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Arbeitsgemeinschaft: Conformal Field Theory

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Abstract. The two major approaches to chiral conformal field theory – one based on operator algebras and one based on vertex algebras – both lead to representation categories which are tensor categories and, in the case of rational chiral conformal field theories, more specifically modular tensor categories. In this Arbeitsgemeinschaft, we have studied algebraic structures related to tensor categories arising in conformal field theory. The notion of a module category over this tensor category is central in the construction of a full local conformal field theory in various frameworks.

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Introduction by the Organisers

The Arbeitsgemeinschaft mit aktuellem Thema "Algebraic structures in conformal field theories", organized by Y. Kawahigashi (University of Tokyo), V. Ostrik (University of Oregon) and C. Schweigert (University of Hamburg), was held from April 1 to 7, 2007.

Two-dimensional conformal field theory plays a fundamental role in the theory of two-dimensional critical systems of classical statistical mechanics, in quasi one-dimensional condensed matter physics and in string theory. The study of defects in systems of condensed matter physics, of percolation probabilities and of (open) string perturbation theory in the background of certain solitonic solutions of string theory, the so-called D-branes, forces one to analyze conformal field theories on surfaces that may have boundaries and / or can be non-orientable. This study has recently led to deeper insight into the mathematical structure of conformal field theory.

Many mathematical disciplines have contributed to a better understanding of conformal field theory and have received stimulating input from questions arising in conformal field theories. There are two major approaches to chiral conformal field theory: one that is based on operator algebras and one based on vertex algebras. In both approaches, chiral conformal field theory is described by a certain infinite dimensional algebraic structure. In the first approach, conformal field theory is described by a net of von Neumann algebras, where each von Neumann algebra consists of bounded linear operators and is generated by local observables. This approach was initiated by Haag and Kastler more than 40 years ago (and applies also to quantum field theories in dimensions other than two). The latter is based on algebraic axiomatization of quantum fields in chiral conformal field theory and was initiated by Frenkel, Lepowsky, Meurman and Borcherds in the 1980s.

Both algebraic structures lead to representation categories which are tensor categories and, in the case of rational chiral conformal field theories, more specifically modular tensor categories. In the operator algebraic approach, the representation category consists of representations of a net of von Neumann algebras on a Hilbert space; while its original form is due to Doplicher, Haag and Roberts, for chiral conformal field theory certain adaptations have to be made. The representation category of vertex algebras consists of modules over (conformal) vertex algebras. In this Arbeitsgemeinschaft, we have studied algebraic structures related to tensor categories arising in conformal field theory.

These tensor categories also encode the monodromy representations of the vector bundles of conformal blocks for rational vertex algebras, objects that are of interest for algebraic geometry. Moreover, modular tensor categories are a crucial ingredient in the construction of three-dimensional topological field theories.

While chiral conformal field theories have certain physical applications in the description of quantum Hall systems, full local conformal field theories are relevant for the physical applications referred to in the first paragraph. Recently, it has been understood that the construction of a full local conformal field theory is best described using the structure of a module category over the tensor category that describes the chiral data.

In view of the above background, we started the Arbeitsgemeinschaft with a general introduction (Svegstrup) to tensor categories and module categories to provide an oecumenic language for both approaches. Then we had three talks on how tensor categories naturally arise in various approaches to conformal field theory: through operator algebras (Bartels), loop groups (Bunke), and Frobenius algebras (Grossman).

The notion of a fusion category provides an abstract framework for various kinds of representation categories. The quantum double construction, originally due to Drinfeld, is a very important method to produce a modular tensor category. It applies to a finite group and also to a more general tensor category. Whenever a finite group acts on a conformal field theory by symmetries, one can pass to a new theory fixed by such a symmetry. This new theory is called an orbifold, and constructions of this type are studied in many different approaches. Doubles of finite groups, their representation categories and orbifold theories in the operator algebraic approach were introduced in the talks by Müller and Gray.

Certain module categories can be classified and the A-D-E classification for the case of the quantum $SL(2)$ due to Kirillov and Ostrik is a basic example. Another type of classification for fusion categories is the one for given fusion rules or a given number of simple objects. Various such classification results have been obtained by Ostrik and collaborators; they have been the subject of two talks (Phung, Cuntz). The operator algebraic approach based on theory of subfactors by Jones has produced a few exceptional tensor categories (the topic of Peters' talk) which have not been obtained by other approaches such as theory of quantum groups. Our understanding of such structures in the framework of conformal field theory is still very poor and further developments are expected.

There have been various concrete constructions of $(2+1)$-dimensional topological quantum field theory in the sense of Atiyah. Two of such constructions, due to Reshetikhin-Turaev and Turaev-Viro, are particularly closely related to tensor categories. In these constructions, a three-dimensional closed manifold is realized through Dehn surgery and triangulation, respectively, and combinatorial data arising from a tensor category produce a number for each closed manifold which is a topological invariant of the manifold. This was explained in a first talk by Suszek; in a second talk (Schommer-Pries), the construction of three-dimensional topological field theories from subfactors was presented.

Conformal blocks play a fundamental role in conformal field theory. They were defined in the concrete context of Wess-Zumino-Witten models in Graziano's talk and the Knizhnik-Zamolodchikov connection on them was introduced in Nieper-Wißkirchen's talk.

Correlation functions, symmetries and dualities are studied in the categorical framework of conformal field theory. Techniques from topological field theory provide tools to study such objects. This was the topic of two talks (Lehn, Zito). In this context, a tensor functor from a tensor category to a category of bimodules called α-induction plays a prominent role; this was the topic of a Asaeda's talk. A study of boundary conformal field theory in the operator algebraic approach, based on recent results of Longo-Rehren was presented by Bahns. It gives a concrete realization of the general abstract structure.

We had 17 talks by the participants and two sessions where 12 participants gave 10-minute presentations on their work. We also had two short supplemental presentations by the organizers. We had a sufficient amount of time for free discussions among the participants.

The meeting had 54 participants from various countries in Europe and the U.S., Canada, Japan and India.

Arbeitsgemeinschaft: Conformal Field Theory

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Abstracts

Tensor Categories and Module Categories

Rolf Dyre Svegstrup

In the following we will introduce the notions of tensor categories, braidings and duality for tensor categories, and module categories.

In order to conserve space many definitions have been written in the form of equations rather than commutative diagrams. However, commutative diagrams undoubtedly provide a clearer picture and the reader is encouraged to write the commutative diagrams corresponding to the relevant equations on his own.

1. Tensor Categories

To motivate our definition of a tensor category we briefly consider the category of finite-dimensional complex vector spaces having linear maps as morphisms, $\mathsf{Vec}_{\mathsf{fin}}$. On this category the usual tensor product of vector spaces and linear maps defines a bifunctor $\otimes : \mathsf{Vec}_{\mathsf{fin}} \times \mathsf{Vec}_{\mathsf{fin}} \to \mathsf{Vec}_{\mathsf{fin}}$. Besides defining a bifunctor, the tensor product has such properties as $V \otimes \mathbb{C} \cong V \cong \mathbb{C} \otimes V$ for $V \in \mathsf{Vec}_{\mathsf{fin}}$ as well as $(V \otimes W) \otimes X \cong V \otimes (W \otimes X)$ for any $V, W, X \in \mathsf{Vec}_{\mathsf{fin}}$.

We will define a tensor category as being a category equipped with a bifunctor with properties similar to those indicated above.

Definition. *A* tensor category *is a six-tuple* $(\mathcal{C},\otimes,I,a,l,r)$ *where* $\mathcal{C}$ *is a category,* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *a bifunctor,* I *an object in* $\mathcal{C}$*, and* $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$, $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$ *are natural isomorphisms such that*

(1) $(\mathrm{id}_U \otimes a_{V,W,X}) \circ a_{U,V\otimes W,X} \circ (a_{U,V,W} \otimes \mathrm{id}_X) = a_{U,V,W\otimes X} \circ a_{U\otimes V,W,X}$
(2) $(\mathrm{id}_V \otimes l_W) \circ a_{V,I,W} = r_V \otimes \mathrm{id}_W$

The natural isomorphism a *is called the associativity constraint and* l *(resp.* r*) is called the left (resp. right) unit constraint. The object* I *is called the unit of the tensor category. If* a*,* l *and* r *are all identities of the category, the tensor category is said to be* strict.

The requirements on the associativity constraint and the left and right unit constraints above are called the Pentagon Axiom and the Triangle Axiom, respectively, refering to the shape of the commutative diagrams representing them. They ensure that the exact manner of moving parentheses around is of no concern.

It is worth noting that what we call a tensor category above is also sometimes called a monoidal category in the literature. Further adding to the confusion both terms are sometimes taken to refer to a stricter definition in which the category is furthermore required to be abelian and the bifunctor $\otimes$ additive. We will get back to this in Section 4.

Lastly, we mention that any tensor category is actually equivalent to a strict tensor category. This is known as Mac Lane's Coherence Theorem and allows us to assume strictness of our tensor categories.

2. Braided Tensor Categories

Returning to the example of the category of finite-dimensional complex vector spaces, we note that for any two vector spaces V and W the tensor products $V \otimes W$ and $W \otimes V$ are isomorphic. Generalizing this notion to tensor categories leads to the concept of braidings.

For any category $\mathcal{C}$ we define the flip functor $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by $\tau(U, V) = (V, U)$ on objects and $\tau(f, g) = (g, f)$ on morphisms.

A *commutativity constraint* on a tensor category is a natural isomorphism $c : \otimes \to \otimes\tau$.

Definition. *A* braiding *in a tensor category* $(\mathcal{C},\otimes,I,a,l,r)$ *is a commutativity constraint* c *such that*

(1) $(\mathrm{id}_V \otimes c_{U,W}) \circ a_{V,U,W} \circ (c_{U,V} \otimes \mathrm{id}_W) = a_{V,W,U} \circ c_{U,V\otimes W} \circ a_{U,V,W}$
(2) $(c_{U,W} \otimes \mathrm{id}_V) \circ a^{-1}_{U,W,V} \circ (\mathrm{id}_U \otimes c_{V,W}) = a^{-1}_{W,U,V} \circ c_{U,V\otimes W} \circ a_{U,V,W}$

A braided tensor category $(\mathcal{C},\otimes,I,a,l,r,c)$ *is a tensor category* $(\mathcal{C},\otimes,I,a,l,r)$ *with a braiding* c*.*

The two requirements on the associativity and commutative constraints are called the Hexagon Axioms.

In our example $\mathsf{Vec}_{\mathsf{fin}}$ the braiding satisfies $c_{V,W} \circ c_{W,V} = \mathrm{id}_{V\otimes W}$ and the one hexagon axiom above therefore follows from the other. However, not all braided tensor categories have this property. In case that $c_{V,W} \circ c_{W,V} = \mathrm{id}_{V\otimes W}$ holds true, we say that the tensor category is *symmetric*.

3. Duality

To motivate the definition of duality in tensor categories, we return once more to our current example $\mathsf{Vec}_{\mathsf{fin}}$. Let V be a finite-dimensional complex vector space and denote by V^* its dual space. We can then define an evaluation map $\mathsf{ev} : V^* \otimes V \to \mathbb{C}$ and the natural isomorphism $\lambda : V \otimes V^* \to \mathsf{Hom}(V, V)$. Furthermore we have coevalution, $\delta : \mathbb{C} \to V \otimes V^*$ defined by setting its value on 1 to be $\delta(1) = \lambda^{-1}(\mathrm{id}_V)$.

Identifying V with $V \otimes \mathbb{C}$, it is easy to check that for instance $(\mathrm{id}_V \otimes \mathsf{ev}) \circ (\delta \otimes \mathrm{id}_V) = \mathrm{id}_V$ holds true. For general tensor categories, we will take the concept of duality to mean the existence of an object V^* for every object V along with morphisms playing the role of evaluation and coevalution. However, as earlier noted, not every tensor category is symmetric and we therefore have to distinguish between left and right duality. This will be made clear below.

Definition. *A strict tensor category* $(\mathcal{C},\otimes,I)$ *has* left duality *if for each object* V *in* $\mathcal{C}$ *there exists an object* V^* *in* $\mathcal{C}$ *along with morphisms* $b_V : I \to V \otimes V^*$ *and* $d_V : V^* \otimes V \to I$ *such that* $(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$ *and* $(d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$.

A useful mnemonic for the letters b and d is to take them to mean *birth* and *death*, respectively. For a given object V the dual object V^* is defined uniquely up to isomorphism. Working with duality and braidings is most easily done through pictorial representations as can be found in [1].

Right duality can similarly be defined as the existence of an object ${}^*V$ for each object V along with morphisms $b'_V : I \to^* V \otimes V$ and $d'_V : V \otimes^* V \to I$ satisfying equations similar to those of b_V and d_V.

4. Module Categories

The definition of a module category is more technical than what we have seen in the previous sections and we will not define all the relevant terms. However, all omitted definitions can be found in [2].

Henceforth, we only consider categories which are semi-simple abelian categories over an algebraically closed field k with finite-dimensional Hom-spaces. Also, all our functors will be additive from now on. This would include, for instance, the category $\mathsf{Vec}_{\mathsf{fin}}$ along with the tensor product thereon.

We mentioned in Section 1 that there are varying definitions of tensor categories. In this section we will be using a definition differing from that previously given. In order to avoid confusion we will follow [3] and call this one a monoidal category.

We define a *monoidal category* exactly as in Section 1 but with the caveat that our category satisfies the requirements stated above and that all involved functors, amongst these the tensor functor itself, are additive. For convenience we will assume that our monoidal categories are strict.

Definition. *A* module category *over a monoidal category* $(\mathcal{C},\otimes,I)$ *is a category* $\mathcal{M}$ *along with an exact bifunctor* $\boxtimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ *and natural isomorphisms* $m : \boxtimes \circ (\otimes \times \mathrm{id}_{\mathcal{M}}) \to \boxtimes \circ (\mathrm{id}_{\mathcal{C}} \times \boxtimes)$, $l^{\mathcal{M}} : \boxtimes(I \times \mathrm{id}_{\mathcal{M}}) \to \mathrm{id}_{\mathcal{M}}$ *such that*

(1) $(\mathrm{id}_X \boxtimes m_{Y,Z,M}) \circ m_{X,Y\otimes Z,M} \circ (a_{X,Y,Z} \boxtimes \mathrm{id}_M) = m_{X,Y,Z\boxtimes M} \circ m_{X\otimes Y,Z,M}$

(2) $(\mathrm{id}_X \boxtimes l^{\mathcal{M}}_M) \circ m_{X,I,Y} = m_{X,I,Y}$

The two equations above allow the moving around of parentheses. An easy example of a module category is taking a monoidal category, such as $\mathsf{Vec}_{\mathsf{fin}}$, to be a module category over itself. For more details and examples, the reader can consult [3].

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Tensor Categories from Factors

Arthur Bartels

A net of von Neumann algebras over $\mathbb{R}$ is an inclusion preserving assignment

$$\mathcal{I} \ni I \mapsto A(I) \in \mathcal{N}$$

where $\mathcal{I}$ is the set of bounded open intervals in $\mathbb{R}$ and $\mathcal{N}$ is the set of von Neumann algebras on a fixed Hilbert space H. Such a net is called *additive* if $A(I \cup J) =$

$A(I) \vee A(J)$ whenever $I \cap J \neq \emptyset$. (Here $A(I) \vee A(J)$ denotes the von-Neumann algebra generated by $A(I)$ and $A(J)$.) It is called *local* if $A(J)$ is contained in the commutant of $A(I)$ whenever $I \cap J = \emptyset$. It is said to satisfy *Haag-duality* if $A(J)$ is the commutant of the von-Neumann algebra generated by all $A(I)$ with $I \cap J = \emptyset$.

An endomorphism ρ of the C^*-algebra $\mathcal{A}(\mathbb{R})$ generated by all $A(K)$ is said be localized in $I \in \mathcal{I}$ if $\rho|_{A(J)} = id_{A(J)}$ whenever $J \cap I = \emptyset$. Such an endomorphism is said to be transportable if it can be localized (up to unitary equivalence) in any given $I \in \mathcal{I}$. The *sectors* of A are the transportable endomorphism of $\mathcal{A}(\mathbb{R})$. The tensor product of sectors is defined by composition of endomorphisms. If the net A is additive and satisfies Haag-duality, then the sectors form a braided tensor category.

Further assumptions on the net A lead to further properties of the category of sectors. A net A is *irreducible* if the von-Neumann algebra generated by the $A(I)$ is the von-Neumann algebra of all bounded operators. consider $I, J, K \in \mathcal{I}$ such that I and J are the components of $K - \{x\}$ for some $x \in I$. A is *strongly additive* if in this situation $A(K) = A(I) \vee A(J)$. A has the *split property* if whenever $I, J \in \mathcal{I}$ with $\overline{I} \cap \overline{J} = \emptyset$ then $A(I) \vee A(J)$ is the tensor product $A(I) \otimes A(J)$ of von-Neumann algebras. The net A is *completely rational* if in addition to the above properties the following condition is satisfied. Consider $E = I \cup J$ where $I, J \in \mathcal{I}$ and $\overline{I} \cap \overline{J} = \emptyset$. Consider the commutant $A(E')'$ of the von-Neumann algebra generated by all $A(K)$ with $K \cap E = \emptyset$ and let $A(E) := A(I) \vee A(J)$. By locality $A(E) \subseteq A(E')'$. It is required that the Jones index $[A(E')' : A(E)]$ of this inclusion (of factors) is finite.

The following result is due to Kawahigashi-Longo-Müger [1].

Theorem. *Let A be a completely rational net with a modular PCT-symmetry. Then the sectors of A form a modular tensor category with a finite number of irreducible objects $\rho_1, \ldots, \rho_n$ (localized in I). Moreover,*

$$[A(E')' : A(E)] = \sum_{i=1}^{n} [A(I) : \rho_i(A(I))].$$

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Tensor Categories for Loop Groups

Ulrich Bunke

A characteristic feature of the WZW-model based on a group G is the presence of a loop group symmetry. On the infinitesimal level this symmetry is reflected by an action of a central extension $\hat{\mathfrak{g}}$ of the loop algebra $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ of the Lie algebra $\mathfrak{g}$ of G. We refer to [5] for an introduction to the WZW-model.

The goal of the talk at the Arbeitsgemeinschaft was to explain the category of integrable highest weight representations of $\hat{\mathfrak{g}}$ with particular emphasis on the fusion product. We start with a simple complex Lie algebra $\mathfrak{g}$. We choose a Cartan algebra $\mathfrak{h} \subset \mathfrak{g}$ and a system $\Delta^+ \subset \mathfrak{h}^*$ of positive roots. For $\alpha \in \Delta^+$ let $H_\alpha \in \mathfrak{h}$ denote the corresponding coroot. We fix the unique invariant bilinear form $(\dots,\dots)$ on $\mathfrak{g}$ such that $(H_\theta, H_\theta) = 2$ for the longest root $\theta \in \Delta^+$. We the form the central extension $\hat{\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{C}K$ of loop algebra with commutator $[X \otimes f, Y \otimes g] = [X,Y] \otimes fg - K(X,Y)\mathtt{Res}(fdg)$. We then form the Lie algebra $\tilde{\mathfrak{g}} := \hat{\mathfrak{g}} \oplus d\mathbb{C}$ where $[d, X \otimes f] = X \otimes t\frac{d}{dt}f$ and $[d,K] = 0$. The Lie algebra $\tilde{\mathfrak{g}}$ is a Kac-Moody algebra and is called the affinization of $\mathfrak{g}$. A reference for the theory of Kac-Moody algebras and their representations is [3].

The sum $\tilde{\mathfrak{h}} := \mathfrak{h} \oplus K\mathbb{C} \oplus d\mathbb{C}$ is a Cartan algebra of $\tilde{\mathfrak{g}}$. Let $\alpha_1, \dots, \alpha_r$ be a system of simple roots of $\mathfrak{g}$. We use the same symbols to denote their extensions to $\tilde{\mathfrak{h}}$. Then $\alpha_0, \dots, \alpha_n$ is a system of simple roots of $\tilde{\mathfrak{g}}$, where $\alpha_0 := \delta - \theta$ and δ is dual to d.

Every coroot H_{α_i} fits into a $sl(2)$-triple $\mathfrak{g}_i := (H_{\alpha_i}, E_i, F_i)$, $i = 0, \dots r$. A representation V of $\tilde{\mathfrak{g}}$ is called integrable if each vector $v \in V$ spans a finite-dimensional representation of $\mathfrak{g}_i$ for all i. If V has a highest weight, then integrability imposes a strong restriction on this weights.

Theorem. *The isomorphism classes of integrable highest weight representations of $\tilde{\mathfrak{g}}$ are in bijection with the set of integral (i.e. $\lambda(H_i) \in \mathbb{Z}$, $i = 0, \dots, r$) and dominant (i.e. $\lambda(H_i) \geq 0$, $i = 0, \dots, r$) highest weights $\lambda \in \tilde{\mathfrak{h}}^*$.*

Let V_λ denote the integrable irreducible representation with highest weight λ. The integer $l := \lambda(K) \in \mathbb{N}_0$ is called its level. From now on we will only consider the case where $\lambda(d) = 0$ and let A_l denote the finite set of those integral dominant λ of level l.

For the rest of this note we fix a level $l \in \mathbb{N}$. The category C_l of integrable highest weight representations of $\tilde{\mathfrak{g}}$ at level l is semisimple, and its simple objects are labeled by A_l. Its Grothendieck group is therefore given by $K^0(C_l) := \mathbb{Z}[A_l]$. The category C_l has actually the structure of a modular tensor category. The details of this assertion form a long and complicated story, and we refer to [1] for more information. The braided tensor structure $\otimes_{C_l}$ on C_l induces on $K^0(C_l)$ the structure of a based ring. In our presentation we focussed on a description of this ring structure following the exposition [2].

Given $\mu, \nu \in A_l$, then we have a decomposition $V_\mu \otimes_{C_l} V_\nu = \oplus_{\lambda \in A_l} N^\lambda_{\mu,\nu} V_\lambda$, where the integers $N^\lambda_{\mu,\nu}$ are called fusion coefficients. We are going to *define* the fusion coefficients using the notion of conformal blocks.

Let Σ be a Riemann surface with n distinct marked points $p = (p_1, \dots, p_n)$ and a choice of coordinates $t_1, \dots, t_n$ at these points. Let $\lambda := (\lambda_1, \dots, \lambda_n)$ be a collection of elements of A_l. We can form the Liealgebra $\mathfrak{g}(\Sigma - p) := \mathfrak{g} \otimes \mathcal{O}(\Sigma - p)$. Using the coordinates we get evaluations $\mathsf{ev}_i : \mathfrak{g}(\Sigma - p) \to \mathfrak{g} \otimes \mathbb{C}[[t]][t^{-1}]$. Note that the action of $\tilde{\mathfrak{g}}$ on an integrable highest weight representation extends uniquely to formal power series. As a consequence of the residue theorem the following prescription defines an action of $\mathfrak{g}(\Sigma - p)$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$:

$$(X \otimes f)(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes \mathsf{ev}_i(X \otimes f)v_i \otimes \cdots \otimes v_n \ .$$

Definition. *The space*

$$V(\Sigma, p, \lambda) := \mathrm{Hom}_{\mathfrak{g}(\Sigma-p)}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, \mathbb{C})$$

is called the space of conformal blocks.

One can show that it does not depend on the choice of the coordinates t_i. For $\lambda \in A_l$ let λ^* be the dual of λ defined as the unique representative in A_l of the orbit of $-\lambda$ under the Weyl group of $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$.

Definition. *We define the fusion coefficients by*

$$N^{\lambda}_{\mu,\nu} := \dim V(\mathbb{CP}^1, (0, 1, \infty), (\mu, \nu, \lambda^*))$$

We have the following non-trivial result:

Theorem. *The fusion coefficients $N^{\lambda}_{\mu,\nu}$ are the structure constants of a commutative associative based ring structure on $\mathbb{Z}[A_l]$.*

For details we refer to [1], [2], [4].

The goal of the final part of our presentation was to explain a way to calculate $N^{\lambda}_{\mu,\nu}$. If λ is integral dominant for $\tilde{\mathfrak{g}}$, then $\lambda_{|\mathfrak{h}}$ is integral dominant for $\mathfrak{g}$ and hence corresponds to a finite-dimensional irreducible representation $V_{\lambda_{|\mathfrak{h}}}$ of $\mathfrak{g}$.

For integral dominant weights $\bar{\mu}, \bar{\nu}$ of $\mathfrak{g}$ we define $L^{\bar{\lambda}}_{\bar{\mu},\bar{\nu}} \in \mathbb{N}_0$ such that

$$V_{\bar{\mu}} \otimes V_{\bar{\nu}} := \sum_{\bar{\lambda}} L^{\bar{\lambda}}_{\bar{\mu},\bar{\nu}} V_{\bar{\lambda}} \ ,$$

where the sum runs over all integral dominant wights. Let $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha \in \mathfrak{h}^*$. Then we have the so-called Racah-Speiser formula

$$N^{\lambda}_{\mu,\nu} = \sum_{w} \epsilon(w) L^{((\lambda+\rho)^w - \rho)_{|\mathfrak{h}}}_{\mu_{|\mathfrak{h}},\nu_{|\mathfrak{h}}} \ ,$$

where the sum runs over all $w \in W_{l+h}$ such that $((\lambda+\rho)^w - \rho)_{|\mathfrak{h}}$ is dominant. Here W_{l+h} is the "affine Weyl group at level $l + h$" , the semidirect product $QW(l + h)$ of the Weyl group W of $\mathfrak{h} \subset \mathfrak{g}$ and the lattice $(l + h)Q$, where [1]$h := \rho(H_\theta) + 1$ is the dual Coxeter number, and $Q \subset \mathfrak{h}^*$ is the lattice generated by the W-orbit of the long root θ. The sign $\epsilon(w)$ is given by the number of reflexions in w.

In the talk we explained that this formula is equivalent to the non-trivial fact, that a certain natural group homomorphism $R(\mathfrak{g}) \to K^0(C_l)$ is multiplicative. A conceptual proof was given in [6], but see also [4], [2].

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Frobenius Algebras and Q-Systems

Pinhas Grossman

The notion of a Q-system was introduced by Longo to characterize the canonical endomorphism of a subfactor. If M is an infinite factor (a von Neumann algebra with trivial center which does not admit a trace), a Q-system on M is a triple (γ, v, w), where γ is a unital endomorphism of M and v and w are elements of M which are intertwiners of (Id_M, γ) and (γ, γ^2), respectively, satisfying the following associativity and compatibility conditions: $\gamma(w)w = w^2$ and $v^*w = w^*\gamma(v) = \frac{1}{d}$ for some positive number d. A Q-system is said to be irreducible if the intertwiner space of (Id_M, γ) is 1-dimensional. Longo showed that the existence of an irreducible Q-system for γ is a necessary and sufficient condition for γ to be the canonical endomorphism of an irreducible finite-index subfactor of M. The dimension d of the Q-system is then equal to the Jones index of the subfactor. Longo and Roberts then showed that Q-systems actually have a more general foundation as Frobenius algebras in abstract tensor categories.

If $\mathfrak{C}(= (\mathfrak{C}, \otimes, I))$ is a strict tensor category, then a monoid in $\mathfrak{C}$ is a triple (A, η, μ), where A is an object in $\mathfrak{C}$, $\eta \in Hom(I, A)$, and $\mu \in Hom(A \otimes A, A)$, satisfying the associativity condition $\mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu)$ (as elements of $Hom(A \otimes A \otimes A, A)$) and the unit condition $\mu \circ (\eta \otimes 1_A) = \mu \circ (1_A \otimes \eta) = 1_A$. Then μ is called the multiplication and η is called the unit. A comonoid is defined exactly the same way but "reversing the arrows." A Frobenius algebra is a quintuple $(A, \eta, \mu, \delta, \kappa)$ such that (A, η, μ) is a monoid, (A, δ, κ) is a comonoid, and μ and κ satisfy the duality relation $(\mu \otimes 1_A) \circ (1_A \otimes \kappa) = (1_A \otimes \mu) \circ (\kappa \otimes 1_A) = \kappa \circ \mu$ (as elements of $End(A \otimes A)$).

A strict C*-tensor category is a strict tensor category $\mathfrak{C}$ with the following additional structure. Each Hom-space has a Banach space structure such that composition is bilinear. Also $\mathfrak{C}$ is endowed with a contravariant endofunctor $*$ which fixes objects and commutes with $\otimes$, such that the C* identity is satisfied: $\|x^* \circ x\| = \|x\|^2$ for all morphisms x. If (A, T, S^*) is a monoid in a C*-tensor category, then (A, T^*, S) is a comonoid (since $*$ is contravariant), and it is natural to ask when (A, T, S^*, T^*, S) is a Frobenius algebra. Longo and Roberts showed that this happens whenever the comultiplication S is a scalar multiple of an isometry, i.e. $S^* \circ S = a1_A$ for some positive scalar a. If $\mathfrak{C}$ is irreducible ($End(I)$ consists only of scalar multiples of 1_I) then T is also necessarily an isometry, since $T^* \circ T \in End(I)$. We can "normalize" such a Frobenius algebra by rescaling S to be an isometry, but this requires rescaling T as well, since the product of the norms of the rescaled S and T is fixed by the unit condition. Then a Q-system is defined as a comonoid in an irreducible C*-tensor category whose comultiplication is an isometry. We can write such a comonoid as (A, dT^*, S), where S and T are isometries and d is a positive scalar called the dimension of the Q-system.

If (A, dT^*, S) is a Q-system, and $R = dST \in Hom(I, A \otimes A)$, then by the Frobenius duality we have the relation $(R^* \otimes 1_A) \circ (1_A \otimes R) = 1_A$. More generally, two objects A and $\bar{A}$ of a C*-tensor category are said to be conjugate if there exist $R \in Hom(I, \bar{A} \otimes A), \bar{R} \in Hom(I, A \otimes \bar{A})$ such that $(\bar{R}^* \otimes 1_A) \circ (1_A \otimes R) = 1_A$ and $(R^* \otimes 1_{\bar{A}} \circ (1_{\bar{A}} \otimes \bar{R}) = 1_{\bar{A}}$. By rescaling if necessary we may assume R is an isometry, and then it is easy to check that $(A \otimes \bar{A}, dR^*, 1_A \otimes R \otimes 1_{\bar{A}})$ is a Q-system, where $d = \|\bar{R}\|$.

To any unital C*-algebra M there is associated a C*-tensor category whose objects are unital endomorphism of M, with tensor product given by composition, and whose morphisms are intertwiners. If $u_1 \in Hom(\rho_1, \sigma_1)$ and $u_2 \in Hom(\rho_2, \sigma_2)$, then $u_1 \otimes u_2$ is given by $u_1\rho_1(u_2) = \sigma_1(u_2)u_1$. It is then easy to check that if M is an infinite factor, the definition of a Q-system given in the first paragraph is the same as the categorical definition for the category of unital endomorphisms of M (up to a uniquely determined scalar).

If M is an infinite factor then inner conjugacy classes of unital endomorphisms of M are in bijective correspondence with isomorphism classes of $M - M$ (Hilbert space) bimodules, with the correspondence given by $\rho \mapsto_M L^2(M)_{\rho(M)}$, i.e. the left action is multiplication (continuously extended) and the right action is multiplication twisted by ρ. Since ${}_{\rho(M)}L^2(M)_M$ is also an $M - M$ bimodule, for each ρ there exists an endomorphism σ, unique up to inner conjugacy, such that ${}_{\rho(M)}L^2(M)_M = {}_M L^2(M)_{\sigma(M)}$.

Longo's results on subfactor Q-systems can now be formulated as follows: if ρ and σ are unital endomorphisms of an infinite factor M such that $\rho(M) \subset M$ has finite index, then ρ and σ are conjugate (in the category of unital endomorphisms of M) iff ${}_{\rho(M)}L^2(M)_M =_M L^2(M)_{\sigma(M)}$. In particular, $\bar{\rho}$ exists. Therefore $\rho\bar{\rho}$, which is the canonical endomorphism for the inclusion $\rho(M) \subset M$, has a Q-system.

Conversely, if (γ, dT^*, S) is an irreducible Q-system on M, then there is an irreducible subfactor $N \subset M$ with index d whose canonical endomorphism is γ. If N is isomorphic to M (a situation which can always be achieved by taking a suitable tensor product of the subfactor with another factor), then γ actually splits as $\rho\bar{\rho}$, where ρ is an endomorphism of M implementing the isomorphism with

N. There is also a notion of cocycle equivalence of Q-systems such that cocycle-equivalence classes of irreducible Q-systems on M are in bijective correspondence with inner conjugacy classes of subfactors of M.

This talk was an exposition of parts of the papers of Longo and Longo-Roberts.

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Modular Tensor Categories and Quantum Doubles

Jürgen Müller

In the first part, we introduce ribbon categories [1] and quantum traces as well as semisimple categories and the fusion rings associated to them, as particular examples of based rings [4]. We then define modular tensor categories, which are semisimple ribbon categories whose additional ingredient is a non-degeneracy condition on the associated S-matrix, essentially encoding how far the braiding deviates from being symmetric.

The name-giving property of modular tensor categories is an action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the fusion ring, where the usual standard generators of $\mathrm{SL}_2(\mathbb{Z})$ act through the S-matrix already mentioned, and through a diagonal matrix T whose diagonal entries are the eigenvalues of the twist map associated to the underlying ribbon structure. By Vafa's Theorem (1988) these eigenvalues are roots of unity. Actually, for the modular tensor categories arising in Lusztig's work on characters of reductive groups over a finite field, these have an intepretation as eigenvalues of the Frobenius endomorphism, acting on certain ℓ-adic cohomology groups.

Moreover, we present Verlinde's Formula, which relates the fusion coefficients to the entries of the S-matrix. The basic idea of the proof is to interpret the structure constants matrices, whose entries are the fusions coefficients, as representing matrices for the regular representation of the fusion ring, to view the S-matrix as the character table of the fusion ring, and to show that the latter provides a base change simultaneously diagonalising all the structure constants matrices. Actually, this phenomenon is also well-known in other disciplines such as algebraic combinatorics and algebraic graph theory.

In the second part, we present the module categories of quantum doubles of finite groups as explicit examples of modular tensor categories. To do so, we first recall the notion of finite-dimensional quasi-triangular Hopf algebras H, possessing invertible universal R-matrices in $H \otimes H$, leading to solutions of the quantum Yang-Baxter equation. In general, for a finite-dimensional Hopf algebra H the bicrossed product

$$D(H) := (H^{\mathrm{op}})^* \bowtie H,$$

being called the quantum double of H, is a quasi-triangular Hopf algebra.

Given a finite group G, the quantum double construction applied to the group ring $K[G]$ yields the crossed product $D(G) := K[G]^* \rtimes K[G]$ with respect to the left coadjoint action $x \cdot \delta_g = \delta_{xgx^{-1}} \cdot x$, for all $x, g \in G$, where the δ_g form the basis of $K[G]^*$ dual to the group basis of $K[G]$. Now $D(G)$ has the universal R-matrix

$$R = \sum_{g \in G} \delta_g \otimes g \in D(G) \otimes D(G).$$

If K is an algebraically closed field such that $\mathrm{char}(K) \not| \, |G|$, then the category of finite-dimensional $D(G)$-modules is a modular tensor category, whose quantum trace coincides with the usual trace. The simple $D(G)$-modules are labelled by pairs (g_i, π), where the g_i are a chosen fixed set of conjugacy class representatives of G, and π is an irreducible representation of the centralizer $C_G(g_i)$ of g_i in G. The associated S-matrix is a non-abelian (exotic) Fourier transform matrix having entries

$$s_{g_i,\pi;g_j,\rho} := \frac{1}{|C_G(g_i)|} \cdot \frac{1}{|C_G(g_j)|} \cdot \sum_{\substack{h \in G \\ hg_jh^{-1} \in C_G(g_i)}} \mathrm{tr}_\pi(hg_j^{-1}h^{-1}) \cdot \mathrm{tr}_\rho(h^{-1}g_i^{-1}h).$$

Actually this matrix occurred in Lusztig's work on characters of reductive groups over a finite field in the framework of decomposing unipotent characters.

In the third part we indicate how the notion of quantum doubles can be generalised in the setting of category theory: Given a tensor category $\mathcal{C}$ the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is defined by having objects $(V, c_{-,V})$, where $V \in \mathcal{C}$ and $c_{X,V} \colon X \otimes V \to V \otimes X$ are natural isomorphisms such that for all $X, Y \in \mathcal{C}$ we have

$$c_{X \otimes Y,V} = (c_{X,V} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,V}),$$

and morphisms $f \colon (V, c_{-,V}) \to (W, c_{-,W})$ where $f \colon V \to W$ is a morphism such that

$$(f \otimes \mathrm{id}_X) c_{X,V} = c_{X,W}(\mathrm{id}_X \otimes f)$$

for all $X \in \mathcal{C}$. Then $\mathcal{Z}(\mathcal{C})$ is a braided tensor category. For any finite-dimensional Hopf algebra H we have $\mathcal{Z}(H\text{-mod}) \cong D(H)\text{-mod}$ as braided tensor categories.

If K is an algebraically closed field such that $\mathrm{char}(K) \not| \, |G|$, then any indecomposable module category over $K[G]$-mod is equivalent to $K_\omega[H]$-mod, where $H \leq G$ and $\omega \in H^2(H, K^*)$ are uniquely determined up to G-conjugacy. The indecomposable module categories over $\mathcal{Z}(K[G]\text{-mod})$ are labelled by $(G \times G)$-conjugacy classes of pairs (H, ω), where $H \leq G \times G$ and $\omega \in H^2(H, K^*)$ [5]. More general, the category vec_ψ^G of G-graded vector spaces, with monoidal structure labelled by $\psi \in H^3(G, K^*)$, is a modular tensor category equivalent to $D_\psi(G)$-mod, where $D_\psi(G)$ is a quasi Hopf algebra, and the indecomposable module categories over $\mathcal{Z}(\mathrm{vec}^G_\psi)$ are labelled by $(G \times G)$-conjugacy classes of pairs (H, ω), where $H \leq G \times G$ such that $\widetilde{\psi}|_H = 0$ and $\omega \in H^2(H, K^*)$.

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Orbifolds

Oliver Gray

As a motivation of the notion of an orbifold, we will extend the analysis of a net A of von-Neumann algebras, as seen in [1], to the case where a finite group G acts on A. We will see that there is a natural “quotient” to take, and then generalise this idea to general crossed-G fusion categories. These notes rely heavily on the papers by Müger [4] and Kirillov [3].

Notation: Let $\mathfrak{K}$ be the collection of bounded open intervals in $\mathbb{R}$. For an interval $I \in \mathfrak{K}$ we write $I' = \mathbb{R} - I$, and, when $\sup I < \inf J$, we write $I < J$. For a von-Neumann algebra A we set $A' := \{x \in \mathcal{B}(\mathcal{H}) | xy = yx \; \forall y \in A(I)\}$. Given two von-Neumann algebras A_1, A_2 we define $A_1 \wedge A_2$ to be the smallest von-Neumann algebra containing both A_1 and A_2.

Definition. *We define a quantum field theory (QFT) on the line $\mathbb{R}$ to be a triple $(\mathcal{H}, A, \Omega)$ where $\mathcal{H}$ is a separable Hilbert space with distinguished non-zero vector Ω, and A is a net of von-Neumann algebras on $\mathbb{R}$ (i.e. to each interval $I \in \mathfrak{K}$, A associates the von-Neumann algebra $A(I) \subset \mathcal{B}(\mathcal{H})$) such that $A(I)$ is a type-III factor $\forall I \in \mathfrak{K}$ satisfying locality, Haag duality and strong additivity. (See* [4] *for details.)*

We denote by A_∞ the C^*-algebra generated by all $A(I)$ with $I \in \mathfrak{K}$. As in the talk of Bartels [1], we will study $\mathrm{End}A_\infty := \{* - \text{algebra homomorphisms } \rho : A_\infty \to A_\infty\}$, the strict tensor category given by the following, where $\rho, \sigma \in \mathrm{End}A_\infty$:

$$\begin{aligned} \mathrm{Hom}(\rho,\sigma) &= \{s \in A_\infty | s\rho(x) = \sigma(x)s \; \forall x \in A_\infty\}, \text{ the intertwiners of } \rho \text{ and } \sigma; \\ \rho \otimes \sigma &= \rho \circ \sigma; \\ s \otimes t &= s\rho(t) = \rho'(t)s \text{ for } s \in \mathrm{Hom}(\rho,\rho'), t \in \mathrm{Hom}(\sigma,\sigma'). \end{aligned}$$

Definition. *Let G be a finite group.*
(a) We say G acts on A (or rather, on the QFT $(\mathcal{H}, A, \Omega)$) iff there exists a unitary representation V of G on $\mathcal{H}$ such that

(i) $\beta_g(A(I)) = A(I)\ \forall g \in G$ *where* $\beta_g \in EndA_\infty$ *is defined by* $\beta_g(x) := V(g)xV(g)^*$*;*
(ii) $V(g)\Omega = \Omega$*; and*
(iii) If $\beta_g|A(I) = id$ *then* $g = 1$.

(b) Let $\rho \in EndA_\infty, I \in \mathfrak{K}$. *Then we say* ρ *is g-localised in I iff*

$$\begin{aligned} \rho(x) &= x\ \forall x \in A(J) \text{ with } J < I; \\ \rho(x) &= \beta_g(x)\ \forall x \in A(J) \text{ with } J < I. \end{aligned}$$

Note that if ρ is g-localised in some I then it is g-localised in any interval containing I.

Lemma. *If* ρ *is both g- and h-localised then* $g = h$.

The proof is straight-forward and can be found in [4].

Definition. *(a) A g-localised* $\rho \in EndA_\infty$ *is transportable if for every* $J \in \mathfrak{K}$ *there exists a* $\rho' \in EndA_\infty$*, g-localised in J, such that* ρ *is unitary equivalent to* ρ' *(i.e. there exists a unitary* $u \in Hom(\rho, \rho')$*).*
(b) $G - LocA$ *is the full subcategory of* $EndA_\infty$ *whose objects are finite direct sums of G-localised transportable objects of* $EndA_\infty$.

Compare this to the definitions of localisation and sectors in [1].
The action of G on A defines a collection $\gamma_g \in \mathrm{Aut}(G - \mathrm{Loc}A)\ \forall g \in G$:

$$\begin{aligned} \gamma_g(\rho) &= \beta_g \rho \beta_g^{-1}\ \forall \rho \in G - \mathrm{Loc}A; \\ \gamma_g(x) &= \beta_g(x)\ \forall x \in \mathrm{Hom}(\rho, \sigma) \subset A_\infty. \end{aligned}$$

Definition. *A strict crossed-G (or G-equivariant) category is a strict tensor category* $\mathcal{C}$ *together with a full tensor sub-category* $\mathcal{C}_{hom} \subset \mathcal{C}$*, a map* $\delta : Obj(\mathcal{C}_{hom}) \to G$ *which is constant on isomorphism classes, and a homomorphism* $\gamma : G \to Aut(\mathcal{C})$ *satisfying: (i)* $\delta(X \otimes Y) = \delta(X)\delta(Y)$*; (ii)* $\gamma_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$*; (iii) every object in* $\mathcal{C}$ *is a direct sum of objects in* $\mathcal{C}_{hom}$.

Proposition. $G - LocA$ *is a G-crossed category.*

Proof. Define $(G - \mathrm{Loc}A)_g$ to be the full subcategory of $G - \mathrm{Loc}A$ consisting of those ρ that are g-localised. By the lemma, $(G - \mathrm{Loc}A)_g$ are distinct, so we can define a map $\delta : (G - \mathrm{Loc}A)_{\mathrm{hom}} := \bigcup_{g \in G}(G - \mathrm{Loc}A)_g \to G$ by assigning $\delta(\rho) = g$ if ρ is g-localised. We already have a tensor product on $G - \mathrm{Loc}A$ and a homomorphism $\gamma : G \to \mathrm{Aut}(\mathcal{C})$, and condition (iii) comes immediately from the definition. It remains to check conditions (i) and (ii). Both are straight-forward, but proofs can be found in [4]. □

Definition. *A braiding for a crossed-G category* $\mathcal{C}$ *is a family of isomorphisms* $c_{X,Y} : X \otimes Y \to \gamma_g(X) \otimes Y$ *satisfying* $c_{X',Y'} \circ (s \otimes t) = (\gamma_g(t) \otimes s) \circ c_{X,Y}$ *(where* $X \in \mathcal{C}_g$, $X' \in \mathcal{C}_{g'}, Y, Y' \in \mathcal{C}$*) and, similarly, compatibility with associativity and the group action. See* [2] *for details.*

Proposition. $G - LocA$ *admits a unitary braiding.*

Proof. This is a direct analogue of the $G = \{1\}$ case treated in detail in [1]. See [4] for a detailed proof of the crossed-G case. □

There is a natural "quotient" to take on $G - \mathrm{Loc}A$. Define $(G - \mathrm{Loc}A)^G$ to be those objects and morphisms fixed by G. It is a rigid, braided, semi-simple crossed-G category. If we consider only finite-dimensional $\rho \in \mathrm{End}A_\infty$ (as defined by the Jones index) then we find $(G - \mathrm{Loc}_f A)^G \cong \mathrm{Loc}_f(A^G)$ (as strict braided tensor categories), where A^G is the QFT given by $(\mathcal{H}, A^G, \Omega)$, and $A^G(I) := (A(I))^G|\mathcal{H}^G$. This gives rise to the natural question: can we form a generalised notion of orbifold on other crossed-G categories?

Recall that a fusion category is a ribbon category (i.e. a rigid, balanced, braided, monoidal category $\mathcal{C}$) which is, in addition, a semi-simple abelian category over $\mathbb{C}$ with finite-dimensional spaces of morphisms, and such that $\mathbf{1}$ is simple. We define a crossed-G analogue:

Definition. *(a) A crossed-G fusion category is a rigid, monoidal, semi-simple, abelian category equipped with a braiding (in the crossed-G sense given above) such that* $\mathbf{1}$ *is a simple object and* γ_g *is a tensor functor, and also with a family of balancing isomorphisms* $\{\theta_V : V \to \gamma_g(V)\}$ *which satisfy the crossed-G analogue of the usual axioms (see Ch. 3.2 of* [2]*).*
(b) If $\mathcal{C}$ *is a crossed-G fusion category then the orbifold category* $\mathcal{C}/G$ *consists of objects* $(X, \{\phi_g\}_{g\in G})$ *where* $X \in \mathcal{C}$, $(\phi_g : \gamma_g(X) \to X) \in Hom_{\mathcal{C}}$ $\forall g \in G$ *satisfying*

$$(*) \qquad \phi_g \gamma_g(\phi_h) = \phi_{gh}, \quad \phi_1 = id;$$

while morphisms $f \in Hom_{\mathcal{C}/G}((X, \{\phi_g\}_{g\in G}),\ (Y, \{\psi_g\}_{g\in G}))$ *are exactly those* $f \in Hom_{\mathcal{C}}(X, Y)$ *satisfying* $f \circ \phi_g = \psi_g \circ \gamma_g(f)$.

Note that $(*)$ allows for a canonical identification of all $\gamma_g(X) \cong X$, since, writing $\phi_{g,h} := \phi_g^{-1}\phi_h$, we have $\phi_{g,h}\phi_{h,f} = \phi_{g,f}$, $\gamma_g(\phi_{a,b}) = \phi_{ga,gb}$. This definition applied to $G - \mathrm{Loc}A$ gives an equivalent category $(G - \mathrm{Loc}A)/G \cong (G - \mathrm{Loc}A)^G$.

Theorem. $\mathcal{C}/G$ *is a fusion category.*

Proof. Abelian-ness is easy to check. The other structures are ensured by the following: Monoidality: $(X, \{\phi_g\}_{g\in G}) \otimes (Y, \{\psi_g\}_{g\in G}) = (X \otimes Y, \{\phi_g \otimes \psi_g\}_{g\in G})$; $\mathbf{1}_{\mathcal{C}/G} = (\mathbf{1}_{\mathcal{C}}, \{\mathrm{id}\})$; Duality: $(X, \{\phi_g\}_{g\in G})^* = (X^*, \{(\phi_g^*)^{-1}_{g\in G})$; Braiding: $c^{\mathcal{C}/G}_{X,Y} = (\psi_g \otimes \mathrm{id}) \circ c^{\mathcal{C}}_{X,Y}$; Twists: If $X = \bigoplus_h X_h, X_h \in \mathcal{C}$ then $\theta^{\mathcal{C}/G}$ is the direct sum of the compositions $\phi_h \circ \theta^{\mathcal{C}}$. □

Examples. *(i) Let* $\mathcal{C} =$ *Vect*$(\mathbb{C})$ *with trivial grading (i.e.* $\mathcal{C}_1 = \mathcal{C}$*) and trivial action of G. Then Vect*$(\mathbb{C}) \cong$ *Rep*(G)*, since if X is a vector space, then* $(X, \{\phi_g\}_{g\in G}) \in Obj\,\mathcal{C}/G$*, where each* $\phi_g : X \to X$ *satisfies* $\phi_g\phi_h = \phi_{gh}$ *and thus defines a representation of G.*
(ii) Let $\mathcal{C} = G\,Vec$*, the category of G-graded vector spaces. It can be explicitly described as the category of simple objects* $X_g, g \in G$ *with* $X_g \otimes X_h = X_{gh}$, $X_g^* = X_{g^{-1}}$, $\gamma_g(X_h) = X_{ghg^{-1}}$*. Then the orbifold category* $G\,Vec/G$ *is the category of finite-dimensional modules over the Drinfeld double* $D(G)$*: recall from* [1]

or from [5] *that a representation of $D(G)$ is the same as a G-graded representation of G that satisfies $gV_h \subset V_{ghg^{-1}}$.*

We close with a remark: One can also define a crossed-G version of the s-matrix and thereby a crossed-G version of modularity. Then we have the result: $\mathcal{C}$ is modular iff $\mathcal{C}/G$ is modular, and when one of these is modular, $\mathcal{C}_1$ is also modular; in this case, $\mathcal{C}, \mathcal{C}/G$ and $\mathcal{C}_1$ all share the same central charge.

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Finite Tensor Categories

Phùng Hô Hai

The aim of this talk is first to present some basic facts about finite tensor categories. Then the Frobenius-Peron dimension is introduced and used to derive an analog of the freeness theorem of finite Hopf algebras for finite tensor categories. In the second part we discuss the notion of exact module categories over a finite tensor category and its properties, in particular the relationship with the center of the given category.

1. Definition and basic properties of finite tensor categories

Let k be an algebraically closed field. A finite tensor category over k is by definition a k-linear abelian rigid monoidal category, which is equivalent as a k-linear abelian category to the category of modules over a finite dimensional k-algebra and has the property that the endomorphism ring of the unit object is equal to k. We refer to [3] for the notion of rigid monoidal categories.

Let $\mathcal{C}$ be a finite tensor category over k. By definition, there exists a projective generator, say P, in $\mathcal{C}$. Thus each simple object in $\mathcal{C}$ is a subquotient of P, in particular, there exist only finitely many simple objects in $\mathcal{C}$. Notice that the unit object of $\mathcal{C}$, denoted by I, is also simple, as by assumption $\mathrm{End}(I) = k$.

The assumption on rigidity of $\mathcal{C}$ implies that the tensor product is exact in both arguments. Hence for a projective object P, and any object X in $\mathcal{C}$, the tensor products $P \otimes X$ and $X \otimes P$ are projective. Indeed this follows from the canonical isomorphism $\mathrm{Hom}(P \otimes X, Y) \cong \mathrm{Hom}(P, X^* \otimes Y)$, where X^* denotes the (left) dual to X. In particular the functor $P \otimes -$ splits exact sequences. Using this properties one can easily show that P^* is also projective. Note on the other hand that the dualizing functor maps projective to injective objects and vice-versa. Hence in $\mathcal{C}$, projective objects are injective and vice-versa. Thus, let $\{S_1, ..., S_n\}$ be the set of non-isomorphic simple objects of $\mathcal{C}$, we can choose for each S_i a projective cover P_i.

2. The Frobenius-Peron dimension

The Grothendieck ring $K(\mathcal{C})$ of $\mathcal{C}$ is generated by the classes of S_i, denoted by $[S_i]$, subject to the relations $[S_i] \otimes [S_j] = [S_i \otimes S_j] = \sum_k N_{ij}^k [S_k]$, where N_{ij}^k are certainly non-negative integers. Therefore, by Frobenius-Peron theorem (see [2]), there exists a unique character on $K(\mathcal{C})$, i.e. a ring homomorphism $d : K(\mathcal{C}) \to \mathbb{R}$, that take positive values on simple objects. This gives a dimension theory for $\mathcal{C}$ with values in the set of algebraic integers.

A functor F between two tensor categories $\mathcal{C}$ and $\mathcal{D}$ is said to be quasi-tensor if there exists a natural isomorphism $F(-) \otimes F(-) \to F(- \otimes -)$, and is said to be surjective if F is exact and any object of $\mathcal{D}$ is a subquotient (in $\mathcal{D}$) of the image under F of an object of $\mathcal{C}$. Thus a quasi-tensor functor preserves Frobenius-Peron dimension. Using this fact one can show that a quasi-tensor surjective functor maps projective objects to projective objects. In particular, assuming that the Frobenius-Peron dimension of $\mathcal{C}$ is integral, define the regular object $R_\mathcal{C}$ of $\mathcal{C}$ to be

$$R_\mathcal{C} := \bigoplus_i P_i^{\oplus d(S_i)}$$

which is projective. Then if $F : \mathcal{C} \to \mathcal{D}$ is a quasi-tensor surjective functor, the Frobenius-Peron dimension of $\mathcal{D}$ is integral as well and $F(R_\mathcal{C})$ is a multiple of $R_\mathcal{D}$. This is a categorical analog of the Freeness theorem for finite dimension Hopf algebras (a finite dimension Hopf algebra is free over its Hopf subalgebra).

3. Exact module categories

Let $\mathcal{C}$ be a tensor category. A module category over $\mathcal{C}$ is an abelian category $\mathcal{M}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, which is exact, and functorial isomorphisms $(X \otimes Y) \otimes M \cong X \otimes (Y \otimes M)$, $I \otimes M \cong M$ satisfying obvious coherence conditions. In what follows we shall assume that $\mathcal{C}$ is finite and consider only module categories over $\mathcal{C}$ which are finite. Such a module category $\mathcal{M}$ is called exact if for P projective in $\mathcal{C}$, $P \otimes M$ is projective in $\mathcal{M}$ for any $M \in \mathcal{M}$. Consequently in $\mathcal{M}$ projective objects are injective and vice-versa. Thus $\mathcal{M}$ is the category of finitely generated modules over a Frobenius k-algebra. There exists an equivalence relation in the set of simple objects of $\mathcal{M}$, any two such objects M, M are equivalent if there exists $X \in \mathcal{C}$ such that M is a subquotient of $X \otimes N$. One can show that $\mathcal{M}$ decomposes into a direct sum of exact module subcategories $\mathcal{M}_i$ in each of which all simple objects are equivalent. Further, by considering the induced action of the Grothendieck ring of $\mathcal{C}$ on the Grothendieck group of its exact module categories, one concludes that there are finitely many indecomposable exact module categories.

On the other hand, let $\mathcal{M}$ be an exact module category over $\mathcal{C}$ and $\mathcal{N}$ be a not necessarily exact module category over $\mathcal{C}$, then any module functor $\mathcal{M} \to N$ is exact. In fact, this property also characterizes exact module categories.

4. Internal Hom

Let $\mathcal{M}$ be a module category over $\mathcal{C}$ and M_1, M_2 be its objects. One defines the internal hom $\underline{\text{hom}}(M_1, M_2)$ to be the object in $\mathcal{C}$ that represents the functor $\text{Hom}(- \otimes M_1, M_2) : \mathcal{C} \to \text{Vect}_k$. One can check that this defines in fact a bifunctor which is exact if $\mathcal{M}$ is an exact module category. The internal hom is compatible with the tensor product in $\mathcal{C}$ in the usual sence, in particular $\underline{\text{end}}(M)$ is an algebra in $\mathcal{C}$. If $\mathcal{M}$ is an exact module category then the functor $\underline{\text{hom}}(M, -) : \mathcal{M} \to \text{Mod}_{\mathcal{C}}(\underline{\text{end}}(M))$ is an equivalence of category if M is a generator of $\mathcal{C}$.

5. Hom functors

Let $\mathcal{M}_1, \mathcal{M}_2$ be exact module categories over $\mathcal{C}$. Then the category $\text{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is a finite abelian category. Consequently, if $\mathcal{M}$ is indecomposable as module category, the category $\mathcal{C}^*_{\mathcal{M}} := \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a finite tensor category where the tensor product is the composition of functors and the duality is given by taking adjoint functor. Notice that $\mathcal{M}$ is again an exact module category over $\mathcal{C}^*_{\mathcal{M}}$. Further one has an equivalence $\mathcal{C} \to (\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}$ of tensor categories.

On the other hand, for any exact module category $\mathcal{M}_1$ over $\mathcal{C}$, the category $\text{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M})$ is an exact module category over $\mathcal{C}^*_{\mathcal{M}}$. This construction yields a correspondence between exact module categories over $\mathcal{C}$ and over $\mathcal{C}^*_{\mathcal{M}}$. Finally consider $\mathcal{M}$ as a module category over $\mathcal{C} \boxtimes \mathcal{C}^*_{\mathcal{M}}$, we obtain an equivalence between $(\mathcal{C} \boxtimes \mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}}$ and the center of $\mathcal{C}$.

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Classification of Fusion Categories

Michael Cuntz

Let k be an algebraically closed field of characterictic 0. A *fusion category* $\mathcal{C}$ over k is a k-linear semisimple rigid monoidal category with finitely many simple objects, finite dimensional spaces of morphisms and such that $\text{End}(\mathbf{1}) \cong k$.

1. Fusion categories of rank 2 and 3

A basic invariant for the classification of fusion categories is the Grothendieck ring. The following theorem is a consequence of Ocneanu's idea that a fusion category cannot be nontrivially deformed.

Theorem ("Ocneanu rigidity" [3])**.** *For a given based ring K there are only finitely many fusion categories $\mathcal{C}$ with $K(\mathcal{C}) = K$.*

The *rank* of category $\mathcal{C}$ is defined to be the rank of $K(\mathcal{C})$ over $\mathbb{Z}$ (i.e. the number of isomorphism classes of simple objects in $\mathcal{C}$).

It is unknown how many fusion categories (or ribbon categories) of a given rank exist. The fact that there are only finitely many semisimple Hopf algebras with a given finite number of irreducible representations leads to the conjecture (Wang, Ostrik) that there are only finitely many of them.

1.1. **Rank 2.** Let $\mathcal{C}$ be a fusion category of rank 2 and $\mathbf{1}$, X be representatives for the isomorphism classes of simple objects. Then the fusion ring $K(\mathcal{C})$ is completely determined by the number $n \in \mathbb{Z}_{\geq 0}$ for which

$$X \otimes X = \mathbf{1} \oplus nX.$$

We denote this ring by K_n.

The classification of fusion categories $\mathcal{C}$ with $K(\mathcal{C}) = K_0, K_1$ is due to Moore and Seiberg (1989).

Theorem ([5])**.** *There are exactly* 4 *fusion categories of rank* 2*. Two of them have Grothendieck ring* K_0*, the other two* K_1*.*

Remark that this is an open question over fields of positive characteristic.

To prove this theorem (compare with [5]), assume that there exists a fusion category $\mathcal{C}$ with Grothendieck ring K_n, $n > 1$. Using results of [3], it is possible to construct an equivalence from $\mathcal{C}$ to a subcategory of its Drinfeld center, hence $\mathcal{C}$ is braided.

We then look at the braiding morphism $\beta_{X,X}$ and show that $\mathcal{C}$ is ribbon. From the fact that the dimension of X cannot be an integer, we deduce that $\mathcal{C}$ is a modular tensor category which finally leads to a contradiction.

1.2. **Rank 3.** For rank 3, it is only known which fusion categories admit a structure of ribbon category.

Theorem (for example [6])**.** *There are exactly* 7 *fusion categories of rank* 3 *admitting a structure of ribbon category.*

Note that there are fusion categories of rank 3 which do not admit a ribbon structure. Belinschi, Rowell, Stong and Wang have classified all modular tensor categories of rank ≤ 4.

2. Symmetric fusion categories

A ribbon category $\mathcal{C}$ is called symmetric if the square of the braiding is the identity. Symmetric fusion categories have been classified by Deligne:

Theorem ([1])**.** *For any symmetric fusion category* $\mathcal{C}$ *there exists a finite group* G *and an equivalence* $\mathcal{C} \simeq \mathcal{R}ep(G)$*.*

This is not an equivalence of braided categories: there are two braided categories equivalent to $\mathcal{R}ep(\mathbb{Z}/2\mathbb{Z})$ as monoidal categories.

3. Tambara-Yamagami categories

The Tambara-Yamagami categories appear for example in the classification of categories of Frobenius-Perron dimension pq (see section 4). The original motivation for their definition [7], was to distinguish the categories of representations of the dihedral group D_8 and the group of quaternions.

Let A be a finite group, $\chi : A \times A \to k^\times$ a symmetric nondegenerate bicharacter and $\tau \in k$ such that $|A|\tau^2 = 1$. The category $\mathcal{C}(\chi, \tau)$ over k is defined by:

(1) Objects are finite direct sums of elements of $S = A \sqcup \{m\}$.

(2) For $s, s' \in S$, $\mathrm{Hom}(s, s') = \begin{cases} k & \text{if } s = s' \\ 0 & \text{if } s \neq s' \end{cases}$.

(3) Tensor products of elements of S are given by

$$a \otimes b = ab, \quad a \otimes m = m \otimes a = m, \quad m \otimes m = \bigoplus_{a \in A} a$$

where $a, b \in A$.

(4) Certain associativities depending on χ and τ.

Theorem ([7]). *Any split semisimple tensor category with fusion algebra $\mathbb{Z}[S]$ is equivalent to $\mathcal{C}(\chi, \tau)$ for some (χ, τ).*

4. Fusion categories of given Frobenius-Perron dimension

Let $\mathrm{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple objects in $\mathcal{C}$ and let $V \in \mathrm{Irr}(\mathcal{C})$. The *Frobenius-Perron (FP) dimension* of V, $\mathrm{FPdim}(V)$, is the largest positive eigenvalue of the matrix of multiplication by V in $K(\mathcal{C})$. The *Frobenius-Perron dimension* (compare [2]) of the category $\mathcal{C}$ is

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{V \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}(V)^2.$$

Theorem ([2]). *Let $\mathcal{C}$ be a category over $\mathbb{C}$ of FP dimension pq, where $p < q$ are distinct primes. Then either $p = 2$ and $\mathcal{C}$ is a Tambara-Yamagami category of dimension $2q$, or $\mathcal{C}$ is group-theoretical.*

Theorem (Drinfeld-Gelaki-Nikshych-Ostrik, unpublished). *If $\mathcal{C}$ has FP dimension p^n, $p > 2$ prime, then $\mathcal{C}$ is group-theoretical. If $p = 2$ and all objects have integer dimensions, then $\mathcal{C}$ is group-theoretical.*

5. Categories of type A

There are also some results about categories having the same Grothendieck ring as the category of representations of certain Lie groups, for example:

Theorem ([4]). *Any rigid semisimple tensor category whose Grothendieck semiring is equivalent to the one of $\mathcal{R}ep(\mathrm{SU}(N))$ must necessarily be equivalent to the category $\mathcal{R}ep(U_q\mathfrak{sl}_N)$ with q not a root of unity, up to N possible choices of a twist.*

There is a similar theorem from Tuba and Wenzl [8] about categories of orthogonal or symplectic type.

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Exceptional Tensor Categories in Subfactor Theory

Emily Peters

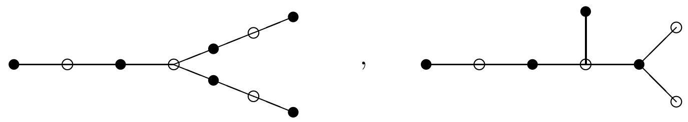
In subfactor theory, some exceptional tensor categories have been constructed by Asaeda, Haagerup and Izumi in [1] and Section 7 of [3]. These examples are not known in other approaches to CFT.

Tensor categories arise from subfactors in the following way: Given a finite-index inclusion of II_1 subfactors $N \subset M$, one of its most important invariants is the *principal graph.* The principal graph is a bipartite graph; it has even vertices $\mathcal{E} = \{$ (isomorphism classes of) irreducible $N-N$ bimodules ${}_NX_N$ which appear in the decomposition of $({}_NL^2(M)_N)^{\otimes n}$ for some $n \in \{0,1,2,\ldots\}\}$, and odd vertices $\mathcal{O} = \{$ (isomorphism classes of) irreducible $N-M$ bimodules ${}_NY_M$ which appear in the decomposition of $({}_NL^2(M)_N)^{\otimes n} \otimes {}_NL^2(M)_M$ for some $n \in \{0,1,2,\ldots\}\}$, with all above tensors being taken over N. The even vertex ${}_NX_N$ connects to the odd vertex ${}_NY_M$ with k edges if $k \cdot {}_NY_M \subset {}_NX_N \otimes {}_NL^2(M)_M$. The bimodules of $\mathcal{E}$ are a tensor category, and the bimodules of $\mathcal{O}$ are a module category over $\mathcal{E}$. Similarly, one can define the *dual principal graph* of a subfactor as the inclusion/reduction graph of irreducible $M-M$ and $M-N$ bimodules contained in tensor powers (over M) of $L^2(M)$; this gives another pair of tensor and module categories. (See [4] for more details.)

If $N \subset M$ is a II_1 subfactor and has finite principal graph Λ, then one can show that the index $[M:N] = \|\Lambda\|^2$ (the norm of a bipartite graph is the operator norm of its adjacency matrix). It is a theorem of Jones ([5]) that if $N \subset M$ is a II_1 subfactor, then $[M:N] \in \{4\cos^2(\pi/n) | n \geq 3\} \cup [4,\infty]$. Haagerup proved in [2] that if additionally, $N \subset M$ is irreducible (i.e., the bimodule ${}_NL^2(M)_M$ is irreducible), and the principal graph of $N \subset M$ is finite, then

$$[M:N] \notin (4, \frac{5+\sqrt{13}}{2} \approx 4.303\ldots).$$

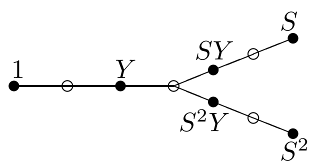
His proof relies on information about the Perron-Frobenius eigenvector of a principal graph, the fact that there are not so many bipartite graphs with index in this interval, and the relation between the dual and principal dual graph. Later, Asaeda and Haagerup proved in [1] that there are exactly two non-isomorphic subfactors of the hyperfinite II_1 factor having index $\frac{5+\sqrt{13}}{2}$; they each have their principal graph as one of



and their dual principal graph the other one. These subfactors are especially interesting because the tensor category they give is not known to come from anywhere else. Unlike most or all previously constructed subfactors, their construction does not start with a group or a quantum group or another known tensor category.

Asaeda and Haagerup's proof has three steps: First, they guess the fusion rules of $N-N$ subfactors by using the symmetries of the first graph. Then they construct a bimodule ${}_NX_M$ satisfying these rules (this is the difficult step); finally, they get a subfactor from ${}_NX_M$ by considering $L_X(N) \subset R_X(M)'$.

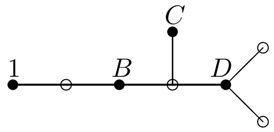
Here are the tensor categories that these graphs are describing. If we label the even vertices of the first graph like so



then the tensor category of even vertices has fusion rules

	1	S	S^2	Y	SY	S^2Y
1	1	S	S^2	Y	SY	S^2Y
S	S	S^2	1	SY	S^2Y	Y
S	1	S	S^2	S^2Y	Y	SY
Y	Y	S^2Y	SY	$Y+SY+S^2Y+1$	$Y+SY+S^2Y+S^2$	$Y+SY+S^2Y+S$
SY	SY	Y	S^2Y	$Y+SY+S^2Y+S$	$Y+SY+S^2Y+1$	$Y+SY+S^2Y+S^2$
S^2Y	S^2Y	SY	Y	$Y+SY+S^2Y+S^2$	$Y+SY+S^2Y+S$	$Y+SY+S^2Y+1$

and if we label the even vertices of the second graph like so



then the tensor category of even vertices has fusion rules

	1	B	C	D
1	1	B	C	D
B	B	$1+B+C+D$	$B+D$	$B+C+2D$
C	C	$B+D$	$1+D$	$B+C+D$
D	D	$B+C+2D$	$B+C+D$	$1+2B+C+2D$

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The Reshetikhin–Turaev Construction – an Example of TQFT

Rafał R. Suszek

Axiomatic Topological Quantum Field Theory (TQFT), as formulated by Atiyah in Ref. [1], renders mathematically rigorous and extends the compass of the earlier constructions, advanced in Refs. [2, 3, 4] where a profound relationship between *physical* TQFT models and the topology of the spaces of their definition was established. In particular, it yields novel topological invariants. The abstract definition of TQFT was explicitly implemented by Reshetikhin and Turaev (RT) in Refs. [5, 6] in the framework of 3d topology. The resulting structure, built on the notion of a modular category, offers useful insights into the nature of the TQFT approach to 2d CFT's and is therefore studied in much detail in the talk, along the lines of Turaev's monograph Ref. [7].

We begin with a brief reminder:

Proposition. *Compact closed smooth oriented $(d-1)$-dim. manifolds together with compact smooth oriented d-dim. cobordisms $(\mathcal{M}, \partial_-\mathcal{M}, \partial_+\mathcal{M})$ considered as morphisms between pairs of such $(d-1)$-dim. manifolds composing their boundary as $\partial\mathcal{M} = \partial_+\mathcal{M} \sqcup \overline{\partial_-\mathcal{M}}$ ($\overline{\Sigma}$ is the orientation conjugate of Σ) form a strict monoidal* cobordism category $(d\text{-Cob}, \sqcup)$, *with composition $\mathcal{M}_1 \cup_f \mathcal{M}_2$ defined by the gluing of the cobordism bases $f : \partial_+\mathcal{M}_2 \longmapsto \partial_-\mathcal{M}_1$ along an orientation-preserving diffeomorphism f and with tensor product given by disjoint union $\sqcup$.*

Our exposition is largely founded on a straightforward extension of the following

Definition. Topological Quantum Field Theory *is a covariant functor*

$$\mathfrak{T} \ : \ (d\text{-Cob}, \sqcup) \longmapsto (\mathcal{V}ect_{fin}(\mathbf{k}), \otimes_{\mathbf{k}})$$

from the topological cobordism category $(d\text{-Cob}, \sqcup)$ *to the algebraic category* $(\mathcal{V}ect_{fin}(\mathbf{k}), \otimes_{\mathbf{k}})$ *of finite-dim. vector spaces over field* $\mathbf{k}$ *of characteristic zero, with* $\mathbf{k}$*-linear operators as morphisms, such that*

- *its object component* $\mathcal{T}$ *is a* modular functor *on the strict monoidal category* $\mathscr{D}iff^{\uparrow}_{d-1}$ *with the same object class as* d-Cob *and with morphisms given by orientation-preserving diffeomorphisms, i.e.* $\mathcal{T}$ *is a covariant functor satisfying the identities:*

$$\mathcal{T}(\Sigma \sqcup \Sigma') = \mathcal{T}(\Sigma) \otimes_{\mathbf{k}} \mathcal{T}(\Sigma')\,, \qquad \forall\, \Sigma, \Sigma' \in \mathrm{Obj}\, \mathscr{D}iff^{\uparrow}_{d-1}\,,$$
$$(f \sqcup g)_\sharp = f_\sharp \otimes_{\mathbf{k}} g_\sharp\,, \qquad \forall\, f, g \in \mathrm{Hom}\, \mathscr{D}iff^{\uparrow}_{d-1}\,,$$

  with $f_\sharp := \mathcal{T}(f)$ *an isomorphism of the* state space $\mathcal{T}(\Sigma)$ *for* $\Sigma = \mathrm{source}(f)$, *and normalised such that* $\mathcal{T}(\emptyset) = \mathbf{k}$*;*
- *its morphism component* τ*, associating the* operator invariant $\tau(\mathcal{M}) \in \mathrm{Hom}_{\mathbf{k}}(\mathcal{T}(\partial_-\mathcal{M}), \mathcal{T}(\partial_+\mathcal{M}))$ *to* $(\mathcal{M}, \partial_-\mathcal{M}, \partial_+\mathcal{M})$*, satisfies the identity:*

$$\tau(\mathcal{M} \sqcup \mathcal{M}') = \tau(\mathcal{M}) \otimes_{\mathbf{k}} \tau(\mathcal{M}')\,, \qquad \forall\, \mathcal{M}, \mathcal{M}' \in \mathrm{Hom}\, d\text{-Cob}\,,$$

  with the normalisation $\tau(\Sigma \times [0,1]) = \mathrm{id}_{\mathcal{T}(\Sigma)}$ *and hence also* $\tau(\emptyset) = \mathbf{1_k}$*,*

and such that the following axioms hold:

1. *(Naturality) Given two cobordisms* $(\mathcal{M}_i, \partial_-\mathcal{M}_i, \partial_+\mathcal{M}_i)\,,\ i = 1, 2$ *and any orientation-preserving diffeomorphism* $f \ : \ \mathcal{M}_1 \longmapsto \mathcal{M}_2$*, we have*
$$\tau(\mathcal{M}_2) \circ (f|_{\partial_-\mathcal{M}_1})_\sharp = (f|_{\partial_+\mathcal{M}_1})_\sharp \circ \tau(\mathcal{M}_1)\,.$$
2. *(Gluing) Given two cobordisms* $(\mathcal{M}_i, \partial_-\mathcal{M}_i, \partial_+\mathcal{M}_i)\,,\ i = 1, 2$ *composable along an orientation-preserving diffeomorphism* $f \ : \ \partial_+\mathcal{M}_2 \longmapsto \partial_-\mathcal{M}_1$*, the functoriality of* $\mathfrak{T}$ *is expressed by the identity:*
$$\tau(\mathcal{M}_1 \cup_f \mathcal{M}_2) = \tau(\mathcal{M}_1) \circ f_\sharp \circ \tau(\mathcal{M}_2)\,.$$

The relevant extension of the above definition, detailed in the talk, consists in replacing the source category d-Cob with the more general category $d\text{-Cob}_{(\mathfrak{B},\mathfrak{A})}$ of $(\mathfrak{B}, \mathfrak{A})$-cobordisms based on a pair $(\mathfrak{B}, \mathfrak{A})$ of concordant involutive space-structures compatible with disjoint union, in the sense of Ref. [7].

The definition of TQFT invites several comments.

Remark. *Inequivalent TQFT's are labeled by isomorphism classes of TQFT functors, where an isomorphism* $\mathfrak{T}_1 \cong \mathfrak{T}_2$ *is understood to be a natural isomorphism between the two functors, commuting with disjoint union, the identifications listed, as well as with the action of homeomorphisms preserving the structure carried by the objects of the cobordism category. We have the result of Ref.* [7]*:*

$$\mathfrak{T}_1 \cong \mathfrak{T}_2 \qquad \Longleftrightarrow \qquad \forall\, \mathcal{M} \in \mathrm{Hom}\, d\text{-Cob}_{(\mathfrak{B},\mathfrak{A})}\,,\ \partial\mathcal{M} = \emptyset \ : \ \tau_1(\mathcal{M}) = \tau_2(\mathcal{M})$$

for a pair $(\mathfrak{T}_1, \mathfrak{T}_2)$ *of TQFT's that are* non-degenerate *in the sense of Turaev.*

Remark. *The Gluing Axiom admits an important generalisation:*

$$\tau(\mathcal{M}_1 \cup_f \mathcal{M}_2) = \alpha(\mathcal{M}_1, \mathcal{M}_2, f)\, \tau(\mathcal{M}_1) \circ f_\sharp \circ \tau(\mathcal{M}_2)$$

which introduces a $\mathbf{k}$*-valued* gluing anomaly $\alpha(\mathcal{M}_1, \mathcal{M}_2, f)$. *The latter yields a projective TQFT functor, reflecting the projectivity of the action of the modular group on* $\mathrm{Obj}\, d\text{-}\mathrm{Cob}_{(\mathfrak{B},\mathfrak{A})}$. *Under certain mild assumptions, the anomaly can be removed at the expense of introducing further structure on* $d\text{-}\mathrm{Cob}_{(\mathfrak{B},\mathfrak{A})}$.

Remark. *By virtue of the Naturality Axiom, a non-anomalous TQFT assigns topological invariants to closed cobordisms.*

In Refs. [5, 6, 7], an explicit example of a 3d TQFT was given that is related to the physical Chern–Simons TQFT of Ref. [3]. The point of departure of the RT construction is the assignment of the topological *RT invariant* to a pair $(\mathcal{M}_{\mathbb{L}}, \Omega)$ consisting of a compact closed smooth oriented d-dim. manifold $\mathcal{M}_{\mathbb{L}}$ obtained by Dehn's surgery on the framed link $\mathbb{L}$ embedded in $\mathbb{S}^3$, and a ribbon graph $\Omega \subset \mathcal{M}_{\mathbb{L}}$ coloured over a modular category $\mathscr{C}$ of rank Δ and dimension $\mathscr{D}$ over field $\mathbf{k}$, with compatible duality and the (finite) set $\{V_i\}_{i\in\mathcal{I}}$ of isoclasses of simple objects V_i, cp Ref. [7]. The assignment rests on the functorial RT realisation F_{RT} in $\mathscr{C}$ of the strict monoidal category of directed $\mathscr{C}$-coloured ribbon graphs in $\mathbb{R}^2 \times [0,1]$ and uses the full modular structure of the latter, cp Turaev's proof of its invariance using a variant of the Kirby–Fenn–Rourke Theorem. The closed formula for the RT invariant reads

$$\bar\tau_{RT}(\mathcal{M}_{\mathbb{L}}, \Omega) = \Delta^{\sigma(\mathbb{L})}\, \mathscr{D}^{-\sigma(\mathbb{L})-m-1} \sum_{\lambda\in\mathrm{col}(\mathbb{L})} \left(\prod_{i=1}^m \underline{\dim}\, V_{\lambda(\mathbb{L}_i)}\right) F_{RT}(\Gamma(\mathbb{L},\lambda)\cup\Omega) \in \mathbf{k}\,,$$

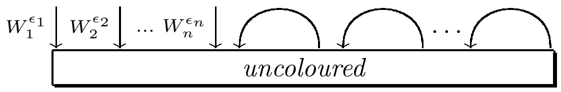
where $\sigma(\mathbb{L})$ is the writhe of $\mathbb{L}$, $\mathrm{col}(\mathbb{L})$ stands for the set of colourings of the connected components $\mathbb{L}_i$, $i = 1, 2, \ldots, m$ of $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \ldots \cup \mathbb{L}_m$ by representatives of isoclasses of simple objects of $\mathscr{C}$, and $\Gamma(\mathbb{L},\lambda)$ denotes the directed $\mathscr{C}$-coloured ribbon link obtained by colouring all $\mathbb{L}_i$ by the respective simple objects $\lambda(\mathbb{L}_i)$.

Remarkably enough, $\bar\tau_{RT}$ extends to a full (projective) TQFT based on

Definition. *Given a modular category* $\mathscr{C}$ *over field* $\mathbf{k}$, *the strict monoidal* $\mathscr{C}$-cobordism category $(3\text{-}\mathrm{Cob}_{\mathscr{C}}, \sqcup)$ *has parameterised* $\mathscr{C}$*-decorated surfaces as objects and* $\mathscr{C}$*-decorated 3d cobordisms as morphisms. Thus, the objects are compact closed smooth oriented 2d manifolds* Σ *of type* $t(\Sigma) = (g, (W_a, \epsilon_a)_{a=1,2,\ldots,n})$, *i.e. of genus* g *and with an ordered sequence of marked points* $P_1 \prec \ldots \prec P_n$, *each coming with a vector tangent to* Σ *and labeled by* $(W_a, \epsilon_a) \in \mathrm{Obj}\,\mathscr{C} \times \{-1, +1\}$, *and with a structure-preserving homeomorphic parameterisation* $\pi_\Sigma : \Sigma_{t(\Sigma)} \longmapsto \Sigma$ *by a* standard $\mathscr{C}$-decorated surface $\Sigma_{t(\Sigma)}$. *The latter is the* $\mathscr{C}$*-decorated* [1] *boundary of*

[1]The decoration is fixed by $(W_a, \epsilon_a)_{a=1,2,\ldots,n}$ and the assignment of a right-directed unital horizontal vector to each free end of an arrow in the diagram. These free ends are attached to $\mathbb{R}^2 \times \{1\} \subset \mathbb{R}^2 \times [0,1]$.

the n-legged genus-g handlebody $H_{t(\Sigma)}$ forming the tubular neighbourhood, in $\mathbb{R}^2 \times [0,1]$, of the partially $\mathscr{C}$-coloured ribbon graph $\Gamma_{t(\Sigma)}$ represented by the diagram:



,

$$W_a^{\epsilon_a} = \begin{cases} W_a & \textit{if} \quad \epsilon_a = 1\,, \\ W_a^* & \textit{if} \quad \epsilon_a = -1\,. \end{cases}$$

The $\mathscr{C}$-decorated cobordisms are compact smooth oriented 3d manifolds, each with an embedded ribbon graph. The graph is coloured over $\mathscr{C}$ and attached to the bases by its free ribbons in a manner determined by the decoration of the bases.

In the last step, we consider the presentation, due to RT, of a $\mathscr{C}$-decorated 3d cobordism $(\mathcal{M}_{\mathbb{L}} \equiv \mathcal{M}, \partial_-\mathcal{M}, \partial_+\mathcal{M})$ with the embedded ribbon graph Ω by the partially $\mathscr{C}$-coloured ribbon graph[2] $\Gamma_{(\mathcal{M}_{\mathbb{L}},\Omega)} = \Gamma^*_{t(\partial_+\mathcal{M})} \circ \Omega \circ \Gamma_{t(\partial_-\mathcal{M})} \cup \mathbb{L}$ determined by the requirement that it yield – upon embedding in $\mathbb{S}^3$, Dehn's surgery on $\mathbb{L}$, and subsequent removal of the tubular neighbourhoods of the base components – the original $\mathscr{C}$-decorated cobordism. We colour $\mathbb{L}$ with $\lambda \in \mathrm{col}(\mathbb{L})$ and the base handles of $\Gamma_{(\mathcal{M}_{\mathbb{L}},\Omega)}$ with sequences of simple objects of $\mathscr{C}$ represented by the multi-labels $\vec{i}^{\pm} \in \mathcal{I}^{g_\pm}$, whereby we get the ribbon graph $\Gamma_{(\mathcal{M}_{\mathbb{L}},\Omega);\lambda;\vec{i}^+;\vec{i}^-}$ with only the base coupons uncoloured. Applying F_{RT} to it and taking the weighted sum over $\mathrm{col}(\mathbb{L})$ as above, we produce an element $F_{RT}(\Gamma_{(\mathcal{M}_{\mathbb{L}},\Omega);\vec{i}^+;\vec{i}^-})^* \in \mathrm{Hom}_{\mathbf{k}}(\mathcal{T}(\partial_-\mathcal{M}), \mathcal{T}(\partial_+\mathcal{M}))$ on dualisation. We thus arrive at the fundamental

Theorem (Reshetikhin & Turaev)**.** *In the hitherto notation, and for*

$$H = \oplus_{i \in \mathcal{I}} V_i \otimes V_i^*,$$

the covariant monoidal functor $\mathfrak{T}_{RT} \;:\; \text{3-Cob}_{\mathscr{C}} \longmapsto \mathcal{V}ect_{fin}(\mathbf{k})$ with the object component (for f a 2d structure-preserving homeomorphism):

$$\begin{cases} \Sigma \longmapsto \mathrm{Hom}_{\mathscr{C}}(\mathbf{1}_{\mathscr{C}}, \otimes_{a=1}^n W_a^{\epsilon_a} \otimes H^{\otimes g})\,, \\ f \longmapsto \mathrm{id}_{\mathcal{T}(source(f))}\,, \end{cases}$$

and with the morphism component (in a self-explanatory matrix notation):

$$((\mathcal{M}_{\mathbb{L}} \equiv \mathcal{M}, \partial_-\mathcal{M}, \partial_+\mathcal{M}), \Omega) \longmapsto [\tau_{RT}(\mathcal{M}_{\mathbb{L}}, \Omega)^{\vec{i}^+}_{\vec{i}^-}]\,, \ \vec{i}^{\pm} \in \mathcal{I}^{g_\pm}\,,$$

$$\tau_{RT}(\mathcal{M}_{\mathbb{L}}, \Omega)^{\vec{i}^+}_{\vec{i}^-} = \mathscr{D}^{1-g_+} \left(\prod_{r=1}^{g_+} \underline{\dim}\, V_{i_r^+} \right) F_{RT}(\Gamma_{(\mathcal{M}_{\mathbb{L}},\Omega);\vec{i}^+;\vec{i}^-})^*$$

defines a TQFT with the gluing anomaly:

$$\alpha(\mathcal{M}_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_2}, f) = (\Delta\, \mathscr{D}^{-1})^{\sigma(\mathbb{L}) - \sigma(\mathbb{L}_1) - \sigma(\mathbb{L}_2)}$$

determined by $\mathcal{M}_{\mathbb{L}} = \mathcal{M}_{\mathbb{L}_1} \cup_f \mathcal{M}_{\mathbb{L}_2}$.

[2]The graph $\Gamma^*_{t(\Sigma)}$ coincides with the upturned $\Gamma_{t(\Sigma)}$ in which all arrows have been reversed.

The RT TQFT admits a straightforward reformulation which removes the gluing anomaly and thus leads to genuine topological invariants of closed cobordisms with embedded $\mathscr{C}$-coloured ribbon links.

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Three-dimensional topological Quantum Field Theories from Subfactors

Chris Schommer-Pries

In 1985 low dimensional topology was revitalized by V. Jones' discovery of the knot polynomial which now bears his name, [4]. This invariant of knots in S^3 is, perhaps, even more remarkable because of its origin in the study of von Neumann algebras and subfactors. Soon after its discovery, E. Witten predicted that the Jones polynomial should generalize to give other invariants of knots and links in 3-manifolds [13]. His predictions were based on physical arguments, where each 3-manifold with boundary is assigned a Hilbert space and these spaces are 'well behaved' when manifolds are glued along their boundaries. The precise sense of 'well behaved' was axiomized by M. Atiyah [1] and the whole structure is now known as a *topological quantum field theory* (TQFT).

In 1991, Reshetikhin and Turaev [10] gave the first rigorous constructions of TQFTs using representations of certain quantum groups at roots of unity. The key structure that their construction required was a braided tensor category (actually the braided tensor category must be ribbon and satisfy a non-degeneracy condition). One of the most important features of a TQFT is that it produces numerical invariants of closed 3-manifolds, and that these invariants can be computed by decomposing the manifold into smaller well known pieces. This can also be reversed: given a framed knot or link in S^3, one can preform surgery on the link to obtain a new 3-manifold, and then evaluate the TQFT on the 3-manifold yielding an invariant of the link. In particular by varying the root of unity, Reshetikhin and Turaev were able to recover the Jones polynomial.

The Reshetikhin-Turaev construction of TQFTs essentially follows this scheme:

(1) Start with a *braided* tensor category.
(2) Use a surgery presentation of a given closed 3-manifold to describe it in terms of braids.
(3) Combine the data from (1) and (2) to construct a number, which is then proven to be independent of all the relevant choices.
(4) Add in the Hilbert spaces to get a full TQFT.

The following year, 1992, Turaev and Viro [12] gave a new construction of TQFTs. The general scheme was identical to the above except in one respect: instead of describing a given 3-manifold in terms of surgery and links, they used a triangulation. Step three was modified too. Given a braided tensor category one extracted certain combinatorial data, known as the *6j-symbols*. Then given a triangulation of a three manifold, step (3) was to label the triangulation by the combinatorial data, extract a number from this, sum over all possible labelings and then prove that the result does not depend on the triangulation. This is known as a state-sum construction of a TQFT. Unfortunately, the resulting TQFTs were not so interesting; they always were just the norm square of the analogous Reshetikhin-Turaev TQFTs.

A year or two later the Turaev-Viro construction was generalized by A. Ocneanu [9] and returned to the realm of subfactors. One of the basic invariants associated to subfactors is a certain (not necessarily braided) tensor category (See [7, 8] for some of the relevant categorical aspects). In fact, it was part of this structure that V. Jones was studying when he discovered the Jones polynomial. Ocneanu's approach started with these tensor categories coming from subfactors and applied the same state-sum recipe employed by Turaev and Viro. He was able to prove that these resulted in TQFTs (and hence 3-manifold invariants) even when the tensor category was not braided.

A detailed exposition of the Ocneanu-Turaev-Viro TQFT construction was given in [2]. My talk is essentially equivalent to the calculation presented there, however I found the calculations more understandable when presented in the language of planar algebras [5]. In this language one merely has to manipulate a number of two-dimensional diagrams via a graphical calculus. Since Ocneanu's work, M. Izumi made several explicit computations of the tensor categories arising from certain exotic and interesting subfactors [3]. This made it possible for N. Sato and M. Wakui to make explicit computations of the corresponding TQFTs and 3-manifold invariants, [11]. Y. Kawahigashi, N. Sato and M. Wakui were later able to express these invariants in terms of the better known surgery presentations of manifolds [6].

A final piece of the story is also discussed in [6]: Given a subfactor, it is possible to construct a new subfactor known as the *asymptotic inclusion*. This new subfactor serves as a sort of 'quantum double' of the original subfactor. In particular the tensor category arising from the new subfactor will be braided. Thus we can compare two TQFTs: We can apply the Ocneanu-Turaev-Viro construction to the original subfactor or we can apply the Reshetikhin-Turaev construction to the new subfactor. These two TQFTs coincide.

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Wess-Zumino-Witten Model: Conformal Blocks

Vincent Graziano

Here we discuss the Wess-Zumino-Witten model. The main purpose of the talk is to give the definition of the conformal blocks in this model. The exposition follows closely the paper of Beauville [1] and the book by Bakalov and Kirillov [2].

Fix the following notation. Denote by $\mathfrak{g}$ a finite-dimensional simple Lie algebra, $\mathfrak{g}((t))$ the loop algebra, $\hat{\mathfrak{g}}$ the affine Lie algebra, and $\hat{\mathfrak{g}}^+ = t\mathfrak{g}[[t]]$. Denote by V_λ^k the Weyl module induced by the irreducible finite dimensional $\mathfrak{g}$-module V_λ with highest weight λ, and level k. In the case $k \in \mathbb{Z}_+$ denote the integrable highest weight representation of weight $\lambda \in P_+^k$ by L_λ^k. Denote by $\mathcal{O}_k$ the category of $\hat{\mathfrak{g}}$-modules of level k which have weight decomposition with finite-dimensional weight subspaces, such that the action of $\hat{\mathfrak{g}}^+$ is locally nilpotent, and the action of $\mathfrak{g}$ is integrable. Let Σ be a compact, connected, nonsingular complex curve. Let $p_1, \dots, p_n$ be marked points on Σ with local parameters $t_1, \dots, t_n$. Denote these data as the triple $(\Sigma, \overrightarrow{p}, \overrightarrow{t})$. We denote by $\mathcal{M}$ the moduli space of such triples.

The WZW model is an example of a conformal field theory, and is based on the category of integrable representations $\mathcal{O}_k^{int}$ of an affine Lie algebra $\hat{\mathfrak{g}}$ at a fixed level $k \in \mathbb{Z}_+$. For any point $(\Sigma, \overrightarrow{p}, \overrightarrow{t})$ in the moduli space associated to each of its marked points p_i a representation V_i of $\hat{\mathfrak{g}}$. The conformal blocks $W(\Sigma, \overrightarrow{p}, \overrightarrow{t}; V_1, \ldots, V_n)$ will be a subspace of $\mathrm{Hom}(\mathbb{C}, V_1 \hat{\otimes} \cdots \hat{\otimes} V_n)$ where $\hat{\otimes}$ is some suitable completion of the tensor product. Rather than define this completion we define the dual space W^*, called the space of coinvariants.

Consider the Lie algebra $\mathfrak{g}(\Sigma - \overrightarrow{p}) \equiv \mathfrak{g} \times \mathcal{O}(\Sigma - \overrightarrow{p})$ of $\mathfrak{g}$-valued functions which are regular outside the points p_i and meromorphic at these points. From this we can define a Lie algebra homomorphism

$$\overrightarrow{\gamma} : \mathfrak{g}(\Sigma - \overrightarrow{p}) \to \mathcal{U}(\hat{\mathfrak{g}})_k \otimes \cdots \otimes \mathcal{U}(\hat{\mathfrak{g}})_k$$

with exactly one copy of the enveloping algebra for each marked point p_i. We can now make the following

Definition. *To the points p_i associate the representations $V_i \in \mathcal{O}_k$. Write $\overrightarrow{V}$ for $V_1 \otimes \cdots \otimes V_n$. The space of* coinvariants *is the vector space*

$$\tau(\Sigma, \overrightarrow{p}, \overrightarrow{V}) \equiv V_{\mathfrak{g}(\Sigma - \overrightarrow{p})} = \overrightarrow{V} / \mathfrak{g}(\Sigma - \overrightarrow{p})\overrightarrow{V}.$$

The space of conformal blocks W is then easily described by the dual space

$$W = \tau(\Sigma, \overrightarrow{p}, \overrightarrow{V})^*.$$

The WZW model has the three important properties.

(1) The spaces of conformal blocks W are finite dimensional.
(2) W is a vector bundle with a projectively flat connection over the moduli space.
(3) Sewing axiom or Factorization property.

To show that the spaces of conformal blocks are finite dimensional we establish first the following.

Lemma. *Let $(\Sigma, \overrightarrow{p}, \overrightarrow{t})$ be a point in $\mathcal{M}$. Let q be a point on the curve Σ distinct from the $\overrightarrow{p}$. Let V_λ be an irreducible $\mathfrak{g}$-module, and V_λ^k the corresponding Weyl module. Associate to the point q the module V_λ^k. Then*

$$(\overrightarrow{V} \otimes V_\lambda)_{\mathfrak{g}(\Sigma - \overrightarrow{p})} \simeq (\overrightarrow{V} \otimes V_\lambda^k)_{\mathfrak{g}(\Sigma - \overrightarrow{p} - q)} = \tau(\Sigma, \overrightarrow{p} \cup q, \overrightarrow{V} \otimes V_\lambda^k),$$

where $\mathfrak{g}(\Sigma - \overrightarrow{p})$ acts on V_λ via the evaluation map at point q. That is, $x \otimes f \mapsto f(q)x$, for $x \in \mathfrak{g}, f \in \mathcal{O}(\Sigma - \overrightarrow{p})$.

Lemma. *Fix $k \in \mathbb{Z}_+$. Let V_i be in $\mathcal{O}_k$. Suppose that at least one of the V_i is integrable. Then*

$$\tau(\Sigma, \overrightarrow{p}, V_1 \otimes \cdots \otimes V_n) = \tau(\Sigma, \overrightarrow{p}, V_1/I_1 \otimes \cdots \otimes V_n/I_n)$$

where I_i is the maximal proper ideal in each V_i.

These two lemmas rely on the Riemann-Roch theorem: we can find a function f on Σ which is regular outside $\overrightarrow{p} \cup q$ that has a simple pole at the point q. See [1] for details.

Corollary. *Let* $L^k_{\overrightarrow{\lambda}} = L^k_{\lambda_1} \otimes \cdots \otimes L^k_{\lambda_n}$. *Then*

$$\tau(\Sigma, \overrightarrow{p}, L^k_{\overrightarrow{\lambda}}) \simeq \tau(\Sigma, \overrightarrow{p} \cup q, L^k_{\overrightarrow{\lambda}} \otimes L^k_0),$$

where L^k_0 *is associated to the point* q, *and denotes the integrable module coming from the trivial representation.*

With these facts in place one can prove the following

Theorem. *Let* $L^k_{\overrightarrow{\lambda}} = L^k_{\lambda_1} \otimes \cdots \otimes L^k_{\lambda_n}$. *Then the spaces of coinvariants*

$$\tau(\Sigma, \overrightarrow{p}, L^k_{\overrightarrow{\lambda}})$$

are finite-dimensional.

As for the other properties, the second is shown using the Sugawara construction. In the case where $\Sigma = \mathbb{P}^1$ the flat connection is described by the Knizhnik-Zamolodchikov equations. The third takes some work to establish. Consider two Riemann surfaces connected by a long narrow tube, then the conformal blocks should be the product of the corresponding blocks associated to each of the surfaces. This construction is carried out by taking the completion of the moduli space $\bar{\mathcal{M}}$ using the Delinge-Mumford compactification. See [8] for the details.

We end the talk with two examples.

Example. *Let* $\Sigma = \mathbb{P}^1$ *and let* $\overrightarrow{p}$ *be* n *distinct points on* Σ. *Let* V_{λ_i} *be* $\mathfrak{g}$*-modules with highest weight* λ_i. *Let* V^k_i *be the Weyl module of* $\hat{\mathfrak{g}}$ *induced by* V_{λ_i}. *Associate to the points* p_i *the Weyl modules* V^k_i. *Then*

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})_{\mathfrak{g}} \simeq \tau(\Sigma, \overrightarrow{p}, V^k_{\lambda_1} \otimes \cdots \otimes V^k_{\lambda_n}).$$

That is to say, in the case where the surface has genus zero the conformal blocks are determined by the highest weights of the finite dimensional $\mathfrak{g}$*-modules. This is not the case for surfaces of other genera.*

Example. *Let* $\Sigma = \mathbb{P}^1$, *and the Lie algebra* $\mathfrak{g}$ *be semisimple. Fix the level* $k \in \mathbb{Z}_+$. *Suppose that the surface is marked with three points* $0, 1,$ *and* ∞. *Associate to these points the integrable representations* $L_{\lambda_p}, L_{\lambda_q}$, *and* L_{λ_r}. *Then*

$$N^r_{p,q} = \dim \tau(\mathbb{P}^1, \overrightarrow{p}, L_{\overrightarrow{\lambda}}),$$

where the $N^r_{p,q}$ *are the multiplicity coefficients for the modular category* $\mathcal{O}^{int}_k$. *See* [7] *for details.*

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The Knizhnik–Zamolodchikov Connection on the Sheaf of Covacua

Marc A. Nieper-Wisskirchen

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra together with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ normalised such that $(\theta,\theta)=2$ for the longest root $\theta\in\mathfrak{h}$. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ together with a decomposition of its set of simple roots into the positive ones and the negative ones. Also fix a non-negative integer l, from now on called the *level*.

The Wess–Zumino–Witten-model for $\mathfrak{g}$ associates then to a marked curve C a certain finite-dimensional vector space V, the *space of covacua*. Here, a marked curve is a smooth compact Riemannian surface with n distinguished marked points $p_1,\ldots,p_n$.

Given a family of such marked curves $\pi\colon X\to S$ over a smooth base S — that is a proper, smooth map of relative dimension 1 of complex manifolds together with non-intersecting sections $s_1,\ldots,s_n$.

In our talk we showed that this family is in fact a vector bundle V of finite rank over S. It carries a natural flat projective connection, i.e. there is a natural flat connection on the projectivisation of V. In the case that the fibres are of genus 0, the connection is induced by the classical Knizhnik–Zamolodchikov connection (and is in fact already defined as a connection on V) [1].

Let us now give precise definitions and a more detailed outline of the talk: Let $\pi\colon X\to S$ be a proper smooth map of relative dimension 1 of complex manifolds. Let $s_1,\ldots,p_n\colon S\to X$ be sections such that the divisors $\Sigma_i:=s_i(S)$ do not intersect each other. Let U be the formal neighborhood of $\Sigma:=\sum_{i=1}^n\Sigma_i$ in X, let $\dot X:=X\setminus\Sigma$, and set $\dot U:=\dot X\cap U$.

We can then make $\underline{\mathfrak{g}}:=\mathfrak{g}\otimes_{\mathbf{C}}\pi_*\mathcal{O}_{\dot U}\oplus\mathcal{O}_S$ a sheaf of Lie algebras over $\mathcal{O}_S$ by defining

$$[(A\otimes f,c),(B\otimes g,d)]:=([A,B]\otimes(fg),(A,B)\ \mathrm{Res}(df\,g)).$$

Let $\lambda\in\mathfrak{h}$ be a dominant weight of $\mathfrak{g}$ (with respect to the chosen set of positive roots). Furthermore assume that λ lies in the so-called *Weyl alcove* defined by

$$(\lambda,\theta)\le l.$$

Then we can define a sheaf of Verma modules $\mathcal{M}_\lambda$ over S to the weight λ with level l in the same way as Verma modules for finite-dimensional semi-simple Lie algebras are defined [5]. In particular, it is generated by $\mathcal{O}_S$. This module has a unique, maximal integrable quotient $\mathcal{L}_\lambda$, which is by construction a quasi-coherent sheaf over S.

By restriction, there is an action of the sheaf $\mathfrak{g} \otimes \pi_* \mathcal{O}_{\dot{X}}$ of Lie algebras on S on $\mathcal{L}_\lambda$. We can then define the sheaf of covacua to be

$$\mathcal{V}_\lambda := \mathcal{L}_\lambda / (\mathfrak{g} \otimes \pi_* \mathcal{O}_{\dot{X}}) \, \mathcal{L}_\lambda.$$

We then showed in our talk that this quasi-coherent sheaf is actually coherent [4].

We continued with constructing a projectively flat connection on $\mathcal{V}_\lambda$. A projectively flat connection on $\mathcal{V}_\lambda$ is nothing else than a projective representation of the tangent sheaf Θ_S on $\mathcal{V}_\lambda$ such that the Leibniz rule holds. From the existence of this connection it follows that $\mathcal{V}_\lambda$ is in fact locally free of finite rank, i.e. the sheaf of sections of a vector bundle [4].

There are two ways to produce this projectively flat connection. One is using a coordinate-free language [5, 3] and actually involves a coordinate-free construction of the Virasoro algebra. The other approach assumes that we are given an isomorphism

$$\pi_* \mathcal{O}_{\dot{U}} = \bigoplus_{i=1}^{n} \mathcal{O}_S[[t_i]].$$

For simplicity, we followed the second approach in our talk.

Our presentation of the connection was based on the Bourbaki talk on [4]. It involves the Sugawara construction, which is a representation of the Virasoro algebra on each integrable $\underline{\mathfrak{g}}$-module $\mathcal{V}_\lambda$. For details, we refer to the Bourbaki talk mentioned.

Let us finish by mentioning that one can measure the failure of the projective connection to be a proper connection. In fact, there is a canonical extension $\mathcal{A}$ of sheaves of Lie algebras of the tangent sheaf Θ_S by the trivial sheaf $\mathcal{O}_S$ such that $\mathcal{A}$ has a canonical representation on $\mathcal{V}_\lambda$ [2, 5].

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The TFT Approach to CFT Correlation Functions

Christian Lehn

In [2] the authors define the notion of a correlation function of a conformal field theory. There a correlation function is an assignment of a multilinear map to a worldsheet satisfying certain compatibility conditions concerning sewings of the worldsheet and Weyl transformations of the metric. In [5] this notion is put in a more abstract setting, which is worked out more explicitly in [4] and [1]. The main idea is to store the conformal structure of the theory in a modular tensor category $\mathcal{C}$, which can be thought of as the representation category of a conformal vertex algebra. After that they define correlation functions for purely topological worldsheets using a topological field theory associated to $\mathcal{C}$ following [6]. Here we take our starting point.
To give the definition of a correlation function (or correlator), we have to introduce some categories and functors between them.

Definition. *An extended surface E is the following data.*

- *A compact, closed, oriented, topological manifold E of dimension 2.*
- *A finite set of marked arcs, i.e. embeddings of an interval $[-\varepsilon,\varepsilon] \hookrightarrow E$. The marked points $p_1,..,p_n$, i.e. the zeros of the arcs, are labelled with pairs $(A_i,\epsilon_i), i = 1,..n$, where $A_i \in Ob\mathcal{C}$ and $\epsilon_i = \pm 1$.*
- *A lagrangian subspace $\lambda \subseteq H_1(E,\mathbb{R})$ w.r.t. the intersection form.*

There are two types of morphisms of extended surfaces: homomorphisms preserving the structure and weighted, oriented cobordisms. The resulting category of extended surfaces $\mathcal{E}$ is a tensor category with disjoint union of surfaces (and obvious extension to the additional structure) as the tensor product. All this is explained in more detail in [6], just like the construction of a tensor functor

$$tft_{\mathcal{C}} : \mathcal{E} \to k - \mathbb{V}ect,$$

where $k - \mathbb{V}ect$ is the category of finite dimensional vector spaces over the ground field k of $\mathcal{C}$, i.e. $k = End_{\mathcal{C}}(\mathbf{1})$ with $\mathbf{1}$ the tensor unit. Throughout the talk $k = \mathbb{C}$ is assumed. In this construction the weight is used to kill the anomaly in the usual construction.
The category $\mathcal{WS}$ of (topological) worldsheets is defined as follows.

Definition. *A (topological) world sheet X is the following data.*

- *A compact, oriented, topological manifold X of dimension 2, possibly with boundary.*
- *An orientation-reversing involution $i : X \to X$, such that $\dot{X} := X/i$ is a manifold.*
- *An orientation-preserving parameterisation $\delta : \partial X \to \mathbb{S}^1 \subseteq \mathbb{C}$, such that for each connected component B of ∂X the restriction $\delta|_B$ is a homeomorphism and compatible with the involution in the sense that*

$$\delta(i(p)) = \overline{\delta(p)} \quad \textit{(complex conjugation)}$$

- *A partition of* $\pi_0(\partial X) = b^{in} \cup b^{out}$ *into "'in-going-boundaries"' and "'out-going-boundaries"' such that* $i_* b^{in} = b^{in}$ *and* $i_* b^{out} = b^{out}$.
- *A section* $s : \dot{X} \to X$ *of the canonical projection* $X \to \dot{X}$.

One should think of $\dot{X}$ as the worldsheet rather than X. The description above can be used to define "'open"', "'closed"' and "'physical"' boundary components and treat them all at once. Morphisms of worldsheets are homeomorphisms compatible with the additional structure and sewings, i.e. the process of sewing an incoming and an outgoing boundary (see [1] for details).
For a fixed $H_{op} \in \mathcal{C}$ and $H_{cl} \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$, the "'spaces of open/closed states"', one constructs an embedding

$$I : \mathcal{WS} \to \mathcal{E}$$

and selects a subspace $Bl(X) \subseteq tft_{\mathcal{C}}(I(X))$. This will constitute the functor of Blocks

$$Bl : \mathcal{WS} \to \mathbb{C} - \mathbb{V}ect,$$

which, together with the trivial functor $One : \mathcal{WS} \to \mathbb{C} - \mathbb{V}ect$, which assigns $\mathbb{C}$ to every object and $id_{\mathbb{C}}$ to every morphism, enables the following

Definition. *A consistent collection of correlators is a monoidal natural transformation*

$$Cor : One \to Bl.$$

Given such a consistent collection of correlators Cor, one can construct a symmetric special Frobenius algebra A_{Cor} in the modular tensor category $\mathcal{C}$ (accomplished in [1]), and conversely given a symmetric special Frobenius algebra A in $\mathcal{C}$, one can construct correlators Cor_A (accomplished in [4]). A lot of the technique is provided in [3]. It is proven in [1] that under rather natural conditions the consistent collection of correlators obtained from an algebra of the form A_{Cor} is equivalent to the original correlator Cor.

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Symmetries and Dualities in Conformal Field Theory

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In the approach to Rational Conformal Field Theory developed in the series of papers ([1]-[6]) the starting point is a set of chiral data as that given by a vertex algebra whose representation category is a modular tensor category $\mathcal{C}$. The basic idea is to use the TFT of Reshetikhin-Turaev type given by $\mathcal{C}$ to establish a correspondence between families of decorated ribbon graphs in 3-d manifolds and correlation functions of the CFT (see [4], sect. 3-6). Worldsheets are assumed to be extended surfaces, which may have boundaries and may not be orientable.

One is furthermore interested in situations in which different parts of a world sheet can be in different phases, i.e. different full CFTs, all with the same underlying rational vertex algebra. Lines along which such phases meet are called defect lines and are supposed to be transparent with respect to the holomorphic and anti-holomorphic components of the stress tensor, i.e. they can be deformed continuously on the world sheet with the field insertion points removed without changing the value of a correlator (see [7], sect. 1).

Each phase of the worldsheet is associated (up to Morita equivalence) to a special symmetric Frobenius algebra in $\mathcal{C}$ (see [1], par. 3). In the case of non-orientable worldsheets the more particular notion of "Jandl" algebra is considered (see [2], sect. 2).

A prominent role is played by the 2-category whose objects are symmetric special (Jandl) Frobenius algebras $\{A,B, \dots\}$ in $\mathcal{C}$, 1-morphisms are A-B-bimodules and 2-morphisms are bimodule morphisms. Each algebra A may be viewed as a bimodule over itself, the corresponding identity 1-morphism. It is easily verified that this 2-category inherits from the initial modular category C the properties of semisimplicity, finiteness and rigidity (see [7], sect. 2).

Boundaries between phases $\mathrm{CFT}(A)$ and $\mathrm{CFT}(B)$ are labelled by A-B-bimodules. The *invisible defect* in phase $\mathrm{CFT}(A)$ is labelled by the algebra A itself, i.e. the identity 1-morphism. *Bulk fields* which transform in the representation $U_i \times U_j$ of the left- and right-moving copies of $\mathcal{V}$ correspond to bimodule morphisms in $\mathrm{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$ (see [4], sect. 3.1 for a precise definition of these spaces). Analogously, *defect fields* which join a defect labelled by an A-B-bimodule X to a defect labelled by another A-B-bimodule Y correspond to elements in $\mathrm{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, Y)$. Boundary conditions associated to, say, phase $\mathrm{CFT}(A)$ and $\mathrm{CFT}(B)$ are labelled by A-modules and B-modules, respectively. A boundary field separating these two boundary conditions and transforming in the U_i representation is labelled by an element in $\mathrm{Hom}_{A|A}(U_i \otimes^+ A, A)$. Defect lines associated to simple bimodules are called "simple" defects.

Information about internal symmetries of the theory is obtained by the study of the Picard groupoid of this 2-category. This consists of isomorphism classes of bimodules of the form ${}_AX_B$ such that ${}_AX_B \otimes_B {}_BX^\vee{}_A \simeq A$ and ${}_BX^\vee{}_A \otimes_A {}_AX_B \simeq B$ (where ${}_BX^\vee{}_A$ is the dual bimodule). In particular, for each symmetric special simple Frobenius algebra A we have the associated Picard group (see [3], def. 2.5) of the tensor category of A-A bimodules. A topological defect is called group-like iff it is labelled by a group-like A-A-bimodule.

A bulk field, passing through a group-like defect line, transforms into a new bulk field (were the defect line not group-like, one would obtain in general a more complicated linear combination of defect fields and defect lines). Thus, one can associate to each group-like defect bimodule a map sending fields into fields. Composition of maps reflects bimodule composition. Non-isomorphic bimodules label distinct defects (see [7], prop. 2.8). In particular, there is a faithful representation of the Picard groupoid acting as a symmetry of the CFT.

By similar reasoning, one shows that Morita equivalent algebras give rise to the same CFT, i.e. equivalent correlation functions (see [7], sect. 3.3).

A B-A-defect Y is called a *duality defect* iff there exists an A-B-defect Y' such that, for every bulk field of $\mathrm{CFT}(A)$, first taking Y past that bulk field and then Y' past the resulting sum over disorder fields, gives a sum over bulk fields of $\mathrm{CFT}(A)$. It is shown (see [7], th. 3.9) that this condition is equivalent to saying that $Y^\vee \otimes_B Y$ is a direct sum of group-like A-A defects. Duality defects are shown to be responsible of high/low temperature dualities (see [7], sect. 3.6; [8]).

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On α-Induction

Marta Asaeda

1. Introduction

This is an expository talk on α-induction. α-induction was first introduced by Longo and Rehren in [6] for a net of subfactors $\mathcal{A} \subset \mathcal{B}$ as a way to extend a localized endomorphism for $\mathcal{A}$ to an endomorphism for $\mathcal{B}$. Xu investigated it further in [11]. Ocneanu introduced chiral projectors of double triangle algebras and showed that they correspond to M-M bimodules of a subfactor $N \subset M$ when N-N, N-M bimodules and $6j$-symbols involving them are given [9]. In [1], Böckenhauer, Evans and Kawahigashi showed that Ocneanu's chiral projectors are essentially the same thing as α-induction and obtained numerous interesting results.

It is known that various operator algebraic phenomena have a lot in common with phenomena occuring in representation theory, vertex operator algebras, and so on. A categorical description yielding a unified view has been developed by Kirillov, Müger, Nikshych, Ostrik, Vainerman as in [8], [5], [7], and [10]. In [10, sec.5] Ostrik gives a categorical description of α-induction. In this talk we will give a survey on [10, sec.5].

2. α-induction in a categorical language

Let $\mathcal{C}$ be a rigid monoidal category, and $\mathcal{M}$ be a module category over $\mathcal{C}$. The definitions of those categories have been given in earlier talks. One may think of $\mathcal{C}$ as a tensor category of N-N bimodules and $\mathcal{M}$ as N-M bimodules for a given subfactor $N \subset M$ if one wishes, as an example. We will indeed mention this example often.

We study $\mathcal{C}^* := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$, a category of module functors, namely a category of functors compatible with $\mathcal{C}$-action on $\mathcal{M}$. $\mathcal{C}^*$ is a monoidal category, with tensor product given by the composition of functors. In subfactor setting we may consider an M-M bimodule V as an object $? \otimes_M V$ in $\mathcal{C}^*$: indeed associativity of bimodule tensor products

$$({}_N X \otimes_N W) \otimes_M V_M \cong_N X \otimes_N (W \otimes_M V_M)$$

shows that $? \otimes_M V$ is a module functor on $\mathcal{M}$, where $X \in \mathcal{C}$, $W \in \mathcal{M}$. Moreover it is known that any $F \in \mathcal{C}^*$ is given by $\otimes_M V$ for some M-M bimodule V.

In a special setting we may construct an object in $\mathcal{C}^*$ out of an object in $\mathcal{C}$. Such is the α-induction. In the following we assume that $\mathcal{C}$ is braided, and that all categories $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{C}^*$ are semisimple, the unit object of $\mathcal{C}$ is irreducible, and that $\mathcal{M}$ is indecomposable. Let $\{\beta_{V,W} : V \otimes W \to W \otimes V\}$ be braidings on $\mathcal{C}$. We define two functors

$$\alpha^{\pm} : \mathcal{C} \to \mathcal{C}^*$$

by $\alpha^{\pm}(X) := X \otimes ?$, with module functor structures

$$c^{\alpha^{\pm}(X)}_{Y,M} : \alpha^{\pm}(X)(Y \otimes M) = X \otimes Y \otimes M \to Y \otimes X \otimes M = Y \otimes \alpha^{\pm}(X)(M)$$

by $c^{\alpha^{\pm}(X)}_{Y,M} = \beta^{\pm 1}_{X,Y} \otimes \mathrm{id}_M$. One may check using hexagon identity (by drawing picture) that this defines module functors, and that they are monoidal functors as well. Note that the braiding $\beta_{X,Y}$ defines an isomorphism of module functors $\alpha^+(X) \circ \alpha^-(Y) \cong \alpha^-(Y) \circ \alpha^+(X)$.

Let $\mathcal{C}^*_{\pm}$ be the monoidal subcategory of $\mathcal{C}$ whose objects are direct summands of $\alpha^{\pm}(X)$'s. Then one may show that the above isomorphism is restricted to a "relative" braiding $F_+ \circ F_- \cong F_- \circ F_+$ for $F_{\pm} \in \mathcal{C}^*_{\pm}$. In particular we obtain a braided category $\mathcal{C}^*_0 := \mathcal{C}^*_+ \cap \mathcal{C}^*_-$.

3. Several results

The following results are proved by Fröhlich, Fuchs, Runkel, and Schweigert in categorical terms, in given references respectively.

3.1. **Modular Invariants.** We assume that $\mathcal{C}$ is also a ribbon category. In such case there is a $SL(2,\mathbb{Z})$ representation on $K_0(\mathcal{C}) \otimes \mathbb{C}$, in particular there are S and T matrices with rows and columns indexed by simple objects $\{X_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{C}$. Let $Z := (Z_{\lambda,\mu})_{\lambda,\mu}$, where $Z_{\lambda,\mu} = \mathrm{dimHom}_{\mathcal{C}^*}(\alpha^+(X_\lambda), \alpha^-(X_\mu))$.

Claim ([1, 3])**.** *The matrix Z commutes with S and T.*

When S is invertible, we say the category $\mathcal{C}$ is modular. Matrices that commute with S and T are called *modular invariants* in such a case.

3.2. **Nondegenerate Case.** Assume further that $\mathcal{C}$ is modular. We have the following results:

Claim ([1, 2])**.** *The category $\mathcal{C}^*$ is generated by $\mathcal{C}^*_{\pm}$, i.e. any object of $\mathcal{C}^*$ is a direct summand of $\alpha^+(X) \circ \alpha^-(Y)$ for some $X, Y \in \mathcal{C}$.*

Claim ([4])**.** *The number of irreducible objects in the category $\mathcal{C}^*$ is* $\mathrm{Tr}(ZZ^t) = \sum_{\lambda,\mu} Z^2_{\lambda,\mu}$. *Moreover the Grothendieck ring $K_0(\mathcal{C}^*)$ is isomorphic to $\oplus_{\lambda,\mu} M_{Z_{\lambda,\mu}}(\mathbb{C})$.*

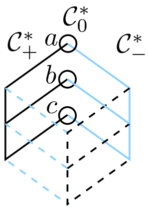
Since $K_0(\mathcal{C})$ is commutative, all its irreducible representations are the characters. It is known that the characters of $K_0(\mathcal{C})$ are given by $\chi_\lambda : [X_\mu] \in K_0(\mathcal{C}) \mapsto S_{\lambda,\mu} \in \mathbb{C}$, where $\lambda, \mu \in \Lambda$, and $S_{\lambda,\mu}$ is an entry of the S-matrix. (Recall that Λ is the index set for the simple objects of $\mathcal{C}$.) Then we have the decomposition $K_0(\mathcal{M}) \otimes \mathbb{C} \cong \oplus_\lambda \mathbb{C}^{n_\lambda}_\lambda$ as $K_0(\mathcal{C}) \otimes \mathbb{C}$-module, where $\mathbb{C}_\lambda$ denotes one-dimensional representation corresponding to χ_λ.

Claim ([3])**.** *The multiplicity n_λ coincides with $Z_{\lambda,\lambda}$.*

4. An example: E_6

Let $\mathcal{C}$ be the category of integrable modules over $\hat{\mathfrak{sl}}_2$ of level $k = 10$. Simple objects of $\mathcal{C}$ are $V_0, V_1, ..., V_{10}$, and $A = V_0 \oplus V_6$ is a rigid commutative $\mathcal{C}$-algebra (see [5] for detail). Let $\mathcal{M} = \mathrm{Rep}A$. Then $\mathcal{M}$ is a module category over $\mathcal{C}$, and itself a monoidal category. The fusion rule of $\mathcal{M}$ is given by the Dynkin diagram E_6. It turns out that the categories $\mathcal{C}^*_{\pm}$ also form E_6 as well. In the following diagram, the vertices on the dark (resp. grey) solid graph are simple objects of $\mathcal{C}^*_+$

(resp. $\mathcal{C}_-^*$), the circled vertices are in $\mathcal{C}_0^*$, and all the vertices, including the ones on broken graphs, compose all the simple objects of $\mathcal{C}^*$.



Modular invariant is given as follows: at the vertices a, b, and c we have indecomposable functors $\alpha^+(V_p) \cong \alpha^-(V_p) \cong \alpha^+(V_q) \cong \alpha^-(V_q)$, where $(p,q) = (0,6),(3,7),(4,10)$ for a, b, c respectively in this order. Thus we have a 11×11 modular invariant matrix $Z = (Z_{i,j})$, where for $i \leq j$, $Z_{i,j} = 1$ if $(i,j) = (0,0),(0,6),(3,3),(3,7),(4,4),(4,10),(6,6),(7,7),(10,10)$, $Z_{i,j} = 0$ otherwise, and for $i > j$, $Z_{i,j} = Z_{j,i}$.

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Classification of Boundary Conformal Field Theories

Dorothea Bahns

The main focus of my talk was a classification result of Longo and Rehren [1, 2] on boundary conformal field theories in the framework of local nets of von Neumann algebras.

Starting point is a net of von Neumann algebras on the real line (called *chiral net*), which is an inclusion preserving map A from the set of bounded open intervals to the set of von Neumann algebras on a fixed Hilbert space $\mathcal{H}_0$ that is local and completely rational (i.e. selfdual, irreducible, strongly additive, split, as well as subject to a certain finiteness condition), and moreover equivariant with respect to an action of the Möbius group.

A local covariant net on the half space $M_+ = \{(t, x) \in \mathbb{R}^2 \mid x > 0\}$ (that is equipped with a quadratic form $\eta = (1, -1)$, the Minkowskian metric), is an inclusion preserving map from double cones $\mathcal{O}$ in M_+ to the set of von Neumann algebras on a fixed Hilbert space $\mathcal{H}$, subject to locality (with respect to causal complements) and Möbius-equivariance. A straightforward prescription allows to construct from a chiral net A an associated local covariant net A_+ (on the half space) of von Neumann algebras on the Hilbert space $\mathcal{H}_0$ of A.

A boundary conformal field theory on the half space M_+ associated to A is then defined to be a local covariant net B_+ (on the half space) of von Neumann algebras on some fixed Hilbert space $\mathcal{H}$ such that there is a representation π of A on $\mathcal{H}$ with an extension to the associated net A_+ such that $\pi(A_+(\mathcal{O})) \subset B_+(\mathcal{O})$ for any double cone $\mathcal{O}$ in M_+ (and some additional technical properties).

By another straightforward construction, such a boundary conformal field theory can then be restricted to yield a (not necessarily local) net on $\mathbb{R}$ (*restriction to the boundary*), a so called chiral extension of A (i.e. a net on $\mathbb{R}$ of von Neumann algebras on $\mathcal{H}$ containing $\pi(A)$ such that the inclusion is relatively local). On the other hand, any chiral extension of A taking values in the set of von Neumann algebras on some fixed Hilbert space $\tilde{\mathcal{H}}$ can be extended to a local boundary conformal field theory, if the extension contains $\tilde{\pi}(A)$ irreducibly (*induction to* M_+).

The theorem of Longo and Rehren then states that, given a chiral net A, there is a bijection between selfdual boundary conformal field theories associated with A and irreducible chiral extensions of A. The theorem's proof relies on techniques of von Neumann subfactor theory and modular theory (Tomita Takesaki Theorem).

As a consequence, one finds that the classification of selfdual boundary conformal field theories is a finite problem, since completely rational theories possess only a finite number of irreducible chiral extensions. For similar reasons also the classification problem of non-selfdual boundary conformal field theories is finite.

Within this framework, the fact that certain inequivalent chiral extensions of a given chiral net A possess the same coupling matrix can be understood as follows. Each of these extensions yields an induced boundary conformal field theory. Employing the technique of Q-systems and studying the algebras of charged intertwiners (see for instance [3, 4]), it can be shown by the split property of A that these boundary conformal field theories are locally isomorphic to *one* local net on $\mathbb{R}^2$ which is obtained from $A \otimes A$ by α-induction.

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