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Mini-Workshop: Geometric Measure Theoretic Approaches to Potentials on Fractals and Manifolds

Organised by
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April 8th – April 14th, 2007

Abstract. The workshop brought together researchers and graduate students from different areas of mathematics, such as analysis, probability theory, geometry, and number theory. The topics of joint interest were motivated by recent problems in potential theory with impacts into these areas:

- discrete approximation to energy minimising measures
- potential theory on fractals and manifolds
- geometric measure theory on fractals
- probabilistic potential theory
- spectral theory on fractals and sets with fractal boundary.

The format of a mini-workshop was especially well-suited for our subject, since it allowed enough time for personal discussions besides the talks given by the participants.

The concept of energy of a charge distribution on a subset of Euclidean space is one of the core subjects of potential theory. Recent generalisations of this concept to hyper-singular energy kernels and discrete N–point distributions exhibit a close connection with ideas from geometric measure theory. A recent article by two of the organisers shows that N–point configurations minimising the discrete energy in the hyper-singular case can be used to characterise the Hausdorff measure on d–dimensional d–rectifiable manifolds embedded in Euclidean space. Such minimal energy point sets can be used for the discretisation of manifolds, which has numerous applications.

On the other hand discretisation by graph structures is a common means for analysis on fractal structures. Usually, a diffusion and an associated Laplace operator are defined by rescaling discrete random walks and their transition operators on the approximating graphs.

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Introduction by the Organisers

The mini-workshop *Geometric Measure Theoretic Approaches to Potentials on Fractals and Manifolds*, organised by Peter Grabner (Graz), Douglas Hardin (Vanderbilt), Edward B. Saff (Vanderbilt), and Martina Zähle (Jena) was held from April 8 to 14, 2007. The meeting had 17 participants from 6 countries. The participants had background from different areas such as fractal geometry, geometric measure theory, stochastic processes, and potential theory. This diversity gave rise to new interactions among the participants. In order to initiate these interactions and to put the focus on the main themes of the workshop, the first two days of the workshop were organised around three introductory lectures:

Edward B. Saff: An overview of discrete minimal energy problems on manifolds
Martina Zähle: Classical potential theory and stochastic processes
Pertti Mattila: Geometric measure theory on fractals.

Potential theory and geometric measure theory have many applications and interactions with various areas of mathematics. The workshop was focussed especially on applications in probability theory, fractal geometry, discrete minimum energy, and harmonic analysis.

Stochastic processes are a classical application of potential theory. Several talks during the workshop were devoted to this area. *Michael Hinz* discussed Dirichlet form techniques for the approximation of jump processes on fractal sets. In particular, he studied the influence of a weight function on the behaviour of the process. *Yimin Xiao* spoke on recent results on the behaviour of α-stable Lévy processes. He discussed the connection between Lévy processes, energy forms, and the corresponding capacities. He suggested using these ideas for the study of fractal properties of more general Markov processes. *Martina Zähle* gave an introductory talk on the interplay between classical potential theory, stochastic processes, and their traces on fractals. She gave special emphasis to Riesz and Bessel potentials, as well as the corresponding function spaces. *Jiaxin Hu* discussed Dirichlet forms on fractals and their domain. On post-critically finite (p. c. f.) fractals rescalings of finite difference operators are used to construct Dirichlet forms. Estimates for the effective resistance in terms of the distance were presented. *Zhenquing Chen* presented a new approach to the definition of reflecting Brownian motion on compact sets with non-smooth boundary. This is based on discrete approximation by random walks on finer and finer grids. It is shown that this definition is equivalent to other less constructive approaches.

Minimisation of discrete and continuous energies is a classical subject of potential theory. Recently, relations to questions originating from geometric measure theory arose, which were one of the motivations for this workshop. *Edward Saff* gave an introductory talk about recent joint work with S. Borodachov and D. Hardin on discrete minimal energy in the hyper-singular case. In contrast to classical potential theory for energy integrals, the discrete energy also exists for Riesz-kernels $\|x-y\|^{-s}$ with $s \geq d$ (d being the dimension of the set). Indeed, it has been proved by D. Hardin and E. Saff that the configurations of minimal energy for such values of the Riesz parameter s are uniformly distributed with respect to normalised d-dimensional Hausdorff measure, as the number of points tends to ∞. *Douglas Hardin* presented the ideas of the proof, which is based on results for d-rectifiable sets, as well as self-similarity considerations. As $s \to \infty$, the minimal energy problem becomes the classical problem of best packing. This relates the determination of the asymptotic main term of minimal energy to packing problems. *Matthew Calef* discussed the limiting case $s = d$, which can be obtained from classical potential theory by taking limits $s \to d-$ of suitably renormalised potentials. In this case the measure of minimal energy equals the normalised Hausdorff measure for compact subsets of $\mathbb{R}^d$ with positive Lebesgue measure. *Johann Brauchart* discussed the support of the equilibrium measure on sets of revolution in $\mathbb{R}^3$ for Riesz-potentials with $0 < s < 1$. In particular, he discussed the question of, when the support is a proper subset of the outer boundary of the surface of revolution. *Abey Lopez* presented asymptotic distribution results for Leja points on the circle. He showed that these points form a uniformly distributed sequence, although their Riesz-s-energy is asymptotically strictly larger than the minimal energy (for $s > 1$).

Geometric measure theory and potential theory provide important techniques in the investigation of properties of fractal sets. *Pertti Mattila* presented methods and results from the geometric measure theory of fractals. Special emphasis was given to projection properties of fractal sets, such as the Hausdorff dimension of the "generic" projection of a set of given dimension. *Daniel Mauldin* discussed constructions of fractal sets based on (possibly infinite) iterated function systems of conformal maps in $\mathbb{R}^n$. He described tools and techniques which have been developed to analyse the properties of the limit set: Hausdorff, packing, Minkowski, or packing dimensions, as well as the quantisation dimension of Gibbs states and equilibrium measures of various potentials.

Classical potential theory Riesz, Bessel, and more general kernels and positive harmonic functions were the subject of three talks. *Volodymyr Andriyevskyy* gave a talk on positive harmonic functions in $\mathbb{C} \setminus E$ depending on geometric properties of the set $E \subset \mathbb{R}$. He discussed necessary and sufficient conditions for the existence of two linearly independent positive harmonic functions. *Natalia Zorii* presented results on the existence of equilibrium measures on non-compact subsets of $\mathbb{R}^n$. *Peter Dragnev* discussed the supports of equilibrium on the sphere under the presence of a Riesz external field. He gave applications of his results to separation of discrete minimal energy point configurations.

Spectral theory on fractal set, its relation to complex dynamics, and corresponding zeta functions were the subject of two talks. *Michel Lapidus* gave an introduction to the theory that he and his coauthors developed of complex dimensions of fractal sets. Relations to tube formulæ of self-similar fractals, Minkowski measurability, and spectral asymptotics on sets with fractal boundary were discussed. *Peter Grabner* presented results on the analytic continuation of the spectral zeta function on certain self-similar fractals and the relation to the classical Poincaré functional equation from complex dynamics.

Mini-Workshop: Geometric Measure Theoretic Approaches to Potentials on Fractals and Manifolds

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Abstracts

Positive Harmonic Functions on Denjoy Domains in the Complex Plane

Volodymyr V. Andriyevskyy

We use the following standard terminology. We denote by Denjoy domain an open subset Ω of the complex plane $\mathbb{C}$ whose complement $E := \overline{\mathbb{C}} \setminus \Omega$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, is a subset of $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, where $\mathbb{R}$ is the real axis. Throughout the lecture we rely on the following assumption. Each point of E (including the point at infinity) is regular for the Dirichlet problem in Ω. Denote by $\mathcal{P}_\infty = \mathcal{P}_\infty(\Omega)$ the cone of positive harmonic functions on Ω which have vanishing boundary values at every point of $E \setminus \{\infty\}$. A classical result by Levin [8], rediscovered independently by Ancona [2] and Benedicks [3], states that either all functions in $\mathcal{P}_\infty$ are proportional or $\mathcal{P}_\infty$ is generated by two linearly independent (minimal) harmonic functions; that is, either dim $\mathcal{P}_\infty = 1$ or dim $\mathcal{P}_\infty = 2$ respectively. In other words, it means that the Martin boundary of Ω has either one or two "infinite" points.

The results in [2] and [3] are proved for positive harmonic functions in domain $\Omega \subset \mathbb{R}^n, n \geq 2$. In this talk we focus on the case $n = 2$ due to its extreme importance in the theory of entire functions, where positive harmonic functions and subharmonic functions in $\mathbb{C}$ which are non-positive on a subset of the real line were the subject of research significantly earlier.

There is a close connection between the dimension of $\mathcal{P}_\infty$ and the behavior of the Green function $g_\Omega(\cdot, z)$ for Ω with pole at $z \in \Omega$.

The problem of finding a geometric description of E such that dim $\mathcal{P}_\infty = 2$ attracted attention of a number of researches (see [1], [3], [7], [10], [6], [11], and [12]).

Theorems 1 and 2 below provide a natural and intrinsic characterization of E with a given dim $\mathcal{P}_\infty$ in terms of the logarithmic capacity cap(S), $S \subset \mathbb{C}$. In these theorems we also connect the dimension of $\mathcal{P}_\infty$ with continuous properties of the Green function g_Ω in a neighborhood of infinity.

Theorem 1. *The following conditions are equivalent:*

(i) There exist points $a_j, b_j \in E$, $-\infty < j < \infty$ *such that*

$$b_{j-1} \leq a_j < b_j \leq a_{j+1}, \quad \lim_{j\to\pm\infty} a_j = \pm\infty,$$

$$\bigcup_{j=-\infty}^{\infty} (a_j, b_j) \supset E^*,$$

$$\inf_{-\infty<j<\infty} \frac{\operatorname{cap}(E \cap [a_j, b_j])}{\operatorname{cap}([a_j, b_j])} > 0,$$

$$\sum_{j=-\infty}^{\infty} \left(\frac{b_j - a_j}{|a_j| + 1}\right)^2 < \infty;$$

(ii) $\dim \mathcal{P}_\infty = 2$*;*
(iii) $\limsup_{\Omega \ni t \to \infty} g_\Omega(t,z)|t| < \infty$ *for any* $z \in \Omega$*.*

For particular results of this kind, see [10, Theorem 2], [11, Theorem 8], [9, Theorem 4], [13] and [4, Theorem 1.11].

Notice that if $(a,\infty) \subset E^*$ or $(-\infty,a) \subset E^*$ for some $a \in \mathbb{R}$, then, by the Benedicks criterion, $\dim \mathcal{P}_\infty = 1$.

Theorem 2. *Let* $E \cap (a,\infty) \neq \emptyset$ *and* $E \cap (-\infty,-a) \neq \emptyset$ *for any* $a > 0$*. The following conditions are equivalent:*
(i) There exist points $\{a_j, b_j\}_{j=-N}^{M}$*, where* $M + N = \infty$*, such that* $a_j, b_j \in E$*,*

$$b_{j-1} \le a_j < b_j \le a_{j+1},$$

$$\sup_j \frac{\operatorname{cap}(E \cap [a_j,b_j])}{\operatorname{cap}([a_j,b_j])} < 1,$$

$$\sum_{j=-N}^{M} \left(\frac{b_j - a_j}{|a_j + b_j| + 1} \right)^2 = \infty;$$

(ii) $\dim \mathcal{P}_\infty = 1$*;*
(iii) $\limsup_{\Omega \ni t \to \infty} g_\Omega(t,z)|t| = \infty$ *for some* $z \in \Omega$*.*

Theorem 1 describes in particular the metric properties of E such that g_Ω has the "highest smoothness" at ∞ (see the recent remarkable result by Carleson and Totik [4, Theorem 1.11] for another description of sets E whose Green's function possesses this property. We do not see any straightforward way to connect the capacity covering condition in Theorem 1 with the capacity condition of Carleson and Totik.

Note that Carroll and Gardiner [5] have independently proved the equivalence (ii)⇔(iii) of Theorem 1. They used a method based on minimal thinness. In our proof of Theorem 1 we discuss the equivalences (i)⇔(ii) and (i)⇔(iii) separately. It makes the proof somewhat longer. However, it presents a new method of investigation of the general metric properties of the Green function on the one side and the Akhiezer-Levin conformal mapping on the other which, in our opinion, is of independent interest.

References

[1] N. I. Akhiezer and B. Ya. Levin, *Generalizations of the Bernstein inequalities for derivatives of entire functions*, in Studies in Modern Problems of the Theory of Functions of a Complex Variable, 125–183. Fizmatgiz, Moskow, 1960 (Russian).

[2] A. Ancona, *Une propriété de la compactification de Martin d'un domaine euclidien*, Ann. Inst. Fourier **29** (1979), 71–90.

[3] M. Benedicks, *Positive harmonic functions vanishing on the boundary of certain domain in* $\mathbb{R}^n$, Ark. Mat. **18** (1980), 53–71.

[4] L. Carleson and V. Totik, *Hölder continuity of Green's functions*, Acta Sci. Math. (Szeged) **70** (2004), 557–608.

[5] T. Carroll and S. Gardiner, *Lipschitz continuity of the Green function in Denjoy domains*, preprint (2006).

[6] S. Gardiner, *Minimal harmonic functions on Denjoy domains*, Proc. Amer. Math. Soc. **107** (1989), 963–975.
[7] P. P. Kargaev, *Existence of the Phragmen-Lindelöf function and some conditions of quasi-analyticity*, Zap Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **126** (1983), 97–108 (Russian); English translation in J. Soviet Math. **27** (1984), 2486–2495.
[8] B. Ya. Levin, *Subharmonic majorants and their applications*, in Proc. All-Union Conference in Function Theory (Kharkov, Ukraine, 1971), 117–120 (Russian).
[9] B. Ya. Levin, V. N. Logvinenko, and M. L. Sodin, *Subharmonic functions of entire degree bounded on subsets of the "real hyperplane"*, Advances in Soviet Methematics **11** (1992), 181–197.
[10] S. Segawa, *Martin boundaries of Denjoy domains*, Proc. Amer. Math. Soc. **103** (1988), 177–183.
[11] S. Segawa, *Martin boundaries of Denjoy domains and quasiconformal mappings*, J. Math. Kyoto Univ. **30** (1990), 297–316.
[12] M. Sodin, *An elementary proof of Benedicks's and Carleson's estimates of harmonic measure of linear sets*, Proc. Amer. Math. Soc. **121** (1994), 1079–1085.
[13] V. Totik, *Metric Properties of Harmonic Measures*, Mem. Amer. Math. Soc. **184** (2006), no 867.

Minimum Riesz s-energy points on compact sets in $\mathbb{R}^3$

Johann S. Brauchart
(joint work with Doug P. Hardin, Edward B. Saff)

Let A be a compact set in the right-half plane $\mathbb{H}^+$ which is identified with the right-half complex plane and $\Gamma(A)$ the set in $\mathbb{R}^3$ obtained by rotating A about the vertical axis. The *discrete energy problem* for Riesz kernels $k_s(\mathbf{x}) := |\mathbf{x}|^{-s}$, $s > 0$, on $\Gamma(A)$ asks to find optimal configurations of N points in the most-stable equilibrium, that is, that minimize the *Riesz s-energy*

$$E_s(X_N) := \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^s}, \qquad s > 0,$$

among all N-point sets $X_N = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ in $\Gamma(A)$. We want to focus on the range $0 < s < 1$. At $s = 1$ the interaction of the points is governed by the Coulomb potential, as s approaches 0 the interaction is of logarithmic character, $k_0(\mathbf{x}) := \log(1/|\mathbf{x}|)$. See [3] for a more detailed discussion of these kind of optimal energy point configurations ($s \geq 0$).

Each X_N also defines a discrete measure $\mu_N := (1/N) \sum_{j=1}^N \delta_{\mathbf{x}_j}$ by placing the point charge $1/N$ at $\mathbf{x}_j$ ($j = 1, \dots, N$). Then $E_s(X_N)$ can be seen as the discrete energy of the counting measure μ_N. We are interested in the limit distribution μ as $N \to \infty$. An essential first step in obtaining this limiting measure is to find its support.

Potential theory yields that the limit distribution μ is given by the equilibrium measure $\mu_{s,\Gamma(A)}$ which uniquely minimizes the s-energy

$$\mathcal{I}[\nu] := \int \int k_s(\mathbf{x} - \mathbf{y}) \mathrm{d}\nu(\mathbf{x}) \mathrm{d}\nu(\mathbf{y})$$

over the class $\mathcal{M}(\Gamma(A))$ of all (Radon) probability measures supported on $\Gamma(A)$. It is assumed that the energy of $\Gamma(A)$,

$$V := V(\Gamma(A)) := \inf \{\mathcal{I}_s[\nu] \mid \nu \in \mathcal{M}(\Gamma(A))\},$$

is always positive. The *equilibrium potential*

$$U_s^{\mu_{s,\Gamma(A)}}(\mathbf{x}) := \int k_s(\mathbf{x}-\mathbf{y}) \mathrm{d}\mu_{s,\Gamma(A)}(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3,$$

satisfies the *variational inequalities* [5, Chap. II, no. 3]

$$U_s^{\mu_{s,\Gamma(A)}} \geq V \qquad \text{quasi-everywhere on } \Gamma(A),$$
$$U_s^{\mu_{s,\Gamma(A)}} \leq V \qquad \text{everywhere on the support } \mathrm{supp}\mu_{s,\Gamma(A)} \text{ of } \mu_{s,\Gamma(A)},$$

which also completely determine the probability measure $\mu_{\Gamma(A)}$. A condition is said to hold *quasi-everywhere* on a Borel set B if it holds everywhere in B except for a subset of infinite energy.

Again, classical potential theory yields that the equilibrium measure on $\Gamma(A)$ is supported on the *outer boundary of* $\Gamma(A)$ which is the boundary of the unbounded component of the complement of $\Gamma(A)$. In the logarithmic case (as $\to 0$) Hardin, Saff, and Stahl [4] showed an even stronger, remarkable, result: *The equilibrium measure is concentrated on the "outer-most" part of* $\Gamma(A)$. The "outer-most" part of a torus $\Gamma(A)$, for example, is the set of revolution generated by rotating the right semi-circle A_+ about the vertical axis. Numerical experiments for a torus for certain $0 < s < 1$ (cf. [6]) show that there are no points on the "inner-most" portion. A natural question arises: *Under what conditions is the support of the equilibrium measure on* $\Gamma(A)$ *a proper subset of the outer boundary of* $\Gamma(A)$*?* We show the following [1, 2]:

- There are compact sets A for which the support of the equilibrium measure on $\Gamma(A)$ is all of $\Gamma(A)$ for every $0 < s < 1$. For example, A is a compact subsets of a horizontal or a vertical line-segment.
- We construct sets of revolution $\Gamma(A)$ such that the support of the equilibrium measure on $\Gamma(A)$ is a proper subset of the outer boundary of $\Gamma(A)$, in contrast to the Coulomb case $s = 1$. We demonstrate this for $0 < s < 1/3$. An example is the circle $\Gamma(\{1/2\})$ on the outer boundary of the "washer" $\Gamma(A)$, where A is the rectangle with lower left corner $1/2 - \mathrm{i}/2$ and upper right corner $1 + \mathrm{i}/2$, which is not in $\mathrm{supp}\mu_{s,\Gamma(A)}$. *We conjecture that one can find for every* $0 < s < 1$ *a compact set* A *for which* $\mathrm{supp}\mu_{s,\Gamma(A)}$ *is a proper subset of the outer boundary of* $\Gamma(A)$.
- For certain sets A the support of the limit distribution on sets of revolution $\Gamma(R + A)$, for the translate $R + A = \{R + z \mid z \in A\}$, tends to the full outer boundary of $\Gamma(A)$ as $R \to \infty$. For example, let the outer boundary of A be a compact subset of a circle with radius r centered at $a > r$ and $0 < s < 1$.
- Related results for the logarithmic case and the case $0 < s < 1$ differ considerably for A being a compact subset of horizontal, vertical line-segments, or circles: (i) Let A be a horizontal line-segment in $\mathbb{H}^+$. Then

supp$\mu_{0,\Gamma(A)}$ is the circle generated by the "outer-most" point of A. However, supp$\mu_{s,\Gamma(A)} = \Gamma(A)$ for all $0 < s < 1$. (ii) The support of the logarithmic equilibrium measure on the translate $\Gamma(R + A)$ of a line-segment A in $\mathbb{H}^+$ with polar angle ϕ to the real axis degenerates to two circles generated by the endpoints of A as $R \to \infty$. The charge of the outer-most circle is $(1 + \cos\phi)/2$. In contrast, the support of the s-equilibrium measure tends to the whole set for every $0 < s < 1$ as $R \to \infty$. The limit measure on A as $R \to \infty$ coincides with the equilibrium measure on A for the kernel $-|z - w|^{1-s}$.

One central idea in proving these results is to take advantage of the rotational symmetry of the problem in $\mathbb{R}^3$. The problem in $\mathbb{R}^3$ for the **singular** kernel k_s is reduced to a problem in $\mathbb{R}^2$ for a related **continuous** kernel $\mathcal{K}_s$ by means of

$$\mathcal{I}_s[\tilde{\mu}] = \int\int \mathcal{K}_s(z,w)\mathrm{d}\mu(z)\mathrm{d}\mu(w) =: \mathcal{J}_{\mathcal{K}_s}[\mu],$$

where the compactly supported *rotational symmetric* measure $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^3)$, admits a decomposition

$$\mathrm{d}\hat{\mu} = \frac{\mathrm{d}\phi}{2\pi}\,\mathrm{d}\mu, \qquad \mu = \hat{\mu} \circ \Gamma \in \mathcal{M}(\mathbb{H}^+),$$

into the normalized Lebesgue measure on the half-open interval $[0, 2\pi)$ and a measure μ on $\mathbb{H}^+$. The kernel $\mathcal{K}_s$ is given by the integral

$$\mathcal{K}_s(z,w) := \frac{1}{2\pi}\int_0^{2\pi} k_s(\mathbf{R}_\phi z - w)\mathrm{d}\phi,$$

where $\mathbf{R}_\phi$ is a rotation by angle ϕ about the axis of revolution. Another central idea is to use convexity of the kernel $\mathcal{K}_s(\gamma(t), w)$ when the outer boundary of A (or its "outer-most" part if $s = 0$) is part of a simple continuous (closed) curve $\gamma : [a, b] \to \mathbb{H}^+$.

References

[1] J.S. Brauchart, D.P. Hardin, and E.B. Saff, *The support of the limit distribution of optimal Riesz energy points on sets of revolutions in* $\mathbb{R}^3$, in preparation.

[2] J.S. Brauchart, D.P. Hardin, and E.B. Saff, *The support of the logarithmic equilibrium measure on sets of revolutions in* $\mathbb{R}^3$*, II*, in preparation.

[3] D.P. Hardin and E.B. Saff, *Discretizing manifolds via minimum energy points*, Notices Am. Math. Soc. **51** (2004), no. 10, 1186–1194.

[4] D.P. Hardin, E.B. Saff, and H. Stahl, *Support of the logarithmic equilibrium measure on sets of revolution in* $\mathbb{R}^3$, J. Math. Phys. **48** (2007), online.

[5] N.S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, New York, 1972, Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[6] R. Womersley, *Visualization of Minimum Energy Points on the Torus*, http://web.maths.unsw.edu.au/~rsw/Torus/, 2005.

A Normalization for the Riesz n-Energy

Matthew Calef

Given a compact set $A \subset \mathbb{R}^n$, let $\omega_N = \{x_1, \ldots, x_N\}$ be a configuration of N distinct points in A. The discrete minimal s-energy of ω_N is

$$E_s(\omega_N) := \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{|x_i - x_j|^s}.$$

The associated minimal energy problem is to find a configuration of N points on A, which we shall denote ω_N^s, such that

$$E_s(\omega_N^s) \leq E_s(\omega_N)$$

for all $\omega_N \subset A$. The compactness of A and a suitable continuous truncation of the kernel $|x-y|^{-s}$ ensure that such an ω_N^s exists for all $s \in (0, \infty)$. The energy of this minimal N-point configuration is denoted

$$\mathcal{E}_s(A, N) := E_s(\omega_N^s).$$

Two questions of interest are, as $N \to \infty$, how do the points in the minimal energy configurations arrange themselves, and how does $\mathcal{E}_s(A, N)$ grow? If s is less than the Hausdorff dimension of A, which we shall denote as d, then this problem has a connection to the following continuous problem:

Let $\mathcal{M}_A$ be the set of all Borel probability measures supported on A. For $s < d$, the continuous s-energy of a measure $\mu \in \mathcal{M}_A$ is defined as

$$I_s(\mu) := \int \int \frac{1}{|x-y|^s} d\mu(y) d\mu(x).$$

The associated minimal energy problem is to find a measure $\mu_s \in \mathcal{M}_A$ such that

$$I_s(\mu_s) \leq I_s(\mu)$$

for all $\mu \in \mathcal{M}_A$. It is known (cf. [1]) that such a μ_s exists and is unique. The above question regarding the growth of the discrete minimal energy is addressed by the fact that

$$\lim_{N \to \infty} \frac{\mathcal{E}_s(A, N)}{N^2} = I_s(\mu_s).$$

Further, by defining a Borel probability measure based on ω_N^s as

$$\gamma_s^{(N)} := \frac{1}{N} \sum_{x \in \omega_N^s} \delta_x,$$

where δ_x is the Dirac-mass measure centered at x, the following convergence in the weak-star sense occurs:

$$\gamma_s^{(N)} \xrightarrow{*} \mu_s.$$

This addresses the above question of how points in minimal energy configurations are distributed as $N \to \infty$.

In the case $s \geq d$ the continuous minimal energy problem is no longer meaningful because $I_s(\mu)$ is not convergent for any $\mu \in \mathcal{M}_A$ (cf. [2]). In response we develop a minimization problem for the case $s = d$ for a certain class of sets. For $A \subset \mathbb{R}^n$ such that $\mathcal{L}^n(A) \in (0, \infty)$, we define a normalized n-energy as

$$\tilde{I}_n(\mu) := \lim_{s \uparrow n} (n-s) \int \int \frac{1}{|x-y|^s} d\mu(y) d\mu(x).$$

With this normalization we are able to prove the following two theorems (cf. [4]):

Theorem 1. *Let $A \subset \mathbb{R}^n$ be compact with $0 < \mathcal{L}^n(A)$, and let λ_n denote the normalized Lebesgue measure restricted to A, i.e. $\lambda_n = \mathcal{L}^n(\cdot \cap A)/\mathcal{L}^n(A)$. Then $\tilde{I}_n(\lambda_n) < \tilde{I}_n(\mu)$ for any Borel measure $\mu \in \mathcal{M}_A$ not equal to λ_n.*

Theorem 2. *Let $A \subset \mathbb{R}^n$ be compact with $0 < \mathcal{L}^n(A)$, and let λ_n denote the normalized Lebesgue measure restricted to A, and let μ_s denote the unique minimizer in $\mathcal{M}_A$ of I_s. Then $\mu_s \xrightarrow{*} \lambda_n$ as s approaches n from below.*

The choice of the normalizing factor $n - s$ and the associated limit were motivated by the following: In the distributional sense, the Fourier transform of a Riesz potential (cf. [3], [1]) is

$$(|x|^{-s})\hat{}(k) = c(s,n)|k|^{n-s}$$

where $c(s,n)$ is a constant depending only on s and n. I_s can be written as (cf. [2])

$$I_s(\mu) = c(s,n) \int_{\mathbb{R}^n} |x|^{s-n} |\hat{\mu}|^2 dx.$$

where $\hat{\mu}$ is the Fourier transform of the measure μ. Finally,

$$\lim_{s \uparrow n} (n-s) c(s,n) = k_n,$$

where k_n is a constant depend only on n.

References

[1] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, New York, NY (1973).
[2] P. Mattila, *Geometry of Sets and Measures in Euclidian Spaces*, Cambridge University Press, Cambridge, UK (1995).
[3] J. Duoandikoetxea *Fourier Analysis*, American Mathematical Society, Providence, RI, (2001)
[4] M. Calef *Continuous and Discrete Problems Involving Riesz Potentials*, In preparation

Discrete Approximations to Reflected Brownian Motion

Zhen-Qing Chen

Let $n \geq 1$ and $D \subset \mathbb{R}^n$ be a domain (connected open set) with compact closure. Consider a reflected Brownian motion (RBM in abbreviation) Y in D. Heuristically, RBM in D is a continuous Markov process Y taking values in $\overline{D}$ that behaves like a Brownian motion in $\mathbb{R}^n$ when $Y_t \in D$ and is instantaneously pushed back along the inward normal direction when $Y_t \in \partial D$. RBM on smooth domains can be constructed in various ways, including the deterministic Skorokhod problem method, martingale problem method, or as a solution to a stochastic differential equation with reflecting boundary conditions (see the Introduction of [3]). When D is non-smooth, all the methods mentioned above cease to work. On non-smooth domains, the most effective way to construct RBM is to use the Dirichlet form method. The RBM constructed using a Dirichlet form coincides with RBM constructed using any other standard method in every smooth domain (see [3], [4] and the references therein).

It is natural to try to construct RBM in a non-smooth domain using a sequence of approximations that can be easily constructed themselves. One such approximating scheme was studied in [1] and [3], where the processes approximating RBM in a non-smooth domain D were RBM's in smooth domains increasing to D. In this talk, we consider processes approximating RBM in D that are defined on the same state space D, or a discrete subspace of D. More specifically, we investigate three discrete or semi-discrete approximation schemes for reflected Brownian motion. The first two approximations involve random walks and we prove that they converge to reflected Brownian motion in a class of bounded non-smooth domains in $\mathbb{R}^n$ that includes all bounded Lipschitz domains and bounded uniform domains (for example, the von Koch snowflake domain). The third scheme is based on "myopic" conditioning and it converges to the reflected Brownian motion in all bounded domains. We now describe these schemes in more detail.

Let D be a bounded domain in $\mathbb{R}^n$ whose boundary ∂D has zero Lebesgue measure. Without loss of generality, we may assume that $0 \in D$. Let D_k be the connected component of $D \cap 2^{-k}\mathbb{Z}^n$ that contains 0 with edge structure inherited from $2^{-k}\mathbb{Z}^n$ (see the next section for a precise definition). We will use $v_k(x)$ to denote the degree of a vertex x in D_k. Let X^k and Y^k be the discrete and continuous time simple random walks on D_k moving at rate 2^{-2k} with stationary initial distribution m_k, respectively, where $m_k(x) = \frac{v_k(x)}{2n} 2^{-kn}$. We show that the laws of both $\{X^k, k \geq 1\}$ and $\{Y^k, k \geq 1\}$ are tight in the Skorokhod space $\mathbb{D}([0,\infty), \mathbb{R}^n)$ of right continuous functions having left limits. We show that if D satisfies an additional condition

$$(1) \qquad C^1(\overline{D}) \text{ is dense in the Sobolev space } (W^{1,2}(D), \|\cdot\|_{1,2}),$$

which is satisfied by all bounded Lipschitz domains and all bounded uniform domains, then both $\{X^k, k \geq 1\}$ and $\{Y^k, k \geq 1\}$ converge weakly to the stationary reflected Brownian motion on D in the Skorokhod space $\mathbb{D}([0,1],\mathbb{R}^n)$. Here

$$W^{1,2} := \left\{ u \in L^2(D,dx) : \nabla u \in L^2(D) \right\} \text{ and}$$

$$\|u\|_{1,2} := \left(\int_D (u(x)^2 + |\nabla u(x)|^2) dx \right)^{1/2}.$$

The last of our main theorems is concerned with "myopic conditioning." We say that a Markov process is conditioned in a myopic way if it is conditioned not to hit the boundary for a very short period of time, say, 2^{-k} units of time, where k is large. At the end of this period of time, we restart the process at its current position and condition it to avoid the boundary for another period of 2^{-k} units of time. We repeat the conditioning step over and over again. Intuition suggests that when 2^{-k} is very small and the process is far from the boundary, the effect of conditioning is negligible. On the other hand, one expects that when the process is very close to the boundary, the effect of conditioning is a strong repulsion from the boundary. These two heuristic remarks suggest that for small 2^{-k}, the effect of myopic conditioning is similar to that of reflection. A more precise description of myopic conditioning of Brownian motion is the following. For every integer $k \geq 1$, let $\{Z^k_{j2^{-k}}, j = 0,1,2,\dots\}$ be a discrete time Markov chain with one-step transition probabilities being the same as those for the Brownian motion in D conditioned not to exit D before time 2^{-k}. The process Z^k_t can be defined for $t \in [(j-1)2^{-k}, j2^{-k}]$ either as the conditional Brownian motion going from $Z^k_{(j-1)2^{-k}}$ to $Z^k_{j2^{-k}}$ without leaving the domain D or as a linear interpolation between $Z^k_{(j-1)2^{-k}}$ and $Z^k_{j2^{-k}}$. We prove that for *any* bounded domain D, the laws of Z^k (defined in either way) converge to that of the reflected Brownian motion on D. The myopic conditioning approximation of reflected Brownian motion is proved for every starting point $x \in D$ so these theorems demonstrate explicitly that the symmetric reflected Brownian motion on D is completely determined by the absorbing Brownian motion in D.

This talk is based on a joint work with Krzysztof Burdzy [2].

References

[1] K. Burdzy and Z.-Q. Chen, *Weak convergence of reflecting Brownian motions.* Elect. Comm. Probab. **3** (1998), 29-33.

[2] K. Burdzy and Z.-Q. Chen, *Discrete approximations to reflected Brownian motion.* To appear in Ann. Probab.

[3] Z.-Q. Chen, *On reflecting diffusion processes and Skorokhod decompositions.* Probab. Theory Relat. Fields, **94** (1993), 281-351.

[4] M. Fukushima (1967), *A construction of reflecting barrier Brownian motions for bounded domains.* Osaka J. Math., **4**, 183-215.

On an energy problem with Riesz external field

Peter D. Dragnev
(joint work with Johann S. Brauchart and Edward B. Saff)

Let $\mathbf{S}^d := \{\mathbf{x} \in \mathbf{R}^{d+1} : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbf{R}^{d+1}$, where $|\cdot|$ denotes the Euclidean norm, and let σ be the unit Lebesgue surface measure on $\mathbf{S}^d$. Given a compact set $E \subseteq \mathbf{R}^{d+1}$, consider the class $\mathcal{M}(E)$ of unit positive Borel measures supported on E. For $0 < s < d$ the *Riesz potential* and *Riesz s-energy* of a measure $\mu \in \mathcal{M}(E)$ are given respectively by

$$U_s^{\mu}(\mathbf{x}) := \int |\mathbf{x} - \mathbf{y}|^{-s} \, d\mu(y), \qquad \mathcal{I}_s(\mu) := \int \int |\mathbf{x} - \mathbf{y}|^{-s} \, d\mu(\mathbf{x}) \, d\mu(\mathbf{y}),$$

The *s-energy* of E is $W_s(E) := \inf\{\mathcal{I}_s(\mu) : \mu \in \mathcal{M}(E)\}$. Its reciprocal gives the *s-capacity of* E, $\operatorname{cap}_s(E) := 1/W_s(E)$ for $s > 0$. A property is said to hold *quasi-everywhere* (q.e.), if the exceptional set has s-capacity zero (see [3, Chapter II] for more details).

Given a non-negative lower semi-continuous *external field* Q we define the *equilibrium measure associated with* $Q(\mathbf{x})$ as the unique minimizer $\mu_Q \in \mathcal{M}(E)$ of the weighted energy

$$V_Q := \inf \left\{ \mathcal{I}_s(\mu) + 2 \int Q(\mathbf{x}) \, d\mu(\mathbf{x}) : \mu \in \mathcal{M}(E) \right\}.$$

The measure μ_Q is characterized by the Gauss variational inequalities

$$U_s^{\mu_Q}(\mathbf{x}) + Q(\mathbf{x}) \geq F_Q \text{ q.e. on E}, \qquad \mathrm{U_s^{\mu_Q}(\mathbf{x}) + Q(\mathbf{x}) \leq F_Q \text{ on } supp(\mu_Q)}, \tag{1}$$

where $F_Q := V_Q - \int Q(x) \, d\mu_Q(x)$. The existence, uniqueness, and characterization related questions in the most general setting can be found in [5]. We remark that logarithmic potentials with external fields are a classical topic in modern function theory (see [4]). In the case when $Q \equiv 0$ and $\operatorname{cap}_s(E) > 0$ the extremal measure is the equilibrium measure of E and is denoted by μ_E.

In [1] *Riesz external fields* $Q_{\mathbf{a},q}(\mathbf{x}) := q|\mathbf{x} - \mathbf{a}|^{-s}$ on $E = \mathbf{S}^d$, $d - 2 < s < d$, were considered, where $q > 0$ and $\mathbf{a}$ is a fixed point on $\mathbf{S}^d$. The goal was to obtain new separation results for minimal s-energy points on the sphere.

In this work we extend the investigation for Riesz external fields $Q = Q_{\mathbf{a},q}$ with $a \notin \mathbf{S}^d$ and develop a technique for finding the equilibrium measure associated with more general external fields. We note that for $d = 2$ and $s = 1$ it is a standard electrostatic problem to find the charge density (signed measure) on a charged, insulated, conducting sphere in the presence of a point charge q placed off the sphere (see [2, Chapter 2]). This motivates us to give the following definition.

Definition 1. Given a compact subset $E \subset \mathbf{R}^{d+1}$ and an external field Q, we call a signed measure η_E supported on E and of total mass $\eta_E(E) = 1$ *a signed equilibrium on E associated with* Q if its weighted potential is constant on E, i.e.

$$U_s^{\eta_E}(\mathbf{x}) + Q(\mathbf{x}) = F_E \qquad \text{for all } \mathbf{x} \in E. \tag{2}$$

The choice of the normalization $\eta_E(E) = 1$ is just for convenience in the applications here. One can show that if a signed equilibrium η_E exists, then it is unique. Our first result establishes existence. We assume that $\mathbf{a}$ lies above the North pole $\mathbf{n} = (\mathbf{0}, 1)$, i.e. $\mathbf{a} = (\mathbf{0}, R)$ and $R > 1$ (the case $R < 1$ is handled by inversion). Observe, that in our notation the last coordinate of a point in $\mathbf{R}^{d+1}$ indicates its altitude.

Theorem 1. *Let $0 < s < d$. The signed equilibrium $\eta_{\mathbf{a}}$ on $\mathbf{S}^d$ in the presence of Riesz external filed $Q_{\mathbf{a},q}(\mathbf{x})$ is given by*

$$d\eta_{\mathbf{a}}(\mathbf{x}) = \left\{ 1 + \frac{qU_s^\sigma(\mathbf{a})}{U_s^\sigma(\mathbf{n})} - \frac{q(R^2-1)^{d-s}}{U_s^\sigma(\mathbf{n})|\mathbf{x}-\mathbf{a}|^{2d-s}} \right\} d\sigma(\mathbf{x}). \tag{3}$$

The next corollary explicitly shows the relationship between q and R, so that μ_Q coincides with the signed equilibrium and has a support the entire sphere. Let $F(a, b; c; x)$ denote the hypergeometric function.

Corollary 1. *Let $0 < s < d$. If*

$$\frac{U_s^\sigma(\mathbf{n})}{q} \geq \frac{(R+1)^{d-s}}{(R-1)^d} - R^{-s} F\left(\frac{s}{2}, \frac{s+d+1}{2}; \frac{d+1}{2}; R^{-2} \right), \tag{4}$$

then $\mu_Q = \eta_a$ and $\operatorname{supp}(\mu_Q) = \mathbf{S}^d$.

Example 1. If $d = 2$, $s = 1$, $q = 1$, then $\mu_Q = \eta_{\mathbf{a}}$ iff $R \geq (3 + \sqrt{5})/2$.

When $\eta_{\mathbf{a}}$ is not a positive measure, we write $\eta_{\mathbf{a}} = \eta_{\mathbf{a}}^+ - \eta_{\mathbf{a}}^-$. Observe, that $\operatorname{supp}(\eta_{\mathbf{a}}^+)$ is a spherical cap centered at the South pole

$$\Sigma_t := \{\mathbf{x} = (\mathbf{y}, u) \in \mathbf{S}^d : -1 \leq u \leq t\},$$

where t is the altitude of the boundary hypercircle. Using (1) and (2), and the principle of domination we derive that $\operatorname{supp}(\mu_Q) \subset \Sigma_t$. Since μ_Q is also an equilibrium measure associated with the external field Q, restricted on the subset Σ_t, we consider the signed equilibrium η_t of Σ_t for varying altitude t. Using M. Riesz approach to s-balayage (see [3, Chapter IV]), we introduce the measures

$$\epsilon_t := \operatorname{Bal}_s(\delta_{\mathbf{a}}, \Sigma_t), \qquad \nu_r := \operatorname{Bal}_s(\sigma, \Sigma_t), \tag{5}$$

where $\delta_{\mathbf{a}}$ is the Dirac-delta measure at $\mathbf{a}$. Our main result is the following.

Theorem 2. *Let $d - 2 < s < d$. The signed equilibrium η_t for the spherical cap Σ_t is given by*

$$\eta_t \quad := \quad \frac{1 + q\|\epsilon_t\|}{\|\nu_t\|} \nu_t - q\epsilon_t. \tag{6}$$

If $t_0 := \max\{t : \eta_t \geq 0\}$, then $\mu_Q = \eta_{t_0}$ and $\operatorname{supp}(\mu_Q) = \Sigma_{t_0}$.

Remark 1. *The restriction on the parameter s is forced by the application of the balayage and the principle of domination. It is a topic of further research to extend the range of s.*

Explicit formulas in terms of q and R, involving hypergeometric functions, are given for the density η_t and the number t_0. Using the superposition representation of the balayage of a measure in terms of the original measure and the balayage of the Dirac-delta measure, we extend the result to external fields that are s-potentials.

References

[1] P. D. Dragnev and E. B. Saff, *Riesz spherical potentials with external fields and minimal energy points separation* , Potential Anal., **26**, (2007), 139-162.

[2] J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1999.

[3] N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1972.

[4] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag, New York, 1997.

[5] N. V. Zorii, *Equilibrium potentials with external fields*, Ukr. Math. Zh., **55**, no. 9 (2003), 1423-1444.

Poincaré functional equations, harmonic measures on Julia sets, and fractal zeta functions

Peter J. Grabner

(joint work with Gregory Derfel and Fritz Vogl)

Let

$$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x$$

be a real polynomial with $a_1 = p'(0) = \lambda > 1$. Then the functional equation

$$f(\lambda z) = p(f(z)), \quad f(0) = 0, \quad f'(0) = 1 \tag{1}$$

has been studied by H. Poincaré [12, 13]. Its solution entire f provides a local linearisation of the polynomial function p around the repelling fixed point $x = 0$, and therefore gives a local normal form for the complex dynamics of p (cf.[2, 11]).

In the case of the super-attracting fixed point ∞ the local normal form is described by the Böttcher function [3], which is the solution of the functional equation

$$g(z)^d = g(p(z)).$$

The function g is holomorphic in some neighbourhood of ∞ and has an analytic continuation to any simply connected subset of

$$\mathcal{F}_\infty(p) = \left\{ z \in \mathbb{C} \mid \lim_{n\to\infty} p^{(n)}(z) = \infty \right\},$$

the Fatou component of ∞. This analytic continuation can be obtained by the integral representation

$$g(z) = \exp\left(\int_{\mathcal{J}(p)} \log(z - x)\, d\mu(x) \right),$$

where $\mathcal{J}(p)$ denotes the Julia set of p and μ is the harmonic measure on $\mathcal{J}(p)$ (cf. [4]).

In [6] we could prove several theorems, which relate the asymptotic behaviour of the Poincaré functions to geometric properties of the Julia or Fatou sets.

Theorem 1. *Let f be the entire solution of the Poincaré equation* (1) *for a real polynomial p with $\lambda = p'(0) > 1$. Assume further that the Fatou component of ∞, $\mathcal{F}_\infty(p)$ contains an angular region $W_{\alpha,\beta}$. Then the following asymptotic expansion for f is valid for all $z \in W_{\alpha,\beta}$ large enough*

$$f(z) = \exp\left(z^\rho F\left(\log_\lambda z\right)\right) + \sum_{n=0}^{\infty} c_n \exp\left(-nz^\rho F\left(\log_\lambda z\right)\right), \tag{2}$$

where F is a periodic function of period 1 *holomorphic in the strip*

$$\left\{z \in \mathbb{C} \mid \frac{\alpha}{\log\lambda} < \Im z < \frac{\beta}{\log\lambda}\right\}$$

and $\rho = \log_\lambda d$. The coefficients c_n are the Laurent series coefficients of the inverse of the Böttcher function g. Furthermore,

$$\forall z \in W_{\alpha,\beta} : \Re z^\rho F(\log_\lambda z) > 0 \tag{3}$$

holds.

Theorem 2. *The periodic function F occurring in the asymptotic expression* (2) *for f is constant, if and only if the polynomial p is either linearly conjugate to z^d or the Chebyshev polynomial of the first kind $T_d(z)$.*

Theorem 3. *Let p be a polynomial of degree $d > 1$ with real Julia set $\mathcal{J}(p)$. Then for any fixed point ξ of p with $\min\mathcal{J}(p) < \xi < \max\mathcal{J}(p)$ we have $|p'(\xi)| \geq d$. Furthermore, $|p'(\min\mathcal{J}(p))| \geq d^2$ and $|p'(\max\mathcal{J}(p))| \geq d^2$. Equality in one of these inequalities implies that p is linearly conjugate to the Chebyshev polynomial T_d of degree d.*

Remark 1. *This theorem can be compared to* [5, Theorem 2] *and* [9, 14]*, where estimates for the derivative of p for connected Julia sets are derived. Furthermore, in* [8] *estimates for $\frac{1}{n}\log|(p^{(n)})'(z)|$ for periodic points of period n are given.*

We could show the existence of a meromorphic continuation of the Dirichlet generating function ("zeta function")

$$\zeta_f(s) = \sum_{\substack{f(-\xi) \\ \xi\neq 0}} \xi^{-s} \tag{4}$$

to the whole complex plane. Furthermore, we could relate the residues at the poles of ζ_f to the residues at the poles of the Mellin transform of the harmonic measure μ. The measure μ could also be related to the distribution of zeros of f, namely let

$$N_f(x) = \sum_{\substack{|\xi|<x \\ f(\xi)=0}} 1.$$

Then we have the following theorem.

Theorem 4. *Let f be the entire solution of* (1). *Then the following are equivalent*

(1) *the limit* $\lim_{x\to\infty} x^{-\rho} N_f(x)$ *does not exist.*
(2) *the limit* $\lim_{t\to 0} t^{-\rho}\mu(B(0,t))$ *does not exist.*

We conjecture that the existence of one of these limits characterises polynomials linearly conjugate to Chebyshev polynomials or pure powers.

Diffusion on fractals has been studied extensively as a generalisation of usual Brownian motion on manifolds. After its introduction in the physics literature (cf. [15]) M. Barlow and E. Perkins [1] gave a very detailed study of the properties of the diffusion on the Sierpiński gasket. Later T. Lindstrøm [10] generalised these results to nested fractals. Especially, he derived results on the distribution of the eigenvalues of the Laplacian associated to the diffusion. Here the Laplacian is seen as the infinitesimal generator of Brownian motion.

In [7] we could relate the spectrum of the Laplacian Δ on self-similar fractals with spectral decimation to the value distribution of a Poincaré function. This yielded the analytic continuation of the spectral zeta function

$$\zeta_\Delta(s) = \sum_{\substack{\Delta u = -\mu u\\ \mu\neq 0}} \mu^{-s}$$

to the whole complex plane. Furthermore, we could give analytic expressions for certain values and residues of the zeta function.

References

[1] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Relat. Fields **79** (1988), 543–623.
[2] A.F. Beardon, *Iteration of rational functions*, Springer, Berlin, New York, 1991.
[3] L. Böttcher, *Beiträge zur Theorie der Iterationsrechnung*, Bull. Kasan Math. Soc. **14** (1905), 176.
[4] H. Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144.
[5] X. Buff, *On the Bieberbach conjecture and holomorphic dynamics*, Proc. Amer. Math. Soc. **131** (2003), 755–759 (electronic).
[6] G. Derfel, P. J. Grabner, and F. Vogl, *Complex asymptotics of Poincaré functions and properties of Julia sets*, submitted, 2007, available at `http://arxiv.org/abs/0704.3952`.
[7] G. Derfel, P. J. Grabner, and F. Vogl, *The Zeta function of the Laplacian on certain fractals*, Trans. Amer. Math. Soc. (2007), to appear, available at `http://arxiv.org/abs/math/0508315`.
[8] A. È. Erëmenko and G. M. Levin, *Estimation of the characteristic exponents of a polynomial (Russian)*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. (1992), no. 58, 30–40 (1993), translation in J. Math. Sci. (New York) **85** (1997), 2164–2171.
[9] G. M. Levin, *On Pommerenke's inequality for the eigenvalues of fixed points*, Colloq. Math. **62** (1991), 167–177.
[10] T. Lindstrøm, *Brownian motion on nested fractals*, Mem. Amer. Math. Soc., vol. 420, Amer. Math. Soc., 1990.
[11] J. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.
[12] H. Poincaré, *Sur une classe étendue de transcendantes uniformes*, C. R. Acad. Sci. Paris **103** (1886), 862–864.
[13] H. Poincaré, *Sur une classe nouvelle de transcendantes uniformes*, J. Math. Pures Appl. IV. Ser. **6** (1890), 316–365.

[14] Ch. Pommerenke, *On conformal mapping and iteration of rational functions*, Complex Variables Theory Appl. **5** (1986), 117–126.
[15] R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters*, J. Physique Lettres **44** (1983), L13–L22.

Discrete Minimal Energy Problems on Rectifiable Sets

Douglas P. Hardin and Edward B. Saff

Let A be an infinite compact set in $\mathbb{R}^{d'}$ whose d-dimensional Hausdorff measure[1], $\mathcal{H}_d(A)$, is finite and positive (hence, d is the Hausdorff dimension of A). For a collection of $N(\geq 2)$ distinct points $\omega_N := \{x_1, \ldots, x_N\} \subset A$, and $s > 0$, the *Riesz s-energy of ω_N* is defined by

$$E_s(\omega_N) := \sum_{1 \leq i \neq j \leq N} \frac{1}{|x_i - x_j|^s} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{|x_i - x_j|^s},$$

while the *N-point Riesz s-energy of A* is defined by

$$\mathcal{E}_s(A, N) := \inf\{E_s(\omega_N) : \omega_N \subset A, |\omega_N| = N\}, \tag{1}$$

where $|X|$ denotes the cardinality of a set X. Since A is compact, there must be at least one N-point configuration $\omega_{s,N} \subset A$ such that $\mathcal{E}_s(A, N) = E_s(\omega_{s,N})$.

This class of minimal discrete s-energy problems can be considered as a bridge between logarithmic energy problems and best-packing ones. Indeed, when $s \to 0$ and N is fixed, the minimal energy problem turns into the problem for the logarithmic potential energy

$$\sum_{1 \leq i \neq j \leq N} \log \frac{1}{|x_i - x_j|},$$

which is minimized over all N-point configurations $\{x_1, \ldots, x_N\} \subset A$.

On the other hand, when $s \to \infty$, and N is fixed, we get the best-packing problem (cf. [5], [3]); i.e., the problem of finding N-point configurations $\omega_N \subset A$ with the largest separation radius:

$$\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} |x_i - x_j|. \tag{2}$$

We are interested in the geometrical properties of optimal s-energy N-point configurations for a set A; that is, sets ω_N for which the infimum in (1) is attained. Indeed, these configurations are useful in statistical sampling, weighted quadrature, and computer-aided geometric design where the selection of a "good" finite (but possibly large) collection of points is required to represent a set or manifold A. Since the exact determination of optimal configurations seems, except in a handful of cases, beyond the realm of possibility, our focus is on the asymptotics

[1] If $d = d'$, then we pick a normalization so that Hausdorff measure and Lebesgue measure agree.

of such configurations. Specifically, we consider the following questions.

(i) What is the asymptotic behavior of the quantity $\mathcal{E}_s(A,N)$ as N gets large?
(ii) How are optimal point configurations $\omega_{s,N}$ distributed as $N \to \infty$?

In the case $s < \dim A$ (the Hausdorff dimension of A), answers to questions (i) and (ii) are determined by the *equilibrium measure* $\lambda_{s,A}$ that minimizes the continuous energy integral

$$I_s(\mu) := \iint\limits_{A\times A} \frac{1}{|x-y|^s}\, d\mu(x)\, d\mu(y)$$

over the class $\mathcal{M}(A)$ of (Radon) probability measures μ supported on A. Specifically (cf. [11, Section II.3.12]), we have

$$\lim_{N\to\infty} \mathcal{E}_s(A,N)/N^2 = I_s(\lambda_{s,A})$$

and (in the weak-star sense)

$$\frac{1}{N}\sum_{x\in\omega_{s,N}} \delta_x \xrightarrow{*} \lambda_{s,A},$$

where δ_x denotes the atomic measure centered at x. In the case when $A = S^d$, the unit sphere in $\mathbb{R}^{d+1}$, the equilibrium measure is simply the normalized surface area measure and it follows that optimal energy points on the sphere are uniformly distributed in this sense.

If $s \geq \dim A$, then $I_s(\mu) = \infty$ for every $\mu \in \mathcal{M}(A)$ and the above potential theoretic methods cannot be applied. However, using the translation and scaling invariance of these Riesz kernels together with measure theoretic techniques, we obtain answers to (i) and (ii) for compact sets in $\mathbb{R}^d$.

Theorem 1 ([6]). *Let A be a compact set in $\mathbb{R}^d$. Then*

$$\lim_{N\to\infty} \frac{\mathcal{E}_d(A,N)}{N^2\log N} = \frac{\beta_d}{\mathcal{H}_d(A)}, \tag{3}$$

where β_d is the volume of the d-dimensional unit ball.

For $s > d$, we have

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}}, \tag{4}$$

where $C_{s,d}$ is a positive constant independent of A.

Furthermore, if $\mathcal{H}_d(A) > 0$ and $s \geq d$, then optimal s-energy configurations $\omega_{s,N}$ for A are asymptotically uniformly distributed with respect to $\mathcal{H}_d$; that is,

$$\frac{1}{N}\sum_{x\in\omega_{s,N}} \delta_x \xrightarrow{*} \frac{\mathcal{H}_d|_A}{\mathcal{H}_d(A)}, \quad N\to\infty. \tag{5}$$

When $0 < \mathcal{H}_d(A) < \infty$ we observe that the minimum energy experiences a transition in order of growth; namely, as s increases from values less than d to values greater than d the energy switches from order N^2 to order $N^{1+s/d}$ as $N \to \infty$.

At the transition value $s = d$, the order of growth is $N^2 \log N$.

To describe our results for d-dimensional sets A embedded in higher dimensional Euclidean spaces we need the following definitions. Following [4] a set $A \subset \mathbb{R}^{d'}$ is called *d-rectifiable* if it is an image of a bounded set from $\mathbb{R}^d$ with respect to a Lipschitz mapping. A set $A \subset \mathbb{R}^{d'}$ is called *$(\mathcal{H}_d, d)$-rectifiable*, if $\mathcal{H}_d(A) < \infty$ and A is a union of at most a countable collection of d-rectifiable sets and a set of $\mathcal{H}_d$-measure zero. Also see [10] for relevant definitions.

Theorem 2 ([1, 2, 6])**.** *Let A be a compact set in $\mathbb{R}^{d'}$ and $s \geq d$. When $s = d$ we assume that A is a subset of a C^1 d-dimensional manifold, while if $s > d$ we merely assume that A is an $(\mathcal{H}_d, d)$-rectifiable set whose d-dimensional Minkowski content exists and equals its d-dimensional Hausdorff measure. Then conclusions* (3) *and* (4) *from Theorem 1 hold. Furthermore, if $\mathcal{H}_d(A) > 0$, then* (5) *from Theorem 1 holds.*

We remark that Theorem 2 applies, for $s > d$, to any d-rectifiable set A.

It is also shown in [1] that there exist Sierpinski-like compact sets A of positive $\mathcal{H}_d$ measure with d integer such that the limit in (4) fails to exist, at least for each s sufficiently large.

The results of Theorem 2 have been extended in [2] to weighted Riesz s-energy of the form

$$E_s^w(\omega_N) := \sum_{1 \leq i \neq j \leq N} \frac{w(x_i, x_j)}{|x_i - x_j|^s},$$

for suitable weight functions $w : A \times A \to [0, \infty)$. In this case, the limiting distribution of weighted s-optimal Riesz configurations is given by the *weighted Hausdorff measure* $\mathcal{H}_d^{s,w}$ defined on Borel sets $B \subset A$ by

$$\mathcal{H}_d^{s,w}(B) := \int_B (w(x,x))^{-d/s} d\mathcal{H}_d(x), \tag{6}$$

suitably normalized.

We remark that the constant $C_{s,d}$ of Theorems 1 and 2 can be represented using the energy for the unit cube in $\mathbb{R}^d$ via formula (4):

$$C_{s,d} = \lim_{N \to \infty} \frac{\mathcal{E}_s([0,1]^d, N)}{N^{1+s/d}}, \quad s > d.$$

For $d = 1$ and $s > 1$, it was shown in [12] that $C_{s,1} = 2\zeta(s)$, where $\zeta(s)$ is the classical Riemann zeta function. However, for other values of d, the constant $C_{s,d}$ is as yet unknown. For the case $d = 2$, it is a consequence of results in [9] that

$$C_{s,2} \leq \left(\sqrt{3}/2\right)^{s/2} \zeta_L(s), \tag{7}$$

where $\zeta_L(s)$ is the zeta function for the planar triangular lattice L consisting of points of the form $m(1,0) + n(1/2, \sqrt{3}/2)$ for $m, n \in \mathbb{Z}$. It is conjectured in [9] that in fact equality holds in (7).

References

[1] S.V. Borodachov, D.P. Hardin, E.B. Saff, *Asymptotics of Best-Packing on Rectifiable Sets*, Proc. Amer. Math. Soc., to appear.
[2] S.V. Borodachov, D.P. Hardin, E.B. Saff, *Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets*, Trans. Amer. Math. Soc., to appear.
[3] J.H. Conway, N.J.A. Sloane, *Sphere Packings, Lattices and Groups,* Springer Verlag, New York: 3rd ed., 1999.
[4] H. Federer, *Geometric measure theory,* Springer-Verlag, Berlin-Heidelberg-New York, 1969, 680 pages.
[5] L. Fejes Toth, *Regular Figures* Pergamon Press, Berlin–Göttingen–Heidelberg, 1953.
[6] D.P. Hardin, E.B. Saff, *Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds*, Adv. Math. **193** (2005), 174–204.
[7] D.P. Hardin, E.B. Saff, *Discretizing manifolds via minimum energy points*, Notices of the AMS. **51** (2004), no. 10, 1186–1194.
[8] A.V. Kolushov, V.A. Yudin, *On the Korkin-Zolotarev construction*, Discrete Mathematics **6** (1994), no. 1, 155–157 (in Russian).
[9] A.B.J. Kuijlaars, E.B. Saff, *Asymptotics for minimal discrete energy on the sphere*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 523–538.
[10] P. Mattila, *Geometry of sets and measures in Eucledian spaces. Fractals and Rectifiability,* Cambridge Univ. Press, 1995, 344 pages.
[11] N.S. Landkof, *Foundations of modern potential theory.* Springer-Verlag, Berlin-Heidelber-New York, 1972, 426 pages.
[12] A. Martinez-Finkelstein, V. Maymeskul, E.A. Rakhmanov, E.B. Saff, *Asymptotics for minimal discrete Riesz energy on curves in* $\mathbb{R}^d$ Canad. Math. J., **56** (2004), 529–552.

Approximation of jump processes on fractals

Michael Hinz

The presented work provides approximations of pure jump processes on fractal subsets of $\mathbb{R}^n$. On d-sets and some generalizations such processes have been studied during the recent years, see e.g. [17], [14], [4] or [5]. For jump processes on $\mathbb{R}^n$ recent results on approximation can be found in [12]. Our method is related to an approach proposed in [11].

We consider two different settings. First we consider a *d-set*, i.e. a compact subset $F \subset \mathbb{R}^n$ which carries a finite Radon measure μ such that $F = supp\, \mu$ and $c_1\, r^d \leq \mu(B(x,r)) \leq c_2\, r^d$ with $c_1, c_2 > 0$ for all $r < r_0$ and $x \in F$ with $0 < d \leq n$. $B(x,r)$ denotes the open ball with radius r and center x. We approximate processes on d-sets by processes whose state spaces are the closed ε-parallel sets of F,

$$F_\varepsilon = \{x \in \mathbb{R}^n : dist(x,F) \leq \varepsilon\} \ ,$$

where $dist(x,F) = \inf_{y\in F} |x-y|$, each F_ε of positive n-dimensional Lebesgue measure. On F_ε we introduce probability measures μ_ε such that for each $\varepsilon > 0$, μ_ε is an n-measure on $\mathbb{R}^n$, equivalent to the restriction of the n-dimensional Lebesgue measure to F_ε. For a function $f \in L_1(F_\varepsilon)$ we have by construction

$$\int f(x)\mu_\varepsilon(dx) = \int (f)_\varepsilon (x)\mu(dx) \ ,$$

where

$$(f)_\varepsilon(x) := \frac{1}{|B(x,2\varepsilon)\cap F_\varepsilon|}\int_{B(x,2\varepsilon)\cap F_\varepsilon} f(y)dy ,$$

$|A|$ stands for the n-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$. Clearly, the measures μ_ε converge weakly to μ on $\mathbb{R}^n$. Now consider the quadratic form given by

$$\mathcal{E}(u,u) = \int\int_{(F\times F)\setminus D} (u(x)-u(y))^2 J(x,y)\mu(dx)\mu(dy) \ , \ u \in L_2(\mu) \ ,$$

where for $x, y \in F$,

$$J(x,y) = \frac{1}{|x-y|^\alpha \mu(B(x,|x-y|))} \ ,$$

with $\alpha \in (0,2)$ and $D = \{(x,x) : x \in F\}$. Set

$$\mathcal{F} := \{u \in L_2(\mu) : \mathcal{E}(u,u) < \infty\} \ ,$$

then $\mathcal{F} = H^{\alpha/2}(F)$, defined as the trace on F of the Bessel potential space $H^{\alpha/2}(\mathbb{R}^n)$, see [13]. $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L_2(\mu)$. Next, define the approximating forms by

$$\mathcal{E}^\varepsilon(w,w) = \int\int_{(F_\varepsilon\times F_\varepsilon)\setminus D} (w(x)-w(y))^2 J_\varepsilon(x,y)\mu_\varepsilon(dx)\mu_\varepsilon(dy)$$

for $w \in L_2(\mu_\varepsilon)$. Here $D = \{(x,x) : x \in F_\varepsilon\}$ and

$$J_\varepsilon(x,y) = \frac{1}{|x-y|^\alpha \mu_\varepsilon(B(x,|x-y|))} .$$

Set $\mathcal{F}_\varepsilon = \{w \in L_2(\mu_\varepsilon) : \mathcal{E}^\varepsilon(w,w) < \infty\}$. Again $\mathcal{F} = H^{\alpha/2}(F_\varepsilon)$ and each $(\mathcal{E}^\varepsilon, \mathcal{F}_\varepsilon)$ is a regular Dirichlet form on $L_2(\mu_\varepsilon)$. By the general theory, cf. [9], there exists a μ_ε-symmetric Hunt processes X^ε on each F_ε and a μ-symmetric Hunt process X on F, uniquely determined by $\mathcal{E}^\varepsilon$ resp. $\mathcal{E}$.

The notions of convergence we make use of were introduced in [15], see this reference for details. In particular, the definitions given there allow to investigate convergence of sequences of functions (operators, forms, etc.) from (on) the $L_2(\mu_\varepsilon)$ to functions (operators, forms, etc.) in (on) $L_2(\mu)$. We prove that the $\mathcal{E}^\varepsilon$ (generalized) Mosco converge to $\mathcal{E}$ as ε tends to zero, the strong convergence of the associated resolvents and semigroups in the sense of Kuwae and Shioya follows, see [15]. As a consequence, the finite dimensional distributions of X^ε with initial distributions μ_ε weakly converge to those of X with initial distribution μ.

The second setting we study is that of a jump process on a self-similar (compact) set $F \subset \mathbb{R}^n$,

$$F = \Psi(F) = \bigcup_{i=1}^N \psi_i(F)$$

with contractive similarities ψ_i, $i = 1,..,N$ all having the contraction ratio $r_1 = ... = r_N = s$ and satisfying the open set condition, see [8]. The associated self-similar probability measure μ is a d-measure making F a d-set. Again the objective is to approximate the process X on F with Dirichlet form $\mathcal{E}$ defined as above. To do so, we introduce certain discrete measures μ_m on F, $m \in \mathbb{N}$, converging weakly to μ as m tends to infinity. We follow a similar strategy as before. For $w \in L_2(\mu_m)$, set

$$\mathcal{E}^m(w,w) = \int\int_{(F\times F)\setminus D} (w(x) - w(y))^2 J_m(x,y)\mu_m(dx)\mu_m(dy).$$

with

$$J_m(x,y) = \frac{1}{|x-y|^\alpha \, \mu_m(B(x,|x-y|))} \ , \quad x,y \in F \ .$$

For $m \in \mathbb{N}$, $(\mathcal{E}^m, L_2(\mu_m))$ is a regular Dirichlet form. It is associated to a continuous time Markov chain Y^m with finite state space $V_m = supp\, \mu_m$. $\mathcal{E}^m$ can be rewritten as

$$\mathcal{E}^m(w,w) = \sum_{x,y\in V_m} (w(x) - w(y))^2 \, C^m_{xy} \ ,$$

the conductivities C^m_{xy} having simple explicit representations. Again we prove the convergence of the spectral structures as m tends to infinity via generalized Mosco convergence, cf. [15], similarly as above the convergence of the finite dimensional distributions follows. Now we additionally obtain the weak convergence of the laws of the approximating Markov chains Y^m to the law of X in the Skorohod space $D_F([0,t_0])$ of right-continuous functions on $[0,t_0]$ with left limits and values in F, considered under initial distributions μ_m and μ, respectively.

References

[1] D. Aldous, *Stopping times and tightness*, Annals of Probability, **6**, 335–340.

[2] R.F. Bass, D.A. Levin, *Transition probabilities for symmetric jump processes*, Trans. AMS **354**, 7 (2002), 2933–2953.

[3] E.A. Carlen, S. Kusuoka, D.W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. IHP **2** (1987), 245–287.

[4] Z.-Q. Chen, T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Proc. and their Appl. 108 (2003), 27–62.

[5] Z.-Q. Chen, T. Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, preprint.

[6] P. Diaconis, L. Saloff-Coste, *Nash inequalities for finite Markov chains*, J. Theoret. Prob. **9** (2) (1996), 459–510.

[7] St. N. Ethier, Th.G. Kurtz, *Markov processes*, Wiley, New York, 1986.

[8] K.J. Falconer, *Fractal Geometry*, Wiley, Chichester, 1990.

[9] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and symmetric Markov processes*, deGruyter, Berlin, New York, 1994.

[10] M. Hinz, *Heat kernel estimates for stable like processes on h-sets*, preprint, Univ. Jena, 2006.

[11] W. Hansen, M. Zähle, *Restricting α-stable Lévy processes from $\mathbb{R}^n$ to fractal sets*, Forum Math. 18 (2006), 171–191.

[12] R. Husseini, M. Kassmann, *Markov chain approximations for symmetric jump processes*, preprint.

[13] A. Jonsson, H. Wallin, *Function spaces on subsets of $\mathbb{R}^n$*, Math. Reports, Vol.2, Academic Publishers, Harwood, 1984.

[14] T. Kumagai, *Some remarks for jump processes on fractals*, In: Grabner, Woess (Eds.), Trends in mathematics: Fractals in Graz 2001, Birkhäuser, Basel.

[15] K. Kuwae, T. Shioya, *Convergence of spectral structures*, Comm. Anal. Geom. **11** (4) (2003), 599–673.
[16] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Analysis 123 (1994), 368–421.
[17] A. Stós, *Symmetric α-stable processes on d-sets*, Bull. Polish Acad. Sci. Math. **48** (2000), 237–245.

Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals

Jiaxin Hu
(joint work with X.S. Wang)

In this paper we consider *post-critically finite* (p.c.f.) self-similar fractals with *strictly recurrent* self-similar Dirichlet forms. We first obtain *effective resistance* estimates in terms of the Euclidean metric, which particularly imply the embedding theorem for the *domains* of the *Dirichlet forms*. We then characterize the domains of the Dirichlet forms. This paper was published in Studia Math. (2006).

The main theorems are the following.

Theorem 1. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ with a strictly recurrent self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ having a weight $\{r_i^{-1}\}_{1=1}^M$. Assume that c_i is the contraction ratio of F_i, that is*

$$|F_i(x) - F_i(y)| \leq c_i|x - y|, \quad \text{for } x, y \in \mathbb{R}^n.$$

Then there exists some $c > 0$ such that, for all $x, y \in K$,

$$c^{-1}|x - y|^{\alpha_1} \leq R(x, y) \tag{1}$$

where $\alpha_1 = \max_{1 \leq i \leq M} \left\{\frac{\ln r_i}{\ln c_i}\right\}$.

In order to obtain the upper bound of R, we propose the following condition.

(C1): There exist a family of numbers $\mathbf{b} = \{b_i\}_{i=1}^M$ with $0 < b_i < 1$ for every i, and a constant $c > 0$ such that, for any $0 < \lambda < 1$,

$$\text{dist}(K_w, K_\tau) \geq c\lambda,$$

if $K_w \cap K_\tau = \emptyset$ for $w, \tau \in \Lambda_{\mathbf{b}}(\lambda)$.

Condition (C1) says that any two disjoint components obtained form any partition $\Lambda_{\mathbf{b}}(\lambda)$ with $0 < \lambda < 1$ is apart away by distance $c\lambda$.

Theorem 2. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ with a strictly recurrent self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ having a weight $\{r_i^{-1}\}_{1=1}^M$. Assume that condition (C1) holds for some $\mathbf{b} = \{b_i\}_{i=1}^M$. Then*

$$R(x, y) \leq c\,|x - y|^{\alpha_2} \tag{2}$$

for all $x, y \in K$, where $\alpha_2 = \min_{1 \leq i \leq M} \left\{\frac{\ln r_i}{\ln b_i}\right\}$ and $c > 0$.

Theorem 3. *Let $(K, \{F_i\}_{i=1}^M)$ be a p.c.f. fractal in $\mathbb{R}^n$ with a strictly recurrent self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ having a weight $\{r_i^{-1}\}_{1=1}^M$. Let μ be a self-similar measure with the weight $\{r_i^\alpha\}_{1=1}^M$ where $\sum_{i=1}^M r_i^\alpha = 1$. Then there exist some $c, c_0 > 0$ such that*

$$c^{-1} W_\alpha(f) \leq \mathcal{E}(f) \leq c\, W_\alpha(f) \tag{3}$$

for all $f \in \mathcal{F}$, where

$$W_\alpha(f) := \sup_{0<\lambda<1} \lambda^{-(2\alpha+1)} \int_K \int_{B_R(x, c_0\lambda)} |f(x) - f(y)|^2 d\mu(y) d\mu(x).$$

In particular, we have that $\mathcal{F} = \{f \in C(K) : W_\alpha(f) < \infty\}$.

References

[1] M.T. Barlow, *Diffusions on fractals*, Lect. Notes Math. **1690**, Springer, 1998, 1-121.
[2] K.J. Falconer, *Fractal Geometry- Mathematical Foundation and Applications*, John Wiley, 1990.
[3] A. Grigor'yan, J. Hu and K.-S. Lau, *Heat kernels on metric-measure spaces and an application to semilinear elliptic equations*, Trans. Amer. Math. Soc. **355** (2003), 2065–2095.
[4] J. E. Hutchinson, *Fractals and self-similarity, Indiana Univ.* Math. J. **30** (1981), 713–747.
[5] A. Jonsson, *Brownian motion on fractals and function spaces*, Math. Zeit. **122** (1996), 495-504.
[6] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.
[7] T. Lindstrøm, *Brownian motion on nested fratals*, Mem. Amer. Math. Soc. **420** (1990).
[8] R. Strichartz, *Differential equations on fractals: A tutorial*, Princeton University Press, 2006.

Complex Fractal Dimensions, Tube Formulas and Zeta Functions

Michel L. Lapidus

We discuss some elements of the theory of fractal complex dimensions developed by the author and his collaborators over the last few years, as presented in the recent research monograph with Machiel van Frankenhuijsen, "Fractal Geometry, Complex Dimensions and Zeta Functions: geometry and spectra of fractal strings" [5]. We will also briefly talk about the higher-dimensional theory (joint with Erin Pearse) in which we obtain an analogue of the Stein-Weyl-Federer tube formula for self-similar tilings naturally associated to self-similar fractals and define a corresponding 'geometric zeta function'. (See [4], building upon an earlier joint work on a tube formula for the Koch snowflake curve and published in 2006 in the J. London Math. Soc. [3])

If time allows, we may mention some extensions of the theory to multifractals [6] and to the p-adic realm (jointly with Hung Lu).

Finally, we note that some aspects of the theory of complex fractal dimensions are used or pursued in a forthcoming book by the author, entitled "In Search of the Riemann Zeros: strings, fractal membranes and noncommutative spacetimes" [2].

For more recent work see the author's homepage [1].

References

[1] M. L. Lapidus. homepage. http://math.ucr.edu/~lapidus.
[2] M. L. Lapidus. *In Search of the Riemann Zeros: strings, fractal membranes and noncommutative spacetimes.* Amer. Math. Soc., Providence, R. I., Oct. 2007 (in press).
[3] M. L. Lapidus and E. P. J. Pearse. *A tube formula for the Koch snowflake curve, with applications to complex dimensions.* London Math. Soc. (2), **74** (2006) 397–414.
[4] M. L. Lapidus and E. P. J. Pearse. *Tube formulas and complex dimensions of self-similar tilings.* available at http://arxiv.org/abs/math/0605527, 2007.
[5] M. L. Lapidus and M. van Frankenhuijsen. *Fractal geometry, complex dimensions and zeta functions, Geometry and spectra of fractal strings.* Springer Monographs in Mathematics. Springer, New York, 2006.
[6] M. L. Lapidus, J. L. Vehel, and J. A. Rock. *Fractal strings and multifractal zeta functions.* available at http://arxiv.org/abs/math-ph/0610015, 2007.

Riesz Energy and Distribution of Leja Sequences

Abey López García

(joint work with Alexander I. Aptekarev and Edward B. Saff)

Given a point set $\omega_N = \{x_1, \dots, x_N\}$ of N distinct points in $\mathbb{R}^{d'}$, and $s \in [0, \infty)$, the **discrete Riesz s-energy** of ω_N is defined as

$$E_s(\omega_N) := \sum_{1 \le i \ne j \le N} K(|x_i - x_j|; s),$$

where $|\cdot|$ denotes the Euclidean norm and

$$K(t; s) = \begin{cases} t^{-s}, & \text{if} \quad s > 0, \\ -\log(t), & \text{if} \quad s = 0 \end{cases}$$

is the Riesz kernel. Let $A \subset \mathbb{R}^{d'}$ be an infinite compact set. We say that $\omega^*_{N,s} \subset A$ is an **optimal N-point configuration** on A if

$$E_s(\omega^*_{N,s}) = \inf_{\omega_N \subset A} E_s(\omega_N).$$

When $s <\dim_{\mathcal{H}}(A)$ (where $\dim_{\mathcal{H}}(A)$ denotes the Hausdorff dimension of A, which will be denoted as d throughout the rest of this report), there is a unique probability measure $\lambda_{A,s}$ supported on A such that

$$\int\int K(|x-y|; s) d\lambda_{A,s}(x) d\lambda_{A,s}(y) = \inf_{\mu} \int\int K(|x-y|; s) d\mu(x) d\mu(y),$$

where the infimum is taken over the class of all probability measures on A. The double integral $\int\int K(|x-y|; s) d\lambda_{A,s}(x) d\lambda_{A,s}(y)$ is the **continuous Riesz s-energy** of the equilibrium measure $\lambda_{A,s}$ and will be represented as $I_s(\lambda_{A,s})$.

The asymptotic behavior of $E_s(\omega^*_{N,s})$ and asymptotic distribution of $\omega^*_{N,s}$ for optimal configurations $\omega^*_{N,s}$ has been studied extensively in recent years (see, for example, [6], [3], [1], and the survey article [2]).

Using standard potential theoretic ideas it is shown in [4] that, for $s < d$,

$$\lim_{N\to\infty} \frac{E_s(\omega^*_{N,s})}{N^2} = I_s(\lambda_{A,s}), \tag{1}$$

$$\frac{1}{N} \sum_{x\in\omega^*_{N,s}} \delta_x \longrightarrow \lambda_{A,s}, \tag{2}$$

where (2) is understood in the weak-star sense. In [1] Hardin and Saff employed geometric measure theoretic tools to obtain, for a class of rectifiable sets A, that

$$\lim_{N\to\infty} \frac{E_d(\omega^*_{N,d})}{N^2\log(N)} = \frac{\mathcal{H}_d(B^d)}{\mathcal{H}_d(A)}, \tag{3}$$

$$\lim_{N\to\infty} \frac{E_s(\omega^*_{N,s})}{N^{1+\frac{s}{d}}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{\frac{s}{d}}}, \quad \text{for} \quad s > d, \tag{4}$$

where $\mathcal{H}_d$ represents d-dimensional Hausdorff measure (normalized by the condition $\mathcal{H}_d([0,1]^d) = 1$), B^d is the unit ball in $\mathbb{R}^d$, and $C_{s,d}$ is some constant independent of A. This constant equals $2\zeta(s)$ when $d = 1$, where $\zeta(s)$ is the classical Riemann zeta function. In [1] it is also shown that if $\mathcal{H}_d(A) > 0$, then

$$\frac{1}{N} \sum_{x\in\omega^*_{N,s}} \delta_x \longrightarrow \frac{\mathcal{H}_d(\cdot)}{\mathcal{H}_d(A)}, \quad s \geq d. \tag{5}$$

Given $s \in [0,\infty)$ and a sequence of point sets $\omega_N \subset A$, we say that $\{\omega_N\}_N$ is **asymptotically s-energy minimizing** on A ($\{\omega_N\}_N \in AEM(A;s)$) if it satisfies, with $\omega^*_{N,s}$ replaced by ω_N, the limit relations (1), (3) or (4), according to the value of s.

The determination of minimal s-energy points is a very difficult problem. This motivates in part the analysis of an alternative construction of points obtained by means of a "greedy" algorithm, analogous to the method described by Leja in [5].

Definition 1. Let $A \subset \mathbb{R}^{d'}$ be a compact set, and $s \in [0,\infty)$. A sequence $(a_n)_{n=1}^{\infty} \subset A$ is called a **Leja s-sequence** if it is formed as follows:

- a_1 is selected arbitrarily on A.
- Assuming that $a_1,\ldots,a_n$ have been selected, a_{n+1} is chosen to satisfy

$$E_s(\{a_1,\ldots,a_n,a_{n+1}\}) = \inf_{x\in A} E_s(\{a_1,\ldots,a_n,x\}).$$

We remark that the choice of a_{n+1} is not unique in general. We will use the notation

$$\alpha_{N,s} := \{a_1,\ldots,a_N\}$$

to denote the set of the first N Leja points.

Do Leja sequences behave like minimal energy configurations, satisfying the properties described by (1)-(5)? We now report on some results which answer partially this general question.

Theorem 1. *If $s \in [d'-2, d)$ then any Leja s-sequence $\{\alpha_{N,s}\}_N$ is $AEM(A;s)$. If $s \in [0, d'-2)$ and $\operatorname{supp}(\lambda_{A,s}) = \partial_\infty(A)$, the outer boundary of A, then the same conclusion follows. Under these conditions,*

$$\frac{1}{N}\sum_{k=1}^{N} \delta_{a_k} \longrightarrow \lambda_{A,s}, \quad as \quad N \to \infty.$$

This result is valid in particular on the unit sphere $S^d \subset \mathbb{R}^{d+1}$, for all $0 \le s < d$. Perhaps surprisingly, Leja sequences are not $AEM(A;s)$ in the case when $A = S^1$ and $s > 1$. Before stating this result we first give a description of these sequences on S^1.

Theorem 2. *Leja s-sequences $\{a_n\}_{n=1}^{\infty} \subset S^1$ are independent of s, and accordingly we write α_N instead of $\alpha_{N,s}$. The sets α_N are obtained iteratively in the following way: $\alpha_1 = \{a_1\}$, $a_1 \in S^1$ is chosen arbitrarily. If the set α_{2^m} is formed for some $m \ge 0$, then*

$$\alpha_{2^m+\kappa} = \alpha_{2^m} \cup e^{i\theta_\kappa} \cdot \alpha_\kappa, \quad for \quad 1 \le \kappa \le 2^m,$$

where θ_κ is some angle satisfying

$$\theta_\kappa \in \left\{ \frac{\pi(2k-1)}{2^m} \right\}_{k=1}^{2^m}.$$

Furthermore,

$$\frac{1}{N}\sum_{k=1}^{N} \delta_{a_k} \longrightarrow \sigma, \quad as \quad N \to \infty,$$

where σ denotes the normalized arclength measure on S^1.

We remark that if $\{x_n\}_{n=1}^{\infty} \subset [0,1)$ represents the Van der Corput sequence (see [7] for definition), then the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n := \exp(2\pi i x_n)$ gives a Leja sequence on S^1. Our main result is the following:

Theorem 3. *Any Leja sequence $\{\alpha_N\}_{N=1}^{\infty} \subset S^1$ is not $AEM(S^1;s)$ for $s \in (1,\infty)$. In fact, the subsequence $\alpha_{3\cdot 2^n}$ satisfies:*

$$\lim_{n\to\infty} \frac{E_s(\alpha_{3\cdot 2^n})}{(3\cdot 2^n)^{1+s}} = f(s)\frac{2\zeta(s)}{(2\pi)^s},$$

where $f(s) = \frac{1}{2}\left(\frac{4}{3}\right)^{1+s} + \left(\frac{1}{3}\right)^{1+s}$, and $f(s) > 1$ for all $s > 1$.

Conjecture: For $s > 1$, there exists no sequence $\{b_n\}_{n=1}^{\infty} \subset S^1$ such that the sequence of configurations $\{b_n\}_{n=1}^{N}$ is $AEM(S^1;s)$.

In the critical case $s = 1$ we obtain:

Theorem 4. *Any Leja sequence $\{\alpha_N\}_{N=1}^{\infty} \subset S^1$ is $AEM(S^1;1)$, i.e.*

$$\lim_{N\to\infty} \frac{E_1(\alpha_N)}{N^2\log(N)} = \frac{1}{\pi}.$$

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References

[1] D.P. Hardin and E.B. Saff, *Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds*, Adv. Math. **193** (2005), 174–204.
[2] D.P. Hardin and E.B. Saff, *Discretizing manifolds via minimum energy points*, Notices Amer. Math. Soc. November (2004), 1186-1194.
[3] A.B.J. Kuijlaars and E.B. Saff, *Asymptotics for minimal discrete energy on the sphere*, Trans. Amer. Math. Soc. **350** (1998), 523–538.
[4] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Heidelberg, 1972.
[5] F. Leja, *Une méthode élémentaire de résolution du problème de Dirichlet dans le plan*, Ann. Soc. Math. Polon. **23** (1950), 230–245.
[6] A. Martinez-Finkelshtein, V.V. Maymeskul, E.A. Rakhmanov, E.B. Saff, *Asymptotics for minimal discrete Riesz energy on curves in* $\mathbb{R}^d$, Canad. J. Math. **56** (2004), 529–552.
[7] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley and Sons, New York, 1974.

Geometric measure theory on fractals

Pertti Mattila

The purpose of this note is to describe some general methods rather than particular results. These methods are such that they can be used to study geometric measure theoretic properties of very general sets and measures in Euclidean spaces, including those of fractal type. We begin with energy-integrals of measures.

For a Borel set $A \subset \mathbb{R}^n$ let $\mathcal{P}(A)$ be the set of all Borel probability measures on $\mathbb{R}^n$ with compact support contained in A. For $0 < s < n$ the s-energy of $\mu \in \mathcal{P}(A)$ is

$$I_s(\mu) = \int \int |x-y|^{-s} d\mu x d\mu y.$$

The s-capacity of A is defined by

$$C_s(A) = \sup\{I_s(\mu)^{-1} : \mu \in \mathcal{P}(A)\}.$$

Then

$$C_s(A) > 0 \text{ if and only is there is } \mu \in \mathcal{P}(A) \text{ such that } I_s(\mu) < \infty.$$

It is well-known that the Hausdorff dimension of $A, \dim A$, is related to the capacities by

$$\dim A = \sup\{s : C_s(A) > 0\}$$

with the interpretation $\sup \emptyset = 0$.

We shall illustrate the use of these potential-theoretic concepts with orthogonal projections. For simplicity we restrict to the plane, for higher dimensions, see for example [2]. For $\theta \in [0, \pi)$ let $p_\theta : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection

$$p_\theta(u, v) = (u \cos\theta, v \sin\theta).$$

We denote by L^1 the 1-dimensional Lebesgue measure. Kaufman proved in 1968 the following result:

Theorem 1. *For any Borel set* $A \subset \mathbb{R}^2$

(1) $C_s(p_\theta(A)) > 0$ *for almost all* $\theta \in [0, \pi)$ *if* $C_s(A) > 0$ *and* $0 < s < 1$,

(2) $L^1(p_\theta(A)) > 0$ *for almost all* $\theta \in [0, \pi)$ *if* $C_1(A) > 0$.

The above relations with capacities and Hausdorff dimension give immediately the following corollary, which was proved in 1954 by Marstrand by direct and more complicated methods:

Corollary 1. *For any Borel set* $A \subset \mathbb{R}^2$

(1) $\dim p_\theta(A) = \dim A$ *for almost all* $\theta \in [0, \pi)$ *if* $\dim A \leq 1$,
(2) $L^1(p_\theta(A)) > 0$ *for almost all* $\theta \in [0, \pi)$ *if* $\dim A > 1$.

The proof of Theorem 1 is rather simple. We give the basic ideas. For $\mu \in \mathcal{P}(A)$ let $p_\theta\mu$ be the image of μ under p_θ, that is,

$$p_\theta\mu(B) = \mu(p_\theta^{-1}(B)) \text{ for } B \subset \mathbb{R}.$$

Then $p_\theta\mu \in \mathcal{P}(p_\theta(A))$ if $\mu \in \mathcal{P}(A)$. If $0 < s < 1$, one checks easily by Fubini's theorem that

$$\int_0^\pi I_s(p_\theta\mu)d\theta \leq c(s)I_s(\mu)$$

with $c(s) < \infty$. This gives immediately (1). To prove (2) one can verify with the help of Fatou's lemma, Fubini's theorem and a little of elementary geometry that

$$\int_0^\pi \int \liminf_{\delta \to 0} (2\delta)^{-1} p_\theta\mu((u - \delta, u + \delta)) dp_\theta\mu u d\theta \leq I_1(\mu).$$

By some general facts on differentiation of measures in $\mathbb{R}^n$ this implies that for almost all $\theta \in [0, \pi)$ $p_\theta\mu$ is absolutely continuous, even with L^2-density, with respect to L^1. This gives (2).

There are many extensions of this basic method to deal with transformation of dimension under various classes of parametrized mappings. A general setting where methods of the above type are combined with delicate Fourier analytic techniques is developed by Peres and Schlag in [5], see also [3] for discussion on these.

The usefulness of Fourier transform stems from the formula

$$I_s(\mu) = c(n, s) \int |x|^{s-n} |\hat{\mu}(x)|^2 dx.$$

The following result was proved by Falconer in 1985:

Theorem 2. *Let* $A \subset \mathbb{R}^n$ *be a Borel set and*

$$D(A) = \{|x - y| : x, y \in A\}$$

its distance set. If $\dim A > (n + 1)/2$, *then* $L^1(D(A)) > 0$.

To prove this one can introduce the distance measure $\delta(\mu) \in \mathcal{P}(D(A))$ for $\mu \in \mathcal{P}(A)$ by

$$\delta(\mu)(B) = \int \mu(\{y : |x - y| \in B\}) d\mu x$$

for Borel sets $B \subset \mathbb{R}$. Then it is rather simple to show that $\delta(\mu)$ is absolutely continuous, even with L^∞-density, if $I_{(n+1)/2}(\mu) < \infty$. Theorem 2 follows from this. Later it was shown in [4] that $D(A)$ has even non-empty interior under the above assumption.

It is believed that when $n \geq 2$, $\dim A > n/2$ should imply $L^1(D(A)) > 0$. The following improvement was proved by Wolff in [6] for $n = 2$ and by Erdogan in [1] for general n:

Theorem 3. *If $A \subset \mathbb{R}^n$ is a Borel set and $\dim A > \frac{n(n+2)}{2(n+1)}$, then $L^1(D(A)) > 0$.*

The idea here is to show that $\delta(\mu)$ is absolutely continuous with L^2-density if $I_s(\mu) < \infty$ for some $s > \frac{n(n+2)}{2(n+1)}$. This is much more difficult and uses both geometric and Fourier-analytic techniques.

References

[1] B. Erdogan *A bilinear Fourier extension theorem and applications to the distance set problem* Internat. Math. Res. Notices **23** (2005), 1411–1425.

[2] P. Mattila *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[3] P. Mattila *Hausdorff dimension, projections, and the Fourier transform*, Publ. Mat. **48** (2004), 3-48.

[4] P. Mattila and P. Sjölin, *Regularity of distance measures and sets*, Math. Nachr. 204 (1999), 157-162.

[5] Y. Peres and W. Schlag, *Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. **102** (2000), 193-251.

[6] T. Wolff *Decay of circular means of Fourier transforms of measures*, Internat. Math. Res. Notices **10** (1999), 547-567.

Exact packing dimension for random recursive constructions

R. Daniel Mauldin

The theory of the limit set generated by the iteration of finitely many similarity maps has been well developed for some time now. A more complicated theory of the limit set generated by the iteration of infinitely many uniformly contracting conformal maps was developed by Mauldin and Urbanski. Several years after that, they explored in their book [10] the geometric and dynamic properties of a far reaching generalization of conformal iterated function systems, called graph directed Markov systems (GDMS). We presented some of the main concepts and results for the study of the geometric and measure theoretic properties of the limit sets generated by iterating a finite or countably infinite family of conformal maps subject to some restrictions of a Markovian type on what maps are allowed in the iteration. Here we also indicate some further developments and possible applications.

The setting is as follows. Let (V, E) be a directed multigraph. We assume the vertex set V is finite and the set of edges E is countable. We consider the functions $i, t : E \mapsto V$ where $i(e)$ is the initial vertex of e and $t(e)$ is the terminal vertex of e. An incidence matrix $A : E^2 \mapsto \{0, 1\}$ is given such that if $A(e, e') = A_{e,e'} = 1$, then $t(e) = i(e')$. The coding space consists of all infinite words describing a walk through the graph subject to A: $E_A^\infty = \{\omega \in E^\infty : A_{\omega_i,\omega_{i+1}} = 1, \forall i\}$. E_A^* is the set of all finite words, E_A^n consists of all words of length n, $|\omega|$ is the length of ω. A Graph Directed Markov System (GDMS) consists of the following additional objects. For each $v \in V$ a compact metric space X_v, a number $s, 0 < s < 1$ and for every e a $1-1$ map $\phi_e : X_{t(e)} \mapsto X_{i(e)}$ with Lipschitz constant s. The limit set is described as follows: for each $\omega = (\omega_1, ..., \omega_n) \in E_A^*$, let $\phi_\omega = \phi_{\omega_1} \circ ... \circ \omega_n : X_{t(\omega_n)} \mapsto X_{i(\omega_1}$. For $\omega \in E_A^\infty$ we define $\pi(\omega)$ to be the single point in the intersection of the descending sequence of sets $\phi_{\omega|n}(X_{t(\omega_n)})$. Thus, π, the coding map, is a continuous map from E_A^∞ onto the limit set $J = J_A$. In order to analyze the geometric and measure theoretic properties of J we make further assumptions about this general construction given in [10]. This allows us to transfer results obtained on the abstract coding space to the geometric limit set J. Of course, we need additional assumptions in order to do this. For examples, we assume that the spaces $X_v \subset R^d$ and the maps ϕ_e are conformal maps. We also make some further technical conditions such as OSC, the open set condition and the bounded distortion property.

One motivating result is the following.

Theorem 1 (Beford, Mauldin-Urbanski)**.** *Suppose we are given a conformal GDMS with finitely many edges and the directed multigraph is strongly connected, i.e, A is irreducible. Then α, the Hausdorff dimension of the limit set J, is the unique zero of the topological pressure function, $P_A(t)$. Moreover, both the α-dimensional Hausdorff and packing measures are positive and finite:*

$$0 < \mathcal{H}^\alpha(J) \leq \mathcal{P}^\alpha(J) < \infty.$$

Also, the upper and lower Minkowski or box counting dimensions of J are both equal to α.

The topological pressure function, P_A, is defined as follows. For each $t \geq 0$ and $n \geq 1$ we set:

$$Z_A(t) = \sum_{\omega \in E_A^n} ||\varphi'_\omega||^t,$$

then

$$P_A(t) = \lim_{n \to \infty} \frac{1}{n} \ln Z_A(t).$$

We note that in the very special case where we are iterating finitely many similarity maps $\phi_i, 1 \leq i \leq n$ with contraction constants r_i, then $P_A(t) = \log(\sum_i r_i^t)$. So, this theorem generalizes the classical theorem of Moran and Hutchinson.

My talk focused on various generalization of this result. For example, various systems of continued fractions show that if the graph is has infinitely many edges, then it can happen that the Hausdorff measure is zero and the packing measure is infinite, see [9]. The study of the residual set in Apollonian packing or the curvilinear Sierpinski gasket shows that it cannot be obtained as the limit set generated by finitely many contracting conformal maps but it can be obtained as the limit set of infinitely many maps, see [11]. It is suggested that the tools described in this talk should be useful for studying Laplacians and other operators on the curvilinear gasket rather than the standard self-similar one.

I discussed some of the main tools used to analyze the limit sets: conformal measures, equilibrium measures and transfer operators or Perron-Frobenius-Ruelle operators.

Finally, I discussed some possible applications of this theory to the topic of quantization dimension. This theory has been well developed by Graf and Luschgy and, in particular, they gave a careful development for self-similar sets and measures, [3]. Mauldin and Lindsay started the corresponding theory for self-conformal measures and related the quantization dimension function to the functions arising from the thermodynamical formalism [6]. All this remains to be developed for the general setting of graph directed systems.

References

[1] A.Haas, D. Molnar, *Metrical Diophantine approximation for continued fraction like maps of the interval*, Trans. Amer. Math. Soc. **356** (2004), 2851-2870.
[2] D.Hensley, *The Hausdorff dimension of some continued fraction Cantor sets*, Journal of Number Theory, **33** (1989), 182-198.
[3] S. Graf, H. Luschgy, *Foundations of Quantization for Probability Distributions*, Lecture Notes in Mathematics 1730, Springer, Berlin, 2000.
[4] Oliver Jenkinson, *On the density of Hausdorff dimensions of bounded type of continued fraction sets: the Texan Conjecture*, Stochastics and Dynamics **4**(1) (2004) 77-84.
[5] Marc Kessebohmer, Sanguo Zhu, *Dimension sets for infinite IFS's: The Texan Conjecture*, Journal of Number Theory, Article in press, Corrected Proof.
[6] L. Lindsay, R. D. Mauldin, *Quantization dimension for conformal iterated functions systems.* Nonlinearity **15** (2002), 189-199.
[7] R.D.Mauldin, M.Urbanski, *Parabolic iterated function systems*, Erg. Th. Dynam. Sys. **20**, (2000), 1423-1447.
[8] R.D.Mauldin, M.Urbanski, *Dimensions and measures in iterated function systems*, Proc. London Math. Soc. (3) **73** (1996), 105-154.
[9] R.D.Mauldin, M.Urbanski, *Conformal iterated function systems with applications to the geometry of continued fractions.* Trans. Amer. Math. Soc. **351** (1999), 4995-5025.
[10] R.D.Mauldin, M.Urbanski, *Graph Directed Markov Systems*, volume 148 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2003.
[11] R.D.Mauldin, M.Urbanski, *Dimension and measures for a curvilinear Sierpinski gasket or Apollonian packing*, Adv. Math. **136** (1998), 26-38.
[12] Mariusz Urbanski, *Hausdorff dimension estimates for infinite conformal IFSs*, Nonlinearity **15** (2002) 727-734.

Potential Theory of Additive Lévy Processes

Yimin Xiao
(joint work with Davar Khoshnevisan)

Additive Lévy processes arise naturally in the studies of the Brownian sheet, intersections of Lévy processes and form a natural class of random fields with certain Markov structure. This talk is concerned with potential theory of additive Lévy processes and their applications in studying random fractals. In Part 1 we present some recent results on connections between an arbitrary additive Lévy process X in $\mathbb{R}^d$ and a class of energy forms and their corresponding capacities. In the second part, we apply the results in Part 1 to solve some long-standing problems in the folklore of the theory of Lévy processes. These methods may also be useful for studying fractal properties of more general Markov processes; see [11] for further information.

1. Potential theory for additive Lévy processes

Let $X_1, \ldots, X_N$ be independent Lévy processes that take their values in $\mathbb{R}^d$ and $X_j(0) = 0$ for $j = 1, \ldots, N$. The N-parameter stochastic process $X = \{X(t), t \in \mathbb{R}^N_+\}$ defined by $X(t) = \sum_{j=1}^N X_j(t_j)$ $(t \in \mathbb{R}^N_+)$ is called an *(N, d)-additive Lévy process*. When $X_1, \ldots, X_N$ are isotropic stable processes in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$, then X is called an *additive stable process*. One can also define an N-parameter multiplicative Lévy process Y with values in $\mathbb{R}^{Nd}$ by $Y(t) = \big(X_1(t_1), \ldots, X_N(t_N)\big)$ $(t \in \mathbb{R}^N_+)$. Note that Y is a special case of additive Lévy processes.

We consider the following questions for an additive Lévy process $X = \{X(t), t \in \mathbb{R}^N_+\}$ in $\mathbb{R}^d$:

(a). Given a Borel set $F \subseteq \mathbb{R}^d$, when can $\mathbb{P}\{X(\mathbb{R}^N_+) \cap F \neq \emptyset\} > 0$?

(b). Given a Borel set $F \subseteq \mathbb{R}^d$, when does $F \oplus X(\mathbb{R}^N_+)$ have positive Lebesgue measure?

(c). For any fixed $a \in \mathbb{R}^d$ and a Borel set $E \subset \mathbb{R}^N_+$, when can $\mathbb{P}\{X^{-1}(a) \cap E \neq \emptyset\} > 0$? Here $X^{-1}(a) = \{t \in (0, \infty)^N : X(t) = a\}$ is the level set of X.

Questions (a) and (b) were answered in [9] under some sector-type condition, which is removed by Khoshnevisan and Xiao [8].

Theorem 1. *Let X be an (N, d)-additive Lévy process whose Lévy exponent $\Psi = (\Psi_1, \ldots, \Psi_N)$. Then, for any nonrandom compact set $F \subset \mathbb{R}^d$, $\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+) \oplus F)\} > 0$ if and only if*

$$\inf_{\mu \in \mathcal{P}(F)} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi)}\right) d\xi < \infty,$$

where $\widehat{\mu}$ denotes the Fourier transform of μ.

By taking $F = \{0\}$ we have

Corollary 1. *Let X be an (N, d)-additive Lévy process with exponent Ψ. Then*

$$\mathbb{E}\{\lambda_d(X(\mathbb{R}^N_+))\} > 0 \iff \int_{\mathbb{R}^d} \prod_{j=1}^N \mathrm{Re}\left(\frac{1}{1 + \Psi_j(\xi)}\right) d\xi < +\infty.$$

Applying Theorem 1 to additive stable Lévy processes, we have the following result which is useful for studying fractal properties of (additive) Lévy processes. See Khoshnevisan [3] and the reference therein for related results.

Corollary 2. *Suppose $X = X_1 \oplus \cdots \oplus X_N$ is an additive stable process in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$. Then for any Borel set $F \subset \mathbb{R}^d$, the following are equivalent:*

(i) $\mathrm{Cap}_{d-N\alpha}(F) > 0$;
(ii) $\mathbb{P}\{\lambda_d(F \oplus X(\mathbb{R}_+^N)) > 0\} > 0$;
(iii) *F is not polar for X in the sense that* $\mathbb{P}\{F \cap X(\mathbb{R}_+^N) \neq \emptyset\} > 0$.

Questions (c) was considered by Khoshnevisan and Xiao [5]. We say that an additive Lévy process X is *absolutely continuous* if for each $t \in (0,\infty)^N$, the function $\xi \mapsto \exp\{-\sum_{j=1}^N t_j \Psi_j(\xi)\} \in L^1(\mathbb{R}^d)$. In this case, for every $t \in (0,\infty)^N$, $X(t)$ has a density function $p(t;\bullet)$ that is given by the formula

$$p(t;x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} \exp\Big(-\sum_{j=1}^N t_j \Psi_j(\xi)\Big)\, d\xi, \quad x \in \mathbb{R}^d.$$

Let Φ be the function defined by $\Phi(s) = p(\overline{s}, 0)$ $(s \in \mathbb{R}^N)$, where $\overline{s} = \langle |s_j| \rangle \in \mathbb{R}_+^N$. Khoshnevisan and Xiao [5, Theorem 2.9] showed that

$$\mathbb{P}\{X^{-1}(a) \neq \emptyset\} > 0 \iff \Phi \in L^1_{loc}(\mathbb{R}^N).$$

See Khoshnevisan, Shieh and Xiao [4] for further results.

2. Applications to Lévy processes

The results in Section 1 can be applied to study the Hausdorff dimension, exact capacity of the ranges and the self-intersections of ordinary Lévy processes, solving several open problems in theory of Lévy processes. We refer to Khoshnevisan and Xiao [5, 6, 7], Khoshnevisan, Xiao and Zhong [9] for details of these results. In the following, we state a new result on intersections of Lévy processes.

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a d-dimensional Lévy process. A point $x \in \mathbb{R}^d$ is called a *k-multiple point* of X if there exist k distinct times $t_1, t_2, \cdots, t_k \in (0,\infty)$ such that $X(t_1) = \cdots = X(t_k) = x$. The set of k-multiple points is denoted by $M_k^{(d)}$ and the set of *k-multiple times* is denoted by

$$L_k^{(d)} = \left\{ (t_1, \cdots, t_k) \in \mathbb{R}_+^k \;\middle|\; \begin{array}{c} t_1, \ldots, t_k \text{ are distinct and} \\ X(t_1) = \cdots = X(t_k) \end{array} \right\}.$$

The existence of k-multiple points of X has been solved by LeGall et al. [10], Evans [1], Fitzsimmons and Salisbury [2]. Khoshnevisan and Xiao [8] proved the following refined result.

Theorem 2. *Let $X_1, \ldots, X_k$ be k independent Lévy processes on $\mathbb{R}^d$, and assume that each X_j has a one-potential density u_j that is strictly positive a.e. Then, for all nonempty Borel sets $F \subseteq \mathbb{R}^d$,*

$$\mathbb{P}\{X_1(t_1) = \cdots = X_k(t_k) \in F \ \ \text{for distinct } t_1, \ldots, t_k > 0\} > 0$$

if and only if there exists a $\mu \in \mathcal{P}(F)$ such that

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \left|\hat{\mu}\left(\xi^1 + \cdots + \xi^k\right)\right|^2 \prod_{j=1}^k \mathrm{Re}\left(\frac{1}{1+\Psi_j(\xi^j)}\right) d\xi^1 \cdots d\xi^k < \infty.$$

Further results on the Hausdorff dimensions of $M_k^{(d)}$ and $L_k^{(d)}$ are obtained in Khoshnevisan and Xiao [8].

References

[1] Evans, S. N. (1987), *Multiple points in the sample paths of a Lévy process.* Probab. Th. Rel. Fields **76**, 359–367.

[2] Fitzsimmons, P. J. and Salisbury, T. S. (1989), *Capacity and energy for multiparameter Markov processes.* Ann. Inst. H. Poincaré Probab. Statist. **25**, 325–350.

[3] Khoshnevisan, D. (2002), *Multiparameter Processes: An Introduction to Random Fields.* Springer, New York.

[4] Khoshnevisan, D., Shieh, N.-R. and Xiao, Y. (2007), *Hausdorff dimension of the contours of symmetric additive Lévy processes.* Probab. Th. Rel. Fields, to appear.

[5] Khoshnevisan, D. and Xiao, Y. (2002), *Level sets of additive Lévy process.* Ann. Probab. **30**, 62–100.

[6] Khoshnevisan, D. and Xiao, Y. (2003), *Weak unimodality of finite measures, and an application to potential theory of additive Lévy processes.* Proc. Amer. Math. Soc. **131**, 2611–2616.

[7] Khoshnevisan, D. and Xiao, Y. (2005), Lévy processes: capacity and Hausdorff dimension. *Ann. Probab.* **33**, 841–878.

[8] Khoshnevisan, D. and Xiao, Y. (2007), *Harmonic analysis of additive Lévy processes.* Preprint.

[9] Khoshnevisan, D., Xiao, Y. and Zhong, Y. (2003), *Measuring the range of an additive Lévy processes.* Ann. Probab. **31**, 1097–1141.

[10] Le Gall, J.-F., Rosen, J. S. and Shieh, N.-R. (1989), Multiple points of Lévy processes. *Ann. Probab.* **17**, 503–515.

[11] Xiao, Y. (2004), *Random fractals and Markov processes.* In: Fractal Geometry and Applications, (Michel L. Lapidus and Machiel van Frankenhuijsen, editors), pp. 261–338, American Mathematical Society, Rhode Island.

Classical Potential Theory and Stochastic Processes

Martina Zähle

The classical *Riesz potential operator* I^s of order $s > 0$ in $\mathbb{R}^n$ (with $n \geq 3$ for simplicity) is determined by its Fourier transform

$$(I^s u)\hat{} = |\cdot|^{-s}\hat{u}\,,\ u \in L_2(\mathbb{R}^n)\,.$$

For $s \neq n + 2k$ one obtains the Riesz kernel representation

$$I^s u(x) = \mathrm{const}(n,s) \int \frac{1}{|x-y|^{n-s}} u(y)\, dy\,.$$

The inverse operator $I^{-s} v = D^s v = (|\cdot|^s\, \hat{v})\check{}$ is a special pseudodifferential operator with symbol $|\cdot|^s$, $s > 0$. These operators define the positive powers of minus the Laplace operator:

$$D^s = (-\Delta)^{s/2}\,.$$

The associated quadratic form in $L_2(\mathbb{R}^n)$,

$$\mathcal{E}^{2s}(u,v) := \langle D^s u, D^s v\rangle_{L_2(\mathbb{R}^n)} = \int |\xi|^{2s} \hat{u}(\xi)\, \overline{\hat{v}(\xi)}\, d\xi$$

with domain $\{u \in L_2(\mathbb{R}^n) : D^s u \in L_2(\mathbb{R}^n)\}$ for $0 < \alpha = 2s < 2$ admits the difference representation

$$\mathcal{E}^{\alpha}(u,v) = \mathrm{const}(n,\alpha) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}} dxdy\,.$$

For $\alpha = 2$ we have

$$\mathcal{E}^2(u,v) = \mathrm{const}(n) \int_{\mathbb{R}^n} \langle \mathrm{grad} u(x), \mathrm{grad} v(x) \rangle \, dx \, .$$

The *Bessel potential operator* G^s of order $s > 0$ determined by the Fourier transform

$$(G^s u)\hat{} = (1 + |\cdot|^2)^{-s/2} \, \hat{u} \, , \; u \in L_2(\mathbb{R}^n) \, ,$$

has the interpretation

$$G^s = (\mathrm{id} - \Delta)^{-s/2} \, .$$

In the set $H^s(\mathbb{R}^n)$ of s-Bessel potentials a *Hilbert space structure* is introduced by the scalar product

$$\langle u, v \rangle_{H^s(\mathbb{R}^n)} = \int (1 + |\xi|^2)^s \, \hat{u}(\xi) \, \overline{\hat{v}(\xi)} \, d\xi \, .$$

Note that

$$\mathcal{E}_1^{2s}(u,v) := \langle u, v \rangle_{L_2(\mathbb{R}^n)} + \mathcal{E}^{2s}(u,v) = \int (1 + |\xi|^{2s}) \, \hat{u}(\xi) \, \overline{\hat{v}(\xi)} \, d\xi$$

is an equivalent scalar product in $H^s(\mathbb{R}^n)$. ($H^s(\mathbb{R}^n)$ coincides with the Besov space $B^s_{22}(\mathbb{R}^n)$.)

Let R^α_λ be the *resolvent* of $-D^\alpha$, i.e., the operator

$$R^\alpha_\lambda := (\lambda \mathrm{id} + D^\alpha)^{-1} \, .$$

From the general theory of operator semigroups it follows that for any $\alpha > 0$ the negative self-adjoint operator $-D^\alpha$ generates a strongly continuous semigroup

$$T^\alpha_t := e^{-D^\alpha t} \, , \; t > 0 \, ,$$

of contractive operators on $L_2(\mathbb{R}^n)$ with I^α as potential operator. Moreover, the resolvent relationship reads:

$$R^\alpha_\lambda u = \int_0^\infty e^{-\lambda t} \, T^\alpha_t u \, dt$$

and $\mathcal{E}^\alpha$ is the corresponding closed and regular quadratic form in $L_2(\mathbb{R}^n)$ with domain $H^{\alpha/2}(\mathbb{R}^n)$.

For $0 < \alpha \leq 2$, $\mathcal{E}^\alpha$ is Markovian, hence a regular Dirichlet form. Therefore the associated Markov process X^α is a Hunt process, the so-called *symmetric α-stable Lévy process.*

This approach remains valid for arbitrary *Lévy processes in $\mathbb{R}^n$ subordinate to Brownian motion*: Replace $D^\alpha = (-\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$, by the pseudodifferential operator $f(-\Delta)$ for a Bernstein function f. The corresponding *f-Riesz potential operator* U^f has a kernel $K^f = (f(|\cdot|^2)^{-1})^\vee$ which is equivalent to $|\cdot|^{-n} f(|\cdot|^{-2})^{-1}$ (under some additional condition on f for the lower estimate). The scalar product in the (f,s)-*Bessel potential space* $H^{f,s}(\mathbb{R}^n)$ is introduced by

$$\langle u, v \rangle_{H^{f,s}(\mathbb{R}^n)} = \int (1 + f(|\xi|^2))^s \, \hat{u}(\xi) \, \overline{\hat{v}(\xi)} \, d\xi$$

and the domain of the associated *Dirichlet form*

$$\mathcal{E}^f(u,v) := \int f(|\xi|^2)\,\hat{u}(\xi)\,\overline{\hat{v}(\xi)}\,d\xi$$

agrees with $H^{f,1}(\mathbb{R}^n)$. (Note that $H^{f,s}(\mathbb{R}^n)$ may be interpreted as some Besov space of generalized smoothness.)
The associated *Markov semigroup* is representable by the *subordination formula*

$$T_t^f u(x) = \int_0^\infty \int_{\mathbb{R}^n} u(y)\,p_s(x-y)\,dy\,\eta_t^f(ds)\,.$$

Here $(p_t)_{t\geq 0}$ are the transition densities of *Brownian motion* and $(\eta_t^f)_{t\geq 0}$ is the convolution semigroup of measures associated with f, which determines the *subordinator*. The *subordinated Markov process* X^f is given by $(T_t^f)_{t\geq 0}$.

These classical notions and results are extended to related versions on *fractal h-sets* F with h-measures μ (equivalent to Hausdorff measure $\mathcal{H}^h$ on F). Under the *trace condition*

$$\int_0^1 h(r)\,f(r^{-2})^{-1}\,r^{-(n+1)}\,dr < \infty$$

on the Bernstein function f and the gauge function h traces on F of the potential spaces $H^{f,s}(\mathbb{R}^n)$, the potential operator U^f, the Dirichlet form $\mathcal{E}^f$, and the Lévy process X^f are considered.

References

[1] M. Fukushima, Y. Oshima, M. Takeda: *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin, 1994.

[2] N. Jacob: *Pseudo-Differential Operators and Markov Processes. Vol.1: Fourier Analysis and Semigroups*, Imperial College Press, London, 2001, *Vol.2: Generators and Their Potential Theory*, Imperial College Press, London, 2002.

[3] H. Triebel: *Theory of Function Spaces III*, Birkhäuser, Basel, 2006.

[4] M. Zähle: *Potential spaces and traces of Lévy processes on h-sets*, Preprint.

Necessary and sufficient conditions of the solvability of the Gauss variational problem

Natalia Zorii

Let $\mathbf{X}$ denote a locally compact Hausdorff space, and $\mathfrak{M}$ the linear space of all real-valued Radon measures ν on $\mathbf{X}$. A *kernel* κ on $\mathbf{X}$ is meant as a lower semicontinuous function $\kappa : \mathbf{X}\times\mathbf{X} \to [0,\infty]$. Let $\kappa(\nu,\nu)$ and $\kappa(\,\cdot\,,\nu)$ denote respectively the *energy* and the *potential* of a measure ν with respect to a kernel κ (certainly, provided the corresponding integral in the definition is well defined; see, e. g., [1, 2]). Write $\mathcal{E} := \bigl\{\,\nu\in\mathfrak{M}:\ -\infty<\kappa(\nu,\nu)<\infty\,\bigr\}$.

Let $\mathcal{A} = (A_i)_{i \in I}$ be an ordered finite collection of closed sets in $\mathbf{X}$ such that each A_i is assigned the sign $+$ or $-$, and $A_i \cap A_j = \varnothing$ whenever $\operatorname{sign} A_i \neq \operatorname{sign} A_j$. Given $\mathcal{A}$, let $\mathfrak{M}(\mathcal{A})$ consist of all *linear combinations*

$$\mu = \sum_{i \in I} (\operatorname{sign} A_i)\, \mu^i,$$

where μ^i, $i \in I$, is a nonnegative measure supported on A_i. We call $\mu \in \mathfrak{M}(\mathcal{A})$ a *measure associated* with $\mathcal{A}$, and μ^i the *i-coordinate* of μ. Any two elements in $\mathfrak{M}(\mathcal{A})$ are regarded as *identical* if and only if all their coordinates coincide.

For measures associated with $\mathcal{A}$, it is convenient to determine the so-called $\mathcal{A}$-*vague* convergence as the vague convergence by coordinates. (Recall that the *vague* topology on $\mathfrak{M}$ is the topology of pointwise convergence on the class of all real-valued continuous functions on $\mathbf{X}$ with compact support; see, e. g., [1, 2].)

Given $\mathcal{A}$, let A denote the union of all A_i, $i \in I$. Fix a continuous function $g : A \to (0, \infty)$ and a vector $a = (a_i)_{i \in I}$ with $a_i \in \mathbb{R}_+$ for all $i \in I$, and write

$$\mathcal{E}(\mathcal{A}, a, g) := \Big\{ \mu \in \mathfrak{M}(\mathcal{A}) : \quad \mu \in \mathcal{E}, \quad \int g\, d\mu^i = a_i \quad \text{for all} \quad i \in I \Big\}.$$

Fix also a universally measurable function $f : A \to [-\infty, \infty]$, and write

$$\mathcal{E}_f(\mathcal{A}, a, g) := \Big\{ \mu \in \mathcal{E}(\mathcal{A}, a, g) : \quad \int f\, d\mu \quad \text{is defined} \Big\}.$$

The *Gauss variational problem* is the problem of minimizing of the expression

$$G_f(\mu) := \kappa(\mu, \mu) - 2 \int f\, d\mu$$

over the class $\mathcal{E}_f(\mathcal{A}, a, g)$ (see, e. g., [2, 3]). In the case where the infimum value

$$G_f(\mathcal{A}, a, g) := \inf_{\mu \in \mathcal{E}_f(\mathcal{A}, a, g)} G_f(\mu)$$

is finite, the following problem on the solvability naturally arises.

Problem 1. *In the Gauss variational problem, does there exist a minimizing measure, namely a measure $\lambda \in \mathcal{E}_f(\mathcal{A}, a, g)$ such that $G_f(\lambda) = G_f(\mathcal{A}, a, g)$?*

If all A_i, $i \in I$, are compact, the class $\mathcal{E}(\mathcal{A}, a, g)$ is clearly $\mathcal{A}$-vaguely compact. Therefore, if one chooses κ and f so that G_f be $\mathcal{A}$-vaguely lower semicontinuous, the existence of minimizing measures λ easily follows (see [2]).

But if any A_i is noncompact, the class $\mathcal{E}(\mathcal{A}, a, g)$ is no longer $\mathcal{A}$-vaguely compact, and hence one has to use some additional tools of investigation.

Therefore from now on κ is always required to be *positive definite*, which means that it is symmetric and the energy $\kappa(\nu, \nu)$, $\nu \in \mathfrak{M}$, is nonnegative whenever defined. Then $\mathcal{E}$ is a pre-Hilbert space with the scalar product (ν_1, ν_2), equal to the mutual energy $\kappa(\nu_1, \nu_2)$, and the semi-norm $\|\nu\| := \sqrt{\kappa(\nu, \nu)}$. The topology on $\mathcal{E}$ defined by means of $\|\cdot\|$ is called the *strong* topology. See [1].

Definition [1]**.** A kernel κ is called *perfect* if $\mathcal{E}^+ := \{ \nu \in \mathcal{E} : \ \nu \geqslant 0 \}$, treated as a topological subspace of $\mathcal{E}$, is strongly complete and the strong topology on $\mathcal{E}^+$ is finer than the induced vague topology.

Examples. In $\mathbb{R}^n$, $n \geqslant 3$, the Newtonian kernel $|x-y|^{2-n}$, the Riesz kernels $|x-y|^{\alpha-n}$, $0<\alpha<n$, and the Green kernel g_D (here, D is an open set, and g_D its generalized Green function) are perfect. See, e. g., [1].

The concept of perfect kernels is an efficient tool in extremal problems over classes of *positive* measures (see [1]). But $\mathcal{E}_f(\mathcal{A},a,g)$ consists of *signed* measures, while, by H. Cartan, $\mathcal{E}$ is *strongly incomplete* even for the Newtonian kernel.

Nevertheless, we succeeded in solving Problem 1 exactly for perfect kernels (see [4, 5, 7]). We showed that in the noncompact case the Gauss variational problem is in general *unsolvable*, and this occurs even under extremely natural restrictions on κ, g, and f (in particular, for the Newtonian, Green, or Riesz kernels in $\mathbb{R}^n$, $n \geqslant 3$). Under fairly general assumptions, we obtained *necessary and sufficient* conditions for the problem to be solvable.

In the remainder, κ is supposed to be perfect and bounded on $A_i \times A_j$ whenever $\operatorname{sign} A_i \neq \operatorname{sign} A_j$. Suppose also that $\inf\, g(x) > 0 \quad (x \in A)$.

Let $\operatorname{cap} Q$ denote the interior *capacity* of a set Q relative to a kernel κ.

Theorem 1. [4] *Let either g be bounded or there exist $r \in (1,\infty)$ and $\zeta \in \mathcal{E}$ such that*

$$g^r(x) \leqslant \kappa(x,\zeta), \quad x \in A.$$

Suppose that either $f = \kappa(\,\cdot\,,\omega)$ for some $\omega \in \mathcal{E}$ or $(\operatorname{sign} A_i)\, f\big|_{A_i}$, $i \in I$, are nonpositive (or with compact support) and upper semicontinuous. If moreover

$$\operatorname{cap} A_i < \infty \quad \textit{for all} \quad i \in I, \tag{1}$$

then the Gauss variational problem is solvable for any vector $a = (a_i)_{i\in I}$.

In the following theorem, κ is required to satisfy the *generalized maximum principle* with a constant h. This means that for any measure $\nu \geqslant 0$, $\kappa(x,\nu)$ is bounded from above by $h\,M$ everywhere in $\mathbf{X}$ provided $\kappa(x,\nu) \leqslant M$ on $\operatorname{supp}\nu$.

Theorem 2. [4] Suppose κ is continuous for $x \neq y$, g is bounded, while $f = \kappa(\,\cdot\,,\xi)$, where $\xi \in \mathcal{E}$ is a bounded measure such that $A \cap \operatorname{supp}\xi = \varnothing$ and

$$\sup_{x\in K,\ y \in \operatorname{supp}\xi} \kappa(x,y) < \infty \quad \text{for all compact sets } K \subset A.$$

Under the above assumptions, in order that the Gauss variational problem be solvable for every a, it is necessary and sufficient that *(1) be satisfied.*

Consider now κ, g, and f satisfying all the assumptions of Theorem 2, and suppose that $\operatorname{cap} A_j = \infty$, $j \in I$ being fixed. Then, in accordance with Theorem 2, there exists $a' = (a_i')_{i\in I}$ such that the Gauss variational problem has no solutions. But what is a complete description of the set of all vectors a such that this phenomenon of unsolvability occurs?

Theorem 3. [7] *Suppose all other A_i, $i \neq j$, are of finite capacity and do not intersect A_j, while $\kappa(\,\cdot\,,y) \to 0$ as $y \to \infty$ uniformly on compact sets. Then the Gauss variational problem is solvable for a vector $a = (a_i)_{i\in I}$ if and only if*

$$a_j \leqslant \int g\, d\gamma^j.$$

Here, γ is a solution (it exists) of the problem of minimizing of $G_f(\mu)$ over

$$\Big\{ \mu \in \mathfrak{M}(\mathcal{A}) : \int g\, d\mu^i = a_i \quad \textit{for all} \quad i \neq j \Big\}.$$

Remark 1. *Theorems 1 and 2 were extended to the so-called* constrained *problem, posed by E. Rakhmanov (cf.* [3]*), where admissible measures were additionally supposed to be bounded from above by a fixed constraint. See* [5, 6].

Remark 2. *Theorem 3 implies necessary and sufficient conditions of the solvability of the main minimum problem of the theory of capacities of condensers with respect to perfect kernels (in particular, the Newtonian, Green, or Riesz ones). The corresponding results were obtained by the author in 1986 – 2000; cf.* [4].

References

[1] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139 – 215.

[2] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-1 **25** (1961), 135 – 352.

[3] E. B. Saff, V. Totik, *Logarithmic potentials with external fields*, Springer-Verlag, Berlin (1997).

[4] N. Zorii, *On the solvability of the Gauss variational problem*, Comput. Methods Funct. Theory, **2** (2002), 427 – 448.

[5] N. V. Zorii, *Equilibrium problems for potentials with external fields*, Ukrain. Math. Zh., **10** (2003), 1315 – 1339.

[6] N. Zorii, *On the constrained Gauss variational problem*, Bull. Soc. Sci. Lettr. Łódź **53** Sér. Rech. Déform. **40** (2003), 25 – 40.

[7] N. V. Zorii, *Necessary and sufficient conditions for the solvability of the Gauss variational problem*, Ukrain. Math. Zh., **57** (2005), 60 – 83.

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