Mathematisches Forschungsinstitut Oberwolfach

Report No. 24/2007

Progress in Surface Theory

Organised by
Uwe Abresch (Bochum)
Josef Dorfmeister (München)
Masaaki Umehara (Osaka)

April 29th – May 5th, 2007

Abstract. The theory of surfaces has undergone substantial changes in recent years, with many different active areas at this point in time. It has become mainstream to study minimal surfaces and constant mean curvature surfaces in more general target spaces than space forms. So far, the main strands of development center around methods from the theory of elliptic differential equations, complex analytic methods, and integrable systems techniques. The study of singular surfaces has widened the classical point of view and seems to be a particularly promising direction of research when considering certain moduli space problems.

Mathematics Subject Classification (2000): 17B80,30F15, 35Q53, 37K25, 37K35, 53A04, 53A05, 53A07, 53A10, 53A30, 53C20 ,53C25, 53C40, 53C42, 53C43, 53D12, 57R45, 58A30,58D10, 58E20, 58F07, 58J05, 58J10,,53C20, 53A0, 70H06.

Introduction by the Organisers

The workshop *Progress in Surface Theory*, organised by Uwe Abresch (Bochum), Josef Dorfmeister (München), and Masaaki Umehara (Osaka) was held April 29$^{\text{th}}$–May 5$^{\text{th}}$, 2007. In recent years, Surface Theory has become a very active field of research again. Studying surfaces in more general ambient spaces than space forms and surfaces with certain types of mild singularities has become much more mainstream, even though a complete understanding of all important surface classes in space forms is still an issue. In order to facilitate these developments, it has been crucial to adopt methods from different fields. Methods from the theory of elliptic differential equations, complex analytic methods, and integrable systems techniques are now widely used.

It has been a major goal of this workshop to bring together many of the best experts in all these substantially different fields.

The workshop was attended by 26 participants from seven different countries, with particularly large contingents from Germany, Japan, France, and the US. This is quite an impressive list for a half-size workshop like this. On the other hand, every participant could be given the opportunity to present his/her work.

The official program consisted of 25 talks at 40–50 minutes each. In addition, two evenings were used for more informal presentations concerning the computer visualization of surfaces. Most lectures were well prepared and well presented and provided a lot of useful background information, facilitating the interaction between the different groups. It also provided an opportunity for young, in particular young German, mathematicians to have direct contact with a broad international group of established mathematicians and their work.

While the schedule was quite full, just for this reason the workshop indeed provided a fairly complete picture of the currently active areas of surface theory. Nevertheless, in the afternoons and the evenings there was still quite some time left for individual discussions, time that was extensively and intensively used by the participants. Several (even senior) participants commented to us that they had learned quite a bit from the talks and the interactions that they had with other participants. Certainly, the great setting and the superb atmosphere, for which Oberwolfach is reknowned around the world, making it a unique conference place, provided the environment that generated many fruitful discussions.

We feel that the meeting was exciting and highly sucessful, in particular in view of some startling new results and the extraordinary amount of scientific interaction among the participants.

Here is a brief summary of the mathematical contents: one area of surface theory where a lot of progress has been made concerns qualitative estimates for minimal and constant mean curvature (cmc) surfaces in 3-manifolds with bounded geometry. W. H. Meeks has given a nice survey explaining the key techniques from analysis that these results are based on.

The move to more general ambient spaces than space forms got a big push when U. Abresch and H. Rosenberg discovered a natural holomorphic quadratic differential on cmc surfaces in a wide class of 3-dimensional homogeneous bundles. This discovery has widened the realm for explicit constructions (see, e.g., the generalized Lawson correpondences of B.Daniel).

L. Hauswirth presented interesting consequences of this: translating via the generalized Lawson correspondence minimal surfaces in the Heisenberg group $\mathsf{Nil}(3)$ into cmc $1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and vice versa. (As another high point in this development, I. Fernandez and P. Mira posted a paper shortly after the workshop, on arXiv, in which they solved the Bernstein problem in $\mathsf{Nil}(3)$.) I. Taimanov and his student D. Berdinsky have been working on extending the integrable systems approach to cmc tori in this class of homogeneous bundles.

H. Rosenberg himself explained a particularly nice application of minimal surface theory in $\mathbb{H}^2 \times \mathbb{R}$ that he has made in collaboration with P. Collin. They have constructed a surjective *harmonic diffeomorphism* from the complex plane $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$, thereby disproving a conjecture by S. T. Yau. By the uniformization theorem, this diffeomorphism cannot be conformal. Moreover, it is known that there is no surjective harmonic diffeomorphism from the hyperbolic plane $\mathbb{H}^2$ onto the entire complex plane $\mathbb{C}$.

The *classical theory of minimal surfaces* in euclidean space is not a closed subject yet either. T. Shoda constructed irreducible immersed trigonal minimal surfaces in flat tori $\mathbb{R}^n/\Lambda$ with $n > 3$ that cannot be immersed into some $\mathbb{R}^3/\Lambda'$. Moreover, generalizing earlier work of Nadirashvili, F. Martin has established a number of highly interesting *density results*. The most notable one states that there is an open dense set in the space of Jordan curves such that for each $\Gamma\colon \mathbb{S}^1 \to \mathbb{R}^3$ in this set there exists a complete minimal immersion $f\colon \mathbb{D} \looparrowright \mathbb{R}^3$ that extends continously to the closed disc $\bar{\mathbb{D}}$ in such a way that the extension to the boundary $\partial\mathbb{D}$ coincides with Γ upon identifying $\partial\mathbb{D}$ and $\mathbb{S}^1$ by a suitable homomorphism. Moreover, f is not proper and the surface is hyperbolic.

The topic of cmc surfaces was represented in both its flavours, the qualitative theory and the integrable systems approach. The talks covering the *qualitative theory* centered around bifurcation and deformation arguments. J. Ratzkin presented a nice nondegeneracy result for the spectrum of properly embedded cmc surfaces of genus 0 in a solid slab in euclidean space. By a well-known theorem of Korevaar, Kusner, and Solomon the ends of such a surface are asymptotic to undoloids. The crucial step in Ratzkin's approach was to use an infinitesimal version of the Lawson correspondence in order to translate between even and almost odd Jacobi fields. A somewhat different, though closely related point of view on the deformation theory of Alexandrov embedded k-noids was presented by K. Große-Brauckmann, a talk which created a particular amount of follow-up investigations. F. Pacard on the other hand explained how one can foliate the ends of an asymptotically hyperbolic manifold by cmc hypersurfaces.

The k-noids were also a central theme in the work related to the *integrable systems approach* to cmc surfaces. J. Dorfmeister explained how all cmc trinoids of genus $g = 0$ can be constructed from basic complex potenials. M. Kilian presented a complementary result that he had established in joint work with W. Rossman and N. Schmitt: a holomorphic perturbation of the potential of an undoloid yields an annular cmc immersion that is $\mathcal{C}^\infty$-asymptotic to the reference undoloid. In particular, he gave potentials describing k-noids with $k > 3$. There is hope that a combination of classical deformation theory and integrable systems methods will eventually lead to an understanding of the moduli spaces of k-noids.

The *discrete aspect* of the integrable systems approach was presented in the talk of W. Rossman. He described a 1-parameter family of discrete systems that can be used to study how the shape of asymptotically undoloidal ends of surfaces in $\mathbb{H}^3$ with constant mean curvature $H > 1$ changes when H approaches 1, and one enters the realm of Bryant surfaces.

S. Kobayashi on the other hand used the *inherently complex nature* of the integrable systems method in order to give a uniform description of many 2-dimensional 1-1 integrable systems as real forms of complex 2-dimensional constant mean curvature surfaces.

Integrable systems as a *unifying framework* also played a key role in the talks by D. Brander and C. L. Terng. D. Brander established a link between integrable systems based on 3-involution loop groups and geometric problems in certain classes of high-dimensional manifolds. In a similar spirit, C. L. Terng explained various applications of a fairly general class of systems, the U/K-systems, to surface theory.

F. E. Burstall, on the other hand, focussed on *specific integrable systems* modelled on 4-dimensional 4-*symmetric spaces*, systems that are expected to tie in nicely with the concept of (generalized) Darboux transformations. The generalized Darboux transformations themselves were introduced by K. Leschke, and the Willmore surfaces were the central topic in the lecture by Ch. Bohle. In a similar context, F. Pedit discussed estimates for the Willmore functional for conformal tori of spectral genus 0.

Another extension of the classical approach to surface theory, represented prominently in this workshop, is the inclusion of singularities. Since, as is well known, singularities cannot be avoided in some cases, it is natural to consider a definition of "surfaces" which now admits singularities of certain types.

Cuspidal edges and swallowtails occur quite frequently as singularities. On the other hand, these two types of singularities are known to also occur generically in wave fronts. Therefore, a detailled investigation of the geometry of wave fronts is of great importance.

M. Umehara and M. Kokubu gave talks on flat surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ as wave fronts, and presented several global results, including, the classification theorem of complete flat fronts in $\mathbb{R}^3$ and the orientability of flat surfaces in $\mathbb{H}^3$. On the other hand, K. Yamada gave a talk on constant mean curvature 1 surfaces in de Sitter 3-space, which are the projection of null immersed curves in $\mathsf{Sl}(2, \mathbb{C})$. He explained in a very clear and instructive talk that such surfaces admit cuspidal edges, swallowtails, and cuspidal cross caps in general, and illustrated his results by computer generated pictures.

It is important to note that the extension to "mildly singular" surfaces extends the class of classical surfaces considerably. It has the advantage of leading to more complete moduli spaces and will, as we expect, eventually lead to realistic classes of surfaces in general 3-dimensional manifolds, also reflecting many applications.

At the very end, we would like to mention the talk on flows on spaces of curves U. Pinkall. His talk intertwined classical surface theory, integrable systems methods, discrete surface theory and computer visualization and was particularly well received.

This particular workshop included two evenings with visualization presentations by R. Palais, *Virtual Math Museum*, F. Pedit, *Gallery of Surfaces*, and U. Pinkall, *JReality*.
(http://VirtualMathMuseum.org,
http://www.mathematik.uni-tuebingen.de/ab/Differentialgeometrie/
gallery/index.html,
http://www3.math.tu-berlin.de/jreality/).

Workshop: Progress in Surface Theory

Table of Contents

Abstracts

Entire minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ and the construction of surjective harmonic diffeomorphisms from the complex plane $\mathbb{C}$ to the hyperbolic plane $\mathbb{H}^2$

Harold Rosenberg
(joint work with Pascal Collin)

We study complete minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$, which take asymptotic boundary values $+\infty$ and $-\infty$ on alternating sides of an ideal inscribed polygon Γ in $\mathbb{H}^2$. We give necessary and sufficient conditions on the "lengths" of the sides of the polygon (and all inscribed polygons in Γ) that ensure the existence of such a graph. We then apply this to construct entire minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ that are conformally the complex plane $\mathbb{C}$. The vertical projection of such a graph yields a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}^2$, disproving a conjecture of Rick Schoen.

In 1952, E. Heinz proved that there is no harmonic diffeomorphism from a disk onto the complex plane $\mathbb{C}$, with the euclidean metric [2]. He used this to give another proof of Bernstein's theorem: An entire minimal graph over the euclidean plane is a plane.

Later, R. Schoen and S.T. Yau, asked whether Riemannian surfaces which are related by a harmonic diffeomorphism are quasi-conformally related. In that direction, R. Schoen conjectured there is no harmonic diffeomorphism from $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}^2$ [5], [6] and [4].

In my paper with Pascal Collin [1], we construct harmonic diffeomorphisms from $\mathbb{C}$ onto $\mathbb{H}^2$. We use entire minimal graphs to construct these examples (E. Heinz used the non-existence of harmonic diffeomorphisms from $\mathbb{H}^2$ onto $\mathbb{C}$ to prove the non-existence of non-trivial entire minimal euclidean graphs).

Consider the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ and entire minimal graphs Σ^2 defined over $\mathbb{H}^2$. The vertical projection $\Sigma^2 \longrightarrow \mathbb{H}^2$ is a surjective harmonic diffeomorphism, so we will solve the problem by constructing entire minimal graphs Σ^2 that are conformally $\mathbb{C}$ (Theorem 3). Notice that the horizontal projection $\Sigma^2 \longrightarrow \mathbb{R}$ is a harmonic function on Σ^2 when Σ^2 is a minimal surface. Hence if this height function is bounded on Σ^2, Σ^2 is necessarily hyperbolic (conformally the unit disc). So we must look for unbounded minimal graphs.

Here is the idea of the construction. Let Γ be an ideal geodesic polygon in $\mathbb{H}^2$ with an even number of sides (the vertices of Γ are at infinity). We give necessary and sufficient conditions on the geometry of Γ which ensure the existence of a minimal graph u over the polygonal domain D bounded by Γ, which takes the values $+\infty$ and $-\infty$ on alternate sides of Γ [1, § 3]. This is a Jenkins-Serrin type theorem at infinity [3]. We call such graphs u over D ideal Scherk graphs and we show their conformal type is $\mathbb{C}$ [1, § 5]. These graphs are complete minimal surfaces of finite total curvature, an integer multiple of 2π.

We attach certain ideal quadrilaterals to all of the sides of Γ (outside of D) so that the extended polygonal domain D_1 admits an ideal Scherk graph. We do this so that D_1 depends on a small parameter τ, and a minimal Scherk function $u_1(\tau)$ defined over D_1 satisfies the following. Given a fixed compact disk K_0 in the domain D, $u_1(\tau)$ is as close as we wish to u over K_0, for τ sufficiently small. Also ideal Scherk graphs are conformally $\mathbb{C}$ so there is a compact disk K_1 (containing K_0) in each D_1, so that the conformal type of the annulus in the graph of $u_1(\tau)$, over $K_1 - K_0$, is greater than one. Now fix τ and do the same process to enlarge D_1, attaching certain ideal quadrilaterals to all of the sides to obtain a domain, admitting an ideal Scherk graph $u_2(\tau')$, which is as close as we want to $u_1(\tau)$ on K_1 for τ' sufficiently small; then the conformal moduli of the annulus in the graph of $u_2(\tau')$ over $K_1 - K_0$ remains greater than one. As before take K_2 with the same condition on modulus for $u_2(\tau')$ over $K_2 - K_1$. The entire minimal graph Σ^2 is obtained by continuing this process and choosing a convergent subsequence. The conformal type of Σ^2 is $\mathbb{C}$ because we write $\mathbb{H}^2$ as the union of an increasing sequence of compact disks K_n, and each annulus in Σ^2 over each $K_{n+1} - K_n$ has conformal modulus at least one.

We mention some related problems and results. What can be said of the asymptotic values of an entire minimal graph? What sort of Fatou-type theorems can one expect? It is not hard to construct an entire minimal graph that has radial limits only on a countable set. However, we do conjecture that radial limits exist almost everywhere if the function is bounded.

The Jenkins-Serrin theorem at infinity Pascal and I prove provides examples of complete simply connected minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, the values of the total curvature of these graphs is an integer multiple of 2π. Laurent Hauswirth and I have proved that any complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ has total curvature an integer multiple of 2π. For the moment, the only examples we know are simply connected. It is an interesting problem to construct such examples that are annuli, or more complicated topology.

References

[1] P. Collin, H. Rosenberg, *Construction of harmonic diffeomorphisms and minimal graphs*, preprint, Univ. Paris 7, Jan. 2007.

[2] E. Heinz, *Über die Lösungen der Minimalflächengleichung*, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. **1952** (1952), 51–56.

[3] H. Jenkins, J. Serrin, *Variational problems of minimal surface type*, II. *Boundary value problems for the minimal surface equation*, Arch. Rational Mech. Anal. **21** (1966), 321–342.

[4] V. Markovic, *Harmonic diffeomorphisms and conformal distortion of Riemann surfaces*, Comm. Anal. Geom. **10** (2002), 847–876.

[5] R. Schoen, *The role of harmonic maps in rigidity and deformation problems*, in: Complex Geometry, Proceedings of the Osaka Intern. Conf. 1988, 179–200, Lecture Notes in Pure and Applied Mathematics **143**, Marcel Dekker Inc. 1993.

[6] R. Schoen, S. T. Yau, *Lectures on harmonic maps*, in: Conference Proceedings and Lecture Notes in Geometry and Topology II, Intern. Press, Cambridge MA 1997.

Twistors, 4-symmetric spaces and integrable systems

Francis Burstall
(joint work with Idrisse Khemar)

An integrable system. Let us begin by describing an integrable system that has arisen in the general theory of Terng [8] and work of Hélein–Romon [4, 5, 6] and Khemar [7].

Our first ingredient is a Lie algebra $\mathfrak{g}$ together with an automorphism $\tau \in \mathrm{Aut}(\mathfrak{g})$ with $\tau^4 = 1$. We have an eigenspace decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1} \tag{1}$$

so that τ has eigenvalue $e^{2\pi ik/4}$ on $\mathfrak{g}_k$.

Now let Σ be a Riemann surface and $\alpha \in \Omega^1_\Sigma \otimes \mathfrak{g}$ be a $\mathfrak{g}$-valued 1-form on Σ. The complex structure J^Σ of Σ gives a type decomposition $\alpha = \alpha^{1,0} + \alpha^{0,1}$ while (1) gives a second decomposition:

$$\alpha = \alpha_0 + \alpha_2 + \alpha_1 + \alpha_{-1}.$$

With this in hand, our integrable system comprises the following equations on α:

$$\alpha_1^{0,1} = 0 \tag{2a}$$

$$\mathrm{d}\,\alpha_2^{1,0} + [\alpha_0 \wedge \alpha_2^{1,0}] = 0 \tag{2b}$$

$$\mathrm{d}\,\alpha + \tfrac{1}{2}[\alpha \wedge \alpha] = 0. \tag{2c}$$

The equations (2) have a *zero-curvature* formulation: for $\lambda \in \mathbb{C}^\times$, define $\alpha_\lambda \in \Omega^1_\Sigma \otimes \mathfrak{g}^{\mathbb{C}}$ by

$$\alpha_\lambda = \lambda^2\alpha_2^{1,0} + \lambda\alpha_1^{1,0} + \alpha_0 + \lambda^{-1}\alpha_{-1}^{0,1} + \lambda^{-2}\alpha_2^{0,1}.$$

Then, in the presence of (2a), equations (2b) and (2c) are equivalent to the demand that $\mathrm{d}\,\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ (thus $\mathrm{d} + \alpha_\lambda$ is a flat connection), for all $\lambda \in \mathbb{C}^\times$.

Thus the methods of integrable system theory (see, for example, [2]) apply to give generalised Weierstrass formulae, algebro-geometric solutions, spectral deformations and so on.

My purpose in this note is to describe the geometry behind this integrable system.

4-symmetric spaces and twistor spaces. Let us integrate our set-up: let G be a Lie group with Lie algebra $\mathfrak{g}$ admitting $\tau \in \mathrm{Aut}(G)$ with $\tau^4 = 1$. Set $H = \mathrm{Fix}(\tau)$ and contemplate the 4-symmetric space G/H. A solution α of (2) integrates, by virtue of (2c), at least locally, to give a map $g : \Sigma \to G$ with $g^{-1}\,\mathrm{d}\,g = \alpha$ and thus a map $j = gH : \Sigma \to G/H$. Since the system is gauge-invariant, it is the map j that carries the geometry.

Now set $\sigma = \tau^2$ so that $\sigma^2 = 1$ and let $K = \mathrm{Fix}(\sigma)$. We assume henceforth that K is compact so that G/K is a Riemannian symmetric space. Our 4-symmetric space G/H therefore fibres over a Riemannian symmetric space G/K and I claim that this fibration factors through the twistor fibration of G/K.

For this, recall that the twistor space $J(N)$ of an even-dimensional Riemannian manifold is the bundle of orthogonal almost complex structures on TN:

$$J(N) = \{j \in \mathfrak{o}(TN) : j^2 = -1\}.$$

In the case at hand, $(T_{eK}G/K)^{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ so that $\tau_{|T_{eK}G/K}$ is such an almost complex structure. Extending by equivariance, we see that the fibration $\pi : G/H \to G/K$ factors $G/H \hookrightarrow J(G/K) \to G/K$.
Punchline: *any* 4*-symmetric space is a submanifold of the twistor space of a Riemannian symmetric space.*

Now, with $N = G/K$, a solution α of (2) delivers $j : \Sigma \to G/H \subset J(N)$ and then $\phi = \pi \circ j : \Sigma \to N$. We may therefore view j as an orthogonal almost complex structure on $\phi^{-1}TN$. Any frame $g : \Sigma \to G/H$ with $j = gH$ gives an isomorphism $\phi^{-1}TN^{\mathbb{C}} \cong \Sigma \times (\mathfrak{g}_1 \oplus \mathfrak{g}_1)$ under which j corresponds to τ; $\mathrm{d}\,\phi$ with $\alpha_1 + \alpha_{-1}$ and $\mathrm{d} + \alpha_0 + \alpha_2$ with $\overline{\nabla}$, the Levi-Civita connection of N, pulled backed by ϕ. It follows that α_2 corresponds with $\frac{1}{2}j(\overline{\nabla}j)$ so that $\mathrm{d} + \alpha_0$ corresponds to the connection $D = \overline{\nabla} - \frac{1}{2}j(\overline{\nabla}j)$ on $\phi^{-1}TN$. We can now read off the significance of the equations (2):

First (2a) is exactly the assertion that ϕ is holomorphic with respect to j: $\mathrm{d}\,\phi \circ J^{\Sigma} = j \circ \mathrm{d}\,\phi$. This means that j is a *twistor lift* of ϕ. As a consequence, ϕ is a (branched) conformal immersion.

Equation (2b) amounts to the demand that $\mathrm{d}^D j(\overline{\nabla}j)^{1,0} = 0$. This is the vertical part of a harmonic map equation for j and, at least when $\dim N = 4$, is exactly the condition that j be a harmonic section in the sense of C.M. Wood [9]. We say that such a twistor lift is *vertically harmonic.*

It is interesting that our integrable system is solely concerned with the geometry of j *qua* map into twistor space. The only role played by the 4-symmetric space G/H is to provide a (possibly empty) algebraic constraint on j.

4 dimensions. We have seen that our theory is one about twistor lifts of conformal immersions in a symmetric space. In general, there are many such lifts but, in favourable circumstances, there are distinguished lifts and then our theory is one of the conformal immersions themselves.

In particular, suppose that $\phi : \Sigma \to N$ is a conformal immersion into an oriented 4-manifold. The twistor space of N has two components $J_\pm(N)$, each a S^2-bundle over N, and there are unique twistor lifts $j_\pm : \Sigma \to J_\pm(N)$ given by choosing one of the two orthogonal almost complex structures on the normal bundle of ϕ. The vertical harmonicity of these twistor lifts has been studied by Hasegawa [3].

Examples. Let us enumerate the 4-symmetric spaces that fibre over a 4-dimensional Riemannian symmetric space. There are three cases:

1. N has constant sectional curvatures. Here both $J_+(N)$ and $J_-(N)$ are themselves 4-symmetric so there is no algebraic constraint on j to take into account. Using the Codazzi equation, one sees that vertical harmonicity of j is equivalent to the holomorphicity of the mean curvature vector of ϕ (where we use j to view

$N\Sigma$ as a complex line bundle equipped with the Koszul–Malgrange holomorphic structure). Thus, we conclude, as has Hasegawa [3],

Theorem. *ϕ has vertically harmonic twistor lift if and only if it has holomorphic mean curvature vector.*

It is intriguing that these surfaces already solve an integrable system in conformal geometry: they are constrained Willmore surfaces (see Bohle [1] for the case $N = \mathbb{R}^4$).

2. N is Hermitian symmetric. Here we fix orientations so that the ambient complex structure J^N is a section of $J_+(N)$ and contemplate the subbundle $Z = \{j \in J_+(N) : \{j, J^N\} = 0\}$ of almost complex structures that anti-commute with J^N. This is a circle bundle over N (it is the unit circle bundle in the canonical bundle of N) and is 4-symmetric.

It is easy to see that j_+ takes values in Z if and only if ϕ is Lagrangian and then that j_+ is vertically harmonic if and only if ϕ is also Hamiltonian stationary:

Theorem. *$j_+ : \Sigma \to Z$ is vertically harmonic if and only if ϕ is a Hamiltonian stationary Lagrangian immersion.*

This provides a conceptual explanation for the integrable system appearing in the work of Hélein–Romon.

3. $N = \mathbb{C}P^2$. In this case, there is a second 4-symmetric space fibering over N since $J_-(N)$ is also 4-symmetric. Here there are ambient curvature terms in the Codazzi equation but the part which anti-commutes with j_- vanishes and one is led once more to holomorphic mean curvature vector:

Theorem. *j_- is vertically harmonic if and only if ϕ has holomorphic mean curvature vector.*

References

[1] C. Bohle, *Constrained Willmore tori in the 4-sphere*, Preprint, 2006.

[2] D. Brander and J. Dorfmeister, *Generalized DPW method and an application to space forms*, arXiv:math.DG/0604247, 2006.

[3] K. Hasegawa, *On surfaces whose twistor lifts are harmonic sections*, J. Geom. Phys. **57** (2007), 1549–1566.

[4] F. Hélein and P. Romon, *Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces*, Differential geometry and integrable systems (Tokyo, 2000), Contemp. Math., vol. 308, Amer. Math. Soc., Providence, RI, 2002, pp. 161–178.

[5] ———, *Hamiltonian stationary Lagrangian surfaces in* $\mathbb{C}^2$, Comm. Anal. Geom. **10** (2002), no. 1, 79–126.

[6] ———, *Hamiltonian stationary tori in the complex projective plane*, Proc. London Math. Soc. (3) **90** (2005), 472–496.

[7] I. Khemar, *Surfaces isotropes de* $\mathbb{O}$ *et systèmes intégrables*, arXiv:math.DG/0511258, 2005.

[8] C.-L. Terng, *Geometries and Symmetries of Soliton equations and Integrable Elliptic equations*, arXiv:math/0212372, 2002.

[9] C.M. Wood, *Harmonic sections of homogeneous fibre bundles*, Differential Geom. Appl. **19** (2003), 193–210.

Density theorems for complete minimal surfaces and applications

Francisco Martín
(joint work with A. Alarcón, L. Ferrer, N. Nadirashvili)

The study of complete minimal surfaces of hyperbolic type has come up with some absolutely extraordinary counterexamples to show that the things most of us thought were true are not true. This report contains some density results that the author recently obtained in two different papers; one of them with Nikolai Nadirashvili [11], and the other one with Antonio Alarcón and Leonor Ferrer [1]. These results and their applications contradict our former intuition about the behavior of minimal surfaces in Euclidean space.

The first theorem asserts that there exists a family of Jordan curves in $\mathbb{R}^3$ such that the solution of the Plateau problems associated to these curves are complete minimal disks. By the convex hull property the above minimal disks are bounded. One can think that curves of this kind (spanned by complete minimal disks) are rare, but it turns out that they are dense in the space of all Jordan curves endowed with the Hausdorff distance. The precise statement of the theorem is the following:

Theorem 1 (—, Nadirasvili)**.** *There exist complete, conformal, minimal immersions*

$$f : \mathbb{D} \longrightarrow \mathbb{R}^3$$

so that they admit a continuous extension to the closed disk, $F : \overline{\mathbb{D}} \longrightarrow \mathbb{R}^3$. The map $F_{|\mathbb{S}^1}$ is an embedding and $F(\mathbb{S}^1)$ is a non-rectifiable Jordan curve with Hausdorff dimension 1.

Moreover, for any Jordan curve Γ in $\mathbb{R}^3$ and any $\xi > 0$, we can find a minimal immersion satisfying the above conditions and such that $\delta^H(\Gamma, F(\mathbb{S}^1)) < \xi$, where δ^H means the Hausdorff distance.

It is clear that the examples provided by the above theorem are counterexamples to the Calabi-Yau problem for minimal surfaces, whose original statement was given in 1965 by E. Calabi [2] (see also [3] and [14]). This author conjectured that *a complete minimal hypersurface in $\mathbb{R}^n$ must be unbounded.*

The second theorem is related with the *properness* of complete bounded minimal surfaces. Recall that an immersed submanifold of $\mathbb{R}^n$ is proper if the pre-image through the immersion of any compact subset of $\mathbb{R}^n$ is compact in the submanifold. It is clear from the definition that a proper minimal surface in $\mathbb{R}^3$ must be unbounded, so the minimal disks in Theorem 1 are not proper in $\mathbb{R}^3$. Morales and the author [7, 8, 12] proved that every convex domain $B \subseteq \mathbb{R}^3$ (not necessarily bounded or smooth) admits a complete properly immersed minimal disk, $f : \mathbb{D} \to B$, where by proper we mean in this case that $f^{-1}(C)$ is compact for any $C \subset B$ compact. These examples disproved a longstanding conjecture, which asserted that a complete minimal surface (without boundary) with finite topology and which is properly immersed in $\mathbb{R}^3$ should be parabolic. Recently [9] these same authors improved on their original techniques and were able to show that every bounded domain with $C^{2,\alpha}$-boundary admits a complete properly immersed minimal disk whose limit set is close to a prescribed simple closed curve on the boundary of the domain. In contrast to these existence results for complete properly immersed minimal disks in bounded domains, Meeks, Nadirashvili and the author [6] proved the existence of bounded open regions of $\mathbb{R}^3$ which do not admit complete properly immersed minimal surfaces with an annular end. In particular, these domains do not contain a complete properly immersed minimal surface with finite topology.

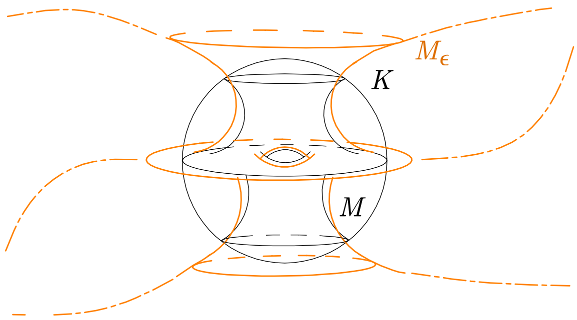


FIGURE 1. The minimal surface M can be approximated on the compact K by a properly immersed minimal surface M_ϵ.

The above line of results is the frame of our following theorem:

Theorem 2 (Alarcón, Ferrer, —). *Properly immersed, hyperbolic minimal surfaces of finite topology are dense in the space of all properly immersed minimal surfaces in $\mathbb{R}^3$, endowed with the topology of smooth convergence on compact sets.*

Note that the best understood families of minimal surfaces in $\mathbb{R}^3$ (properly embedded, periodic, finite total curvature, finite type,...) are included in the statement of Theorem 2. Furthermore, if we do not care about properness, then we can prove that:

Theorem 3 (Alarcón, Ferrer, —). *Complete (hyperbolic) minimal surfaces are dense in the space of minimal surfaces in $\mathbb{R}^3$ (without boundary) endowed with the topology of C^k convergence on compact sets, for any $k \in \mathbb{N}$.*

In the case of hyperbolic minimal surfaces we have an infinite number of linearly independent Jacobi fields. This is the key point in the proof of the above theorem. This enormous capability of deformation allows us to "model" a given compact piece of a hyperbolic minimal surface in order to approximate any other minimal surface with the same topological type (see Figure 1).

One of the most interesting applications of Theorem 2 is the construction of the first example of a complete minimal surface properly immersed in $\mathbb{R}^3$ with an uncountable number of ends.

Theorem 4. *There exists a domain $\Omega \subset \mathbb{C}$ and a complete proper minimal immersion $\psi : \Omega \to \mathbb{R}^3$ whose space of ends is a Cantor set.*

At this point, it is important to note that complete proper minimal surfaces in $\mathbb{R}^3$ with uncountably many ends cannot be embedded as a consequence of a result by Collin, Kusner, Meeks and Rosenberg [4]. A more general application of Theorem 2 is the following:

Theorem 5. *Any planar domain can be properly and minimally immersed in* $\mathbb{R}^3$.

Again, embeddedness creates a dichotomy in the global theory of minimal surfaces, because it has been recently proved by W. H. Meeks, J. Pérez and A. Ros [10] that:

Theorem 6 (Meeks, Pérez, Ros)**.** *The only properly embedded, non-flat, minimal planar domains in* $\mathbb{R}^3$ *are the catenoid, the helicoid and Riemann's minimal examples.*

Finally, we would like to point out that Theorem 5 represents a partial answer to a more general conjecture by Meeks:

Conjecture 1 (Meeks)**.** *Any open surface can be properly and minimally immersed in* $\mathbb{R}^3$.

This conjecture was proven subsequently to the workshop by L. Ferrer, W.H. Meeks and the author [5]. More generally, they have demonstrated that:

Theorem 7 (Ferrer, Martín, Meeks)**.** *Let* $\mathcal{D}$ *be a domain which is convex (possibly* $\mathcal{D} = \mathbb{R}^3$*) or smooth and bounded. Given any open surface* M*, there exists a complete proper minimal immersion* $f\colon M \to \mathcal{D}$.

The above work was conceived during the Oberwolfach workshop.

References

[1] A. Alarcón, L. Ferrer, F. Martín, *Density theorems for complete minimal surfaces in* $\mathbb{R}^3$, to appear in G.A.F.A.

[2] E. Calabi, *Problems in differential geometry.* Ed. S. Kobayashi and J. Ells, Jr., Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965. Nippon Hyoronsha Co., Ltd., Tokyo (1966) 170.

[3] S. S. Chern, *The geometry of G-structures.* Bull. Amer. Math. Soc. **72** (1966), 167–219.

[4] P. Collin, R. Kusner, W. H. Meeks III, and H. Rosenberg, *The topology, geometry and conformal structure of properly embedded minimal surfaces.* J. Differential Geom. **27** (2004), 377–393.

[5] L. Ferrer, F. Martín and W. H. Meeks, III. *The existence of proper minimal surfaces of arbitrary topological type.* Preprint.

[6] F. Martín, W. H. Meeks III, N. Nadirashvili, *Bounded domains which are universal for minimal surfaces.* American J. Math. **129** (2007), 455-461.

[7] F. Martín and S. Morales, *On the asymptotic behavior of a complete bounded minimal surface in* $\mathbb{R}^3$. Trans. Amer. Math. Soc. **356** (2004), no. 10, 3985–3994.

[8] F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of* $\mathbb{R}^3$. Duke Math. J. **128** (2005), 559–593.

[9] F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of* $\mathbb{R}^3$ *(II): The behavior of the limit set.* Comment. Math. Helv. **81** (2006), 699–725.

[10] W. H. Meeks III, J. Pérez, A. Ros. *Properly embedded minimal planar domains.* Preprint.

[11] F. Martín, N. Nadirashvili, *A Jordan curve spanned by a complete minimal*, Arch. Ration. Mech. Anal. **184** (2007), 285-301.
[12] S. Morales, *On the existence of a proper minimal surface in* $\mathbb{R}^3$ *with a conformal type of disk.* G.A.F.A. **13** (2003), 1281–1301.
[13] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces.* Invent. Math. **126** (1996), 457–465.
[14] S.-T. Yau, *Problem section. Seminar on Differential Geometry,* pp. 669–706, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J., 1982.

Harmonic maps and minimal surfaces

Laurent Hauswirth

We consider $X : \Sigma \to \mathbb{H}\times\mathbb{R}$ a minimal surface conformally embedded in $\mathbb{H}\times\mathbb{R}$, $\mathbb{H}$ the hyperbolic plane. We denote by $X = (F, h)$ the immersion where $F : \Sigma \longrightarrow \mathbb{H}$ is the vertical projection to $\Sigma = \Sigma \times (0)$, and $h : \Sigma \longrightarrow \mathbb{R}$ the horizontal projection. The metric induced by the immersion is of the form $ds^2 = \lambda^2(z)|dz|^2$.

F is a harmonic map and h is a real harmonic function. We denote the disk with the hyperbolic metric by $(D, \sigma^2(u)|du|^2)$. The harmonic map equation in the complex coordinate $u = u_1 + iu_2$ of D is

$$F_{z\bar{z}} + 2(\log \sigma \circ F)_u F_z F_{\bar{z}} = 0$$

where $2(\log \sigma \circ F)_u = 2\bar{F}(1 - |F|^2)^{-1}$. In the theory of *harmonic maps* there is the holomorphic *quadratic Hopf differential* associated to F:

$$Q(F) = (\sigma \circ F)^2 F_z \bar{F}_z (dz)^2 := \phi(z)(dz)^2$$

The function ϕ depends on z, whereas $Q(F)$ does not.

Since we consider conformal immersions, we have e $(h_z)^2(dz)^2 = -Q(F)$. Then the zeroes of Q are double and we can define η as the holomorphic one form $\eta = \pm 2i\sqrt{Q}$. The sign is chosen so that:

$$h = \mathrm{Re} \int \eta$$

Then we define the function ω on Σ by $n_3 = \tanh \omega$ (n_3 is the third coordinate of the normal). By identification we have

$$\omega = \frac{1}{2} \ln \frac{|F_z|}{|F_{\bar{z}}|}$$

The metric $ds^2 = \lambda|w|^2$ is given in a local coordinate z by $ds^2 = 4\cosh^2 \omega |Q|$ We remark that the zeroes of Q correspond to the poles of ω so that the immersion is well defined. Moreover the zeroes of Q are points of Σ, where the tangent plane is horizontal.

It is a well known fact that harmonic mappings satisfy the Böchner formula (R. Schoen-S.T. Yau):

$$\triangle_0 \omega = 2\sinh(2\omega)|Q|$$

where $\triangle_0$ denote the laplacian in the euclidean metric $|dz|^2$.

To construct minimal surfaces it suffices to obtain the harmonic map. T.Y. Wan derive from Q, an harmonic map F. He solve the *sinh*-Gordon equation (10). It is the Gauss equation of a $H = 1/2$ surface in the Minkovski space $M^{2,1}$. By Gauss-Codazzi equation he obtain F as the Gauss map of the constant mean curvature surface $H = 1/2$ in $M^{2,1}$:

Theorem 1. *(-,Sa Earp, Toubiana) Let $\Omega \subset \mathbb{C}$ be a simply connected open set and consider two conformal minimal immersions $X, X^* : \Omega \to \mathbb{M} \times \mathbb{R}$ which are isometric each other. Assume $Q = Q^*$. Then X and X^* differ from an isometry of $\mathbb{H} \times \mathbb{R}$. In summary $\{\omega, Q\}$ define uniquely (up to an isometry) the minimal surface. If $Q^* = e^{2i\theta} Q$ then X and X^* are θ-associate.*

We can generalize the Krust's theorem on minimal surfaces of $\mathbb{R}^3$:

Theorem 2. *(-,Toubiana, Sa Earp) If we consider a minimal graph $X(\Omega)$ on a convex domain $F(\Omega)$ in $\mathbb{H}$, then the associate surface $X^\theta(\Omega)$ is a graph.*

Examples of genus zero We consider minimal surfaces transverse to horizontal section. The curve $\mathbb{H} \times \{t\} \cap X$ are of constant curvature curves $k_{\mathbb{H}}(t)$. Since the third coordinate $dh \neq 0$, the holomorphic quadratic differential has no zeroes. Then we can assume $Q = \frac{1}{4}(dz)^2$. To construct surfaces we need to solve the system

$$\begin{cases} \Delta_0 \omega = \sinh \emptyset \cosh \emptyset \\ \lambda(\partial_x k_{\mathbb{H}}(y)) = 0 = \omega_{xy} - \tanh \omega \omega_x \omega_y \end{cases}$$

These solutions gives doubly-periodic solutions on $\mathbb{R}^2$ similar to the solution of sinh-Gordon equation of U. Abresh for Wente tori. The lines where ω has values $\pm\infty$ parametrize the values of the surface at ininity of $\mathbb{H} \times \mathbb{R}$. We can now integrate ω to get a harmonic map F which give the minimal surface $X = (F, y)$. It is a 2-parameters family where we can found annulus (catenoidal types surfaces), helicoidal surfaces and Riemann type surfaces. This system classify all minimal surfaces foliated by constant curvature curves in $\mathbb{H} \times \{t\}$. The function $u = \lambda(\partial_x k_{\mathbb{H}}(y))$ is a jacobi field on the surface. Using this property we have

Theorem 3. *(-,Meeks) Let A be a compact annulus embedded in $\mathbb{H} \times \mathbb{R}$, bounded by two compact constant curvature curves in $\mathbb{H} \times \{t_1\}$ and $\mathbb{H} \times \{t_2\}$. Then $u = 0$ on A and the annulus is foliated by constant horizontal curve.*

Graphs: If $\omega \geq 0$, the harmonic map F is a harmonic diffeomorphism. H. Rosenberg and P. Collin construct graphs on ideal polygons of $\mathbb{H}$ by the Jenkin-Serrin's method. Their examples are conformally $\mathbb{C}$. If the associated $Q = z^{2n}(dz)^2$ then the polygon has $2n + 2$ vertices. The minimal surfaces of finite total curvature are asymptotic at infinity to these canonical examples:

Theorem 4. *(-,Rosenberg) Let X be a complete minimal immersion of Σ in $\mathbb{H}\times\mathbb{R}$ with finite total curvature. Then*

a) Σ is conformally $\overline{M} - \{p_1...., p_n\}$, a Riemann surface punctured in a finite number of points.

b) Q is holomorphic on M and extends meromorphically to each puncture. If we parameterize each puncture p_i by the exterior of a disk of radius R_0, and if $Q(z) = z^{2m_i}(dz)^2$ at p_i then $m_i \geq -1$.

c) The third coordinate of the unit normal vector $n_3 \to 0$ uniformly at each puncture and each ends of X is uniformly asymptotic to a graph on an ideal polygon with $2m_i + 2$ vertices.

d)The total curvature is a multiple of 2π: $\int(-KdA) = 2\pi(2-2g-2k-\sum_{i=1}^{n} m_i)$

Constant mean curvature $H = 1/2$ of $\mathbb{H}\times\mathbb{R}$: U. Abresch and H. Rosenberg constructed a holomorphic quadratic differential Q_0 associated to the surface; this Q_0 generalizes the Hopf differential associated to constant mean curvature surfaces of $\mathbb{R}^3$. When $H = 1/2$ and the surface is a graph, Fernandez-Mira proved there exists a harmonic map from the surface to $\mathbb{H}$ whose associated holomorphic quadratic differential is $Q = -Q_0$.

In addition, given a harmonic map F from a surface to $\mathbb{H}$ they construct multi-graphs $H = 1/2$ on $\mathbb{H}\times\mathbb{R}$ with this harmonic map as Gauss map. We have

Theorem 5. *(-, Rosenberg, Spruck) Let Σ be a complete immersed surface in $\mathbb{H}\times\mathbb{R}$ of constant mean curvature $H = 1/2$. If Σ is transverse to Z then Σ is an entire vertical graph over $\mathbb{H}$*

Theorem 6. *(-, Rosenberg, Spruck) Let Σ be a properly embedded constant mean curvature $\frac{1}{2}$ surface in $\mathbb{H}\times\mathbb{R}$. Suppose Σ is asymptotic to a horocylinder C, and disjoint from C. If the mean curvature vector of Σ has the same direction as that of C at points of Σ converging to C, then Σ is equal to C (or a subset of C if $\partial\Sigma \neq \emptyset$).*

Darboux transforms of Willmore surfaces

Katrin Leschke

We report on work in progress on Darboux transforms of Willmore surfaces.

1. Darboux transforms

Recently, the classical Darboux transformation [5] on isothermic surfaces has been generalized to a transformation on conformal maps [3]. Geometrically, a classical Darboux transformation is given by an enveloping condition:

Definition 1 ([5])**.** *Given two conformal immersions $f, f^\sharp : M \to \mathbb{R}^3$ from a Riemann surface M into 3–space, $f^\sharp$ is called a* classical Darboux transform *of f if there exists a sphere congruence S enveloping both f and $f^\sharp$. Here, we say that a sphere congruence S envelopes f if for each point $p \in M$ there exists a sphere*

$S(p)$ so that $f(p) \in S(p)$ and the oriented tangent spaces of $S(p)$ at $f(p)$ and of f at p coincide.

In this case $(f, f^\sharp)$ are called a classical Darboux pair*.*

Fact 1. *If $(f, f^\sharp)$ form a classical Darboux pair then f and $f^\sharp$ are isothermic that is f and $f^\sharp$ allow conformal curvature line parametrizations.*

Because of this fact, it is necessary to consider transforms $f^\sharp : M \to \mathbb{R}^4$ into 4–space when generalizing the above transformation to general conformal immersions $f : M \to \mathbb{R}^3$. The condition for a sphere congruence to envelope a conformal map $f : M \to \mathbb{R}^4$ reads in terms of the Gauss map $(N, R) : M \to S^2 \times S^2 = \mathrm{Gr}_2(\mathbb{R}^4)$ as $N_f = N_S$, $R_f = R_S$. We relax the enveloping condition and say that S *left–envelopes* f if $N_f = N_S$.

Definition 2 ([3])**.** *Given two conformal maps $f, f^\sharp : M \to \mathbb{R}^4$ from a Riemann surface M into 3–space, $f^\sharp$ is called a* Darboux transform *of f if there exists a sphere congruence S enveloping f and left-enveloping $f^\sharp$.*

To obtain generalized Darboux transforms of a conformal map one has to find holomorphic sections of a quaternionic holomorphic bundle, that is sections in the kernel of an elliptic operator. Locally, there exist infinitely many independent holomorphic sections. However, when considering global Darboux transforms there is only a Riemann surface worth of closed Darboux transforms, the *spectral curve* of f, at least in the case when $M = T^2$ is a 2-torus. In the presence of harmonicity, essentially all Darboux transforms are given by solutions of ODEs. To explain this, we first consider the special case of Darboux transforms of constant mean curvature surfaces (CMC).

2. Constant mean curvature surfaces

By the well–known theorem of Ruh-Vilms [8] we know that $f : M \to \mathbb{R}^3$ is CMC if and only if the Gauss map $N : M \to S^2 = \{N \in \mathrm{Im}\,\mathbb{H} \mid N^2 = -1\}$ is harmonic, which is equivalent to the fact that the (1,0) part of dN (with respect to the complex structure N)

$$(dN)' = \frac{1}{2}(dN - N * dN)$$

is closed. Here $*$ is the negative Hodge star operator, and the multiplication is within the quaternions. This allows to introduce a family of flat connections: Considering $M \times \mathbb{C}^2 = (M \times \mathbb{H}, I)$ where the complex structure on $M \times \mathbb{H}$ is given by $Iv := vi$, we see that

$$d^\mu = d + (dN)'(N(a-1) + b)$$

is a flat complex connection on $M \times \mathbb{C}^2$ for all $\mu \in \mathbb{C}_*$ if and only if N is harmonic. Here

$$a = \frac{\mu + \mu^{-1}}{2}, b = \frac{\mu^{-1} - \mu}{2} I$$

so that $a^2 + b^2 = 1$. We consider a local transformation theory so we assume from now on without loss of generality M to be simply connected.

Definition 3. *Let* $\alpha \in \Gamma(M \times \mathbb{C}^2)$ *be a parallel secion of* d^μ *for some* $\mu \in \mathbb{C}_* \setminus \{1\}$, *and* $T^{-1} = N\alpha\frac{a-1}{2}\alpha^{-1} + \alpha\frac{b}{2}\alpha^{-1}$. *Then*

$$f^{\alpha,\mu} : M \to \mathbb{R}^4, f^{\alpha,\mu} = f + T$$

is called a μ–Darboux transform *of* f.

Theorem 1 ([4]). *Let* $f : M \to \mathbb{R}^3$ *be a constant mean curvature surface. Then*

(1) $\{f^{\alpha,\mu} \mid \mu \in \mathbb{C}_* \setminus \{1\}, f^{\alpha,\mu}$ *is* μ*–Darboux transform of* $f\}$
$\subsetneq \{f^\sharp \mid f^\sharp$ *is Darboux transform of* $f\}$.

(2) *A* μ*–Darboux transform of* f *is Möbius equivalent to a classical Darboux transform if and only if* $\mu \in \mathbb{R}_* \cup S^1$.

(3) *Every* μ*-Darboux transform* $f^{\alpha,\mu}$ *is, up to Möbius equivalence, a CMC surface in* $\mathbb{R}^3$. *In particular, the Gauss map* $N^{\alpha,\mu} = -TNT^{-1}$ *of* $f^{\alpha,\mu}$ *is harmonic.*

3. Darboux transforms of Willmore surfaces

Willmore surfaces in $f : M \to \mathbb{R}^4$, that is critical points of the Willmore functional $W(f) = \int_M |H|^2 - K - K^\perp$, are well-known to be given by a harmonicity condition: f is Willmore if and only if its conformal Gauss map is harmonic. The conformal Gauss map can be interpreted (see [2]) as a complex structure S on the trivial $\mathbb{H}^2$–bundle so that we have an analogous description of Willmore surfaces in terms of complex flat connections on a trivial $\mathbb{C}^4 = (\mathbb{H}^2, I)$ bundle:

Lemma 1. $f : M \to \mathbb{R}^4$ *is Willmore if and only if the family of connections*

$$d^\mu = d + *A(S(a-1) + b)$$

is flat for all $\mu \in \mathbb{C}_*$, *where again* $a = \frac{\mu+\mu^{-1}}{2}, b = \frac{\mu^{-1}-\mu}{2}I$. *Moreover, the Hopf field of the conformal Gauss map* S *is given by* $-2 * A = (dS)' = \frac{1}{2}(dS - S * dS)$.

From [3] it follows again, that a parallel section of such a connection gives a Darboux transform.

Proposition 1. *If* $\psi \in \Gamma(\tilde{M} \times \mathbb{H}^2)$ *is a parallel section of* d^μ *on the universal cover* $\tilde{M}$ *of* M *then* $L^\mu = \psi\mathbb{H} : M \to \mathbb{HP}^1 = S^4$ *is a Darboux transform of* f.

Preliminary investigations indicate that the resulting Darboux transforms are not Willmore. We conjecture them to be constrained Willmore. However, there is an induced transformation (see also [1] for a different approach) which preserves the Willmore property: Using the gauge G between the induced quaternionic flat connection $\hat{d}^\mu$ and the trivial connection, we obtain:

Theorem 2. *For* $\mu \in \mathbb{C}_* \setminus \{1\}$ *and* $a = \frac{\mu+\mu^{-1}}{2}, b = \frac{\mu^{-1}-\mu}{2}I$ *let*

$$T^{-1} = SG\frac{a-1}{2}G^{-1} + G\frac{b}{2}G^{-1}.$$

Then the line bundle $L^\mu = (G^{-1}(a-1)GT)^{-1}L : M \to \mathbb{HP}^1 = S^4$ *gives a Willmore surface in* S^4 *whose conformal Gauss map is given by* $S^\mu = TST^{-1}$.

Remark 1. (1) *This construction extends the Darboux transformation defined in* [2] *for Willmore surfaces. Burstall and Calderbank* [1] *obtain a similar generalization of* [2] *for conformal maps into* S^n*, however without the explicit formulas for the transforms.*

(2) *In general, the Darboux transform on the Willmore surface and the Darboux transforms on the conformal Gauss map do not coincide. It is an open problem how these two transformations are related.*

(3) *This method can also be applied to other surface classes which are given by harmonicity: e.g., in the case of Hamiltonian Stationary Lagrangians in* $\mathbb{C}^2$*, that is conformal maps into* $\mathbb{R}^4$ *with harmonic left normal* $N : M \to S^1$*, the Darboux transformation on the conformal map has again harmonic left normal* [7] *and thus is at least constrained Willmore* [6]*. We expect the transformation to give Hamiltonian stationary surfaces. In particular,* [3] *gives a geometric interpretation of the spectral curve of an Hamiltonanian stationary torus.*

References

[1] F. Burstall, D. Calderbank, *Conformal submanifold geometry*, in preparation.
[2] F. Burstall, D. Ferus, K. Leschke, F. Pedit, *Darboux transforms of constant mean curvature surfaces*, in preparation.
[3] C. Bohle, K. Leschke, F. Pedit, U. Pinkall, *Conformal maps of a 2-torus into the 4-sphere*, in preparation.
[4] E. Carberry, K. Leschke, F. Pedit, *Darboux transforms of constant mean curvature surfaces*, in preparation.
[5] G. Darboux, *Sur les surfaces isothermiques*, C. R. Acad. Sci. Paris 128 (1899), 1299-1305.
[6] K. Leschke, *"Freitagsvorlesung": Notes on Quaternionic Holomorphic Geometry*, unpublished.
[7] K. Leschke, P. Romon, *Darboux transforms of Hamiltonian Stationary Lagrangians*, in preparation.
[8] E. Ruh, J. Vilms, *The tension field of the Gauss map*, Trans. Am. Math. Soc. 149 (1970), 569-573.

Complex constant mean curvature surfaces via integrable system methods

Shimpei Kobayashi

1. Introduction

This report is a digest of the papers [4] and [3].

It is known that complex constant mean curvature (CMC for short) immersions in $\mathbb{C}^3$ are natural extensions of CMC-immersions in $\mathbb{R}^3$ (see [3]). In this report, conversely we consider "real forms surfaces" of a complex CMC-immersion (more precisely we consider the real forms surfaces of the parallel complex constant Gaußian curvature immersion of the complex CMC-immersion), which are associated from real forms of the twisted $\mathfrak{sl}(2,\mathbb{C})$ loop algebra $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$, and classify all such surfaces according to the classification of real forms of $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$ (see [1], [2]). We obtain seven classes of surfaces, which we call *integrable surfaces*.

The generalized Weierstraß type representation for complex CMC-immersions is a procedure to construct complex CMC-immersions in $\mathbb{C}^3$ from pairs of holomorphic potentials, which are pairs of holomorphic 1-forms $\check{\eta} = (\eta, \tau)$ with $\eta = \sum_{j\geq -1}^{\infty} \eta_j \lambda^j$ and $\tau = \sum_{-\infty}^{j\leq 1} \tau_j \lambda^j$. Here η_j and τ_j are diagonal (resp. off-diagonal) holomorphic 1-forms depending only on one complex variable if j is even (resp. j is odd). We recall that a real form $\mathfrak{a}$ of a complex Lie algebra $\mathfrak{g}$ is the invariant subspace given by a semi-linear involution ρ on $\mathfrak{g}$, i.e. $\mathfrak{a} = \{x \in \mathfrak{g} \mid \rho(x) = x\}$. Since each class of integrable surfaces is given by a real form of $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$, there exists an unique semi-linear involution ρ corresponding to each class of integrable surfaces. Using these semi-linear involutions, we define pairs of semi-linear involutions on the set of pairs of holomorphic potentials $\check{\eta} = (\eta, \tau)$. Then the generalized Weierstraß type representation for each class of integrable surfaces can be formulated by the above construction with a pair of holomorphic potentials which is invariant under the pair of semi-linear involutions (Theorem 3.1). Therefore we give an unified theory for all integrable surfaces.

2. Real forms of $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$

In this section, we quote the classification of real forms for the twisted $\mathfrak{sl}(2,\mathbb{C})$ loop algebra $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$ (see [4] for more details).

Theorem 2.1 ([1], [2]). *Let* $\mathfrak{c}_j$, $j \in \{1,2,3,4\}$, *be the following involutions on* $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$

$$
\begin{aligned}
&\mathfrak{c}_1 : g(\lambda) \mapsto -\overline{g(-1/\bar{\lambda})}^t, \quad \mathfrak{c}_2 : -\mathrm{Ad}\begin{pmatrix} 1/\sqrt{i} & 0 \\ 0 & \sqrt{i} \end{pmatrix} \overline{g(i/\bar{\lambda})}^t, \\
&\mathfrak{c}_3 : g(\lambda) \mapsto -\overline{g\left(1/\bar{\lambda}\right)}^t, \quad \mathfrak{c}_4 : g(\lambda) \mapsto \overline{g\left(-1/\bar{\lambda}\right)},
\end{aligned} \tag{1}
$$

where $g(\lambda) \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$. *Then up to the conjugations, the almost compact real forms of* $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$ *are the following Lie subalgebras of* $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$*:*

$$
\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},j)} = \{g(\lambda) \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma \mid \mathfrak{c}_j \circ g(\lambda) = g(\lambda)\} \quad \textit{for } j \in \{1,2,3,4\}. \tag{2}
$$

Theorem 2.2 ([1], [2]). *Let* $\mathfrak{s}_j$, $j \in \{1,2,3,4\}$, *be the following involutions on* $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$

$$
\begin{aligned}
&\mathfrak{s}_1 : g(\lambda) \mapsto -\overline{g(-\bar{\lambda})}^t, \\
&\mathfrak{s}_3 : g(\lambda) \mapsto -\overline{g\left(\bar{\lambda}\right)}^t, \quad \mathfrak{s}_4 : g(\lambda) \mapsto \overline{g\left(-\bar{\lambda}\right)},
\end{aligned} \tag{3}
$$

where $g(\lambda) \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$. *Then up to the conjugations, the almost split real forms of* $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$ *are the following Lie subalgebras of* $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$*:*

$$
\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{s},j)} = \{g(\lambda) \in \Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma \mid \mathfrak{s}_j \circ g(\lambda) = g(\lambda)\} \quad \textit{for } j \in \{1,3,4\}. \tag{4}
$$

Let $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$ be the set of $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$-valued 1-forms. Analogously to the involutions for $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$, we define the involutions $\tilde{\mathfrak{c}}_j$ (resp. $\tilde{\mathfrak{s}}_j$) on $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$

as follows:

$$(5)\qquad \begin{cases} \tilde{\mathfrak{c}}_1 : g(\lambda) \mapsto -\overline{g(-1/\bar{\lambda})}^t, & \tilde{\mathfrak{c}}_2 : -\mathrm{Ad}\begin{pmatrix} 1/\sqrt{i} & 0 \\ 0 & \sqrt{i} \end{pmatrix} \overline{g(i/\bar{\lambda})}^t, \\ \tilde{\mathfrak{c}}_3 : g(\lambda) \mapsto -\overline{g\left(1/\bar{\lambda}\right)}^t, & \tilde{\mathfrak{c}}_4 : g(\lambda) \mapsto \overline{g\left(-1/\bar{\lambda}\right)}, \end{cases}$$

$$\begin{cases} \tilde{\mathfrak{s}}_1 : g(\lambda) \mapsto -\overline{g(-\bar{\lambda})}^t, \\ \tilde{\mathfrak{s}}_3 : g(\lambda) \mapsto \overline{-g\left(\bar{\lambda}\right)}^t, \\ \tilde{\mathfrak{s}}_4 : g(\lambda) \mapsto \overline{g\left(-\bar{\lambda}\right)}, \end{cases}$$

where $g(\lambda) \in \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$.

3. The generalized Weierstrass type representation for integrable surfaces

The generalized Weierstraß type representation for complex CMC-immersions (or equivalently CGC-immersions by the parallel immersions) is divided into the following 4 steps (see also [3] for more details):

Step1: Let $\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda))$ be a pair of holomorphic potentials of the following forms

$$(6)\qquad \check{\eta} = (\eta(z,\lambda),\ \tau(w,\lambda)) = \left(\sum_{k=-1}^{\infty} \eta_k(z)\lambda^k,\ \sum_{m=-\infty}^{1} \tau_m(w)\lambda^m\right),$$

where $(z,w) \in \mathfrak{D}^2$ and where $\mathfrak{D}^2$ is some holomorphically convex domain in $\mathbb{C}^2$, $\lambda \in \mathbb{C}^*$, $|\lambda| = r$, and η_k and τ_m are $\mathfrak{sl}(2,\mathbb{C})$-valued holomorphic differential 1-forms. Moreover $\eta_k(z)$ and $\tau_k(w)$ are diagonal (resp. off-diagonal) matrices if k is even (resp. odd). We also assume that the upper right entry of $\eta_{-1}(z)$ and the lower left entry of $\tau_1(w)$ do not vanish for all $(z,w) \in \mathfrak{D}^2$.

Step2: Let C and L denote the solutions to the following linear ordinary differential equations

$$(7)\qquad dC = C\eta \ \text{ and } \ dL = L\tau \ \text{ with } \ C(z_*,\lambda) = H(w_*,\lambda) = \mathrm{id},$$

where $(z_*, w_*) \in \mathfrak{D}^2$ is a fixed base point.

Step3: We factorize the pair of matrices (C, L) via the generalized Iwasawa decomposition as follows (see [3] for more details):

$$(8)\qquad (C,\ L) = (F,\ F)(\mathrm{id},\ W)(V_+,\ V_-),$$

where $V_+ \in \Lambda^+ SL(2,\mathbb{C})_\sigma$ and $V_- \in \Lambda^- SL(2,\mathbb{C})_\sigma$.

Step4: The Sym formula

$$\Psi = -\frac{1}{2H}\left(i\lambda\partial_\lambda F(z,w,\lambda)\cdot F(z,w,\lambda)^{-1} + \frac{i}{2}F(z,w,\lambda)\sigma_3 F(z,w,\lambda)^{-1}\right),$$

represents a complex CMC-immersion in $\mathfrak{sl}(2,\mathbb{C}) \cong \mathbb{C}^3$.

Let $\tilde{\mathfrak{c}}_j$ (resp. $\tilde{\mathfrak{s}}_j$), $j \in \{1, 2, 3, 4\}$ (resp. $j \in \{1, 3, 4\}$), be the involutions defined in (5). Using $\tilde{\mathfrak{c}}_j$ and $\tilde{\mathfrak{s}}_j$, we now define the following pairs of involutions on $\check{\eta} = (\eta, \tau) \in \Omega(\Lambda\mathfrak{sl}(2, \mathbb{C})_\sigma) \times \Omega(\Lambda\mathfrak{sl}(2, \mathbb{C})_\sigma)$:

$$\mathfrak{r}_j : (\eta, \tau) \to (\tilde{\mathfrak{c}}_j \tau,\ \tilde{\mathfrak{c}}_j \eta) \quad \text{and} \quad \mathfrak{d}_j : (\eta, \tau) \to (\tilde{\mathfrak{s}}_j \eta,\ \tilde{\mathfrak{s}}_j \tau). \tag{9}$$

Then we have the following theorem.

Theorem 3.1 ([4]). *Let $\check{\eta} = (\eta(z, \lambda), \tau(w, \lambda))$ be a pair of holomorphic potentials defined as in* (6)*, and let $\mathfrak{r}_j$ (resp. $\mathfrak{d}_j$) $j \in \{1, 2, 3, 4\}$ (resp. $j \in \{1, 3, 4\}$) be the pairs of involutions defined in* (9)*. Then we have the followings:*

- (C, 1) *If $\mathfrak{r}_1(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are spacelike constant negative Gaußian curvature surfaces in $\mathbb{R}^{2,1}$,*
- (C, 2) *If $\mathfrak{r}_2(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are constant sectional curvature surfaces in $\mathbb{C}^3$,*
- (C, 3) *If $\mathfrak{r}_3(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are constant positive Gaußian curvature surfaces in $\mathbb{R}^3$,*
- (C, 4) *If $\mathfrak{r}_4(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are timelike constant negative Gaußian curvature surfaces in $\mathbb{R}^{2,1}$,*
- (S, 1) *If $\mathfrak{d}_1(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are spacelike constant positive Gaußian curvature surfaces in $\mathbb{R}^{2,1}$,*
- (S, 3) *If $\mathfrak{d}_3(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are constant negative Gaußian curvature surfaces in $\mathbb{R}^3$.*
- (S, 4) *If $\mathfrak{d}_4(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are timelike constant positive Gaußian curvature surfaces in $\mathbb{R}^{2,1}$,*

It is known that for three classes of surfaces in the above seven classes of surfaces, there exist the parallel constant mean curvature surfaces in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$.

Corollary 3.2. *We retain the assumptions in Theorem 3.1. Then we have the followings:*

- $(C, 1M)$ *For $(C, 1)$ case in Theorem 3.1, there exist the parallel spacelike constant mean curvature surfaces in $\mathbb{R}^{2,1}$.*
- $(C, 3M)$ *For $(C, 3)$ case in Theorem 3.1, there exist the parallel constant mean curvature surfaces in $\mathbb{R}^3$.*
- $(S, 4M)$ *For $(S, 4)$ case in Theorem 3.1, there exist the parallel timelike constant mean curvature surfaces in $\mathbb{R}^{2,1}$.*

We call the surfaces defined in Theorem 3.1 and Corollary 3.2 *integrable surfaces.*

Surfaces class	Gauss curvature	Gauss curvature	Parallel CMC
Surfaces in $\mathbb{R}^3$	$K^{(\mathfrak{s},3)} = -4\|H\|^2$	$K^{(\mathfrak{c},3)} = 4\|H\|^2$	$\|H\|$
Spacelike surfaces in $\mathbb{R}^{2,1}$	$K^{(\mathfrak{s},1)} = 4\|H\|^2$	$K^{(\mathfrak{c},1)} = -4\|H\|^2$	$\|H\|$
Timelike surfaces in $\mathbb{R}^{2,1}$	$K^{(\mathfrak{c},4)} = -4\|H\|^2$	$K^{(\mathfrak{s},4)} = 4\|H\|^2$	$\|H\|$
Surfaces in $\mathbb{C}^3$	$K^{(\mathfrak{c},2)}_{\mathrm{int}} = 4H^2$	$\times$	$\times$

Table 1. Integrable surfaces defined by the real forms of $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$

References

[1] Valérie Back-Valente, Nicole Bardy-Panse, Hechmi Ben Messaoud, and Guy Rousseau. Formes presque-déployées des algèbres de Kac-Moody: classification et racines relatives. *J. Algebra*, 171(1):43–96, 1995.
[2] Hechmi Ben Messaoud and Guy Rousseau. Classification des formes réelles presque compactes des algèbres de Kac-Moody affines. *J. Algebra*, 267(2):443–513, 2003.
[3] J. Dorfmeister, S.-P. Kobayashi, and F. Pedit. Complex surfaces of constant mean curvature fibered by minimal surfaces. *Preprint*, 2006.
[4] Shimpei Kobayashi. Real forms of complex surfaces of constant mean curvature. *In preparation*, 2007.

The space of closed curves in a conformal 3-manifold

Ulrich Pinkall
(joint work with Falk Henrich)

The space $\mathcal{M}$ of all (unparametrized) oriented regular closed curves in a 3-manifold M with a conformal structure (an equivalence class of Riemannian metrics) is an infinite dimensional Frechet-manifold that carries some interesting geometric structures:

1. Length functional

On the tangent space $T_\gamma\mathcal{M} = \{\text{normal vector fields along } \gamma\}$ we can define a real valued function L (positive homogeneous of degree one) as

$$L(Y) = 1/\oint 1/|Y(s)|ds.$$

Here the norm of $Y(s)$ as well as the arclength ds is taken with respect to a compatible Riemannian metric on M. Obviously L is invariant under conformal changes of the metric. Note that $L(Y) = 0$ if the normal vector field Y has zeroes. For a curve $t \mapsto \gamma_t \in \mathcal{M}, \quad t \in [a,b]$ we define its length to be

$$\mathcal{L}(\gamma) = \int_a^b L(\dot{\gamma}).$$

2. Affine connection

We define in a conformally invariant way an affine connection ∇ on $\mathcal{M}$ whose geodesics are precisely the critical points of $\mathcal{L}$. Moreover, the Lagrangian L on $T\mathcal{M}$ satisfies $\nabla L = 0$. Viewed as cylindical surfaces

$$f : S^1 \times [a, b] \to M$$

those geodesics are isothermic surfaces on which the curves $s \mapsto f(s, t)$ make an angle of $45°$ with the curvature lines of f.

3. Complex structure

Rotating a normal vector field $Y \in T_\gamma\mathcal{M}$ by $90°$ defines an almost complex structure

$$J : T\mathcal{M} \to T\mathcal{M}.$$

The Nijenhuis tensor of J vanishes and for any compatible Riemannian metric on M the corresponding Riemannian metric of $\mathcal{M}$ obtained by taking the L^2-norm of normal vector fields $Y \in T_\gamma\mathcal{M}$ is Kähler in the sense that

$$\nabla^{LC} J = 0$$

for its Levi-Civita connection ∇^{LC}. Due to infinite dimensional phenomena, $\mathcal{M}$ fails to be a complex manifold in the sense of admitting complex coordinate charts [1]. Nevertheless, in many respects $\mathcal{M}$ behaves like a finite dimensional complex manifold.

We study holomorphic curves in $\mathcal{M}$. Locally they correspond to conformal submersions

$$\pi : M \to \Sigma$$

of M onto a Riemann surface Σ.

4. Total torsion

The monodromy in the normal bundle of a closed curve γ is conformally invariant and is called the *total torsion* $\mathcal{T}(\gamma)$. This defines a function

$$\mathcal{T} : \mathcal{M} \to S^1 = \mathbb{R}/2\pi.$$

The critical points of $\mathcal{T}$ are given by the conformally invariant differential equation

$$R(N, JN)\gamma' + J\mathcal{H}' = 0.$$

Here N is any unit normal vector field along γ, R the curvature tensor of M and $\mathcal{H}$ the mean curvature vector field along γ. Critical points of $\mathcal{T}$ can be considered as "round circles" in M.

References

[1] L. Lempert, *Loop Spaces as Complex Manifolds*, J. Differential Geometry **38** (1993), 519–543.

Discrete constant mean curvature surfaces via conserved quantities in any space form

Wayne Rossman

(joint work with F. Burstall, U. Hertrich-Jeromin, S. Santos)

Our purpose here is to present a definition for discrete constant mean curvature (CMC) h surfaces[1] in any of the three space forms Euclidean 3-space R^3, spherical 3-space S^3 and hyperbolic 3-space H^3. This new definition is equivalent to the previously known definitions [2] in the case of R^3. It also satisfies a Calapso transformation relation (the Lawson correspondence), suggesting the definition is also natural for the space form S^3, and for CMC surfaces with $h \geq 1$ in H^3. The definition is the first one for CMC surfaces with $-1 < h < 1$ in H^3.

To motivate this definition for discrete CMC surfaces, we first consider the case of smooth surfaces, and we begin by describing the 3-dimensional space forms using the 5-dimensional Minkowski space $R^{4,1}$.

Minkowski 5-space. We give a 2×2 matrix formulation for Minkowski 5-space. Let H denote the quaternions and $\operatorname{Im} H$ the imaginary quaternions.

$$\mathbb{R}^{4,1} = \left\{ X = \begin{pmatrix} x & x_\infty \\ x_0 & -x \end{pmatrix} \middle| \; x \in \operatorname{Im} H, x_0, x_\infty \in \mathbb{R} \right\}$$

with signature $(+,+,+,+,-)$ Minkowski metric $\langle X, Y \rangle$ such that $\langle X, Y \rangle \cdot I = -\frac{1}{2}(XY + YX)$, $I =$ identity matrix. The 4-dimensional light cone is $\mathbb{L}^4 = \{X \in \mathbb{R}^{4,1} \,|\, ||X||^2 = 0\}$. We can make the 3-dimensional space forms as follows: A space form M is $M = \mathbb{L}^4 \cap \{X \,|\, \langle X, Q \rangle = -I\}$ for any nonzero $Q \in \mathbb{R}^{4,1}$. It turns out that M has curvature κ, where $Q^2 = \kappa \cdot I$, so without loss of generality we can obtain any space form by choosing

$$Q = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix} \;, \quad \text{and then} \quad M = \left\{ X = \frac{2}{1 - \kappa x^2} \cdot \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \right\} \;,$$

which is equivalent to $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \cup \{\infty\} \,|x_1^2 + x_2^2 + x_3^2 \neq -\kappa^{-1}\}$, where $x = x_1 i + x_2 j + x_3 k \in \operatorname{Im} H$. Note that when $\kappa < 0$, M becomes two copies of $H^3(\kappa)$.

Smooth surfaces in space forms. Let

$$x = x_1(u,v)i + x_2(u,v)j + x_3(u,v)k \approx X \in M$$

be a surface in M. Assume (u, v) is a conformal curvature-line coordinate system (every CMC surface can be parametrized this way). First we define the Christoffel transformation x^*, which for a CMC surface in R^3 gives the parallel CMC surface:

Definition. *The Christoffel transformation of x is any x^* (defined in R^3 up to translation) such that $dx^* = x_u^{-1} du - x_v^{-1} dv$.*

In the next definition, the nonzero real constant c can be chosen freely:

[1]2000 Mathematics Subject Classification: 53A10, 53C42, 37K35, 37K25

Definition. *For some $c \in R\backslash\{0\}\}$, we set $\tau = c\begin{pmatrix} xdx^* & -xdx^*x \\ dx^* & -dx^*x \end{pmatrix}$. If there exist smooth Q and Z in $R^{4,1}$ depending on (u,v) such that*

$$d(Q+\lambda Z) = (Q+\lambda Z)\lambda\tau - \lambda\tau(Q+\lambda Z) \tag{1}$$

holds for all $\lambda \in R$, then we call $Q+\lambda Z$ a linear conserved quantity *of x.*

Some properties of linear conserved quantities are immediate: Q and Z^2 are constant; $F\tau = \tau F = 0$; $F \perp Z$ and $F \perp dZ$. Properties like this can be utilized to prove the following theorem:

Theorem 1. [1] *The surface x is constant mean curvature in a space form M (produced by $Q \neq 0$) if and only if there exists (for that Q) a linear conserved quantity $Q+\lambda Z$.*

Isothermic discrete surfaces and their Christoffel transforms. Consider a discrete surface $\mathfrak{f}_p \in \operatorname{Im} H$, where p is any point in a discrete lattice domain. Consider one quadrilateral in the lattice with vertices p, q, r, s ordered counterclockwise about the quadrilateral. We define the cross ratio of this quadrilateral as

$$q_{pqrs} = (\mathfrak{f}_q - \mathfrak{f}_p)(\mathfrak{f}_r - \mathfrak{f}_q)^{-1}(\mathfrak{f}_s - \mathfrak{f}_r)(\mathfrak{f}_p - \mathfrak{f}_s)^{-1} .$$

When, for every quadrilateral, we can write the cross ratio as

$$q_{pqrs} = a_{pq}/a_{ps} \in \mathbb{R}$$

so that the function a_{pq} defined on the edges of $\mathfrak{f}$ satisfies

$$a_{pq} = a_{sr} \in \mathbb{R} \text{ and } a_{ps} = a_{qr} \in \mathbb{R} ,$$

then we say that $\mathfrak{f}$ is *isothermic*.

We can define the Christoffel transform $\mathfrak{f}^*$ of $\mathfrak{f}$ by

$$d\mathfrak{f}^*_{pq} d\mathfrak{f}_{pq} = a_{pq} .$$

We can then prove the following:

Lemma 2. [2] *If $\mathfrak{f}$ is isothermic, then there exists a discrete surface $\mathfrak{f}^*$ satisfying the above equation for $d\mathfrak{f}^*$, and $\mathfrak{f}^*$ is isothermic with the same cross ratios as $\mathfrak{f}$.*

Linear conserved quantities. We can now discretize (1) to obtain

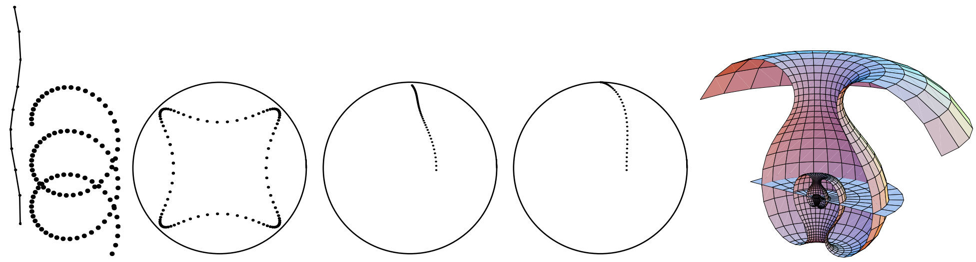
$$(1+\lambda\tau_{pq})(Q+\lambda Z)_q = (Q+\lambda Z)_p(1+\lambda\tau_{pq}) , \tag{2}$$

where $\lambda \in \mathbb{R}$ and $Q, Z \in R^{4,1}$ are functions on the lattice domain, and

$$\tau_{pq} = \begin{pmatrix} \mathfrak{f}_p d\mathfrak{f}^*_{pq} & -\mathfrak{f}_p d\mathfrak{f}^*_{pq}\mathfrak{f}_q \\ d\mathfrak{f}^*_{pq} & -d\mathfrak{f}^*_{pq}\mathfrak{f}_q \end{pmatrix} .$$

We now come to the goal of this talk:

Definition. *If a linear conserved quantity $Q+\lambda Z$, $Q \neq 0$, exists for an isothermic discrete surface $\mathfrak{f}$, we say that $\mathfrak{f}$ is of constant mean curvature in the space form M determined by Q.*



Equation (2) can be extended to define polynomial conserved quantities.

In the figure, we show discrete CMC surfaces of revolution. The first two curves are profile curves for discrete nonminimal CMC surfaces of revolution in R^3, the first being unduloidal and the second nodoidal. (For each of these two curves, the axis of rotation producing the surface is a vertical line drawn to the left of the curve, and is not shown in the figure.) The third picture shows the profile curve for a discrete CMC surface of revolution in S^3, where S^3 is stereographically projected to R^3, and the shown circle is a geodesic of S^3 that is also the axis of the surface – and furthermore, this example has a periodicity that causes it to close on itself and form a torus. The final three pictures show discrete CMC surfaces of revolution in H^3. The first two, with $H > 1$ and $H = 1$ respectively, are shown in the Poincare model, and the first is unduloidal while the second looks similar to a smooth embedded catenoid cousin. (For these two curves, the corresponding axis of revolution is the vertical line between the uppermost and lowermost points of the circle shown, and this circle lies in the boundary sphere at infinity of H^3.)

The last picture is a minimal surface that lies in both copies of $M = H^3$, and the horizontal plane shown here is the virtual boundary at infinity of two copies of the halfspace model for H^3. This example was not known before, because the notion of discrete CMC was not defined before in this case.

References

[1] F. Burstall, D. Calderbank, *Conformal submanifold geometry*, in preparation.

[2] A. Bobenko, U. Pinkall, *Discretization of surfaces and integrable systems*, Oxford Lect. Ser. Math. Appl. **16** (1999), 3–58.

Birfurcations of the nodoids

Karsten Grosse-Brauckmann
(joint work with Yong He)

The Delaunay surfaces are the non-compact surfaces of revolution with constant mean curvature. They are simply periodic and represent proper immersions of a sphere minus two points. They come in two families: The embedded unduloids and the non-embedded nodoids each form one-parameter families, provided the mean curvature is normalized to $H \equiv 1$. We write $\{N_r : r > 0\}$ for the nodoid family, using their neckradius $r > 0$ as a parameter. The *neckradius* is the minimal distance of the surface from its axis of revolution.

Mazzeo and Pacard [2] prove that branches of constant mean curvature surfaces with j-fold dihedral symmetry bifurcate from the nodoid family $\{N_r : r > 0\}$ at certain neckradii $r = r(j)$, where $r(j) \to \infty$ as $j \to \infty$. Rossman shows by numerical computation [3] that the lowest bifurcation occurs at $r(2) = 1/2$.

The PhD thesis of He [1] establishes existence for the families of Mazzeo and Pacard, continued up to until the surfaces become degenerate. Moreover, the nodoid parameter of bifurcation is explicit:

Theorem 1. *For each $j \in \mathbb{N}$, $j \geq 2$, there exists a continuous family $t \mapsto M_{j,t}$, parameterized with $t \in [0, 1)$, of proper immersions of $\mathbb{S}^2$ minus two points. The surfaces $M_{j,t}$ have constant mean curvature $H \equiv 1$ and satisfy:*
1. Bifurcation: $M_{j,0}$ agrees with a nodoid N_r of neckradius $r = (j-1)/2$.
2. Symmetries: Each surface $M_{j,t}$ is invariant under rotation by an angle $2\pi/j$; it is also invariant under some translation depending on (j,t).
3. Degenerate limit: As $t \to 1$, the surfaces $M_{j,t}$ have unbounded principal curvatures. Suitable intrinsic disks of large fixed radius contained in $M_{j,t}$ converge to a (multiply covered) union of j spheres which intersect.

I thank Martin Kilian for bringing the surfaces to my attention. Let me add a few remarks:
1. There are similar bifurcation families setting off from k-fold covers of nodoids.
2. The family $M_{j,t}$ can be continued across $t = 0$ by the rotated surfaces $M_{j,-t}$. This corresponds to the continuation of the unduloid family across the cylinder by translated unduloids.
3. Mazzeo and Pacard have further bifurcation surfaces which have a discrete screw symmetry. They parameterize them with an additional parameter $\alpha \in [-\pi/j, \pi/j]$. Presumably, these surfaces give rise to a two-parameter family $M_{j,t,\alpha}$. Then $M_{j,t,0}$ would agree with our family $M_{j,t}$, and for $t = 0$ the surfaces would agree with nodoids of some neckradius $r = r(j, \alpha)$. However, these surfaces do not have the mirror symmetries necessary for our methods to apply.
4. In the space forms $\mathbb{S}^3$ and $\mathbb{H}^3$ there are analogues of the nodoid bifurcations: In $\mathbb{S}^3$ covers of distance tori of constant mean curvature H bifurcate for certain values of H to (non-embedded) "unduloids" with j lobes. These are still surfaces of revolution. In $\mathbb{H}^3$, the catenoid cousins bifurcate to surfaces with j-fold rotational symmetry, called *warped catenoid cousins* by Rossman, Umehara, and Yamada. I

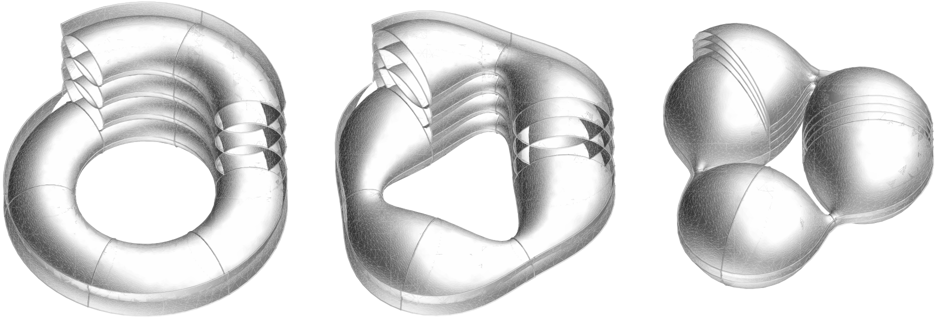


FIGURE 1. Three surfaces of the family $M_{3,t}$, with t increasing from 0 (nodoid, left) to almost 1 (right). Depicted is a translational fundamental domain, together with three partial translational copies (out of infinitely many). Curves of planar reflection, contained in mirror planes of the surface, are thickened. Observe that the vertical translation decreases to 0 when $t \to 1$.

thank them and Frank Pacard for pointing this out.
5. After the talk, Uwe Abresch and Ulrich Pinkall indicated that their integrable systems methods should perhaps lead to an alternative existence proof for $M_{j,t}$.

Our proof of the theorem employs the conjugate cousin technique. Invoking the desired mirror symmetries, it constructs geodesic polygons in $\mathbb{S}^3$ for which the Plateau problem can be solved. In our case, the polygons are quadrilaterals and the bifurcations show up in a bifurcation of the families of these quadrilaterals. The quadrilaterals are entirely explicit with known edgelengths. This makes the bifurcation neckradius explicit.

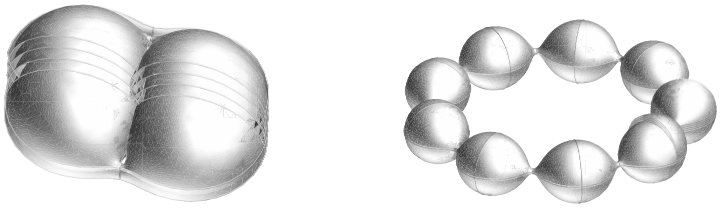


FIGURE 2. Surfaces $M_{2,t}$ and $M_{9,t}$ for t close to 1.

References

[1] Y. He, PhD thesis, TU Darmstadt, in preparation
[2] R. Mazzeo, F. Pacard *Bifurcating nodoids*, Contempory Mathematics **314** (2002), 169–186.
[3] W. Rosman, *The first bifurcation point for Delaunay nodoids*, Exp. Math. **14** (2005), 331–342.

Existence of constant mean curvature foliations in asymptotically hyperbolic spaces

Frank Pacard

1. Introduction

An influential paper by Huisken and Yau [3] proved the existence of a foliation near infinity in an asymptotically Euclidean manifold using a geometric heat flow. There are analogous results in the asymptotically hyperbolic setting. Existence of CMC foliations on high order perturbations of the Anti-de-Sitter-Schwarzschild space was proved by Rigger [6], again using mean curvature flow, and quite recently Neves and Tian [5] have established uniqueness.

Our goal is to revisit this problem in the asymptotically hyperbolic case. Let M be the interior of a smooth $n+1$ dimensional compact manifold $\overline{M}$ with boundary ∂M. We will say that a metric g defined on M is *conformally compact* if $g = x^{-2}\overline{g}$, where $\overline{g}$ is a metric on $\overline{M}$, the closed manifold with boundary, which is smooth and nondegenerate up to the boundary, and x is a smooth defining function for ∂M, i.e. $x = 0$ only on ∂M and $dx \neq 0$ there.

The conformal infinity of a conformally compact metric (M, g) is the conformal class $\mathfrak{c}(g)$ on ∂M defined by $\mathfrak{c}(g) = [\overline{g}|_{T\partial M}]$. This correspondence between the interior Riemannian geometry of g and conformal geometry on the boundary is one of the principal motivations for studying conformally compact metrics, see [1] for more on this.

Typically we assume that $\overline{g}$ and x are $\mathcal{C}^\infty$, but most everything we do below requires lower regularity. A brief calculation shows that the curvature tensor of g has the form

$$R_{ijk\ell} = -|d\log x|_g^2\,(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) + \mathcal{O}(x^{-3}).$$

We will say that a conformally compact metric g defined on M is *asymptotically hyperbolic* (AH) if the boundary defining function x satisfies $|d\log x|_g = 1$ on ∂M.

There is a useful normal form for any AH metric, proved by Graham and Lee [2], which is valid in a collar neighborhood of infinity. If (M, g) is AH and if $h^{(0)}$ is any (smooth) metric on ∂M which represents the conformal class $\mathfrak{c}(g)$, there exists a defining function x defined in a collar neighbourhood of ∂M such that

$$x^2\, g = dx^2 + h(x), \tag{1}$$

where $h(x)$ is a family of metrics on ∂M which depends smoothly on $x \geq 0$ and satisfies

$$h(0) = h^{(0)}.$$

Such a boundary defining function will be referred to as a *special boundary defining function.*

2. Statement of the results

We assume that g is AH metric and x is a special boundary defining function. The mean curvature H_ε of the hypersurface $\{x = \varepsilon\}$ is then given explicitely by

$$H_\varepsilon = \mathrm{Tr}_h \left(h - \tfrac{1}{2}\, x h' \right) |_{x=\varepsilon} \tag{2}$$

If we assume that $h(x)$ admits a second order expansion

$$x^2\, g = dx^2 + h^{(0)} + x\, h^{(1)} + x\, h^{(2)} + \mathcal{O}(x^3). \tag{3}$$

we find that

$$H_\varepsilon = n - \varepsilon\, \kappa_1 - \varepsilon^2\, \kappa_2 + \mathcal{O}(\varepsilon^3)$$

where the functions κ_j are defined by

$$\kappa_1(g, h^{(0)}) = \tfrac{1}{2}\, \mathrm{Tr}_{h^{(0)}} h^{(1)} \quad \text{and} \quad \kappa_2(g, h^{(0)}) = \tfrac{1}{2}\, \mathrm{Tr}_{h^{(0)}} (2h^{(2)} - h^{(1)} \circ h^{(1)}).$$

It should be clear that the functions κ_1 and κ_2 depend not only on the metric g but also on the choice of the representative $h^{(0)}$ of $\mathfrak{c}(g)$ and in fact, if $\hat{h}^{(0)} = e^{2\phi_0} h^{(0)}$ for some function ϕ_0 defined on ∂M, then we have

$$\kappa_1(g, \hat{h}^{(0)}) = e^{-\phi_0}\, \kappa_1(g, h^{(0)})$$

and

$$\kappa_2(g, \hat{h}^{(0)}) = e^{-2\phi_0} \left(\kappa_2(g, h^{(0)}) + \Delta_{h^{(0)}} \phi_0 + \tfrac{n-2}{2}\, |d\phi_0|^2_{h^{(0)}} \right).$$

Observe that, if $\kappa_1(g, h^{(0)})$ does not change sign, choosing appropriately the conformal factor $e^{2\phi_0}$, we can always reduce to the case where $\kappa_1(g, h^{(0)}) = \pm 1$. Similarly, if $\kappa_1(g, h^{(0)}) \equiv 0$, then this remains the case for any $h^{(0)} \in \mathfrak{c}(g)$.

Our first result handles the case of AH metrics for which $\kappa_1(g, h^{(0)}) > 0$.

Theorem 1. *Assume that x is a special boundary defining function associated to $h^{(0)}$ and further assume that $\kappa_1(g, h^{(0)}) = +1$. Then, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the hypersurface $x = \varepsilon$ can be perturbed into S_ε, a constant mean curvature hypersurface with mean curvature $H = n - \varepsilon$. The hypersurfaces S_ε constitute the unique local foliation of a collar neighborhood of ∂M by constant mean curvature hypersurfaces.*

Surprisingly, when $\kappa_1(g, h^{(0)}) < 0$, we are not able to prove the existence of a foliation by constant mean curvature hypersurfaces of a collar neighborhood of ∂M because of some resonance phenomena that prevents our construction to work in general.

Theorem 2. *Assume that x is a special boundary defining function associated to $h^{(0)}$ and further assume that $\kappa_1(g, h^{(0)}) = -1$. Then there exists $I \subset (0, \infty)$ such that, for all $\varepsilon \in I$, the hypersurface $x = \varepsilon$ can be perturbed into S_ε, a constant mean curvature hypersurface with mean curvature $H = n + \varepsilon$. The hypersurfaces S_ε constitute a lamination of a collar neighborhood of ∂M by constant mean curvature hypersurfaces. Moreover, given $q \geq 1$, there exists $c_q > 0$ such that $|\, |I \cap (0, \varepsilon)| - \varepsilon| \leq c_q\, \varepsilon^q$ as ε tends to 0.*

Finally, we also consider the case where $\kappa_1(g, h^{(0)}) \equiv 0$ and $\kappa_2(g, h^{(0)})$ is a constant (non zero) function.

Theorem 3. *Assume that x is a special boundary defining function associated to $h^{(0)}$ and further assume that $\kappa_1(g, h^{(0)}) \equiv 0$ and that the function $\kappa_2(g, h^{(0)}) = \pm 1$.*

(i) *When $\kappa_2(g, h^{(0)}) = +1$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the hypersurface $x = \varepsilon$ can be perturbed into S_ε, a constant mean curvature hypersurface with mean curvature $H = n - \varepsilon^2$.*

(ii) *When $\kappa_2(g, h^{(0)}) = -1$, if we further assume that $\Delta_{h^{(0)}} + 2$ is injective, then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the hypersurface $x = \varepsilon$ can be perturbed into S_ε, a constant mean curvature hypersurface with mean curvature $H = n + \varepsilon^2$.*

In addition the hypersurfaces S_ε constitute a local foliation of a collar neighborhood of ∂M by constant mean curvature hypersurfaces.

In the case where $\kappa_1(g, h^{(0)}) \equiv 0$, the condition that ensures the existence of a constant mean curvature foliation is precisely that there exists $\hat{h}^{(0)} \in \mathfrak{c}(g)$ such that the function $\kappa_2(g, \hat{h}^{(0)})$ is constant (not equal to 0). Writing $\hat{h}^{(0)} = e^{2\phi_0}\, h^{(0)}$, this can be translated into the existence of ϕ_0, a nondegenerate solution of the following Yamabe type problem

$$e^{-2\phi_0} \left(\kappa_2(g, h^{(0)}) + \Delta_{h^{(0)}} \phi_0 + \frac{n-2}{2} \, |d\phi_0|^2_{h^{(0)}} \right) = \textit{constant}$$

Observe that, when $n \geq 3$, we may write $e^{2\phi_0} = u_0^{\frac{4}{n-2}}$ and we get the more familiar equation

$$\Delta_{h_0} u_0 + \frac{n-2}{2}\, \kappa_2(g, h^{(0)})\, u_0 - \textit{constant}\, u_0^{\frac{n+2}{n-2}} = 0 \tag{4}$$

It is easy to check that this equation is always (uniquely) solvable when $\kappa_2(g, h^{(0)}) > 0$ but otherwise, its solvability is a rather delicate problem which is intimately related to the solvability of the Yamabe problem.

Recall that, given a metric h defined on a n dimensional manifold N, the Schouten tensor A_h is defined by

$$A_h = \frac{1}{n-2} \left(\mathrm{Ric}(h) - \frac{R_h}{2(n-1)}\, h \right)$$

where $\mathrm{Ric}(h)$ is the Ricci tensor and R_h is the scalar curvature of h. A special subclass of AH metrics is of special importance since it includes the set of AH Einstein metrics. An AH metric g is said to be *Asymptotically Einstein* (AHAE) if it can be expanded as in (3) with $h^{(1)} \equiv 0$ and $h^{(2)} = -A_{h^{(0)}}$.

This notion does not depend on the choice of the special boundary defining function. Observe that, for a AH metric one has $\mathrm{Ric}(g) + n\, g = \mathcal{O}_g(x)$ whereas for any AHAE metric one has $\mathrm{Ric}(g) + n\, g = \mathcal{O}_g(x^3)$ showing that, in some sense, the metric approximates an Einstein metric to higher order.

In the case where the metric g is AHAE (and hence $h^{(1)} \equiv 0$), the equation (4) is nothing but the Yamabe equation

$$\Delta_{h_0} u_0 - \frac{n-2}{4(n-1)} R_{h^{(0)}} u_0 + constant\, u_0^{\frac{n+2}{n-2}} = 0$$

where $R_{h^{(0)}}$ is the scalar curvature of the boundary metric $h^{(0)}$. Then, Theorem 3 shows that constant mean curvature foliations of a collar neighborhood of ∂M are associated to Yamabe metrics on ∂M. More precisely, assume that $n \geq 3$ and that g is AHAE. Then, there are at least as many geometrically distinct such foliations as the number of nondegenerate constant scalar curvature metrics in $\mathfrak{c}(g)$.

References

[1] C. Fefferman and C.R. Graham, *Conformal invariants*, in Elie Cartan et les Mathématiques d'Adjourd'hui, Astérisque (1985), 95-116.

[2] C.R. Graham and J. Lee, *Einstein metrics with prescribed conformal infinity on the ball.* Adv. Math. **87** (1991), 186-225.

[3] G. Huisken and S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature.* Invent. Math. **124** (1996), 281–311.

[4] R. Mazzeo and F. Pacard, *Constant curvature foliations in asymptotically hyperbolic spaces.* Preprint.

[5] A. Neves and G. Tian, *Existence and uniquenes of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds.* arXiv.org:math/0610767

[6] R. Rigger, *The foliation of asymptotically hyperbolic manifolds by surfaces of constant mean curvature.* Manuscripta Math. **113** (2004), 403–421.

Nondegeneracy of constant mean curvature surfaces.

J. Ratzkin

(joint work with K. Grosse–Brauckmann, N. Korevaar, R. Kusner, and J. Sullivan)

Let $f : \Sigma \to \mathbb{R}^3$ be a proper, constant mean curvature embedding of a surface Σ which has genus g and k ends. We assume the mean curvature H is not 0 and rescale so that $H \equiv 1$. By a theorem of Korevaar, Kusner and Solomon [KKS], an end E_j of $f(\Sigma)$ is asymptotic to a fixed Delaunay surface D_{n_j}, which is rotationally symmetric, periodic, and determined up to rigid motion by its necksize $n_j \in (0, \pi]$, the length of the shortest closed geodesic.

Using this asymptotics theorem, one can assign asymptotic data to such a CMC surface $f(\Sigma)$, consisting of the asymptotic axes, asymptotic necksizes, and asymptotic neck phases. The asymptotics map brings up a natural question: how well does the asymptotic data determine the CMC surface? For instance, one can ask for $f(\Sigma)$ to be locally rigid, in the sense that there are no one-parameter families of CMC surfaces $f^\epsilon(\Sigma)$, with $f^0(\Sigma) = f(\Sigma)$ with $f^\epsilon(\Sigma)$ all having the same asymptotic data.

The existence of such a one-parameter family would yield a solution u to the linearized equation

$$\mathcal{L}_f(u) := \frac{1}{2}(\Delta_f + |A_f|^2)u = 0$$

where u decays along each end. Here Δ_f is the Laplace–Beltrami operator and $|A_f|$ is the norm of the second fundamental form. Solutions to $\mathcal{L}_f(u) = 0$ are called Jacobi fields of $f(\Sigma)$. We call $f(\Sigma)$ *nondegenerate* if $u \in L^2$ and $\mathcal{L}_f(u) = 0$ implies $u \equiv 0$. It is a theorem of Kusner, Mazzeo and Pollack [KMP] that if $f(\Sigma)$ is nondegenerate then $f(\Sigma)$ is locally rigid in the sense described above. Moreover, in this case the moduli space of nearby CMC surfaces is parameterized by a $3k$–dimensional subset of the Jacobi fields with at most linear growth.

Our main theorem is

Theorem. *If $f(\Sigma)$ has genus zero and is contained in a solid slab then $f(\Sigma)$ is nondegenerate.*

There are six basic steps to our proof. First, that $f(\Sigma)$ is contained in a solid slab implies one can decompose $f(\Sigma)$ into two graphs which are interchanged under reflection through a plane contained in the slab [KKS]. We normalize the reflection plane to be the xy–plane. The symmetry allows us to decompose Jacobi fields u into an even part u_+, which solves a Neumann boundary value problem, and an odd part u_-, which solves a Dirichlet boundary value problem. In the second step we show that any bounded, odd Jacobi field must be a multiple of the Jacobi field generating vertical translation, *i.e.* vertical part of the normal ν. This Jacobi field is not in L^2.

Our third step recalls the interchange of Dirichlet and Neumann data in conjugate harmonic functions. We linearize the conjugate cousin equation for the surface $f(\Sigma)$ [GKS] to obtain a new vector field V along the top half of the surface. The vector fields V and $U = u\nu$ (where ν is the normal to $f(\Sigma)$) are related by

$$dV = 2V \times df \circ J + d(u\nu) \circ J + df \circ \dot{J}.$$

Unfortunately, starting with an even Jacobi field u doesn't guarantee an odd Jacobi field $v := \langle V, \nu\rangle$. In our fourth step we show that the conjugate field V we construct satisfies the condition

$$(\partial_t V - 2V \times \eta) \parallel e_3, \qquad (\partial_n V + 2V \times \tau) \perp e_3$$

along the boundary of the top half of $f(\Sigma)$. We call this condition almost odd. There is a corresponding almost even condition on U:

$$\partial_n U \parallel e_3, \qquad \partial_t U \perp e_3$$

along the boundary of the top half of $f(\Sigma)$, and we show that U is almost even if and only if its conjugate V is almost odd.

We use this correspondence of almost even Jacobi fields to almost odd Jacobi fields and some local computations in our fifth step, where we show that the classifying map of [GKS] has an injective differential. Let $\mathcal{V}$ be the space of even Jacobi fields with at most linear growth on all ends of $f(\Sigma)$. The injectivity result stated above implies $\dim(\mathcal{V}) \leq 2k$. In our sixth (and final) step, we adapt a relative index computation of [KMP] to show that $\dim(\mathcal{V}) \geq 2k$, with equality if and only if $f(\Sigma)$ is nondegenerate. Putting these last two steps together completes the proof of our main theorem.

References

[GKS] K. Grosse–Brauckmann, R. Kusner, and J. Sullivan. *Coplanar constant mean curvature surfaces*, preprint.

[KKR] N. Korevaar, R. Kusner, and J. Ratzkin. *On the nondegeneracy of constant mean curvature surfaces*, Geom. Funct. Anal. **16** (2006), 891–923.

[KKS] N. Korevaar, R. Kusner, and B. Solomon. *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), 465–503.

[KMP] R. Kusner, R. Mazzeo, and D. Pollack. *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. **6** (1996), 120–137.

An application of a 3-involution loop group to reflective submanifolds

David Brander

1.1. **Overview.** We discuss some results from the recent preprint [1]. The goal is to generalize the 3-involution loop group defined by Ferus and Pedit in [2], to arbitrary commuting involutions of any Lie group, and identify the associated special submanifolds. Results: generalizations, to arbitrary reflective submanifolds, of known results concerning isometric immersions of space forms; in case of previous results, new proofs; and new special submanifolds as integrable systems.

1.2. **Reflective submanifolds.** A *reflective submanifold*, N, of a Riemannian manifold, $\bar{N}$, is a totally geodesic symmetric submanifold. If $\bar{N} = \bar{U}/\bar{K}$ is a connected symmetric space, then N is characterized by a pair of commuting involutions, $\bar{\tau}$ and $\hat{\sigma}$, of the Lie algebra $\bar{\mathfrak{u}}$ of $\bar{U}$, and $\bar{K} = \bar{U}_{\bar{\tau}}$ is the fixed point subgroup of $\bar{\tau}$. That is, N is determined by a triple: $(\bar{\mathfrak{u}}, \bar{\tau}, \hat{\sigma})$. One has two canonical decompositions $\bar{\mathfrak{u}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}} = \hat{\mathfrak{k}} \oplus \hat{\mathfrak{p}}$. Setting $\mathfrak{p} := \bar{\mathfrak{p}} \cap \hat{\mathfrak{p}}$, the reflective submanifold is given by: $N = \pi_{\bar{N}} \exp(\mathfrak{p})$.

Reflective submanifolds of symmetric spaces have been classified by DSP Leung [3, 4], and there are clearly many cases. In space forms, the reflective submanifolds are just the complete totally geodesic submanifolds.

1.3. **Isometric immersions of space forms.** An isometric immersion

$$f : M^k(c) \to M^n(\tilde{c}),$$

of space forms with constant sectional curvature c and $\tilde{c}$ respectively, has *negative extrinsic curvature* if $c < \tilde{c}$. We only discuss this case, as there are always umbilic hypersurfaces for positive extrinsic curvature. There are two basic questions: existence of a local solution, and existence of a complete solution. For these it is known:

(1) Local solutions exist iff $n \geq 2k - 1$ (Cartan).

(2) Theorem (JD Moore): If $0 < c < 1$, there is no complete isometric immersion with flat normal bundle of $S^k(c)$ into S^n for any $k > 1$ and any n.
(3) Conjecture: If $c < -1$ there is no complete isometric immersion with flat normal bundle of $H^k(c)$ into $H^n(-1)$ for any $k > 1$ and any n.

Moore's proof of item (2) was actually in codimension $k - 1$, but this can be replaced with flat normal bundle. Item (3) is equivalent to the non-immersibility of H^k into $\mathbf{R}^n$, and this is usually stated in codimension $k - 1$ also; again, the best results to date concerning this problem only depend on flatness of the normal bundle, not the codimension. Note: $S^k(c)$ is a reflective submanifold, with flat normal bundle, of $S^n(c)$, and the analogue holds for $H^k(c)$ in $H^n(c)$.

1.4. **The generalization to other reflective submanifolds.** If M is a Riemannian manifold, we let M_R denote the same manifold with the metric scaled by a factor $R > 0$. Since S^k and H^k are (the only k-dimensional) reflective submanifolds of S^n and H^n respectively, items (1)-(3) can be generalized to arbitrary reflective submanifolds of symmetric spaces, by asking whether, given $N \subset \bar{N}$, is there, for appropriate choices of R, a local/global isometric immersion of N_R into $\bar{N}$? Since a reflective submanifold does not, in general, have flat normal bundle, we must replace this condition with an appropriate one:

If $N \subset \bar{N}$ (Riemannian immersed submanifold), and $\mathrm{Dim}(\bar{N}') = \mathrm{Dim}(\bar{N})$, then an isometric immersion $f : N \to \bar{N}'$ is *normal curvature preserving*) if $(T^{\perp}N, \nabla^{\perp})$ is isomorphic to $(T'^{\perp}N, \nabla'^{\perp})$, i.e. there is a bundle isomorphism between the corresponding normal bundles which pulls back the normal bundle connections.

We also ask that our immersions be $\mathcal{V}_{\mathfrak{p}}$*-submanifolds*, that is they belong to the Grassmann geometry of submanifolds which are are tangent to $\mathfrak{p} \subset \bar{\mathfrak{p}}$ at each point, if we identify the tangent bundle of $\bar{N}$ with $\bar{\mathfrak{p}}$, via left translation to the origin of the symmetric space. This condition is automatic in space forms.

The generalization of items (1)-(3) above which we consider is:
Problem A: Suppose given a simply connected, immersed, reflective submanifold $N := \exp(\mathfrak{p})$ of a semisimple Riemannian symmetric space $\bar{N}$. Thus, N_R is a reflective submanifold of $\bar{N}_R$. Does there exist a (local or global) isometric immersion of N_R into $\bar{N}$ as a normal curvature preserving $\mathcal{V}_{\mathfrak{p}}$-submanifold? More specifically, we ask this for:

(1) $R > 1$, if $\bar{N}$ is of compact type,
(2) $R < 1$, if $\bar{N}$ is of non-compact type.

1.5. **Results.** Concerning the compact case, we prove:

Theorem 1. *The following list contains the geometric interpretations of all possible solutions to Problem A for the case $R > 1$ and $\bar{N}$ is a simply connected, compact, irreducible, Riemannian symmetric space. In all cases, local solutions exist and can be constructed by loop group methods. In all cases where* $\mathrm{Dim}(N_R) > 1$, *there is no solution which is geodesically complete.*

(1) $N_R = S^k_R$ *is an isometric immersion with flat normal bundle of a* k*-sphere of radius* $\sqrt{R}$ *into the unit sphere* S^n*, with* $0 < k \leq (n+1)/2$*, and* $n \geq 2$.
(2) $N_R = S^n_R$ *is an isometric totally real immersion of an* n*-sphere of radius* $\sqrt{R}$ *into complex projective space* $\mathbf{C}P^n$*, with* $n \geq 2$.

In particular, the second case, of Lagrangian immersions of a sphere into $\mathbf{C}P^n$ is a new example of a special submanifold as an integrable system.

For the non-compact case we obtain the analogue, replacing the compact symmetric spaces with their non-compact duals as appropriate. The difference is, we do not obtain the global non-existence result, which remains an open problem.

1.6. **Outline of proof.** We describe here the compact case. Given $(\bar{\mathfrak{u}}, \bar{\tau}, \hat{\sigma})$, let $G = \bar{U}^{\mathbf{C}}$ and let $\bar{U} := G_\rho$ be the real form determined by the anti-linear involution ρ, with corresponding Lie algebra $\bar{\mathfrak{u}}$. We extend the involutions to ΛG, the group of smooth maps from the unit circle into G, by the rules: $(\rho X)(\lambda) = \rho(X(\bar{\lambda}))$, $(\hat{\sigma} X)(\lambda) = \hat{\sigma}(X(-\lambda))$, $(\bar{\tau} X)(\lambda) = \bar{\tau}(X(-1/\lambda))$, for $\lambda \in S^1$. Consider the subgroup of elements fixed by all three involutions:

$$\mathcal{H} = (\Lambda G)_{\rho \bar{\tau} \hat{\sigma}}.$$

We consider certain maps into $\mathcal{H}/\mathcal{H}^0$, where $\mathcal{H}^0 = \bar{U}_{\bar{\tau}\hat{\sigma}} = \bar{K} \cap \widehat{K}$.

A $\mathcal{V}^1_{-1}$*-compatible immersion* $f : M \to \mathcal{H}/\mathcal{H}^0$, is a regular map such that if $F : M \to \mathcal{H}$ is a frame for F then the λ-Fourier expansion of its Maurer-Cartan form has highest and lowest exponents 1 and -1 respectively:

$$F^{-1}\mathrm{d}F = \alpha_{-1}\lambda^{-1} + \alpha_0 + \alpha_1 \lambda.$$

Because elements of $\mathcal{H} = (\Lambda G)_{\rho\bar{\tau}\hat{\sigma}}$ are loops with values in $\bar{U}$ for values of λ in $\mathbf{R}$, a $\mathcal{V}^1_{-1}$-immersion gives, for fixed real values of λ, an immersion $f_\lambda : M \to \bar{U}/(\bar{K} \cap \widehat{K})$. We can then take projections of f_λ to either $\bar{U}/\bar{K}$ or $\bar{U}/\widehat{K}$ or, more generally, $\bar{U}/L$ where $\bar{K} \cap \widehat{K} \subset L$. We have the following results for projections to $\bar{U}/\bar{K}$ and $\bar{U}/\widehat{K}$:

Proposition 1. *Let* $f : M \to \mathcal{H}/\mathcal{H}^0$ *be a* $\mathcal{V}^1_{-1}$*-compatible immersion. For* $\lambda \in \mathbf{R}^* \setminus \{\pm 1\}$*, let* $\bar{f} : M \to \bar{U}/\bar{K}$ *be the projection of* f_λ*. Suppose that* $\mathrm{Dim}(M) = \mathrm{Dim}(\mathfrak{p})$*, and that* $\bar{f}$ *is regular. Then* $\bar{f}$ *is a solution of Problem A (for* $R > 1$*). Conversely, any solution of Problem A, corresponds to such a* $\mathcal{V}^1_{-1}$ *immersion.*

Proposition 2. *Let* $f : M \to \mathcal{H}/\mathcal{H}^0$ *be a* $\mathcal{V}^1_{-1}$*-compatible immersion. For* $\lambda \in \mathbf{R}^*$*, let* $\hat{f} : M \to \bar{U}/\widehat{K}$ *be the projection of* f_λ*. Then:*

- $\hat{f}$ *is a curved flat in* $\bar{U}/\widehat{K}$*.*
- *If* $\bar{f}$ *is regular then so is* $\hat{f}$ *(but not conversely). Hence:*
- *Local regular solutions to Problem A exist* $\Rightarrow \mathrm{Dim}(\mathfrak{p}) \leq \mathrm{Rank}(\bar{U}/\widehat{K})$*.*
- *Global regular solutions to Problem A do not exist if* N *is not flat.*

Theorem 2. *Conversely, if* $\mathrm{Dim}(\mathfrak{p}) \leq \mathrm{Rank}(\bar{U}/\widehat{K})$*, then local solutions to Problem A exist and can be constructed from curved flats in* $\bar{U}/\widehat{K}$*.*

Finally, to prove Theorem 1 one can, in principal, go through the entire list of reflective submanifolds and check this rank condition.

1.7. **Other projections.** For the cases where $\mathrm{Dim}(\mathfrak{p}) > \mathrm{Rank}(\bar{U}/\widehat{K})$, lower dimensional solutions exist, and can be described as certain deformations of *sub*-manifolds of N. Alternatively, we can take other projections: we consider an example where $\bar{U} = G_2$. Instead of projecting to $\bar{U}/\bar{K}$, we project to $G_2/SU(3) = S^6$. The resulting surfaces are deformations of totally geodesic complex curves in S^6, with the property that the restriction of TS^6 to the surface decomposes canonically into three 2-dimensional sub-bundles, each of which is invariant under the almost complex structure of S^6.

References

[1] D. Brander. Grassmann geometries in infinite dimensional homogeneous spaces and an application to reflective submanifolds. *arxiv preprint.*
[2] D. Ferus and F. Pedit. Isometric immersions of space forms and soliton theory. *Math. Ann.*, 305:329–342, 1996.
[3] D. S. P. Leung. On the classification of reflective submanifolds of Riemannian symmetric spaces. *Indian Univ. Math. J.*, 24:327–339, 1974.
[4] D. S. P. Leung. Reflective submanifolds. III. congruency of isometric reflective submanifolds and corrigenda to the classification of reflective submanifolds. *J. Differential Geom.*, 14, 1979.

On flat fronts in hyperbolic 3-space

Masatoshi Kokubu

We report some results from joint works ([3], [4], [5]) with Wayne Rossman, Masaaki Umehara, and Kotaro Yamada.

We investigate flat surfaces in hyperbolic 3-space H^3 in the class of wave fronts (fronts, for short) and/or projective wave fronts (p-fronts, for short), which is wider than the class of immersions. It is well-known that the horospheres and the hyperbolic cylinders are the only complete flat surfaces immersed in H^3. It means that the global differential geometry of flat surfaces immersed in H^3 have completed. However, global studies for flat (p-)fronts in H^3 are not known well and they excite interest for us.

A *front* (or a *p-fronts*) $f\colon M^2 \to H^3$ is, by definition, the projection of a Legendre immersion $L\colon M^2 \to T_1H^3$ (or $P(TH^3)$) into the unit tangent bundle (or the projective bundle associated to tangent bundle). Note that, since T_1H^3 naturally projects $P(TH^3)$, any front is a p-front. For an orientable surface immersed in H^3, the pair (f, n) of the immersion f itself and the unit normal n defines a Legendre immersion into T_1H^3. Hence, any immersion is a front.

As the case of CMC-1 surface theory, we mainly regard H^3 as the matrix model $\mathrm{SL}(2,\boldsymbol{C})/\mathrm{SU}(2) = \{xx^* \,;\, x \in \mathrm{SL}(2,\boldsymbol{C})\}$ or $\mathrm{PSL}(2,\boldsymbol{C})/\mathrm{PSU}(2)$.

1. Flat fronts

Our investigation for flat fronts is based on the existence of a holomorphic lift to $\mathrm{SL}(2,\boldsymbol{C})$, which is originally shown by Gálvez, Martínez and Milán [1] in case of immersions. Precisely speaking, a flat front $f\colon M^2 \to H^3$ is given by $f = \mathcal{E}\mathcal{E}^*$

where $\mathcal{E}\colon \widetilde{M}^2 \to \mathrm{SL}(2, \boldsymbol{C})$ is a holomorphic non-singular Legendre curve from the universal covering $\widetilde{M}^2$. Note that the complex structure on M^2 is compatible to the second fundamental form. Moreover, Legendre curves, hence flat fronts, have the following representation formulas:

Theorem 1. *For an arbitrary pair (G, ω) of a non-constant meromorphic function G and a non-zero meromorphic 1-form ω on a Riemann surface M^2, the meromorphic map*

$$\mathcal{E} = \begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix} \qquad \left(C = i\sqrt{\frac{\omega}{dG}},\ A = GC \right) \tag{1}$$

is a meromorphic Legendrian curve in $\mathrm{PSL}(2, \mathrm{C})$ whose hyperbolic Gauss map and canonical form are G and ω, respectively. Conversely, if $\mathcal{E}$ is a meromorphic Legendrian curve in $\mathrm{PSL}(2, \boldsymbol{C})$ defined on M^2 with non-constant hyperbolic Gauss map G and non-zero canonical form ω, then $\mathcal{E}$ is as in (1).

Theorem 2. *Let G and G_* be non-constant meromorphic functions on a Riemann surface M^2 such that $G(p) \neq G_*(p)$ for all $p \in M^2$. Assume that*

$$\int_\gamma \frac{dG}{G - G_*} \in i\boldsymbol{R}$$

for every loop γ on M^2. Set

$$\xi(z) = c \cdot \exp \int_{z_0}^{z} \frac{dG}{G - G_*},$$

where $z_0 \in M^2$ is a base point and $c \in \boldsymbol{C} \setminus \{0\}$ is an arbitrary constant. Then

$$\mathcal{E} = \begin{pmatrix} G/\xi & \xi G_*/(G - G_*) \\ 1/\xi & \xi/(G - G_*) \end{pmatrix}$$

is a non-constant meromorphic Legendrian curve defined on $\widetilde{M}^2$ in $\mathrm{SL}(2, \boldsymbol{C})$ whose hyperbolic Gauss maps are G and G_*, and the projection $f = \mathcal{E}\mathcal{E}^*$ is single-valued on M^2. Moreover, f is a front if and only if G and G_* have no common branch points. Conversely, any non-totally-umbilic flat front can be constructed this way.

These representation formulas play an important role in studying flat fronts like the Weierstrass formula in minimal surface theory. Geometrically, $G(p)$ and $G_*(p)$ are the points in the ideal boundary $\partial H^3 (\cong S^2)$ so that the hyperbolic line orthogonal to $f(M^2)$ at $f(p)$ meets ∂H^3 at $G(p)$ and $G_*(p)$.

We say that a flat front f is *weakly complete* if $L^*(ds_S^2)$ is a complete metric, where ds_S^2 is the Sasakian metric on $T_1 H^3$. Moreover, f is said to be of *finite type* if $L^*(ds_S^2)$ has finite total curvature.

Theorem 3. *A weakly complete flat front of finite type has finite topology.*

Each end is said to be *regular* if the hyperbolic Gauss maps G G_* extend meromorphically to the end. As an analogue of the Osserman inequality for minimal surfaces in $\boldsymbol{R}^3$, we have:

Theorem 4. *A weakly complete flat front* $f\colon \overline{M^2}\setminus\{p_1,\dots,p_n\}\to H^3$ *of finite type with regular ends satisfies the inequality*

$$\deg G + \deg G_* \geq n.$$

Moreover, equality holds if and only if all ends are embedded.

As other applications of Theorem 2, we also gave

- the classification of weakly complete flat fronts of finite type which has at most three embedded regular ends,
- the construction of weakly complete flat fronts of finite type which has embedded regular ends for a given finite number of points in ∂H^3,
- the construction of weakly complete flat fronts of finite type of arbitrary genus $k \geq 1$, which has $4k+1$ embedded regular ends.

2. Caustics, flat p-fronts

For an immersed surface, the *caustic* is, by definition, the loci of the centers of principal curvature. The following are well-known:

- Parallel surfaces have the same caustic.
- A parallel surface f_t of f has a singularity at p if t equals to radius of principal curvature.
- Any parallel surface of a flat surface is also flat.

Therefore the following definition does make sense.

Definition 1. *For a flat front* $f\colon M^2\to H^3$*, we call the loci of singularities of* f_t $(-\infty < t < \infty)$ *the caustic.*

It is known that, for a flat immersed surface in H^3, the caustic is also flat (so long as it is an immersion). Similarly, we have:

Proposition 3. *For a flat front in* H^3*, the caustic is locally a flat front.*

It should be noted that a caustic may fail to be globally a front, but a *p-front*. It is verified from Roitman's representation formula [6] for caustics of flat fronts:

Theorem 5. *Let* $f: M^2\to H^3$ *be a flat front with hyperbolic Gauss maps* G, G_**. Then its caustic* C_f *is defined on* $M^2\setminus\{\text{umbilical points}\}$*, and is given by*

$$C_f = \frac{1}{|G-G_*|}\begin{pmatrix} |G|^2|\alpha|^{-1/2}+|G_*|^2|\alpha|^{1/2} & G|\alpha|^{-1/2}+G_*|\alpha|^{1/2} \\ \bar{G}|\alpha|^{-1/2}+\bar{G}_*|\alpha|^{1/2} & |\alpha|^{-1/2}+|\alpha|^{1/2}\end{pmatrix}$$

where $\alpha = dG/dG_*$.

One can show that an unit normal n for C_f is given by

$$n = \frac{i}{|G-G_*||\alpha|^{1/2}}\begin{pmatrix} G_*\bar{G}\sqrt{\alpha}-G\bar{G}_*\sqrt{\bar{\alpha}} & G_*\sqrt{\alpha}-G\sqrt{\bar{\alpha}} \\ \bar{G}\sqrt{\alpha}-\bar{G}_*\sqrt{\bar{\alpha}} & \sqrt{\alpha}-\sqrt{\bar{\alpha}}\end{pmatrix}.$$

The term $\sqrt{\alpha}$ may cause C_f not to be a front but a p-front.

In general, there are orientable p-fronts and non-orientable p-fronts. However, this is not true for flat p-fronts in H^3. Indeed, we have:

Theorem 6. *Any flat p-front in H^3 is orientable.*

3. Addendum

Finally, we remark that we have also studied the singularities of flat fronts in [2] with Kentaro Saji, for example, we gave criteria for a singularity to be a cuspidal edge or swallowtail.

References

[1] J. A. Gálvez, A. Martínez and F. Milán, *Flat surfaces in hyperbolic 3-space*, Math. Ann., **316** (2000), 419–435.
[2] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221** (2005), no. 2, 303–351.
[3] M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space and their caustics*, J. Math. Soc. Japan, **59** (2007), no. 1, 265–299.
[4] M. Kokubu, M. Umehara and K. Yamada, *An elementary proof of Small's formula for null curves in* $\mathrm{PSL}(2, \boldsymbol{C})$ *and an analogue for Legendrian curves in* $\mathrm{PSL}(2, \boldsymbol{C})$, Osaka J. Math., **40** (2003), no. 3, 697–715.
[5] M. Kokubu, M. Umehara and K. Yamada, *Flat fronts in hyperbolic 3-space*, Pacific J. Math., **216** (2004), no. 1, 149–175.
[6] P. Roitman, *Flat surfaces in hyperbolic 3-space as normal surfaces to a congruence of geodesics*, Tohoku Math. J., **59** (2007), no. 1, 21–37.

Flat surfaces in $\mathbf{R}^3$

Masaaki Umehara
(joint work with Satoko Murata)

An immersed surfaces in the Euclidean 3-space $\mathbf{R}^3$ is called *flat* if its Gaussian curvature vanishes. The following result is classical.

Fact. (Hartman-Nirenberg [1]) *A complete flat immersed surface is a plane or a cylinder.*

This implies that the study of global behaviour of flat surfaces requires the study of singular points as well.

(Definition of wave fronts) Let M^2 be a 2-manifold and $f : M^2 \to \mathbf{R}^3$ a C^∞-map. A point $p \in M^2$ is called *regular* if f is an immersion on a sufficiently small neighborhood of p, and is called *singular* if it is not regular. Moreover, $f : M^2 \to \mathbf{R}^3$ is called a (*wave*) *front* if

(1) there exists a unit vector field ν along f such that ν is perpendicular to the image of tangent spaces $f_*(TM)$. (ν is called the *unit normal vector field* of f, which can be identified with the Gauss map $\nu; M^2 \to \mathbf{R}^3$.)
(2) Then the pair of maps
$$L := (f, \nu) : M^2 \to \mathbf{R}^3 \times S^2 (\cong T_1^* \mathbf{R}^3)$$
gives an immersion.

(Definition of flat fronts) A (wave) front f is called *flat* if its Gauss map degenerate everywhere.

If f is an immersion, the above definition is equivalent that the flatness of the Gaussian curvature. The circular cone or the tangential developable of a helix is a typical example of flat front with singularities.

Let f be a front. For each real number t, the map

$$f_t := f + t\nu : M^2 \to \mathbf{R}^3 \qquad (t \in \mathbf{R})$$

is also a front, which is called the *parallel front* of f. Since the Gauss map of f_t is the same as that of f, the parallel front of flat fronts are also flat. On the other hand, the set

$$\mathcal{C}_f := \bigcup_{t\in\mathbf{R}} \{f_t(p) \in \mathbf{R}^3\,;\, p \text{ is a singular point of } f_t\}$$

is called the *caustic* (or the *focal surface*) of f. If f is a flat front, then so is $\mathcal{C}_f$. In this sense, it is natural to consider flat surfaces in the category of wave front .

By definition, front has globally defined unit normal vector field. However, it seems too strong for the global study of flat surfaces. So we also give the following definition:

(Definition of p-fronts) $f : M^2 \to \mathbf{R}^3$ is called a *p-front* if it is locally a front, that is, for each $p \in M^2$, there exists a neighborhood U_p of p such that the restriction of the map $f|_{U_p}$ is a (wave) front.

Any front is a p-front, but converse may not be true in general. The flatness of p-front is also canonically defined. Now we define the completeness for p-fronts.

(Definition) A p-front $f : M^2 \to \mathbf{R}^3$ is called *complete* if there exists a symmetric covariant tensor T such that

- supp(T) is compact,
- the sum $T + ds^2$ gives a complete Riemannian metric on M^2.

If f is an immersion, the above completeness is equivalent to the completeness of the first fundamental form. We get the following result which will play a crucial role for the classification of complete flat fronts:

(Theorem A.) ([4]) *If a complete flat p-front admits a singular point, then it does not admit umbilics.*

On the other hand, we can prove that there exists an umbilic line on any non-orientable developable p-front. So we get the following corollary:

(Corollary.) ([4]) *Let $f : M^2 \to \mathbf{R}^3$ be a complete flat p-front. Then M^2 is orientable.*

It should be remarked that any flat p-front in hyperbolic 3-space is orientable. (cf. [3]). On the other hand, there exist flat Möbius strips in $\mathbf{R}^3$ and S^3 (cf. [5] and [2]). So the above corollary need the assumption "completeness".

By Theorem A, we get the following classification theorem of complete flat fronts with singularity in $\mathbf{R}^3$:

(Theorem B.) ([4]) *Let $\xi : S^1 \to S^2$ be a regular curve without inflection points, and α a 1-form on S^1 such that $\xi\alpha$ is exact, namely $\int_{S^1} \xi\alpha = 0$. Then*

$$f : S^1 \times \mathbf{R} \ni (t, u) \mapsto \gamma(t) + u \int_0^t \xi\alpha \in \mathbf{R}^3,$$

gives a complete flat front with singularity. Conversely, any complete flat fronts with a singular point are given in this manner.

As an application of Theorem B, we get the following:

(Theorem C.) ([4]) *Let f be a complete flat front with a singular point. Then the image of its Gauss map Im(ν) consists of a locally convex regular curve in S^2. Moreover, the following two assertions are equivalent:*

(1) *Im(ν) is convex,*
(2) *Each end of f is properly embedded.*

References

[1] P. Hartman and L. Nirenberg, *On spherical image whose Jacobians do not change signs*, Amer. J. Math. 81 (1959) 901–920.
[2] J. A. Gálvez and P. Mira, *Isometric immersions of $\mathbf{R}^2$ into $\mathbf{R}^4$ and perturbation of Hopf tori*, preprint.
[3] M. Kokubu, W. Rossman, M. Umehara and K. Yamada: *Flat fronts in hyperbolic 3-space and their caustics*, J. Math. Soc. Japan 59 (2007) 265–299.
[4] S. Murata and M. Umehara, *Flat Surfaces with singularities in Euclidean 3-space* , preprint (math.DG/0605604).
[5] W. Wunderlich, *Über ein abwickelbares Möbiusband*, Monatsh. Math. 66 (1962) 276–289.

Spacelike mean curvature one surfaces in de Sitter 3-space

Kotaro Yamada
(joint work with S. Fujimori, W. Rossman, M. Umehara and S.-D. Yang)

We denote by S^3_1 the *de Sitter* 3*-space*, which is the 3-dimensional Lorenzian space form of constant sectional curvature 1 defined as

$$\begin{aligned} S^3_1 &:= \{x \in L^4 \mid \langle x, x\rangle = 1\} = \mathrm{SL}(2, \boldsymbol{C})/\,\mathrm{SU}(1,1) \\ &= \{a e_3 {}^t\bar{a} \mid a \in \mathrm{SL}(2,\boldsymbol{C})\} \qquad \left(e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \end{aligned}$$

where L^4 is the Lorenz-Minkowski 4-space with the inner product $\langle\ ,\ \rangle$ of signature $(-,+,+,+)$. We denote by π_S the canonical projection

$$\pi_S\colon\ \mathrm{SL}(2,\boldsymbol{C}) \ni a \longmapsto a e_3 {}^t\bar{a} \in S^3_1.$$

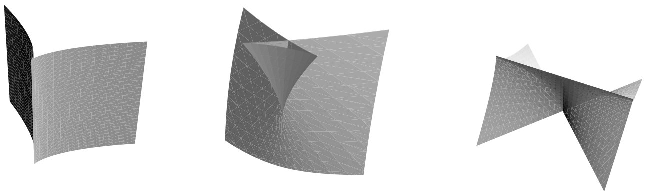


FIGURE 1. Generic singularities of CMC-1 faces

Similarly, we denote by π_H the projection onto the hyperbolic 3-space H^3:

$$\pi_H\colon \mathrm{SL}(2,\boldsymbol{C}) \ni a \longmapsto a^t\bar{a} \in \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{SU}(2) = H^3.$$

A holomorphic map $F : M^2 \to \mathrm{SL}(2,\boldsymbol{C})$ of a Riemann surface M^2 into the complex Lie group $\mathrm{SL}(2,\boldsymbol{C})$ is called *null* if $\det(dF/dz)$ vanishes identically, where z is a local complex coordinate of M^2. As a well-known fact [1], the projection $\pi_H \circ F$ of a holomorphic null *immersion* $F\colon M^2 \to \mathrm{SL}(2,\boldsymbol{C})$ gives a mean curvature one (CMC-1) immersion into the hyperbolic 3-space. On the other hand, $\pi_S \circ F$ gives a spacelike mean curvature one surfaces in S^3_1 which may have *singular points* even if F is an immersion. In fact, it is known that the only complete spacelike surface of constant mean curvature one is the totally geodesic surface.

So it is natural to generalize CMC-1 surface theory to that with singular points. In fact, a notion of *CMC*-1 *faces* is defined in [2] as an analogy of *maxfaces*, which is a certain class of maximal surfaces with singularities in the Lorenz-Minkowski 3-space [7]. CMC-1 faces are obtained by Weierstrass-type representation formula:

Theorem 1 (The representation formula [2])**.** *Let $f\colon M^2 \to S^3_1$ be a smooth map defined on an oriented 2-manifold M^2. Then f is a CMC-1 face if and only if there exists a complex structure on M^2 and a null holomorphic immersion $F\colon \widetilde{M}^2 \to \mathrm{SL}(2,\boldsymbol{C})$ such that $f = \pi_S \circ F$, where $\widetilde{M}^2$ is the universal cover of M^2.*

We call F the *holomorphic lift* of f.

Definition 1. *In the situations as in Theorem 1, we define*

$$G = \frac{dF_{11}}{dF_{12}} = \frac{dF_{21}}{dF_{22}}, \qquad g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}, \qquad \omega = F_{11}dF_{21} - F_{21}dF_{11}.$$

The holomorphic maps $G\colon M^2 \to \boldsymbol{C} \cup \{\infty\}$ and $g\colon \widetilde{M}^2 \to \boldsymbol{C} \cup \{\infty\}$ are called the hyperbolic *(*secondary*) Gauss maps, respectively. The pair (g, ω) is called the* Weierstrass data *of f.*

A point $p \in M^2$ is a singular point of f if and only if $|g(p)| = 1$ holds. In fact, the induced metric is represented as $ds^2 = (1 - |g|^2)^2|\omega|^2$.

Theorem 2 ([4])**.** *Generic singularities of CMC-1 faces are cuspidal edges, swallowtails and cuspidal cross caps, see Figure 1.*

Remark that a CMC-1 face may not be considered as a wave front (see M. Umehara's report in this volume). In fact, the cuspidal cross cap is not a wave front.

Though it has singularities, CMC-1 faces are represented by a holomorphic object. Thus it is natural to consider global problems on it.

Definition 2 (Completeness)**.** *A CMC-1 face $f\colon M^2 \to S^3_1$ is* complete *if there exists a compact set C and a symmetric 2-tensor T on M^2 such that $ds^2 + T$ is a complete Riemannian metric on M^2, where ds^2 is the induced metric of f.*

By definition, the set of singular points of a complete CMC-1 face is compact.

Proposition 4 ([3])**.** *Let $f\colon M^2 \to S^3_1$ be a complete CMC-1 face and consider M^2 as a Riemann surface with the complex structure in Theorem 1. Then there exist a compact Riemann surface $\overline{M}^2$ and a finite set of points $\{p_1, \dots, p_n\} \subset \overline{M}^2$ such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{p_1, \dots, p_n\}$. Each p_j corresponds to an end of f.*

The end p_j is said to be *regular* if G is meromorphic at p_j. Now, we can state our main theorem:

Theorem 3 (Osserman-type inequality [3])**.** *Let G be the hyperbolic Gauss map of a complete CMC-1 face $f\colon M^2 = \overline{M}^2 \setminus \{p_1, \dots, p_n\} \to S^3_1$, where $\overline{M}^2$ is a compact Riemann surface. Then*

$$2 \deg G \geq -\chi(M^2) + \#\{ends\} = -\chi(\overline{M}^2) + 2\#\{ends\}$$

holds. Moreover, the equality holds if and only if all ends are regular and properly embedded.

For a minimal surface in $\boldsymbol{R}^3$ with the Gauss map G, $4\pi \deg G$ is the total absolute curvature, and the inequality in Theorem 3 is the well-known Osserman inequality.

The proof of the inequality in Theorem 3 is done in the similar way as the case of CMC-1 surfaces in H^3 [6]. To show the second part of the theorem, we have to look at the properties of each end, that is a CMC-1 *immersion*

$$(*) \qquad f\colon \Delta^* = \{z \,|\, 0 < |z| < 1\} \longrightarrow S^3_1.$$

Though the holomorphic lift F is defined only on the universal cover $\widetilde{\Delta}^*$ of Δ^*, $f = \pi_S \circ F = F e_3 {}^t\overline{F}$ is well-defined on Δ. Hence there exists $\rho_f \in \mathrm{SU}(1,1)$ such that $F \circ \tau = F \rho_f$, where τ is a deck transformation of the universal cover $\widetilde{\Delta}^* \to \Delta^*$ corresponds to a loop surrounding the origin. Note that $\mathrm{SU}(1,1)$ is isomorphic to $\mathrm{SL}(2, \boldsymbol{R})$, that is the double cover of the connected component of the isometry group of the hyperbolic plane H^2.

Definition 3. *An end p_j is said to be* elliptic, parabolic, *or* hyperbolic *if the corresponding matrix ρ_f is elliptic, parabolic, or hyperbolic, respectively as an isometry of H^2.*

Lemma 3. *The secondary Gauss map g of the end $(*)$ can be chosen as follows*:

- $g(z) = z^{\mu}h(z)$ $(\mu \in \boldsymbol{R})$, *if the end is elliptic,*
- $\hat{g}(z) = h(z) + \log z$, *if the end is parabolic, or*
- $\hat{g}(z) = z^{i\mu}h(z)$ $(\mu \in \boldsymbol{R} \setminus \{0\})$ *if the end is hyperbolic,*

where $\hat{g} = -i(g+1)/(g-1)$ *and* $h\colon \Delta^* \to \boldsymbol{C}$ *is a (single-valued) holomorphic function.*

Definition 4. *An end as in the previous lemma is called* of finite type *if the function h in the lemma is meromorphic at* 0. *Moreover, a parabolic end of finite type is called of the* first *(resp.* second*)* kind *if $h(0) \neq \infty$ (resp. $h(0) = \infty$).*

Then, we have

Theorem 4 ([3])**.** *A complete end is of finite type, and either elliptic or parabolic of the first kind. In particular, hyperbolic end will never occur.*

Thus, the second part of Theorem 3 is obtained by analyzing a behavior of elliptic ends and parabolic ends of the first kind.

References

[1] R. Bryant, *Surfaces of mean curvature one in hyperbolic space*, Astérisque, **154–155** (1987), 321–347.
[2] S. Fujimori, *Spacelike CMC* 1 *surfaces with elliptic ends in de Sitter* 3*-Space*, Hokkaido Math. J., **35** (2006), 289–320.
[3] S. Fujimori, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang, *Spacelike mean curvature one surfaces in de Sitter* 3*-space*, in preparation.
[4] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, preprint, math.DG/0510366.
[5] S. Lee and S.-D. Yang, *Spacelike constant mean curvature* 1 *trinoids in de Sitter three-space*, Osaka J. Math., **43** (2006), 641–663.
[6] M. Umehara and K. Yamada, *A duality on CMC-1 surface in the hyperbolic* 3*-space and a hyperbolic analogue of the Osserman Inequality*, Tsukuba J. of Math., **21** (1997), 229–237.
[7] M. Umehara and K. Yamada, *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J., **35** (2006), 13–40.

Embedded minimal and constant mean curvature surfaces in locally homogeneous three-manifolds

William H. Meeks, III

Recently there have appeared a number of deep results complete embedded minimal surfaces in R^3 and constant mean curvature $H = 1$ surfaces in hyperbolic three-space H^3, when the surfaces have finite topology. Before discussing these results and their generalizations to other three-manifolds we make some notation. Let $\mathcal{N}$ be the set of complete locally homogeneous three-manifolds. For a fixed $H \geq 0$ and $N \in \mathcal{N}$, let $\mathcal{M}^H(N)$ denote set of complete embedded surfaces of constant mean curvature H in N. A recent result of Colding and Minicozzi [1] states that any surface in $\mathcal{M}^0(R^3)$ of finite topology is properly embedded. Results of Collin [2] and Meeks and Rosenberg [4] then imply that the ends of such surfaces are asymptotic to ends of planes, catenoids or helicoids and in particular, these surfaces have bounded second fundamental form.

In [5], Meeks and Rosenberg generalized the results of Colding and Minicozzi by proving the next theorem.

Theorem 1 (Minimal Lamination Closure Theorem)**.** *If N is a three-manifold and M is a connected, complete, embedded minimal surface in N with positive injectivity radius, then the closure $\overline{M}$ of M has the structure of a $C^{1,\alpha}$-minimal lamination. Furthermore, if $N = R^3$, then such an M is properly embedded.*

As an application of the above theorem and the application of a delicate blow-up procedure, Meeks, Perez and Ros [3] recently proved the following theorem.

Theorem 2 (Finite Topology Bounded Geometry Theorem)**.** *If $N \in \mathcal{N}$ is compact, then the surfaces of finite topology in $\mathcal{M}^0(N)$ have bounded second fundamental form. If $N \in \mathcal{N}$ is not necessarily compact and $M \in \mathcal{M}^0(N)$ has finite topology, then the closure $\overline{M}$ has the structure of a minimal lamination.*

The following corollary answers a long standing question as to whether or not every finite topology minimal surface in $\mathcal{M}^0(S^3)$ is compact, the answer being in the affirmative.

Corollary 1. *If $N \in \mathcal{N}$ has nonnegative scalar curvature and is not flat, then any surface $M \in \mathcal{M}^0(N)$ of finite topology is properly embedded in N.*

Recently G. Tinaglia and the author have been working on generalization the above theorems to the constant mean curvature setting where $H > 0$. The results in this case appear to be stronger but this is work in progress.

References

[1] T.H. Colding and W.P. Minicozzi II, *The Calabi-Yau conjectures for embedded surfaces*, To appear in Annals of Math. (2007).

[2] P. Collin, *Topologie et courbure des surfaces minimales de R^3*, Annals of Math, 2-nd Series, **145–1** (1997), 1–31.

[3] W. H. Meeks III, J. Perez, and A. Ros, *Embedded minimal surfaces: removable singularities, local pictures and parking garage structures, the dynamics of dilation invariant collections and the characterization of examples of quadratic curvature decay*, Preprint.

[4] W. H. Meeks III and H. Rosenberg, *The Uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology*, Annals of Math., **161** (2005), 723–754.

[5] W. H. Meeks III and H. Rosenberg, *The minimal lamination closure theorem*, Duke Math. Journal, **133(3)** (2006), 467–497.

Constant mean curvature surfaces with Delaunay ends

Martin Kilian
(joint work with Wayne Rossman, Nick Schmitt)

Delaunay surfaces play a prominent role in the theory of non-compact complete constant mean curvature CMC surfaces because they constitute the simplest possible end behavior.

A famous result by Korevaar, Kusner and Solomon [11], building on results of Meeks [13], asserts that a properly embedded annular end of a CMC surface is a Delaunay end. The study of Delaunay ends by the conjugate surface methods of Grosse-Brauckmann, Kusner, and Sullivan [4, 5] require the additional assumption of Alexandrov embeddedness, and are restricted to embedded (unduloidal) Delaunay ends. The gluing techniques of Mazzeo and Pacard [12] are limited to attaching Delaunay ends with small asymptotic necksizes, but allow little control over how the resulting surface actually looks like.

The methods used in our paper [9], based on the generalized Weierstrass representation of Dorfmeister, Pedit and Wu [3], provide a means to study both embedded unduloidal and non-Alexandrov-embedded nodoidal type ends of arbitrary asymptotic necksize, and allow explicit control over the resulting surface.

The generalized Weierstrass representation describes CMC immersions locally via *potentials* - loops of holomorphic 1-forms with values in $\mathfrak{sl}_2(\mathbb{C})$. The relation between the potential and the immersion involves a loop group valued ODE and a loop group factorization, and thus in general provides only an indirect relation between the geometric properties of the induced immersion and its potential. On the other hand the method has been very useful in proving the existence of many new classes of non-compact CMC surfaces with non-trivial topology. Particular progress has been made when the surface is homeomorphic to an n-punctured sphere [8, 10, 15, 16]. In this case, the punctures correspond to poles of the potential. Graphics of these surfaces have long suggested that simple poles with appropriate residues yield the asymptotic end behavior of a Delaunay surface. We prove this correlation between simple poles and Delaunay ends in our preprint [9], thus providing the first instance of how the potential explicitly encodes the geometry of the surface. More specifically, given a potential $A\, z^{-1} dz$ of a Delaunay surface (see [6, 1] for discussions of Delaunay surfaces via loop group methods), consider a holomorphic perturbation $\xi = A\, z^{-1} dz + \mathrm{O}(z^0) dz$. Our main result is the following

Theorem: An annular CMC immersion induced by a holomorphic perturbation of a Delaunay potential is C^∞-asymptotic to the underlying (unperturbed) half-Delaunay surface. In particular, it is properly immersed. Moreover, if the half-Delaunay surface is embedded, then the end of the immersion is properly embedded.

The surface induced by a perturbed Delaunay potential may gain topology or geometric complexity — see for example the n-noids [2, 16, 14, 15] and higher genus examples with ends [7]. Nonetheless at $z = 0$, the perturbed surface is

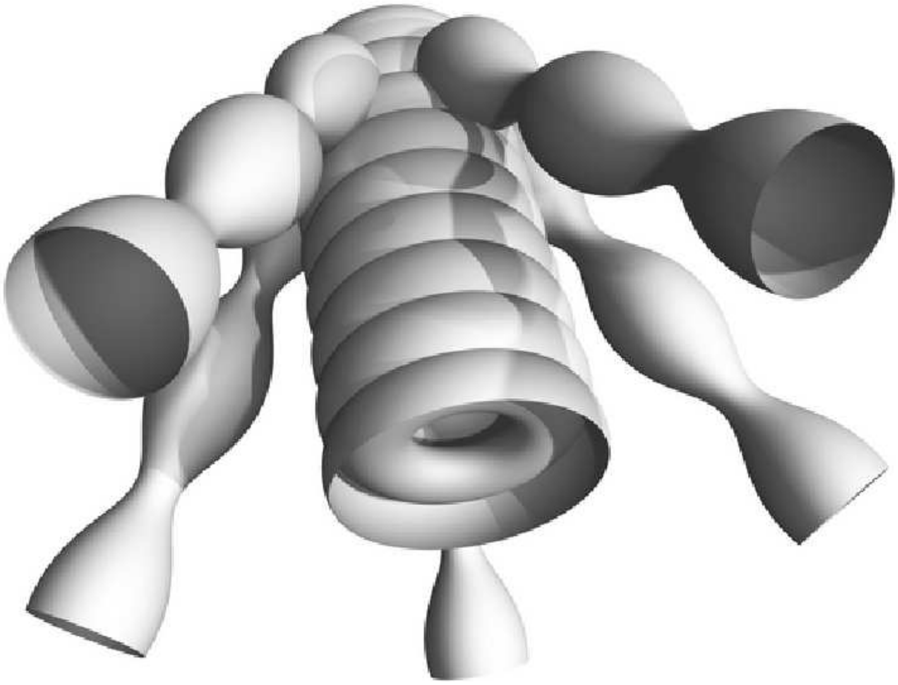


Figure 1. A cmc immersion of the six-punctured sphere with asymptotically Delaunay ends and pyramidal symmetry [15]. Five of the six ends are unduloidal; the sixth is nodoidal with large negative weight.

asymptotic to the underlying Delaunay surface. In particular, since the n-noids described in [16, 14, 15] are obtained from potentials that have at each end the form of a perturbed Delaunay potential, they all have Delaunay ends (see e.g the surface in Figure 1).

The convergence of the surfaces is obtained by showing that their moving frames and metrics converge. More specifically, let Φ_0 and Φ be the respective solutions to the ODE $d\Phi_0 = \Phi_0 A\, z^{-1} dz$ and its perturbation $d\Phi = \Phi\xi$. The convergence of the ratio of Φ to Φ_0 is shown using the holomorphic gauge relating them at the regular singularity $z = 0$. The periodicity of the Delaunay surface provides growth rate estimates on the positive part of Φ_0 by Floquet analysis. This leads to the convergence of their unitary and positive factors, in turn implying C^1-convergence of the surfaces. An elliptic regularity bootstrap argument on the Gauss-Codazzi data strengthens this to C^∞-convergence.

In the second part of our paper [9] we deal with the situation in which the initial condition to the ODE does not extend holomorphically from the r-circle to the unit circle, even though the monodromy of the solution is still unitary. We show that in this setting the solution has acquired singularities that arise from Bianchi-Bäcklund transforms. Thus the second part accommodates the additional singularities that appear from dressing by simple factors, and proves that adding bubbles to a surface with a Delaunay end preserves this Delaunay end.

References

[1] F. E. Burstall and M. Kilian, *Equivariant harmonic cylinders*, Quart. J Math. **57** (2006), 449–468.

[2] J. Dorfmeister and M. Kilian, *Dressing preserving the fundamental group*, Differential Geom. Appl. **23** (2005), 176-204.

[3] J. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), no. 4, 633–668.

[4] K. Große-Brauckmann, R. Kusner, and J. M. Sullivan, *Triunduloids: embedded constant mean curvature surfaces with three ends and genus zero*, J. Reine Angew. Math. **564** (2003), 35–61.

[5] K. Große-Brauckmann, R. Kusner, and J. M. Sullivan, *Coplanar constant mean curvature surfaces*, DG/0509210, 2005.

[6] M. Kilian, *On the associated family of Delaunay surfaces*, Proc. Amer. Math. Soc **132** (2004), no. 10, 3075–3082.

[7] M. Kilian, S.-P. Kobayashi, W. Rossman, and N. Schmitt, *Constant mean curvature surfaces of any positive genus*, J. London Math. Soc. (2) **72** (2005), no. 1, 258–272.

[8] M. Kilian, I. McIntosh, and N. Schmitt, *New constant mean curvature surfaces*, Experiment. Math. **9** (2000), no. 4, 595–611.

[9] M. Kilian, W. Rossman, and N. Schmitt, *Delaunay ends of constant mean curvature surfaces*, Compositio Math., to appear. DG/0701082.

[10] M. Kilian, N. Schmitt, and I. Sterling, *Dressing CMC n-noids*, Math. Z. **246** (2004), no. 3, 501–519.

[11] N. J. Korevaar, R. Kusner, and B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), no. 2, 465–503.

[12] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. **9** (2001), no. 1, 169–237.

[13] W. H. Meeks, III, *The topology and geometry of embedded surfaces of constant mean curvature*, J. Differential Geom. **27** (1988), no. 3, 539–552.

[14] W. Rossman and N. Schmitt, *Simultaneous unitarizability of $SL_n(\mathbb{C})$-valued maps, and constant mean curvature k-noid monodromy*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), (2006), 549–577.

[15] N. Schmitt, *Constant mean curvature n-noids with Platonic symmetries*, DG/0702469.

[16] N. Schmitt, M. Kilian, S.-P. Kobayashi, and W. Rossman, *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms*, to appear in J. London Math. Soc., 2007.

Geometric properties of 3-dimensional homogeneous bundles

Uwe Abresch

In earlier work conducted jointly with H. Rosenberg, we have constructed new holomorphic quadratic differentials for constant mean curvature (cmc) surfaces in a certain class of homogeneous 3-manifolds (see [1, 3] and [2]), thereby in effect establishing this class of 3-manifolds as a natural class of ambient spaces for the global theory of minimal and constant mean curvature surfaces. Here our goal is to describe the geometric properties of this new class of ambient spaces.

When working with *homogeneous* 3*-manifolds* (M^3, g), it is convenient to observe that the isotropy group of any point $p \in M^3$ can be identified with a subgroup of $\mathsf{O}(3) \cong \mathsf{O}(T_pM^3)$, and hence $\dim \mathrm{Iso}(M^3, g) \in \{3, 4, 6\}$. It is natural to handle these *three cases* separately.

A Riemannian 3-manifold with a 6*-dimensional isometry group* is a space of constant curvature κ, and so, in the simply-connected case, it is up to scaling isometric to the sphere $\mathbb{S}^3$, to euclidean space $\mathbb{R}^3$, or to the hyperbolic space $\mathbb{H}^3$. In other words, the spaces of constant curvature cover 3 out of the 8 maximal homogeneous structures in the sense of Thurston.

Recall that the Riemannian curvature tensor R of a 3-manifold is determined by its Ricci tensor; more precisely, $\langle R(X,Y)Y, X\rangle = -\langle X \times Y, G\,(X \times Y)\rangle$ where $G := Ric - \frac{1}{2}\mathbb{1}$ denotes the Einstein tensor. Thus a Riemannian 3-manifold (M^3, g) has constant curvature κ, if and only if $-G = \kappa\mathbb{1}$.

The presence of a 4*-dimensional isometry group* $\mathrm{Iso}(M^3, g)$ is an equally restrictive condition. In this case, the Einstein tensor G must have a 2- and a 1-dimensional eigenspace, and so there is a natural splitting $TM^3 = \mathcal{H}^{(2)} \oplus \mathcal{V}^{(1)}$. Since G is divergence-free, the integral curves of the unit vector field ζ generating the 1-dimensional eigenspace $\mathcal{V}^{(1)}$ are geodesics. Each of these geodesics is actually a component of the fixed point set of a 1-parameter group of rotations. It follows that the unit vector field ζ is in fact a Killing field. Hence, there is a Riemannian submersion $\pi\colon (M^3, g) \to (N^2, \bar{g}) := (M^3, g)/\zeta$ and a homomorphism $\hat{\pi}\colon \mathrm{Iso}(M^3, g) \to (N^2, \bar{g})$ such that $\hat{\pi}(\psi) \circ \pi = \pi \circ \psi$ for all $\psi \in \mathrm{Iso}(M^3, g)$. This means that the quotient $(N^2, \bar{g})$ is a surface of constant curvature κ and that the submersion π itself has constant bundle curvature τ. In fact, the numbers κ and τ determine the (local) geometry of (M^3, g) uniquely.

In order to see this, one observes that $\nabla_X \zeta = -\frac{1}{2}\tau\,\zeta \times X$ where $\times$ denotes the cross product of the oriented 3-manifold (M^3, g). Moreover, the Einstein tensor and the Cotton tensor of (M^3, g) are given by

$$\begin{aligned} -G &= \tfrac{1}{4}\tau^2\,\mathbb{1} + (\kappa - \tau^2)\,\langle\zeta,\,.\,\rangle\zeta\;, \\ \mathit{Cotton} &= -\tfrac{3}{2}\tau\,G_0 = -\tfrac{1}{2}\tau\,\big(3G - \mathrm{tr}(G)\mathbb{1}\big)\;. \end{aligned}$$

Given this information, it is not hard to compute the Jacobi fields along horizontal geodesics. As a result one obtains the following expression for the metric in Fermi coordinates around any vertical geodesic:

$$g_{\kappa,\tau} = \mathrm{d}r^2 + \mathrm{sn}_\kappa(r)^2\,\mathrm{d}\varphi^2 + \big(\mathrm{d}z - 2\tau\,\mathrm{sn}_\kappa(\tfrac{1}{2}r)^2\,\mathrm{d}\varphi\big)^2$$

where sn_κ denotes the generalized sine function, i.e. the solution of the differential equation $\mathrm{sn}_\kappa'' + \kappa\,\mathrm{sn}_\kappa = 0$ with initial data $\mathrm{sn}_\kappa(0) = 0$ and $\mathrm{sn}_\kappa'(0) = 1$. Conversely, given $\kappa, \tau \in \mathbb{R}$, it is not hard to verify that the projection $(r, \varphi, \zeta) \mapsto (r, \varphi)$ is a Riemannian submersion from the given coordinate chart onto a surface of constant curvature κ, and that this submersion has totally-gedesic fibers and constant bundle curvature τ. Moreover, there always exist a complete simply-connected

Riemannian 3-manifold $M^3_{\kappa,\tau}$ modeled on such a chart and a Riemannian submersion $M^3_{\kappa,\tau} \to N^2_\kappa$ onto a simply-connected surface N^2_κ of constant curvature.

These considerations are the key to the following result:

Theorem (cf. [4, 5, 6]). *A simply-connected oriented Riemannian* 3*-manifold* (M^3, g) *is a homogeneous space with a* 4*-dimensional isometry group, if and only if it is the total space of a Riemannian submersion* $M^3_{\kappa,\tau} \to N^2_\kappa$ *with* $\kappa \neq \tau^2$*.*

When classifying the manifolds $M^3_{\kappa,\tau}$ by the isomorphism class of their full isometry group, one needs to distinguish the following cases:

	$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
$\tau = 0$	$\mathbb{H}^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{S}^2 \times \mathbb{R}$
$\tau \neq 0$	$\widetilde{\mathsf{Sl}}(2, \mathbb{R})$	$\mathsf{Nil}(3)$	$\mathbb{S}^3_{\text{Berger}}$

Note that $\mathbb{R}^3$ is just Euclidean space splitted as $\mathbb{R}^2 \times \mathbb{R}$ and that the Berger spheres are not maximal homogeneous unless they are standard spheres, i.e., unless $\kappa = \tau^2$. So the manifolds $M^3_{\kappa,\tau}$ with $\kappa \neq \tau^2$ cover precisely 4 additional maximal homogeneous structures in the sense of Thurston.

If $\kappa = \tau^2$, the manifold $M^3_{\kappa,\tau}$ turns out to be a space of constant curvature κ, and the isometries that are compatible with the submersion $M^3_{\kappa,\tau} \to N^2_\kappa$ define a 4-dimensional subgroup in the full isometry group.

Using the expression for $\nabla\zeta$ given above, it is straightforward to compute that the Einstein tensor G of all the spaces $M^3_{\kappa,\tau}$ solves the differential equation $\nabla_X G = \frac{1}{2}\tau\big[\star X, G\big]$ for all vector fields X. Here $\star X$ denotes the skew-symmetric endomorphism $Y \mapsto X \times Y$. Analyzing the integrablity conditions in detail, one finds that the preceding identity can also be used for recognition:

Theorem. *Let* $\tau_0 \in \mathbb{R}$*. The Einstein tensor* G *of a simply-connected oriented Riemannian* 3*-manifold* (M^3, g) *solves the overdetermined system*

$$\nabla_X G = \tfrac{1}{2}\tau_0\big[\star X, G\big] \ ,$$

if and only if (M^3, g) *is isometric to some space* M^3_κ *of constant curvature or to some space* $M^3_{\kappa,\tau}$ *with* $\tau = \tau_0$*.*

Recall that we have constructed holomorphic quadratic differentials for immersed constant mean curvature surfaces in all the $M^3_{\kappa,\tau}$ [3, 2]. In case $\tau = 0$, the manifolds $M^3_{\kappa,\tau}$ are symmetric spaces, and — like $\mathbb{R}^3$ and $\mathbb{H}^3$ — they are conformal coverings of suitable domains in the standard sphere $\mathbb{S}^3$. So the products $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ indeed fit the current picture of surface theory very well. However, the spaces $M^3_{\kappa,\tau}$ with $\tau \neq 0$ have non-trivial Cotton tensors, and thus they are not even locally conformally flat. One can therefore expect to find some interesting new phenomena when studying cmc surfaces in these target spaces.

Proposition 1. *For all* $\kappa, \tau \in \mathbb{R}$ *with* $\kappa \neq \tau^2$ *the full isometry group of* $M^3_{\kappa,\tau}$ *contains* 180^{o}*-rotations around any horizontal geodesic.*

If $\tau \neq 0$, the isotropy group of any point $p \in M^3_{\kappa,\tau}$ is generated by such rotations and is thus isomorphic to the diagonal subgroup $\mathsf{O}(2) \subset \mathsf{O}(\mathcal{H}^{(2)}_p) \times \mathsf{O}(\mathcal{V}^{(1)}_p)$. In case $\tau = 0$, the isotropy group $\mathrm{Iso}(M^3_{\kappa,\tau})_p$ contains additional hyperplane reflections and is thus isomorphic to the product $\mathsf{O}(\mathcal{H}^{(2)}_p) \times \mathsf{O}(\mathcal{V}^{(1)}_p)$ itself.

The impact for surface theory is that the full isometry group $\mathrm{Iso}(M^3_{\kappa,\tau})$ always contains enough 180°-rotations in order to construct complete minimal surfaces $\Sigma^2 \looparrowright M^3_{\kappa,\tau}$ by first solving a Plateau problem for some polygonal contour and then extending the resulting minimal surface by means of Schwarz reflection.

A simply-connected homogeneous space (M^3, g) with a 3-*dimensional isometry group* is in effect a 3-dimensional Lie group with a left-invariant metric. The isometry groups of all the geometries discussed beforehand, except $\mathbb{S}^2 \times \mathbb{R}$, contain 3-dimensional subgroups that still act transitively, and so they show up again.

However, on the 3-dimensional Lie groups one finds an *additional* 1-*parameter family* of left-invariant metrics g such that the Einstein tensor G has an eigenvalue of multiplicity 2 everywhere. They represent the 8^{th} maximal homogeneous structure known as $\mathsf{Solv}(3)$. The subbundle of $T\mathsf{Solv}(3)$ consisting of the 2-dimensional G-eigenspaces is integrable; in fact, there is a natural Riemannian submersion $\mathsf{Solv}(3) \to \mathbb{R}$. The integral curves of the horizontal unit vector field ζ are geodesics. Intrinsically, the fibers of the submersion $\mathsf{Solv}(3) \to \mathbb{R}$ are Euclidean planes. Extrinsically, they are minimal surfaces, but they are not totally-geodesic. Finally, the Einstein tensor of $\mathsf{Solv}(3)$ is a multiple of $\mathbb{1} - 2\langle \zeta, . \rangle \zeta$.

The *key difference* between the homogeneous bundles $\mathsf{Solv}(3) \to \mathbb{R}$ and the bundles $M^3_{\kappa,\tau} \to N^2_\kappa$ is that the Cotton tensor of $\mathsf{Solv}(3)$ has 3 distinct eigenvalues and is thus not a multiple of the traceless part of G. It seems to be an interesting question how much of the conformal type of the spaces $M^3_{\kappa,\tau}$ is determined when requiring the Cotton tensor to have an eigenvalue of multiplicity 2 everywhere.

References

[1] U. Abresch, H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Acta Math. **193** (2004), 141–174.

[2] U. Abresch, *Holomorphic quadratic differentials*, Oberwolfach Report **47** (2004), 2506–2508.

[3] U. Abresch, H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. **28** (2005), 1–28.

[4] J.W. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), 293–329.

[5] P.A. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

[6] W.P. Thurston, *Three-dimensional geometry and topology*, Princeton Math. Series **35**, Princeton Univ. Press, 1997.

Surfaces in three-dimensional Lie groups and the corresponding Dirac operators

Iskander A. Taimanov
(joint work with Dmitry A. Berdinsky)

We study the Weierstrass representation for surfaces in three-dimensional Lie groups endowed with left-invariant metrics. Therewith the derivational equations for a surface

$$f : M \to G$$

with a conformal parameter z are written in the form

$$\mathcal{D}\psi = 0$$

where

$$\mathcal{D} = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right],$$

$$f^{-1} f_z = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2) e_1 + \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2) e_2 + \psi_1 \bar{\psi}_2 e_3$$

and $\{e_1, e_2, e_3\}$ is the orthonormal base for the Lie algebra of the group G. The induced metric takes the form

$$e^{2\alpha} dz d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)^2 dz d\bar{z}.$$

The particular case of such a representation is the classical Weierstrass representation for minimal surfaces in the commutative group $\mathbb{R}^3$ (in this case $U = V = 0$). In general for surfaces in $\mathbb{R}^3$ we have $U = \bar{U} = V$ and it appears that the spectral properties of the corresponding Dirac operator have geometrical meanings. We started to study this connection

$$\text{the spectrum of } \mathcal{D} \longleftrightarrow \text{ a surface in } \mathbb{R}^3$$

in the middle of 1990s and, in particular, introduced the spectral curve approach to proving the Willmore conjecture (see the survey [4]). Although this approach attracted a substantial attention and was far developed it is not completely realized until recently.

The basic observation which relates the Weierstrass representation and the Willmore functional $\mathcal{W}$ for surfaces in $\mathbb{R}^3$ is the equality

$$\int_M U^2 dx \wedge dy = \frac{1}{4}\mathcal{W} = \frac{1}{4}\int_M H^2 d\mu \tag{1}$$

where $z = x + iy$, H is the mean curvature and $d\mu$ is the induced measure.

We study the Weierstrass representation in detail for surfaces in the three-dimensional lie group Nil and in the group $\widetilde{SL}_2$ and establish that

- $$U = V = \frac{He^\alpha}{2} + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)$$
  for $G = \mathrm{Nil}$,
  $$U = \frac{He^\alpha}{2} + i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right),$$
  $$V = \frac{He^\alpha}{2} + i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right)$$
  for $G = \widetilde{SL}_2$;
- for a minimal surface the spinor ψ meets the equations
  $$\bar\partial\psi_1 = \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_2,$$
  $$\partial\psi_2 = -\frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_1$$
  for $G = \mathrm{Nil}$,
  $$\bar\partial\psi_1 = i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right)\psi_2,$$
  $$\partial\psi_2 = -i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right)\psi_1$$
  for $G = \widetilde{SL}_2$ (the analogs of the classical Weierstrass representation);
- for a constant mean curvature surface the quadratic differential which is equal to
  $$\left(\langle\nabla_{f_z} f_z, N\rangle + \frac{{Z_3}^2}{2H+i}\right)dz^2 \quad \text{for } G = \mathrm{Nil};$$
  $$\left(\langle\nabla_{f_z} f_z, N\rangle + \frac{5}{2(H-i)}Z_3^2\right)dz^2 \quad \text{for } G = \widetilde{SL}_2$$
  is holomorphic. [1] Moreover the converse is true for surfaces in Nil;
- although the integrand of the spinor energy
  $$E(M) = \int UV\,dx \wedge dy$$
  is complex-valued for closed surfaces in Nil and $\widetilde{SL}_2$ the functional is real-valued and equals
  $$\frac{1}{4}\int_M \left(H^2 + \frac{\widehat{K}}{4} - \frac{1}{16}\right)d\mu \quad \text{for } G = \mathrm{Nil},$$
  $$\frac{1}{4}\int_M \left(H^2 + \frac{5}{16}\widehat{K} - \frac{1}{4}\right)d\mu \quad \text{for } G = \widetilde{SL}_2.$$

[1] Here N is the normal vector to a surface, $Z_3 = \langle f^{-1}f_z, e_3\rangle$ and in both cases the basic vector e_3 corresponds to the axis of a rotational symmetry. This gives another proof of the Abresch–Rosenberg theorem [1].

where $\widehat{K}$ is the sectional curvature of the ambient space along the tangent plane.

It appears that the spinor energy introduced in [2] as the generalization of the Willmore functional (1) at least for surfaces in Nil resembles the geometrical properties of the classical Willmore functional better than the usual generalization which for any ambient space is equal to $\int(H^2+\widehat{K})d\mu$ (see [3] and the talk of D.A. Berdinsky at this meeting).

We would like to mention that the general method of deriving such a representation works for surfaces in all three-dimensional Lie groups with left-invariant metrics. In particular, we think that it would be interesting to study the persistence of special classes of surfaces (minimal, constant mean curvature) under deformations of left-invariant metrics and Lie group structures. For example, let us take a family of Lie groups G_μ with the Lie algebras with generators e_1, e_2, and e_3 which meet the commutation relations

$$[e_1,e_2]=e_2, \quad [e_1,e_3]=\mu e_3, \quad [e_2,e_3]=0,$$

and form the orthogonal base for the Lie algebra. These groups are pairwise non-isomorphic for $-1 \le \mu \le 1$ and have the following types in the Bianci classification:
$\mu=-1$: VI_0;
$-1<\mu<0$: VI_a, $0<a<1$, $\mu=\frac{a-1}{a+1}$;
$\mu=0$: III;
$0<\mu<1$: VI_a, $1<a<\infty$, $\mu=\frac{a-1}{a+1}$;
$\mu=1$: V.

In particular, for $\mu=-1,0$, and 1 we have three homogeneous spaces with Thurston's geometries:

$$G_{-1}=\mathrm{Sol}, \quad G_0=H^2\times\mathbb{R}, \quad G_1=H^3,$$

i.e., the solvable group, the product of the Lobachevsky plane and the line, and the three-dimensional Lobachevsky space.

References

[1] U. Abresch, H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. **28** (2005), 1–28.
[2] D.A. Berdinsky, I.A. Taimanov, *Surfaces in three-dimensional Lie groups*, Siberian Math. Journal **46** (2005), 1005-1019.
[3] D.A. Berdinsky, I.A. Taimanov, *Surfaces of revolution in the Heisenberg group and the spectral generalization of the Willmore functional*, Siberian Math. Journal **48** (2007), 395-407.
[4] I.A. Taimanov, *Two-dimensional Dirac operator and surface theory*, Russian Mathematical Surveys **61 (2006), no. 1, 79–159.**

Surfaces of revolution in the Heisenberg group and the spectral generalization of the Willmore functional

Dmitry A. Berdinsky
(joint work with Iskander A. Taimanov)

We study the Weierstrass representation for surfaces in the three-dimensional Lie group Nil endowed with the Thurston geometry. It was introduced in [2] and it appears that the main quantity, the potential U of the representation, has many interesting geometrical properties.

Given a surface $f : M \to \mathrm{Nil}$, the Weierstrass representation describe the derivational equations in terms of $\psi = (\psi_1, \psi_2)^\top$ where

$$f^{-1}f_z = \frac{i}{2}(\bar\psi_2^2 + \psi_1^2)e_1 + \frac{1}{2}(\bar\psi_2^2 - \psi_1^2)e_2 + \psi_1\bar\psi_2 e_3,$$

$z = x + iy$ is a conformal parameter on M, and $\{e_1, e_2, e_3\}$ is the orthonormal base for the Lie group of Nil. Therewith the derivational equations reduce to the Dirac equation

$$\mathcal{D}\psi = 0$$

where

$$\mathcal{D} = \left[\begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right],$$

$$U = \frac{He^\alpha}{2} + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)$$

and

$$e^{2\alpha}dzd\bar z = (|\psi_1|^2 + |\psi_2|^2)^2 dzd\bar z.$$

is the induced metric.

1. By [2], we have the following formula for the *spinor energy*

$$E(M) = \int_M U^2 dx \wedge dy = \frac{1}{4}\int_M \left(H^2 + \frac{\widehat{K}}{4} - \frac{1}{16}\right) d\mu$$

where $d\mu$ is the induced measure, H is the mean curvature and $\widehat{K}$ is the sectional curvature of Nil along the tangent plane to the surface.

In [3] it is showed that

- for all constant mean curvature (CMC) spheres in Nil the spinor energy is equal to π;
- for a closed surface obtained by revolving a curve γ in the half-lane $\{\rho \geq 0, z \in \mathbb{R}\} = \mathrm{Nil}/SO(2)$ around the z-axis we have

(1) $$E(M) = \frac{\pi}{8}\int_\gamma \left(\dot\sigma - \frac{\sin\sigma}{\rho}\right)^2 \sqrt{4\rho^2 + \rho^4}\, ds + \frac{\pi\chi(M)}{2}$$

  where σ is the angle between γ and $\frac{\partial}{\partial\rho}$ (in the metric $d\rho^2 + \frac{4\rho^2}{4\rho^2+\rho^4}dz^2$ for which the projection $\mathrm{Nil} \to \mathrm{Nil}/SO(2)$ is a submersion) and $\chi(M)$ is the Euler characteristic of M. Moreover if $\dot\sigma = \frac{\sin\sigma}{u}$ everywhere on the surface then it is a CMC sphere.

The former implies

- for spheres of revolution $E(M) \geq \pi$ and the equality is attained exactly at CMC spheres;
- for tori of revolution $E(M) > 0$.

For surfaces in $\mathbb{R}^3$ the CMC spheres are the round spheres and they are exactly the isoperimetric profiles for all volumes. The Willmore functional attains on these spheres minimal possible value which is 4π, and these are exactly umbilic spheres.

By [1], all CMC spheres in Nil are CMC spheres of revolution and conjecturally they are again isoperimetric profiles for all volumes (and this is true for small volumes). The spinor energy is constant on the CMC spheres, attains on them its minimal possible value for surfaces of revolution (and we think for all surfaces) and these spheres are not umbilic but another similar quantity which is $\left(\dot{\sigma} - \frac{\sin\sigma}{\rho}\right)$ vanishes exactly on them.

Therefore the spinor energy sounds to be the right generalization of the Willmore functional for surfaces in Nil since it respects solutions to the isoperimetric problem (at least for small volumes) as the Willmore functional for surfaces in $\mathbb{R}^3$.

We think that in general E is bounded from below for closed surfaces however until recently we can not prove that.

2. Let us put

$$U = e^v.$$

Then we have

$$\partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} v_z - \frac{1}{2}H_z e^{-v}e^{\alpha} & Be^{-v} \\ -e^v & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & e^v \\ -\overline{B}e^{-v} & v_{\bar{z}} - \frac{1}{2}H_{\bar{z}} e^{-v}e^{\alpha} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where Bdz^2 is a certain quadratic differential which is holomorphic exactly for CMC surfaces ($B = \frac{1}{4}(2H+i)\widetilde{A}$ in the notation of [2]). For $H = \mathrm{const}$ we have $B_{\bar{z}} = 0$ and the compatibility condition for these equations implies

$$v_{z\bar{z}} + e^{2v} - |B|^2 e^{-2v} = 0.$$

For instance, if there exist CMC tori in Nil then one may reduce this compatibility equation by rescaling the parameter z to the elliptic sinh-Gordon equation:

$$\Delta u + \sinh u = 0, \qquad u = 2v,$$

which describes the CMC tori in $\mathbb{R}^3$. However in the latter case it is written for the logarithm of the conformal factor of the metric.

It would be interesting to describe CMC tori in Nil by using methods of integrable systems as it was done before for CMC tori in $\mathbb{R}^3$ (Pinkall–Sterling, Bobenko).

References

[1] U. Abresch, H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. **28** (2005), 1–28.
[2] D.A. Berdinsky, I.A. Taimanov, *Surfaces in three-dimensional Lie groups*, Siberian Math. Journal **46** (2005), 1005-1019.
[3] D.A. Berdinsky, I.A. Taimanov, *Surfaces of revolution in the Heisenberg group and the spectral generalization of the Willmore functional*, Siberian Math. Journal **48** (2007), 395-407.

Constrained Willmore Surfaces

Christoph Bohle
(joint work with Paul Peters, Ulrich Pinkall)

In this report we present three results obtained in the framework of the DFG SPP 1154 project "Constrained Willmore Surfaces", see [1, 2, 4] for details.

1.) Definition of constrained Willmore surfaces. The notion of constrained Willmore surfaces generalizes that of Willmore surfaces, the critical points of the conformally invariant Willmore functional $\mathcal{W}(f) = \int |\mathrm{II}^\circ|^2 dA + 2\chi(M)$ (where II° denotes the trace–free second fundamental form of the immersion $f\colon M \to \bar{M}$ and $\chi(M)$ the Euler–characteristic of M). Intuitively, constrained Willmore surfaces are the "critical points" of $\mathcal{W}$ on the space of conformal immersions of a Riemann surface M into a Riemannian manifold $\bar{M}$. One way to make this idea precise is:

Definition. A conformal immersion $f\colon M \to \bar{M}$ is *constrained Willmore* if it is critical point of $\mathcal{W}$ under compactly supported infinitesimal conformal variations.

There are two alternative notions of constrained Willmore immersions (which in general are probably not equivalent to our definition—in contrast to the variational problem without the conformal constraint): a "weak" one of critical points with respect to genuine conformal variations with compact support and a "strong" one of immersions satisfying an Euler–Lagrange equation with a holomorphic quadratic differential as Lagrange multiplier. In [1] we prove that, if M is compact, all constrained Willmore immersions satisfy the stronger condition. Moreover, we prove that all three notions coincide for immersions of compact surfaces that are not strongly isothermic, i.e., that are no critical point of the projection to Teichmüller space. The "weak" notion is interesting because minimizers of $\mathcal{W}$ in a conformal class are a priori only "weakly" constrained Willmore. The "strong" notion is interesting because of its integrable systems interpretation.

2.) CMC surfaces and Bryant spheres with smooth ends. Constant mean curvature (CMC) immersions into 3–dimensional space forms are examples of constrained Willmore immersions (in the "strong" sense). At the same time they are strongly isothermic. A CMC–H immersion is Willmore iff $H = 0$.

By the maximum principle, compact CMC–H surfaces can only exist in space forms of curvature $\kappa > -H^2$. However, one can still obtain Willmore immersions of compact surfaces into the conformal 3–sphere by completing Euclidean minimal surfaces with planar ends in $\mathbb{R}^3 = S^3 \backslash \{\infty\}$ through the point ∞ at infinity. In fact, all Willmore sphere in the conformal S^3 are obtained this way [6]. Their Willmore energies are quantized and take all values $\mathcal{W} \in 4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$, see [7]. It turns out [5] that Willmore spheres are special examples of soliton spheres and that all soliton spheres in S^3 have Willmore functional $\mathcal{W} \in 4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$.

In the context of soliton sphere it is interesting that there is the following direct analogue to the "conformal compactification" of Euclidean minimal surfaces with planar ends: it is possible to obtain immersions of compact surfaces by taking the extension through the ideal boundary of suitable Bryant surfaces, i.e., CMC–H surfaces in hyperbolic space of curvature $\kappa = -H^2$. For example, among the real 1–parameter family of catenoid cousins, the immersions corresponding to positive integer parameters extend smoothly through the ideal boundary and are soliton spheres (à la Taimanov [15]) related to 1–solitons of the mKdV–equation. Using [9, 16], smooth ends of Bryant surfaces can be characterized [2] as those catenoidal or horospherical Bryant ends for which the Bryant representation is single valued.

In case of immersions of the sphere there is a far reaching analogy between minimal surfaces with planar ends and Bryant surfaces with smooth ends [2]: both types of immersions can be obtained as projection of rational immersed null curves in the non–degenerate quadric $Q^3 \subset \mathbb{CP}^4$. Moreover, like for Willmore spheres (i.e., minimal spheres with planar ends) we have:

Theorem. *Bryant spheres with smooth ends are soliton spheres. Their possible Willmore energies are* $\mathcal{W} \in 4\pi(\mathbb{N}^* \setminus \{2,3,5,7\})$.

Similar to the example at the end of Section 3 in [2], the rational null immersion

$$F = \begin{pmatrix} \frac{2(1+n)z^n}{2\sqrt{1+2n}-nz^{1+n}} & \frac{4\sqrt{1+2n}+4nz^{-1-n}-nz^{1+n}}{4\sqrt{1+2n}-2nz^{1+n}} \\ \frac{2nz^{1+n}}{2\sqrt{1+2n}-nz^{1+n}} & \frac{2(1+n)z^{-n}}{2\sqrt{1+2n}-nz^{1+n}} \end{pmatrix}$$

into $SL(2,\mathbb{C})$ is obtained by applying a suitable projective transformation to the Bryant representation of a catenoid cousin. The corresponding Bryant sphere has one smooth catenoid cousin end and (n+1) smooth horospherical ends and Willmore energy $\mathcal{W} = 4\pi(2n+2)$ (Figure 1 shows the case $n = 6$).

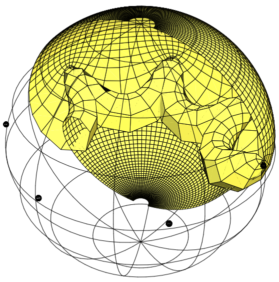


FIGURE 1: Bryant sphere with 8 ends: a soliton sphere obtained by smoothly extending a Bryant surface through the ideal boundary.

Bryant spheres with smooth ends are constrained Willmore as punctured surfaces, but not as compact surfaces: because there is only one conformal structure on the 2–sphere, all constrained Willmore immersions from S^2 into the conformal 3–sphere S^3 are Willmore (and hence Euclidean minimal with planar ends for some point at infinity).

3.) Constrained Willmore tori in S^4. A harmonic map $f\colon T^2 \to S^2$ is either conformal (i.e. (anti)–holomorphic) and has $\deg(f) \neq 0$ or it is of finite type and

$\deg(f) = 0$. An elegant way to prove this result of Eells/Wood and Pinkall/Sterling is Hitchin's integrable systems approach [11]. In [4] we use a similar approach combined with methods of quaternionic holomorphic geometry [10, 3] to prove:

Theorem *Let $f\colon T^2 \to S^4$ be a constrained Willmore immersion that is not Euclidean minimal. Then, for $\perp_f$ the normal bundle seen as a complex line bundle,*

- $\deg(\perp_f) \neq 0 \Rightarrow f$ *is super–conformal*
- $\deg(\perp_f) = 0 \Rightarrow f$ *is of finite type (i.e. has finite spectral genus).*

In brief, the idea to prove the theorem (and in particular several previous partial results [13, 14, 12]) is to analyze the holonomy representation $H^\mu\colon \Gamma \to SL(4,\mathbb{C})$ of the associated family ∇^μ of flat connections of a constrained Willmore immersions $f\colon T^2 = \mathbb{C}/\Gamma \to S^4$ and to check that

- trivial holonomy H^μ implies "holomorphic type" (i.e., the immersion is super–conformal or Euclidean minimal) and
- non–trivial holonomy H^μ implies finite type (i.e., the eigenline or "spectral curve" is of finite genus).

An interesting special case is the case that the holonomy representation reduces to a non–trivial $SL(2,\mathbb{C})$–representation. This happens for example for conformally immersed tori with the property that there is a point at infinity $\infty \in S^4$ such that one of the Euclidean normal vectors of the immersion into $\mathbb{R}^4 = S^4\backslash\{\infty\}$ is harmonic but not conformal.

All conformal immersions for which one of the Euclidean normals is harmonic are constrained Willmore (generalizing this, Burstall proved [8] that all immersions into 4–dimensional space forms whose mean curvature vector is holomorphic are constrained Willmore). Examples include CMC–tori in $\mathbb{R}^3$ and S^3, Hamiltonian stationary Lagrangian tori in $\mathbb{R}^4$ and Lagrangian tori with conformal Maslov form in $\mathbb{R}^4$. A conformally immersed torus with a non–conformal, harmonic Euclidean normal belongs to the finite type case and the constrained Willmore spectral curve coincides with the harmonic map spectral curve (and is hence hyper elliptic).

References

[1] C. Bohle, G. P. Peters, and Ulrich Pinkall, Constrained Willmore surfaces. `math.DG/0411479`.
[2] C. Bohle and G. P. Peters, Bryant surfaces with smooth ends. `math.DG/0411480`.
[3] C. Bohle, K. Leschke, F. Pedit, and U. Pinkall, Conformal maps from a 2–torus to the 4–sphere. Preprint.
[4] C. Bohle, Constrained Willmore tori in the 4–sphere. Preprint.
[5] C. Bohle and G. P. Peters, Soliton spheres. In preparation.
[6] R. L. Bryant, A duality theorem for Willmore surfaces. *J. Diff. Geom.* **20** (1984), 23–53.
[7] R. L. Bryant, Surfaces in conformal geometry, *Proc. Sympos. Pure Math.* **48** (1988) 227–240.
[8] F. Burstall, private communication (see also Oberwolfach Report for this meeting).
[9] P. Collin, L. Hauswirth, and H. Rosenberg, The geometry of finite topology Bryant surfaces. *Ann. of Math.* **153** (2001), 623–659.
[10] D. Ferus, K. Leschke, F. Pedit and U. Pinkall, Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2–tori. *Invent. Math.* **146** (2001), 507–59.
[11] N. Hitchin, Harmonic maps from a 2–torus to the 3–sphere. *J. Diff. Geom.* **31** (1990), 627–710.

[12] K. Leschke, F. Pedit and U. Pinkall, Willmore tori in the 4–sphere with nontrivial normal bundle. *Math. Ann.* **332** (2005), 381–394.
[13] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori. *Ann. of Math.* **130** (1989), 407–451.
[14] M. U. Schmidt, A proof of the Willmore conjecture. `math.DG/0203224`, 2002.
[15] I. A. Taimanov, The Weierstrass representation of spheres in R^3, the Willmore numbers, and soliton spheres. *Proc. Steklov Inst. Math.* **225** (1999), 322–343.
[16] M. Umehara and K. Yamada, Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space. *Ann. of Math.* **137** (1993), 611–638.

Conformally immersed tori in 4–space of spectral genus zero

Franz Pedit
(joint work with Christoph Bohle and Ulrich Pinkall)

Recently there has been significant progress in describing conformal immersions $f\colon T^2 \to S^4$ of 2–tori in 4–space with topologically trivial normal bundle [1, 2, 6, 11, 10]. Each such conformal immersion comes with an auxiliary Riemann surface, its *spectral curve* Σ_f. Generally, the spectral curve has infinite genus, but there are interesting classes of conformal immersions f, including constant mean curvature tori [9, 7, 5] and Willmore tori [10, 3], when Σ_f can be compactified to a finite genus curve. In this situation algebro–geometric methods yield explicit theta function formulas for the conformal immersion f.

This prompts the question whether these methods can be applied to the Willmore problem, that is, whether the infimum of the Willmore energy, the average $W = \int H^2$ of the mean curvature square, over a torus T^2 immersed in 3–space is $2\pi^2$? A tempting strategy could be to show that Willmore tori of large spectral genera have large Willmore energy (which is known for minimal tori in the 3–sphere [4]) so that one is left to check conformal immersions of low spectral genera.

The simplest conformally immersed tori $f\colon T^2 \to S^4$ are those where the spectral curve $\Sigma_f = \mathbb{P}^1$ has genus zero. Examples of such tori include the isothermic Darboux transformations of the Clifford torus, Dirac tori ($H^2|df|^2$ is a flat metric), Hamiltonian stationary tori in $\mathbb{R}^4$, rotational tori whose profile is a Darboux transformed circle in the upper half plane and Hopf lifts of Darboux transformed circles on the 2–sphere. All those surfaces are parameterized by rational expressions in trigonometric functions. A reasonable guess is that conformal tori of spectral genus zero in the 4–sphere are precisely those which can be conformally parameterized by rational expressions in trigonometric functions.

Our main result is a sharp estimate for the Willmore energy for conformally immersed tori of spectral genus zero in S^4 which, among other things, implies the Willmore conjecture for these tori.

Theorem 1. *Let $f\colon T^2 \to S^4$ be a conformal immersion of spectral genus zero. Then there is an n–fold conformal cover $T^2 \to \tilde{T}^2$ to a rectangular torus of width $\widetilde{w}$ and hight $\tilde{h}$ and the Willmore energy*

$$W(f) = n\pi^2(\frac{\tilde{h}}{\tilde{w}} + \frac{\tilde{w}}{\tilde{h}})\,.$$

Since there is only a countable collection of real curves in the moduli space of tori which are finite covers of rectangular tori, we see that only countable families of moduli can be realized by conformal immersions of spectral genus zero.

The theorem has an immediate corollary giving an estimate of the Willmore energy in terms of the conformal type of the torus T^2:

Corollary 1. *Let $f\colon T^2 \to S^4$ be a conformal immersion of spectral genus zero. Then the Willmore energy*

$$W(f) \geq \pi^2(\operatorname{Im}\tau + \frac{1}{\operatorname{Im}\tau}),$$

where the lattice Γ of the torus T^2 is spanned by 1 and τ.

It is worthwhile to compare our estimates for spectral genus zero tori to those by Li and Yau [8]: we get better estimates near the hexagonal lattice and obviously estimates for rectangular tori. Note also that one can always find a rectangular conformally immersed torus with Willmore energy arbitrarily close to 8π: take to concentric spheres and connect them by two catenoids at the north and south poles. But such a torus, whose fundamental domain has large τ, can never have spectral genus zero. It is possible though to make such a torus with spectral genus one, as the 2–lobed Delaunay torus in S^3 shows.

References

[1] C. Bohle, K. Leschke, F. Pedit, and U. Pinkall, Conformal maps from a 2–torus to the 4–sphere. Preprint.

[2] C. Bohle, F. Pedit, and U. Pinkall, Spectral curves of quaternionic holomorphic line bundles over a 2–torus. Preprint.

[3] C. Bohle, Constrained Willmore tori in the 4–sphere. Preprint.

[4] D. Ferus, K. Leschke, F. Pedit and U. Pinkall, Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2–tori. *Invent. Math.* **146** (2001), 507–59.

[5] D. Ferus, F. Pedit, U. Pinkall and I. Sterling, Minimal Tori in S^4, *J. reine angew. Math.* **429** (1992), 1–47.

[6] P. G. Grinevich and M. U. Schmidt, Conformal invariant functionals of immersions of tori into R^3. *J. Geom. Phys.* **26** (1998), 51–78.

[7] N. Hitchin, Harmonic maps from a 2-torus to the 3-sphere, *J. Differential Geom.* **31** (1990), 627–710.

[8] P. Li and S. T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Invent. Math.* **69** (1982), 269–291.

[9] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori. *Ann. of Math.* **130** (1989), 407–451.

[10] M. U. Schmidt, A proof of the Willmore conjecture. `arxiv.org/math.DG/0203224`, 2002.

[11] I. A. Taimanov, The Weierstrass representation of closed surfaces in R^3. *Funct. Anal. Appl.* **32** (1998), 49–62.

Trigonal minimal surfaces in flat tori

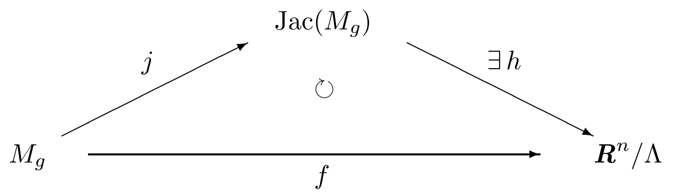
Toshihiro Shoda

Abstract We report the following topics of trigonal minimal surfaces in flat tori which are announced at the Oberwolfach conference entitled "Progress in Surface Theory": (i) a topological obstruction similar to that of hyperelliptic minimal surfaces, (ii) an explicit example in the higher codimensional case. This surface satisfies good properties distinct from traditional ones.

Our object is a minimal immersion f from a compact surface M_g of genus g into an n-dimensional flat torus $\mathbf{R}^n/\Lambda$ which does not lie in any subtorus of $\mathbf{R}^n/\Lambda$ (it can be replaced by an n-periodic minimal surface in $\mathbf{R}^n$). The conformal structure induced by the isothermal coordinates makes M_g a Riemann surface and f is called a conformal minimal immersion.

One of the beautiful classical theorems on compact Riemann surfaces states that every compact Riemann surface of positive genus is holomorphically embedded in the Jacobian by the Abel-Jacobi map. The Jacobian satisfies the following universal property:

Proposition 1.
Given $f : M_g \longrightarrow \boldsymbol{R}^n/\Lambda$ *a conformal minimal immersion, we may assume* $f(p_0) = 0$*. Then there exists a real homomorphism* h *from* $\mathrm{Jac}(M_g)$ *to* $\boldsymbol{R}^n/\Lambda$ *so that* $f = h \circ j$*, where* j *is the Abel-Jacobi map*:



The Abel-Jacobi map plays an important role in the theory of algebraic curves (e.g. Torelli's Theorem, Schottoky problem, etc.) and so, by the above Proposition, it is useful to study minimal surfaces from the point of view of the theory of algebraic curves.

Algebraic curves can be divided into non-hyperelliptic curves and hyperelliptic curves, and in particular, there is a topological obstruction to hyperelliptic minimal surfaces in 3-dimensional flat tori. Actually, a hyperelliptic minimal surface of even genus cannot be minimally immersed into any 3-dimensional flat torus.

Note that every compact Riemann surface can be represented as a branched d-cover of the sphere for some $d \geq 1$, and it is reasonable to ask whether there is a topological obstruction or not for $d > 2$ (a Riemann surface with $d = 2$ is a hyperelliptic curve, and hence we omit the case $d = 2$). Now we consider this problem for $d = 3$. Recall that a non-hyperelliptic curve with $d = 3$ is called a trigonal curve.

The first result is the following topological obstruction:

Theorem ([7]).
Let $f : M_g \longrightarrow \boldsymbol{R}^3/\Lambda$ be a conformal minimal immersion from a trigonal Riemann surface M_g of genus g into a 3-dimensional flat torus. Then, $g = 3r+1$ holds for some $r \geq 1$. Therefore, a trigonal Riemann surface of genus 0 or 2 (mod 3) cannot be minimally immersed into any 3-dimensional flat torus.

Remark.
There are some examples of trigonal minimal surfaces in 3-dimensional flat tori: (i) *A previous example* [1], [6] ($r = 1$), (ii) *Schoen's I-WP surface* [2], [5] ($r = 3$), *where r is given in Main Theorem 1.*

Next, we consider the higher codimensional case. In 1976, Nagano-Smyth [3] constructed compact minimal surfaces in n-dimensional flat tori abstractly. On the other hand, there are few explicit examples, and thus we will construct an example of trigonal minimal surfaces in the simplest case, that is, in case $n = 4$. Actually, an example of trigonal minimal surfaces of genus 10 in 4-dimensional flat tori is given.

Now, we consider our example in view of Nagano-Smyth's discussion [4]. In fact, this example satisfies the following properties: (a) the conjugate surface $f_{\pi/2} : M_{10} \longrightarrow \mathbf{R}^4/\Lambda_{\pi/2}$ is well-defined for some torus $\mathbf{R}^4/\Lambda_{\pi/2}$, (b) the associate surfaces $f_\theta : M_{10} \longrightarrow \mathbf{R}^4/\Lambda_\theta$ are also well-defined for a countable dense set of angles $e^{i\theta} \subset S^1$ and some torus $\mathbf{R}^4/\Lambda_\theta$, (c) this example is homologous to 0 in the 4-torus.

Here, we review the Nagano-Smyth's argument. Let $S_f(M_g)$ be a subgroup of the automorphism group of M_g, and say f has symmetry $S_f(M_g)$ if and only if $S_f(M_g)$ extends under f to a group of affine transformations of $\mathbf{R}^n/\Lambda$. When the corresponding linear representation of $S_f(M_g)$ is irreducible, we say f has irreducible symmetry $S_f(M_g)$. If the complexification of this representation is also irreducible, then we say f has absolutely irreducible symmetry $S_f(M_g)$. Nagano-Smyth showed that (i) if f has absolutely irreducible symmetry, then f satisfies (c) (Theorem 2 in [4]), (ii) if we assume the above irreducible conditions by Weyl group and some conditions, then f satisfies (a) and (b) (Theorem 5 in [4]). However, our example satisfies (a), (b), (c), and has only reducible symmetry. Therefore, the converse propositions of Nagano-Smyth's results do not hold respectively.

Finally, we state the above results as follows:

Theorem ([7]).
There exists a trigonal minimal surface of genus 10 in a 4-dimensional flat torus satisfying the following properties: (i) *the conjugate surface $f_{\pi/2} : M_{10} \longrightarrow \boldsymbol{R}^4/\Lambda_{\pi/2}$ is well-defined,* (ii) *the associate surfaces $f_\theta : M_{10} \longrightarrow \boldsymbol{R}^4/\Lambda_\theta$ are well-defined for a countable dense set of angles $e^{i\theta} \subset S^1$ and some torus $\boldsymbol{R}^4/\Lambda_\theta$,* (iii) *this surface is homologous to 0 in the torus and has only reducible symmetry.*

References

[1] S. Fujimori and T. Shoda, *Triply periodic minimal surfaces of genus 4*, EG-Models (2005).

[2] H. Karcher, *The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions*, Manuscripta Math 64 (1989), 291–357.
[3] T. Nagano and B. Smyth, *Minimal surfaces in tori by Weyl groups I*, Proc. Amer. Math. Soc. 61 (1976), 102–104.
[4] T. Nagano and B. Smyth, *Periodic minimal surfaces and Weyl groups*, Acta. Math 145 (1980), 1–27.
[5] A. Schoen, *Infinite periodic minimal surfaces without self-intersections*, Technical Note D-5541, N. A. S. A, Cambridge, Mass, May (1970).
[6] T. Shoda, *New components of the Moduli space of minimal surfaces in 4-dimensional flat tori*, J. London. Math. Soc (2) 70 (2004), no. 3, 797–816.
[7] T. Shoda, *Trigonal minimal surfaces in flat tori*, to appear in Pacific. J. Math, math.DG/0703622.

Coarse Classification of CMC-trinoids of genus $g = 0$ with embedded ends

Josef Dorfmeister

1. Introduction

Similar to the theory of minimal surfaces there are two main themes for the construction of surfaces in $\mathbb{R}^3$ of constant mean curvature $H \neq 0$, called CMC-surfaces here, namely methods primarily based on analysis at one hand, and loop group based methods on the other hand. The later ones include algebro-geometric techniques, primarily applicable to "finite type immersions", and the generally applicable generalized Weierstraß representation.

Many examples of CMC immersions have been constructed by largely analytic methods by Kapouleas, Mazzeo, Pacard and others. A special class of CMC-trinoids of genus $g = 0$, namely the Alexandrov embedded ones, have been classified by Große-Brauckmann, Kusner and Sullivan [5].

Loop group methods have been used very successfully in the range of finite type CMC-immersions. The generalized Weierstraß representation has been used to construct cylinders and tori [3], [1] and certain trinoids of genus $g = 0$ [4],[6].

The present contribution presents a coarse classification of all CMC-trinoids of genus $g = 0$ with embedded ends in the sense that it describes how all such surfaces can be constructed in a specific way by the generalized Weierstraß representation.

The class of immersion under consideration includes the Alexandrov embedded CMC-immesion cassified in [5]. It also includes non-Alexandrov embedded CMC-immersions of genus $g = 0$ with embedded ends (see [7]). It would be very interesting to have a fine classification of these CMC-immersions, namely one describing the type of immersion in detail via the generalized Weierstraß data.

2. Embedded ends and conformal coordinates

Concerning embedded CMC-ends, Korevaar, Kusner and Salomon have proven a beautiful result [8]:

Theorem 2.1. *1) Every embedded CMC-end φ can be parametrized in the form*
(1)
$$\varphi : \mathbb{H}_r = \{x + iy \in \mathbb{C}; x > -\ln(r) \to \mathbb{R}^3, \varphi(x, y) = (x, r(x, y)cos(y), r(x, y)sin(y))$$

2) Moreover, there exists some embedded Delaunay surface (unduloid)

(2) $$\varphi_0 : \mathbb{H}_r \to \mathbb{R}^3, \varphi(x, y) = (x, r_0(x)cos(y), r_0(x)sin(y))$$

such that

(3) $$|| \, \partial_x{}^n \partial_y{}^m \varphi(x, y) - \partial_x{}^n \partial_y{}^m \varphi_0(x, y) \, ||< C(n, m)e^{-A(n,m)x}$$

uniformly in y for certain positive constants $C(n, m)$ and $A(n, m)$.

Remark 2.2. *The type of convergence just described we will refer to as "strong exponential convergence"*

Recall that the generalized Weierstraß representation requires conformal coordinates. Thus we ask the question

Can one obtain the KKS-inequalities also in conformal coordinates ?

Thus the procedure is to convert to conformal coordinates and to make sure that the KKS-inequalities stay preserved.

Step1: Choose conformal coordinates for φ_0 and change φ accordingly.

It is easy to see hat this can be done such that the KKS-inequalities are preserved.

Step2 : Change of coordinates such that the end is now at $z = 0$: This is achieved by the change of coordinates $w = -\ln(z)$ which yields the immersions

(4) $$\tilde{\varphi}, \tilde{\varphi_0} : \mathbb{D}_r^* \to \mathbb{R}^3.$$

We will call these immersions the "finite end picture" of the embedded end and the original immersions as the "infinite end picture" of the embedded ends under consideration.

It is straightforward to prove that the KKS-inequalities translate into the finite end picture in the form:

Proposition 1.

(5) $$|| \, \partial_z{}^n \partial_{\overline{z}}{}^m \varphi(z, \overline{z}) - \partial_z{}^n \partial_{\overline{z}}{}^m \varphi_0(z, \overline{z}) \, ||< D(n, m)| \, z \, |^{E(n,m)}$$

for certain positive constants $D(n, m)$ and $E(n, m)$.

Remark 2.3. *1. Note, we can always assume that $E(n, m) > 1$ holds.*

2. We will also call this behaviour as "strong exponential convergence".

We want to find a change of coordinates to conformal coordinates in the finite end picture.

Thus we need to solve the Beltrami equation in the finite end picture.

$$\omega_{\overline{z}} = \mu(z, \overline{z})\omega_{\overline{z}} \tag{6}$$

where

$$\mu = \frac{\tilde{G} - \tilde{E} - 2i\tilde{F}}{\tilde{G} + \tilde{E} + 2\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}} \tag{7}$$

and where $\tilde{E}, \tilde{G}$ and $\tilde{F}$ denote the entries of the first fundamental form of the given end in the fnite end picture.

Remark 2.4. *1. Note that $\mu_0 = 0$ and $\lim_{z\to 0} = 0$ hold.*

2. It is straightforward to prove that μ is Hölder continuous on the full disk $\mathbb{D}_r$.

The main result in this context is

Theorem 2.5. *The Beltrami equation $\omega_{\overline{z}} = \mu(z, \overline{z})\omega_{\overline{z}}$ has a solution such that*

- $\omega(0, 0) = 0$
- $\partial_z \omega(0, 0) \neq 0$
- $\omega_{n,m}(z, \overline{z}) = {\partial_z}^n {\partial_{\overline{z}}}^m \omega(z, \overline{z})$ *is Hölder continuous on $\mathbb{D}_s$ for some sufficiently small $s > 0$*
- $\omega_{n,m}(0, 0) = 0$.

From this one can derive the following

Theorem 2.6. *Set $\tilde{\psi} \circ \omega^{-1}$ and $\tilde{\psi}_0 = \tilde{\varphi}_0$, then $\tilde{\psi} - \tilde{\psi}_0$ converges to 0 strongly exponentially.*

3. The associated family of one end

As a consequence of the results described above one obtains hat the appropriate differences of the conformal factors, the coefficients of the Hopf differentials, the frames and their Maurer-Cartan forms converge to 0 strongly exponentially in the finite end picture and also in the infinite end picture.

Moreover, for the extended frames F and F_0 of the given end and the asymptotic Delaunay surface of the associated families one obtains:

Theorem 3.1. *1. $F(z, \overline{z}, \lambda) = A(\lambda)S(z, \overline{z}, \lambda)F_0(z, \overline{z}, \lambda)$, where S-I and $F_0^{-1}SF_0$ coverge to 0 strongly exponentially .*

2. The monodromy matrix of F is the conjugate by A of the monodromy matrix of F_0.

3. The coefficients of the Hopf differentials differ by a holomorphic function which vanishes at the end.

4. Coarse classification of CMC-trinoids of genus $g = 0$ with embedded ends

The conformal description of embedded ends outlined above yields with some effort a decription of the potentials (the basic parameters of the generalized Weierstrass representation) of embedded ends and finally permits one to prove that every trinoid of genus $g = 0$ with embedded ends can be obtained form the potentials considered in [4].

References

[1] J. Dorfmeister, G. Haak, *On constant mean curvature surfaces with periodic metric*, Pacific J. Math. , **182** (1998), 229–287.

[2] J. Dorfmeister, F. Pedit and H. Wu, *Weierstraß type representations of harmonic maps into symmetric spaces*, Comm. Analysis and Geom. **6** (1998), 633–668.

[3] J. Dorfmeister, S. Kobayashi *Coarse classification of CMC-cylinders* , Trans.Amer.Math.Soc. 359 (2007), 2483 - 2500

[4] J. Dorfmeister, H. Wu, *Construction of Constant Mean Curvature N-Noids from Holomorphic Potentials*, to appear in Math.Z.

[5] K. Grosse-Brauckmann, R. Kusner, J. Sullivan, *Triunduloids: embedded constant mean curvature surfaces with three ends and genus zero*, J. Reine Angew. Math. **564** (2003), 35-61.

[6] M.Kilian , S. Kobayashi, W. Rossman, N. Schmitt. *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms*, J. London Math. Soc., to appear. math.DG/0403366

[7] M. Kilian, N. Schmitt, I. Sterling *Dressing CMC n-noids*, Math.Z. 246, No. 3, (2004) 501-519

[8] N.J. Korevaar, R. Kusner, B. Solomon, *The Structure of Complete Embedded Surfaces With Constant Mean Curvature*, J. Diff. Geom. **30** (1989), 465-503

U/K-systems and submanifold geometry

Chuu-Lian Terng

In this talk, I will explain some methods of constructing solutions of the U/K-systems and how these systems arise in submanifold geometry.

Let σ be an involution of U that gives the rank n symmetric space U/K, and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ its Cartan decomposition. Let $a_1, \ldots, a_n$ be a basis of a maximal abelian subalgebra $\mathcal{A}$ in $\mathcal{P}$, and $\mathcal{A}^\perp = \{x \in \mathcal{P} \mid [x, \mathcal{A}] = 0\}$. The *$U/K$-system* (cf. [5]) is the following first order system for $v : \mathbb{R}^n \to \mathcal{A}^\perp$:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \qquad 1 \le i < j \le n. \tag{1}$$

Different choices of basis for $\mathcal{A}$ amounts to linear changes of coordinates of $\mathbb{R}^n$. Note that v is a solution of the U/K-system if and only if

$$\theta_\lambda = \sum_{i=1}^{n} (a_i \lambda + [a_i, v]) dx_i$$

is a flat $\mathcal{U}_\mathbb{C}$-connection on $\mathbb{R}^n$ for all parameter $\lambda \in \mathbb{C}$. We call θ_λ the **Lax pair** of the U/K-system and the unique $E : \mathbb{R}^n \times \mathbb{C} \to U_\mathbb{C}$ satisfying $E(0, \lambda) = \mathrm{I}$ and

$E^{-1}dE = \theta_\lambda$ the **parallel frame**. Moreover, θ_λ and E satisfies the *U/K-reality condition*:

$$\overline{\theta_{\bar\lambda}} = \theta_\lambda, \quad \sigma(\theta_\lambda) = \theta_{-\lambda},$$
$$\overline{E(x,\bar\lambda)} = E(x,\lambda), \quad \sigma(E(x,\lambda)) = E(x,-\lambda).$$

Next we give some methods of constructing solutions of the U/K-systems.

• **Dressing action** ([6]). Let $F(x,\lambda)$ be the frame for the Lax pair of a solution v of U/K-system, and $g \in L^\sigma(U)$, where $L^\sigma(U)$ is the group of local $U_{\mathbb{C}}$-valued holomorphic germs g at $\lambda = \infty$ satisfying the U/K-reality condition and $g(\infty) = \mathrm{I}$.

(1) We can factor $g(\lambda)F(x,\lambda) = \tilde F(x,\lambda)\tilde g(x,\lambda)$ for x in a neighborhood $\mathcal{O}$ of $0 \in \mathbb{R}^n$ such that $\tilde F(x,\lambda)$ is holomorphic for $\lambda \in \mathbb{C}$ and $\tilde g(x,\cdot) \in L^\sigma(U)$.
(2) Write $\tilde g(x,\lambda) = \mathrm{I} + m_1(x)\lambda^{-1} + \cdots$, and $m_1^\perp$ the projection onto $\mathcal{A}^\perp$. Then $g * v := v + m_1^\perp$ is a solution of the U/K-system.
(3) $g * v$ defines an action of $L^\sigma(U)$ on solutions of U/K-system.
(4) If g is rational, then $g*v$ can be computed explicitly in terms of poles and residues of g and the parallel frame F of v.
(5) Notice that $v = 0$ is a solution of the U/K-slystem. If g is rational, then the explicit solution $g*0$ is a *soliton* solution and the number of poles of g is the soliton number of $g*0$. If $g \in L^\sigma(U)$ so that ga_ig^{-1} are polynomials in λ^{-1}, then $g*0$ can be constructed by solving ODE and we obtain the finite type solutions of Burtall, Ferus, Pedit and Pinkall [2].
(6) If U/K is compact and solution v of the U/K-system is smooth and rapidly decaying at infinity, then so is $g*v$.

• **Cauchy problem** ([5]). Let $a_1 \in \mathcal{A}$ be regular. Then there exists an open dense subset $\mathcal{S}_0$ of the space of rapidly decaying smooth maps from $\mathbb{R}$ to $\mathcal{A}^\perp$ such that given $v_0 \in \mathcal{S}_0$, there exists a unique smooth solution $v : \mathbb{R}^n \to \mathcal{A}^\perp$ of the U/K-system such that $v(x_1, 0, \ldots, 0) = v_0(x_1)$ and v is rapidly decaying in x_1. Moreover, $\mathcal{S}_0$ contains the unit ball in L^1-norm.

Below we give some geometric problems whose equations are U/K-systems.

(1) **Curved flats** ([4]). A n-dim submanifold M in a rank n symmetric space U/K is a *curved flat* if M is tangent to a flat at p for all $p \in M$. Let v be a solution of the U/K-system, and E its parallel frame. Then given $0 \neq r \in \mathbb{R}$, $x \to E(x,r)\sigma(E(x,r))^{-1}$ is a curved flat in U/K. Here we use the Cartan embedding U/K in U as a totally geodesic submanifold via $gK \mapsto g\sigma(g)^{-1}$. Conversely, all curved flats are obtained this way.

(2) **Isothermic surfaces** ([1], [3]). A surface in $\mathbb{R}^3$ is *isothermic* if there exists a conformal line of curvature coordinates. The Gauss-Codazzi equation is the $O(4,1)/(O(3)\times O(1,1))$-system and the dressing action of a rational g with two simple poles give the Ribaucour transformations.

(3) **Flag Lagrangian submanifolds in** $\mathbb{C}^n$ ([7]). We will give more detail for this example and show how dressing actions can be used to construct these submanifolds. We embed $U(n)\ltimes\mathbb{C}^n \subset GL(2n+1,\mathbb{R})$ via $(A+iB, u+iv) \to \begin{pmatrix} A & -B & u \\ B & A & v \\ 0 & 0 & 1 \end{pmatrix}$.
Let σ be the involution for the symmetric space $(U(n) \ltimes \mathbb{C}^n)/(O(n) \ltimes \mathbb{R}^n)$:

$$\sigma(g) = TgT^{-1}, \qquad T = \mathrm{diag}(\mathrm{I}_n, -\mathrm{I}_n, 1).$$

(i) We can use Codazzi equations to construct line of curvature coordinates x on a flat Lagrangian submanifold of $\mathbb{C}^n$ with flat and non-degenerate normal bundle such that $I = \varphi_{x_i} dx_i^2$ and $\mathrm{II} = \sum dx_i^2 \otimes J(\partial f/\partial x_i)$, where f is the immersion, J is the standard complex structure of $\mathbb{C}^n$, and φ is called the *Egoroff potential*. Then the Gauss-Codazzi equation for these immersions is the $(U(n)\ltimes\mathbb{C}^n)/(O(n)\ltimes\mathbb{R}^n)$-system. Moreover, if $F = \begin{pmatrix} E & X \\ 0 & 1 \end{pmatrix}$ is the parallel frame of θ_λ then $X(\cdot,\lambda)$ is a flat Lagrangian immersion in $\mathbb{C}^n$ for each $\lambda \in \mathbb{R}$ and f is $X(\cdot,1)$. We call $X(\cdot,\cdot)$ the *associated family of flat Lagrangian immersions.*

(ii) Given a Hermitian projection π of $\mathbb{C}^n$ and $\alpha \in \mathbb{C}$, define

$$g_{\alpha,\pi}(\lambda) = \mathrm{I} + \frac{\alpha - \bar\alpha}{\lambda - \alpha}\,\pi, \quad f_{\alpha,\pi} = g_{-\bar\alpha,\rho}\, g_{\alpha,\pi},$$

where ρ is the Hermitian projection of $\mathbb{C}^n$ onto $g_{\alpha,\pi}(-\bar\alpha)(\mathrm{Im}\pi)$. Then

(a) The set $\{g_{i\alpha,\pi} \,|\, \alpha \in \mathbb{R}, \bar\pi = \pi\} \cup \{f_{\alpha,\pi} \,|\, \alpha \in \mathbb{C}, \pi\}$ generates the group of rational maps $g : S^2 \to GL(n,\mathbb{C})$ that satisfy the $U(n)/O(n)$-reality condition and $g(\infty) = \mathrm{I}$.

(b) Let $g'_{\alpha,\pi}(\lambda) = \begin{pmatrix} g_{\alpha,\pi}(\lambda) & 0 \\ 0 & \frac{\lambda-\bar\alpha}{\lambda-\alpha} \end{pmatrix}$, and $X(x,\lambda)$ an associated family of flat Lagrangian immersions with Egorff potential φ. Then

$$g'_{i\alpha,\pi} * X = g_{i\alpha,\pi^\perp}(X + \frac{2i\alpha}{\lambda - i\alpha}\, E\tilde\pi\eta)$$

is an associated family of flat Lagrangian immersions with Egoroff potential $g'_{i\alpha,\pi} * \varphi = \varphi - 2i\alpha\eta^t\tilde\pi^t$, where $\tilde\pi(x)$ is the projection onto

$$E(x,i\alpha)^{-1}(\mathrm{Im}\pi) \text{ and } \eta(x) = E(x,-i\alpha)^{-1}X(x,-i\alpha).$$

(iii) Given $\alpha, b \in \mathbb{R}$, define $k_{\alpha,b}(\lambda) = \begin{pmatrix} \mathrm{I}_n & \frac{ib}{\lambda - i\alpha} \\ 0 & 1 \end{pmatrix}$ If $X(x,\lambda)$ is an associated family of flat Lagrangian immersions in $\mathbb{C}^n$, then

$$k_{\alpha,b} * X = X + i\,\frac{(\mathrm{I} - E(x,\lambda)E(x,i\alpha)^{-1})b}{\lambda - i\alpha}$$

is an associated family of flat Lagrangian immersions.

Some open problems

(1) Identify geometric problems whose PDE is a U/K-system.
(2) Find generators for the group of rational maps from S^2 to $U_{\mathbb{C}}$ that satisfy the U/K-reality condition.
(3) Read geometry and topology of the corresponding submanifold for $g * 0$ in terms of holomorphic germs g.
(4) Understand the geometry and topology of submanifolds (with singularities) corresponding to smooth decaying solutions of the U/K-system.

References

[1] Brück, M., Du, X., Park, J., and Terng, C.L., *The submanifold geometries associated to Grassmannian systems*, Mem. Amer. Math. Soc., **155** (2002), no. 735, viii+95 pp.

[2] Burstall, F. E., Ferus, D., Pedit, F., Pinkall, U, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. (2) 138 (1993), no. 1, 173–212.

[3] Burstall, F. E., *Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems*, Integrable systems, geometry, and topology, 1–82, AMS/IP Stud. Adv. Math., 36, Amer. Math. Soc., Providence, RI, 2006.

[4] Ferus, D., Pedit, F., *Curved flats in symmetric spaces*, Manuscripta Math., **91** (1996), 445–454

[5] Terng, C-L., *Soliton equations and differential geometry*, J. Differential Geom. **45** (1997), 407-445.

[6] Terng, C-L., Uhlenbeck, K., *Bäcklund transformations and loop group actions*, Comm. Pure. Appl. Math. **53** (2000), no. 1, 1-75.

[7] Terng, C-L., Wang, E., *Transformations of flat Lagrangian immersions and Egoroff nets*, arXiv:math/0611063

The Virtual Math Museum

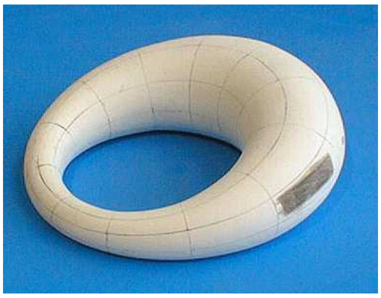
Richard S. Palais

The field of computer-aided mathematical visualization is now roughly a quarter of a century old. At first, mathematical visualizations were mostly created by research mathematicians using custom software to display some special and poorly understood mathematical object in order to help better understand its properties. The development of general purpose mathematical programs, like Matlab, Maple and Mathematica, not only made this much easier, but also led to a second important use; the visualization of mathematical objects for pedagogical purposes, to help students better understand abstract mathematical concepts by seeing concrete visual representations of special examples.

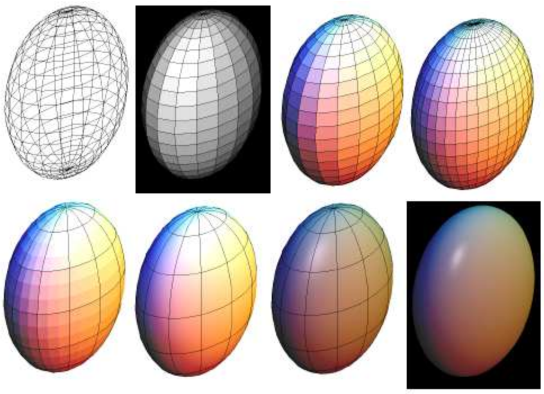
The 3D-XplorMath Consortium is an international group of mathematicians who have been cooperating for many years in developing a mathematical visualization program, 3D-XplorMath, that can make the highest quality images of many types of mathematical objects. In our presentation at the Oberwolfach, we demonstrated this software and explained a new initiative of the Consortium; namely the creation of an online Virtual Mathematical Museum, where Galleries of well documented interactive mathematical exhibits will be on permanent display for everyone to use freely in teaching and research, or just to appreciate their innate beauty.

By a *mathematical museum* we understand a collection of *mathematical objects* arranged in galleries of similar or related objects. But, mathematical objects are abstract things, so what goes into such a museum cannot be the objects themselves, but rather some sort of *representations* of the objects—and how these representations or *renderings* are chosen and constructed is an important and interesting part of the story. In fact a given mathematical object can have many different representations with different ones being better adapted for different purposes, and we will call a collection of representations of the same object an *Exhibit*.

Until recently, the only feasible way to represent mathematical objects was as a physical model, for example these two plaster and string models:



But as computer power and computer graphics based techniques gradually improved, mathematical visualizations became increasingly "realistic". Here is a progression of images of an ellipsoid, from wireframe through raytraced, that shows what was possible at various stages along the way.



In recent years (in part as a consequence of the great popularity and profitability of computer games) the sophistication of 3D rendering programs has increased enormously. As a result it has become possible to create visual representations of a mathematical object that can convince most viewers that they are looking at a real physical object. Here for example is the cover picture of the September

22, 2006 issue of Science Magazine, showing the winner of the 2006 NSF/Science Visualization Challenge—a still-life that depicts glass models of five famous mathematical surfaces, lying next to each other on a glass-covered wooden tabletop where they reflect each other and are reflected in the tabletop.

These "glass models" in fact never existed as physical objects. They were created as computer-based models by the graphic artist, Luc Benard, using the mathematical visualization program 3D-XplorMath, and then exported by him to the 3D modeling program Bryce, where he completed the creation of the still-life.

If you would like to try out the 3D-XplorMath visualization program, you can go to its home page (the URL is: http://3D-XplorMath.org) and click on the Download link. If you are a Macintosh user you can download the original Macintosh only program (which is more feature rich and has many more exhibits since it has been under development much longer) but you can also download the newer Java-based 3D-XplorMath-J which is cross-platform. You can also play with 3D-XplorMath-J by clicking on a link at the Download page, or by going directly to the URL http://3D-XplorMath.org/Applets

You are also invited to visit the Virtual Math Museum. You will find it at the URL http://VirtualMathMuseum.org

Reporter: Katrin Leschke

Participants

Prof. Dr. Uwe Abresch
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum

Dr. Dmitry A. Berdinsky
Sobolev Inst. of Mathematics
Siberian Branch of the Academy of Sciences
pr. Koptjuga N4
630090 Novosibirsk
RUSSIA

Dr. Christoph Bohle
Fakultät II-Institut f. Mathematik
Technische Universität Berlin
Sekr. MA 3-2
Straße des 17. Juni 136
10623 Berlin

Dr. phil. David Brander
Dept. of Mathematics
Faculty of Science
Kobe University
Okamoto
Kobe 657
JAPAN

Prof. Dr. Francis E. Burstall
School of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Prof. Dr. Josef Dorfmeister
Zentrum Mathematik
TU München
85747 Garching

Prof. Dr. Karsten Große-Brauckmann
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. Laurent Hauswirth
Equipe d'Analyse et de
Mathematiques Appliquees
Universite de Marne-la-Vallee
5, bd Descartes
F-77454 Marne-la-Vallee Cedex 2

Dr. Martin Kilian
Fakultät für Mathematik und
Informatik
Universität Mannheim
68131 Mannheim

Prof. Dr. Shimpei Kobayashi
School of Information Environement
Tokyo Denki University
Musai Gakuendai
2-1200 Insai
Chiba 270-1382
JAPAN

Prof. Dr. Masatoshi Kokubu
Department of Mathematics
Tokyo Denki University
School of Engineering
Kanda-Nishikicho 2-2, Chiyodaku
Tokyo 101-8457
JAPAN

Dr. Katrin Leschke
Institut für Mathematik
Universität Augsburg
86159 Augsburg

Prof. Dr. Francisco Martin
Dept. of Geometry and Topology
University of Granada
Fuente Nueva s/n.
E-18071 Granada

Prof. Dr. William H. Meeks
Dept. of Mathematics
University of Massachusetts
Amherst , MA 01003-9305
USA

Prof. Dr. Frank Pacard
Centre de Mathematiques
Faculte de Sciences et Technologie
Universite Paris XII, Val de Marne
61 avenue du General de Gaulle
F-94010 Creteil Cedex

Prof. Dr. Richard S. Palais
Department of Mathematics
University of California at Irvine
Irvine , CA 92697-3875
USA

Prof. Dr. Franz Pedit
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Prof. Dr. Ulrich Pinkall
Fakultät II-Institut f. Mathematik
Technische Universität Berlin
Sekr. MA 3-2
Straße des 17. Juni 136
10623 Berlin

Dr. Jesse Ratzkin
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs , CT 06269-3009
USA

Prof. Dr. Harold Rosenberg
U. F. R. de Mathematiques
Case 7012
Universite Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Wayne Rossmann
Dept. of Mathematics
Faculty of Science
Kobe University
Okamoto
Kobe 657
JAPAN

Prof. Dr. Toshihiro Shoda
Dept. of Mathematics
Faculty of Culture and Education
Saga University
1 Honjo-cho, Saga-City
Saga 840
JAPAN

Prof. Dr. Iskander A. Taimanov
Sobolev Inst. of Mathematics
Siberian Branch of the Academy of Sciences
pr. Koptjuga N4
630090 Novosibirsk
RUSSIA

Prof. Dr. Chuu-Lian Terng
Department of Mathematics
University of California at Irvine
Irvine , CA 92697-3875
USA

Prof. Dr. Masaaki Umehara
Dept. of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-16, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Kotaro Yamada
Faculty of Mathematics
Kyushu University
Fukuoka 812-8581
JAPAN