset{\mathrm{cyc}}{\sim} g$.

(One may even replace f by $f + \lambda$ for $\lambda > 0$ and still get the same conclusion.) One particular example of such a polynomial is

$$\begin{aligned} f \;:=\; & X^6Y^6 + X^4Y^2X^2Y^4 + X^4Y^4X^2Y^2 + X^4Y^6X^2 + X^2Y^2X^4Y^4 + \\ & X^2Y^2X^2Y^2X^2Y^2 + X^2Y^2X^2Y^4X^2 + X^2Y^4X^4Y^2 + \\ & X^2Y^4X^2Y^2X^2 + X^2Y^6X^4 + Y^2X^6Y^4 + Y^2X^4Y^2X^2Y^2 + \\ & Y^2X^4Y^4X^2 + Y^2X^2Y^2X^4Y^2 + Y^2X^2Y^2X^2Y^2X^2 + \\ & Y^2X^2Y^4X^4 + Y^4X^6Y^2 + Y^4X^4Y^2X^2 + Y^4X^2Y^2X^4 + Y^6X^6. \end{aligned}$$

This polynomial is related to the so-called Bessis-Moussa-Villani (BMV) conjecture [BMV] from theoretical physics. An easily accessible equivalent formulation due to Lieb and Seiringer [LS] is the following:

Bessis-Moussa-Villani Conjecture. *Whenever $A, B \in \mathrm{Sym}(\Bbbk^{s\times s})$ are positive semidefinite, the polynomial $tr\big((A + tB)^m\big) \in \mathbb{R}[t]$ has nonnegative coefficients.*

The coefficient of t^k in p is the trace of $S_{m,k}(A, B)$, the sum of all words of length m in A and B, in which exactly k B's appear. It is known that the trace of $S_{m,k}(A, B)$ is nonnegative for all positive semidefinite A, B in the following cases:

(1) $k \le 2$ or $m - k \le 2$ (2) $s \le 2$
(3) $m = 6$ [Hi] (4) $m = 7$ [Hae].

In [KS2] we use sums of hermitian squares to study the conjecture, e.g. with

$$\begin{aligned} f_1 &:= 3X^3Y^4X^2 \in \Bbbk\langle X, Y\rangle, \\ f_2 &:= 3X^5Y^4 + 3X^3Y^2X^2Y^2 + 3XY^2X^4Y^2 \in \Bbbk\langle X, Y\rangle, \\ f_3 &:= 3XY^2X^2Y^2X^2 + 3XY^4X^4 \in \Bbbk\langle X, Y\rangle, \\ S_{9,4}(X^2, Y^2) &\overset{\text{cyc}}{\sim} f_1^*f_1 + f_2^*f_2 + f_3^*f_3, \end{aligned}$$

showing that the trace of $S_{9,4}(A, B)$ is nonnegative for positive semidefinite A, B. On the other hand, $S_{9,3}(X^2, Y^2)$ is *not* cyclically equivalent to an element of $\Sigma^2(\Bbbk\langle X, Y\rangle)$. Nevertheless, we were able to prove:

Theorem. *The BMV conjecture holds for $m \le 9$.*

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On the Schur-Szegö composition of polynomials in one variable

Vladimir Petrov Kostov

(joint work with Boris Shapiro)

Definition 1. A polynomial in one variable is called *hyperbolic* if it has only real roots. We write HP for "hyperbolic polynomial".

The Schur-Szegö composition of two polynomials $P(x) = \sum_{i=0}^{n} C_n^i a_i x^i$ and $Q(x) = \sum_{i=0}^{n} C_n^i b_i x^i$ is given by $P * Q(x) = \sum_{i=0}^{n} C_n^i a_i b_i x^i$. Let Pol_n denote the linear space of all polynomials in x of degree $\leq n$. We always use its basis $\mathcal{B} := (x^n, x^{n-1}, \ldots, 1)$. To $P \in Pol_n$ one can associate the operator T_P determined by the condition: $T_P(1+x)^n = P(x)$. One has $T_P(x^i) = a_i$, $i = 0, 1, \ldots, n$. We refer to the sequence $\{a_i\}$ as to the *diagonal sequence* of P. Any two such operators T_P and T_Q commute and their product $T_P T_Q$ corresponds to $P * Q$. The Schur-Szegö composition theorem reads:

Theorem 2. *Given any linear-fractional image $\mathcal{K}$ of the unit disk containing all the roots of P one has that any root of $P * Q$ is the product of some root of Q by $-\gamma$ where $\gamma \in \mathcal{K}$.*

Denote by $Hyp_n \subset Pol_n$ the set of all HPs and by Hyp_n^+ (resp. Hyp_n^-) its subset of HPs with all positive (resp. all negative) roots. Denote by $H_{u,v,w} \subset Hyp_n$ ($u, v, w \in \mathbf{N} \cup 0$, $u + v + w = n$) the set of all HPs with u negative and w positive roots and a v-fold zero root.

Proposition 3. *If $P, Q \in Hyp_n$ and if $Q \in Hyp_n^+$ or $Q \in Hyp_n^-$, then $P * Q \in Hyp_n$. Moreover, all roots of $P * Q$ lie in $[-M, -m]$ where M is the maximal and m is the minimal pairwise product of roots of P and Q.*

A diagonal sequence, (or an operator $T : Pol_n \to Pol_n$ acting diagonally in $\mathcal{B}$) is called a *finite multiplier sequence (FMS)*, if it sends Hyp_n into Hyp_n. The set $\mathcal{M}_n$ of all FMS is a semigroup. The following theorem defines a linear diffeomorphism of $\mathcal{M}_n$ and $\overline{Hyp_n^+} \cup \overline{Hyp_n^-}$.

Theorem 4. *For $T \in End(Pol_k^{\mathbf{R}})$ the following two conditions are equivalent:*

(i) T is a finite multiplier sequence;
(ii) All different from 0 roots of the polynomial $P_T(x) = \sum_{j=0}^{k} C_k^j \gamma_j x^j$ are of the same sign.

Proposition 5. *(see* [KoSh]*) Given P, $Q \in Pol_n^{\mathbf{C}}$ such that x_P, x_Q are roots respectively of P, Q of multiplicity m_P, m_Q with $m_P + m_Q \geq n$, one has that $-x_P x_Q$ is a root of $P * Q$ of multiplicity $m_P + m_Q - n$. (If $m_P + m_Q = n$, then $-x_P x_Q$ is not a root of $P * Q$.)*

Remark 6. If $m_P > 0$, $m_Q > 0$ and $m_P + m_Q < n$, then $-x_P x_Q$ might or might not be a root of $P * Q$. Example: -1 is a root of $((x-1)(x-2)(x-3)) * ((x-1)(x-4)(x-d))$ iff $d = 17/23$.

Proposition 7. *(see* [KoSh]*) For any $P \in H_{u,v,w}$ and any $Q \in Hyp_n^-$ one has $P * Q \in H_{u,v,w}$. In particular, Hyp_n^- is a semigroup w.r.t. the Schur-Szegö composition.*

The roots of P, Q and $P * Q$ involved in Proposition 5 (i.e. of the form $-x_P x_Q$ where $m_P + m_Q > n$) are called *A-roots*, the remaining roots of P, Q, $P * Q$ are called B-roots. With one exception – if 0 is a root of P, then it is considered as A-root of $P * Q$. Associate to $P \in Hyp_n$ its *multiplicity vector* (i.e. the vector whose components are the multiplicities of the distinct roots of P listed in the increasing order). For a root α of $P \in Hyp_n$ denote by $[\alpha]_-$ (resp. $[\alpha]_+$) the total number of roots of P to the left (resp. to the right) of α and by sign(α) the sign of α.

Theorem 8. *(see* [KoSh]*) For any $P \in Hyp_n$ and $Q \in Hyp_n^-$ the multiplicity vector of $P * Q$ is uniquely determined by Proposition 7 and the following conditions:*

(i) For any A-root $\alpha \neq 0$ of P and any A-root β of Q one has $[-\alpha\beta]_- = [\alpha]_- + [\beta]_{sign(\alpha)}$.
(ii) Every B-root of $P * Q$ is simple.

Corollary 9. *The Schur-Szegö composition restricted to Hyp_n^- induces a semigroup structure on the set of all multiplicity vectors considered as ordered partitions of n. E.g.* $(2,14,1) * (5,6,6) = (1,1,2,1,1,1,3,1,1,1,3,1)$.

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Real algebraic morphisms represent few homotopy classes

Wojciech Kucharz
(joint work with Jacek Bochnak)

Let Y be a compact nonsingular real algebraic set. We define a numerical invariant $\beta(Y)$ to be the supremum of all nonnegative integers n with the following property: for every n-dimensional compact connected nonsingular real algebraic set X, every continuous map from X into Y is homotopic to a regular map. One easily sees that if d is an integer satisfying $0 \leq d \leq \beta(Y)$, then every continuous map from any d-dimensional compact connected nonsingular real algebraic set into Y is homotopic to a regular map. There is a subtle connection between $\beta(Y)$ and the subgroups $H_i^{\mathrm{alg}}(Y, \mathbb{Z}/2)$ of $H_i(Y, \mathbb{Z}/2)$, $i \geq 0$, generated by the homology classes represented by i-dimensional algebraic subsets of Y.

Theorem 1. *If* $\dim Y = p$ *and* $H_{p-k}^{\mathrm{alg}}(Y, \mathbb{Z}/2) \neq 0$ *for some* $k \geq 1$*, then* $\beta(Y) \leq k$*. In particular,* $\beta(Y) \leq \dim Y$*, provided* $\dim Y \geq 1$.

This result has the following two consequences.

Corollary 2. *If the* k*-th Stiefel-Whitney class of* Y *is nonzero for some* $k \geq 1$*, then* $\beta(Y) \leq k$*. In particular,* $\beta(Y) = 0$ *or* $\beta(Y) = 1$*, provided* Y *is nonorientable.*

Corollary 3. *If* $\dim Y = p$ *and* $H_k(Y, \mathbb{Z}/2) \neq 0$ *for some* k *satisfying* $0 < k < p$*, then*

$$\beta(Y) \leq \begin{cases} \max\{k, p-k\} - 1 \leq p - 2 & \text{for } k \neq \frac{p}{2} \\ \frac{p}{2} & \text{for } k = \frac{p}{2} \end{cases}$$

The computation of the exact value of $\beta(Y)$ is hard, except in some special cases. If $\dim Y = 0$, then $\beta(Y) = \infty$. For $\dim Y = 1$, we have $\beta(Y) = 1$ if Y is rational, and $\beta(Y) = 0$ otherwise. The case $\dim(Y) = 2$ is more complicated.

Theorem 4. *Assume* $\dim Y = 2$*. Then* $\beta(Y) = 0$ *or* $\beta(Y) = 1$*, and either value can occur. If* Y *is either rational or homeomorphic to the* 2*-sphere, then* $\beta(Y) = 1$.

There are several conjectures related to the problem under consideration. We only mention two of them.

Conjecture 5. *If* $\dim Y \geq 2$*, then* $\beta(Y) \leq \dim Y - 1$*. In particular,* $\beta(S^n) = n-1$ *for* $n \geq 2$*, where* S^n *is the unit* n*-sphere.*

It follows from Theorem 1 and [2] that $\beta(S^n) = n - 1$ if n is even.

Conjecture 6. *For any pair* (n, p) *of nonnegative integers, every continuous map from* S^n *into* S^p *is homotopic to a regular map.*

This conjecture is true for some pairs (n, p), c.f. [1, 5, 6]

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Positive polynomials and fibre products of varieties

Salma Kuhlmann

Approximation of positive polynomials by sums of squares originates around Hilbert's 17th problem, and has today important applications to polynomial optimisation.

In the first part of my talk, I will survey the main recent results achieved on that topic: I will consider positive (respectively, non-negative) polynomials on compact (respectively, unbounded) semi-algebraic sets. I will discuss representations in the associated preorderings (respectively, linear representations in the associated quadratic module). The representation often depends on the dimension of the semi-algebraic set; I will present stronger results in the low dimensional case.

In the second part of the talk, I will consider two special situations: (i) when the positive polynomials under consideration are invariant under the action of a linear reductive group (ii) when the positive polynomials under consideration are sparse (that is, satisfy some separation and overlap conditions on the variables appearing in the monomials).

The problem of representing invariant polynomials by sums of squares is reduced to the usual representation problem by virtue of the orbit map. We discuss examples where the orbit map is exploited to this end.

The problem of sparse representations of sparse polynomials was first considered by Kojima and Lasserre. They were interested in optimization under the sparsity relaxation. They were able to give a sparse version of Putinar's theorem for Archimedean modules, provided that the overlap of the variables satisfies the so-called running intersection property. We present an algebraic interpretation (in the language of fibre products of algebraic varieties) of the Kojima-Lasserre results. We give a sparse version of the moment problem, and of the various other approximations of the cone of positive semi-definite polynomials.

We present some open problems.

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Trajectories of Subriemannian gradients of polynomials

Krzysztof Kurdyka
(joint work with Din Si-Tiep, Patrice Orro)

Let $X_1, \dots, X_{n-1}$ be polynomial vector fields on $\mathbb{R}^n$. Assume that they are linearly independent at any $x \in \mathbb{R}^n$. We denote by Δ_x the hyperplane generated by $X_1(x), \dots, X_{n-1}(x)$. We define now the powers of Δ in the following way

$$\Delta^1 = \Delta,\ \Delta^{r+1} = span(\{[X,Y],\ X \in \Delta^r,\ Y \in \Delta\} \cup \Delta^r),$$

where $[X, Y]$ stands for Lie brackets. We say that the distribution Δ is *totally non-holonomic* (or that it satisfies *Hörmander's condition*) if there exists an integer r such that

$$\Delta^r_x = \mathbb{R}^n, x \in \mathbb{R}^n.$$

A subriemannian metric g (see e.g. [2]) on the distribution Δ is a scalar product g_x on each Δ_x which varies smoothly with x. In the case we consider it will be chosen in such a way that $X_1(x), \dots, X_{n-1}(x)$ is an orthonormal basis. For $\xi \in \Delta_x$ we write $|\xi| = g_x(\xi,\xi)^{1/2}$. We say that an absolutely continuous curve $\gamma : [a,b] \to \mathbb{R}^n$ is *horizontal* if

$$\gamma'(t) \in \Delta_{\gamma(t)},$$

for any $t \in (a,b)$ such that $\gamma'(t)$ exists. We define subriemannian length of a horizontal curve γ as

$$l(\gamma) = \int_a^b |\gamma'(t)|dt.$$

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, we define *horizontal gradient* of f as

$$\nabla^h f(x) = \sum_{i=1}^{n-1} \frac{\partial f}{\partial X_i}(x) X_i(x).$$

Of course $\nabla^h f$ is a vector field on $\mathbb{R}^n$. It depends on the distribution Δ and the metric g. We study some properties of trajectories of $\nabla^h f$ that is the curves $\gamma : [a,b] \to \mathbb{R}^n$ such that

$$\gamma'(t) = \nabla^h f(\gamma(t)),\ t \in (a,b).$$

Finally we define *horizontal critical set* of f as $V_f = \{\nabla^h f = 0\}$.

Example 1. Heisenberg's distribution on $\mathbb{R}^3$ is defined by

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3},\ X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}$$

For $f(x) = x_3$ trajectories of $\nabla^h f$ are spiraling around $V_f = \{x_1 = x_2 = 0\}$. When we consider their length between two levels of f, say between $\{x_3 = 0\}$ and $\{x_3 = 1\}$, we can compute that their length equals $\frac{2}{r}$, where r is distance of $\gamma(t)$ to V_f,which is constant in this case. Hence, unlike the Riemannian case studied by Łojasiewicz [4] and others [3], [1], these lengths are not uniformly bounded.

For any $\gamma > 0$ we define a subriemannian metric $g_\gamma(x) = dist(x, V_f)^\gamma g(x)$ which is degenerate on V_f. It induces a distance δ_γ on $\mathbb{R}^n \setminus V_f$, as infimum of length (in g_γ metric) of horizontal curves joining two given points. For general distribitions of the above type we have

Theorem 2. *For a given polynomial $f : \mathbb{R}^n \to \mathbb{R}$ there exists $\gamma > 0$ such that if $x(t), t \in [t_1, t_2]$ is a trajectory of $\nabla^h f$ in compact subset B of $\mathbb{R}^n$, then the length (in g_γ metric) of $x(t)$ is finite. Precisely is bounded by $c|f(x(t_2)) - f(x(t_1))|$ for some $c = c(B) > 0$.*

We obtain more precise results for some special type of distributions defined by

$$X_i = \frac{\partial}{\partial x_1} + P_1(x_1, .., x_{n-1})\frac{\partial}{\partial x_n},\ i = 1, \dots, n-1, \tag{1}$$

where P_i are polynomials such that $(P_1, \dots, P_{n-1})$ is not a gradient. Note that the classical Heisenberg's and Martinet's flat distribution are of that type. Let $\mathbb{R}_d[x]$ denote the space of polynomials in n variables od degree $\leq d$.

Theorem 3. *There is an open dense semialgebraic set $\mathcal{L}_d \subset \mathbb{R}_d[x]$ such that if $f \in \mathcal{L}_d$, then V_f is a smooth algebraic curve (or an empty set) moreover f is a Morse function on V_f. Clearly $\mathcal{L}_d$ depends on the distribution (1).*

Theorem 4. *If $f \in \mathcal{L}_d$, then Theorem 1 holds with $\gamma = 1$, moreover the distance δ_1 extends to a distance on $\mathbb{R}^n$.*

Theorem 5. *Assume $f \in \mathcal{L}_d$, if $x(t), t \in [t_1, t_2)$ is a trajectory of $\nabla^h f$ in a compact subset B of $\mathbb{R}^n$, then $\lim_{t\to t_2} x(t) = x_0$ exists. Moreover, $x_0 \in V_f$ if and only if $t_2 = \infty$.*

We have also an exemple of a distribution of type (1) with a polynomial function $f \notin \mathcal{L}_d$ such that some trajectory $\nabla^h f$ has no limit, it accumulates on a cycle. This cannot happen in the Riemmnnian case [4].

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Real singular Del Pezzo surfaces and threefolds fibred by rational curves

Frédéric Mangolte
(joint work with Fabrizio Catanese)

1. Introduction

Let $f\colon W \to X$ be a real smooth projective threefold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable. Then, by [3, Theorem 1.1], a connected component $M \subset W(\mathbb{R})$ is obtained from a Seifert fibred manifold or a connected sum of lens spaces by taking connected sums with a finite number of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number of copies of $S^1 \times S^2$.

Let $\nu := \nu(M)$ be the integer defined as follows:

(1) If $g\colon M \to F$ is a Seifert fibration, ν denotes the number of multiple fibres of g
(2) If M is a connected sum of lens spaces, ν denotes the number of lens spaces

Theorem 1. If X is a geometrically rational surface, then $\nu \leq 4$.

This result answers in affirmative a question of Kollár who proved in 1999 that $\nu \leq 6$ and suggested that 4 would be the sharp bound.

We derive Theorem 1 from a careful study of real singular Del Pezzo surfaces with only rational double points as singularities (see Theorem 2).

Thanks to the Minimal Model Program over $\mathbb{R}$, the original setting $f: W \to X$ is replaced by the following: W is a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$ and $f: W \to X$ is a rational curve fibration over $\mathbb{R}$ such that $-K_W$ is f-ample.

Let M be a connected component of the topological normalization $\overline{W(\mathbb{R})}$ (see next section) and assume that M is a Seifert fibred 3-dimensional manifold or a connected sum of lens spaces. Then, by [2, Thm. 2.6], there exists a Werther fibration $g: M \to F$ over a 2-manifold with boundary. Werther fibrations are defined in [2], but for our purpose it is sufficient to recall that $g|_{g^{-1}(F\setminus\partial F)}$ is a Seifert fibration. Now there is an injection from the set of multiple fibres of $g|_{g^{-1}(F\setminus\partial F)}$ to the set of singular points of X contained in $f(M)$ which are of type A^+ and globally separating (see next section). Under this injection, the multiplicity of the Seifert fibre equals $\mu + 1$ if the singular point is of type A^+_μ.

Theorem 1 is now a corollary of the following:

Theorem 2. Let X be a projective surface defined over $\mathbb{R}$. Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of $\overline{X(\mathbb{R})}$ contains at most 4 Du Val singular points which are not of type A^- and globally nonseparating.

2. Rational surfaces with Du Val singularities

On a surface, a rational double point is called a Du Val singularity. Over $\mathbb{C}$, these singularities are classified by their Dynkin diagrams, namely A_μ, $\mu \geq 1$, D_μ, $\mu \geq 4$, E_6, E_7, E_8.

Over $\mathbb{R}$, there are more possibilities. In particular, a surface singularity will be said to be *of type* A^+_μ if it is real analytically equivalent to

$$x^2 + y^2 - z^{\mu+1} = 0, \ \mu \geq 1 \ ;$$

and *of type* A^-_μ if it is real analytically equivalent to

$$x^2 - y^2 - z^{\mu+1} = 0, \ \mu \geq 1 \ .$$

The type A^+_1 is real analytically isomorphic to A^-_1; otherwise, singularities with different names are not isomorphic.

Definition 1. Let V be a symplicial complex with only a finite number of points $x \in V$ where V is not a manifold. Define the *topological normalization*

$$\overline{n}: \overline{V} \to V$$

as the unique proper continuous map such that $\overline{n}$ is a homeomorphism over the set of points where V is a manifold and $\overline{n}^{-1}(x)$ is in one-to-one correspondence with the connected components of a good punctured neighborhood of x in V otherwise.

Definition 2. Let X be a real algebraic surface with isolated singularities, and let $x \in X(\mathbb{R})$ be a singular point of type $A^{\pm}_{\mu}$ with μ odd. The topological normalization $\overline{X(\mathbb{R})}$ has two connected components locally near x. We will say that x is *globally separating* if these two local components are on different connected components of $\overline{X(\mathbb{R})}$ and *globally nonseparating* otherwise.

Definition 3. Let X be a projective surface with Du Val singularities, let

$$\mathcal{P} := \operatorname{Sing} X \setminus \left\{ x \text{ is of type } A^{-}_{\mu}, \text{ and } x \text{ is globally nonseparating} \right\} .$$

Lemma 1 (Kollár)**.** Let $\overline{n}\colon \overline{X(\mathbb{R})} \to X(\mathbb{R})$ be the topological normalization, and define $M_1, M_2, \ldots, M_r$ be the connected components of $\overline{X(\mathbb{R})}$. The unordered sequence of numbers $\#(\overline{n}^{-1}(\mathcal{P}) \cap M_i) := m_i$, $i = 1, 2, \ldots, r$ is an invariant of extremal birational contractions of projective surfaces with Du Val singularities.

Applying the Minimal Model Program over $\mathbb{R}$ to a geometrically rational surface with Du Val singularities, we reduce the proof of Theorem 2 to the study of singular Del Pezzo surfaces of degree one.

Recall that a Del Pezzo surface X is by definition a surface whose anticanonical divisor is ample. We add the adjective *Du Val* to emphasize that we allow X to have Du Val singularities. Let X be a real Du Val Del Pezzo surface and let $S \to X$ be the minimal resolution of singularities. The smooth surface S has nef anticanonical divisor and is called a *weak Del Pezzo surface* by many authors.

We obtain a Del Pezzo surface X of degree 1 by blowing up a finite number of pairs of conjugate imaginary smooth points and some real smooth point (there are several choices to do this).

The anticanonical model of a Del Pezzo surface X of degree 1 is a ramified double covering $q\colon X \to Q$ of a quadric cone $Q \subset \mathbb{P}^3$ whose branch locus is the union of the vertex of the cone and a cubic section not passing through the vertex (see e.g. [1, Exposé V]).

Let X' be the singular elliptic surface obtained from X by blowing up the pull-back by q of the vertex of the cone.

The surface X' is a ramified double covering of the Hirzebruch surface $\mathbb{F}_2$ whose branch curve is the union of the unique section of negative selfintersection, the section at infinity Σ_∞, and a trisection B disjoint from Σ_∞.

If the trisection is irreducible, then it has at most 4 singular points because it has genus 4. The heart of the proof is the study of normal forms for the reducible B. After studying the normal forms for the reducible branch curves, it appears that in almost all cases, the number of singular points is less than or equal to 4. There will remain only two cases to examine separately. In one case, the fifth point turns out to be of type $A^{+}_{1} \cong A^{-}_{1}$ globally nonseparating and in the second case, the fifth point turns out to be of type A^{-}_{2}.

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Fixed points of automorphisms of real algebraic curves

Jean-Philippe Monnier

In this note, a real algebraic curve X is a proper geometrically integral scheme over $\mathbb{R}$ of dimension 1. Let g denote the genus of X; throughout the paper we assume $g \geq 2$ and $X(\mathbb{R}) \neq \emptyset$.

An automorphism φ of X is an isomorphism of schemes of X with itself. Seeing X as a compact Klein surface, May proved in that the order of a group of automorphisms of X is bounded above by $12(g-1)$. Moreover, he also proved that the maximum possible order of an automorphism of X is $2g+2$. The corresponding results for Riemann surfaces are well-known; Hurwitz proved that a compact Riemann surface of genus $g \geq 2$ cannot have more than $84(g-1)$ automorphisms and Wiman showed that the order of a cyclic group of automorphisms of a compact Riemann surface of genus $g \geq 2$ is at most $4g+2$.

If $\varphi \in \mathrm{Aut}(X)$ then φ extends to an automorphism $\varphi_{\mathbb{C}}$ of $X_{\mathbb{C}} = X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$ such that $\overline{\varphi_{\mathbb{C}}(Q)} = \varphi_{\mathbb{C}}(\bar{Q})$ for any closed point Q of $X_{\mathbb{C}}$. We will denote by $\mu(\varphi_{\mathbb{C}})$ (resp. $\mu_{\mathbb{R}}(\varphi)$) the number of closed (resp. real closed) fixed points of $\varphi_{\mathbb{C}}$ (resp. φ). The Riemann-Hurwitz inequality for the number of fixed points of an automorphism gives

$$\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g - 2|\varphi| g'}{|\varphi| - 1}.$$

Let s be the number of coonected components of $X(\mathbb{R})$, we give new Riemann-Hurwitz inequalities for real curves depending on g and s.

Theorem 1. *Let φ be a non-trivial automorphism of order N of a real curve. If $\pi : X \to X/\langle\varphi\rangle$ has at least one real ramification point then $\mu(\varphi_{\mathbb{C}}) \leq 2 + \frac{2g-2s+\mu_{\mathbb{R}}(\varphi^{\frac{N}{2}})}{|\varphi|-1}$. Consequently, $\mu(\varphi_{\mathbb{C}}) - \mu_{\mathbb{R}}(\varphi^{\frac{N}{2}}) \leq 2 + \frac{2g-2s}{|\varphi|-1} \leq 2(g+1-s)$. If $|\varphi| \geq g+2-s$ then $\mu(\varphi_{\mathbb{C}}) - \mu_{\mathbb{R}}(\varphi^{\frac{N}{2}}) \leq 2$.*

Theorem 2. *Let φ be a non-trivial automorphism of X such that $\pi : X \to X' = X/\langle\varphi\rangle$ is without real ramification points. Then*

$$\mu(\varphi_{\mathbb{C}}) \leq 4 + 2\frac{g+1-s}{|\varphi|-1} \text{ if } s > 1$$

and

$$\mu(\varphi_{\mathbb{C}}) \leq 4 + 2\frac{g-1}{|\varphi|-1} \text{ if } s = 1.$$

Consequently, $\mu(\varphi_{\mathbb{C}}) \leq 4$ *if* $|\varphi| \geq g+3-s$ *(resp.*$|\varphi| \geq g+1$*) and if* $s > 1$ *(resp.* $s = 1$*).*

We give some nice consequences of the previous theorems.

Corollary 3. *Let* φ *be an automorphism of order* $N > 1$ *of an M-curve.*

1. *If one of the fixed points of* $\varphi_{\mathbb{C}}$ *is real then all are real.*
2. *If* $N > 2$ *and if* $\pi : X \to X/\langle\varphi\rangle$ *has at least one real ramification point then* $\varphi_{\mathbb{C}}$ *is fixed point free.*
3. *If* $\pi : X \to X' = X/\langle\varphi\rangle$ *is without real ramification points then* $\mu(\varphi_{\mathbb{C}}) \leq 4$*.*

Using these new Riemann-Hurwitz inequalities , we derive some consequences concerning the maximum order of an automorphism and the maximum order of an abelian group of automorphisms of a real curve. We also bound the full group of automorphisms of a real hyperelliptic curve.

Theorem 4. *Let* G *be a group of automorphisms of* X *of order* $N > 1$ *such that* $\pi : X \to X/G$ *has at least one real ramification point.*

1. *If* G *is cyclic then* $|G| \leq 2s \leq 2g+2$*.*
2. *If* G *is abelian then* $|G| \leq \inf\{4s, 2g+2+4(g+1-s)\} \leq 3g+3$*.*

Theorem 5. *Let* G *be a group of automorphisms of* X *of order* $N > 1$ *such that* $\pi : X \to X/G$ *is without real ramification point.*

1. *If* G *is cyclic then* $|G| \leq \sup\{2g+4-s, 2g+2-\frac{2}{3}s\}$ *if* $s > 1$ *and* $|G| \leq 2g+2$ *if* $s = 1$*.*
2. *If* G *is abelian then* $|G| \leq g+3+2(g+1-s) \leq 3g+3$*.*

Theorem 6. *Let* X *be a real hyperelliptic curve.*

1. *If the hyperelliptic involution* $\imath$ *has at least a real fixed point (e.g. if* $s \geq 3$*) then* $|\mathrm{Aut}(X)| \leq 4s \leq 4g+4$*.*
2. *If the hyperelliptic involution* $\imath$ *does not have any real fixed point then* $|\mathrm{Aut}(X)| \leq 4g+4$*.*

Blow-analytic equivalence of two variable real analytic function germs

Adam Parusiński

(joint work with Satoshi Koike)

We give a complete characterisation of blow-analytic equivalence classes of 2 variable function germs.

Theorem 1. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *and* $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be real analytic function germs. Then the following conditions are equivalent*

1. f *and* g *are blow-analytically equivalent.*
2. f *and* g *have weakly isomorphic minimal resolution spaces.*
3. *The real tree models of* f *and* g *are isomorphic.*

This theorem can be stated in the oriented and non-oriented case. In this report we consider only the non-oriented case.

We say that $f(x,y)$ and $g(x,y)$ have *weakly isomorphic resolution spaces* if there exist resolutions $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^2, 0)$, $\tilde\mu : (\tilde M, \tilde\mu^{-1}(0) \to (\mathbb{R}^2, 0)$ of $f(x,y)$ and $g(x,y)$, respectively, and a homeomorphism $\Phi : M \to \tilde M$ such that:

(1) $\Phi(\mu^{-1}(0)) = \tilde\mu^{-1}(0)$ and $\Phi((f \circ \mu)^{-1}(0)) = (g \circ \tilde\mu)^{-1}(0)$.
(2) If C is a component of $(f \circ \mu)^{-1}(0)$ then $\operatorname{mult}_C f \circ \mu = \operatorname{mult}_{\phi(C)} g \circ \tilde\mu$.
(3) $f \circ \mu(p) > 0$ iff $g \circ \tilde\mu(\Phi(p)) > 0$.

Blow-analytic equivalence. The blow-analytic equivalence was introduced by Tzee-Char Kuo at the end of 1970's [5], [6], [7]. For survey articles see [1], [2].

We say that two real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exist real modifications $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^n, 0)$, $\mu' : (M', \mu'^{-1}(0)) \to (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (M', \mu'^{-1}(0))$ which induces a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ h$, that is the following diagram is commutative:

$$\begin{array}{ccccc} (M, \mu^{-1}(0)) & \xrightarrow{\mu} & (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle h}\downarrow & & \| \\ (\tilde M, \tilde\mu^{-1}(0)) & \xrightarrow{\tilde\mu} & (\mathbb{R}^n, 0) & \xrightarrow{g} & (\mathbb{R}, 0) \end{array}$$

Let X, Y be a real analytic connected manifolds of pure dimension n. We say that a proper real analytic mapping $\sigma : X \to Y$ is a *real modification* if there exist complexifications $X_{\mathbb{C}}, Y_{\mathbb{C}}$ of X and Y, respectively, and a holomorphic extension $\sigma_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ of σ, such that $\sigma_{\mathbb{C}}$ *is an isomorphism in the complement of a closed nowhere dense subset* B *of* $X_{\mathbb{C}}$. (That is $\sigma_{\mathbb{C}}$ resticted to $X_{\mathbb{C}} \setminus B$ is open and an isomorphism onto its image.)

Note that a real analytic map that is an isomorphism in the complement of a closed nowhere dense subset is not necessarily a real modification, as for instance $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma(x) = x^3$. Using an argument similar to the classical elimination of indeterminacy of the rational maps between algebraic surfaces, we show that

Theorem 2. *Let X, Y be connected nonsingular real analytic surfaces and let $\sigma : X \to Y$ be a proper surjective real analytic map. Then σ is a real modification if and only if it is a composition of point blowings-up.*

Idea of proof of Theorem 1. If $f(x,y)$ and $g(x,y)$ are blow-analytically equivalent then, by Theorem 2, there exists a commutative diagram

$$\begin{array}{ccccccc} (M, \mu'^{-1}(E_1)) & \xrightarrow{\mu'} & (\widetilde{\mathbb{R}^2}, E_1) & \xrightarrow{\pi} & (\mathbb{R}^2, 0) & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle \exists h_1 ?}\downarrow & & {\scriptstyle h}\downarrow & & \| \\ (\tilde M, \tilde{\mu'}^{-1}(\tilde E_1)) & \xrightarrow{\tilde\mu'} & (\widetilde{\mathbb{R}^2}, \tilde E_1) & \xrightarrow{\tilde\pi} & (\mathbb{R}^2, 0) & \xrightarrow{g} & \mathbb{R} \end{array}$$

where π and $\tilde\pi$ denote the blowing-up of the origin, μ' and $\tilde\mu'$ are compositions of point blowings-up, Φ is an analytic isomorphism and h is a homeomorphism such that $f = g \circ h$. Using the combinatorial properties of dual graphs of real resolutions we show the existence a homeomorphism h_1.

Therefore, by induction, f and g are *cascade blow-analytic equivalent*, i. e.

$$\begin{array}{ccccccccccc} M & \xrightarrow{b_k} & M_{k-1} & \xrightarrow{b_{k-1}} & \cdots & \xrightarrow{b_2} & M_1 & \xrightarrow{b_1} & \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R} \\ \Phi\downarrow & & \downarrow h_{k-1} & & & & \downarrow h_1 & & \downarrow h & & \| \\ \tilde{M} & \xrightarrow{\tilde{b}_k} & \tilde{M}_{k-1} & \xrightarrow{\tilde{b}_{k-1}} & \cdots & \xrightarrow{\tilde{b}_2} & \tilde{M}_1 & \xrightarrow{b_1} & \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R}, \end{array} \tag{1}$$

where $b_i, \tilde{b}_i$ are point blowings-up and h_i are homeomorphisms.

A function blow-analytically equivalent to normal crossings is normal crossings (we show it for two variable function germs, the general case is open) and hence we may get rid of unnecessary blowings-up and assume that both $b_1 \circ \cdots \circ b_k$ and $\tilde{b}_1 \circ \cdots \circ \tilde{b}_k$ are the minimal resolutions of f and g resp.. This shows the implication (1) $\Longrightarrow$ (2) of Theorem 1. The implication (2) $\Longrightarrow$ (1) is fairly standard.

The following properties of *cascade blow-analytic homeomorphisms* (i.e. the homeomorphisms like h in (1)), are used in the proof of (1)$\Longleftrightarrow$ (3) of Theorem 1.

Theorem 3. *Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a cascade blow-analytic homeomorphism that does not change the orientation. Then*

- *(a) h preserves the Puiseux characteristics sequence of real analytic demi-branches $\gamma : [0, \varepsilon) \to \mathbb{R}^2$, the signs of coefficients at the Puiseux characteristic exponents, and the order of contact between two real analytic demi-branches.*
- *(b) There exists a constant $C > 0$ such that for (x, y) close to the origin*

$$C^{-1}\|(x,y)\| \le \|h(x,y)\| \le C\|(x,y)\|, \qquad C^{-1} \le Jac(h)(x,y) \le C.$$

Remark 1. Kobayashi and Kuo [3] constructed an example of a blow-analytic homeomorphism that sends a smooth curve to a singular one and vice versa. By Theorem 3 such a homeomorphism cannot be a cascade one.

Real tree model. The real tree model is an analog of the tree model of [8] of a complex analytic function germ and, for a given real analytic $f(x, y)$, encodes the information about the order contact and the Puiseux characteristic exponents of the Newton-Puiseux roots of f. For a complex root of f

$$x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N}, + \cdots \qquad N \le n_1 < n_2 < \cdots .$$

it takes into account its part up to the first non-real coefficient. The real tree model contains also the information about the signs of coefficients at the Puiseux characteristic exponents. Thanks to Theorem 3 the proof of (1) $\Longleftrightarrow$ (3) of Theorem 1 is based on a fairly straightforward computation of the change of the tree model by a blowing-up.

Examples. Abderrahmane showed that blow-analytically equivalent weighted homogeneous singular $f(x, y)$ and $g(x, y)$ have the same weights. The blow-analytic classification of Brieskorn two variable singularities $\pm x^p \pm y^q$ was obtained in [4].

These results can be also easily verified by Theorem 1. Here are examples that cannot be distinguished by the previously known methods

(1) $x^3 + y^5$ and $x^3 - y^5$ are not blow-analytically equivalent by an orientation preserving homeomorphism.
(2) $x(x^3 - y^5)(x^3 + y^5)$, $x(x^3 - y^5)(x^3 - 2y^5)$ are not blow-analytically equivalent.

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Pólya's Theorem with Zeros

Victoria Powers

Let $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_n]$ and let $\mathbb{R}^+[X]$ denote polynomials in $\mathbb{R}[X]$ with nonnegative coefficients.

Pólya's Theorem [2] says that if p is a homogeneous polynomial in n variables which is positive on the standard n-simplex, then for a sufficiently large exponent N, $(x_1 + \cdots + x_n)^N p \in \mathbb{R}^+[X]$. This elegant and beautiful result has many applications, both in pure and applied mathematics.

In [3], the second and third authors gave an explicit bound for the exponent N in terms of the size of the coefficients and the minimum value of p on the simplex. This result has been used by other authors in applications; for example M. Schweighofer [5] used this quantitative Pólya's Theorem to give an algorithmic proof of Schmüdgen's Positivstellensatz, and de Klerk and Pasechnik [1] used it to give results on approximating the stability number of a graph.

One can ask the following: When does Pólya's Theorem holds if the condition "positive on Δ_n" is relaxed to "nonnegative on Δ_n"? Previously, in joint work with B. Reznick [4], we extended Pólya's theorem with bound to forms which are positive on the simplex apart from a certain type of zero on the corners of the simplex.

In recent joint work with M. Castle and B. Reznick (unpublished), we study the question further. In particular, we give a constructive version, with degree bounds, of a result of M. Schweighofer from [6] which is, roughly speaking, a localized version of Pólya's Theorem:

Proposition 1. *Given $p \in \mathbb{R}[X]$ and suppose there exists closed $S \subseteq \Delta_n$ such that there are homogeneous $g_1, \dots, g_m \in \mathbb{R}[X]$, and $h_1, ..., h_m \in \mathbb{R}[X]^+$ with*

(1) $p = g_1 h_1 + \cdots + g_m h_m$
(2) $g_i(x) > 0$ for all $x \in S$.

Suppose further that there exists a closed set T with $T \subseteq S \subseteq \Delta_n$ and $B \in \mathbb{N}$ with the following property: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \operatorname{supp}(h_i)$ for some i, and $|\beta| \geq B$, then $\frac{\beta}{|\beta|} \in S$. Then there exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, the coefficient of X^α in $(X_1 + ... + X_n)^N f$ is nonnegative.

In particular, for each i, let $k(i)$ be the bound from Lemma 1 for g_i on S, i.e.,

$$k(i) = \frac{d_i(d_i - 1)}{2} \frac{L(g_i)}{\lambda_i} - d_i,$$

where λ_i is the minimum of g_i on S and $d_i = \deg g_i$. Then we can take

$$N = \max\{k(g_1), \dots, k(g_m), B\}.$$

We can apply the proposition to give an improvement of the results in [4]:

Suppose p is positive on Δ_n except for a zero at one $v_1 := (1, 0, \dots 0)$. Let $p_d(x_2, \dots, x_n)$ be the leading coefficient of p as a polynomial in x_1 and $e = \deg_{x_1} p$, so that

$$p = p_d(x_2, \dots, x_n) x_1^e + q(x_1, \dots, x_n)$$

where $\deg_{x_1} q < e$. Note that our assumptions on p imply that p_d is positive definite. Let $\mathbb{R}[\tilde{X}]$ denote $\mathbb{R}[x_2, \dots, x_n]$ and for $\alpha = (\alpha_2, \dots, \alpha_n) \in \mathbb{N}^{n-1}$, let $\tilde{X}^\alpha$ denote $x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}$.

Theorem 1. *Given p as above, suppose that $\operatorname{supp}(p_d)$ contains a multiple of every monomial in $\operatorname{supp}(q)$ which appears with a negative coefficient. Then p satisfies the conclusion of Pólya's Theorem if and only if every coefficient in p_d is nonnegative.*

In particular, suppose every coefficient of p_d is nonnegative. Then we can find an expression

$$p = \sum_{i=1}^{m} \tilde{X}^{\gamma_i}(c_i x_1^e + \phi_i),$$

with $\gamma_i \in \mathbb{N}^{n-1}$, $0 < c_i \in \mathbb{R}$, and $\phi_i \in \mathbb{R}[X]$ with $\deg_{x_1} \phi_i < e$. Let $c = \min_i\{c_i\}$ and U the sum of the absolute values of the coefficients of p. Now let $r := \frac{c}{c+2U}$ and $s := \frac{c}{2}\left(\frac{c}{c+2U}\right)^e$ and let λ be the minimum of p on the closure of $\Delta_n \setminus \Delta_n(1, r)$. Then for

$$N > \max\left\{\frac{e(e-1)}{2}\frac{L(p)}{s}, \frac{d(d-1)}{2}\frac{L(p)}{\lambda}\right\},$$

all coefficients of $(x_1 + \cdots + x_n)^n p$ are nonnegative.

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Hilbert's construction of psd polynomials that are not a sum of squares

Bruce Reznick

Summary: In 1888, Hilbert described how to find real polynomials which take only non-negative values but which are not a sum of squares of polynomials. His construction was so restrictive that no explicit examples appeared until the late 1960s. We revisit and generalize Hilbert's construction and discuss old and new examples.

A real (not necessarily homogeneous) polynomial $p(x_1,\dots,x_n)$ is *psd* or *positive* if $p(a)\ge 0$ for all $a\in\mathbb{R}^n$; it is *sos* or a *sum of squares* if there exist polynomials h_j so that $p=\sum h_j^2$. Let $P_{n,m}$ denote the cone of real psd forms of degree m in n variables and $\Sigma_{n,m}$ its subcone of sos forms and let $\Delta_{n,m}=P_{n,m}\smallsetminus\Sigma_{n,m}$. It was well-known by the mid-19th century that $\Delta_{2,m}=\Delta_{n,2}=\emptyset$. Hilbert showed in 1888 that $P_{3,4}=\Sigma_{3,4}$, and that this is the only other case in which $P_{n,m}=\Sigma_{n,m}$: it suffices to find forms in $\Delta_{3,6}$ and $\Delta_{4,4}$ and multiply them by powers of linear forms if necessary. Throughout this talk, we toggle between forms in n variables and polynomials in $n-1$ variables. We first present Hilbert's construction in $\Delta_{3,6}$, dehomogenized to two variables.

Suppose $f_1(x,y)$ and $f_2(x,y)$ are two real cubic polynomials with precisely nine distinct real common zeros – $\{\pi_i\}$, indexed arbitrarily – so that no three of the π_i's lie on a line and no six lie on a quadratic. Then there exists a quadratic $\phi(x,y)$ with zeros at $\{\pi_1,\dots,\pi_5\}$ and a quartic $\psi(x,y)$ with the same zeros and which is singular at $\{\pi_6,\pi_7,\pi_8\}$; the sextic $\phi\psi$ is singular at $\{\pi_1,\dots,\pi_8\}$. Hilbert showed that we can take $(\phi\psi)(\pi_9)>0$ and that there exists $c>0$ so that $p=f_1^2+f_2^2+c\phi\psi$ is positive. If $p=\sum h_j^2$, then each h_j is a cubic which vanishes on $\{\pi_1,\dots,\pi_8\}$, so that (by Cayley-Bacharach) $h_j(\pi_9)=0$ as well. It follows that $0=p(\pi_9)=c(\phi\psi)(\pi_9)$, a contradiction.

Hilbert's restriction on the common zeros of the polynomials meant that no very simple example could be constructed, and the first explicit element in any $\Delta_{n,m}$

was not produced for many decades. In 1965, Motzkin presented a specific sextic polynomial $m(x,y)$ which is positive by the arithmetic-geometric inequality and which is not sos by the arrangement of monomials in its Newton polytope. (Olga Taussky-Todd, who had a lifelong interest in sums of squares and was Hilbert's last assistant, heard Motzkin speak, and informed him that $m(x,y)$ was the first example.) After homogenization,

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \in \Delta_{3,6}. \tag{1}$$

Around the same time and independently, R. M. Robinson wrote that he saw "an unpublished example of a ternary sextic worked out recently by W. J. Ellison using Hilbert's method. It is, as would be expected, very complicated. After seeing this, I discovered that an astonishing simplification would be possible by dropping some unnecessary assumptions made by Hilbert." Robinson observed that the cubics $f_1(x,y) = x^3 - x$ and $f_2(x,y) = y^3 - y$ have nine common zeros; namely, the 3×3 square $\{-1,0,1\}^2$. There are eight different lines which contain three of these points. Nonetheless, the sextic $(x^2-1)(y^2-1)(1-x^2-y^2)$ is singular at the outer eight points and is positive at $(0,0)$. By taking the maximum value for c in Hilbert's construction and homogenizing, Robinson showed that

$$R(x,y,z) = x^6+y^6+z^6-x^4y^2-x^4z^2-x^2y^4-x^2z^4-y^4z^2-y^2z^4+3x^2y^2z^2 \tag{2}$$

is in $\Delta_{3,6}$. The papers of Motzkin and Robinson renewed interest in the subject, leading to more examples, including one by M. D. Choi and T. Y. Lam:

$$S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \in \Delta_{3,6}. \tag{3}$$

The talk presented an analysis of the underlying part of Hilbert's Method, allowing for considerable generalization. There are two main parts to the Method: a general perturbation of a given positive polynomial with fixed zeros by an arbitrary form which is singular at those zeros. We need the zeros to be "singular non troppo" or "round": the lowest order terms in the Taylor series at each zero must be a positive definite quadratic form. This perturbation preserves positivity and the zeros, but the perturbed polynomial might not be sos. The key point of the argument is that, under certain circumstances, there are polynomials of degree $2d$ which are singular on a set $\mathcal{A} \subseteq \mathbb{R}^n$, but are not in the vector space generated by products of polynomials of degree d which vanish on $\mathcal{A}$.

As a result, we are able to show that Robinson's simplification always works; that is, the Hilbert construction is valid when f_1 and f_2 are ternary cubics with exactly nine real intersections, whether or not three are on a line or not. We also show that if seven points are chosen in the plane, no four on a line, not all on a quadratic, then there is a positive polynomial p which is not sos and has those seven points as zeros. The proofs require Bezout's Theorem and a considerable amount of old-fashioned curve theory.

For example, let

(4)
$$\mathcal{A} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1)\},$$
$$F_1(x,y,z) = x(y^2 - z^2), \quad F_2(x,y,z) = y(z^2 - x^2), \quad F_3(x,y,z) = z(x^2 - y^2),$$
$$G(x,y,z) = (x^2 - y^2)(x^2 - z^2)(y^2 - z^2).$$

It is easy to show that the F_k's span the set of ternary cubics which vanish on $\mathcal{A}$ and that G is singular on $\mathcal{A}$ and not in the span of the F_jF_k's. It follows that for some $c > 0$, $P_c := F_1^2 + F_2^2 + F_3^2 + cG$ is psd and not sos. In fact, $P_1 = 2S$, providing a new derivation of (3).

Choi, Lam and Reznick showed in 1980 that if $P \in \Delta_{3,6}$, then P has finitely many zeros, and the maximum possible number is 10. At that time, the Robinson form R was essentially the only known example of such a sextic. Using Hilbert's Method, we are able to construct many new examples.

For $t > 0$, let

(5)
$$R_t(x,y,z) := \left(\frac{t^4 + 2t^2 - 3}{3}\right)(x^3 - xz^2)^2 + \left(\frac{1 + 2t^2 - 3t^4}{3t^4}\right)(y^3 - yz^2)^2 + R(x,y,z);$$

R_t is psd with ten zeros and not sos. For $t^2 < \frac{1}{2}$, let

(6)
$$M_t(x,y,z) = (1 - 2t^2)(x^4y^2 + x^2y^4) + t^4(x^4z^2 + y^4z^2) - (3 - 8t^2 + 2t^4)x^2y^2z^2 - 2t^2(x^2 + y^2)z^4 + z^6;$$

M_t is psd with ten zeros and not sos. Let

(7)
$$S_t(x,y,z) = t^4(x^6 + y^6 + z^6) + (1 - 2t^6)(x^4y^2 + y^4z^2 + z^4x^2) + (t^8 - 2t^2)(x^2y^4 + y^2z^4 + z^2x^4) - 3(1 - 2t^2 + t^4 - 2t^6 + t^8)x^2y^2z^2;$$

S_t is always psd and not sos; it has ten zeros if $t > 0$. Note that $S_0 = M$, $S_1 = S$, so S_t provides a "homotopy" between S and M in $\Delta_{3,6}$ in the class of positive forms with ten zeros. Finally, though it only has seven zeros, we note that

(8)
$$U_c(x,y,z) = x^2y^2(x-y)^2 + y^2z^2(y-z)^2 + z^2x^2(z-x)^2 + cxyz(x-y)(y-z)(z-x)$$

is psd when $|c| \le 4\sqrt{\sqrt{2} - 1}$ and sos only if $c = 0$.

As one example in higher degree, for $m \in \mathbb{N}$, let

(9)
$$(t)_m = \prod_{j=0}^{m-1} (t - j).$$

Then for $d \ge 3$ and some $c_d > 0$, the polynomial

(10)
$$(x)_d^2 + (y)_d^2 + c_d(x)_2(y)_2(x + y - 2)_{d-1}(x + y - 4)_{d-3}$$

is psd and not sos, and has at least $\binom{d+2}{2} - 2$ zeros.

We can also answer a question of Robinson, and show that $(ax^2 + by^2 + cz^2)R(x, y, z)$ is sos if and only if $a, b, c \geq 0$ and $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the sides of a (possibly degenerate) triangle.

Further examples and proofs will soon appear in a much longer publication.

Certificates of Positivity in the Bernstein basis

Marie-Françoise Roy
(joint work with Fatima Boudaoud, Fabrizio Caruso, Richard Leroy)

Let $P \in \mathbb{Z}[\mathbb{X}]$ be a polynomial of degree p with coefficients of bitsize bounded by τ. If P is positive on $[-1, 1]$, we obtain a certificate of positivity (i.e. a description of the polynomial making obvious that it is positive) of bitsize $O(p^4(\tau + \log_2 p))$. Previoussimilar results had a bitsize complexity exponential in p and τ. We extend this technique to the multivariate case (no complexity estimate so far).

1. What is a certificate of positivity in the Bernstein basis?

$$\mathrm{Bern}_{p,i}(\ell, r) = \binom{p}{i} \frac{(X - \ell)^i (r - X)^{p-i}}{(r - \ell)^p}, \tag{1}$$

for $i = 0, \ldots, p$.

Think of

$$1 = \left(\frac{r - X}{r - \ell} + \frac{X - \ell}{r - \ell} \right)^p$$

- take positive values on (ℓ, r),
- $\mathrm{Bern}_{p,0}(\ell, r)$ is positive at ℓ, $\mathrm{Bern}_{p,p}(\ell, r)$ is positive at r,
- basis of the vector-space of polynomials of degree $\leqslant p$

$b(P, p, \ell, r)$ is the ordered list of coefficients of P in the Bernstein basis.

$b(P, p, \ell, r)_0$ is the value of P at ℓ and $b(P, p, \ell, r)_p$ is the value of P at r.

$\mathrm{Cert}(P, d, \ell, r)$: **all the elements of $b(P, d, \ell, r)$ are non negative, with** $b(P, d, \ell, r)_0 > 0$, $b(P, d, \ell, r)_d > 0$.

$\mathrm{Cert}(P, d, \ell, r)$ guarantees the existence of a **certificate of positivity for P on** $[\ell, r]$, i.e. a description of P making obvious that it is positive on $[\ell, r]$: $b(P, d, \ell, r)$

If $\mathrm{Cert}(P, p, -1, 1)$ holds, P is positive on $[-1, 1]$.

Reciprocal not true !

$$P = 5X^2 - 4X + 1.$$

P is positive on $[-1, 1]$, but $b(P, 2, -1, 1) = [10, -4, 2]$ and $\mathrm{Cert}(P, 2, -1, 1)$ does not hold. However, since $b(P, 21, -1, 1)$ is

$$[210, 182, 156, 132, 110, 90, 72, 56, 42, 30, 20, 12, 6, 2, 0, 0, 2, 6, 12, 20, 30, 42] \tag{2}$$

$\mathrm{Cert}(P, 21, -1, 1)$ does hold. 21 is the smallest natural number with this property.

Bernstein (1915): if P is positive on $[-1,1]$, there exists $d \geqslant p$ such that $\mathrm{Cert}(P,d,-1,1)$ holds. **Bernstein degree**: smallest natural number d such that $\mathrm{Cert}(P,d,-1,1)$ holds:

Powers and Reznick (2001): quantitative bound on the Bernstein degree exponential in p and τ.

Wanted: different , smaller, certificate of positivity for P on $[-1,1]$ using also the Bernstein basis.

Idea: keep the initial degree, refine the interval; look for certificates of positivity on subintervals.

$\mathrm{Cert}(P,d,L)$ means: L: $-1=\ell_0<\ell_1<\ldots<\ell_{n-1}<\ell_n=1$ is a subdivision: for each $i=1,\ldots,n$ $\mathrm{Cert}(P,d,\ell_{i-1},\ell_i)$ holds.

If P ***is positive on*** $[-1,1]$***, find a subdivision*** L ***such that*** $\mathrm{Cert}(P,p,L)$ ***holds.***

Tool: a litte geometry

Notation: Var is the number of sign changes in a sequence of numbers

Theorem 1. *P square free*

If P has no root in the circle of diameter (ℓ,r), then $\mathrm{Var}(b(P,p,\ell,r))=0$.

Why shorter ? *Adaptativity* ! Subintervals of different length, short intervals being concentrated on parts of $|-1,1]$ where the sign of P is less obvious.

Consider $P=5X^2-4X+1$, its certificate of positivity is:

$[[[-1,0],1,[10,3,1]],[[0,1/2],4,[4,0,1]],[[1/2,1],4,[1,2,8]]]$

which reads as

- subdivision $-1<0<1/2<1$
- $b(P,2,-1,0)=[10,3,1]$,
- $b(4P,2,0,1/2)=[4,0,1]$,
- $b(4P,2,1/2,1)=[1,2,8]$.

2. Certificate of positivity

- certificate of positivity if Q positive on $[-1,1]$,
- or point x of $[-1,1]$ such that $Q(x)\leqslant 0$ otherwise.

Idea: try to isolate real roots of Q, no roots implies proof of positivity (or negativity).

Algorithm 1. (De Casteljau)

- **Input:** $([\ell,r],\ b(P,p,\ell,r))$ and m.
- **Output:** $([\ell,m],\ b(P,p,\ell,m))$ and $([m,r],\ b(P,p,m,r))$.
- **Procedure:**
  - Initialization: $b_j^{(0)} := b(cP,p,\ell,r)_j$, for $j=0,\ldots,p$.
  - For $i=1,\ldots,p$,
    - For $j=0,\ldots,p-i$, compute $b_j^{(i)} := ((r-m)b_j^{(i-1)}+(m-\ell)b_{j+1}^{(i-1)})/(r-\ell)$.
    - Output

$$\begin{aligned} b' &= b_0^{(0)},\ldots,b_0^{(j)},\ldots,b_0^{(p)},\\ b'' &= b_0^{(p)},\ldots,b_j^{(p-j)},\ldots,b_p^{(0)} \end{aligned}$$

Algorithm 2. [Square free certificate of positivity]

- **Input:** square-free $Q \in \mathbb{Z}[X]$ of degre q.
- **Output:**
  - POS if $Q > 0$ on $[-1, 1]$, certified by $b(Q, q, L)$ where L is a rational subdivision of $[-1, 1]$ of length n,
  - NEG similar
  - NO and a point x such that $Q(x) = 0$, or a segment $[\ell, r]$ such that $Q(\ell)Q(r) < 0$.
- **Procedure:**
  - Initialization: $M := \{([-1, 1], b(Q, q, -1, 1))\}$, $\mathrm{Pos}(Q) := \emptyset, \mathrm{Neg}(Q) := \emptyset$.
  - While M is non-empty,
    - Remove an element $([\ell, r], b)$ from M.
    - If $b_0 = 0$ return NO and $[\ell]$.
    - If $b_p = 0$ return NO and $[r]$.
    - If $b_0 b_p < 0$, return NO and $[\ell, r]$.
    - If $\mathrm{Var}(b) = 0$,
      - $\rightarrow$ if $b_0 > 0$ and $b_p > 0$, add $([\ell, r], b)$ to $\mathrm{Pos}(Q)$,
      - $\rightarrow$ if $b_0 < 0$ and $b_p < 0$, add $([\ell, r], b)$ to $\mathrm{Neg}(Q)$.
    - Else $m = (\ell + r)/2$, compute b' and b'', using Algorithm 2 with input $b(Q, \ell, r)$, add
    $$([\ell, m], b'), ([\mathrm{mr}], b'')$$
    to M.
  - If $\mathrm{Pos}(Q) := \emptyset$, return NEG and Neg(Q).
  - If $\mathrm{Neg}(Q) := \emptyset$, return POS and Pos(Q).

Proposition 2. *The number of nodes of the binary tree output by Algorithm 2 is at most* $2(2\tau + 3\nu + 3)q$.

Example of $P = X^4 + 64X^2 - 16X + 1$.

- $\mathrm{Var}(b(3P/2, 4, -1, 1) = [123, 12, -29, -12, 752]) = 2$: refine $[-1, 1]$.
- $\mathrm{Var}(b(3P, 4, -1, 0) = [246, 135, 59, 15, 3]) = 0$, $\mathrm{Var}(b(3P, 4, 0, 1) = [3, -9, 11, 63, 150])$=2: refine $[0, 1]$.
- $\mathrm{Var}(b(32^4P, 4, 0, 1/2) = [48, -48, -16, 144, 435]) = 2$: refine $[0, 1/2]$, while $\mathrm{Var}(b(32^4P, 4, 1/2, 1) = [435, 726, 1148, 1704, 2400]) = 0$.
- $\mathrm{Var}(b(32^8P, 4, 0, 1/4) = [768, 0, -256, 0, 771]) = 2$: refine $[0, 1/4]$, while $\mathrm{Var}(b(32^8P, 4, 1/4, 1/2) = [771, 1542, 2828, 4632, 6960]) = 0$.
- Isolation phase stops: $\mathrm{Var}(b(32^{12}P, 4, 0, 1/8) = [12288, 6144, 2048, 0, 3]) = 0$, and $\mathrm{Var}(\ b(32^{12}P, 4, 1/8, 1/4) = [3, 6, 2060, 6168, 12336]) = 0$.

3. Comparison

3.1. **Minimum of a polynomial.**

Theorem 3. *The minimum of P is at least* $2^{-2p(1+\tau+\nu)+(\tau+1)}$.

Proof (indication): Use resultants to estimate the bitsize of the polynomial whose roots are the values of P at the roots of P'

Is this bound accurate ?

Bugeaud and Mignotte:

$$A(k,p) = X^{2p} + (2^k X - 1)^2.$$

minimum close to the estimation of Theorem 3 [3]: if $\tau = 2k$, the minimum of $A(k,p)$ smaller than the value $2^{-p\tau}$ obtained at $x = 2^{-k}$, depends exponentially on p and τ.

3.2. **Size of certificates of positivity.** Bernstein degree is estimated by

$$p(p-1)2^{p(2\tau+3\nu+3)+\nu-2}, \tag{3}$$

quantitative bound on the Bernstein degree

$$\leqslant \frac{p(p-1)}{2}\frac{L}{\lambda} \tag{4}$$

where λ minimum of $P(x)$ on $[-1,1]$, L maximum value of the elements of $b(P,p,-1,1)$: estimation on λ given by Theorem 3, estimation on L by bitsize of Bernstein coefficients.

3.2.1. *With respect to the bit size.* Powers and Reznick's bound is sharp

For the family of polynomials of degree 2 indexed by $k \in \mathbb{N}$, $P_k = (2^k - 1)X^2 + 1$, the Bernstein degree is precisely $2^k - 1$. Our certificate of positivity for P_k : linear

- $b(P_k, 2, -1, 0) = [2^k, 1, 1]$,
- $b(P_k, 2, 0, 1) = [1, 1, 2^k]$.

3.2.2. *With respect to the degree.* Consider $A(k,p) = X^{2p} + (2^k X - 1)^2$ Numerical experiments performed using SARAG

- when $p = 1$, $A(1,1) = 5X^2 - 4X + 1$, first example
- when $p = 2$, the Bernstein degree is 82, while our certificate of positivity is much smaller

$$[[[-1,0],3,[30,18,11,6,3]],[[0,1/2],48,[48,24,8,0,3]],\\ [[1/2,1],48,[3,6,20,48,96]]].$$

Bernstein degree of $A(1,p)$ exponential in p, bigger than $2^{2p} + 2p - 1$. It is not hard to prove that $b(A(1,p), 2N, 0, 1)_N < 0$ for any $N < 2^{2p-1} + p$.

4. Multivariate polynomials

Bersntein's basis exists: k variables, L simplexdefined by $k+1$ linear inequalities $\ell_i \geqslant 0$, $i = 0, \ldots, k$, normalized by $1 = \ell_0 + \ldots + \ell_k$. Let $i = (i_0, \ldots, i_k)$, $i_0 + \ldots + i_k = p$, $\binom{p}{i}$ multinomial coefficient, $\ell^i = \prod_{j=0}^{k} \ell_j^{i_j}$

$$\mathrm{Bern}_{p,i}(L) = \binom{p}{i} \ell^i \tag{5}$$

Think of

$$1 = \Big(\ell_0 + \ldots + \ell_k\Big)^p$$

– take positive values on L,
– gives values at the vertices of the simples
– basis of the vector-space of polynomials of degree $\leqslant p$

Global and local certificates of positivity exist. Bernstein theorem holds: increase degree. Local certificates exist: subdivide until there are no sign variations

Generalizaiton of De Casteljau exists.

Algorithm 3. (Multivariate De Casteljau)

- **Input:** $(L,\ b(P,p,L))$ and m, a baryrenter of the vertices with weight $\alpha = (\alpha_0, \ldots, \alpha_k)$, $L_0, \ldots, L_k$ the $k+1$ simplices subdivided
- **Output:** $L_0, \ldots, L_k$ the $k+1$ simplices after subdivision, $(L_i,\ b(P,p,L_i))$, $i = 0, \ldots, k$.
- **Procedure:**
  - Initialization: $b_j^{(0)} := b(P,p,\ell,r)_j$, for $j = (j_0, \ldots, j_k)$, $j_0 + \ldots + j_k = p$.
  - For $i = 1, \ldots, p$,
    - Let $e_s = (0, \ldots, 0, 1, 0, \ldots, 0)$, 1 at place number s.
    - For $j = (j_0, \ldots, j_k)$, $j_0 + \ldots + j_k = p$-j, compute $b_j^{(i)} := \sum_{s=0}^{k} \alpha_s b_{j+e_s}^{(i-1)}$.
    - Output coefficients on right face $b_{s,j} = b_{j-j_s e_s}^{(j_s)}$

Algorithm for certificate of positivity exisits: subdivide until all coefficients are positive.

Problem : quantitative bounds ? in the univariate case, polynomial in d and τ, here ?

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A connectedness theorem for real spectra of polynomial rings, in connection with Pierce-Birkhoff conjecture

Daniel Schaub
(joint work with Franҫis Lucas, James J. Madden, Mark Spivakovsky)

Throughout, R will denote a real closed field and A the polynomial ring $R[x_1, \dots, x_n]$.

The Pierce-Birkhoff conjecture asserts that any piecewise-polynomial function $f : R^n \to R$ can be expressed as a maximum of minima of a finite family of polynomials. In this lecture we describe where we are in our program for a proof of the Pierce-Birkhoff conjecture in its full generality (the best results up to now are due to Louis Mahé [13], who proved the conjecture for $n = 2$, as well as some partial results for $n = 3$).

We start by stating the Pierce–Birkhoff conjecture in its original form as it was first stated by M. Henriksen and H. Isbell in the early nineteen sixties.

Definition 1. A function $f : R^n \to R$ is said to be **piecewise polynomial** if R^n can be covered by a finite collection of closed semi-algebraic sets P_i such that for each i there exists a polynomial $f_i \in A$ satisfying $f|_{P_i} = f_i|_{P_i}$. If f is a piecewise polynomial function, we say that f is **defined by** r polynomials if r is the number of distinct polynomials among the f_i above.

Clearly, any piecewise polynomial function is continuous. Piecewise polynomial functions form a ring, containing A, which is denoted by $PW(A)$.

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from A by iterating the operations of sup and inf. Since applying the operations of sup and inf to polynomials produces functions which are piecewise polynomial, this ring is contained in $PW(A)$ (the latter ring is closed under sup and inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture is:

Conjecture 2. (Pierce-Birkhoff) *If $f : R^n \to R$ is in $PW(A)$, then there exists a finite family of polynomials $g_{ij} \in A$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in R^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

In 1989 J.J. Madden [11] reformulated this conjecture in terms of the real spectrum of A and separating ideals. Let us recall Madden's formulation together with the relevant definitions.

First, we need to recall the notion of a point of the real spectrum of a ring. Let B be a ring. A point α in the real spectrum of B is, by definition, the data of a prime ideal $\mathfrak{p}$ of B, and a total ordering $\leq$ of the quotient ring $B/\mathfrak{p}$, or, equivalently, of the field of fractions of $B/\mathfrak{p}$. Another way of defining the point α is as a homomorphism from B to a real closed field, where two homomorphisms are identified if they have the same kernel $\mathfrak{p}$ and induce the same total ordering on $B/\mathfrak{p}$.

The ideal $\mathfrak{p}$ is called the support of α and denoted by $\mathfrak{p}_\alpha$, the quotient ring $B/\mathfrak{p}_\alpha$ by $B[\alpha]$, its field of fractions by $B(\alpha)$ and the real closure of $B(\alpha)$ by $k(\alpha)$. The total ordering of $B(\alpha)$ is denoted by $\leq_\alpha$. Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$.

Definition 3. The real spectrum of B, denoted by Sper B, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of B and $\leq_\alpha$ is a total ordering of $B/\mathfrak{p}_\alpha$.

The real spectrum Sper B is endowed with two natural topologies, the spectral (or Harrison) topology and the constructible topology.

Next we define the notion of **separating ideal**, introduced by Madden in [11] :

Definition 4. Let B be a ring. For $\gamma, \delta \in$ Sper B, the **separating ideal** of γ and δ, denoted by $<\gamma, \delta>$, is the ideal of B generated by all the elements $f \in B$ which change sign between γ and δ, that is, all the f such that $f(\gamma) \geq 0$ and $f(\delta) \leq 0$ (or vice versa).

Let f be a piecewise polynomial function on R^n and $\alpha \in$ Sper A. Let the notation be as in Definition 1. The covering $R^n = \bigcup\limits_i P_i$ induces a corresponding covering Sper $A = \bigcup\limits_i \tilde{P}_i$ of the real spectrum. Pick and fix an i such that $\alpha \in \tilde{P}_i$. We set $f_\alpha := f_i$. We refer to f_α as a local polynomial representative of f at α. In general, the choice of i is not uniquely determined by α. Implicit in the notation f_α is the fact that one such choice has been made.

In [11], Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he showed that the Pierce-Birkhoff conjecture is equivalent to

Conjecture 5. (Pierce-Birkhoff conjecture, the abstract version) *Let f be a piecewise polynomial function and α, β points in Sper A. Let $f_\alpha \in A$ be a local representative of f at α and $f_\beta \in A$ a local representative of f at β. Then $f_\alpha - f_\beta \in <\alpha, \beta>$.*

We now state a conjecture which implies the Pierce-Birkhoff conjecture:

Conjecture 6. (the Connectedness conjecture) *Let $\alpha, \beta \in$ Sper A and let $g_1, \dots, g_s$ be a finite collection of elements of A, not belonging to $< \alpha, \beta >$. Then there exists a connected set $C \subset$ Sper A such that $\alpha, \beta \in C$ and $C \cap \{g_i = 0\} = \emptyset$ for $i \in \{1, \dots, s\}$ (in other words, α and β belong to the same connected component of the set Sper $A \setminus \{g_1 \dots g_s = 0\}$).*

The special case when $s = 1$ is Madden's "separation conjecture"; it is equivalent to the special case of Pierce-Birkhoff conjecture when the number of semi-algebraic pieces defining the piecewise polynomial function f is precisely 2.
This reduces the Pierce-Birkhoff conjecture to constructing, for each $\alpha, \beta \in$ Sper A, connected sets in Sper A having certain properties.

The following theorem, though not in itself enough for the proof of the Pierce-Birkhoff conjecture, is the first and simplest example of the sort of connected sets we really need.

We first define the valuation ν_α of $B(\alpha)$ associated to a point α of the real spectrum. The valuation ν_α is defined using the order $\leq_\alpha$. It has the following properties :
(1) $\nu_\alpha(B[\alpha]) \geq 0$
(2) given positive elements $x, y \in B(\alpha)$,

$$\nu_\alpha(x) < \nu_\alpha(y) \Rightarrow x > Ny, \ \forall N \in \mathbb{N}. \tag{1}$$

Finally, we state the main theorem. Let $A = R[x_1, \dots, x_n]$ be a polynomial ring and denote by ν_δ be the valuation associated to the point $\delta \in \text{Sper}(A)$. Let $\omega_{ij}, \theta_{il} \in \mathbb{Q}, i \in \{1, \dots, n\}, j \in \{1, \dots, q\}, l \in \{1, \dots, u\}$.
Let $h_j(\nu_\delta(x)) = \sum\limits_{i=1}^{n} \omega_{ij}\nu_\delta(x_i)$ for $j \in \{1, \dots, q\}$ and $z_l(\nu_\delta(x)) = \sum\limits_{i=1}^{n} \theta_{il}\nu_\delta(x_i)$ for $l \in \{1, \dots, u\}$, where we write x in place of $(x_1, \dots, x_n)$ and $\nu_\delta(x)$ for the n-tuple $(\nu_\delta(x_1), \dots, \nu_\delta(x_n))$.

Theorem 7. *The set*

$$S = \{\delta \in \mathit{Sper}(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, \ h_j(\nu_\delta(x)) > 0, j \in \{1, \dots, q\}$$

$$z_l(\nu_\delta(x)) = 0, l \in \{1, \dots, u\}\} \tag{2}$$

is connected in the spectral topology.

In other words, subsets of Sper A defined by finitely many $\mathbb{Q}$-linear equations and *strict* inequalities on $\nu(x_1), \dots, \nu(x_n)$ are connected.

To understand the relation between this theorem and the Connectedness Conjecture, we need to introduce the theory of approximate roots of a valuation. Given a ring A and a valuation ν, non-negative on A, a family of approximate roots is a collection $\{Q_i\}$, finite or countable, of elements of A. A generalized monomial (with respect to a given collection $\{Q_i\}$ of approximate roots) is, by definition, an element of A of the form $\prod\limits_j Q_j^{\gamma_j}$, $\gamma_j \in \mathbb{N}$. The main defining properties of the approximate roots are the fact that every ν-ideal I in A is generated by generalized monomials contained in it, that is, generalized monomials $\prod\limits_j Q_j^{\gamma_j}$ satisfying $\sum\limits_j \gamma_j \nu(Q_j) \geq \nu(I)$, and the fact that for each i, Q_i is described by an explicit formula in terms of $Q_1, ..., Q_{i-1}$. In particular, the valuation ν is completely determined by the set $\{Q_i\}$ and the values $\nu(Q_i)$.

We then show that every element $g \in A$ can be written as a finite sum of the form

$$g = c\mathbf{Q}^{\theta} + \sum_{j=1}^{N} c_j \mathbf{Q}^{\delta_j}, \tag{3}$$

where c and c_j are elements of A such that $\nu(c) = \nu(c_j) = 0$ and $\mathbf{Q}^{\theta}$ and $\mathbf{Q}^{\delta_j}$ are generalized monomials such that

$$\nu\left(\mathbf{Q}^{\theta}\right) < \nu\left(\mathbf{Q}^{\delta_j}\right) \text{ for } 1 \leq j \leq N. \tag{4}$$

Now let $\delta \in \mathrm{Sper}(A)$ and let $\nu = \nu_\delta$. Then, by (1) and (4) the sign of g with respect to $\leq_\delta$ is determined by the sign of its leading coefficient c.

Let $\alpha, \beta \in \mathrm{Sper}(A)$ having a common specialization. We will give in [10] an explicit description of the set of generalized monomials (with respect to approximate roots Q_i common to ν_α and ν_β) which generate the separating ideal $< \alpha, \beta >$. Furthermore we show that all the approximate roots Q_i for ν_α such that $Q_i \notin < \alpha, \beta >$ are also approximate roots for ν_β and vice versa.

Now let $g \in \mathrm{Sper}(A) \setminus < \alpha, \beta >$, as in the separation conjecture. The fact that $g \notin < \alpha, \beta >$ implies the existence of an expression (3) in which all the approximate roots Q_i are common for ν_α and ν_β and the inequalities (4) hold for both $\nu = \nu_\alpha$ and $\nu = \nu_\beta$.

The inequalities (4) can be viewed as linear inequalities on $\nu(Q_1), \dots, \nu(Q_t)$ with integer coefficients.

To prove this conjecture, one of our main ideas is to look for a connected set $C \subset \mathrm{Sper}(A)$ having the following properties :

(1) $\alpha, \beta \in C$;

(2) the Q_i appearing in (3) are approximate roots simultaneously for all the ν_δ, $\delta \in C$;

(3) the inequalities (4) hold for $\nu = \nu_\delta$, for all $\delta \in C$;

(4) the leading coefficient c keeps a constant sign on C.

Once such a C is found, (1) and (4) imply that the sign of g on C is constant, which proves that α and β lie in the same connected component of $\mathrm{Sper}(A) \setminus \{g = 0\}$.

(5) The set $C(g,\alpha,\beta) = \left\{ \delta \in \mathrm{Sper}(A) \;\middle|\; Q_i > 0, i \in \{1,\dots,t\}, \right.$

$$\left. \sum_{i=1}^{t} \omega_{ij}\nu_\delta(Q_i) > 0, j \in \{1,\dots,q\}, \sum_{i=1}^{t} \lambda_{ij}\nu_\delta(Q_i) = 0, j \in \{1,\dots,l\} \right\},$$

where $\omega_{ij}, \lambda_{ij} \in \mathbb{Z}$, all the inequalities (4) described above appear among the $\sum\limits_{i=1}^{t} \omega_{ij}\nu_\delta(Q_i) > 0$ and the remaining equalities and inequalities on the right hand side of (5) encode the fact that $Q_1,\dots,Q_t$ are approximate roots for ν_δ for all δ belonging to $C(g,\alpha,\beta)$. If $g_1,\dots,g_s$ is a finite collection of elements of $A \setminus <\alpha,\beta>$, we put $C(g_1,\dots,g_s,\alpha,\beta) = \bigcap\limits_{i=1}^{s} C(g_i,\alpha,\beta)$. By construction, both sets $C(g,\alpha,\beta)$ and $C(g_1,\dots,g_s,\alpha,\beta)$ contain α and β, the element g does not change sign on $C(g,\alpha,\beta)$ and none of the elements $g_1,\dots,g_s$ change sign on $C(g_1,\dots,g_s,\alpha,\beta)$. Thus to prove the separation conjecture it is sufficient to prove that
$C(g,\alpha,\beta)$ is connected and to prove the Pierce–Birkhoff conjecture it is sufficient to prove that $C(g_1,\dots,g_s,\alpha,\beta)$ is connected. This makes that the set S of theorem 7 is a special case of the kind of connected sets we are looking for.

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Gradient tentacles and sums of squares

Markus Schweighofer

This report is mainly based on [Sch] where more details can be found.

Let always $f \in \mathbb{R}[\bar{X}] := \mathbb{R}[X_1, \dots, X_n]$. Our motivation is the problem to compute (numerically to a given precision) the infimum

$$f^* := \inf f(\mathbb{R}^n) \in \mathbb{R} \cup \{-\infty\}$$

of a polynomial f on $\mathbb{R}^n$. This problem is anyway very hard but it seems to be even harder if f is bounded from below but does not attain a minimum on $\mathbb{R}^n$, e.g. $n = 2$, $\bar{X} = (X, Y)$ and $f = (1 - XY)^2 + Y^2$. Whereas

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } \mathbb{R}^n\},$$

is very difficult to compute,

$$\sup\{a \in \mathbb{R} \mid f - a \in \sum \mathbb{R}[\bar{X}]^2\}$$

can easily be computed by solving a semidefinite program (SDP). The problem is however that for fixed degree and large number of variables, very few nonnegative f are sums of squares. The idea in this talk is to combine sums of squares with calculus. This idea is originally due to Demmel, Nie and Sturmfels who proved the following theorem [NDS].

Theorem 1 (Demmel, Nie, Sturmfels)**.** *Consider the gradient ideal*

$$I := \left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right)$$

of f and suppose $f^* \in f(\mathbb{R}^n)$. Denoting by $V(I) \subseteq \mathbb{C}^n$ the reduced affine variety defined by I (the gradient variety of f), we have that the following are equivalent:

(i) $f \geq 0$ *on* $\mathbb{R}^n$
(ii) $f \geq 0$ *on* $V(I) \cap \mathbb{R}^n$
(iii) $f \in \sum \mathbb{R}[\bar{X}]^2 + \sqrt{I}$
(iv) $f + \varepsilon \in \sum \mathbb{R}[\bar{X}]^2 + I$ *for all* $\varepsilon > 0$

As a corollary, we get that $f^* = \sup\{a \in \mathbb{R} \mid f - a \in \sum \mathbb{R}[\bar{X}]^2 + I\}$ can be computed by solving a *sequence* of SDPs provided that $f^* \in f(\mathbb{R}^n)$. In the very hard case that $f^* \notin f(\mathbb{R}^n)$, this method might however yield to completely wrong results. For example, this method would output 1 for $f = (1 - XY)^2 + Y^2$ since the only critical point of f is the origin and $f(0) = 1$. But in fact, we have $f^* = 0$ for this f.

A way out of this problem would be to prove the following conjecture. In all practical examples we computed the corresponding SDP relaxations worked very well and led quickly to f^*.

Conjecture 2. *In Theorem 1, the assumption $f^* \in f(\mathbb{R}^n)$ can be removed if one replaces the gradient ideal by*

$$I := \left(\frac{\partial f}{\partial X_i} X_j, \ldots, \frac{\partial f}{\partial X_j} X_i \mid 1 \leq i < j \leq n \right).$$

In the following, we will replace the real part of the gradient variety by larger semialgebraic subsets defined by polynomial inequalities depending only on ∇f.

Definition 3. The polynomial

$$g := 1 - \left(\sum_{i=1}^{n} \left(\frac{\partial f}{\partial X_i} \right)^2 \right) \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[\bar{X}]$$

defines the *gradient tentacle*

$$S(\nabla f) := \{ x \in \mathbb{R}^n \mid \|\nabla f(x)\| \|x\| \leq 1 \} \subseteq \mathbb{R}^n.$$

Theorem 4. *Suppose the condition $(*)$ below holds and f is bounded from below on $\mathbb{R}^n$. Then the following are equivalent:*

(i) $f \geq 0$ *on* $\mathbb{R}^n$
(ii) $f \geq 0$ *on* $S(\nabla f)$
(iii) $f + \varepsilon \in \sum \mathbb{R}[\bar{X}]^2 + g \sum \mathbb{R}[\bar{X}]^2$

Here $(*)$ is a condition from Parusiński [Par], namely that there are only finitely many $z \in \mathbb{P}^{n-1}(\mathbb{C})$ such that $\nabla f_d(z) = f_{d-1}(z) = 0$ where $d := \deg f$ and f_k is the k-homogeneous part of f. This condition is true generically. In fact, generically there is even *no* such z. Of course, $(*)$ is always satisfied for $n = 2$. Moreover, we do not know if $(*)$ is really necessary for Theorem 4 to hold.

Definition 5. For $S \subseteq \mathbb{R}^n$, we introduce the set $R_\infty(f, S)$ of *asymptotic values* of f on S which consists of all $y \in \mathbb{R}$ such that there is a sequence $(x_k)_{k \in \mathbb{N}}$ of points $x_k \in S$ with $\lim_{k \to \infty} \|x_k\| = \infty$ and $\lim_{k \to \infty} f(x_k) = y$.

We now can formulate one of the ingredients of the proof of Theorem 4, namely the following theorem which relies on the theory of iterated rings of bounded elements [Sch].

Theorem 6. *Let $f, g_1, \ldots, g_m \in \mathbb{R}[\bar{X}]$ and set*

$$S := \{ x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \}.$$

Suppose that

(a) *f is bounded on S,*
(b) *$R_\infty(f, S)$ is a finite subset of $\mathbb{R}_{>0}$ and*
(c) *$f > 0$ on S.*

Then f is in the preordering generated by $g_1, \ldots, g_m$ in $\mathbb{R}[\bar{X}]$.

We will apply this theorem only for the case $m = 1$. In this case, the preordering in question is just $\sum \mathbb{R}[\bar{X}]^2 + g \sum \mathbb{R}[\bar{X}]^2$ where $g := g_1$. To prove Theorem 4 from Theorem 6, it suffices to show that f is bounded on $S(\nabla f)$ (we refer to [Spo]),

(A) $R_\infty(f, S(\nabla f))$ is finite if $(*)$ holds and that
(B) $f^* \in R_\infty(f, S(\nabla f))$ if f is bounded from below and $f^* \notin f(\mathbb{R}^n)$.

Definition 7. Denote by $B(f)$ the *bifurcation set* of f, i.e., the smallest subset of $\mathbb{R}$ such that for its complement A in $\mathbb{R}$, $f|_{f^{-1}(A)} : f^{-1}(A) \to A$ is a locally trivial C^∞ fiber bundle. The set $K(f)$ of *generalized critical values* of f consists of all $y \in \mathbb{R}$ for which there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\lim_{k\to\infty} \|\nabla f(x_k)\|(1 + \|x_k\|) = 0 \qquad \text{and} \qquad \lim_{k\to\infty} f(x_k) = y.$$

We make use the following well-known theorem (see, e.g., [KOS, Theorem 3.1]).

Theorem 8. *Suppose $f \in \mathbb{R}[\bar{X}]$. Then $B(f) \subseteq K(f)$ and $K(f)$ is finite.*

From this follows easily (B) above. The following result is surprising and can be derived from [Par, Theorem 1.4].

Theorem 9. $(*) \implies R_\infty(f, S(\nabla f)) \subseteq K(f)$

This implies of course (A) above. Without $(*)$, the previous theorem is in general not true. As already said, it is an open problem if one can avoid assumption (*) in Theorem 4. Anyway, there is a way of getting rid of this condition by using what we call *higher gradient tentacles*. These gradient tentacles have the computational disadvantage that the degree of the defining inequality can be much bigger. Also it is not a priori clear how big the degree has to be chosen.

Definition 10. The polynomial

$$g_N := 1 - \|\nabla f(x)\|^{2N}(1 + \|x\|^2)^{N+1} \in \mathbb{R}[\bar{X}]$$

where $N \in \mathbb{N}$ defines the N-th *higher gradient tentacle* of f

$$S(\nabla f, N) := \{x \in \mathbb{R}^n \mid \|\nabla f(x)\|^{2N}(1 + \|x\|^2)^{N+1} \le 1\}.$$

Clearly, we have

$$V(I) \cap \mathbb{R}^n \subseteq S(\nabla f, 1) \subseteq S(\nabla f, 2) \subseteq \cdots \subseteq S(\nabla f)$$

where I is the gradient ideal of f. Now the analog of (A) is easier to show than before but for (B) we now need results of Kurdyka, Orro and Simon [KOS].

Theorem 11. *Suppose f is bounded from below. Then there is $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$, the following are equivalent:*

(i) $f \ge 0$ *on* $\mathbb{R}^n$
(ii) $f \ge 0$ *on* $S(\nabla f, N)$
(iii) $f + \varepsilon \in \sum \mathbb{R}[\bar{X}]^2 + g_N \sum \mathbb{R}[\bar{X}]^2$ *for all* $\varepsilon > 0$

If $f^* \in f(\mathbb{R}^n)$, then these conditions are equivalent for all N.

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Upper bounds on the number of homotopy types of sets defined by fewnomials

Nicolai Vorobjov
(joint work with Saugata Basu)

We understand the *complexity* of a polynomial $P \in \mathbb{R}[X_1, \dots, X_m]$ as a measure of the size of its defining formula. There are several natural measures, among them:

(i) the degree $d = \deg(P)$;
(ii) the number r of monomials in P;
(iii) the *additive complexity* a of P.

(We say that P has additive complexity a if, starting with variables $X_1, \dots, X_m$ and constants in $\mathbb{R}$, and applying additions and multiplications, a formula representing P can be obtained using at most a additions and an unlimited number of multiplications, see the formal definition in [2, 4, 1].)

In their book [2], Benedetti and Risler, generalizing examples (i)–(iii), introduced an axiomatic definition of the complexity of a polynomial, and consequently, of a semi-algebraic set. Axioms include the analogy of *Bezout inequality*, i.e., an upper bound on the number of non-degenerate isolated solutions of any system of m polynomial equations, as an explicit function of the complexity of polynomials. In the case (i), this is the usual Bezout inequality, in the case (ii) this is the Khovanskii's bound [6], the latter also easily implies an upper bound in the additive complexity case (iii) (see [2]).

Benedetti and Risler conjectured that there is a finite number of of homeomorphism types of semi-algebraic sets of the given complexity. They also formulated their conjecture for particular cases (ii) and (iii) above (the truth of the conjecture in the case (i) easily follows from Hardt's triviality theorem [5]). In these cases the conjecture was proved independently by van den Dries [4] and Coste [3] using o-minimality, without producing any explicit upper bounds on the number of homeomorphism types as functions of the complexity.

We prove a weaker but effective versions of cases (ii), (iii) of the conjecture. Namely, we give explicit, single exponential upper bounds on the number of possible homotopy types of semi-algebraic sets.

Let ϕ be a Boolean formula with variables in $\{a_i, b_i, c_i| \ 1 \leq i \leq s\}$. For an ordered list $\mathcal{P} = (P_1, \dots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \dots, X_m]$, we denote by $\phi_{\mathcal{P}}$ the formula obtained from ϕ by replacing for each i, $1 \leq i \leq s$, the variable a_i (respectively, b_i and c_i) by $P_i = 0$ (respectively, by $P_i > 0$ and by $P_i < 0$).

Definition 1. We say that two ordered lists $\mathcal{P} = (P_1, \dots, P_s)$, $\mathcal{Q} = (Q_1, \dots, Q_s)$ of polynomials $P_i, Q_i \in \mathbb{R}[X_1, \dots, X_m]$ have the same homotopy type if for any Boolean formula ϕ, the semialgebraic sets defined by $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{Q}}$ are homotopy equivalent.

Let $\mathcal{M}_{m,r}$ be the family of ordered lists $\mathcal{P} = (P_1, \dots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \dots, X_m]$ with the total number of monomials in all polynomials in $\mathcal{P}$ not exceeding r.

Theorem 2 ([1])**.** *The number of different homotopy types of ordered lists in* $\mathcal{M}_{m,r}$ *does not exceed*

$$2^{O(mr)^4}. \tag{1}$$

In particular, the number of different homotopy types of semi-algebraic sets defined by a fixed formula $\phi_{\mathcal{P}}$*, where* $\mathcal{P}$ *varies over* $\mathcal{M}_{m,r}$*, does not exceed (1).*

A similar bound is true for the additive complexity.

The proof of Theorem 1 is based on a new single exponential upper bound on the number of homotopy types of fibres of the projection map of a semi-Pfaffian set. To simplify the notations we formulate here this bound for a special case of semi-algebraic sets.

Let $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_m, Y_1, \dots, Y_n]$, and let ϕ be a Boolean formula with atoms of the form $P = 0,\ P > 0,\ P < 0$, where $P \in \mathcal{P}$. We call the semi-algebraic set $S \subset \mathbb{R}^{m+n}$ defined by ϕ, a $\mathcal{P}$-semi-algebraic set. Let π_S be the restriction to S of the projection map $\pi :\ \mathbb{R}^{m+n} \to \mathbb{R}^n$.

Theorem 3 ([1])**.** *Let the cardinality* $\#\mathcal{P} = s$*, and for every* $P \in \mathcal{P}$*, the degree* $\deg(P) < d$*. Then there exists a finite set* $A \subset \mathbb{R}^n$*, with*

$$\#A \leq (2^m snd)^{O(mn)},$$

such that for every $\mathbf{y} \in \mathbb{R}^n$ *there exists* $\mathbf{z} \in A$ *such that for every* $\mathcal{P}$*-semi-algebraic set* $S \subset \mathbb{R}^{m+n}$*, the set* $\pi_S^{-1}(\mathbf{y})$ *is homotopy equivalent to* $\pi_S^{-1}(\mathbf{z})$*. In particular, for any fixed* $\mathcal{P}$*-semi-algebraic set* S*, the number of different homotopy types of fibres* $\pi_S^{-1}(\mathbf{y})$ *for various* $\mathbf{y} \in \pi(S)$ *is also bounded by*

$$(2^m snd)^{O(mn)}.$$

A similar theorem is true for finite families $\mathcal{P}$ of real Pfaffian functions, and projections π_S of $\mathcal{P}$-semi-Pfaffian sets S. It implies Theorem 1, using a technique for representing fewnomials as Pfaffian functions, introduced in § 3, Chapter 9 of [4].

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Lower bounds, sharpness in real enumerative geometry

Jean-Yves Welschinger

Complex plane rational curves are images of holomorphic maps from the Riemann sphere to the complex projective plane. Plane rational curves of a given degree form a $(3d-1)$-parameter family. It is a classical result of complex enumerative geometry that the number N_d of degree d rational curves passing through $3d-1$ generic points in the plane does not depend on the choice of the points. These curves are all immersed with only ordinary double points as singularities. This is no more true over the real numbers, basically because the field of real number is not algebraically closed. Namely, choose r generic points $x_1, \dots, x_r$ in the real projective plane and s generic pairs of complex conjugated points $\xi_1, \overline{\xi}_1, \dots, \xi_s, \overline{\xi}_s$, where $r + 2s = 3d - 1$. Denote by $\mathcal{R}_d(\underline{x}, \underline{\xi})$ the finite set of degree d real rational curves containing $\underline{x}, \underline{\xi}$. The cardinality of this set depends on the choice of $\underline{x}, \underline{\xi}$. However these real rational curves have two kinds of real double points. *Non-isolated* real double points are local intersection of two real branches, like $x^2 - y^2 = 0$, whereas *isolated* real double points are local intersection of two complex conjugated branches, like $x^2 + y^2 = 0$. For $C \in \mathcal{R}_d(\underline{x}, \underline{\xi})$, denote by $m(C)$ its number of real isolated double points and set

$$\chi^d_r(\underline{x}, \underline{\xi}) = \sum_{C \in \mathcal{R}_d(\underline{x}, \underline{\xi})} (-1)^{m(C)} \in \mathbb{Z}.$$

Theorem 1. *The integer $\chi^d_r(\underline{x}, \underline{\xi})$ does not depend on the generic choice of $\underline{x}, \underline{\xi}$.* □

Hence, the complex projective plane together with its complex conjugation comes with a family of enumerative invariants $(\chi^d_r)_{d \in \mathbb{N}^*, 0 \leq r \leq 3d-1, r = 3d-1 \mod (2)}$. They provide lower bounds in real enumerative geometry, namely.

Corollary 2. $|\chi^d_r| \leq_{(*)} \# \mathcal{R}_d(\underline{x}, \underline{\xi}) \leq N_d$.

I did obtain these results couple of years ago. They hold more generally for any symplectic four-manifold equipped with an antisymplectic involution, see [1]. The main result I did explain here was the following sharpness result.

Theorem 3. *For every $d \in \mathbb{N}^*$ and $0 \leq r \leq 1$, the lower bounds $(*)$ is sharp. Moreover, $sign(\chi_r^d) = (-1)^{\frac{(d-1)(d-2)}{2}}$.*

This means that there exists a configuration $\underline{x}, \underline{\xi}$ of points such that all elements of $\mathcal{R}_d(\underline{x}, \underline{\xi})$ are counted with respect to the same sign in χ_r^d, the parity of the smooth genus $\frac{(d-1)(d-2)}{2}$. Such configuration of points are obtained when $\underline{\xi}$ is very closed to a real imaginary conic -of equation $x^2 + y^2 + z^2 = 0$-. This sharpness result also holds in a much more general situation, it is announced in [2]. The tool to prove it comes from symplectic field theory, one stretches the manifold near its real locus.

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Reporter: Daniel Plaumann

Participants

Prof. Dr. Francesca Acquistapace
Dip. di Matematica "L.Tonelli"
Universita di Pisa
Largo Bruno Pontecorvo,5
I-56127 Pisa

Prof. Dr. Carlos Andradas
Facultad de Matematicas
Depto. de Algebra
Universidad Complutense de Madrid
E-28040 Madrid

Doris Augustin
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Benoit Bertrand
Departement de Mathematiques
Universite de Geneve
Case Postale 64
2-4 rue du Lievre
CH-1211 Geneve 4

Prof. Dr. Frederic Bihan
Lab. de Mathematiques
Universite de Savoie
F-73376 Le Bourget du Lac Cedex

Prof. Dr. Ludwig Bröcker
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Fabrizio Broglia
Dip. di Matematica "L.Tonelli"
Universita di Pisa
Largo Bruno Pontecorvo,5
I-56127 Pisa

Prof. Dr. Erwan Brugalle
Inst. de Mathematiques de Jussieu
Universite Paris VI
175 rue du Chevaleret
F-75013 Paris

Prof. Dr. Fabrizio Catanese
Lehrstuhl für Mathematik VIII
Universität Bayreuth
NW - II
95440 Bayreuth

Prof. Dr. Jaka Cimpric
FMF - Matematika
University of Ljubljana
Jadranska 19
1111 Ljubljana
SLOVENIA

Prof. Dr. Michel Coste
Departement de Mathematiques
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Alexander Degtyarev
Department of Mathematics
Bilkent University
06800 Bilkent, Ankara
TURKEY

Prof. Dr. Max A. Dickmann
U. F. R. de Mathematiques
Case 7012
Universite Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Jose F. Fernando
Facultad de Matematicas
Depto. de Algebra
Universidad Complutense de Madrid
E-28040 Madrid

Prof. Dr. Andreas Fischer
Dept. of Mathematics
University of Saskatchewan
106 Wiggins Road
Saskatoon Sask. S7N 5E6
CANADA

Prof. Dr. Andrei Gabrielov
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette IN 47907-2067
USA

Prof. Dr. Riccardo Ghiloni
Dipartimento di Matematica
Universita di Trento
Via Sommarive 14
I-38050 Povo (Trento)

Prof. Dr. Joost van Hamel
Departement Wiskunde
Faculteit der Wetenschappen
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Leuven

Prof. Dr. Johannes Huisman
Dept. de Mathematiques
Universite de Bretagne Occidentale
6, Avenue Victor Le Gorgeu
F-29287 Brest Cedex

Prof. Dr. Ilia Itenberg
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Dr. Tobias Kaiser
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Igor Klep
FMF - Matematika
University of Ljubljana
Jadranska 19
1111 Ljubljana
SLOVENIA

Prof. Dr. Manfred Knebusch
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Vladimir Kostov
Laboratoire J.-A. Dieudonne
Universite de Nice
Sophia Antipolis
Parc Valrose
F-06108 Nice Cedex 2

Prof. Dr. Wojciech Kucharz
Dept. of Mathematics and Statistics
University of New Mexico
Albuquerque , NM 87131
USA

Prof. Dr. Salma Kuhlmann
Dept. of Mathematics
University of Saskatchewan
106 Wiggins Road
Saskatoon Sask. S7N 5E6
CANADA

Prof. Dr. Krzysztof Kurdyka
Lab. de Mathematiques
Universite de Savoie
F-73376 Le Bourget du Lac Cedex

Richard Leroy
I.R.M.A.R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Louis Mahe
U. F. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Valery Mahe
School of Mathematics
University of East Anglia
GB-Norwich NR4 7TJ

Dr. Frederic Mangolte
Lab. de Mathematiques
Universite de Savoie
F-73376 Le Bourget du Lac Cedex

Prof. Dr. Murray Marshall
Dept. of Mathematics
University of Saskatchewan
106 Wiggins Road
Saskatoon Sask. S7N 5E6
CANADA

Prof. Dr. Jean-Philippe Monnier
Dept. de Mathematiques
Faculte des Sciences
Universite d'Angers
2, Boulevard Lavoisier
F-49045 Angers Cedex

Tim Netzer
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz

Prof. Dr. Adam Parusinski
Dept. de Mathematiques
Faculte des Sciences
Universite d'Angers
2, Boulevard Lavoisier
F-49045 Angers Cedex

Prof. Dr. Dmitrii V. Pasechnik
School of Physical and Mathematical
Sciences
Nanyang Technological University
1 Nanyang Walk, Blk 5
Singapore 637616
SINGAPORE

Daniel Plaumann
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz

Prof. Dr. Victoria Powers
Dept. of Mathematics and
Computer Science
Emory University
400, Dowman Dr.
Atlanta , GA 30322
USA

Prof. Dr. Alexander Prestel
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz

Prof. Dr. Nicolas Puignau
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Prof. Dr. Bruce Reznick
Dept. of Mathematics, University of Illinois at Urbana-Champaign
273 Altgeld Hall MC-382
1409 West Green Street
Urbana , IL 61801-2975
USA

Prof. Dr. Marie-Francoise Roy
U. F. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Daniel Schaub
Dept. de Mathematiques
Faculte des Sciences
Universite d'Angers
2, Boulevard Lavoisier
F-49045 Angers Cedex

Prof. Dr. Claus Scheiderer
Fakultät für Mathematik
Universität Konstanz
78457 Konstanz

Prof. Dr. Niels Schwartz
Fakultät für Mathematik
und Informatik
Universität Passau
Innstr. 33
94032 Passau

Markus Schweighofer
Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz

Prof. Dr. Boris Shapiro
Department of Mathematics
Stockholm University
S-10691 Stockholm

Alexandre Sine
Dept. de Mathematiques
Faculte des Sciences
Universite d'Angers
2, Boulevard Lavoisier
F-49045 Angers Cedex

Prof. Dr. Luis Felipe Tabera
Departamento de Matematicas
Facultad de Ciencias
Universidad de Cantabria
Avda. de los Castros, s/n
E-39005 Santander

Marcus Tressl
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Prof. Dr. Nicolai N. Vorobjov
Dept. of Computer Sciences
University of Bath
GB-Bath BA2 7AY

Prof. Dr. Jean-Yves Welschinger
Unite de Mathematiques Pures et Appliquees
Ecole Normale Superieure de Lyon
46 Allee d'Italie
F-69364 Lyon Cedex 07

Etienne Will
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex