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Algebraic Groups

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Abstract. The workshop dealt with a broad range of topics from the structure theory and the representation theory of algebraic groups (in the widest sense). There was emphasis on the following areas: structure and classification of wonderful varieties, finite reductive groups and character sheaves, quantum cohomology of homogeneous varieties, representation categories and their connections to orbits and flag varieties.

Mathematics Subject Classification (2000): 20Gxx.

Introduction by the Organisers

The workshop continued a series of Oberwolfach meetings on algebraic groups, started in 1971 by Tonny Springer and Jacques Tits who both attended the present conference. This time, the organizers were Michel Brion, Jens Carsten Jantzen, and Raphaël Rouquier.

During the last years, the subject of algebraic groups (in a broad sense) has seen important developments in several directions, also related to representation theory and algebraic geometry. The workshop aimed at presenting some of these developments in order to make them accessible to a "general audience" of algebraic group-theorists, and to stimulate contacts between participants.

Each of the first four days was dedicated to one area of research that has recently seen decisive progress:

- structure and classification of wonderful varieties,
- finite reductive groups and character sheaves,
- quantum cohomology of homogeneous varieties,
- representation categories and their connections to orbits and flag varieties.

The first three days started with survey talks that will help to make the subject accessible to the next generation. The talks on the last day introduced to several recent advances in different areas: arithmetic groups, eigenvalue problems, counting orbits over finite fields, quivers and reflection functors. In order to leave enough time for fruitful discussions, the number of talks (generally of one hour) was limited to four per day.

Besides the scientific program, the participants enjoyed a piano recital on Thursday evening, by Harry Tamvakis.

The workshop was attended by 53 participants, coming mainly from Europe and North America. This includes 6 PhD students, supported by the Marie Curie program of the European Union. The organizers are grateful to the EU for this support, and to the MFO for providing excellent working conditions.

Workshop: Algebraic Groups

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Abstracts

A survey on wonderful varieties

Guido Pezzini

Let G be a reductive connected complex algebraic group, and let X be a normal irreducible G-variety: X is *spherical* if it has a dense B-orbit where B is a Borel subgroup of G. This definition represents a sort of common generalization of many families studied in the literature about reductive groups, such as toric varieties (where G is a complex torus), Grassmannians and flag varieties, symmetric varieties. Spherical varieties have also links with the theory of hamiltonian actions on real symplectic manifolds, being the “algebraic analogue” of multiplicity-free manifolds.

Wonderful varieties are a special class of spherical ones, and their definition comes from the properties of the compactifications of symmetric homogeneous spaces given by De Concini and Procesi in [8]. An irreducible G-variety X is *wonderful* if:

(1) X is smooth and projective;
(2) X has an open G-orbit, whose complement is the union of prime divisors $X_1, \ldots, X_r$ which are smooth, with normal crossings, and $X_1 \cap \ldots \cap X_r \neq \emptyset$;
(3) $x, y \in X$ are on the same G-orbit if and only if $\{i \mid X_i \ni x\} = \{j \mid X_j \ni y\}$.

The theory of *embeddings*, developed by Luna and Vust in [15] and described in the spherical case in [10], shows that wonderful varieties are precisely those spherical varieties being smooth, projective, having only one closed G-orbit, and such that all B-stable prime divisors containing a G-orbit are also G-stable.

From this point of view one extracts from a wonderful variety X several discrete invariants. They come mainly from the action of B on X:

(1) the (finite) set Δ_X of all B-stable but not G-stable prime divisors, called *colors*;
(2) the B-weights of the rational functions on X being B-eigenvectors; these weights are a sublattice Ξ_X of the group of characters of T, a chosen maximal torus inside B;
(3) the B-weights appearing in the T-module $T_z(X)/T_z(G.z)$, where z is the unique fixed point of B_- (the opposite of B with respect to T); these are called the *spherical roots*, and are a basis (denoted Σ_X) of Ξ_X.
(4) the set of simple roots S^p_X associated to the stabilizer of the open B-orbit on X; this is a parabolic subgroup containing B.

These invariants are involved in a number of “tools” used to study these varieties, such as for example “generalizations” of the Cartan matrix and the open cell (as in the theory of reductive groups), of the little Weyl group (as for symmetric varieties). Colors here are considered as elements of an abstract set, each equipped with an associated element in $\mathrm{Hom}_{\mathbb{Z}}(\Xi_X, \mathbb{Q})$.

Wonderful varieties play a significative role in the classification of spherical varieties. The paper [13] shows this relation: it proves that if the triple of invariants $(S^p_\bullet, \Sigma_\bullet, \Delta_\bullet)$ classify wonderful $G/Z(G)$-varieties for a given group G, then it is possible to classify all spherical G-varieties. The triple $(S^p_X, \Sigma_X, \Delta_X)$ is called the *spherical system* of X.

In the same paper Luna conjectured a set of axioms defining admissible triples; these axioms are inspired by known classification of varieties having rank 1 and 2 (see [1], [19]), the rank being the number r in the definition. Using these axioms, spherical systems are considered as combinatorial objects, and represented by diagrams attached to the Dynkin diagram of G. The standard conjecture is then:

Conjecture. Spherical systems classify wonderful varieties.

In [13] Luna proves the conjecture for all semisimple G of type A, Bravi and Pezzini for G of mixed type $\mathsf{A}-\mathsf{C}$ (partially) and $\mathsf{A}-\mathsf{D}$ in [4], Bravi for G simply laced $(\mathsf{A}-\mathsf{D}-\mathsf{E})$ in [2]. The conjecture for all G is still an open problem, although recently Losev has shown in [12] that spherical systems at least separate wonderful varieties.

A variety can be wonderful under the action of many different groups, with the same G-stable divisors or not. For example, symmetric rank 1 wonderful varieties are homogeneous under the action of a group bigger than G. In [6] Brion shows that for any wonderful variety X the full group $\mathrm{Aut}^0(X)$ is semisimple, and X is wonderful under its action too.

Other researches about the geometry of wonderful varieties are under development; for example the cohomology of line bundles. Here one can look for a generalization of the classical Borel-Weil theorem: the Picard group has a basis given by the classes of colors (see [5]), the global sections of a line bundle generated by global sections is not an irreducible G-module but it is multiplicity-free and it is completely described (see [5]), higher cohomology of such a line bundle is zero (see [7]), all ample line bundles are very ample (see [16]). A complete description of the cohomology of all line bundles is accomplished only for varieties of *minimal rank*, i.e. $\operatorname{rank} X = \operatorname{rank} G - \operatorname{rank} H$, H being a generic stabilizer, by Tchoudjem (see [18]). Such varieties are also completely classified, by Ressayre (see [17]).

Another problem is to construct wonderful varieties in some projective space, and give their equations. The compactifications of De Concini and Procesi were found in the projective space of suitable irreducible G-modules, but this can be done only when the stabilizers of all points of X are equal to their normalizers (see [16]). Moreover, Losev has shown in [11] that when the generic stabilizer H is equal to its normalizer, then the *Demazure embedding* produces a wonderful variety. The construction is the following: one considers the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H, and the grassmannian $\mathrm{Gr}(\mathfrak{g}, \dim \mathfrak{h})$; the closure of the orbit $G \cdot [\mathfrak{h}]$ gives a variety G-isomorphic to X.

Wonderful varieties have been used in the recent theory of invariant Hilbert schemes developed by Alexeev and Brion. This scheme parametrizes G-stable affine subvarieties of a given G-module V; more precisely all subvarieties $Y \subseteq V$

whose coordinate ring $\mathbb{C}[Y]$ has the same structure as a G-module (this plays the role of the Hilbert polynomial for classical Hilbert schemes). In particular, if $\mathbb{C}[Y]$ has no multiplicities then Y is spherical, and $\mathbb{C}[Y]$ is simply described by the monoid of highest weights appearing in it.

Luna has worked (see [14]) on the case where $\mathbb{C}[Y]$ is the sum of one copy of each irreducible G-module: the so-called *model* varieties. He shows that for any G there exists a wonderful G-variety whose orbits parametrize G-isomorphism classes of such varieties.

Jansou has worked (see [9]) on the case where V^* is irreducible of highest weight λ and the monoid Γ of highest weights of $\mathbb{C}[Y]$ is $\mathbb{N}\lambda$. Here he obtains a similar result, and proves that in this case the invariant Hilbert scheme is a reduced point or an affine line, corresponding resp. to the cases where the wonderful variety involved has rank 0 or 1.

Bravi and Cupit-Foutou in [3] extended this result to any V, in the case where the monoid Γ is *saturated*, i.e. $\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$, where Λ^+ is the set of dominant weights. Here the invariant Hilbert scheme will be an affine space, of dimension equal to the rank of associated wonderful variety.

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Equations defining symmetric varieties and affine Grassmannians I and II

Peter Littelmann, Andrea Maffei
(joint work with Rocco Chirivì)

The talks are reports on joint work [1] with Rocco Chirivì (Pisa, Italy).
Let G be a connected semisimple algebraic group over the complex numbers, let σ be an involution of G and let H be the subgroup of points fixed by σ. We assume σ to be simple, this means that the action of $G \rtimes \{\mathrm{id}, \sigma\}$ on the Lie algebra of G is irreducible. Let $\bar{H}$ be the normalizer of H in G and let X be the wonderful compactification of $G/\bar{H}$ constructed by De Concini and Procesi [5]. We have a G equivariant map $\pi : G/H \longrightarrow X$ factoring through the quotient $G/\bar{H}$.

We are interested in the study of the coordinate ring of the affine variety G/H and in the coordinate rings given by projective immersions of X; they are strictly related through the map π.

Let Ω be the set $\{\mathcal{L} \in \mathrm{Pic}(X) : \pi^*\mathcal{L}$ is isomorphic to the trivial line bundle$\}$; it is a free lattice and any line bundle on Ω has a G linearization. So the vector space $\Gamma_X = \bigoplus_{\mathcal{L}\in \mathrm{Pic}(X)} \Gamma(X, \mathcal{L})$ is a G algebra and we have an equivariant morphism of algebras $\pi^* : \Gamma_X \longrightarrow \mathbb{C}[G/H]$. The complement of $G/\bar{H}$ in X is the union of ℓ smooth divisors $X_1, \dots, X_\ell$ which intersect transversally; and for each divisor there exists a G invariant section s_i of $\Gamma(X, \mathcal{O}(X_i))$ whose associated divisor is equal to X_i. These sections can be normalized in such a way that $\pi^*(s_i) = 1$.

Making use of the results of De Concini and Procesi [5] and Helgason and Vust [6, 14], it is easy to check from the decomposition of Γ_X and of $\mathbb{C}[G/H]$ into G modules that π^* induces an isomorphism

$$\frac{\Gamma_X}{(s_i - 1 \,:\, i = 1, \dots, \ell)} \simeq \mathbb{C}[G/H].$$

Let Δ be the subset of Ω given by the isomorphism classes of $\mathcal{O}(X_1), \dots, \mathcal{O}(X_\ell)$. By [12] it is known that Δ is a simple basis of an irreducible root system Φ and, by [6, 14], it is known that Ω is a set of possible weights of Φ (i.e. Ω is a lattice containing the root lattice and contained in the weight lattice). In particular the submonoid Ω^+ given by line bundes generated by global sections corresponds to the set of dominant weights in Ω w.r.t. Δ.

When Ω^+ is a free monoide, Γ_X and $\mathbb{C}[G/H]$ have a natural choice of generators which correspond through the map π^*; let us denote by $\mathbb{V}^*$ the vector space spanned by such generators. In [2] a SMT in these generators has been constructed. The relation among these generators have not been computed in [2] but there it is proved that such relations can be written in a certain form. Using this rough description one may easily prove the following result:

Proposition. *If Φ is of type A, BC or C and G is simply connected or if Φ is of type B and G is adjoint, then Ω^+ is a free monoid and the relations between the generators of $\mathbb{C}[G/H]$ are quadratic.*

The aim of the paper [1] is to make a further step and give a precise description of the relations among these generators in the cases of the Proposition above by introducing some new symmetry into the problem. More precisely we introduce a group L containing G as the semisimple part of a maximal Levi of G, and we show that the relations in the generators of $\mathbb{C}[G/H]$ may be deduced by the Plücker relations of a Grassmannian of L. In particular the relations are determined by the representation theory of L.

The construction of this extended group L is uniform and goes as follows. Fix a suitable spherical dominant weight ϵ, add a node n_0 to the Dynkin diagram of G and, for all simple roots α, join n_0 with the node n_α of the simple root α by $\epsilon(\alpha^\vee)$ lines, further put an arrow in the direction of n_α if $\epsilon(\alpha^\vee) \geq 2$. In the cases of the Proposition above this extended diagram is of finite or affine type.

Then one takes $\mathcal{L}$ to be the ample generator of the Picard group of the Grassmann variety $\mathcal{G}r = L/P$, where P is the maximal parabolic subgroup corresponding to the new node n_0. We show that in this Grassmann variety there exists a Richardson variety $\mathcal{R}$ such that, on the level of G-modules, $\oplus_{n\geq 0} H^0(\mathcal{R}, \mathcal{L}^n) = \mathbb{C}[G/H]$; in particular $H^0(\mathcal{R}, \mathcal{L}) \simeq \mathbb{V}^*$.

We need to recall a few facts about the generalized Plücker relations. In [8], a basis $\mathbb{F} \subset \Gamma(\mathcal{G}r, \mathcal{L})$ has been constructed together with a partial order "$\geq$", such that the monomials $\mathbb{F}^2 = \{ff' \mid f, f' \in \mathbb{F}, f \leq f'\} \subset \Gamma(\mathcal{G}r, \mathcal{L}^{\otimes 2})$ form a basis. For a pair $f, f' \in \mathbb{F}$ of non comparable elements, let $R_{f,f'} \in \mathsf{S}^2(\Gamma(\mathcal{G}r, \mathcal{L}))$ be the relation expressing the product ff' as a linear combination of elements in $\mathbb{F}^2$. It was shown in [7] that the $R_{f,f'}$'s generate the defining ideal of $\mathcal{G}r \hookrightarrow \mathbb{P}(\Gamma(\mathcal{G}r, \mathcal{L})^*)$; moreover such basis and relations are comparable with Richardson varieties. So in particular there exists a (finite) set $\mathbb{F}_0$ of $\mathbb{F}$ such that the subvariety $\mathcal{R}$ of $\mathcal{G}r$ is defined by the vanishing of all the elements of $\mathbb{F} \setminus \mathbb{F}_0$.

In order to analyse the algebra structure of $\mathbb{C}[G/H]$ we construct a G-equivariant ring homomorphism $\varphi : \Gamma_{\mathcal{G}r} \longrightarrow \mathbb{C}[G/H]$. If Φ is of finite type, then the morphism φ is just the pull back of a canonical G equivariant map $G/H \to \mathcal{G}r$. In the general case, the underlying idea is the same, but the construction is more involved.

Furthermore we can define a G equivariant injection $i : \mathbb{V}^* \hookrightarrow \Gamma(\mathcal{G}r, \mathcal{L})$ such that $\varphi \circ i : \mathbb{V}^* \to \mathbb{C}[G/H]$ is an isomorphism onto the image and $i(\mathbb{V}^*) = \mathbb{F}_0$.

Notice, however, that the relations $R_{f,f'}$ for $f, f' \in \mathbb{F}_0$ involve also elements in $\mathbb{F} - \mathbb{F}_0$. Let $\mathbb{F}_1 \sqcup \mathbb{F}_0$ be the (finite) set of functions appearing in some polynomial $R_{f,f'}$ for $f, f' \in \mathbb{F}_0$. Denote by $\hat{R}_{f,f'} \in \mathsf{S}^2(\mathbb{V}^*)$ the relation obtained from $R_{f,f'}$ by replacing a generator $h \in \mathbb{F}_0$ by $g_h \in \mathbb{G}$ and a generator $h \in \mathbb{F}_1$ by the function $F_h = \varphi(h)$ of $\mathbb{G}$.

Theorem (1)**.** *The relations $\{\hat{R}_{f,f'} | f, f' \in \mathbb{F}_0$ not comparable$\}$ generate the ideal Rel of the relations among the generators $\mathbb{G}$ of $\mathbb{C}[G/H]$.*

Theorem (2)**.** *Consider $\mathbb{G} = \{g_f \mid f \in \mathbb{F}_0\}$ as a partially ordered set with the same partial order as on $\mathbb{F}$. Then $\mathbb{G}$ is a basis of $\mathbb{V}^* \subset \mathbb{C}[G/H]$, the set $\mathbb{SM}_0$ of ordered monomials in $\mathbb{G}$ realizes a standard monomial theory for $\mathbb{C}[G/H]$ and the relations $\hat{R}_{f,f'}$ for the non standard ff' are a set of straightening relations.*

If L is of finite type (or, equivalently, the restricted root system is of type A) we can show that $\mathbb{F}_1$ is given by just two elements f_0, f_1 and that

$$F_{f_0} = F_{f_1} = 1.$$

In particular, in these cases the explicit relations may be summarized in the following description of the coordinate ring of the symmetric variety:

$$\mathbb{C}[G/H] \simeq \frac{\Gamma_{\mathcal{G}r}}{(f_0 = f_1 = 1)}.$$

In some special cases a standard monomial theory for $\mathbb{C}[G/H]$ had been developed before

- for $G/H = SL(n)$, corresponding to the involution $(x, y) \mapsto (y, x)$ of the group $SL(n) \times SL(n)$ and whose restricted root system is of type A, here our construction gives the same as the construction of De Concini, Eisenbud and Procesi [4];
- for G/H ='symmetric quadrics', corresponding to the involution $x \mapsto (x^{-1})^t$ of the group $SL(n)$ and whose restricted root system is of type A, a theory of standard monomials has been introduced by Strickland [13] and Musili [10, 9]; however, we do not know whether their SMT is equivalent to ours;
- for $G/H = Sp(2\,n)$, corresponding to the involution $(x, y) \mapsto (y, x)$ of the group $Sp(2\,n) \times Sp(2\,n)$ and whose restricted root system is of type C, a theory of standard monomials has been introduced by De Concini in [3]. Also in this case we do not know whether this SMT is equivalent to ours.

The results above cover almost all cases with restricted root system of type A; there are only two families missing whose restricted root system is of type A_1 (and hence they are very simple), the 'symplectic quadrics' and an involution of E_6 which is briefly discussed at the end of [1].

Finally we want to stress that the condition on the restricted root system to be of type A, B, C or BC, while looking strong, is actually fulfilled for many involutions. In the Tables in [11] it holds for 12 families of involutions out of a total of 13 families and in 4 exceptional cases out of a total of 12. Moreover one should add to such list of families the involutions such that $G = H \times H$, H is simple and the involution is given by $(x, y) \mapsto (y, x)$; for these cases $\mathbb{C}[G/H]$ is the coordinate ring of H and our condition is equivalent to H equals to $SL(n)$ or $Sp(2n)$ or $SO(2n + 1)$.

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On maximal Poisson-commutative subalgebras in the symmetric algebra of a Lie algebra

Dmitri Panyushev
(joint work with Oksana Yakimova)

Let $\mathfrak{q}$ be an algebraic Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. The symmetric algebra $\mathcal{S}(\mathfrak{q})$ has a natural structure of Poisson algebra, and our goal is to present a sufficient condition for the maximality of Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{q})$ obtained by the argument shift method. In what follows we write "commutative (subalgebra)" in place of "Poisson-commutative". Write $\mathcal{Z}(\mathfrak{q})$ for the Poisson centre of $\mathcal{S}(\mathfrak{q})$ and $\operatorname{ind}\mathfrak{q}$ for the index of $\mathfrak{q}$. Then $\operatorname{trdeg}\mathcal{Z}(\mathfrak{q}) \leqslant \operatorname{ind}\mathfrak{q}$.

It is well known and easily seen that if $\mathcal{A}$ is a commutative subalgebra of $\mathcal{S}(\mathfrak{q})$, then $\operatorname{trdeg}(\mathcal{A}) \leqslant b(\mathfrak{q}) := (\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$. We say that $\mathcal{A}$ is of *maximal dimension*, if the equality holds; $\mathcal{A}$ is called *maximal* if it is maximal with respect to inclusion among the commutative subalgebras. Clearly, any maximal commutative subalgebra contains $\mathcal{Z}(\mathfrak{q})$.

Let $\mathfrak{q}^*_{reg}$ denote the set of all *regular* elements of $\mathfrak{q}^*$. That is,

$$\mathfrak{q}^*_{reg} = \{\xi \in \mathfrak{q}^* \mid \dim\mathfrak{q}{\cdot}\xi \geqslant \dim\mathfrak{q}{\cdot}\eta \text{ for all } \eta \in \mathfrak{q}^*\} .$$

As is well-known, $\mathfrak{q}^*_{reg}$ is a dense open subset of $\mathfrak{q}^*$.

Definition 1. The coadjoint representation of $\mathfrak{q}$ is said to have the *codim–n property* if $\operatorname{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{reg}) \geqslant n$.

The argument shift method. This method was proposed by Mishchenko and Fomenko as a tool for constructing commutative subalgebras of maximal dimension. Let $f \in \mathcal{S}(\mathfrak{q})$ be a polynomial of degree d. For any $\xi \in \mathfrak{q}^*$, we may consider a shift of f in direction of ξ: $f_{a,\xi}(\mu) = f(\mu + a\xi)$, where $a \in \Bbbk$. Expanding the right hand side as polynomial in a, we obtain the expression $f_{a,\xi}(\mu) = \sum_{j=0}^{d} f_\xi^j(\mu) a^j$. Associated with this shift of argument, we obtain the family of polynomials f_ξ^j, where $j = 0, 1, \dots, d-1$. (Since $\deg f_\xi^j = d - j$, the value $j = d$ is not needed.) We will say that the polynomials $\{f_\xi^j\}$ are ξ-*shifts* of f. Notice that $f_\xi^0 = f$ and f_ξ^{d-1} is a linear form on $\mathfrak{q}^*$, i.e., an element of $\mathfrak{q}$. Actually, $f_\xi^{d-1} = (df)_\xi$. There is also an obvious symmetry with respect to ξ and μ: $f_\xi^j(\mu) = f_\mu^{d-j}(\xi)$.

The following observation appears in [2].

Lemma 1. *If $f, g \in \mathcal{Z}(\mathfrak{q})$, then $\{f_{a,\xi}, g_{b,\xi}\} = 0$ for any $\xi \in \mathfrak{q}^*$ and $a, b \in \Bbbk$.*

Given $\xi \in \mathfrak{q}^*$, let $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ denote the subalgebra of $\mathcal{S}(\mathfrak{q})$ generated by the ξ-shifts of all elements of $\mathcal{Z}(\mathfrak{q})$. By Lemma 1, $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ is commutative, and it is a natural candidate on the role of commutative subalgebras of maximal dimension. In [2], it was proved that if $\mathfrak{g}$ is reductive and $\xi \in \mathfrak{g} \simeq \mathfrak{g}^*$ is regular semisimple, then $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{g}))$ is of maximal dimension. Later, Tarasov proved that such subalgebras $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{g}))$ are actually maximal [6].

Bolsinov's criterion. A general criterion for algebras $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ to be of maximal dimension is found by Bolsinov.

Theorem 1 (cf. Bolsinov [1, Theorem 3.1])**.** *Suppose that $(\mathfrak{q}, \mathrm{ad}^*)$ has the codim–2 property and $\operatorname{trdeg} \mathcal{Z}(\mathfrak{q}) = \operatorname{ind} \mathfrak{q}$. Then $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ is of maximal dimension for any $\xi \in \mathfrak{q}^*_{reg}$.*

Notice that the codim–2 property is equivalent to that there is a plane $P \subset \mathfrak{q}^*$ such that all nonzero elements of P are regular. Our main result is

Theorem 2. *Let $\mathfrak{q}$ be an algebraic Lie algebra and $l = \operatorname{ind} \mathfrak{q}$.*

- (i) *Suppose that $(\mathfrak{q}, \mathrm{ad}^*)$ has the codim–2 property and $\mathcal{Z}(\mathfrak{q})$ contains algebraically independent homogeneous polynomials $f_1, \dots, f_l$ such that $\sum_{i=1}^{l} \deg f_i = b(\mathfrak{q})$. Then, for any $\xi \in \mathfrak{q}^*_{reg}$, $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ is a polynomial algebra of Krull dimension $b(\mathfrak{q})$;*
- (ii) *Furthermore, if $(\mathfrak{q}, \mathrm{ad}^*)$ has the codim–3 property, then $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ is a maximal commutative subalgebra of $\mathcal{S}(\mathfrak{q})$.*

The main technical assertion needed in the proof of Theorem 2 is the following extension of Bolsinov's criterion:

Theorem 3. *Suppose $\operatorname{trdeg} \mathcal{Z}(\mathfrak{q}) = \operatorname{ind} \mathfrak{q}$ and $(\mathfrak{q}, \mathrm{ad}^*)$ has the codim–2 property. Let $P \subset \mathfrak{q}^*$ be a plane such that $P \setminus \{0\} \subset \mathfrak{q}^*_{reg}$. Suppose that*

$$(*) \qquad \dim \operatorname{span}\{(df)_{\xi_0} \mid f \in \mathcal{Z}(\mathfrak{q})\} = \operatorname{ind} \mathfrak{q} \quad \text{for some } \xi_0 \in P.$$

Then $\dim \operatorname{span}\{(df)_\eta \mid f \in \mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))\} = b(\mathfrak{q})$ for any linearly independent ξ and η in P.

Applications. Here we point out several classes of Lie algebras satisfying the assumptions of Theorem 2:

1) $\mathfrak{g}$ reductive.

2) Generalised Takiff Lie algebras (this follows from results of Tauvel–Raïs [4]). In particular, Takiff Lie algebras, i.e., the semi-direct products of the form $\mathfrak{q} = \mathfrak{g} \ltimes \mathfrak{g}$, where $\mathfrak{g}$ is reductive.

3) Centralisers of nilpotent elements in $\mathfrak{sl}_n$ (Yakimova).

4) Truncated parabolic subalgebras of maximal index in $\mathfrak{sl}_n$ (Joseph).

5) Some $\mathbb{Z}_2$-contractions of simple Lie algebras [3].

Example 1. Let us show that the *codim–3* property is essential in Theorem 2(ii). Let σ be a Weyl involution of a simple Lie algebra $\mathfrak{g}$. Consider the corresponding $\mathbb{Z}_2$-contraction $\mathfrak{q} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$. Here $\mathfrak{g}_1$ contains a Cartan subalgebra of $\mathfrak{g}$, $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0} \simeq \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, $\operatorname{ind}\mathfrak{q} = \operatorname{ind}\mathfrak{g}$ and hence $b(\mathfrak{q}) = b(\mathfrak{g})$. If $\xi \in \mathfrak{g}_1$ is a regular semisimple element of $\mathfrak{g}$, then it remains regular as element of $\mathfrak{q}^*$. The corresponding commutative subalgebra $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ is a proper subalgebra of $\mathcal{S}(\mathfrak{g}_1)$. (They both are polynomial algebras of Krull dimension $b(\mathfrak{q})$, but any minimal generating system of the former contains the generators of $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0}$, i.e., polynomials of degree > 1.) Since $\mathcal{S}(\mathfrak{g}_1)$ is a (maximal) commutative subalgebra of $\mathcal{S}(\mathfrak{q})$, $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$ is not maximal. In this case one can prove directly that $(\mathfrak{q}, \mathrm{ad}^*)$ has the *codim–2*, but not *codim-3* property.

Some interesting open problems:

1) Is it true that every maximal commutative subalgebra of $\mathcal{S}(\mathfrak{q})$ is of maximal dimension?

2) Under the assumptions of Theorem 2, one obtains the natural morphism $\pi : \mathfrak{q}^* \to \mathbb{A}^{b(\mathfrak{q})} = \operatorname{Spec}(\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q})))$. Is it true that π flat? At least, we can prove it if $\mathfrak{q} = \mathfrak{sl}_n$ and ξ is regular semisimple.

3) Is it possible to quantise the subalgebras of the form $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{q}))$? If $\mathfrak{q}$ is reductive, then the affirmative answer is obtained by Rybnikov [5]. Futhermore, there is a unique quantisation!

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Characters of finite reductive groups: a survey

Cédric Bonnafé

According to the classification, there are three types of finite simple groups: alternating groups, *finite groups of Lie type*, or sporadic groups. The determination of the character table of these groups is a natural and important question. For alternating groups (in fact, for symmetric groups), the answer goes back to Frobenius [3]. The case of sporadic groups has been settled around thirty years ago. For finite groups of Lie type, the problem is not yet solved in complete generality, despite a lot of recent progress. This talk is an attempt to give the flavour of the different methods used for these types of group: algebraic methods (Harish-Chandra theory), geometric methods (Deligne-Lusztig theory, character sheaves).

A finite simple group of Lie type is "almost" a *finite reductive group*, so we have considered in this talk the problem of computing the character table of these groups. A finite reductive group is constructed as follows. Let G be a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$ and let $F : G \to G$ denote an isogeny on G such that some power of F is a Frobenius endomorphism of G. The fixed point subgroup

$$G^F = \{g \in G \mid F(g) = g\}$$

is a finite group, called a finite reductive group.

Example - If $G = GL_n(\overline{\mathbb{F}}_p)$ and if $F : G \to G$, $(a_{ij}) \mapsto (a_{ij}^q)$ where q is some power of p, then $G^F = GL_n(\mathbb{F}_q)$.

The character table of the small linear groups ($GL_n(\mathbb{F}_q)$ for $n \leqslant 4$) has been obtained by Steinberg in 1951 [13], [14]. The case of $GL_n(\mathbb{F}_q)$ was solved in 1955 by Green [4]. In 1968, Srinivasan determined the character table of $Sp_4(\mathbb{F}_q)$ for p odd [12]. The character tables of some other small groups were also obtained using only algebraic methods. Among these algebraic methods, the Harish-Chandra theory reduces the parametrization of irreducible characters to the determination of the so-called *cuspidal* characters.

However, these cuspidal characters remained mysterious until the middle seventies. In 1976, inspired by an example of Drinfeld, Deligne-Lusztig have built a geometric theory (now called *Deligne-Lusztig theory*) for constructing a lot of representations of G^F on the ℓ-adic cohomology of varieties on which G^F acts [2]. In an impressive series of papers, Lusztig obtained finally the parametrization of the irreducible characters of G^F, together with their degrees and an algorithm for computing the character values at semisimple elements [6] (at least for groups with connected centre; the non-connected centre case can be reduced to this one, even though this can lead to technical difficulties, see [1] for details).

Drinfeld's example (1974 ? 1975 ?) - Let $\mathcal{C} = \{(x,y) \in \mathbf{A}^2(\overline{\mathbb{F}}_p) \mid (xy^q - yx^q)^{q-1} = 1\}$. The group $GL_2(\mathbb{F}_q)$ acts linearly on $\mathbf{A}^2(\overline{\mathbb{F}}_p)$ and stabilizes $\mathcal{C}$. The group $\mathbb{F}_{q^2}^\times$ acts also on $\mathbf{A}^2(\overline{\mathbb{F}}_p)$ by multiplication and stabilizes $\mathcal{C}$. Both actions commute. Drinfeld showed that, if θ is a linear character of $\mathbb{F}_{q^2}^\times$ such that $\theta^q \neq \theta$, then the θ-isotypic component of the first étale cohomology group with compact support $H^1(\mathcal{C}, \overline{\mathbb{Q}}_\ell)$ is a cuspidal $GL_2(\mathbb{F}_q)$-module, and all cuspidal $GL_2(\mathbb{F}_q)$-modules can be obtained in this way.

In the course of the parametrization of irreducible characters, Lusztig introduced another orthonormal basis of the space of class functions, whose elements are called the *almost characters*. Computations in small examples suggest that the *almost character table* should be much simpler than the character table. Moreover, the transition matrix between almost characters and irreducible characters has been explicitly given by Lusztig and is relatively simple. So, roughly speaking, determining the almost character table is almost equivalent to determining the character table.

In the middle eighties, Lusztig built another theory for explaining the relative simplicity of the almost character table. This geometric theory (theory of *character sheaves*) involves perverse sheaves and produces a third basis of the space of class functions, whose elements are called the *characteristic functions* of "F-stable" character sheaves. These F-stable character sheaves are parametrized by the same set as the almost characters and Lusztig conjectured that the transition matrix between the almost characters and the characteristic functions should be diagonal. Moreover, Lusztig described a theoretical algorithm for computing these characteristic functions, up to certain normalizations. Therefore, if one can:

- prove Lusztig's conjecture,
- determine the diagonal coefficients involved in this conjecture,
- determine the normalizations involved in Lusztig's algorithm,

then the character table of G^F can be, at least theoretically, determined.

We do not give here a precise list of all cases in which these three problems have been solved but let us describe briefly some of the results. For this, let us assume that p and q are large enough for avoiding a boring list of subtle cases. Then, the question of the normalization of characteristic functions has been solved in many cases by Shoji [10], [11]. Lusztig's conjecture and the computation of the diagonal coefficients has been solved

- for groups with connected centre by Shoji [8];
- for symplectic and orthogonal groups by Waldspurger [15];
- for $SL_n(\mathbb{F}_q)$ by Shoji [9];
- for $SL_n(\mathbb{F}_q)$ and $SU_n(\mathbb{F}_q)$ by Bonnafé [1];

For instance, a complete algorithm for computing the character table of $SL_n(\mathbb{F}_q)$ has been written in [1].

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On the unipotent support of character sheaves and a conjecture of Kawanaka

Meinolf Geck

Let G be a connected reductive group defined over a finite field with q elements. Let $F\colon G \to G$ be the corresponding Frobenius map. Then $G^F = \{g \in G \mid F(g) = g\}$ is a finite group of Lie type. We assume throughout that q is a power of a sufficiently large prime. In the 1980s, Kawanaka showed how to associate to each unipotent element $u \in G^F$ a representation Γ_u of G^F which he called a *generalized Gelfand–Graev representation* (GGGR for short); see [Kaw] for a survey. In the extreme case where u is the identity, we obtain the regular representation of G^F; if u is regular unipotent, then Γ_u is an ordinary Gelfand–Graev representation. It is known, for example, that the latter ones are multiplicity–free—this is no longer true for arbitrary GGGR's. In general, Γ_u is obtained by inducing a certain irreducible representation from the unipotent radical of a parabolic subgroup of G^F, which is associated to u using the Dynkin–Kostant theory of unipotent classes of G.

Lusztig [Lu2] gave a geometric interpretation of the GGGR's in the framework of his theory of *character sheaves*. In particular, he expressed the characters of the GGGR's in terms of intersection cohomology complexes of closures of unipotent classes with coefficients in various local systems. Lusztig [Lu2] used this to define the notion of a "unipotent support" for irreducible representations of G^F and for character sheaves on G. Lusztig's results on the "unipotent support" hold under the above assumpotions on q. In [GM] these restrictions are removed as far as the unipotent support of irreducible representations of G^F is concerned.

Here are some further examples of applications of GGGR's to the representation theory of G^F:

- Since Γ_u is induced from a unipotent representation, it is a projective representation in the sense of modular representation theory, where we work over a discrete valuation ring with residue field of characteristic ℓ where ℓ is a prime not dividing q. This idea has been exploited, for example, in [GHM] where it shown that the reduction modulo ℓ of the unique cuspidal unipotent representation of $\mathrm{GU}_n(q)$ (when it exists) remains irreducible.
- Since Γ_u has an explicit decsription as an induced representation, it is possible (at least in certain cases) to find the smallest field extension over $\mathbb{Q}$ where Γ_u can be realized. This idea can then be used to obtain results about the Schur indices of irreducible representations of G^F; see [Ge2] and the references there.

In his PhD thesis (Université Lyon 1, 2004), Hézard [Hez] used the geometric interpretation of GGGR's to prove *Kawanaka's conjecture* [Kaw], which states that the characters of the GGGR's form a $\mathbb{Z}$-basis of the abelian group of all unipotently supported virtual characters of G^F.

One of the main ingredients in the proof is a result (which may be of independent interest) about the support of character sheaves with non-zero restriction to the unipotent variety of G. Under a certain technical condition—which is formulated in [Ge1], following a suggestion of Lusztig— the restriction of such a character sheaf to its "unipotent support" is just a G-equivariant irreducible local system (up to shift). That a statement of this kind should hold had been indicated much earlier by Lusztig [Lu1].

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Lusztig's conjecture for finite classical groups of characteristic 2

Toshiaki Shoji

Let G be a connected reductive group over a finite filed $\mathbf{F}_q$ of characteristic p with Frobenius map F. Irreducible representations of the finite reductive group G^F over $\bar{\mathbf{Q}}_l$ ($l \neq p$) have been classified by Lusztig in 1980's. The remaining important problem is the computation of the character table of G^F. For this purpose, he founded the theory of character sheaves, and proposed the following conjecture connecting character sheaves with irreducible characters of G^F. Let $\widehat{G}^F$ be the set of F-stable character sheaves. For each $A \in \widehat{G}^F$ we denote by χ_{A,φ_A} the characteristic function of A with respect to an isomorphism $\varphi_A : F^*A \to A$. By the basic result on the theory of character sheaves, the set $\{\chi_{A,\varphi_A} \mid A \in \widehat{G}^F\}$ gives rise to an orthonormal basis of the space $\mathcal{V}$ of class functions of G^F, under a mild condition on p (p is almost good) and under an appropriate normalization of φ_A. Moreover there exists an algorithm of computing χ_{A,φ_A}. (However, this algorithm, which is essentially an algorithm of computing generalized Green functions, involves certain ambiguity of scalars.) On the other hand, he constructed another basis $\{R_x \mid x \in X(G^F)\}$ of $\mathcal{V}$, called almost characters of G^F, based on the classification of irreducible characters. Lusztig's conjecture asserts that these two bases coincide up to root of unities, i.e., $\widehat{G}^F$ is parametrized by the set $X(G^F)$ in such a way that $\chi_{A_x,\varphi_x} = \zeta_x R_x$, where A_x is an F-stable character sheaf corresponding to $x \in X(G^F)$, and ζ_x is a root of unity. Since the transition matrix between almost characters and irreducible characters is explicitly known by its construction, Lusztig's conjecture, followed by the determination of scalars ζ_x and the determination of generalized Green functions, provides us a uniform method of computing the character table of G^F.

In the case where the center of G is connected, Lusztig's conjecture was solved by the author. For finite classical groups Sp_{2n}, SO_{2n+1} and O_{2n}, Waldspurger solved the conjecture (and its generalization for disconnected groups) and determined the scalars ζ_x explicitly, under the condition that p, q are large enough, completing the partial results obtained by Lusztig, and then by the author. The case of SL_n (split type) was solved by the author under the condition that p is large enough, and independently Bonnafé solved the case of SL_n and SU_n under the condition that q is large enough.

In this talk, we discuss the conjecture in the case of classical groups with characteristic 2. The method (based on the idea of Lusztig) employed by Waldspurger cannot be applied to the case of characteristic 2. Instead of it, we appeal to the theory of symmetric spaces over finite fields due to Kawanaka and Lusztig. For a connected reductive group G over $\mathbf{F}_q$, we consider the G^{F^2}-module G^{F^2}/G^F, which is regarded as an analogy of the symmetric space to the case of finite fields. Now G^{F^2}/G^F is isomorphic to $\mathrm{Ind}_{G^F}^{G^{F^2}} 1$, and we define $m_2(f)$ for any class function

f of G^{F^2} by

$$m_2(f) = \langle \mathrm{Ind}_{G^F}^{G^{F^2}} 1, f\rangle_{G^{F^2}} = \langle 1, f\rangle_{G^F} = \frac{1}{|G^F|}\sum_{x\in G^F} f(x).$$

The determination of $m_2(\rho)$ for any irreducible character ρ of G^{F^2} is the starting point for the theory of symmetric spaces, and is an important problem. In the case where G has the connected center, Kawanaka [1] determined $m_2(\rho)$ for all cases, and then Lusztig [2] gave a closed formula for them. In particular, $m_2(R_x)$ can be computed for almost characters R_x. We can show that

Theorem 1. *Let G be a classical group with connected center such that $DG \neq \mathrm{Spin}_N$. Then for any p, the scalars occurring in the Lusztig's conjecture can be determined for G^{F^2}.*

In fact, in the case where G has the connected center, the determination of scalars is reduced to the case where the corresponding character sheaf A_x is a cuspidal character sheaf. Since we know $m_2(R_x)$ (in fact $m_2(R_x) = 1$) for the cuspidal case, it is enough to compute $m_2(\chi_{A_x,\varphi_x})$ for this case. Assume that A_x is a cuspidal character sheaf. Then A_x is given by the intersection cohomology complex as $A_x = \mathrm{IC}(\overline{C}, \mathcal{E})[\dim C]$, where C is a single conjugacy class in G and $\mathcal{E}$ is a certain G-equivariant simple local system on C. In order to describe χ_{A_x,φ_x}, we prepare some general fact. Let C be an F-stable conjugacy class in a connected reductive group G, and take $g \in C^{F^2}$. One can choose g such that F^2 acts trivially on the component group $A_G(g) = Z_G(g)/Z_G^0(g)$. The set of G^{F^2}-conjugacy classes in C^{F^2} is in bijective correspondence with the set of conjugacy classes of $A_G(g)$. We denote by g_a a representative of the G^{F^2}-conjugacy class in C^{F^2} corresponding to the class a in $A_G(g)$. For any irreducible character τ of $A_G(g)$, we define a class function f_τ on G^{F^2} by

$$f_\tau(h) = \begin{cases} \tau(a) & \text{if } h \text{ is } G^{F^2}\text{-conjugate to } g_a, \\ 0 & \text{if } h \notin C^{F^2}. \end{cases}$$

We have the following lemma.

Lemma 2. *Assume that C is F-stable, and $g \in C^F$ such that F acts trivially on $A_G(g)$. Assume further that τ satisfies the property that $\tau(a^2) = 1$ for any $a \in A_G(g)$. Then we have $m_2(f_\tau) = |C^F|/|G^F|$.*

Returning to our setting, let τ be the irreducible character of $A_G(g)$ corresponding to the local system $\mathcal{E}$. It is known, by a deep result of Lusztig, that $\chi_{A_x,\varphi_x} = q^{\mathrm{codim}\, C} f_\tau$ under some choice of φ_x. Since $A_G(g) \simeq (\mathbf{Z}/2\mathbf{Z})^k$, we have $a^2 = 1$ for any a. Thus the lemma can be applied, and the theorem follows.

Next we will pass from G^{F^2} to G^F. Then we have the following result.

Theorem 3. *Let $G = Sp_{2n}$ with $p = 2$. Then the scalars occurring in the Lusztig's conjecture can be determined for G^F.*

The idea of the proof of Theorem 2 is to compare the case G^F and G^{F^2}, and to show that the scalar ζ_x is independent of the extension of $\mathbf{F}_q$. Assume that $G = Sp_{2n}$ with $p = 2$, and let W be the Weyl group of G. In this case, the cuspidal character sheaf has a support on a unipotent class C, and the corresponding almost character is a linear combination of unipotent characters. The set of unipotent characters is in 1:1 correspondence with the set $\Phi_{n,\mathrm{odd}}$ of symbols of rank n and odd defects. We denote by ρ_Λ the unipotent character of G^F corresponding to $\Lambda \in \Phi_{n,\mathrm{odd}}$, and denote by R_Λ the corresponding almost character. Unipotent characters contain the principal series characters. We denote by ρ_E the unipotent (principal series) character corresponding to $E \in W^\wedge$, and by $\Lambda(E)$ the corresponding symbol. Let $V_q = \mathrm{Ind}_{B^F}^{G^F} 1$ be the induced representation from the Borel subgroup B^F. Then V_q is a $G^F \times \mathcal{H}_q$-module, where $\mathcal{H}_q$ is the Iwahori-Hecke algebra corresponding to W, and it is decomposed as $V_q \simeq \bigoplus_{E \in W^\wedge} \rho_E \otimes E(q)$, where $E(q)$ is an irreducible representation of $\mathcal{H}_q$ corresponding to E. We evaluate the character of V_q at $u \in C^F$ and $T_w \in \mathcal{H}_q$ (T_w is a standard basis of $\mathcal{H}_q$ corresponding to $w \in W$). Then we have

$$\begin{aligned} \mathrm{Tr}((u, T_w), V_q) &= \sum_{E \in W^\wedge} \mathrm{Tr}(u, \rho_E)\, \mathrm{Tr}(T_w, E(q)) \\ &= f_{\Lambda_0,w}(q) R_{\Lambda_0}(u) + \sum_{\Lambda \neq \Lambda_0} f_{\Lambda,w}(q) R_\Lambda(u), \end{aligned} \tag{1}$$

where Λ_0 is the cuspidal symbol (i.e., ρ_{Λ_0} is a cuspidal character), $f_{\Lambda,w}(q)$ is a certain polynomial in q (note that we fix $u \in C^F$, and consider all extensions F^k simultaneously), One can show that there exists $w \in W$ such that $f_{\Lambda_0,w}(q) \neq 0$ and that $\mathrm{Tr}((u, T_w), V_q)$ is a polynomial in q. By induction on the rank of G and by our result

Theorem 4 ([3]). *Assume that $G = Sp_{2n}$ with $p = 2$. Then the generalized Green function can be computed explicitly. In particular, for a good choice of unipotent elements, it turns out to be a polynomial in q,*

we see that $R_\Lambda(u)$ is a polynomial in q for any $\Lambda \neq \Lambda_0$. It follows, by (1) and by our choice of w, that $R_{\Lambda_0}(u)$ is also a polynomial in q. This implies that the scalars ζ_x are independent of the extensions F^k, and Theorem 3 follows.

Remark. It seems likely that the method employed for Sp_{2n} will work also for SO_{2n} of split type. This case is under progress. However, the case SO_{2n} of non-split type can not be treated by our method. Also our method is not so useful for exceptional groups since $m_2(R_x) = 0$ for almost all cuspidal character sheaves A_x.

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Character sheaves and Cherednik algebras

Victor Ginzburg
(joint work with Michael Finkelberg)

1. Associated with an integer $n \geq 1$ and an algebraic curve C, there is an interesting family, $\mathsf{H}_{\kappa,\psi}$, of sheaves of associative algebras on $C^{(n)} = C^n/\mathbb{S}_n$, the n-th symmetric power of C. The algebras in question, referred to as *global Cherednik algebras* are natural deformations of the cross-product $\mathscr{D}_\psi(C^n) \rtimes \mathbb{S}_n$, of the sheaf of (twisted) differential operators on C^n and the symmetric group $\mathbb{S}_n$ that acts on $\mathscr{D}_\psi(C^n)$. The algebras $\mathsf{H}_{\kappa,\psi}$ were introduced by P. Etingof as 'global counterparts' of rational Cherednik algebras studied by Etingof-Ginzburg.

The global Cherednik algebra $\mathsf{H}_{\kappa,\psi}$ contains an important *spherical subalgebra* $\mathsf{eH}_{\kappa,\psi}\mathsf{e}$, where e denotes the symmetriser idempotent in the group algebra of the group $\mathbb{S}_n$. We prove that the algebra $\mathsf{eH}_{\kappa,\psi}\mathsf{e}$ may be obtained as a quantum Hamiltonian reduction of $\mathscr{D}_{n\kappa,\psi}(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})$, a sheaf of twisted differential operators on $\mathrm{rep}^n_C \times \mathbb{P}^{n-1}$.

Our result provides a strong link between categories of $\mathscr{D}_{n\kappa,\psi}(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})$-modules and $\mathsf{H}_{\kappa,\psi}$-modules, respectively. Specifically, following the strategy of Gan-Ginzburg we construct an exact functor

$$\mathbb{H}: \ \mathscr{D}_{n\kappa,\psi}(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})\text{-}\mathsf{mod} \ \longrightarrow \ \mathsf{H}_{\kappa,\psi}\text{-}\mathsf{mod}, \tag{1}$$

called the functor of Hamiltonian reduction.

2. In mid 80's, G. Lusztig introduced an important notion of *character sheaf* on a reductive algebraic group G. In more detail, write $\mathfrak{g}$ for the Lie algebra of G and use the Killing form to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let $\mathcal{N} \subset \mathfrak{g}^*$ be the image of the set of nilpotent elements in $\mathfrak{g}$, and let $G \times \mathcal{N} \subset G \times \mathfrak{g}^* = T^*G$ be the *nil-cone* in the total space of the cotangent bundle on G. Recall further that, associated with any perverse sheaf M on G, one has its characteristic variety $SS(M) \subset T^*G$. A character sheaf is, by definition, an $\mathrm{Ad}\, G$-equivariant perverse sheaf M, on G, such that the corresponding characteristic variety is nilpotent, i.e., such that we have $SS(M) \subset G \times \mathcal{N}$. We will be interested in the special case $G = GL_n$. Motivated by the geometric Langlands conjecture, G. Laumon generalized the notion of character sheaf on GL_n to the 'global setting' involving an arbitrary smooth algebraic curve. Given such a curve C, Laumon replaces the adjoint quotient stack $G/\mathrm{Ad}\, G$ by Coh^n_C, a certain stack of length n coherent sheaves on C. He then defines a *global nilpotent subvariety* of the cotangent bundle $T^*Coh^n_C$, and considers the class of perverse sheaves M on Coh^n_C such that $SS(M)$ is contained in the global nilpotent subvariety. In this paper, we introduce character sheaves on $\mathrm{rep}^n_C \times \mathbb{P}^{n-1}$. Here, the scheme rep^n_C is an appropriate Quot scheme of length n sheaves on C, a close cousin of Coh^n_C, and $\mathbb{P}^{n-1}$ is an $n-1$-dimensional projective space. We define a version of 'global nilpotent variety' $\mathbb{M}_{\mathsf{nil}} \subset T^*(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})$, and introduce a class of $\mathscr{D}$-modules on $\mathrm{rep}^n_C \times \mathbb{P}^{n-1}$, called *character $\mathscr{D}$-modules*, which have a nilpotent characteristic variety, i.e., are such that $SS(M) \subset \mathbb{M}_{\mathsf{nil}}$. The group $G = GL_n$ acts on both rep^n_C and $\mathbb{P}^{n-1}$ in a natural way. In analogy with the theory studied by Lusztig and Laumon, perverse sheaves associated to character $\mathscr{D}$-modules via the Riemann-Hilbert correspondence are locally constant along G-diagonal orbits in $\text{rep}_C^n \times \mathbb{P}^{n-1}$. A very special feature of the G-variety $\text{rep}_C^n \times \mathbb{P}^{n-1}$ is that the corresponding nilpotent variety, $\mathbb{M}_{\mathsf{nil}}$, turns out to be a *Lagrangian* subvariety in $T^*(\text{rep}_C^n \times \mathbb{P}^{n-1})$. This follows from a geometric result saying that the group GL_n acts diagonally on $\mathcal{N} \times \mathbb{C}^n$ with *finitely many* orbits. These orbits may be parametrised by the pairs (λ, μ), of arbitrary partitions $\lambda = \lambda_1 + \ldots + \lambda_p$ and $\mu = \mu_1 + \ldots + \mu_q$, with total sum $\lambda + \mu = n$ (R. Travkin).

3. Character $\mathscr{D}$-modules play an important role in representation theory of the global Cherednik algebra $\mathsf{H}_{\kappa,\psi}$. In more detail, there is a natural analogue, $\mathcal{O}(\mathsf{H}_{\kappa,\psi})$, of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for the global Cherednik algebra. We show that the Hamiltonian reduction functor (1) sends character $\mathscr{D}$-modules to objects of the category $\mathcal{O}(\mathsf{H}_{\kappa,\psi})$, moreover, the latter category gets identified, via the functor $\mathbb{H}$, with a quotient of the former by the Serre subcategory Ker$\mathbb{H}$.

In the special case of the curve $C = \mathbb{C}^\times$, the global Cherednik algebra reduces to H_κ, the *trigonometric* Cherednik algebra of type $\mathbf{A}_{n-1}$. and the sheaves considered by Laumon become Lusztig's character sheaves on the group GL_n. Similarly, our character $\mathscr{D}$-modules become (twisted) $\mathscr{D}$-modules on $GL_n \times \mathbb{P}^{n-1}$.

Given a character sheaf on GL_n in the sense of Lusztig, one may pull-back the corresponding $\mathscr{D}$-module via the first projection $GL_n \times \mathbb{P}^{n-1} \to GL_n$. The resulting $\mathscr{D}$-module on $GL_n \times \mathbb{P}^{n-1}$ is a character $\mathscr{D}$-module in our sense. However, there are many other quite interesting character $\mathscr{D}$-modules on $GL_n \times \mathbb{P}^{n-1}$ which do not come from Lusztig's character sheaves on GL_n.

Sometimes, it is more convenient to replace GL_n by its subgroup SL_n. In that case, we prove that *cuspidal* character $\mathscr{D}$-modules correspond, via the Hamiltonian functor, to finite dimensional representations of the corresponding Cherednik algebra.

Quantum cohomology of homogeneous varieties: a survey

Harry Tamvakis

Let G be a semisimple complex algebraic group and P a parabolic subgroup of G. The homogeneous space $X = G/P$ is a projective complex manifold. My aim in this lecture is to survey what is known about the (small) quantum cohomology ring of X. Here is a brief historical introduction, with no claim of completeness. About 15 years ago, ideas from string theory and mirror symmetry led physicists to make some startling predictions in enumerative algebraic geometry (see e.g. [18, 19]). This involved the notion of *Gromov-Witten invariants*, which are certain natural intersection numbers on the moduli space of degree d holomorphic maps from a compact complex curve C of genus g (with n marked points) to X.

When the genus g is arbitrary, computing these invariants is a rather difficult problem. The case when X is a point was a conjecture of Witten, proved by Kontsevich. Later, Okounkov and Pandharipande examined the case when $X =$

$\mathbb{P}^1$. The genus zero theory led to the so called big quantum cohomology ring, and to work on mirror symmetry by Givental, Yau, and their collaborators. I will specialize further to the case of $n = 3$ marked points, when we obtain the small quantum cohomology ring $\mathrm{QH}^*(X)$. Although much has been understood here, still many open questions remain.

1. Cohomology of G/B and G/P

We begin with the Bruhat decomposition $G = \bigcup_{w\in W} BwB$, where B is a Borel subgroup of G, and W is the Weyl group. The Schubert cells in $X = G/B$ are the orbits of B on X; their closures $Y_w = \overline{BwB/B}$ are the Schubert varieties. For each $w \in W$, let $w^\vee = w_0 w$ and $X_w = Y_{w^\vee}$, so that the complex codimension of X_w is given by the length $\ell(w)$. Using Poincaré duality, we obtain the Schubert classes $\sigma_w = [X_w] \in \mathrm{H}^{2\ell(w)}(X)$, which form a free $\mathbb{Z}$-basis of $\mathrm{H}^*(X)$. This gives the additive structure of the cohomology ring.

For the multiplicative structure, if the group W is generated by the simple reflections s_i for $1 \leq i \leq r$, we obtain the Schubert divisor classes $\sigma_{s_i} \in \mathrm{H}^2(X)$ which generate the ring $\mathrm{H}^*(X)$. Moreover, we have Borel's presentation [2]

$$\mathrm{H}^*(G/B, \mathbb{Q}) = \mathrm{Sym}(\Lambda(B))/\mathrm{Sym}(\Lambda(B))^W_{>0}$$

where $\Lambda(B)$ denotes the character group of B, and $\mathrm{Sym}(\Lambda(B))^W_{>0}$ is the ideal generated by W-invariants of positive degree in the symmetric algebra of $\Lambda(B)$.

For any parabolic subgroup P, if W_P is the corresponding subgroup of the Weyl group W, we have $\mathrm{H}^*(G/P) = \bigoplus \mathbb{Z}\,\sigma_{[w]}$, the sum over all cosets $[w] \in W/W_P$. The corresponding Borel presentation has the form

$$\mathrm{H}^*(G/P, \mathbb{Q}) = \mathrm{Sym}(\Lambda(B))^{W_P}/\mathrm{Sym}(\Lambda(B))^W_{>0}.$$

2. Quantum cohomology of G/B and G/P

Let r be the rank of $\mathrm{H}^2(G/P)$, and $q = (q_1, \ldots, q_r)$ a finite set of formal variables. The ring $\mathrm{QH}^*(X)$ is a graded $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathrm{H}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of each variable q_i is given by $\deg(q_i) = \int_X \sigma_{s_i^\vee} \cdot c_1(TX)$. Note that our grading of cohomology classes will be with respect to their *complex* codimension.

A holomorphic map $f : \mathbb{P}^1 \to X$ has degree $d = (d_1, \ldots, d_r)$ if $f_*[\mathbb{P}^1] = \sum_i d_i \sigma_{s_i^\vee}$ in $\mathrm{H}_2(X)$. The quantum product in $\mathrm{QH}^*(X)$ is defined by

$$\sigma_u\,\sigma_v = \sum \langle \sigma_u, \sigma_v, \sigma_{w^\vee}\rangle_d\, \sigma_w\, q^d \tag{1}$$

where the sum is over $d \geq 0$ and elements $w \in W$ such that $\ell(w) = \ell(u) + \ell(v) - \sum d_i \deg(q_i)$. The nonnegative integer $\langle \sigma_u, \sigma_v, \sigma_{w^\vee}\rangle_d$ is a 3-point, genus 0 Gromov-Witten invariant, and can be defined enumeratively as the number of degree d holomorphic maps $f : \mathbb{P}^1 \to X$ such that $f(0) \in \widetilde{X}_u$, $f(1) \in \widetilde{X}_v$, $f(\infty) \in \widetilde{X}_{w^\vee}$, where the tilde in $\widetilde{X}_u$, $\widetilde{X}_v$, $\widetilde{X}_{w^\vee}$ means that the respective Schubert varieties are taken to be in general position. In most cases, counting the number of such maps f is equivalent to counting their images, which are degree d rational curves in X.

Alternatively, one may realize $\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d$ as

$$\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d = \int_{\overline{M}_{0,3}(X,d)} \mathrm{ev}_1^*(\sigma_u)\, \mathrm{ev}_2^*(\sigma_v)\, \mathrm{ev}_3^*(\sigma_{w^\vee}),$$

an intersection number on Kontsevich's moduli space $\overline{M}_{0,3}(X,d)$ of stable maps. A stable map is a degree d morphism $f : (C, p_1, p_2, p_3) \to X$, where C is a tree of $\mathbb{P}^1$'s with three marked smooth points p_1, p_2, and p_3, and the stability condition is such that the map f admits no automorphisms. The evaluation maps $\mathrm{ev}_i : \overline{M}_{0,3}(X,d) \to X$ are given by $\mathrm{ev}_i(f) = f(p_i)$.

One observes that each degree zero Gromov-Witten invariant

$$\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_0 = \#\, \widetilde{X}_u \cap \widetilde{X}_v \cap \widetilde{X}_{w^\vee} = \int_X \sigma_u\, \sigma_v\, \sigma_{w^\vee}$$

is a classical structure constant in the cohomology ring of X, showing that $\mathrm{QH}^*(X)$ is a deformation of $\mathrm{H}^*(X)$. The surprising point is that the product (1) is associative; see [9] for a proof of this. We will be interested in extending the classical understanding of Schubert calculus on G/P to the quantum cohomology ring.

3. The Grassmannian $G(m,N)$

One of the first spaces where this story was worked out was the Grassmannian $X = G(m,N) = SL_N/P_m$ of m dimensional linear subspaces of $\mathbb{C}^N$. Here the Weyl groups $W = S_N$, $W_{P_m} = S_m \times S_n$, where $n = N - m$, and there is a bijection between the coset space W/W_P and the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ whose Young diagram is contained in an $m \times n$ rectangle $R(m,n)$. The latter objects will index the Schubert classes in X. Since $\mathrm{H}^2(X)$ has rank one, there is only one deformation parameter q, of degree N in $\mathrm{QH}^*(X)$.

3.1. **Presentation** [17, Siebert and Tian]**.** There is a universal short exact sequence of vector bundles

$$0 \to S \to E \to Q \to 0$$

over $G(m,N)$, with S the tautological rank m subbundle of the trivial vector bundle E, and $Q = E/S$ the rank n quotient bundle. Then

$$\mathrm{QH}^*(G(m,N)) = \mathbb{Z}[c(S), c(Q), q] / \langle c(S)c(Q) = 1 + (-1)^m q \rangle$$
$$= \mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n, q]^{S_m \times S_n} / \langle e_r(x,y) = 0, r < N;\ e_N(x,y) = (-1)^m q \rangle\,.$$

The new relation $c_m(S^*)c_n(Q) = q$ is equivalent to $\sigma_{1^m}\sigma_n = q$, and contains the enumerative geometric statement that $\langle \sigma_{1^m}, \sigma_n, [\mathrm{pt}] \rangle_1 = 1$. This latter can be checked directly from geometry, or deduced from the Pieri rule which follows.

3.2. **Quantum Pieri rule** [1, Bertram]**.** The special Schubert classes $\sigma_p = c_p(Q)$ for $1 \le p \le n$ generate the ring $\mathrm{QH}^*(X)$. Moreover, we have

$$\sigma_p\, \sigma_\lambda = \sum_{\mu \subset R(m,n)} \sigma_\mu + \sum_{\mu \subset R(m+1,n)} \sigma_{\widehat{\mu}}\, q,$$

where both sums are over μ obtained from λ by adding p boxes, no two in a column, and $\widehat{\mu}$ is obtained from μ by removing a hook of length N from its rim.

This means that the only $\mu \subset R(m+1,n)$ that contribute to the second sum are those which include the northeast-most and southwest-most corners in their diagram. For example, in $X = G(3,8)$, we have $\sigma_3\,\sigma_{422} = \sigma_{542} + \sigma_{21}\,q + \sigma_{111}\,q$.

3.3. **Quantum Littlewood-Richardson numbers.** These are the Gromov-Witten invariants in the equation

$$\sigma_\lambda\,\sigma_\mu = \sum_{d,\nu} C^{\nu,d}_{\lambda,\mu}\,\sigma_\nu\,q^d \tag{2}$$

in $\mathrm{QH}^*(G(m,N))$. The quantum Pieri rule gives an algorithm to compute the quantum Littlewood-Richardson numbers $C^{\nu,d}_{\lambda,\mu}$, however not a positive combinatorial rule extending the classical one. A puzzle based conjectural rule for these numbers was given by Buch, Kresch, and the author [4], and recently a 'geometric' and positive combinatorial rule was proved by Coskun.

As one of the many combinatorial offshoots of this theory, I mention a clever reformulation of the algorithm determining the numbers $C^{\nu,d}_{\lambda,\mu}$ due to Postnikov [15]. When $d = 0$, if $s_\mu(x_1,\ldots,x_m)$ denotes the Schur polynomial in m variables, and we alter the summation in (2) to be over μ instead of ν, then we get

$$\sum_\mu C^{\nu,0}_{\lambda,\mu}\,s_\mu(x_1,\ldots,x_m) = \sum_{T \text{ on } \nu/\lambda} x^T$$

where the second sum is over all semistandard Young tableaux T on the skew shape ν/λ with entries no greater than m. For each fixed $d \geq 0$, Postnikov defines a *toric shape* $\nu/d/\lambda$ which is a subset of the torus $T(m,n)$, the rectangle $R(m,n)$ with opposite sides identified; the role of d in the description of the shape is a shift by d squares in the southeast direction. One then shows that

$$\sum_\mu C^{\nu,d}_{\lambda,\mu}\,s_\mu(x_1,\ldots,x_m) = \sum_{T \text{ on } \nu/d/\lambda} x^T,$$

the second sum over all Young tableaux on the toric shape $\nu/d/\lambda$. One nice application of this result is determining exactly which powers q^d occur in a quantum product $\sigma_\lambda\,\sigma_\mu$ with a non-zero coefficient.

4. Flag varieties for SL_n

We set $X = SL_n/B$ to be the complex manifold parametrizing complete flags of linear subspaces $0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n$, with $\dim E_i = i$. We then have $\mathrm{QH}^*(X) = \bigoplus \mathbb{Z}\,\sigma_w\,q^d$, the sum over permutations $w \in S_n$ and multidegrees d, while $q^d = q_1^{d_1}\cdots q_{n-1}^{d_{n-1}}$, with each variable q_i of degree 2.

4.1. **Presentation** [11, Givental and Kim]**.** Let E_i also denote the corresponding tautological vector bundle over X, and $x_i = -c_1(E_i/E_{i-1})$. The Borel presentation of $\mathrm{H}^*(X)$ is a quotient of $\mathbb{Z}[x_1,\ldots,x_n]$ by the ideal generated by the elementary symmetric polynomials $e_i(x_1,\ldots,x_n)$ for $1 \leq i \leq n$. For the quantum cohomology ring, we have

$$\mathrm{QH}^*(X) = \mathbb{Z}[x_1,\ldots,x_n,q_1,\ldots,q_{n-1}]/\langle E_i(x,q) = 0,\ 1 \leq i \leq n\rangle$$

where the *quantum elementary symmetric polynomials* $E_i(x, q)$ are the coefficients of the characteristic polynomial

$$\mathtt{det}(A + tI_n) = \sum_{i=0}^{n} E_i(x, q) t^{n-i}$$

of the matrix

$$A = \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix}.$$

4.2. **Quantum Monk/Chevalley formula** [8, Fomin, Gelfand, and Postnikov]**.** This is a formula for the quantum product $\sigma_{s_i} \sigma_w$. It was extended by Peterson to any G/B; see section 5.2.

4.3. **Quantum cohomology of** SL_n/P**.** Ciocan-Fontanine [7] obtained analogues of the above results for any homogeneous space for SL_n. We remark that quantum cohomology is *not* functorial, and so one has to work on each parabolic subgroup P separately. The conclusion of this discussion is that the quantum cohomology of SL_n flag varieties is fairly well understood; one can also recognize each Schubert class σ_w in the presentation of $\mathrm{QH}^*(X)$ using *quantum Schubert polynomials* [8].

5. Lie types other than A

5.1. **General** G/B**.** A presentation of $\mathrm{QH}^*(G/B)$ for general G was given by Kim[12]. It is notable because the relations come from the integrals of motion of the Toda lattice associated to the Langlands dual group $G^\vee$. In his 1997 MIT lectures, D. Peterson announced a presentation of $\mathrm{QH}^*(G/P)$ for any parabolic subgroup P of G. This result remains unpublished; moreover, it is difficult to relate Peterson's presentation to the Borel presentation of $\mathrm{H}^*(G/P)$ given earlier. For work in this direction when $G = SL_n$, see Rietsch [16], which includes a connection with the theory of total positivity. Recently, Cheong [6] has made a corresponding study of the Grassmannians LG and OG of maximal isotropic subspaces.

5.2. **Peterson's quantum Chevalley formula** [10]**.** Let R be the root system for G and R^+ the positive roots. For $\alpha \in R^+$ we denote by s_α the corresponding reflection in W. To any root α there corresponds the coroot $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ in the Cartan subalgebra of Lie(G). For any positive coroot $\alpha^\vee$ with $\alpha^\vee = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$, define $|\alpha^\vee| = \sum_i d_i$ and $q^{\alpha^\vee} = \prod_i q_i^{d_i}$. Then we have

$$\sigma_{s_i} \cdot \sigma_w = \sum_{\ell(ws_\alpha) = \ell(w)+1} \langle \omega_i, \alpha^\vee \rangle \, \sigma_{ws_\alpha} + \sum_{\ell(ws_\alpha) = \ell(w) - 2|\alpha^\vee| + 1} \langle \omega_i, \alpha^\vee \rangle \, \sigma_{ws_\alpha} \, q^{\alpha^\vee}$$

in $\mathrm{QH}^*(G/B)$, where the sums are over $\alpha \in R^+$ satisfying the indicated conditions, and ω_i is the fundamental weight corresponding to s_i. Using this result, one can recursively compute the Gromov-Witten invariants on any G/B space.

5.3. **Peterson's comparison theorem** [20]**.** Every Gromov-Witten invariant $\langle\sigma_u, \sigma_v, \sigma_w\rangle_d$ on G/P is equal to a corresponding number $\langle\sigma_{u'}, \sigma_{v'}, \sigma_{w'}\rangle_{d'}$ on G/B. The exact relationship between the indices is explicit, but not so easy to describe; see [20] for further details and a complete proof. Combining this with the previous result allows one to compute any Gromov-Witten invariant on any G/P space.

5.4. **Grassmannians in other Lie types** [13, 14, Kresch and T.]**.** Let $X = Sp_{2n}/P_n$ be the Grassmannian $\mathrm{LG}(n, 2n)$ parametrizing Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with a symplectic form. The Schubert varieties on LG are indexed by strict partitions λ with $\lambda_1 \le n$, and the degree of q this time is $n+1$.

5.4.1. *Presentation of* $\mathrm{QH}^*(\mathrm{LG})$*.* If $0 \to S \to E \to Q \to 0$ denotes the tautological sequence of vector bundles over LG, then we may identify Q with S^*, and the special Schubert classes $\sigma_p = c_p(S^*)$ again generate the ring $\mathrm{QH}^*(\mathrm{LG})$. The Whitney sum formula $c_t(S)c_t(S^*) = 1$ gives the classical relations

$$(1 - \sigma_1 t + \sigma_2 t^2 - \cdots)(1 + \sigma_1 t + \sigma_2 t^2 + \cdots) = 1$$

or equivalently $\sigma_r^2 + 2\sum_{i=1}^{n-r}(-1)^i \sigma_{r+i}\sigma_{r-i} = 0$ for $1 \le r \le n$. For the quantum ring, we have the presentation

$$\mathrm{QH}^*(\mathrm{LG}) = \mathbb{Z}[\sigma_1, \ldots, \sigma_n, q]/\langle \sigma_r^2 + 2\sum_{i=1}^{n-r}(-1)^i \sigma_{r+i}\sigma_{r-i} = (-1)^{n-r}\sigma_{2r-n-1}q\rangle.$$

Observe that if we identify q with $2\sigma_{n+1}$, the above equations become classical relations in the cohomology of $\mathrm{LG}(n+1, 2n+2)$. Looking for the enumerative geometry which lies behind this algebraic fact, we find that

$$\langle\sigma_\lambda, \sigma_\mu, \sigma_\nu\rangle_1 = \int_{\mathrm{IG}(n-1,2n)} \sigma_\lambda^{(1)}\, \sigma_\mu^{(1)}\, \sigma_\nu^{(1)} = \frac{1}{2}\int_{\mathrm{LG}(n+1,2n+2)} \sigma_\lambda\, \sigma_\mu\, \sigma_\nu.$$

The first equality is an example of a "quantum = classical" result; here the isotropic Grassmannian $\mathrm{IG}(n-1, 2n) = Sp_{2n}/P_{n-1}$ is the parameter space of lines on $\mathrm{LG}(n, 2n)$, and $\sigma_\lambda^{(1)}$, $\sigma_\mu^{(1)}$, $\sigma_\nu^{(1)}$ are certain Schubert classes in $\mathrm{H}^*(\mathrm{IG})$.

5.4.2. *Symmetries of Gromov-Witten invariants.* Kresch and the author [14] also studied the quantum cohomology of the maximal orthogonal Grassmannians $\mathrm{OG} = \mathrm{OG}(n, 2n+1) = SO_{2n+1}/P_n$. There are quantum Pieri rules for LG and OG which extend the known ones in classical cohomology. Using them, one shows that the Gromov-Witten invariants on these spaces enjoy a $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry, which implies that the tables of Gromov-Witten invariants for $\mathrm{LG}(n-1, 2n-2)$ and $\mathrm{OG}(n, 2n+1)$ coincide, after applying an involution. Similar symmetries were observed by Postnikov [15] and others for type A Grassmannians; recently, Chaput, Manivel, and Perrin have extended them to all hermitian symmetric spaces.

The original proofs of all the above results relied on intersection theory on $\overline{M}_{0,3}(X,d)$ or Quot schemes. A technical breakthrough was found by Buch [3]; his 'Ker/Span' ideas greatly simplified most of the arguments involved. Using this approach, Buch, Kresch, and the author have made a corresponding analysis of $\mathrm{QH}^*(G/P)$ when G is a classical group and P any maximal parabolic subgroup.

6. "Quantum = Classical" results

The title refers to theorems which equate any Gromov-Witten invariant on a hermitian symmetric Grassmannian with a classical triple intersection number on a related homogeneous space. These results were discovered in joint work of the author with Buch and Kresch [4]. More recently, Chaput, Manivel, and Perrin [5] have presented the theory in a uniform framework which includes the exceptional symmetric spaces E_6/P_6 and E_7/P_7. There follows a summary of this story.

Assume that $X = G/P$ is a hermitian symmetric space. For $x, y \in X$, let $\delta(x,y)$ be the minimum $d \geq 0$ such that there exists a rational curve of degree d passing through the points x and y. The invariant $\delta(x,y)$ parametrizes the G orbits in $X \times X$. If $\delta(x,y) = d$, then define $Z(x,y) = \overline{\bigcup C_{x,y}}$, where the union is over all rational curves $C_{x,y}$ of degree d through the points x and y. Then $Z(x,y)$ is a homogeneous Schubert variety X_{w_d} in X. Now G acts transitively on the set of translates $\{gX_{w_d} \mid g \in G\}$; therefore the variety Y_d parametrizing all such X_{w_d} in X is a homogeneous space G/P_d for some (generally non maximal) parabolic subgroup P_d of G. To each Schubert class σ_λ in $\mathrm{H}^*(X)$ there corresponds naturally a Schubert class $\sigma_\lambda^{(d)}$ in $\mathrm{H}^*(Y_d)$. Then we have

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{Y_d} \sigma_\lambda^{(d)}\, \sigma_\mu^{(d)}\, \sigma_\nu^{(d)}.$$

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Equivariant Gromov-Witten invariants of Grassmannians

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(joint work with Leonardo Mihalcea)

This project continues work with A. Kresch and H. Tamvakis. Let $X = \mathrm{Gr}(m, \mathbb{C}^n)$ be the Grassmannian of m-dimensional vector subspaces V in $\mathbb{C}^n$, and set $k = n - m$. Each integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$ with $\lambda_1 \leq k$ defines a Schubert variety X_λ in X, consisting of all points $V \in X$ for which $\dim(V \cap \mathbb{C}^{k+i-\lambda_i}) \geq i$ for $1 \leq i \leq m$. The codimension of this Schubert variety is $|\lambda| = \sum \lambda_i$, and it defines a Schubert class $[X_\lambda]$ in $H^{2|\lambda|}(X;\mathbb{Z})$. The Schubert classes form an additive basis for the cohomology ring $H^*(X) = H^*(X;\mathbb{Z})$.

Define a rational curve in X to be the image C of any morphism of varieties $\mathbb{P}^1 \to X$. The degree of the curve is defined by $\deg(C) = \int_X [C] \cdot [X_{(1)}]$. Notice that a single point in X is considered as a rational curve of degree zero, according to this definition.

Given three partitions λ, μ, and ν such that $|\lambda|+|\mu|+|\nu| = \dim(X)+nd$ for some degree d, the (three-point, genus zero) Gromov-Witten invariant $\langle X_\lambda, X_\mu, X_\nu \rangle_d$ is defined as the number of rational curves of degree d that meet all of the Schubert varieties $g_1.X_\lambda$, $g_2.X_\mu$, and $g_3.X_\nu$, where $g_1, g_2, g_3 \in \mathrm{GL}_n(\mathbb{C})$ are general group elements that are fixed in advance. The small quantum ring of X is the $\mathbb{Z}[q]$-algebra defined by $\mathrm{QH}(X) = H^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module, and with multiplicative structure given by the quantum product

$$[X_\lambda] * [X_\mu] = \sum_{\nu, d \geq 0} \langle X_\lambda, X_\mu, X_{\nu^\vee} \rangle_d \, [X_\nu] \, q^d \,,$$

where $\nu^\vee = (k-\nu_m, \dots, k-\nu_1)$ is the Poincare dual partition of ν. We refer to [5] for details and references.

The kernel and span of a curve $C \subset X$ is defined as the intersection and linear span of the m-planes $V \subset \mathbb{C}^n$ corresponding to points on the curve:

$$\mathrm{Ker}(C) = \bigcap_{V \in C} V \quad \text{and} \quad \mathrm{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n .$$

These definitions were introduced in [2], where they were used to given elementary proofs of Bertram's structure theorems for the quantum ring of X [1]. If $C \subset X$ is a rational curve, then $\dim \mathrm{Ker}(C) \geq m - \deg(C)$ and $\dim \mathrm{Span}(C) \leq m + \deg(C)$, and these inequalities are satisfied with equality for general curves.

In [3], the Gromov-Witten invariants of X were computed by using the two-step flag variety $Y_d = \mathrm{Fl}(m-d, m+d; \mathbb{C}^n) = \{(A,B) \mid A^{m-d} \subset B^{m+d} \subset \mathbb{C}^n\}$, which is the variety of kernel-span pairs of expected dimension for curves of degree d. Notice that the kernel-span pair of any general curve meeting the Schubert variety $X_\lambda \subset X$ must lie on the modified Schubert variety $Y_\lambda = \{(A,B) \in Y_d \mid \exists\, V \in X_\lambda : A \subset V \subset B\}$.

Theorem (B-Kresch-Tamvakis)**.** *The map $C \mapsto (\mathrm{Ker}(C), \mathrm{Span}(C))$ gives an explicit bijection between the rational curves $C \subset X$ of degree d meeting $g_1.X_\lambda$, $g_2.X_\mu$, and $g_3.X_\nu$, and the points in the intersection $g_1.Y_\lambda \cap g_2.Y_\mu \cap g_3.Y_\nu \subset Y_d$.*

As a consequence, any three-point, genus zero Gromov-Witten invariant on a Grassmannian can be expressed as a classical triple intersection number on a two-step flag variety. The theorem has analogues for Lagrangian and maximal orthogonal Grassmannians [3]. More recent work by Chaput, Manivel, and Perrin has generalized the theorem for all minuscule and cominuscule homogeneous spaces [4].

Let $\overline{M} = \overline{M}_{0,3}(X,d)$ denote Kontsevich's moduli space of 3-pointed stable maps $f : C \to X$ of degree d [8]. This means that C is a tree of projective lines with three marked non-singular points, $f_*[C] = d\,[\text{line}]$, and any component of C that is mapped to a single point by f must have at least 3 points that are marked or singular. The evaluation map $\mathrm{ev}_i : \overline{M} \to X$ sends a stable map f to the image of the i-th marked point. The Gromov-Witten invariants on X are equal to the intersection numbers

$$\langle X_\lambda, X_\mu, X_\nu \rangle_d = \int_{\overline{M}} \mathrm{ev}_1^*[X_\lambda] \cdot \mathrm{ev}_2^*[X_\mu] \cdot \mathrm{ev}_3^*[X_\nu] .$$

Let $T \subset \mathrm{GL}_n(\mathbb{C})$ be the torus of diagonal matrices. Then each Schubert variety $X_\lambda \subset X$ is T-stable, and the same is true for the opposite Schubert variety $X_\lambda^{\mathrm{op}} = w_0 X_\lambda$, where $w_0 \in S_n$ is the longest permutation. These varieties therefore define T-equivariant Schubert classes $[X_\lambda], [X_\lambda^{\mathrm{op}}] \in H_T^*(X)$. In contrast to ordinary Schubert classes, the equivariant class of X_λ is different from the class of X_λ^{op}.

Givental and Kim [6] defined equivariant Gromov-Witten invariants of X by

$$\langle X_\lambda, X_\mu, X_\nu^{\mathrm{op}}\rangle_d^T = \int_{\overline{M}}^T \mathrm{ev}_1^*[X_\lambda]\cdot \mathrm{ev}_2^*[X_\mu]\cdot \mathrm{ev}_3^*[X_\nu^{\mathrm{op}}]\,,$$

where $\int_{\overline{M}}^T$ denotes the proper pushforward $H_T(\overline{M}) \to H_T(\text{point}) = \mathbb{Z}[t_1,\dots,t_n]$. The invariant $\langle X_\lambda, X_\mu, X_\nu^{\mathrm{op}}\rangle_d^T$ is a homogeneous polynomial of degree $|\lambda| + |\mu| + |\nu| - \dim(X) - nd$ in the variables $t_1,\dots,t_n$. When this polynomial degree is zero, it specializes to an ordinary Gromov-Witten invariant. The equivariant small quantum ring of X is the $\mathbb{Z}[q,t_1,\dots,t_n]$-algebra defined by $\mathrm{QH}_T(X) = H^*(X)\otimes_{\mathbb{Z}}\mathbb{Z}[q,t_1,\dots,t_n]$ as a module, and with product given by $[X_\lambda] * [X_\mu] = \sum_\nu \langle X_\lambda, X_\mu, X_{\nu^\vee}^{\mathrm{op}}\rangle_d^T\, q^d\,[X_\nu]$. Mihalcea has proved a Chevalley type formula for the equivariant quantum ring of any homogeneous space G/P [9]. He has also proved [11] that the equivariant Gromov-Witten invariants satisfy the same positivity property that Graham proved for the ordinary equivariant Schubert structure constants [7]. For Grassmannians, Mihalcea has proved a presentation and a Giambelli formula for the equivariant quantum ring [10], and a Pieri formula for $\mathrm{QH}_T(X)$ can be obtained from Robinson's Pieri rule for double Schubert polynomials [12]. The main new result of this talk is the following.

Theorem (B-Mihalcea)**.** *The equivariant Gromov-Witten invariants on Grassmannians are given by*

$$\langle X_\lambda, X_\mu, X_\nu^{op}\rangle_d^T = \begin{cases} \int_{Y_d}^T [Y_\lambda]\cdot[Y_\mu]\cdot[Y_\nu^{op}] & \textit{if } \lambda,\mu,\nu \textit{ contain a } d\times d \textit{ rectangle;} \\ 0 & \textit{otherwise.}\end{cases}$$

The role of the $d \times d$ rectangle was known for ordinary Gromov-Witten invariants [13]. Since the equivariant Gromov-Witten invariants have no enumerative interpretation, and also because they are defined relative to Schubert varieties that are not in general position, the proof of the equivariant theorem must be based on intersection theory. The main new construction is a blow-up of Kontsevich's moduli space that makes it possible to assign a kernel-span pair of the expected dimensions to every curve. We note that our theorem generalizes to arbitrary (co)minuscule homogeneous spaces by exploiting the constructions in [3] and [4].

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Generalised Gelfand-Graev models for primitive ideals

Alexander Premet

Let G be a simple, simply connected algebraic group over $\mathbb{C}$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let (e,h,f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $(\,\cdot\,,\,\cdot\,)$ be the G-invariant bilinear form on $\mathfrak{g}$ with $(e,f)=1$ and define $\chi = \chi_e \in \mathfrak{g}^*$ by setting $\chi(x) = (e,x)$ for all $x \in \mathfrak{g}$. Denote by $\mathcal{O}_\chi$ the coadjoint G-orbit of χ.

Let $\mathcal{S}_e = e + \mathrm{Ker}\,\mathrm{ad}\,f$ be the Slodowy slice at e through the adjoint orbit of e and let H_χ be the enveloping algebra of $\mathcal{S}_e$; see [9, 7, 4]. Recall that $H_\chi = \mathrm{End}_{\mathfrak{g}}\,(Q_\chi)^{\mathrm{op}}$ where Q_χ is the generalised Gelfand–Graev module for $U(\mathfrak{g})$ associated with the triple (e,h,f). The $\mathfrak{g}$-module Q_χ is induced from a one-dimensional module $\mathbb{C}_\chi$ over of a nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\dim\mathfrak{m} = \frac{1}{2}\dim\mathcal{O}_\chi$. The subalgebra $\mathfrak{m}$ is $(\mathrm{ad}\,h)$-stable, all weights of $\mathrm{ad}\,h$ on $\mathfrak{m}$ are negative, and χ vanishes on $[\mathfrak{m},\mathfrak{m}]$. The action of $\mathfrak{m}$ on $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is given by $x(1_\chi) = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$; see [9, 7, 4, 10] for more detail. According to recent results of D'Andrea–De Concini–De Sole–Heluany–Kac [5] and De Sole–Kac [6], the algebra H_χ is isomorphic to the Zhu algebra of the vertex W-algebra associated with $\mathfrak{g}$ and e. The latter algebra is, in turn, isomorphic to the finite W-algebra $W^{\mathrm{fin}}(\mathfrak{g},e)$ obtained from the Poisson algebra $\mathrm{gr}\,H_\chi$ by BRST quantisation; see [6] for detail. Thus, $H_\chi \cong W^{\mathrm{fin}}(\mathfrak{g},e)$ as predicted in [9, 1.10].

Let $\mathcal{C}_\chi$ denote the category of all $\mathfrak{g}$-modules on which $x - \chi(x)$ acts locally nilpotently for every $x \in \mathfrak{m}$. Given a $\mathfrak{g}$-module M we set

$$\mathrm{Wh}_\chi(M) := \{m \in M \mid x.m = \chi(x)m \quad \forall\, x \in \mathfrak{m}\}.$$

The algebra H_χ acts on $\mathrm{Wh}_\chi(M)$ via a canonical isomorphism $H_\chi \cong \big(U(\mathfrak{g})/N_\chi\big)^{\mathrm{ad}\,\mathfrak{m}}$ where N_χ denotes the left ideal of the universal enveloping $U(\mathfrak{g})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}$. By Skryabin's theorem [11], the functors $V \rightsquigarrow Q_\chi \otimes_{H_\chi} V$ and $M \rightsquigarrow \mathrm{Wh}_\chi(M)$ are mutually inverse equivalences between the category of all H_χ-modules and the category $\mathcal{C}_\chi$; see also [7, Theorem 6.1]. Skryabin's equivalence implies that for any irreducible H_χ-module V the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$

is a primitive ideal of $U(\mathfrak{g})$. By the Irreducibility Theorem, the associated variety $\mathcal{VA}(I)$ of any primitive ideal I of $U(\mathfrak{g})$ is the closure of a nilpotent orbit in $\mathfrak{g}^*$.

Given a finitely generated $U(\mathfrak{g})$-module M we denote by $\mathrm{Dim}(M)$ the Gelfand–Kirillov dimension of M. By [10, Theorem 3.1], for any irreducible H_χ-module V the associated variety of $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ contains $\mathcal{O}_\chi$, and if $\dim V < \infty$, then

$$\mathcal{VA}\big(\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)\big) = \overline{\mathcal{O}}_\chi \quad \text{and} \quad \mathrm{Dim}(Q_\chi \otimes_{H_\chi} V) = \frac{1}{2}\dim \overline{\mathcal{O}}_\chi.$$

It was conjectured in [10, 3.4], that for any primitive ideal $\mathcal{I}$ of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_\chi$ there exists a finite dimensional irreducible H_χ-module V with the property that $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$. In [10, 5.6], this was proved under the assumption that e belongs to the minimal nilpotent orbit of $\mathfrak{g}$. The main goal of my talk was to announce that the conjecture holds in full generality:

Theorem 1. *If $\mathcal{I}$ is a primitive ideal of $U(\mathfrak{g})$ such that $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_\chi$, then $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ for some finite dimensional irreducible H_χ-module V.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and Φ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let Π be a basis of simple roots in Φ and W the Weyl group of Φ. Given $\lambda \in \mathfrak{h}^*$ we let $L(\lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight λ. By Duflo's theorem, for any primitive ideal $\mathcal{J}$ of $U(\mathfrak{g})$ there exists an irreducible highest weight module $L(\mu)$ such that $\mathcal{J} = \mathrm{Ann}_{U(\mathfrak{g})} L(\mu)$. Generically, the number of such $\mu \in \mathfrak{h}^*$ equals the order of W, but there are instances where μ is uniquely determined by $\mathcal{J}$.

Let $\mathcal{I}$ be a primitive ideal with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_\chi$ and choose $\lambda \in \mathfrak{h}^*$ such that $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda)$. Let A_0 denote the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by all $\langle \lambda, \alpha^\vee \rangle$ with $\alpha \in \Pi$. To prove Theorem 1 we work with a suitable localisation $A = S^{-1}A_0$ of A_0 and consider natural A-forms $\mathfrak{g}_A$ and $L_A(\lambda)$ of $\mathfrak{g}$ and $L(\lambda)$, respectively. We denote by $\pi(A)$ the set of all primes $p \in \mathbb{Z}$ for which there is an algebra epimorphism $A \twoheadrightarrow \mathbb{F}_p$. Given $p \in \pi(A)$ we pick a maximal ideal $\mathfrak{P}$ of A with $A/\mathfrak{P} \cong \mathbb{F}_p$ and consider the natural highest weight module $L_p(\lambda) := L_A(\lambda) \otimes_A \Bbbk$ over the restricted Lie algebra $\mathfrak{g}_\Bbbk = \mathfrak{g}_A \otimes_A \Bbbk$, where $\Bbbk$ is the algebraic closure of $\mathbb{F}_p$.

Let $G_\Bbbk$ be the simply connected algebraic $\Bbbk$-group with $\mathrm{Lie}(G_\Bbbk) = \mathfrak{g}_\Bbbk$. If p is sufficiently large, then $(\,\cdot\,,\,\cdot\,)$ induces a nondegenerate $G_\Bbbk$-invariant symmetric form on $\mathfrak{g}_\Bbbk$ and there is a natural bijection between the nilpotent G-orbits in $\mathfrak{g}$ and the nilpotent $G_\Bbbk$-orbits in $\mathfrak{g}_\Bbbk$. Let $Z_p = \langle x^p - x^{[p]} \mid x \in \mathfrak{g}_\Bbbk \rangle$, the p-centre of the universal enveloping algebra $U(\mathfrak{g}_\Bbbk)$. Given $\eta \in \mathfrak{g}_\Bbbk^*$ we denote by $\Bbbk_\eta = \Bbbk 1_\eta$ the 1-dimensional Z_p-module such that $(x^p - x^{[p]})(1_\eta) = \eta(x)^p 1_\eta$ for all $x \in \mathfrak{g}_\Bbbk$, and we set $U_\eta(\mathfrak{g}_\Bbbk) := U(\mathfrak{g}_\Bbbk) \otimes_{Z_p} \Bbbk_\eta$ and $L_p^\eta(\lambda) := L_p(\lambda) \otimes_{Z_p} \Bbbk_\eta$. The associative algebra $U_\eta(\mathfrak{g}_\Bbbk)$ is called the reduced enveloping algebra of $\mathfrak{g}_\Bbbk$ corresponding to η.

We show that for all sufficiently large $p \in \pi(A)$ the $U_\eta(\mathfrak{g}_\Bbbk)$-module $L_p^\eta(\lambda)$ has dimension $\leqslant Dp^{d(e)}$, where $D = D(\lambda)$ is independent of p and $d(e) = \frac{1}{2}\dim \mathcal{O}(\chi)$. Let $\mathcal{O}_\Bbbk$ be the nilpotent $G_\Bbbk$-orbit in $\mathfrak{g}_\Bbbk$ corresponding to the G-orbit containing e. We prove that there is a linear function $\bar{\chi} = (\bar{e},\,\cdot\,)$ on $\mathfrak{g}_\Bbbk$ with $\bar{e} \in \mathcal{O}_\Bbbk$ such that the $U_{\bar{\chi}}(\mathfrak{g}_\Bbbk)$-module $L_p^{\bar{\chi}}(\lambda)$ is nonzero. Combining the Kac–Weisfeiler conjecture proved in [8] with the above results, we then show that the algebra $U_{\bar{\chi}}(\mathfrak{g}_{\Bbbk})$ has a simple module of dimension $lp^{d(e)}$ for some $l \leqslant D = D(\lambda)$. On the other hand, it is established in [9] that $U_{\bar{\chi}}(\mathfrak{g}_{\Bbbk}) \cong \mathrm{Mat}_{p^{d(e)}}\left(H_{\bar{\chi}}^{[p]}\right)$, where $H_{\bar{\chi}}^{[p]}$ is a "restricted" version of the finite W-algebra $H_\chi \cong W^{\mathrm{fin}}(\mathfrak{g}, e)$. We thus deduce that for all sufficiently large $p \in \pi(A)$ the algebra $H_{\bar{\chi}}^{[p]}$ has a simple module of dimension $l \leqslant D = D(\lambda)$.

The algebra H_χ relates to $H_{\bar{\chi}}^{[p]}$ in the same way as $U(\mathfrak{g})$ relates to the restricted enveloping algebra $U^{[p]}(\mathfrak{g}_{\Bbbk})$. We construct a natural A-form $H_{\chi,A}$ in H_χ and show that $H_{\bar{\chi}}^{[p]}$ is a homomorphic image of the $\Bbbk$-algebra $H_{\chi,A} \otimes_A \Bbbk$. Next we introduce certain affine varieties $\mathcal{Y}_n$ of matrix representations of H_χ and use reduction modulo p to prove that $\mathcal{Y}_l(\mathbb{C}) \neq \emptyset$ for some $l \leqslant D$. The definition of $\mathcal{Y}_l$ then implies that H_χ has a finite dimensional irreducible module V such that $\mathcal{I} \subseteq \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$. Since $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ and $\mathcal{I}$ have the same associated variety, a well-known result of Borho–Kraft [3] yields $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$.

Corollary 1. *All finite W-algebras $W^{\mathrm{fin}}(\mathfrak{g}, e)$ possess finite-dimensional irreducible representations.*

Corollary 2. *If $p \gg 0$, then for every linear function η on $\mathfrak{g}_{\Bbbk}$ the reduced enveloping algebra $U_\eta(\mathfrak{g}_{\Bbbk})$ has a simple module of dimension $lp^{(\dim G_{\Bbbk}\cdot\,\eta)/2}$ where $l < p$.*

To deduce Corollaries 1 and 2 from Theorem 1 we have to rely in a crucial way on the main results of Barbasch–Vogan [1, 2]. It is worth remarking that for a general η Corollary 2 gives the best to date upper bound for the minimal dimension of irreducible $U_\eta(\mathfrak{g}_{\Bbbk})$-modules.

We conjecture that two irreducible finite dimensional H_χ-modules V_1 and V_2 are isomorphic if and only if

$$\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V_1) \;=\; \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V_2).$$

We also conjecture that for any irreducible finite dimensional H_χ module V the Goldie rank of the primitive quotient $U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ equals $\dim V$.

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The rough structure of generalised Verma modules

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(joint work with Volodymyr Mazorchuk)

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}$. Let $\mathfrak{p} \supset \mathfrak{b}$ be a parabolic subalgebra with reductive part $\mathfrak{a}$. For any Lie algebra L we denote its universal enveloping algebra by $\mathcal{U}(L)$.

Let V be an *arbitrary* simple $\mathfrak{a}$-module. We make it into a $\mathfrak{p}$-module by extending the action trivially. The $\mathfrak{g}$-module

$$\Delta(\mathfrak{p}, V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V$$

is called the *generalised Verma module* with respect to the data $(\mathfrak{g}, \mathfrak{p}, V)$.

A natural question is the following: What is the structure of this module? For instance: simple composition factors, their multiplicities etc.

This is a difficult problem and to some extent not reasonable because of the following facts:

- There is no classification of simple $\mathfrak{a}$-modules, so that the possible simple modules V we start with are not classified! (Except when $\mathfrak{a}$ is of type A_1 where Block ([Bl]) classified the simple modules.)
- Consider the "easiest" case when $\mathfrak{p} = \mathfrak{b}$ and V is a one-dimensional module, then $\Delta(\mathfrak{p}, V)$ is a usual Verma module and the answer is given by Kazhdan-Lusztig theory.
- For arbitrary $\mathfrak{p}$ but finite dimensional V, the module $\Delta(\mathfrak{p}, V)$ is a so-called parabolic Verma module and the multiplicities of simple composition factors are given by parabolic Kazhdan-Lusztig polynomials.
- In general, the multiplicities might not be finite. (This follows from the example of a non-holonomic simple $\mathcal{D}$-module constructed by Stafford in [St]).

Concerning arbitrary simple $\mathfrak{a}$-modules, the only tool we have available is the theorem of Duflo: *The set of primitive ideals in* $\mathcal{U}(\mathfrak{a})$ *is the set of annihilators of simple highest weight modules.*

In other words: Annihilators of simple $\mathcal{U}(\mathfrak{a})$-modules are classified and for any arbitrary simple module V there is always a simple highest weight module $L(\lambda)$ which has the same annihilator.

Before we proceed we recall the (from our point of view) principal idea behind Kazhdan-Lusztig theory: To determine for instance the composition factors of the Verma module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}$ we consider this module as a "standard" object in some category, namely the principal block $\mathcal{O}_0$ of the corresponding BGG-category $\mathcal{O}$.

The three different bases of the Grothendieck group of $\mathcal{O}_0$ given by the isomorphism classes of indecomposable projectives, Vermas and simple modules respectively correspond to the three bases in the Hecke algebra (Kazhdan-Lusztig basis, standard basis and dual Kazhdan-Lusztig basis). The transformation matrix between standard modules and simple modules is then described via Kazhdan-Lusztig-polynomials. Our approach towards understanding generalised Verma modules is heavily built on these results.

Let me outline the main steps and results in our work: depending just on the annihilator of V we construct an abelian category $\mathcal{C}(V)$ of $\mathfrak{g}$-modules such that

- the generalised Verma module $\Delta(\mathfrak{p}, V)$ is an object of $\mathcal{C}(V)$
- $\mathcal{C}(V)$ is equivalent to a module category over some properly stratified algebra
- there is an equivalence of categories Φ from $\mathcal{C}(V)$ to some category $\mathcal{C}(L(\lambda))$ of $\mathfrak{g}$-modules constructed from $L(\lambda)$ via translation functors
- $\mathcal{C}(L(\lambda))$ is equivalent to a module category over some properly stratified algebra and the combinatorics can be described via induced cell modules for the Hecke algebra
- The equivalence Φ maps properly stratified modules to such and hence gives a formula for the simple composition factors of $\Delta(\mathfrak{p}, V)$ (as an object in $\mathcal{C}(V)$) via the combinatorics of $\mathcal{C}(L(\lambda))$

There is however a price to pay: the simple modules in $\mathcal{C}(V)$ are not simple as $\mathfrak{g}$-modules, but always have simple heads and all the other occurring simple composition factors have larger annihilator. Hence we in fact describe what we call the *rough structure* of a generalised Verma module, i.e. we consider only simple composition factors with small enough annihilators. (The notion "rough structure" was first introduced and considered in [KM]). For the simple modules which can be detected by the rough structure, the multiplicities will turn out to be finite. Our final result is a complete (combinatorial) description of the rough structure of generalised Verma modules for an arbitrary simple $\mathfrak{a}$-module with trivial central character *in case* $\mathfrak{a}$ *is of type* A.

Our techniques generalise the ideas and work of Milicic and Soergel on induced Whittaker modules ([MS]).

The categories mentioned above categorify induced cell modules for the Hecke algebra. More precisely the statement is as follows: Let W be the Weyl group of $\mathfrak{g}$ and $\mathcal{H} := \mathcal{H}(W)$ the corresponding generic Hecke algebra defined over $\mathbb{Z}[v, v^{-1}]$ as defined in [So]. Let $W_{\mathfrak{p}}$ be the parabolic subgroup corresponding to $\mathfrak{p}$ with Hecke algebra $\mathcal{H}' = \mathcal{H}(W_{\mathfrak{p}}) \subset \mathcal{H}$. We have the induced sign module $sgn \otimes_{\mathcal{H}'} \mathcal{H} = \mathcal{N}^{\mathfrak{p}}$. Let $\mathcal{O}_0^{\mathfrak{p}}$ be the principal block of the parabolic category $\mathcal{O}$ for $\mathfrak{g}$ with its graded version $\tilde{\mathcal{O}}_0^{\mathfrak{p}}$ as introduced in [BGS]. For an abelian category $\mathcal{A}$ let $K_0(\mathcal{A})$ be the Grothendieck group of $\mathcal{A}$. The following is a well-known result from Kazhdan-Lusztig-theory

Theorem 1. *There is an isomorphism of $\mathcal{H}$-modules*

$$K_0(\tilde{\mathcal{O}}_0^{\mathfrak{p}}) \quad \cong \quad \mathcal{N}^{\mathfrak{p}}$$

such that the isomorphism classes of (the standard lifts of)

- *parabolic Verma modules correspond to the standard basis elements,*
- *indecomposable projective modules correspond to the Kazhdan-Lusztig basis elements*
- *simple modules correspond to the dual Kazhdan-Lusztig basis elements,*

and the action of $\mathcal{H}$ on the right hand side is induced by (graded versions of) translation functors through the wall.

The sign module for $\mathcal{H}'$ is only a very special case of a *(right) Cell module* as defined by Kazhdan and Lusztig. It comes along with the Kazhdan-Lusztig basis and its dual Kazhdan-Lusztig basis. Following the original construction in a categorical way we construct to each cell module R of $\mathcal{H}'$ a subquotient category of $\tilde{\mathcal{O}}$ for the Lie algebra $\mathfrak{a}$, and a graded version $\mathcal{O}(\mathfrak{p}, \mathcal{C}(R))_0$ of a category of $\mathfrak{g}$-modules such that the following holds:

Theorem 2.
$\mathcal{O}(\mathfrak{p}, \mathcal{C}(R))$ is equivalent to a category of modules over a finite dimensional graded algebra A, where A is properly stratified, hence we have the natural notion of standard and proper standard objects.
There is an isomorphism of $\mathcal{H}$-modules

$$K_0(\mathcal{C}(R)) \quad \cong \quad R \otimes_{\mathcal{H}'} \mathcal{H}$$

such that the isomorphism classes of (the standard lifts of)

- *the standard objects correspond to the induced Kazhdan-Lusztig basis elements,*
- *the proper standard objects correspond to the induced dual Kazhdan-Lusztig basis elements,*
- *indecomposable projective modules correspond to the Kazhdan-Lusztig basis elements*
- *simple modules correspond to the dual Kazhdan-Lusztig basis elements,*

and the action of $\mathcal{H}$ on the right hand side is induced by translation functors through the wall.

In case $\mathfrak{a}$ is of type A, we prove that if we have two isomorphic right cell modules, then the corresponding categories are equivalent. In other words, we categorified induced cell modules. Finally we show that each generalised Verma module can be realised as a proper standard object in some category which is equivalent to one of the $\mathcal{O}(\mathfrak{p}, \mathcal{C}(R))_0$ above. Hence a combinatorial description of its rough structure is given by the theorem. To establish the necessary equivalence of categories one has to study the so-called Kostant's problem: Let M be a $\mathfrak{g}$-module and consider the space E_M of locally-finite vectors of $\mathrm{End}_{\mathbb{C}}(M)$ under the adjoint action. This is a $\mathcal{U}(\mathfrak{g})$ bimodule. In which case is the natural map $\mathcal{U}(\mathfrak{g})/\operatorname{Ann}_{U(\mathfrak{g})} M \to E_M$ surjective? We give a partial answer to this for simple modules M if $\mathfrak{g}$ is of type

A which is enough to ensure the equivalence of categories we were looking for. Details can be found in [MS].

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Pursuing the double affine Grassmannian

Alexander Braverman

Let G be a reductive group and K a local non-archimedian field. Let $\mathcal{O}$ be its ring of integers. The spherical Hecke algebra $\mathcal{H}_G$ is the algebra of functions on $G(K)$ which are compactly supported and $G(\mathcal{O})$-biinvariant.

The classical Satake isomorphism asserts that $\mathcal{H}_G$ is isomorphic to $K(\operatorname{Rep} G^\vee) \otimes \mathbb{C}$ where $\operatorname{Rep} G^\vee$ is the category of finite-dimensional representations of the Langlands dual group $G^\vee$. Lusztig (followed by Kato and Brylinski) has produced an explicit formula for this isomorphism using q-analogs of the weight multiplicity for representations of $G^\vee$.

In a joint work with D. Kazhdan we generalize the story to the case of the corresponding affine Kac-Moody group G_{aff}. We define the Hecke algebra $\mathcal{H}_{G_{\mathrm{aff}}}$ and show that it is isomorphic to some completion of $K(\operatorname{Rep} G^\vee_{\mathrm{aff}}) \otimes \mathbb{C}$ where $G^\vee_{\mathrm{aff}}$ is the affine dual group and $\operatorname{Rep} G^\vee_{\mathrm{aff}}$ is the category of integrable modules over $G^\vee_{\mathrm{aff}}$. We conjecture a formula analogous to that of Lusztig.

Finite dimensional modules of DAHA

Eric Vasserot
(joint work with Michela Varagnolo)

Double affine Hecke algebras have been introduced 20 years ago by Cherednik to prove some conjecture of Macdonald. The understanding of their representations has progressed very much recently. Simple modules in the category O are classified. The DAHA admits a rational degeneration whose category O is quasi-hereditary. In both cases (DAHA and rational DAHA) the finite dimensional modules are very difficult to classify. It has been claimed that both sets are in one-to-one correspondence. This turns out to be false.

We classify all finite dimensional spherical simple modules of rational DAHA. They are labelled by elliptic numbers, i.e., by orders of regular (in the sense of Springer) elliptic elements in Weyl groups. The basic tools in the proof are the K-theoretic construction of the simple DAHA's modules (by the second author) and Kazhdan-Lusztig's work on affine Springer fibers.

For (non rational) DAHA's the picture is more complicated and involves another (discrete) parameter. It is related, in a mysterious way, to the tamely ramified Langlands correspondence.

Geometric constructions of cohomology classes for arithmetic groups

Joachim Schwermer

The cohomology of an arithmetic group Γ in a connected reductive algebraic group G/k, k an algebraic number field, can be interpreted in terms of the automorphic spectrum of the underlying group. This context in place, I reviewed various geometric approaches to construct non-trivial cohomology classes (special cycles à la Millson-Raghunathan resp. Rohlfs-Schwermer, modular symbols etc.) for some classical groups and drew some consequences for the existence of certain automorphic representations in these cases. In conclusion, examples of exceptional groups were discussed and some open problems in the case of arithmetically defined Kleinian groups, i. e., arithmetically defined 3-dimensional hyperbolic manifolds, were posed.

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Comparison of eigencone, saturation and cohomology of flag varieties under diagram automorphisms

Shrawan Kumar

This is a report of my joint work with P. Belkale. Let G be a simple, connected complex algebraic group with maximal compact subgroup K. Let $\mathfrak{h}_+$ be the positive Weyl chamber of G. It is known that there is a bijection $C : \mathfrak{k}/K \to \mathfrak{h}_+$, where $\mathfrak{k}$ is the Lie algebra of K and K acts on $\mathfrak{k}$ by the adjoint representation. Define the *eigencone* (for any $s \geq 1$):

$$\Gamma(s,K) = \{(h_1,\dots,h_s) \in (\mathfrak{k}/K)^s : \exists k_j \in \mathfrak{k}, k_j \sim h_j, j = 1,\dots,s, \text{ and } \sum_{j=1}^{s} k_j = 0.\}$$

Positive Weyl chambers in $\mathfrak{h}^{\mathrm{Sp}(2n)}$ and $\mathfrak{h}^{\mathrm{SO}(2n+1)}$. Let $\mathfrak{h}^{\mathrm{Sp}(2n)}$ be the Cartan subalgebra for $\mathrm{Sp}(2n)$ and similarly for $SO(2n+1)$. We first describe $\mathfrak{h}_+^{\mathrm{Sp}(2n)}$: It is given by n-tuples $(x_1,\dots,x_n)$ so that

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$

Choose the standard flag on $\mathbb{C}^{2n}$ to be induced from the following ordering of the basis vectors $e_1,\dots,e_n,f_n,\dots,f_1$. Take the symplectic form on $\mathbb{C}^{2n}$ defined by $\langle e_i,f_j\rangle = -\delta_{i,j}, \langle e_i,e_j\rangle = \langle f_i,f_j\rangle = 0$. Consider the map (which corresponds to the natural embedding $\mathrm{Sp}(2n) \to \mathrm{SL}(2n)$)

$$\theta : \mathfrak{h}_+^{\mathrm{Sp}(2n)} \to \mathfrak{h}_+^{\mathrm{SL}(2n)},\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,-x_n,\dots,-x_1).$$

We similarly describe $\mathfrak{h}_+^{\mathrm{SO}(2n+1)}$: It is given by n-tuples $(x_1,\dots,x_n)$ so that

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$

Choose the flag on $\mathbb{C}^{2n+1}$ to be induced from the following ordering of the basis vectors: $e_1,\dots,e_n,g,f_n,\dots,f_1$. Take the symmetric form on $\mathbb{C}^{2n+1}$ so that g is perpendicular to all of e_i,f_j and $(e_i,f_j) = \delta_{i,j}$, $(e_i,e_j) = (f_i,f_j) = 0, (g,g) = 1$. Again, there is a map (which corresponds to the natural embedding $\mathrm{SO}(2n+1) \to \mathrm{SL}(2n+1)$)

$$\theta' : \mathfrak{h}_+^{\mathrm{SO}(2n+1)} \to \mathfrak{h}_+^{\mathrm{SL}(2n+1)},\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0,-x_n,\dots,-x_1).$$

We state our first main theorem:

Theorem 1. (a) *For* $h_1,\dots,h_s \in \mathfrak{h}_+^{\mathrm{Sp}(2n)}$,

$$(h_1,\dots,h_s) \in \Gamma(s,\mathrm{Sp}(2n)) \Leftrightarrow (\theta(h_1),\dots,\theta(h_s)) \in \Gamma(s,SU(2n)).$$

(b) *For* $h_1,\dots,h_s \in \mathfrak{h}_+^{\mathrm{SO}(2n+1)}$,

$$(h_1,\dots,h_s) \in \Gamma(s,\mathrm{SO}(2n+1)) \Leftrightarrow (\theta'(h_1),\dots,\theta'(h_s)) \in \Gamma(s,SU(2n+1)).$$

Our proof of this theorem is a consequence of a surprising result in intersection theory described below.

Intersection theory. Let V be a $2n$-dimensional complex vector space equipped with a nondegenerate symplectic form $\langle\, ,\, \rangle$. Let $A^1, \dots, A^s$ be subsets of $[2n] := \{1, \dots, 2n\}$ each of cardinality m. Let $E^j_\bullet, j = 1, \dots, s$, be complete isotropic flags on V in general position. The second main theorem is a key technical result that underlies the proof of the first main theorem.

Recall that if A is a subset of $[2n]$ of cardinality m and $F_\bullet$ a complete flag on $\mathbb{C}^{2n}$, then, by definition, the corresponding Schubert cell is $\Omega^o_A(F_\bullet) = \{X \in \mathrm{Gr}(m, \mathbb{C}^{2n}) : \dim X \cap F_u = \ell \text{ for } a_\ell \le u < a_{\ell+1}, \ell = 1, \dots, m\}$, where $A = \{a_1 < a_2 < \cdots < a_m\}$ and $\mathrm{Gr}(m, \mathbb{C}^{2n})$ is the ordinary Grassmannian of m-dimensional subspaces of $\mathbb{C}^{2n}$.

Theorem 2. *The intersection* $\cap_{j=1}^s \Omega^o_{A^j}(E^j_\bullet)$ *of subvarieties of* $\mathrm{Gr}(m, V)$ *is proper (possibly empty) for complete isotropic flags* $E^j_\bullet$ *on* V *in general position.*

Similarly for $\mathrm{SO}(2n+1)$, suppose V' is a $2n+1$ dimensional vector space equipped with a nondegenerate symmetric bilinear form. Let $A^1, \dots, A^s$ be subsets of $[2n+1]$ each of cardinality m. Let $E^j_\bullet, j = 1, \dots, s$, be isotropic flags on V' in general position.

Theorem 3. *The intersection* $\cap_{j=1}^s \Omega^o_{A^j}(E^j_\bullet)$ *of subvarieties of* $\mathrm{Gr}(m, V')$ *is proper.*

The saturation conjecture. Any dominant weight λ of $\mathrm{SL}(2n, \mathbb{C})$ restricts to a dominant weight μ of the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. Similarly, any dominant weight λ of $\mathrm{SL}(2n+1, \mathbb{C})$ restricts to a dominant weight ν of the orthogonal group $\mathrm{SO}(2n+1, \mathbb{C})$. The following theorem is proved geometrically by the method of "theta sections":

Theorem 4. *Let* $V_{\lambda_1}, \dots, V_{\lambda_s}$ *be irreducible representations of* $\mathrm{SL}(2n)$ *(with highest weights* $\lambda_1, \dots, \lambda_s$ *respectively) such that their tensor product has a nonzero* $\mathrm{SL}(2n)$*-invariant. Then, the tensor product of the representations of* $\mathrm{Sp}(2n)$ *with highest weights* $\mu_1, \dots, \mu_s$ *has a nonzero* $\mathrm{Sp}(2n)$*-invariant.*

A similar property holds for the odd orthogonal group $\mathrm{SO}(2n+1)$.

Theorem 5. *Let* $V_{\lambda_1}, \dots, V_{\lambda_s}$ *be representations of* $\mathrm{SL}(2n+1)$ *such that their tensor product has a nonzero* $\mathrm{SL}(2n+1)$*-invariant. Then, the tensor product of the representations of* $\mathrm{SO}(2n+1)$ *with highest weights* $\nu_1, \dots, \nu_s$ *has a nonzero* $\mathrm{SO}(2n+1)$*-invariant.*

Derivation of saturation property from Theorems 4 and 5. The natural map θ on the duals (induced by the restriction of line bundles) takes $(y_1, \dots, y_{2n}) \in \mathfrak{h}^{\mathrm{SL}(2n)} = (\mathfrak{h}^{\mathrm{SL}(2n)})^*$ to $(y_1 - y_{2n}, y_2 - y_{2n-1}, \dots, y_n - y_{n+1}) \in \mathfrak{h}^{\mathrm{Sp}(2n)} = (\mathfrak{h}^{\mathrm{Sp}(2n)})^*$.

Now, suppose we are given irreducible representations $W_{\mu_1}, \dots, W_{\mu_s}$ of $\mathrm{Sp}(2n)$ with highest weights $\mu_1, \dots, \mu_s$. Set $\mu_j = (y^j_1, y^j_2, \dots, y^j_n)$ with $y^j_i \in \mathbb{Z}$. Assume that the tensor product $W_{N\mu_1} \otimes \cdots \otimes W_{N\mu_s}$ has a nonzero $\mathrm{Sp}(2n)$-invariant for some $N > 0$. Then, $(\mu_1, \dots, \mu_s) \in \Gamma(s, \mathrm{Sp}(2n))$ and hence $(\theta(\mu_1), \dots, \theta(\mu_s)) \in \Gamma(s, SU(2n))$. Clearly, $\lambda_j = \theta(\mu_j) = (y^j_1, \dots, y^j_n, -y^j_n, \dots, -y^j_1)$, which is in the root lattice of $\mathrm{SL}(2n)$.

Using the saturation theorem of Knutson-Tao, the tensor product of these has a nonzero $\mathrm{SL}(2n)$-invariant. The weights λ_j of $\mathrm{SL}(2n)$ restrict (as above) to the weights $2\mu_j$ of $\mathrm{Sp}(2n)$. Therefore, Theorem 4 gives us the following improvement of the Kapovich-Millson saturation theorem for $\mathrm{Sp}(2n)$.

Theorem 6. *Given dominant integral weights $\mu_1, \dots, \mu_s$ of $\mathrm{Sp}(2n)$, the following are equivalent:*

(1) *For some $N \geq 1$, the tensor product of representations of $\mathrm{Sp}(2n)$ with highest weights $N\mu_1, \dots, N\mu_s$ has a nonzero $\mathrm{Sp}(2n)$-invariant.*
(2) *The tensor product of representations with highest weights $2\mu_1, \dots, 2\mu_s$ has a nonzero $\mathrm{Sp}(2n)$-invariant.*

By carrying out a similar analysis for the odd orthogonal groups, we obtain

Theorem 7. *Given dominant integral weights $\nu_1, \dots, \nu_s$ of $\mathrm{SO}(2n+1)$, the following are equivalent:*

(1) *For some $N \geq 1$, the tensor product of representations with highest weights $N\nu_1, \dots, N\nu_s$ has a nonzero $\mathrm{SO}(2n+1)$-invariant.*
(2) *The tensor product of representations wih highest weights $2\nu_1, \dots, 2\nu_s$ has a nonzero $\mathrm{SO}(2n+1)$-invariant.*

Conjugacy in algebraic groups

Simon Goodwin
(joint work with Gerhard Röhrle)

Let G be a linear algebraic group defined over $\mathbb{F}_p$ and let X be a G-variety defined over $\mathbb{F}_p$. For q a power of p, we write $G(q)$ for the group of $\mathbb{F}_q$-rational points of G, and $X(q)$ for the set of $\mathbb{F}_q$-rational points of X. We consider questions about uniformity in q of the number $k(G(q), X(q))$ of $G(q)$-orbits in $X(q)$. Examples of such questions are given below. Thanks to arguments due to M. Du Sautoy in [3], it is known that there always exists a linear recurrence relation for the values of $k(G(p^s), X(p^s))$.

A special case is the situation where $X = G$ and G is acting by conjugation. It is well-known, and easy to prove, that the number $k(\mathrm{GL}_n(q))$ of conjugacy classes of the finite general linear group $\mathrm{GL}_n(q)$, over the field $\mathbb{F}_q$ of q elements is given by a polynomial in q. In contrast, the analogous question for the number of conjugacy classes in the group $\mathrm{U}_n(q)$ of upper unitriangular matrices over $\mathbb{F}_q$ is known to be very difficult. It is a conjecture of G. Higman [9] that this number is a polynomial in q. This conjecture has attracted much research interest. For example, in [10] G. R. Robinson showed that the zeta function

$$\zeta_{\mathrm{U}_n}(t) = \exp\left(\sum_{s=1}^{\infty} \frac{k(\mathrm{U}_n(p^s))}{s} t^s\right)$$

is a rational function in t and J. Thompson has given a strategy for attacking the conjecture in his manuscript [12]. The conjecture has been verified for $n \leqslant 13$ by computer calculation by A. Vera-Lopez and J. M. Arregi, see [13].

In order to discuss more general related problems we need to introduce some notation. Let G be a split connected reductive algebraic group defined over $\mathbb{F}_p$. We write $G(q)$ for the finite group of $\mathbb{F}_q$-rational points of G. Let B be a Borel subgroup of G defined over $\mathbb{F}_p$ and let U be the unipotent radical of B. We assume throughout that p is good for G.

It is well-known that the number of $G(q)$-conjugacy classes is given by Polynomials On Residue Classes (PORC). In view of Higman's Conjecture, it is natural to ask whether $k(U(q))$ is a polynomial in q. In ongoing research, G. Röhrle and the author are writing a computer program to calculate the number of $U(q)$-conjugacy classes. Thus far it has been observed that if G is of type B_5, C_5, F_4 or E_6, then the number of $U(q)$-conjugacy is only PORC and not a polynomial in q. This computer program is based on algorithm described in [4], which is similar to that given by H. Bürgstein and W. Hesselink in [2].

Using ideas from the paper [4], the author constructed, in [5], a family of varieties that parameterize the conjugacy classes of U. By applying Dwork's theorem (1st Weil conjecture) to each of the varieties in this family, and using the fact that $C_U(x)$ is connected for all $x \in U$, the author deduced that the zeta function

$$\zeta_U(t) = \exp\left(\sum_{s=1}^{\infty} \frac{k(U(p^s))}{s} t^s\right)$$

is a rational function in t.

In [5], the author goes on to give a family of varieties parameterizing the conjugacy classes of B contained in U. The centralizers $C_B(x)$ are not in general connected for $x \in U$. However, the family of varieties parameterizing the B-conjugacy classes in U are constructed in such a way that the Galois cohomology of $C_B(x)$ is constant for x in a fixed variety in the family. This allows the author to prove that the zeta function $\zeta_{B,U}(t)$ is a rational function in t; this function is defined analogously to $\zeta_U(t)$, but the coefficients are given by the number $k(B(p^s), U(p^s))$ of $B(p^s)$-conjugacy classes in $U(p^s)$. Using Jordan decompositions the author is then able to deduce that the zeta function $\zeta_B(t)$ is a rational function in t.

In [1], J. Alperin proved that the number $k(\mathrm{U}_n(q), \mathrm{GL}_n(q))$ of $\mathrm{U}_n(q)$-conjugacy classes in all of $\mathrm{GL}_n(q)$ is a polynomial in q. The proof of this involved a counting argument allowing one to express $k(\mathrm{U}_n(q), \mathrm{GL}_n(q))$ in terms of characters for $\mathrm{GL}_n(q)$ and consequently Green functions; the work of J. A. Green in [8] implies that these Green functions are polynomials in q.

Let P be a parabolic subgroup of G defined over $\mathbb{F}_p$ and let U_P be the unipotent radical of P. In [6], G. Röhrle and the author generalized Alperin's result by showing that, if $Z(G)$ is connected, then the number $k(U_P(q), G(q))$ of $U_P(q)$-conjugacy classes in $G(q)$ is a polynomial in q, except when G has a simple component of type E_8 in which case there are two polynomials depending on q mod 3. In case $Z(G)$ is not connected, $k(U_P(q), G(q))$ is PORC. There is also a version of this result for non-split G, but it is more complicated to state. The proof generalizes that of Alperin using results from the representation theory of $G(q)$ to express $k(U_P(q), G(q))$ in terms of Green functions; thanks to work of T. Shoji in [11] these Green functions are given by polynomials in q.

The methods used in [6] lead to interesting results about the variety $\mathcal{P}_u^0$ of conjugates of P whose unipotent radical contains the unipotent element $u \in G$. More precisely, it is shown that the number of $\mathbb{F}_q$-rational points in $\mathcal{P}_u^0$ is a polynomial in q; moreover, when u is *split*, as defined by Shoji [11] the coefficients are given by the Betti numbers of the variety $\mathcal{P}_u^0$. We note that the analogous results hold for the variety $\mathcal{P}_u$ of conjugates of P containing u.

For $G = \mathrm{GL}_n$, the author and Röhrle have shown in [7] that the number of $P(q)$-conjugacy classes in $G(q)$ is a polynomial in q. In general, one can only hope to prove that $k(P(q), G(q))$ is PORC.

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Mutations for quivers with potentials

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This talk is based on [1]. We study quivers with relations given by non-commutative analogs of Jacobian ideals in the complete path algebra. This framework allows us to give a quiver-theoretic interpretation of matrix mutations from the theory of cluster algebras. This gives a far-reaching generalization of Bernstein-Gelfand-Ponomarev reflection functors. The motivations for this work come from several sources: superpotentials in physics, Calabi-Yau algebras, cluster algebras.

We think of a *quiver* as a quadruple $Q = (Q_0, Q_1, h, t)$ consisting of a pair of finite sets Q_0 (*vertices*) and Q_1 (*arrows*), and two maps $h : Q_1 \to Q_0$ (*head*) and $t : Q_1 \to Q_0$ (*tail*). It is represented as a directed graph with the set of vertices Q_0, and directed edges $a : ta \to ha$ for $a \in Q_1$. We fix Q_0 once and for all, but Q_1 will be allowed to vary. We also assume that Q has no loops, that is, $ha \neq ta$ for all $a \in Q_1$.

For every $k \in Q_0$, the *quiver mutation* at k transforms any quiver Q without oriented 2-cycles into another quiver $\overline{Q} = \mu_k(Q)$ of the same kind. The quiver $\overline{Q}$ is obtained from Q by the following three-step procedure:

(1) For every incoming arrow $a : j \to k$ and every outgoing arrow $b : k \to i$ in Q, create a "composite" arrow $[ba] : j \to i$ in $\overline{Q}$.
(2) Reverse all arrows at k; that is, replace each arrow $a : j \to k$ with $a^\star : k \to j$, and $b : k \to i$ with $b^\star : i \to k$.
(3) Remove any maximal disjoint union of oriented 2-cycles.

Note that μ_k is an involution in the following sense: $\mu_k^2(Q)$ is isomorphic to Q (non-canonically because of a choice one has to make in Step (3)).

Our goal is to find a representation-theoretic extension of quiver mutations at arbitrary vertices. Recall that a *representation* M of Q is a family of finite-dimensional vector spaces (over some fixed field K) $(M(i))_{i\in Q_0}$ and a family of linear maps $(M(a) : M(ta) \to M(ha))_{a\in Q_1}$. If a vertex k is a *sink* (i.e., there are no arrows b with $tb = k$), or a *source* (i.e., there are no arrows a with $ha = k$), then the mutation μ_k amounts to Step (2). In this case, μ_k can be extended to an operation on representations, with the help of reflection functors. Namely, the representation $\overline{M} = \mu_k(M)$ of $\overline{Q}$ is defined as follows. We set $\overline{M}(i) = M(i)$ for $i \neq k$, and $\overline{M}(c) = M(c)$ for any arrow c not incident to k. If k is a sink, we set $M_{\rm in} = \bigoplus_{ha=k} M(ta)$, assemble all maps $M(a)$ for $ha = k$ into one linear map $\alpha : M_{\rm in} \to M(k)$, and set $\overline{M}(k) = \ker\alpha$. For every $a \in Q_1$ with $ha = k$, we define the map $\overline{M}(a^\star) : \overline{M}(k) \to M(ta)$ as the embedding $\overline{M}(k) \to M_{\rm in}$ followed by the projection $M_{\rm in} \to M(ta)$. If k is a source, we set $M_{\rm out} = \bigoplus_{tb=k} M(hb)$, assemble all maps $M(b)$ for $tb = k$ into one linear map $\beta : M(k) \to M_{\rm out}$, and set $\overline{M}(k) = \operatorname{coker}\beta$. For every $b \in Q_1$ with $tb = k$, we define the map $\overline{M}(b^\star) : M(hb) \to \overline{M}(k)$ as the embedding $M(hb) \to M_{\rm out}$ followed by the projection $M_{\rm out} \to \overline{M}(k)$.

Note that μ_k^2 is *not* the identity operation. If k is a sink then $\overline{\overline{M}}_k = \operatorname{im}\alpha$, so in passing from M to $\mu_k^2(M)$ we "lose" coker α. If k is a source then $\overline{\overline{M}}_k = \operatorname{im}\beta$, so in passing from M to $\mu_k^2(M)$ we "lose" $\ker\beta$. To remedy this, we define (following

[2]) a *decorated* representation of Q as a pair $\mathcal{M} = (M, V)$, where M is an ordinary representation of Q, and $V = (V(i))_{i \in Q_0}$ is just a collection of finite-dimensional vector spaces attached to the vertices (with no maps attached). We extend the reflection functors to decorated representations by defining $\mu_k(\mathcal{M}) = \overline{\mathcal{M}} = (\overline{M}, \overline{V})$ (for k a sink or a source) as follows: $\overline{M}(i) = M(i)$ and $\overline{V}(i) = V(i)$ for $i \neq k$, while

$$\overline{M}(k) = \ker \alpha \oplus V(k), \quad \overline{V}(k) = \mathrm{coker}\ \alpha \quad (k \text{ a sink});$$

$$\overline{M}(k) = \mathrm{coker}\ \beta \oplus V(k), \quad \overline{V}(k) = \ker \beta \quad (k \text{ a source}).$$

To extend this construction to the case of an arbitrary vertex k, we encode Q algebraically by the two vector spaces $R = K^{Q_0}$ (the *vertex span* of Q) and $A = K^{Q_1}$ (the *arrow span* of Q). The space R is a commutative algebra with the K-basis $\{e_i \mid i \in Q_0\}$ of minimal orthogonal idempotents adding up to 1. The space A is an R-bimodule, with the bimodule structure defined so that $A_{i,j} = e_i A e_j$ has the K-basis $\{a \mid ha = i, ta = j\}$. The *complete path algebra* $R\langle\langle A \rangle\rangle$ is defined as $\prod_{d \geq 0} A^{\otimes_R d}$ (with the convention $A^{\otimes_R 0} = R$); thus, the elements of $R\langle\langle A \rangle\rangle$ are (possibly infinite) K-linear combinations of *paths* $a_1 \cdots a_d$ such that all a_k are arrows, and $t(a_k) = h(a_{k+1})$ for $1 \leq k < d$. The algebra $R\langle\langle A \rangle\rangle$ is equipped with the $\mathfrak{m}$-adic topology, where $\mathfrak{m} = \mathfrak{m}(A) = \prod_{d \geq 1} A^{\otimes_R d}$.

A *potential* S on A is a (possibly infinite) linear combination of *cyclic* paths $a_1 \cdots a_d$ (that is, those with $t(a_d) = h(a_1)$); since Q is assumed to have no loops, $S \in \mathfrak{m}(A)^2$. Let Pot(A) denote the space of potentials up to the following *cyclical equivalence*: $S \sim S'$ if $S - S'$ lies in the closure of the K-span of the elements $a_1 \cdots a_d - a_2 \cdots a_d a_1$ for all cyclic paths $a_1 \cdots a_d$. We call a pair (A, S) with $S \in$ Pot(A) a *quiver with potential* (QP for short). By a *right-equivalence* between QPs (A, S) and (A', S') we mean an isomorphism of R-algebras $\varphi : R\langle\langle A \rangle\rangle \to R\langle\langle A' \rangle\rangle$ such that $\varphi(S)$ is cyclically equivalent to S'.

For any K-linear form $\xi \in A^\star$, the *cyclic derivative* ∂_ξ is the continuous K-linear map Pot(A) $\to$ R⟨⟨A⟩⟩ acting on cyclic paths by

$$\partial_\xi(a_1 \cdots a_d) = \sum_{k=1}^{d} \xi(a_k) a_{k+1} \cdots a_d a_1 \cdots a_{k-1}.$$

We define the *Jacobian ideal* $J(A, S)$ as the closure of the (two-sided) ideal in $R\langle\langle A \rangle\rangle$ generated by the elements $\partial_\xi(S)$ for all $\xi \in A^\star$. We call the quotient $R\langle\langle A \rangle\rangle / J(A, S)$ the *Jacobian algebra* of S, and denote it by $\mathcal{P}(A, S)$. We prove that any right-equivalence between (A, S) and (A', S') sends $J(A, S)$ onto $J(A', S')$, hence induces an isomorphism of the Jacobian algebras $\mathcal{P}(A, S)$ and $\mathcal{P}(A', S')$.

We say that a QP (C, T) is *trivial* if C is the arrow span of the disjoint union of oriented 2-cycles $\{a_1, b_1\}, \ldots, \{a_N, b_N\}$, and $T = b_1 a_1 + \cdots + b_N a_N$. Then we have $J(C, T) = \mathfrak{m}(C)$, hence $\mathcal{P}(C, T) = R$.

We say that a QP (A, S) is *reduced* if $S \in \mathfrak{m}(A)^3$. Our main tool in dealing with QPs and their mutations is the following Splitting Theorem ([1, Theorem 4.6]).

Theorem 1. *Every QP (A, S) is right-equivalent to the direct sum of a trivial QP $(A, S)_{\mathrm{triv}}$ and a reduced QP $(A, S)_{\mathrm{red}}$. The embedding of $(A, S)_{\mathrm{red}}$ into (A, S) induces an isomorphism of Jacobian algebras. Furthermore, the right-equivalence classes of $(A, S)_{\mathrm{triv}}$ and $(A, S)_{\mathrm{red}}$ are determined by the right-equivalence class of (A, S).*

Now we are ready to introduce the mutation of QPs at any vertex k. Assume that a QP (A, S) is reduced and such that A has no oriented 2-cycles involving k; replacing S if necessary by a cyclically equivalent potential, we can also assume that no cyclic path occurring in S begins (and ends) at k. We define $\mu_k(A, S) = (\overline{A}, \overline{S}) = (\widetilde{A}, \widetilde{S})_{\mathrm{red}}$, where

- $\widetilde{A}$ is the arrow span of the quiver obtained from Q by the first two steps of the above three-step mutation procedure;
- $\widetilde{S} = [S] + \Delta$, where $\Delta = \sum_{h(a)=t(b)=k} [ba]a^\star b^\star$, and $[S]$ is obtained from S by replacing each occurrence of a factor ba (with $ha = tb = k$) in a cyclic path with $[ba]$.

Theorem 2. *The correspondence $\mu_k : (A, S) \to (\overline{A}, \overline{S})$ induces an involution on the set of right-equivalence classes of reduced QPs without oriented 2-cycles through k; that is, the right-equivalence class of $(\overline{A}, \overline{S})$ is determined by that of (A, S), and $\mu_k^2(A, S)$ is right-equivalent to (A, S).*

Note that even if we assume that (A, S) has no oriented 2-cycles, this may be no longer true for $(\overline{A}, \overline{S})$. We prove however that, for every A without oriented 2-cycles, a generic choice of a potential S guarantees that an arbitrary sequence of mutations can be applied to (A, S) without creating oriented 2-cycles.

A *decorated representation* of a QP (A, S) is a pair $\mathcal{M} = (M, V)$, where V is a finite-dimensional R-module, and M is a finite-dimensional $\mathcal{P}(A, S)$-module. We extend the mutations of QPs to the level of their decorated representations. If A has no oriented cycles then $S = 0$, and the decorated representations of $(A, 0)$ are just the decorated quiver representations as defined above; furthermore, in this case, the mutation at every sink or source coincides with the one defined above. A right equivalence for decorated representations is defined in a natural way. We prove that every mutation μ_k sending (A, S) to $(\overline{A}, \overline{S})$ establishes a bijection between the right-equivalence classes of indecomposable decorated representations of (A, S) and $(\overline{A}, \overline{S})$; furthermore, $\mu_k^2(\mathcal{M})$ is right-equivalent to $\mathcal{M}$. Unfortunately, the construction of $\mu_k(\mathcal{M})$ is too long to present it here.

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