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Poisson Geometry and Applications

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Abstract. This workshop concentrated on some of the major areas of Poisson geometry and its applications. Topics covered included moment maps, global Poisson geometry (Poisson-Lie groups, homogeneous spaces), Poisson topology (deformations, integrability) and symplectic groupoids. Applications to Schubert calculus, moduli spaces, Dirac structures, cluster varieties, Lie algebroids and groupoids, generalized complex structures, and deformation quantization, were also discussed.

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Introduction by the Organisers

The half-size workshop was organized by Anton Alekseev (Geneve), Rui Loja Fernandes (Lisboa), Eckhard Meinrenken (Toronto) and Markus Pflaum (Frankfurt). Marius Crainic (Utrecht) also acted as an unofficial organizer. The programme consisted of 17 lectures and covered a range of areas in Poisson Geometry and its applications where significant progress has been achieved recently. The aim of the workshop was to emphasize the main themes in Poisson Geometry that play the role of driving forces and organizing principles of the field. A significant number of young researchers, who have made already important contributions to the field, participated in this meeting. During the workshop all participants were involved in a great number of informal discussions, some of which gave rise to new collaborations. In total, 27 researchers have participated in this meeting from institutions in 9 different countries in Europe, USA and Canada, including 4 researchers from German institutions.

The organizers and participants thank the Mathematisches Forschungsinstitut Oberwolfach for providing a truly inspiring atmosphere for this conference. In the following we include the abstracts in alphabetical order.

Workshop: Poisson Geometry and Applications

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Abstracts

Classifying spaces and representations up to homotopy

Camilo Arias Abad
(joint work with Marius Crainic)

Given a Lie groupoid G, the classifying space BG is a topological space which should be thought of as a homotopic substitute for the orbit space. For instance, if T is the transformation groupoid associated to the action of a Lie group G on a manifold M then $B(T)$ is the homotopic quotient of the action and its cohomology is the equivariant cohomology of the action:

$$H^{\bullet}(T) \cong H^{\bullet}_{G}(M)$$

The homotopy type of an orbifold represented by a proper étale groupoid G is defined to be the homotopy type of BG [8]. BG is defined as the thick geometric realization of the simplicial manifold $G_{\bullet}$, the nerve of G. This simply means that BG is the quotient space:

$$BG = (\coprod_{k \geq 0} G_k \times \Delta^k)/\sim$$

where " $\sim$ " is the equivalence relation generated by:

$$(x, \delta_i(y)) \sim (d_i(x), y)$$

Here Δ^k denotes the standard k-simplex, δ_i is the inclusion of the k-simplex as the i-th face of the $k+1$-simplex and $d_i : G_p \to G_{p-1}$ is the map that forgets the i-th component.
When a Lie group acts properly on a manifold, the equivariant cohomology can be computed using the Cartan model [4], the Weil model [6] or the model of Kalkman [7]. Getzler [5] constructed a model for the equivariant cohomology in the non-compact case and later Behrend [2] generalized Getzler's construction to compute the cohomology of classifying spaces of more general groupoids. For an arbitrary Lie groupoid G, the cohomology of BG can be computed using the Bott-Shulman

double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^2(M) & \xrightarrow{\delta} & \Omega^2(G_1) & \xrightarrow{\delta} & \Omega^2(G_2) & \xrightarrow{\delta} & \cdots \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^1(M) & \xrightarrow{\delta} & \Omega^1(G_1) & \xrightarrow{\delta} & \Omega^1(G_2) & \xrightarrow{\delta} & \cdots \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^0(M) & \xrightarrow{\delta} & \Omega^0(G_1) & \xrightarrow{\delta} & \Omega^0(G_2) & \xrightarrow{\delta} & \cdots
\end{array}$$

in which the vertical differentials are the De-Rham operators and the horizontal differentials are the sum of pullback maps:

$$\delta = \sum_{i=0}^{p} (-1)^i {d_i}^*$$

By computing the horizontal cohomology of the above double complex, Bott [3] described the relationship between the classifying space BG and the representations of a Lie group G. More precisely, he constructed a spectral sequence

$$E_1^{pq} = H_{\text{diff}}^{p-q}(G, S^q(\mathfrak{g}^*)) \Rightarrow H^{p+q}(BG)$$

which generalizes the Chern-Weil homomorphism. In case G is compact, the spectral sequence degenerates at the first stage and immediately gives:

$$H^{2q}(BG) \cong S^q(\mathfrak{g}^*)^G$$

It turns out that the spectral sequence of Bott exists for arbitrary Lie groupoids. The first step in this construction is to make sense of the adjoint representation of a Lie groupoid. A representation of a Lie groupoid G over M is a map (i.e. smooth functor) $\phi : G \to Gl(E)$ where E is a vector bundle over M and $Gl(E)$ is the groupoid of linear isomorphism between the fibers. This notion of representation is too restrictive and does not allow for a good defintion of the adjoint representation. Instead, one has to consider representations of Lie groupoids in complexes of vector bundles, called representations up to homotopy. In a representation up to homotopy an arrow

$$g : x \to y,$$

acts as a map of complexes

$$\tilde{g} : E_x^\bullet \to E_y^\bullet,$$

but we allow this action not to respect the associativity. That is, in general $\widetilde{(g_1 g_2)}$ and $\tilde{g}_1 \circ \tilde{g}_2$ are not the same map of chain complexes. However, they are homotopic maps, and there is a controlled and coherent way of choosing the homotopies.

Instead of giving a general definition, we describe here the case of representations of length one.

Definition 1. Let $E = (E^0 \oplus E^1, D^0)$ be a complex of vector bundles of length one. A representation up to homotopy of G on E is given by the following:

(1) A non-associative action: i.e. a smooth map form G to $GL(E)$ which commutes with the structure maps except possibly with the composition. We denote the action of g on E by $\tilde{g}$.

(2) An operator $D^2 \in \Gamma(G_2, \mathrm{Hom}(s^*(E^1), t^*(E^0)))$, where s and t denote the source and target map.

This data is subject to the equations:

- For $v \in E^p_{s(g)}$
$$D^0(\tilde{g}(v)) = \tilde{g}(D^0(v))$$
This means that each element $g \in G$ acts on the complex $(E^\bullet, D^0)$ as a map of complexes.
- For $v \in E^p_{s(g_2)}$
$$\widetilde{(g_1 g_2)}(v) - \tilde{g_1}(\tilde{g_2}(v)) = D^2(g_1, g_2)(D^0(v)) + D^0(D^2(g_1, g_2)(v)),$$
This says that the action may fail to respect the composition, but this failure is controlled by the operator D^2 in the sense that the map
$$D^2(g_1, g_2) : E^1_{s(g_2)} \to E^0_{t(g_1)},$$
is a homotopy between the maps of complexes
$$\widetilde{g_1 g_2} : E^\bullet_{s(g_2)} \to E^\bullet_{t(g_1)}$$
and
$$\tilde{g_1} \circ \tilde{g_2} : E^\bullet_{s(g_2)} \to E^\bullet_{t(g_1)}$$
- For $v \in {E^p}_{s(g_3)}$
$$\tilde{g_1}(D^2(g_2, g_3)(v)) \quad - \quad D^2(g_1 g_2, g_3)(v) + D^2(g_1, g_2 g_3)(v) - D^2(g_1, g_2)(\tilde{g_3}(v) = 0$$

There is a differentiable cohomology with coefficients in representations up to homotopy which is denoted by $H^\bullet_{\mathrm{diff}}(G, E)$.

The notion of representation up to homotopy is the global analog of Quillen's superconnection [9]. The infinitesimal version is described in [1]. Representations up to homotopy are to be thought of as representations in the cohomology vector bundle. Since the rank of the cohomology may vary, the pointwise cohomology is not a vector bundle and one is forced to work at the level of complexes. In case the pointwise cohomology has constant rank the representation up to homotopy descends to the cohomology vector bundle:

Proposition 2. *Let (E, D) be a regular representation up to homotopy of a groupoid G. Then, there is a representation up to homotopy structure $D_{\mathcal{H}(E)}$ in the cohomology complex $\mathcal{H}(E)$ and a quasi-isomorphism:*
$$\Phi : (E, D) \to (\mathcal{H}(E), D_{\mathcal{H}(E)})$$

In particular, a representation up to homotopy in an acyclic complex has zero cohomology.
A Lie groupoid G comes with an obvious choice of a complex of vector bundles on which one can represent it: $A \to TM$. This complex does have the structure of a representation of G, this is the adjoint representation.

Theorem 3. *Let G be a Lie groupoid and σ^s an Ehresmann connection on G.*

1. *The choice of σ^s gives the adjoint complex $A \to TM$ the structure of a representation up to homotopy.*
2. *If γ^s is any other Ehresmann connection on G, then the representations up to homotopy $\mathrm{Ad}(G)_{\sigma^s}$ and $\mathrm{Ad}(G)_{\gamma^s}$ are isomorphic.*

This representation is called the adjoint representation of G and is denoted by $\mathrm{Ad}(G)$.

The adjoint representation of a Lie groupoid is an intermediate notion between the tangent bundle of a manifold and the adjoint representation of a Lie group. We think of it as the space of vector fields on the orbit space.

Example 4.

1. When G is a Lie group one recovers the usual adjoint representation.
2. If M is a manifold seen as a unit groupoid, then the complex corresponding to the adjoint representation of M is nontrivial only in degree one and we have $C(M, \mathrm{Ad})^1 = \mathfrak{X}(M)$.
3. Assume that G is a Lie group and $\pi : P \to B$ is a principal G-bundle. The cohomology with coefficients in the adjoint representation of the transformation groupoid is just the space $\mathfrak{X}(B)$ of vector fields on the base.

Once we have a good notion of adjoint representation we can take duals and symmetric powers, morally, to obtain differential forms out of vector fields. In this way we obtain new representaions up to homotopy $S^q(\mathrm{Ad}^*)$. Exactly as Bott did for the case of groups, one can compare this representations with the horizontal cohomology of the Bott-Shulman double complex:

Theorem 5. *Let G be a Lie groupoid. Then:*

$$H^p_\delta(\Omega^q(G_\bullet)) \cong H^{p-q}_{\mathrm{diff}}(G, S^q(\mathrm{Ad}^*))$$

An immediate corollary of this formula is:

Theorem 6. *Let G be a Lie groupoid. There is a spectral sequence converging to the cohomology of BG:*

$$E_1^{pq} = H^{p-q}_{\mathrm{diff}}(G, S^q(\mathrm{Ad}^*)) \Rightarrow H^{p+q}(BG)$$

In the case of Lie groups the shift in degree immediately implies that the spectral sequence vanishes above the diagonal. This is not the case for arbitrary groupoids: here the grading in the complex may be negative and there may be cohomology in negative degrees. This should be no surprise because already in the case of compact group actions the spectral sequence is non-zero above the diagonal: it becomes a spectral sequence obtained from a filtration on the Cartan model. For compact groups the spectral sequence vanishes below the diagonal. One expects that the spectral sequence also vanishes below the diagonal for proper groupoids. At the moment we do not have the proof of this in general. We do know it is true in two special cases:

Proposition 7. *If G is either a regular proper groupoid or the transformation groupoid of a proper action then the spectral sequence vanishes below the diagonal.*

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Deformations of algebroid stacks

Paul Bressler

(joint work with A.Gorokhovsky, R. Nest, B. Tsygan)

Suppose that k is an algebraically closed field of characteristic zero.

Suppose that R is a commutative k-algebra. A (unital, associative) R algebra A determines in a natural way a R-linear category, denoted A^+, with one object whose R-algebra of endomorphisms is (the opposite algebra) A^{op}. The category A^+ is an example of an R-algebroid. Precisely, an R-algebroid is a non-empty R-linear category $\mathcal{C}$ such that any two objects are isomorphic.

Suppose that X is a space and $\mathcal{A}$ is a sheaf of R algebras on X. The assignment $X \supseteq U \mapsto \mathcal{A}(U)^+$ determines a prestack in R-linear categories, denoted $\mathcal{A}^+$, on X. The associated stack $\widetilde{\mathcal{A}^+}$ is naturally equivalent to the stack of locally free $\mathcal{A}$-modules of rank one. The stack $\widetilde{\mathcal{A}^+}$ is an example of an *R-algebroid stack*. Precisely, an R-algebroid stack is a stack in R-linear categories $\mathcal{C}$ such that

- every $x \in X$ has a neighborhood U such that the category $\mathcal{C}(U)$ is non-empty

- for every $U \subseteq X$, $x \in U$, $A, B \in \mathcal{C}(U)$ there exists a neighborhood $x \in V \subseteq U$ such that $A|_V \cong B|_V$.

Suppose that M is a manifold with the structure sheaf of *complex valued* functions denoted $\mathcal{O}_M$. Since the sheaf of $\mathbb{C}$-algebras $\mathcal{O}_M$ does not admit any nontrivial automorphisms, it follows that it does not have any twisted forms (i.e. any sheaf of $\mathbb{C}$-algebras locally isomorphic to $\mathcal{O}_M$ is, in fact, globally isomorphic to $\mathcal{O}_M$, and such an isomorphism is unique). However, the $\mathbb{C}$-algebroid stack $\widetilde{\mathcal{O}_M^+}$ does have twisted forms. These are in a natural bijective correspondence with $\mathcal{O}_M^\times$-gerbes and are classified by $H^2(M; \mathcal{O}_M^\times)$ (which, in the C^∞-setting, is canonically isomorphic to $H^3(M; \mathbb{Z})$).

Jointly with A. Gorohovsky, R. Nest, and B. Tsygan we consider the formal deformation theory of twisted forms of $\widetilde{\mathcal{O}_M^+}$. Let $\mathcal{S}$ be one such. For each Artin $\mathbb{C}$-algebra R with maximal ideal $\mathfrak{m}_R$ one has the 2-groupoid $\mathrm{Def}(\mathcal{S})(R)$ of flat R-deformations of $\mathcal{S}$ (viewed as a $\mathbb{C}$-algebroid stack). The assignment $R \mapsto \mathrm{Def}(\mathcal{S})(R)$ defines a functor on the category of Artin $\mathbb{C}$-algebras.

Examples of (groupoid-valued) functors on the category of Artin k-algebras are obtained in the following way. Suppose that $\mathfrak{g}$ is a nilpotent differential graded Lie algebra (DGLA) such that $\mathfrak{g}^i = 0$ for $i < -1$. Then, one has the 2-groupoid $\mathrm{MC}^2(\mathfrak{g})$ whose objects are Maurer-Cartan elements of $\mathfrak{g}$.

Suppose that $\mathfrak{g}$ is a DGLA such that $\mathfrak{g}^i = 0$ for $i < -1$. An Artin k-algebra R determines the nilpotent DGLA $\mathfrak{g} \otimes_k \mathfrak{m}_R$, hence the 2-groupoid $\mathrm{MC}^2(\mathfrak{g} \otimes_k \mathfrak{m}_R)$. The assignment $R \mapsto \mathrm{MC}^2(\mathfrak{g} \otimes_k \mathfrak{m}_R)$ defines a functor on the category of Artin $\mathbb{C}$-algebras.

Our principal result is a construction of a DGLA which "controls" the formal deformation theory of a twisted form $\mathcal{S}$ of $\widetilde{\mathcal{O}_M^+}$, M – a C^∞-manifold (and, more generally, an étale groupoid equipped with an integrable complex distribution). Namely, to $\mathcal{S}$ as above we associate a DGLA $\mathfrak{g}_\mathcal{S}$ with $\mathfrak{g}_\mathcal{S}^i = 0$ for $i < -1$ and a canonical equivalence of formal deformation theories (2-groupoid valued functors on the category of Artin $\mathbb{C}$-algebras)

$$R \mapsto \mathrm{Def}(\mathcal{S})(R) \cong \mathrm{MC}^2(\mathfrak{g}_\mathcal{S} \otimes_\mathbb{C} \mathfrak{m}_R)$$

The DGLA $\mathfrak{g}_\mathcal{S}$ is given explicitly as follows. As a graded Lie algebra it is equal to $\Gamma(M; \Omega_M^\bullet \otimes_{\mathcal{O}_M} \overline{C}^\bullet(\mathcal{J}_M)[1])$, where $\Omega_M^\bullet$ is the algebra of complex-valued differential forms on M, $\mathcal{J}_M$ is the sheaf of (infinite) jets of functions, and $\overline{C}^\bullet(\mathcal{J}_M)[1]$ is the graded Lie algebra (under the Gerstenhaber bracket) of $\mathcal{O}_M$-linear, continuous, normalized Hochschild cochains on $\mathcal{J}_M$. The differential is given by $\nabla^{can} + \delta + \iota_{\overline{F}}$, where ∇^{can} is the derivation induced by the canonical flat connection on $\mathcal{J}_M$, δ is the Hochschild differential, and $\iota_{\overline{F}}$ denotes the adjoint action (with respect to the Gerstenhaber bracket) of $F \in \Gamma(M; \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{J}_M)$ whose image $\overline{F} \in \Gamma(M; \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{J}_M/\mathcal{O}_M)$ satisfies $\nabla^{can}\overline{F} = 0$ and represents the image of the class of $\mathcal{S}$ under the natural map $H^2(M; \mathcal{O}_M^\times) \to H^2(M; \mathtt{DR}(\mathcal{J}_M/\mathcal{O}_M))$ (the de Rham cohomology of the canonical flat connection on $\mathcal{J}_M/\mathcal{O}_M$).

Generalized Kähler and hyper-Kähler quotients

Henrique Bursztyn
(joint work with G. Cavalcanti and M. Gualtieri)

My talk discussed the reduction of Courant algebroids and generalized complex structures in the presence of symmetries, with special focus on generalized Kähler and hyper-Kähler quotients. The content of the talk was based on the papers [1, 2].

1. Actions and reduction data

Since generalized geometrical structures are defined in terms of $TM \oplus T^*M$, we consider actions which give rise to symmetries of the Courant algebroid structure on this bundle.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and consider $TM \oplus T^*M$ equipped with its natural symmetric pairing and Courant bracket

$$[\![(X,\xi),(Y,\eta)]\!] = ([X,Y], \mathcal{L}_X\eta - i_Y d\xi + i_{X\wedge Y}H),$$

where $H \in \Omega^3_{cl}(M)$ is a closed 3-form. We say that $\widetilde{\psi}: \mathfrak{g} \to \Gamma(TM \oplus T^*M)$ is a *lifted action* if it is a bracket-preserving map whose image is isotropic. We assume that this action integrates to a G-action making $TM \oplus T^*M$ into a G-equivariant bundle. The projection of $\widetilde{\psi}$ to TM defines an action $\psi: \mathfrak{g} \to \Gamma(TM)$ in the usual sense. If G is compact, the problem of lifting an action is equivalent to finding an equivariant closed extension of the 3-form H.

The set-up to describe our reduction procedure for generalized geometrical structures is a triple $(\widetilde{\psi}, \mathfrak{h}, \mu)$, where $\widetilde{\psi}: \mathfrak{g} \to \Gamma(TM \oplus T^*M)$ is a lifted action, $\mathfrak{h}$ is a G-module and $\mu : M \to \mathfrak{h}^*$ is an equivariant map. We refer to such triple as *reduction data*. We will always assume that 0 is a regular value of μ, and that the G-action on $\mu^{-1}(0)$ is free and proper, so that $M_{red} = \mu^{-1}(0)/G$ is a smooth manifold. Our goal is to transport generalized geometrical structures from M to M_{red}. We start with the Courant bracket.

2. Reduction of Courant brackets

Given reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$, we consider the distribution

$$K := \{\widetilde{\psi}(u) + d\langle \mu, w\rangle,\ u \in \mathfrak{g}, w \in \mathfrak{h}\} \subset TM \oplus T^*M.$$

It follows from the reduction data axioms that $K|_{\mu^{-1}(0)}$ and $K^{\perp}|_{\mu^{-1}(0)}$ are G-equivariant vector bundles over $\mu^{-1}(0)$, and $K|_{\mu^{-1}(0)} \subseteq K^{\perp}|_{\mu^{-1}(0)}$.

Proposition 1. *The vector bundle*

$$E_{red} := \frac{K^{\perp}|_{\mu^{-1}(0)}}{K|_{\mu^{-1}(0)}} \Big/ G \longrightarrow \mu^{-1}(0)$$

inherits the structure of an exact Courant algebroid.

If $H = 0$ and $\widetilde{\psi} = \psi : \mathfrak{g} \to \Gamma(TM)$, then $E_{red} = T(\mu^{-1}(0)/G) \oplus T^*(\mu^{-1}(0)/G)$ with no 3-form; in general, E_{red} is isomorphic to $T(\mu^{-1}(0)/G) \oplus T^*(\mu^{-1}(0)/G)$ equipped with a possibly nontrivial 3-form H_{red}, but this identification may not be canonical.

3. Reduction of Dirac and generalized complex structures

Let us consider reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$ and the associated reduced Courant algebroid E_{red} over $\mu^{-1}(0)/G$. Suppose that $L \subset TM \oplus T^*M$ is a G-invariant Dirac structure. We can try to transport it to E_{red} by considering the distribution

$$L_{red} := \frac{(L \cap K^{\perp} + K)|_{\mu^{-1}(0)}}{K|_{\mu^{-1}(0)}} \Big/ G \tag{1}$$

The operation above reduces to pull-back of Dirac structures when $\mathfrak{g} = \{0\}$, and push-forward when $\mathfrak{h} = \{0\}$, $\mu = 0$. The distribution (1) defines a maximal isotropic subspace of E_{red} pointwise, but it may not be a smooth vector bundle (it is smooth, e.g., if $L \cap K^{\perp}|_{\mu^{-1}(0)}$ has constant rank). In case L_{red} is smooth, then it is automatically integrable and defines a reduced Dirac structure in E_{red}. The same conclusions are valid for complex Dirac structures if one replaces K by $K \otimes \mathbb{C}$.

One can now adapt this procedure for generalized complex structures on M, recalling that these are defined by complex Dirac structures $L \subset TM \oplus T^*M \otimes \mathbb{C}$ satisfying the additional transversality condition $L \cap \overline{L} = \{0\}$. We can state a simple condition in terms of the associated endomorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ which is sufficient to guarantee that L_{red} will be smooth and define a generalized complex structure on $\mu^{-1}(0)/G$:

Theorem 2. *If $\mathcal{J}K = K$ over $\mu^{-1}(0)$, then L_{red} defines a generalized complex structure on $\mu^{-1}(0)/G$.*

Extreme examples of this result include holomorphic quotients of complex manifolds (in this case $\mathfrak{h} = \{0\}$ and $\mu = 0$) and Hamiltonian reduction of symplectic manifolds (in this case $\mathfrak{h} = \mathfrak{g}$ and μ is the moment map). The condition $\mathcal{J}K = K$ interpolates between the condition for a complex group action to be holomorphic and the moment map condition in symplectic geometry. There are also "exotic" examples e.g. reducing symplectic structures to complex structures, see e.g. [1].

4. Generalized Kähler and hyper-Kähler reduction

A generalized Riemannian metric $\mathcal{G}$ is *compatible* with a generalized complex structure $\mathcal{J}$ if $\mathcal{J}\mathcal{G} = \mathcal{G}\mathcal{J}$. In this case, the pair $(\mathcal{J}, \mathcal{G})$ is called a *generalized hermitian structure*. In this case, one can weaken the hypothesis in Theorem 2 by considering the distribution $K^{\mathcal{G}} := \mathcal{G}K^{\perp} \cap K^{\perp}$. Given reduction data $(\widetilde{\psi}, \mathfrak{h}, \mu)$ and assuming that $\mathcal{J}$ and $\mathcal{G}$ are G-invariant, we have:

Theorem 3. *If $\mathcal{J}K^{\mathcal{G}} = K^{\mathcal{G}}$ over $\mu^{-1}(0)$, then $\mathcal{J}$ can be reduced to E_{red} and $\mathcal{G}$ induces a compatible generalized metric.*

This theorem has as direct corollaries the reductions for generalized Kähler and generalized hyper-Kähler structures, and allows one to view usual hyper-Kähler quotients in terms of generalized Kähler reduction. A nontrivial example that fits into this framework is the construction of generalized Kähler/hyper-Kähler structures on moduli spaces of instantons.

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Coisotropic Submanifolds, Reduction and Applications

Alberto S. Cattaneo

A Poisson algebra is a commutative algebra endowed with a Lie bracket which is also a biderivation. Deformation quantization is a formal deformation of the product in the direction of the Poisson bracket. New Poisson algebras may be obtained as subquotients. Namely, given an ideal for the commutative product, one gets a Poisson subalgebra by considering its Lie normalizer. This Poisson subalgebra contains a Poisson ideal given by its intersection with the original ideal. The quotient is called a reduction.

A Poisson manifold is a smooth manifold whose algebra of functions is a Poisson algebra. Equivalently, it is a smooth manifold endowed with a Maurer–Cartan element in the graded Lie algebra of multivector fields, i.e., a bivector field whose Schouten-Nijenhuis bracket with itself vanishes. By a celebrated theorem of Kontsevich every Poisson manifold admits a deformation quantization. This follows from the existence of an L_∞-quasi-isomorpshim from multivector fields to multidifferential operators.

Reduction of Poisson manifolds requires some regularity constraints. The Poisson bivector field yields a morphism from the cotangent to the tangent bundle of the manifold. By restriction and projection, it yields a morphism from the conormal bundle of every submanifold to its normal bundle. The submanifold is called pre-Poisson if this bundle map has constant rank. One extreme case is when this map is zero, and the submanifold is called cosymplectic since there is an induced symplectic structure on the normal bundle; by Dirac's formula this induces a Poisson structure on the cosymplectic submanifold. The other extreme case is when the morphism is surjective, that is, when the conormal bundle is mapped to the tangent bundle, and the submanifold is called coisotropic.

Pre-Poisson submanifolds are endowed with an integrable distribution (actually with a Lie algebroid structure) such that the algebra of invariant functions has a Poisson structure as described in the first paragraph. As the leaf space may not be a smooth manifold, Kontsevich's method cannot be applied.

If the Poisson manifold is actually symplectic, a pre-Poisson submanifold is the same as a presymplectic submanifold (i.e., a submanifold on which the kernel of the restriction of the sympectic form has constant rank). Cosymplectic submanifolds are the same as the symplectic ones. If a Poisson manifold is integrable and the subgroupoid over a submanifold is smooth, then it is presymplectic if and only if the submanifold is pre-Poisson.

By a result of Calvo and Falceto, improved by the author together with Zambon, for every pre-Poisson submanifold one can find a cosymplectic submanifold which contains it coisotropically. So, in a sense, it is enough to understand coistropic submanifolds.

Deformation quantization of coisotropic submanifolds may be approached by the Batalin–Fradkin–Vilkovisky (BFV) method. Namely, one extends a tubular neighborhood of the coisotropic submanifold to a graded Poisson manifold endowed with a function of degree one which Poisson commutes with itself and such that the cohomology with respect to its Hamiltonian vector field is isomorphic in degree zero to the Poisson algebra of invariant functions. Existence of such a function is guaranteed by homological perturbation theory. One then looks for a deformation quantization of the extended graded manifold and for a deformation of the above function to an element that squares to zero. The deformation quantization of (a certain Poisson subalgebra of) the Poisson algebra of invariant functions is achieved as the zeroth cohomology with respect to the adjoint action of the above element. The existence of such an element is however not guaranteed. Potential obstructions (anomalies) lie in the second cohomology group.

A different way to approach deformation quantization in this setting has been proposed by Lyakhovich and Sharapov and, independently, by the author together with Felder. The main observation is that the sum of the Poisson bivector field and the Hamiltonian function above is a Maurer–Cartan element in the multivector fields of the extended graded manifold. The existence of an L_∞-morphism to the multivector fields associates to this element an A_∞-structure. When this is flat, deformation quantization as above is given by its zeroth cohomology.

One may equivalently consider a smaller graded manifold, namely the odd conormal bundle of the coisotropic submanifold. By a result of Roytenberg its graded Lie algebra of multivector fields is canonically anti-isomorphic to that of a formal neighborhood of the the zero section in the normal bundle. So a choice of embedding of the normal bundle in the origianl Poisson manifold produces the sought-after Maurer–Cartan element. This way one also gets an L_∞-structure associated to the coisotropic submanifold (this structure had already appeared in Oh and Park in relation with the problem of studying deformations of coisotropic submanifolds). By a result of Schätz this L_∞-structure is quasi-isomorphic to the differential graded Lie algebra structure obtained in BFV. By a result of Schätz and the author, different choices of embeddings yield L_∞-quasi-isomorphic structures.

The same procedure for deformation quantization may be obtained via the Poisson sigma model, a two-dimensional topological field theory with target a Poisson manifold. It turns out, by Calvo and Falceto, that boundary conditions respecting the symmetries are labeled by pre-Poisson submanifolds (branes). If in addition one requires compatibility with perturbation theory around the zero Poisson structure, the branes have to be coisotropic.

Working on the disk, yields, in the absence of anomalies, deformation quantization of reductions of coisotropic submanifolds, bimodule structures associated to clean intersections, and bimodule morphisms associated to triple clean intersections. In particular, as the graph of a Poisson map is a coisotropic submanifold, in the absence of anomalies one gets a quantization as the bimodule associated to a morphism of the associative algebras. Composition of Poisson maps gets quantized as composition of bimodules. In terms of the morphisms, the composition of quantizations is the quantization of compositions up to conjugation with respect to a special element.

In the case of a Poisson–Lie group, in the absence of anomalies, the above procedure yields a deformation quantization with the structure of a Hopfish algebra. Conjecturally one gets a true Hopf algebra if the Poisson structure is linearizable at the neutral element. For more details, see [1] and references therein.

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Double of a cluster variety

Vladimir Fock
(joint work with Aleksandr Goncharov)

Cluster variety is an algebraic variety (strictly speaking, a scheme) defined by combinatorial data by explicit set of coordinate charts and transition functions. More precisely, for any collection of combinatorial data, called *seed* one associates three varieties $\mathcal{A}_{|\mathbb{I}|}$, $\mathcal{X}_{|\mathbb{I}|}$, and $\mathcal{D}_{|\mathbb{I}|}$. These varieties posses canonical pre-symplectic, Poisson and symplectic structures, respectively. One defines also a discrete group $\mathfrak{D}_{|\mathbb{I}|}$ acting on all the three types of varieties and preserving the respective structures. The manifolds $\mathcal{X}_{|\mathbb{I}|}$ and $\mathcal{D}_{|\mathbb{I}|}$ admit a quantisation (noncommutative deformation of the algebra of functions) which is also $\mathfrak{D}_{|\mathbb{I}|}$-invariant.

Varieties admitting cluster descriptions are simple Lie groups, moduli spaces of Stokes parameters, moduli of flat connections on Riemann surfaces, configuration spaces of flags, Teichmüller spaces and their generalisations, the spaces of measured laminations and some others. One of the important features of cluster varieties is that they are defined not only over a field but also over semifields (semigroups w.r.t. addition and groups w.r.t. the multiplication). For example, one can consider Teichmüller space, space of measured laminations and the space of flat $PSL(2,\mathbb{F})$-connections over a surface Σ as the same cluster manifold but defined over the semifield $\mathbb{R}_{>0}$ of positive real numbers, tropical semifield $\mathbb{R}^t$ (which is ordinary R as a set with maximum for the addition operation and ordinary addition for the multiplication), and a field $\mathbb{F}$, respectively.

Let us give the precise definitions

A *cluster seed*, or just *seed*, $\mathbf{I}$ is a quadruple (I, I_0, ε, d), where
i) I is a finite set;
ii) $I_0 \subset I$ is its subset;
iii) ε is a matrix ε_{ij}, where $i, j \in I$, such that $\varepsilon_{ij} \in \mathbb{Z}$ unless $i, j \in I_0$.
iv) $d = \{d_i\}$, where $i \in I$, is a set of positive integers, such that the matrix $\widehat{\varepsilon}_{ij} = \varepsilon_{ij} d_j$ is skew-symmetric.

The elements of the set I are called *vertices*, the elements of I_0 are called *frozen vertices*. The matrix ε is called *exchange matrix*, the numbers $\{d_i\}$ are called *multipliers*, and the function d on I whose value at i is d_i is called *multiplier function*. We omit $\{d_i\}$ if all of them are equal to one, and therefore the matrix ε is skew-symmetric, and we omit the set I_0 if it is empty.

An isomorphism σ between two seeds is a map $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ is an isomorphism of finite sets $\sigma : I \to I'$ such that $\sigma(I_0) = I'_0$, $d_{\sigma(i)} = d_i$ and $\varepsilon_{\sigma_i, \sigma_j} = \varepsilon_{ij}$. Observe that the automorphism group of a seed may be nontrivial.

For a seed $\mathbf{I}$ we associate a torus $\mathcal{X}_{\mathbf{I}} = (\mathbb{F}^\times)^I$, called $\mathcal{X}$*-torus*, another torus $\mathcal{A}_{\mathbf{I}} = (\mathbb{F}^\times)^I$, called $\mathcal{X}$*-torus* and the third one $\mathcal{D}_{\mathbf{I}} = (\mathbb{F}^\times)^{I \times I}$ called $\mathcal{D}$*-torus* or a *double torus*. We denote the standard coordinates on these tori by $\{x_i | i \in I\}$, $\{a_i | i \in I\}$ and $\{y_i, b_i | i \in I\}$, respectively.

The $\mathcal{X}$-torus is equipped with the Poisson structure

$$\{x_i, x_j\} = \widehat{\varepsilon}_{ij} x_i x_j \tag{1}$$

The $\mathcal{A}$-torus is equipped with the pre-symplectic structure (closed 2-form ω possibly degenerate)

$$\omega = \frac{1}{2} \sum_{i,j} \widehat{\varepsilon}_{ij} \frac{da_i \wedge da_j}{a_i a_j} \tag{2}$$

The $\mathcal{D}$-torus is equipped with the symplectic form

$$\omega_{\mathcal{D}} = \frac{1}{2} \sum_{i,j} \widehat{\varepsilon}_{ij} \frac{db_i \wedge db_j}{b_i b_j} + \sum_i d_i^{-1} \frac{db_i \wedge dy_i}{b_i y_i} \tag{3}$$

The inverse of this form is a nondegenerate Poisson structure which can be written as

$$\{y_i, y_j\} = \widehat{\varepsilon}_{ij} y_i y_j, \quad \{y_i, b_j\} = \delta^i_j d^i y_i b_j, \quad \{b_i, b_j\} = 0 \tag{4}$$

Observe that these sructures are constant in logarithmic coordinates.

Isomorphism between two $\mathcal{X}$-tori $\mathcal{X}_{\mathbb{I}}$ and $\mathcal{X}_{\mathbb{I}'}$ is a map given in coordinates by $x_{\sigma(i)} = x_i$, where σ is an isomorphism of the seeds. Observe that there are much less isomorphisms of $\mathcal{X}$-tori then just isomorphisms of the corresponding Poisson manifolds. Isomorphisms of $\mathcal{A}$- and $\mathcal{D}$-tori are defined analogously.

There exist the following maps between the tori:

$$\mathcal{A}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = \prod_j a_j^{\varepsilon_{ij}}; \tag{5}$$

(6) $$\mathcal{A}_{\mathbf{I}} \times \mathcal{A}_{\mathbf{I}} \to \mathcal{D}_{\mathbf{I}}, \quad y_i = \prod_j a_j^{\varepsilon_{ij}}, \quad b_i = a_i/\tilde{a}_i,$$

Here $\tilde{a}_i$ are coordinates on the second $\mathcal{A}_{\mathbf{I}}$-factor.

(7) $$\mathcal{D}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = y_i,$$

and

(8) $$\mathcal{D}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = y_i \prod_j b_j^{\varepsilon_{ij}}.$$

All the maps are compatible with the respective symplectic, pre-symplectic and Poisson structures. Namely the map (5) is a composition of the quotient by the kernel of the pre-symplectic form and a symplectic map to a symplectic leaf. The map (6) maps the symplectic form to the pre-symplectic one. The map (7) is Poisson, the map (8) is anti-Poisson (Poisson with the opposite Poisson structure on the $\mathcal{X}$-torus). The maps (7) and (8) are dual to each other in the sense on Poisson pairs.

Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I_0', \varepsilon', d')$ be two seeds, and $k \in I - I_0$. A *mutation in the vertex* k is an isomorphism $\mu_k : I \to I'$ satisfying the following conditions:

(1) $\mu_k(I_0) = I_0'$,
(2) $d'_{\mu_k(i)} = d_i$,
(3) $\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k \text{ otherwise} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} < 0 \\ \varepsilon_{ij} + \varepsilon_{ik}|\varepsilon_{kj}| & \text{if } \varepsilon_{ik}\varepsilon_{kj} \geq 0 \end{cases}$

Two seeds related by a sequence of mutations are called equivalent.

Mutations induce rational maps between the corresponding seed tori, which are denoted by the same symbol μ_k and are given by the formulae

$$x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if} \quad i = k \\ x_i(1 + x_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geq 0 \\ x_i(1 + (x_k)^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \leq 0 \end{cases}.$$

for the $\mathcal{X}$-torus,

$$a_{\mu_k(i)} = \begin{cases} \dfrac{\prod_{j|\varepsilon_{jk}>0} a_j^{\varepsilon_{jk}} + \prod_{j|\varepsilon_{jk}<0} a_j^{-\varepsilon_{jk}}}{a_k} & \text{if } i = k \\ a_i & \text{if } i \neq k \end{cases}$$

for the $\mathcal{A}$-torus and

$$b_{\mu_k(i)} = \begin{cases} \dfrac{(1+x_k)^{-1}\prod\limits_{j|\varepsilon_{jk}>0} b_j^{\varepsilon_{jk}} + (1+(x_k)^{-1})^{-1}\prod\limits_{j|\varepsilon_{jk}<0} b_j^{-\varepsilon_{jk}}}{b_k} & \text{if } i=k \\ b_i & \text{if } i \neq k \end{cases}$$

$$y_{\mu_k(i)} = \begin{cases} y_k^{-1} & \text{if} \quad i=k \\ y_i(1+y_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geq 0 \\ y_i(1+(y_k)^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \leq 0 \end{cases} .$$

for the $\mathcal{D}$-torus.

Since in the sequel we shall extensively use compositions of mutations we would like to introduce a shorthand notation for them. Namely, we denote an expression $\mu_{\mu_i(j)}\mu_i$ by $\mu_j\mu_k$, $\mu_{\mu_{\mu_i(j)}\mu_i(k)}\mu_{\mu_i(j)}\mu_i$ by $\mu_k\mu_j\mu_i$, and so on.

Mutations have the following properties (valid for mutation of seeds as well as for mutations of respective tori):

- Every seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ seed is related to other seeds by exactly $\sharp(I - I_0)$ mutations.

A_1 $\mu_i\mu_i = id$

$A_1 \times A_1$ If $\varepsilon_{ij} = \varepsilon_{ji} = 0$ then $\mu_i\mu_j\mu_j\mu_i = id$.

A_2 If $\varepsilon_{ij} = -\varepsilon_{ji} = -1$ then $\mu_i\mu_j\mu_i\mu_j\mu_i = id$. (This is called the *pentagon relation.*)

B_2 If $\varepsilon_{ij} = -2\varepsilon_{ji} = -2$ then $\mu_i\mu_j\mu_i\mu_j\mu_i\mu_j = id$.

G_2 If $\varepsilon_{ij} = -3\varepsilon_{ji} = -3$ then $\mu_i\mu_j\mu_i\mu_j\mu_i\mu_j\mu_i = id$.

By id we mean here an isomorphism of the seeds or tori. Conjecturally all relations between mutation follow from these ones.

Given a seed one can produce a $\sharp(I - I_0)$ seeds by mutations. Continuing this procedure one obtains a $\sharp(I - I_0)$-valent tree whose vertices are seed (or seed tori) and edges are pairs of mutually inverse mutations. Obviously if we start from any other seed from the tree we obtain the same tree. Every two tori of the tree are related by exactly one composition of mutations. Call two points of two different tori equivalent if they are related by the composition of mutations. The cluster manifold (denoted by $\mathcal{X}_{|\mathbf{I}|}$, $\mathcal{A}_{|\mathbf{I}|}$ or $\mathcal{D}_{|\mathbf{I}|}$ depending on which kind of tori are used) is the affine closure of disjoint union of the tori quotiented by the equivalence relation.

Algebraically one can define this manifold as the spectrum of the intersection of inverse images of Laurent polynomials under all possible compositions of mutations acting on the respective kind of tori.

Each particular seed tori can be considered as a coordinate chart of the corresponding cluster manifolds and compositions of mutations can be considered as transition functions between the charts.

Mutations respect the Poisson structure when acting on $\mathcal{X}$ tori, pre-symplectic structure when acting on $\mathcal{A}$-tori and symplectic when acting on $\mathcal{D}$-tori. Thus the cluster manifolds $\mathcal{X}_{|\mathbf{I}|}$, $\mathcal{A}_{|\mathbf{I}|}$ and $\mathcal{D}_{|\mathbf{I}|}$ acquire the respective structures. (In fact the formula for mutation of the matrix ε can be considered a a corollary of this property and the mutation formulae for, say, $\mathcal{X}$-tori).

Mutations commute with the maps (5),(6),(7) and (8) thus these maps are defined between the respective cluster varieties compatible with pre-symplectic, symplectic and Poisson structures thereof.

Mutations are rational maps with positive integral coefficients and thus the cluster manifold can be defined not only over a field but over any semifield as well. For semifields without -1 (like the semifields of positive real numbers or the tropical semifields) the mutations are isomorphisms and thus the whole manifold is isomorphic to every coordinate torus.

The symmetry group $\mathfrak{D}_{|\mathbf{I}|}$ of a cluster manifold permuting the seed tori is called the (generalised) *mapping class group* of the cluster manifold. The name comes from the case of Teichmüller space, when this group is the actual mapping class group. The group depends on the equivalence class of a seed only and is common for cluster manifolds of types $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{D}$. Every sequence of mutations together with an isomorphism of the initial and the final seed gives an element of the mapping class group. Conversely, given a seed, every mapping class group element can be presented by a sequence of mutations starting from the given seed together with the isomorphism between the final seed and the initial one. Two sequences of mutations different by the relations A_1–G_2 correspond to the same mapping class group elements.

Example.

Let us consider the simplest nontrivial example: the seed $\mathbf{I} = \{I, \varepsilon\}$ with $I = \{1, 2\}$ and $\varepsilon_{12} = 1$. There are exactly 5 isomorphism classes of seed tori equivalent to a given one, however all the five seeds are isomorphic, thus the mapping class group is $\mathbb{Z}/5\mathbb{Z}$.

The simplest geometric meaning has the space X. It is the space of 5-tuples of points $(p_1, \dots, p_5)$ on the projective line P^1 such that $p_i \neq p_{i+1 \pmod 5}$ and modulo the automorphisms of P^1. The 5-tuple of coordinate systems on this space is numerated by triangulations of the pentagon with vertices $1, \dots, 5$. For every internal diagonal one associates the cross-ratio of the four points of the quadrilateral which this diagonal cuts into halves. Mutations correspond to removing a diagonal and replacing it by another one of the quadrilateral. The same variety over $\mathbb{R}_{>0}$ is the configuration space of 5-tuples of points on $\mathbb{R}P^1$ with prescribed cyclic order.

The $\mathcal{A}$-space is the space of collections of 10 nonvanishing vectors $v_1, \dots, v_{10}$ in $\mathbb{F}^2$ equipped with a nonzero bivector ω. The collections are considered up to the action of the group $SL(2, \mathbb{F})$ of linear transformations preserving ω and subject to the relations $v_i = -v_{i+5 \pmod{10}}$ and $v_i \wedge v_{i+1 \pmod{10}} = \omega$. The map $\mathcal{X}_{|\mathbf{I}|} \to \mathcal{X}_{|\mathbf{I}|}$ is given by the obvious projection of $\mathbb{F}^2 - \{0\} \to P^1$. For the internal diagonal of the pentagon with ends i and j one associates the coordinate $v_i \wedge v_j / \omega$.

The $\mathcal{D}$ variety is the space of flat $SL(2, \mathbb{F})$ connections on a sphere with 5 different points on the equator removed with parabolic monodromy around these points. Consider the associated vector bundle and choose a monodromy invariant section about each singular points. Then trivialise the bundle over the northern hemisphere. The five chosen sections give five vectors $v_1, \ldots, v_5$ in $\mathbb{F}^2$. The same procedure over the southern hemisphere gives five vectors $w_1, \ldots, w_5$ in another copy of $\mathbb{F}^2$. Given a triangulation of the pentagon we associate to every internal diagonal two coordinates x and b. The coordinate x is just the cross ratio of four points in P^1 defined by the vectors v_i standing at the corners of the quadrilateral cut by the diagonal (just like for the $\mathcal{X}$-space). The coordinate b is given by $b = (v_i \wedge v_j)/(w_i \wedge w_j)$, where i and j are the ends of our diagonal. The two projections to the $\mathcal{X}$ variety are obviously given by projectivising the collections of vectors $\{v_i\}$ and $\{w_i\}$, respectively. The same manifold over $\mathbb{R}_{>0}$ can be identified with the space of complex structures on a sphere with five punctures on the equator.

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Symplectic Connections

Simone Gutt

A **symplectic connection** on a symplectic manifold (M, ω) is a torsion free linear connection ∇ for which the symplectic 2–form ω is parallel. The space of symplectic connections on (M, ω) is an affine space modelled on the (infinite dimensional) space of symmetric covariant 3–tensorfields on M.

In view of Darboux theorem, symplectic geometry is by essence global. Consideration of symplectic connections is nethertheless interesting, showing deep links between conditions on a symplectic connection (holonomy or curvature conditions for example) and the geometry of the manifold.

Given a symplectic manifold (M, ω), one can choose an almost complex compatible structure J. A symplectic connection ∇ preserves J (in the sense that $\nabla J = 0$) iff it is the Levi Civita connection associated to the pseudo Riemannian metric g defined by $g(X, Y) = \omega(X, JY)$; so it is unique and it only exits in a (pseudo)Kähler situation.

Another example where there is a unique preferred connection is given by a **symmetric symplectic space** (M, ω, S), i.e. a symplectic manifold (M, ω) with symmetries attached to each of their points. This means there is a smooth map $S : M \times M \to M$ $(x, y) \to s_x y$ such that each s_x (called the symmetry at x) squares to the identity $[s_x^2 = \mathrm{Id}]$ and is a symplectomorphism of (M, ω) $[s_x^* \omega = \omega]$, such that x is an isolated fixed point of s_x, and such that $s_x s_y s_x = s_{s_x y}$ $\forall x, y \in M$. On a symmetric symplectic space, the unique symplectic connection for which each symmetry s_x is an affinity is given by $\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x \omega(Y + s_{x\star} Y, Z)$.

The **curvature tensor** R^∇ at a point x belongs to the space $\underline{\mathcal{R}}_x$ of 2-forms on $T_x M$ with values in the Lie algebra $sp(T_x M, \omega_x)$ of the group $Sp(T_x M, \omega_x) =$

$\{ A \in \mathrm{End}(V) \mid \omega_x(Au, Av) = \omega_x(u,v)\ \forall u,v \in V \}$ satisfying Bianchi's first identity. In dimension $2n \geq 4$, the space $\underline{\mathcal{R}}_x$, decomposes under the action of $Sp(T_xM, \omega_x)$ into two irreducible subspaces. The corresponding decomposition of the curvature tensor reads $R^\nabla_x = E^\nabla_x + W^\nabla_x$, where E^∇ is completely determined by the Ricci tensor r^∇ and where W^∇ is traceless. A symplectic connection ∇ on (M, ω) is called **of Ricci-type** if $W^\nabla = 0$; and **Ricci-flat** if $E^\nabla = 0$ (hence iff $r^\nabla = 0$).

The first part of the talk was a survey concerning local and global models for a manifold with a symplectic connection with "special curvature". For a written survey on this subject, I refer to [2]. In particular, local models for any Ricci-type connection are given by elementary symplectic reduction of the flat standard symplectic vector space $(M = \mathbb{R}^{2n+2}, \Omega)$ for the action of $\exp tA$. with A a nonzero element in the symplectic Lie algebra $sp(\mathbb{R}^{2n+2}, \Omega)$.
This reduction procedure yields a symmetric space if and only if the element $A \in sp(\mathbb{R}^{2n+2}, \Omega)$ satisfies $A^2 = \lambda\,\mathrm{Id}$. In that case the quotient has a global structure of manifold and one obtains in this way all connected symmetric symplectic spaces with canonical connection of Ricci type (up to covering and restriction to a connected component). The only compact example is $P_n(\mathbb{C})$. One can interpret this reduction as a reduction of a minimal nilpotent adjoint orbit of $Sp(\mathbb{R}^{2n+2}, \Omega)$. Extending this construction to other simple groups [4] Cahen and Schwachhofer gave local models for all Bochner-Kähler connections, Bochner-Lagrangian connections or connections with "special" holonomies.

In my talk I also presented some new results on three ongoing research projects concerning Ricci-type symmetric symplectic spaces.

The first, joint with S. Waldmann, deals with deformation quantisation of those spaces. Classical algebraic methods to construct a deformation quantization on $P_n(\mathbb{C})$ with nice convergence properties, as developed by Bordemann et al. [1], can be extended to the framework of Ricci-type symmetric spaces for $A^2 = \lambda\,\mathrm{Id}$ with $\lambda \neq 0$. The construction and convergence for $P_n(\mathbb{C})$ yields a reverse procedure from the asymptotic expansion of Berezin-type symbolic calculus we developed with Cahen and Rawnsley in the framework of geometric quantization on some Kähler manifolds,

The second project, joint with M. Cahen, A. Dilawar and J. Rawnsley, deals with a property of some symmetric Ricci type spaces : the existence of a subgroup of the transvection group acting simply transitively on the space. Such subgroups are of course examples of symplectic groups .

A bridge between those two projects is the fact that quantisation of the ambient symmetric space (in particular convergent ones) could provide universal deformation formulas for those special subrgroups.

The third project, joint with M. Cahen, J. Rawnsley and N. Richard, deals with extrinsic symmetric spaces. An extrinsic symmetric spaces (M, ω) is a symplectically embedded submanifold of $(\mathbb{R}^{2n+2p}, \Omega)$ so that $\phi_x M \subset M\ \forall x \in M$, where

ϕ_x is the affine transformation of $\mathbb{R}^{2n+2p}$ given by symmetry with respect to the normal space at x: $\phi_x(x+X+N) = x-X+N$ with $X \in T_xM$ and $N \in (T_xM)^{\perp_\Omega}$. There exist only flat 2-dimensional extrinsic symmetric spaces but one can build higher dimensional examples (a 4-dimensional submanifold in $\mathbb{R}^6$ with nilpotent transvection group).

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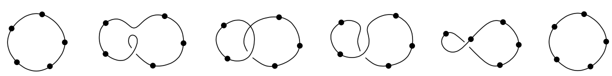
An action of the cactus group

Andre Henriques

Let $\overline{M}_{0,n}(\mathbb{R})$ denote the Deligne-Mumford compactification of the moduli space of real curves of genus zero with n marked points. Its points are the isomorphism classes of stable real curves of genus zero, that is, curves obtained by glueing $\mathbb{RP}^1$'s in a tree-like way, and such that each irreducible component has at least 3 special points. Let $[\overline{M}_{0,n+1}(\mathbb{R})/S_n]$ denote the quotient orbifold of $\overline{M}_{0,n+1}(\mathbb{R})$ by the action permuting the first n marked points. In [3], J. Kamnitzer and the author showed that the *cactus group* $J_n := \pi_1([\overline{M}_{0,n+1}(\mathbb{R})/S_n])$ acts on tensor powers of Kashiwara crystals in a way similar to how the braid group acts on tensor powers of quantum group representations.

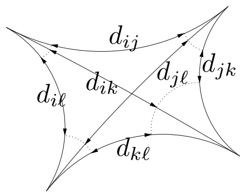
The *big cactus group* J_n' is the fundamental group of $[\overline{M}_{0,n}(\mathbb{R})/S_n]$. It fits into a short exact sequence $0 \to \pi_1(\overline{M}_{0,n}(\mathbb{R})) \to J_n' \to S_n \to 0$, and its elements can be represented by movies, such as the following one:



Let $\mathcal{F}\ell_m := \left(\begin{smallmatrix}1 & & * \\ & \ddots & \\ & & 1\end{smallmatrix}\right)\backslash SL_m$ be the variety of flags $0 \subset V_1 \subset \cdots \subset V_{m-1} \subset \mathbb{R}^m$, equipped with volume forms $\omega_i \in \Lambda^i V_i$. The goal of this note is to construct an action of J_n' on the totally positive part $\mathcal{A}(n)_{>0}$ of the variety $\mathcal{A}(n) := (\mathcal{F}\ell_m)^n/SL_m$. The space $\mathcal{A}(n)_{>0}$ is a certain connected component of the locus $\mathcal{A}(n)_{reg} \subset \mathcal{A}(n)$, where the flags are in generic position. One gets similar actions on $\big((N\backslash G)^n/G\big)_{>0}$ for other reductive groups G.

The space $\mathcal{A}(n)_{>0}$ was introduced by Fock and Goncharov [1]. For $m = 2$, it agrees with the Teichmüller space of decorated ideal n-gons, that is, the space of isometry classes of hyperbolic n-gons with geodesic sides, vertices at infinity, and horocycles around each vertex. It is also an example of a cluster variety, i.e.

it comes with special sets of coordinate systems, whose transition functions are given by *cluster exchange relations* [2]. For $m = 2$, the coordinates are due to Penner [4]. To each pair i, j of vertices of the n-gon, he associates the quantity $\Delta_{ij} := \exp(\frac{1}{2}d_{ij})$, where d_{ij} denotes the hyperbolic length between the intersection points of the horocycles around i and j, and the geodesic from i to j. These coordinates are then subject to the following exchange relations [4]:

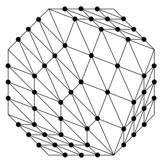
(1) 

$$\Delta_{j\ell} = \frac{\Delta_{ij}\Delta_{k\ell} + \Delta_{jk}\Delta_{i\ell}}{\Delta_{ik}}.$$

For general m, the coordinates on $\mathcal{A}(n)$ are indexed by tuples $(i_1, \ldots, i_n) \in \mathbb{N}^n$ whose sum equals m, and such that at least two entries are non-zero. The coordinate $\Delta_{i_1 \ldots i_n}$ then assigns to $((V^1_\bullet, \omega^1_\bullet), \ldots, (V^n_\bullet, \omega^n_\bullet)) \in (\mathcal{F}\ell_m)^n$ the ratio of $\omega^1_{i_1} \wedge \cdots \wedge \omega^n_{i_n}$ with the standard volume form on $\mathbb{R}^m$. These coordinates satisfy

$$\Delta_{\ldots i \ldots j \ldots k \ldots \ell \ldots} = \big(\Delta_{\ldots i+1 \ldots j \ldots k \ldots \ell-1 \ldots} \cdot \Delta_{\ldots i \ldots j-1 \ldots k+1 \ldots \ell \ldots} + \Delta_{\ldots i \ldots j \ldots k+1 \ldots \ell-1 \ldots} \cdot \Delta_{\ldots i+1 \ldots j-1 \ldots k \ldots \ell \ldots}\big) / \Delta_{\ldots i+1 \ldots j-1 \ldots k+1 \ldots \ell-1 \ldots}\ ,$$

which generalizes (1). Let $\mathcal{A}(n)_{>0}$ be the locus where all the Δ's are > 0. It is a space isomorphic to $\mathbb{R}_{>0}^{(n-2)\cdot\binom{m+1}{2}+(m+1)-n}$, and each triangulation of the n-gon provides such an isomorphism [1]. More precisely, the isomorphism corresponding to a triangulation is given by the coordinates $\Delta_{0 \ldots 0i0 \ldots 0j0 \ldots 0k0 \ldots 0}$, where i, j, k are located at the vertices of the triangles. For example, for $n = 8$, $m = 4$, and the triangulation  of the 8-gon, the corresponding coordinates on $\mathcal{A}(n)_{>0}$ are in natural bijection with the bullets in the following figure:

(2) 

We now explain a general machine for producing actions of J'_n on various spaces. Suppose that we are given two manifolds $X_\triangle$ and X_I, equipped with maps

$$r \circlearrowleft X_\triangle \overset{d_1}{\underset{d_3 = d_0}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{d_2}}}}} X_I \circlearrowright \iota \tag{3}$$

subject to the relations $r^3 = 1$, $\iota^2 = 1$, and $d_i \circ r = r \circ d_{i-1}$. Such data can then be reinterpreted as a contravariant functor $X_\bullet : \mathcal{C} \to \{\text{manifolds}\}$ from the category $\mathcal{C} := \{ \circlearrowleft \triangle \leftleftarrows I \circlearrowright \}$, whose two objects are the oriented triangle "$\triangle$" and the unoriented interval "I", and whose morphisms are the obvious embeddings and automorphisms. Let $\widehat{\mathcal{C}}$ be the category whose objects are the 2-dimensional finite simplicial complexes with oriented 2-faces and connected links, and whose morphisms are the embeddings. There is an obvious inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$, and every object of $\widehat{\mathcal{C}}$ can be written essentially uniquely as the colimit of a diagram in $\mathcal{C}$. Assuming $d_1 \times d_2 \times d_3 : X_\triangle \to X_I^3$ is a submersion, then there is a unique extension of $X_\bullet$ to $\widehat{\mathcal{C}}$ sending colimits to limits. For example, using that extension, we get $X_{\boxslash} \cong X_\triangle \times_{X_I} X_\triangle$.

Theorem 1. *Let $X_\bullet$ be a functor as above, and denote by the same letter its canonical extension to $\widehat{\mathcal{C}}$. Suppose that we are given isomorphisms*

$$\tau : X_{\boxslash} \to X_{\boxbslash} \qquad \textit{and} \qquad \theta : X_\triangle \to X_\triangle$$

making the following diagrams commute:

1) $$\begin{array}{ccc} X_{\boxslash} & \xrightarrow{\tau} & X_{\boxbslash} \\ & {\scriptstyle \prod^4 d_i'} \searrow \quad \swarrow {\scriptstyle \prod^4 d_i''} & \\ & X_I^4 & \end{array}$$
where $d_i' : X_{\boxslash} \to X_I$, $d_i'' : X_{\boxbslash} \to X_I$, $i = 1..4$, are induced by the four face inclusions $I \hookrightarrow \Box$.

2) $$\begin{array}{ccc} X_{\boxslash} & \xrightarrow{\tau} & X_{\boxbslash} \\ {\scriptstyle 1/2}\downarrow & & \downarrow{\scriptstyle 1/2} \\ X_{\boxslash} & \xrightarrow{\tau} & X_{\boxbslash} \end{array}$$
where $1/2 : X_{\boxslash} \to X_{\boxslash}$ and $1/2 : X_{\boxbslash} \to X_{\boxbslash}$ are induced by half turn rotation of the square.

3) $$\begin{array}{ccc} X_{\boxslash} & \xrightarrow{\tau} & X_{\boxbslash} \\ {\scriptstyle 1/4}\downarrow & & \downarrow{\scriptstyle 1/4} \\ X_{\boxbslash} & \xleftarrow{\tau} & X_{\boxslash} \end{array}$$
where $1/4 : X_{\boxslash} \to X_{\boxbslash}$ and $1/4 : X_{\boxbslash} \to X_{\boxslash}$ are induced by rotation by a quarter turn.

4) $$\begin{array}{ccccc} & X_{\pentagon} & \xrightarrow{\tau\times 1} & X_{\pentagon} & \\ {\scriptstyle 1\times\tau}\nearrow & & & & \searrow{\scriptstyle 1\times\tau} \\ X_{\pentagon} & \xrightarrow{\tau\times 1} & X_{\pentagon} & \xrightarrow{1\times\tau} & X_{\pentagon} \end{array}$$
note that "$1 \times \tau$" and "$\tau \times 1$" only become well defined once we have axioms 1) and 2).

5) $d_i \circ \theta = d_{4-i}$

6) $\theta \circ r = r^{-1} \circ \theta$

7) $\theta^2 = 1$,

8) $$\begin{array}{ccc} X_{\boxslash} & \xrightarrow{\tau} & X_{\boxbslash} \\ {\scriptstyle \theta\times\theta}\downarrow & & \downarrow{\scriptstyle \theta\times\theta} \\ X_{\boxslash} & \xrightarrow{\tau\circ 1/2} & X_{\boxbslash} \end{array}$$

then there is a natural action of J_n' on the manifold that $X_\bullet$ associates to a triangulated n-gon. (For example, one gets an action of J_8' on $X_{\octagon}$).

We now use the above theorem to equip $\mathcal{A}(n)_{>0}$ with a J_n' action. Indeed, the manifolds $X_\triangle := \mathcal{A}(3)_{>0}$ and $X_I := \mathcal{A}(2)_{>0}$ fit into a diagram (3), and so provide a functor $\widehat{\mathcal{C}} \to \{\text{manifolds}\}$. The space associated to a triangulated n-gon is $\mathcal{A}(n)_{>0}$, as can be seen from the parameterization (2). We let τ be the composite

$$\tau : X_{\boxslash} \xrightarrow{\sim} \mathcal{A}(4)_{>0} \xrightarrow{\sim} X_{\boxbslash},$$

and θ be the map sending $(F_1, F_2, F_3) \in (\mathcal{F}\ell_m)^3$ to $(F_3^\perp, F_2^\perp, F_1^\perp)$, where the orthogonal of a flag F is given by $(V_1, \ldots, V_{m-1})^\perp := (V_{m-1}^\perp, \ldots, V_1^\perp)$, along with $\pm$ the obvious volume forms. The axioms *1) – 8)* are then easy to check.

Both τ and θ are composites of cluster exchange relations. But the action of J'_n on $\mathcal{A}(n)_{>0}$ is not cluster (it doesn't satisfy the Laurent phenomenon; it doesn't preserve the canonical presymplectic form). The reason is that θ is actually the composite of a cluster map with an automorphism that negates the cluster matrix. In particular, it negates the presymplectic form.

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On homological phase space reduction

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(joint work with Martin Bordemann, Markus Pflaum)

It is known [1] that the BFV-method can be succesfully employed to construct star products on symplectic reduced spaces obtained by regular Marsden-Weinstein reduction. We will see that the method, suitably modified, does also work for certain cases of singular reduced spaces (these results have been published in [2, 3]). In proving that we are able to greatly simplify the the argument of [1] by systematically using the language of homological perturbation theory (see e.g. [4] and references therein). To be more specific, the method applies to Hamiltonian actions of a compact, connected Lie group G on a symplectic manifold (M, ω) with moment map $J : M \to \mathfrak{g}^*$ such that

(1) the components of J generate the ideal I_Z of the zero fibre $Z := J^{-1}(0)$,
(2) the Koszul complex $K_\bullet = K_\bullet(\mathcal{C}^\infty(M), J)$ on J (over the ring $\mathcal{C}^\infty(M)$) is acyclic.

There are several examples of singular moment maps which fulfill these requirements (for regular value $0 \in \mathfrak{g}^*$ of J they are plainly fulfilled). Among them the commuting variety of [5] and the $(1, 1, -1, -1)$-resonance of [6], the latter is known to have a nonorbifold quotient. In order to check condition (1) we use the techniques developed in [7] based on the normal coordinates for J and thereby reduce the question to a basic problem in real algebraic geometry. If condition (1) is true, then it is easy to check whether the Koszul complex is exact. In fact, it is sufficient that $Z_r := \{z \in Z \mid T_z J \text{ is onto}\}$ is dense in Z. This can be proven by using the faithful flatness of the ring of germs of smooth functions over the subring of germs of analytic functions and a theorem of Vasconcelos [8, Theorem 19.9]. In the above situation the Koszul complex on J is a dg commutative algebra resolution of the commutative algebra $\mathcal{C}^\infty(Z)$. Moreover, we may augment the Koszul resolution by the restriction map $\mathrm{res} : K_0 = \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(Z)$. As a final preparatory step, we observe that the augmented Koszul complex (which is exact) admits continuous contracting homotopies: $\mathrm{prol} : \mathcal{C}^\infty(Z) \to K_0 = \mathcal{C}^\infty(M)$ and $h_i : K_i \to K_{i+1}$. This can be shown using the extension theorem and the splitting theorem of Bierstone and Schwarz [9]. If Z is singular one cannot expect the prolongation map prol to be multiplicative. We can summarize what we have done so far using the language of homological perturbation theory:

$$\big(\mathcal{C}^\infty(Z), 0\big) \underset{\mathrm{prol}}{\overset{\mathrm{res}}{\leftrightarrows}} \big(K, \partial\big), h \tag{1}$$

is a *contraction* in the category of complexes of Fréchet spaces, such that res is a map of dg commutative algebras.

Out of the Koszul complex one now constructs a dg Poisson algebra $(\mathscr{A}, \{\,,\}, \mathcal{D} = \{\theta, \,\})$ which is called the *BRST-algebra*. This is done in a standard fashion. First one adjoins variables (of degree 1) which are dual to the Koszul generators (which are now of degree -1) and extends the Poisson structure appropriately. The differential $\mathcal{D} = \{\theta, \,\}$ is the inner derivation corresponding the *charge* θ, which has incorporated in the bracket of $\mathfrak{g}$ and the moment map J. It turns out that $\mathcal{D}$ is the sum of the Koszul differential ∂ and the Lie algebra cohomology differential δ of the $\mathfrak{g}$-modulde K. Using a version of the basic perturbation lemma we obtain a contraction

$$\Big(C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big), d\Big) \underset{\Phi}{\overset{\mathrm{res}}{\leftrightarrows}} (\mathscr{A}^\bullet, \mathcal{D}), H \tag{2}$$

in the category of (bounded) complexes of Fréchet spaces. Here, the left hand side is the complex of Lie algebra cochains of the $\mathfrak{g}$-module $\mathcal{C}^\infty(Z)$. Since res is multiplicative we can transfer the Poisson algebra structure along this contraction and obtain a $\mathbb{Z}$-graded Poisson algebra structure on the Lie algebra cohomology $\mathrm{H}^\bullet(\mathfrak{g}, \mathcal{C}^\infty(Z))$. Note that the subalgebra $\mathrm{H}^0(\mathfrak{g}, \mathcal{C}^\infty(Z))$ can be canonically identified with the Poisson algebra of smooth function on the singular reduced space.

In order to find a continuous star product for the reduced Poisson algebra we proceed as follows. First we construct (again in a standard fashion [1]) a deformation quantization of the right hand side of (2). This is a dg associative $\mathbb{K}[[\nu]]$-algebra $(\mathscr{A}[[\nu]], *, \mathcal{D}_\nu)$, where $\mathcal{D}_\nu$ is a quasi-inner derivation $\nu^{-1}\,\mathrm{ad}_*(\theta_\nu)$ corresponding to *charge* θ_ν. By construction, the charge is made up from the bracket of $\mathfrak{g}$ and a *quantum moment map* $J_\nu : \mathfrak{g} \to \mathcal{C}^\infty(M)[[\nu]]$ (and an unimportant trace term due to operator ordering). Our main result is a deformation of contraction (2). There are formal series of continuuos linear maps d_ν, res_ν, Φ_ν and H_ν wich deform d, res, Φ and H such that

$$(3) \qquad \Big(C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]\big), d_\nu\Big) \underset{\Phi_\nu}{\overset{\mathrm{res}_\nu}{\leftrightarrows}} (\mathscr{A}^\bullet[[\nu]], \mathcal{D}_\nu), H_\nu$$

is a contraction. The proof is essentially done by repeatedly applying two incarnations of the basic perturbation lemma to contraction (1). In principle, we can transfer the dg associative algebra on the right hand side of (3) to the cohomology of the left hand side. Unfortunately, the cohomology of the deformed differential d_ν (which turns to be the Lie algebra cohomology differential of a certain deformed representation of $\mathfrak{g}$ on $\mathcal{C}^\infty(Z)[[\nu]]$) is in general not just $\mathrm{H}^\bullet(\mathfrak{g}, \mathcal{C}^\infty(Z))[[\nu]]$. One way out is to take from the beginning a quantum moment map which corresponds to an *invariant star product* on M. In this case one can show that $d_\nu = d$ and we obtain a $\mathbb{Z}$-graded associative algebra structure on $\mathrm{H}^\bullet(\mathfrak{g}, \mathcal{C}^\infty(Z))[[\nu]]$ deforming the bracket of (2). In particular, in zeroth cohomology this yields a *continuous star product* on the singular reduced space. Another way out is to use the fact that $\mathrm{H}^1(\mathfrak{g}, \mathcal{C}^\infty(Z)) = 0$ for $\mathfrak{g}$ semisimple (this can be shown using [10, Theorem 2.13]). In this case, there is a topological linear isomorphism between the invariants of the classical and the deformed $\mathfrak{g}$-module structures on $\mathcal{C}^\infty(Z)[[\nu]]$.

Finally, let us discuss what happens if one drops assumption (2), i.e., if the Koszul complex is not exact. There are prominent examples for this to happen, e.g., the system of one particle with zero angular momentum in dimension ≥ 3. The standard proposal of the physicists (see e.g. [11]) is to replace the Koszul complex on the right hand side of contraction (1) by what they call a Koszul-Tate resolution (these resolutions are also traded under the name Tate-resolution or dg resolvent). Such a resolution is constructed inductively by adjoining free supercommuting variables in order to kill homology degreewise. What seems to be not so well known is the fact that the adjunction process does not terminate for non complete intersection singularities. For the example above one knows that the number of variables grows exponentially with the degree. The author does not know of any example of a moment map where the Koszul-Tate resolution has been computed to all degrees. It is known that in the Koszul-Tate case one can still construct an analogue of contraction (2) in order to provide a dg Poisson model for the reduced algebra (see e.g. [11, 3]). However, the complex on the left hand side is considerably larger than the Chevalley-Eilenberg complex. The question of quantization of the charge, which in the Koszul-Tate case turns out to be a huge object, is unsolved (for more details see [3]).

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Modular classes of Lie algebroids: recent results

Yvette Kosmann-Schwarzbach

We present recent results and work in progress on the modular classes and relative modular classes of Lie algebroids, a report mainly based on joint papers with Camille-Laurent Gengoux [7], Alan Weinstein [10], Milen Yakimov [11] and Franco Magri [9].

On a Poisson manifold, given a volume form, the map which associates to a function the divergence of the corresponding hamiltonian vector field is a derivation, i.e., a vector field, called a *modular vector field*. It is a 1-cocycle in the Lichnerowicz-Poisson cohomology, and its class, called the *modular class*, is independent of the volume form [12] [15]. If the manifold is not orientable, densities must be used instead of volume forms.

Evens, Lu and Weinstein [4] introduced the notion of a *modular class of a Lie algebroid*, and they observed that the modular class of a Poisson manifold is one-half that of its cotangent Lie algebroid.

It is straightforward [6] to extend the notion of modular class from the case of a Poisson manifold to that of a Lie algebroid A with a Poisson structure, i.e., a section π of $\wedge^2 A$ such that $[\pi,\pi]_A = 0$, where $[\ ,\ ]_A$ is the Schouten-Nijenhuis bracket on $\Gamma(\wedge^\bullet A)$ defining its Gerstenhaber structure. The question that then arises is how to determine what relation exists in general between the modular class $\theta(A,\pi)$ and the modular class $\mathrm{Mod}(A^*)$ of the dual A^* of A equipped with the Lie algebroid structure defined by π.

In order to solve this problem, the notion of *relative modular class*, which also appears in [5] under the name of *modular class of a morphism*, was introduced in [10]. If $\Phi : E \to F$ is a morphism of Lie algebroids over the same base, then $\wedge^\bullet\Phi^*$ is a chain map from the complex $\Gamma(\wedge^\bullet F^*)$ of the Lie algebroid F to the complex $\Gamma(\wedge^\bullet E^*)$ of the Lie algebroid E . Therefore $\mathrm{Mod}\, E - \Phi^*(\mathrm{Mod}\, F)$ is a cohomology class in the Poisson cohomology of E. This is the relative class, denoted $\mathrm{Mod}^\Phi(E, F)$. Then the relation

$$\theta(A,\pi) = \frac{1}{2}(\mathrm{Mod}(A^*) - (\pi^\sharp)^* \mathrm{Mod} A)$$

is valid in general and, since $\mathrm{Mod}(TM) = 0$, it reduces to the fact recalled above in the case of Poisson manifolds. The relative modular classes of general, not necessarily base-preserving, morphisms are treated in [8].

A *twisted Poisson structure*, also called Poisson structure with background [14], on a Lie algebroid A is a pair (π, ψ), where ψ is a 3-cocycle on A and

$$\frac{1}{2}[\pi,\pi]_A = (\wedge^3 \pi^\sharp)\psi \ .$$

A representative of the modular class $\theta(A,\pi) = \frac{1}{2}(\mathrm{Mod}(A^*) - (\pi^\sharp)^* \mathrm{Mod} A)$ is $X+Y$, with $i_X\lambda = -d_A i_\pi \lambda$, where λ is a section of $\wedge^{\mathrm{top}} A^*$, and $Y = \pi^\sharp i_\pi \psi$ [7].

In the *spinor approach* to Poisson and Dirac structures [1] [13], the modular field appears as the obstruction to the existence of a pure spinor defining the graph of π which is closed in the Lie algebroid cohomology of A. This fact extends to the twisted case, replacing d_A by $d_A + \epsilon_\psi$, where ϵ_ψ is exterior product of forms by ψ.

In the case of a *regular* Poisson or twisted Poisson structure, the modular class can be computed in terms of the characteristic class of a representation of the image of $\pi^\sharp$ on the top exterior power of its kernel [11]. This result extends to Lie algebroid extensions with unimodular kernel [8].

These definitions and properties can be applied to Lie algebras, considered as Lie algebroids over a point, whence the notion of *twisted triangular r-matrix*. In [11], we obtained a formula for the modular class of a Lie algebra equipped with a twisted triangular r-matrix in terms of the infinitesimal character of the adjoint representation of $\mathfrak{p}$ in $\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{p}$ is the carrier of the r-matrix, i.e., its image in the Lie algebra. When the carrier of the r-matrix is a Frobenius Lie algebra with respect to a 1-form ξ, the modular class is the unique element X in $\mathfrak{p}$ such that $\mathrm{ad}^*_X \xi$ is equal to the above character. This method is applied to the computation of the class defined by the Gerstenhaber-Giaquinto r-matrix on $\mathfrak{sl}(n,\mathbb{R})$.

Other examples of modular classes appear in the theory of *Poisson-Nijenhuis manifolds* [3] [9]. When a Poisson tensor π and a Nijenhuis tensor N on a manifold are *compatible*, there is a hierarchy of vector fields, $NX^{k-1} - X^k$, $k \geq 1$, where X^k is a modular vector field for the k-th Poisson structure $N^k\pi$, which are cocycles in the Poisson cohomology defined by $N^k\pi$, and independent of the choice of a volume form. Up to a factor of one-half, these modular vector fields coincide with the well-known hierarchy of commuting hamiltonian vector fields defined on a Poisson-Nijenhuis manifold. This construction has been generalized to Lie algebroids with a Poisson-Nijenhuis structure [2].

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Poisson manifolds of compact type

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(joint work with Marius Crainic and Rui Loja Fernandes)

A Poisson structure on a manifold M is a bivector $\Pi \in \mathfrak{X}^2(M)$ closed under the Schouten bracket. Any manifold carries a Poisson structure supported on an arbitrary open set, which means that there is no link between the topology of a manifold and the existence of Poison structures on it. Global questions in Poisson geometry are rather difficult to answer; invariants of a Poisson structure (M, Π), such as the Poisson cohomology groups $H_\Pi(M)$, are in practice of little use.

The complexity of Poisson geometry is already seen at a the local level, due to the huge set of local models. Therefore, it is natural to try to single out a class of Poisson manifolds for which a *Poisson topology* can be developed. By this we mean that the Poisson structure should be linked to the topology of the ambient manifold, this relation resulting in a better behavior of the global Poisson invariants, and constraining the possible (semi)-local models and the way in which these can be glued in order to yield the Poisson structure.

Recall that a Poisson structure (M, Π) induces a Lie algebroid structure $[\cdot,\cdot]_\Pi$ on T^*M. It seems natural to constraint a Poisson structure by putting extra requirements on its induced Lie algebroid structure.

A Poisson structure is called *integrable* if the induced Lie algebroid integrates into a Lie groupoid. If this is the case, the canonical integration $\Sigma(M) \rightrightarrows M$ is endowed with a canonical (multiplicative) symplectic form Ω, via symplectic reduction in an infinite dimensional Banach manifold [2]. Therefore, a natural question is to study under which conditions one can bypass the infinite dimensional setting.

Integrability of a Poisson structure can be understood as a constraint on the *variation of symplectic areas of spheres.* This is better seen in the regular case [2]: for a regular Poisson manifold (M, Π) the isotropy Lie algebra $\mathfrak{g}_x$ at any point $x \in M$ is abelian, its dual being identified with the normal space ν_x. The monodromy lattice $\mathcal{N}_x$ is the image of the map

$$\partial\colon \pi_2(\mathcal{O}_x) \to \mathfrak{g}_x \tag{1}$$

sending each homotopy class $[S]$ in the symplectic leaf $\mathcal{O}_x$ based at x to

$$v \mapsto \frac{d}{dt}\left(\int_{S^2} S^*_{t,v}\omega_t\right)_{|t=0}, \ v \in g^*_x = \nu_x, \tag{2}$$

where $S_{t,v}\colon S^2 \to M$ is a smooth deformation of $S = S_0$ so that each $S_{v,t}$ maps the sphere into a symplectic leaf $(\mathcal{O}_t, \omega_t)$, in such a way that the base point (the north pole say) is sent to a curve representing $v \in \nu_x$.

Since integrability amounts to uniform discreteness of the monodromy lattices [1], we then conclude its relation with the variation of symplectic areas of spheres (and a similar interpretation holds in the non-regular case, as long as the isotropy Lie algebras are of compact type).

Definition 1. A Poisson manifold (M, Π) is said to be of compact type (PMCT) if the Lie algebroid $(T^*M, [\cdot,\cdot]_\Pi)$ is integrable, and its canonical integration $\Sigma(M) \rightrightarrows M$ is a (Hausdorff) compact manifold.

The Poisson manifold is called *proper* if $\Sigma(M) \rightrightarrows M$ is a (Haussdorf) proper manifold. Properness is a semi-local property, meaning that it holds if and only if for some cover of M by open saturated subsets (i.e., one coming from a cover of the leaf space), the induced Poisson structures on all subsets are proper.

If (M, Π) is proper then the isotropy groups G_x, $x \in M$, are compact, the symplectic leaves have finite fundamental group and the leaf space $\underline{M}$ is Haussdorf (see [5]). Moreover the following remarkable linearization result holds:

Theorem 2. [5] *Let* (M,Π) *be a proper Poisson structure, with* $s\colon \Sigma(M)\to M$ *a locally trivial fibration. Let* $x\in M$ *and* Σ_x *a small enough slice to* $\mathcal{O}_x$ *through* x, *open and saturated w.r.t to the transverse Poisson structure. Then*

(1) $s^{-1}(\Sigma_x)$ *is a symplectic submanifold of* $(\Sigma(M),\Omega)$, *the* G_x*-action on* $s^{-1}(x)$ *extends to a Hamiltonian free action on* $(\Sigma_x,\Omega_{\Sigma_x})$, *and there is a Poisson isomorphism*

$$\phi\colon (\mathcal{U},\Pi_{\mathcal{U}})\to (s^{-1}(x)\times_{G_x}\mathfrak{g}^*/G_x,\Pi_{\rm red}),$$

where $\mathcal{U}$ *is the saturation of* Σ_x *w.r.t the Poisson structure, and the target of* ϕ *is (a saturated neighborhood of the zero section of) the reduced Poisson space.*

(2) *The leaf space carries an integral affine structure.*

Notice that the above result gives in particular semi-local models for PMCT, the data being $P\to(\mathcal{O},\omega)$ a principal bundle over a symplectic manifold so that P is compact and $\pi_1(P)=\{1\}$. Moreover, one can show that infinite dimensional reduction to build $(\Sigma(M),\Omega)$ can be avoided.

Corollary 3. *The open subset of* $(\Sigma(M),\Omega)$ *integrating* $(\mathcal{U},\Pi_{\mathcal{U}})$ *is isomorphic to the symplectic reduction at zero of the* G_x*-Hamiltonian space*
$(s^{-1}(\Sigma_x)\times s^{-1}(\Sigma_x),\Omega_{\Sigma_x}\oplus -\Omega_{\Sigma_x})$.

Being of compact type has global consequences for the Poisson structure, as implied by the following result:

Theorem 4. *If* (M,Π) *is a PMCT then*

(1) $[\Pi]\in H^2_\Pi(M)$ *is non vanishing.*

(2) *The Poisson tensor cannot have zeroes (0 dimensional symplectic leaves)*

Notice for example that the linear Poisson structure on the dual of a Lie algebra semi-simple of compact type is proper, but none of the results of theorem 4 hold for it.

It is easy to see that a symplectic manifold is of compact type if and only if it is compact and has finite fundamental group.

Regarding PMCT which are not symplectic, one has the following

Theorem 5. *There is a one to one correspondence between regular PMCT of rank one all whose leaves are simply connected, and* S^1 *quasi-Hamiltonian spaces* (X,ω_X,S^1,μ) *with free action and* $\pi_1(X)=\{1\}$ *(the latter under a suitable equivalence relation). Moreover, the symplectic groupoid integrating the Poisson manifold is built by globalizing the construction in corollary 3, hence avoiding infinite dimensional reduction.*

Examples of Poisson manifolds as in theorem 5 are difficult to construct. The total space M could be described as a mapping torus $Y\times[0,1]/\varphi$, with symplectic leaves (Y_θ,ω_θ), $\theta\in[0,1]/0\sim 1$, and the affine structure on $\underline{M}=[0,1]/0\sim 1$ given by the one on the interval. One observes that (i) the cohomology classes $[\omega_\theta]$ should vary linearly with slope ξ say, since (Y_θ,ω_θ) are the reduced spaces of a S^1-Hamiltonian space and Duistermaat-Heckman theorem applies, (ii) ξ should be non-trivial because otherwise according to equations 1 and 2 the monodromy lattice would be trivial and hence the isotropy groups non-compact, and (iii) $[\omega_\theta]^d \xi^{\dim Y/2-d} = 0$, $d \geq 1$, due to the non-variation of the symplectic volumes of the leaves.

In [4] -addressing a problem in Hamiltonian actions- a PMCT as in theorem 5 is constructed by taking Y to be the K3 surface with an appropriate return map φ. The construction of the leafwise symplectic form is rather delicate and uses very involved results of global analysis.

Theorem 5 can be used to construct new examples out the one of Kotschick, by crossing the associated quasi-Hamiltonian S^1-space of theorem 5 with appropriate Hamiltonian S^1-spaces.

Questions such as the existence (or non-existence) of non-regular PMCT, further study of global properties of PMCT, and the analysis of other notions of "compactness" (e.g. having a compact integration) are the subject of ongoing research [3].

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Rigidity of Poisson group actions

Eva Miranda

(joint work with Philippe Monnier and Nguyen Tien Zung)

In this talk we first review some classical results of rigidity for group actions of compact Lie groups on smooth manifolds and then we prove some rigidity results in the case the group action preserves a Poisson structure. The details of these proofs can be found in the paper [11] and the preprint [10].

In the general case of actions of compact Lie groups on smooth manifold there are two well-known results that entail rigidity. The first one is the theorem of Bochner [1] that says that actions of compact Lie groups can be linearized in a neighbourhood of a fixed point for the action. The second one is the theorem of Palais [12], that establishes that C^1-close actions of compact Lie groups are conjugated via a diffeomorphism close to the identity.

1. Local rigidity in a neighbourhood of a fixed point

Let (P, Π) be a Poisson manifold and let ρ stand for a Poisson action of a compact Lie group G. Ginzburg proved in [6] that Poisson actions are rigid by deformations. In fact using the proof of rigidity by deformations provided in the book [7], we can prove rigidity by deformations for actions preserving additional structures.

In the case when we are not given a path of actions connecting both actions and preserving the Poisson structure, the first attempt is to try to use Moser's path method as can be done in the symplectic case (see [14] and [2]).

Unlike the symplectic case, the path method does not seem to work so well for Poisson structures. Locally, we can construct paths using the smooth geometric data given by the theorem of Vorobjev [13] associated to the Poisson structure.

In order to guarantee that the geometric data are smooth we need to assume an additional hypothesis on the Poisson structure at a point, which we call tameness [11].

In [11] this tameness condition is studied and several examples of tameness and non-tameness are given. In particular, all 2 and 3-dimensional Lie algebras are tame Poisson structures and all semisimple Lie algebras of compact type are tame.

For this class of Poisson structures, we can find Weinstein's splitted coordinates [15] for the Poisson structure such that the group action is locally linear as it is proven in [11]. Namely,

Theorem 1. *Let (P^n, Π) be a smooth Poisson manifold, p a point of P, $2k =$ rank Π(p), and G a compact Lie group which acts on P in such a way that the action preserves Π and fixes the point p. Assume that the Poisson structure Π is tame at p. Then there is a smooth canonical local coordinate system $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_{n-2k})$ near p, in which the Poisson structure Π can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1}$$

and in which the action of G is linear and preserves the subspaces $\{x_1 = y_1 = \ldots x_k = y_k = 0\}$ and $\{z_1 = \ldots = z_{n-2k} = 0\}$.

We can combine this result with Conn's linearization theorem [5] for semisimple Lie algebras of compact type, to obtain the following equivariant linearization result (also contained in [11]).

Theorem 2. *Let (P^n, Π) be a smooth Poisson manifold, p a point of P, $2r =$ rank Π(p), and G a compact Lie group which acts on P in such a way that the action preserves Π and fixes the point p. Assume that the linear part of transverse Poisson structure of Π at p corresponds to a semisimple compact Lie algebra $\mathfrak{k}$. Then there is a smooth canonical local coordinate system*

$$(x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_{n-2r})$$

near p, in which the Poisson structure Π *can be written as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{2}$$

where c_{ij}^k *are structural constants of* $\mathfrak{k}$*, and in which the action of* G *is linear and preserves the subspaces* $\{x_1 = y_1 = \ldots x_r = y_r = 0\}$ *and* $\{z_1 = \ldots = z_{n-2r} = 0\}$*.*

2. Rigidity for close smooth actions.

In the case that we consider C^1-close Hamiltonian actions with moment maps $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$ on a compact Poisson manifold (M, Π), we can prove a rigidity result for the case the Lie algebra $\mathfrak{g}$ is semisimple of compact type. Two moment maps $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$ are C^1-close if they are close in the C^1-topology.

More precisely we can prove [10],

Theorem 3. *Let* $\mu_0 : M \longrightarrow \mathfrak{g}^*$ *and* $\mu_1 : M \longrightarrow \mathfrak{g}^*$ *be* C^1*-close moment maps with* M *compact and* $\mathfrak{g}$ *semisimple of compact type, then there exists a Poisson diffeomorphism* Φ *such that* $\mu_1 = \mu_0 \circ \Phi$*.*

The proof uses the inverse theorem of Nash and Moser [8] via the statement provided by Hamilton in [9] which uses exact sequences.

Roughly speaking, the result of Hamilton says that if a linear-complex defined on graded Fréchet spaces is locally exact via tame homotopy operators, then a non-linear complex which has this associated linear-complex is also locally exact via tame homotopy operators (for details and definitions about graded Fréchet spaces and this tameness condition see [8]).

In our case a Hamiltonian action induces on the set of smooth functions $\mathcal{C}^\infty(M)$ the structure of a a $\mathfrak{g}$-module. We can associate a Chevalley-Eilenberg complex to this $\mathfrak{g}$-module as explained in [3]. The space of cochains is given by multilinear alternating functions from $\mathfrak{g}$ to $\mathcal{C}^\infty(M)$ and the differential is that of Chevalley and Eilenberg [3].

Using an adaptation of a lemma of Conn [5] valid in the case $\mathfrak{g}$ is semisimple of compact type and a weak version of Sobolev lemma, we can prove that this Chevalley-Eilenberg is locally exact via homotopy operators that are tame. We then associate a non-linear complex to this complex and apply Hamilton's statement of Nash-Moser theorem to prove that the new complex is exact.

Exactness of the non-linear complex complex gives $\mu_1 = \mu_0 \circ \Phi$ where Φ is the time-1-map of a Hamiltonian vector field. So indeed Φ is not only a Poisson diffeomorphism but also a Hamiltonian diffeomorphism.

In the case M is not compact but $M = B_R$ is a ball centered at the origin of $\mathbb{R}^n$, we can prove the following result contained in [10]. The proof uses an iterative method inspired by Newton's method explained in [8] to define the equivalence of the nearby Hamiltonian actions.

Theorem 4. *Let* $\mu_0 : B_R \longrightarrow \mathfrak{g}^*$ *and* $\mu_1 : B_R \longrightarrow \mathfrak{g}^*$ *be two* C^1*-close moment maps with* $\mathfrak{g}$ *semisimple of compact type, then there exists a Poisson diffeomorphism* $\Phi : B_{R/2} \longrightarrow B_{R/2}$ *such that* $\mu_1 = \mu_0 \circ \Phi$.

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Orbifolds and their quantizations as noncommutative geometries

Hessel Posthuma

(joint work with M. Pflaum, X. Tang and H-H Tseng)

Orbifolds are a natural generalization of the concept of a manifold with a rich geometric structure. In particular, its orbifold cohomology has many surprising features, most notably a ring structure [2] generalizing, in a nontrivial way, the cup product on the cohomology of a manifold.

As a manifold with singularities, orbifolds can also be viewed as examples of noncommutative spaces. This point of view yields different cohomological tools, such as Hochschild and cyclic (co)homology. The aim of this project is to study the relation between both cohomologies, in particular its ring structure. For this, the

Hochschild cohomology is of particular importance, because it carries a graded product, known as the Yoneda or cup product. To actually relate this to the cohomology of the underlying obifold, we need a deformation quantization, for which we assume the existence of a symplectic structure. Such a space is known as a symplectic orbifold, they very often arise as symplectic quotients with respect to proper group actions on symplectic manifolds.

The set-up is as follows: Let X be an orbifold and G a proper étale groupoid modelling X. We denote the convolution algebra of G by A_{G}. In [3], its Hochschild cohomology was computed to be

$$H^\bullet(A_{\mathsf{G}}, A_{\mathsf{G}}) \cong \Gamma^\infty\left(\tilde{X}, \Lambda^{\bullet-\ell}T_{\tilde{X}} \otimes \Lambda^\ell N_{\tilde{X}}\right). \tag{1}$$

Here $\tilde{X}$ is the so-called inertia orbifold, a disconnected space whose connected components embed into X locally as fixed point sets for the local group actions. An atlas of this orbifold is provided by considering the space of loops

$$\mathsf{B}_0 := \{g \in \mathsf{G},\ s(t) = t(g)\},$$

on which G acts by conjugating loops. The vector bundles $T_{\tilde{X}}$ and $N_{\tilde{X}}$ are the tangent bundle and the normal bundle with respect to the local embedding into X. Finally $\ell : \tilde{X} \to \mathbb{N}$ is the locally constant function given by $\ell = \dim(N_{\tilde{X}})$.

To describe the cup-product on this space of "multivector fields", we need to introduce a third orbifold: consider the space

$$\mathsf{S} := \{(g_1, g_2) \in \mathsf{G}_1 \times \mathsf{G}_1,\ s(g_1) = t(g_1) = s(g_1) = t(g_1)\}.$$

There are three obvious maps $pr_1, pr_2, m : \mathsf{S} \to \mathsf{B}_0$ given by projection onto the first and second component, and multiplication of loops. Again, G acts by conjugating loops and the quotient orbifold is denoted by X_3. With this, the cup product on the Hochschild cohomology (1) is given by:

$$\xi \cup \eta := \int_m pr_1^*\xi \wedge pr_2^*\eta.$$

Here the integral means integration over the fiber of m, which is discrete, and the formula is ultimately understood on the level of germs; recall that G is étale.

There is an important subtlety hidden in the above formula for the cup-product. Recall that ξ and η are actually sections of exterior powers of the tangent bundle to X tensored with the determinant line of the normal bundle. This determinant has the effect that the wedge product is zero if the normal bundles $pr_1^*N_{\tilde{X}}$ and $pr_2^*N_{\tilde{X}}$ have a nontrivial intersection. In other words, the germ of the cup product $\xi \cup \eta$ at a point $x \in \tilde{X}$ is supported on the subset $y \in m^{-1}(x)$ for which

$$\ell(m(y)) = \ell(pr_1(y)) + \ell(pr_2(y)). \tag{2}$$

Next, we consider a formal deformation quantization $A^\hbar_{\mathsf{G}}$ of the convolution algebra A_{G} given by a G-invariant deformation quantization of G_0, which we assume to be symplectic. For this algebra, the Hochschild cohomology is given by

$$H^\bullet(A^\hbar_{\mathsf{G}}, A^\hbar_{\mathsf{G}}) \cong H^{\bullet-\ell}\left(\tilde{X}, \mathbb{C}((\hbar))\right).$$

This result extends [5] to the category of symplectic orbifolds. The proof uses the $\hbar$-filtration to identify the left hand side with the Poisson cohomology of $\tilde{X}$: since $\tilde{X}$ is symplectic- the symplectic form on X pulls back to a symplectic form on $\tilde{X}$- this yields the right hand side.

With this isomorphism, the cup-product is given on the level of differential forms by

$$\alpha \cup \beta := \int_{m_\ell} pr_1^*\alpha \wedge pr_2^*\beta, \tag{3}$$

where m_ℓ is the restriction of m to the sub-orbifold of points satisfying the condition (2). Notice that this condition implies that locally the spaces on which $pr_1^*\alpha$ and $pr_2^*\beta$ are supported, have a transversal intersection, so that the formula is well-defined.

The above formula for the cup product indeed does resemble the product defined in [2] on the cohomology of $\tilde{X}$, but there is an important difference: one easily checks that in the formula for that product, there is no condition on the fibers of m! Therefore one has deal with the non-transversal intersections as well.

To write down a similar formula for a product, but without the assumption (2), one can use the Thom isomorphism to take a wedge product in X to deal with the non-transversal intersections. To make the Thom form invertible, one uses equivariant cohomology with respect to the fiberwise S^1-action on $N_{\tilde{X}}$ associated to the choice of an almost complex structure. As a module over $H_{S^1}(pt.) = \mathbb{C}[t]$ we can complete the equivariant cohomology by taking the tensor product with $\otimes_{\mathbb{C}[t]}\mathbb{C}((t))$. Because the action of S^1 is trivial on $\tilde{X}$ we have of course that $H^\bullet_{S^1}(\tilde{X}) \otimes_{\mathbb{C}[t]} \mathbb{C}((t)) = H^\bullet(\tilde{X}, \mathbb{C}((t)))$. But the S^1 action is crucial for the construction of the ring structure.

If we denote by $Th_{\tilde{N}}$ the equivariant Thom form of the normal bundle $N_{\tilde{X}} \to \tilde{X}$, there is a natural product on $\Omega^\bullet(\tilde{X})((t))$ given by

$$\alpha \wedge_t \beta := \int_m \frac{pr_1^*(\alpha \wedge Th_{\tilde{N}}) \wedge pr_2^*(\beta \wedge Th_{\tilde{N}})}{pr_m^* Th_{\tilde{N}}}.$$

With this product, the equivariant cohomology is an associated graded ring, which turns out to be isomorphic to an equivariant version of Chen–Ruan's orbifold cohomology. In this sense, the above de Rham model is a non-abelian version of the model in [1].

The relation with the Hochschild ring is as follows: the equivariant orbifold cohomology ring has a natural filtration given by

$$\mathcal{F}^k := \{\alpha \in H^\bullet_{S^1}(\tilde{X}) \otimes_{\mathbb{C}[t]} \mathbb{C}((t)),\ \deg(\alpha) - \ell \geq k\}. \tag{4}$$

Taking the graded quotient with respect to this filtration enforces the condition (2), in which case we have $m_\ell^* N_{\tilde{X}} \cong pr_1^* N_{\tilde{X}} \oplus pr_2^* N_{\tilde{X}}$. In this case, the contributions of the equivariant Thom forms cancel, and the product reduces to (3). We therefore have:

Theorem 1. *The Hochschild cohomology ring of a deformation quantization of a symplectic orbifold with product* (3) *is canonically isomorphic to the graded quotient of the equivariant orbifold cohomology ring with respect to the filtration* (4)*.*

For details we refer to [4].

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Convexity in symplectic geometry

Tudor S. Ratiu

(joint work with Petre Birtea, Juan-Pablo Ortega)

This talk is based on some results in joint work with Petre Birtea and Juan-Pablo Ortega contained in [2, 3]. For background material see [10].

The convexity properties of the momentum map provide crucial information on the bifurcation behavior of symmetric Hamiltonian systems. The momentum polytope imposes severe a priori restrictions on the dynamics of such systems. The classical proofs of Atiyah [1] and Guillemin-Kirwan-Sternberg [6, 7, 9] rest on Morse theoretical arguments. In a remarkable paper, Condevaux, Dazord, and Molino [4] show that these convexity properties ultimately rely on pure point set topological arguments. Their approach has been formalized and extended by Hilgert, Neeb, and Plank [8] who formally introduced the "local-to-global principle" on which a large part of the proof of the convexity theorem rests.

In all these theorems the studied object is the momentum map of a symplectic Lie group action. However, the existence of the momentum map is not guaranteed. Worse, if one intends to extend these theorems to the category of Poisson manifolds, then the existence of the momentum map implies that the action necessarily preserves the symplectic leaves, which is a very strong hypothesis; there are many Poisson actions of Lie groups whose orbits cut across symplectic leaves. This is why it is of interests to study general symplectic actions and see what convexity properties are still present in this context.

Let (M,ω) be a connected symplectic manifold and G a Lie group acting properly and canonically on M. We denote by A_G the generalized distribution on M whose leaves are the G-orbits and by $A'_G := \{X_f \mid f \in C^\infty(U)^G, U \text{ open } G\cdot U \subset U\}$ its polar. Both A_G and A'_G are smooth integrable generalized distributions (in the sense of Stefan and Sussmann) and we have $A'_G(m) = (\mathfrak{g}\cdot m)^\omega \cap T_m M^m_{G_m}$, where $M^m_{G_m}$ is the connected component of the isotropy type submanifold $M_{G_m} := \{x \in M \mid G_x = G_m\}$ containing m, $\mathfrak{g}\cdot m$ is the tangent space at m to the orbit $G\cdot m$ and $(\mathfrak{g}\cdot m)^\omega$ is its ω-orthogonal complement in $T_m M$.

The **optimal momentum map** $\mathcal{J} : M \longrightarrow M/A'_G$ is defined as the canonical projection onto the leaf space of A'_G which is, in most cases, not even a Hausdorff topological space, let alone a manifold. This map has the **Noether property**: If $h \in C^\infty(M)^G$ is a G-invariant function, then $\mathcal{J}$ is conserved on the flow of X_h; this is why $\mathcal{J}$ is called a "momentum map".

The G-action on M induces a continuous action on M/A'_G given by $g\cdot\mathcal{J}(m) := \mathcal{J}(g\cdot m)$. This action is smooth in the sense of presheaf spaces. This is the unique G-action on M/A'_G that makes the optimal momentum map G-equivariant and it coincides with the usual smooth G-action on the leaf space of any distribution spanned by G-equivariant vector fields.

The optimal momentum map has the following universality property. For any map $\mathbf{K} : M \to P$, where P is any set, satisfying the Noether property there is a unique $\varphi : M/A'_G \to P$ such that $\mathbf{K} = \varphi \circ \mathcal{J}$.

Assume that the G-action admits a momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ with non-equivariance one-cocycle $\sigma : G \to \mathfrak{g}^*$, where $\sigma(g) := \mathbf{J}(g\cdot m) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(m)$; σ does not depend on $m \in M$ because M is connected. Let $\Theta : G\times\mathfrak{g}^* \to \mathfrak{g}^*$, $\Theta(g,\nu) := \mathrm{Ad}^*_{g^{-1}}\nu + \sigma(g)$ be the associated affine action and let G_μ be the isotropy subgroup of μ with respect to Θ. Then, if $\mathbf{J}(m) = \mu$ and $\mathcal{J}(m) = \rho$, we have $\mathcal{J}^{-1}(\rho) = (\mathbf{J}^{-1}(\mu)\cap M^m_{G_m})^m$, where $(\mathbf{J}^{-1}(\mu)\cap M^m_{G_m})^m$ denotes the connected component of $\mathbf{J}^{-1}(\mu)\cap M^m_{G_m}$ containing $m \in M$. In addition, $G_\rho = N_{G_\mu}(G_m)^{c(m)}$, where $N_{G_\mu}(G_m)^{c(m)}$ is the closed subgroup of $N_{G_\mu}(G_m) := N(G_m)\cap G_\mu$ consisting of all elements in $N_{G_\mu}(G_m)$ that leave $(\mathbf{J}^{-1}(\mu)\cap M^m_{G_m})^m$ invariant, and $N(G_m)$ denotes the normalizer of G_m in G. It turns out that for any $m \in M$, the intersection $\mathbf{J}^{-1}(\mathbf{J}(m))\cap M^m_{G_m}$ is an embedded submanifold of M, even though, in general, the level sets of the optimal momentum map are just initial submanifolds of M. From these results it follows that the reduction procedure can be carried out for the optimal momentum map, that is, $M_\rho := \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth symplectic regular quotient manifold whose symplectic form ω_ρ is characterized by

$$\pi_\rho^*\omega_\rho(m)(X_f(m), X_h(m)) = \{f,h\}(m),$$

for any $f,h \in C^\infty(M)^G$, where $\pi_\rho : \mathcal{J}^{-1}(\rho) \to M_\rho$ is the canonical projection. This formula is valid even if M is only a Poisson manifold, that is, the **optimal reduced spaces** (M_ρ, ω_ρ) are always symplectic. The usual corollaries of the reduction theorem, such as the reduction of dynamics and the Poisson bracket, hold in this case too.

There is a second momentum map that always exists for any symplectic action, namely the **cylinder valued momentum map** introduced in [4]. Its construction goes like this. Let $\pi : M\times\mathfrak{g}^* \to M$ be the trivial principal $(\mathfrak{g}^*,+)$-bundle relative to the action $\nu\cdot(m,\mu) := (m, \mu-\nu)$, with $m \in M$ and $\mu,\nu \in \mathfrak{g}^*$. Define the flat connection one-form $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ by

$$\langle \alpha(m,\mu)(v_m,\nu), \xi\rangle := (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi\rangle,\ v_m \in T_mM,\ \xi \in \mathfrak{g},\ \mu,\nu \in \mathfrak{g}^*.$$

For $(z,\mu) \in M \times \mathfrak{g}^*$, let $\widetilde{M} := (M \times \mathfrak{g}^*)(z,\mu) \subset M \times \mathfrak{g}^*$ be the holonomy bundle through (z,μ); $\widetilde{M}$ consists of all points in $M \times \mathfrak{g}^*$ that can be joined to (z,μ) by a horizontal curve. Let $\mathcal{H} := \mathcal{H}(z,\mu)$ denote the **holonomy group** of α with reference point (z,μ) which is an Abelian zero dimensional Lie subgroup of $(\mathfrak{g}^*, +)$ by the flatness of α. The Bundle Reduction Theorem guarantees that the principal bundle $((M \times \mathfrak{g}^*)(z,\mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z,\mu))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ and that the connection one-form α is reducible to a connection one-form on $(M \times \mathfrak{g}^*)(z,\mu)$. Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Then $C := \mathfrak{g}^*/\overline{\mathcal{H}} \cong \mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$, is a cylinder. Let $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}} = C$ be the quotient projection and $\widetilde{\mathbf{K}} : \widetilde{M} \to \mathfrak{g}^*$ the projection on the second factor. Define the **cylinder valued momentum map** $\mathbf{K} : M \to C$ by $\mathbf{K}(m) = \pi_C(\nu)$, where $\nu \in \mathfrak{g}^*$ is any element such that $(m,\nu) \in \widetilde{M}$. It is a strict generalization of the standard momentum map since the G-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In this case, the cylinder valued momentum map coincides with the standard momentum map. As expected, the choice of (z,μ) in the definition of $\mathbf{K}$ is irrelevant: any two cylinder valued momentum maps differ by a constant in C. Note also that the **Hamiltonian holonomy** $\mathcal{H}$ is the image of the **period homomorphism** $P_\omega : \pi_1(M,z) \to \mathfrak{g}^*$,

$$\langle P_\omega([\gamma]), \xi\rangle := \int_\gamma \mathbf{i}_{\xi_M}\omega,\ \text{for any } \xi \in \mathfrak{g}.$$

The cylinder valued momentum map has all the usual properties of a momentum map. For example, it is is conserved along the flow of the Hamiltonian vector field of any G-invariant function. It turns out that the annihilator $\mathrm{Lie}(\overline{\mathcal{H}})^\circ \subset \mathfrak{g}$ of the Lie algebra $\mathrm{Lie}(\overline{\mathcal{H}}) \subset \mathfrak{g}^*$ of the Lie group $\overline{\mathcal{H}}$ is a Lie subalgebra of $\mathfrak{g}$. Then the Reduction Lemma holds, namely, $\ker(T_m\mathbf{K}) = \left(\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ \cdot m\right)^\omega$. Similarly the Bifurcation Lemma holds: $\mathrm{range}\,(T_m\mathbf{K}) = T_\mu\pi_C\left((\mathfrak{g}_m)^\circ\right)$, where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(m) = \pi_C(\mu)$. The cylinder valued momentum map $\mathbf{K}$ is not equivariant, in general. It turns out that the coadjoint action drops to the cylinder $\mathfrak{g}^*/\overline{\mathcal{H}}$ and that there is an associated cocycle that makes $\mathbf{K}$ equivariant. Thus there is a good reduction theory associated to $\mathbf{K}$. We state the main result only in the regular case [11]; for the singular case see [12].

Let (M,ω) be a connected paracompact symplectic manifold and G a Lie group acting freely and properly on it by symplectic diffeomorphisms. Let $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be a cylinder valued momentum map for this action. Then $\mathfrak{g}^*/\overline{\mathcal{H}}$ carries a natural Poisson structure and there exists a smooth G-action on it with respect to which $\mathbf{K}$ is equivariant and Poisson. Moreover:

(i) The reduced space $M^{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]},\ [\mu] \in \mathfrak{g}^*/\overline{\mathcal{H}}$, inherits a natural Poisson structure from (M,ω) that is, in general, degenerate.

(ii) The symplectic leaves of $M^{[\mu]}$ are the optimal reduced spaces.

(iii) The reduced spaces obtained by foliation reduction equal the orbit spaces $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/N_{[\mu]}$, where N is a normal connected Lie subgroup of G whose Lie algebra is the annihilator $\mathfrak{n} := \left(\mathrm{Lie}\left(\overline{\mathcal{H}}\right)\right)^{\circ} \subset \mathfrak{g}$ of $\mathrm{Lie}\left(\overline{\mathcal{H}}\right) \subset \mathfrak{g}^*$ in $\mathfrak{g}$.
(iv) The quotient Lie group $H_{[\mu]} := G_{[\mu]}/N_{[\mu]}$ acts canonically freely and properly on $M_{[\mu]}$ and the Poisson manifold $M_{[\mu]}/H_{[\mu]}$ is Poisson diffeomorphic to $M^{[\mu]}$.

All these reduced spaces are, in general, distinct. But they are equal if there is a momentum map. Is there a convexity result of $\mathbf{K}$ since reduction works so well?

The answer is positive and the proof of such a result relies on several results. The first one is the following. Let $f : X \to V$ be a continuous map from a connected Hausdorff topological space X to a Banach space V that is open onto its image and has local convexity data. Then the image $f(X)$ is locally convex. If, in addition, $f(X)$ is closed in V then it is convex.

The second pillar of the convexity theorem is a generalization of the local-to-global principle that we state here only in the finite dimensional case; see [2] for an infinite dimensional version. Let $f : X \to V$ be a closed map with values in a convex subset of a finite dimensional Euclidean vector space V and X a connected, locally connected, first countable, and normal topological space. Assume that f has local convexity data and is locally fiber connected. Then all the fibers of f are connected, f is open onto its image and the image $f(X)$ is a closed convex set. We do not go here into the details of what "local convexity data" means and refer for the technical details to the original papers. The point is that if one combines these two results with the Marle-Guillemin-Sternberg normal form one immediately finds generalizations of the usual convexity theorems (for toral and compact non-commutative Lie group actions) by replacing the hypothesis on the compactness of M with the closedness of the momentum map. In addition, the same technique extends the convexity result of Poisson-Lie group actions on compact symplectic manifolds (see [5]) to those whose Poisson-Lie momentum map is closed.

For the cylinder valued momentum map the following result holds. Let (M, ω) be a connected paracompact symplectic manifold, G a connected Abelian Lie group acting properly and canonically on M with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map. If $\mathbf{K}$ is closed then $\mathbf{K}(M) \subset \mathfrak{g}^*/\mathcal{H}$ is a weakly convex subset of $\mathfrak{g}^*/\mathcal{H}$. Here one needs to think of $\mathfrak{g}^*/\mathcal{H}$ as a length metric space with the length metric naturally inherited from $\mathfrak{g}^*$. If $\mathfrak{g}^*/\mathcal{H}$ is uniquely geodesic then $\mathbf{K}(M)$ is convex, $\mathbf{K}$ has connected fibers, and it is open onto its image. In particular, this shows that the image of an Abelian Lie group valued momentum map is a weakly convex subset. The notion of weakly convex subset is discussed in the context of path metric spaces in the original papers.

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Morita equivalence of quantum tori and moduli spaces of flat connections on surfaces

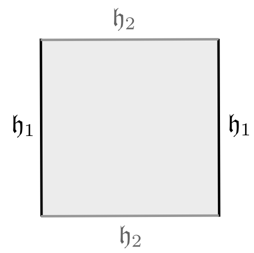
Pavol Ševera

We notice that many interesting symplectic manifolds connected with Poisson-Lie groups are, in fact, moduli spaces of flat connections on surfaces with boundary, with boundary conditions given by Lagrangian subalgebra. Moreover, flat connections that extend from such a surface to a given 3dim body, form a Lagrangian submanifold (at least formally: the formally computed tangent spaces are Lagrangian). These Lagrangian submanifolds then turn the symplectic manifolds into groupoids, modules etc.

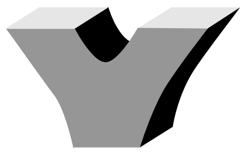
As the basic example, if

$$\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$$

is a Manin triple, and the surface (together with the boundary condition) is this square



then the moduli space is the Lu-Weinstein double symplectic groupoid corresponding to the Manin triple. One of the two products is given by



and the other by the same picture with reversed colours.

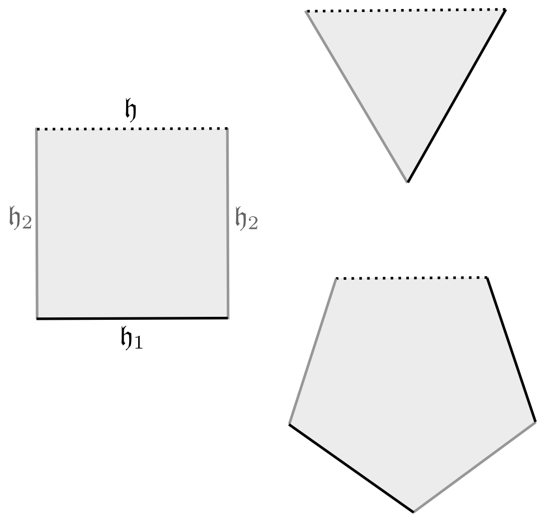
The symplectic form on such a moduli space can be expressed it terms of holonomies of the flat connections (this is just a simple modification of a formula of Alekseev and Malkin to our surfaces with boundaries). The idea is to lift the flat connections to a (possibly non-flat) connections with values in a central extension of $\mathfrak{g}$ by closed 2-forms. The integral of the curvature of this lifted connection over the surface is the symplectic form, and as an integral of curvature, it is equal to the lifted product of holonomies along the boundary (and possible cuts). This product is the mentioned formula for the symplectic form.

Some of these symplectic and Lagrangian manifolds are connected with Morita equivalence of quantum tori. Namely, they provide a symplectic version of the equivalence, and this symplectic version works also for arbitrary Poisson-Lie groups, not just for tori with zero Poisson structures.

We need 3 mutually transversal Lagrangian subalgebras of $\mathfrak{g}$:

$$\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h} \subset \mathfrak{g}.$$

We have a Lie bialgebra as above, plus a new subalgebra $\mathfrak{h}$. Let us consider the following polygons, with boundary conditions indicated by colours, and the corresponding moduli spaces of flat connections.



The two squares give us two symplectic groupoids Γ and Γ' (their bases are Poisson homogeneous spaces). The pentagon gives a symplectic bimodule M_5 of Γ and Γ'. The triangle, on the other hand, gives a symplectic manifold M_3, which is both Γ and Γ' module, but not a bimodule, as these two actions don't commute (they are twisted by the double symplectic groupoid of the Manin triple $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$).

Neither M_3 not M_5 provide a Morita equivalence between the groupoids Γ and Γ' (M_5 is not invertible (indeed, $M_5 \otimes_\Gamma \bar{M}_5$ is given by a hexagon rather than square), while M_3 is not even a bimodule). However, after quantization (if it makes sense) we *will* have Morita equivalence of algebras, provided the double symplectic groupoid is quantized to the trivial Hopf algebra. This is what happens in the case of quantum tori.

Deformation quantization of surjective submersions and principal fiber bundles

Stefan Waldmann

(joint work with Martin Bordemann, Nikolai Neumaier, and Stefan Weiss)

In the last years, in mathematical/theoretical physics it become fashionable to consider *noncommutative space-times* and study (quantum) field theories on such space-times. The idea is that this might be a kind of effective theory to incorporate at least some aspects of quantum gravity. The typical model, which was studied intensely, is the flat Minkowski space-time $\mathbb{R}^{1,3}$ endowed with a noncommutative structure arising from the Weyl-Moyal star product corresponding to a (typically symplectic) constant Poisson structure θ on $\mathbb{R}^{1,3}$. Here one has at least two flavors: formal star products, i.e. treating the deformation parameter λ as a formal parameter and study everything in the context of formal power series, or convergent like e.g. the C^*-algebraic framework of Rieffel. In my talk I will use the formal framework which is technically much simpler.

(Quantum) field theories consist usually of two types of fields: matter fields and gauge fields. Matter fields are usually sections $\Gamma^\infty(E)$ of a certain vector bundle E over space-time M, like e.g. some spinor fields etc. Since the whole structure of a vector bundle is encoded in the module structure of the sections over the algebra of smooth functions over M, it is straightforward to define the analogue of matter fields in a noncommutative setting: one wants to deform this module structure of $\Gamma^\infty(E)$ over $C^\infty(M)$ into a module structure over the deformed algebra $(C^\infty(M)[[\lambda]], \star)$ where $\star$ is the (given) star product on M. In fact, this can always be achieved and is unique up to a certain notion of equivalence, see [1], even including extra structure like positive fiber metrics.

For the gauge fields the question is less obvious: in more geometric terms, gauge theories are described in terms of principal fiber bundles $\mathsf{pr} : P \longrightarrow M$ with some structure Lie group G. In my talk I pointed out by some (counter-) examples that the most reasonable definition of a deformation quantization of a principal fiber bundle is as follows: one looks for a right module structure $\bullet$ for $C^\infty(P)[[\lambda]]$ over $C^\infty(M)[[\lambda]]$ deforming the canonical right module structure via pr^* subject to the condition that $g^*(f \bullet a) = g^*(f) \bullet a$ for all $g \in G$, $f \in C^\infty(P)[[\lambda]]$ and $a \in C^\infty(M)[[\lambda]]$. If one only looks for a right module structure then the definition still makes sense for a surjective submersion $\mathsf{pr} : P \longrightarrow M$, whether it comes from a principal action of some Lie group or not.

The main idea is now to construct such a right module multiplication $\bullet = \sum_{r=0}^{\infty} \lambda^r R_r$ order by order. Since $R_0(f, a) = f \mathsf{pr}^* a$ is already given and a right module structure for the undeformed product, one obtains by the usual argument a cohomological obstruction in the second Hochschild cohomology of $C^\infty(M)$ with values in the bimodule of differential operators $\mathrm{Diffop}(P)$ on P (we require for technical reasons that all operators R_r should be differential). For a principal fiber bundle, one has to require the G-equivariance of the differential operators in addition, whence in this case the bimodule is $\mathrm{Diffop}(P)^G$. In general not much is known about Hochschild cohomology groups with values in some bimodules beside that they tend to be rather large spaces. Here however, the situation is very simple as the bimodules in question turn out to be very *non-symmetric*:

Theorem 1. *For a surjective submersion* $\mathsf{pr} : P \longrightarrow M$ *one has*

$$\mathrm{HH}^k(C^\infty(M); \mathrm{Diffop}(P)) = \begin{cases} \mathrm{Diffop}_{\mathrm{ver}}(P) & k = 0 \\ \{0\} & k \geq 1. \end{cases}$$

If in addition $\mathsf{pr} : P \longrightarrow M$ *is a* G*-principal fiber bundle, one has*

$$\mathrm{HH}^k(C^\infty(M); \mathrm{Diffop}(P)^G) = \begin{cases} \mathrm{Diffop}_{\mathrm{ver}}(P)^G & k = 0 \\ \{0\} & k \geq 1. \end{cases}$$

In both cases "ver" stands for vertical differential operators.

With the vanishing of the second cohomology, the existence is of course a trivial consequence: for all surjective submersions there exists a deformation quantization (for any given star product $\star$ on the base M). Moreover, thanks to the vanishing of the first cohomology group, also the equivalence of such deformations is trivially understood: any two deformations are equivalent. In case of a principal fiber bundle, the additional G-equivariance does not change these results.

Since the module structure is essentially unique, one can show in addition that the commutant, i.e. the module endomorphisms turn out the be a deformation of the classical commutant within all differential operators, i.e. $\mathrm{Diffop}_{\mathrm{ver}}(P)$. Thus one obtains an induced "star product" $\star'$ for $\mathrm{Diffop}_{\mathrm{ver}}(P)[[\lambda]]$ together with a left module structure $\bullet'$ of $(\mathrm{Diffop}_{\mathrm{ver}}(P)[[\lambda]], \star')$ on $C^\infty(P)[[\lambda]]$. In fact, both deformed algebras turn out to be mutual commutants.

There are several applications of this result: first, one can use the deformed principal bundle indeed for an analogue of the construction of associated vector bundles. This is of course at the heart of any noncommutative field theory and reproduces the deformed vector bundles. Second, for the case where the associated vector bundle is a line bundle this construction shines some new light on the Morita theory of star products, see [2] for an overview. Third, the commutant can be seen as the "jet expansion" of a star product $\star'$ on a bigger Poisson manifold in which

P is a coisotropic submanifold. Thus one obtains a phase space reduction picture and gains some new insight in the phase space reduction of star products. Finally, a more explicit construction is available, at least for the symplectic case, by an adapted version of Fedosov's construction [3].

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Poisson Structures on Flag Varieties

Milen Yakimov

(joint work with K. A. Brown, K. R. Goodearl, and B. Webster)

Let G be a complex simple algebraic group and (B, B^-) be a pair of opposite Borel subgroups. Denote by $T = B \cap B^-$ the corresponding maximal torus of G. Let Δ_+ be the set of positive roots of $\mathfrak{g} = \mathrm{Lie}(G)$, related to $T \subset B$. Fix root vectors $e_\alpha \in \mathfrak{g}^\alpha$, $f_\alpha \in \mathfrak{g}^{-\alpha}$, normalized by $\langle e_\alpha, f_\alpha \rangle = 1$, where $\langle ., . \rangle$ is a nondegenerate invariant bilinear form on $\mathfrak{g}$. The standard Poisson structure π_G on G is given by

$$\pi_G = \sum_{\alpha \in \Delta_+} L_{e_\alpha} \wedge L_{f_\alpha} - \sum_{\alpha \in \Delta_+} R_{e_\alpha} \wedge R_{f_\alpha}$$

where L_x and R_x refer to the left and right-invariant vector fields on G, corresponding to $x \in \mathfrak{g}$.

Fix a standard parabolic subgroup $P \supseteq B$ of G and denote the projection $p \colon G \to G/P$. The push-forward $\pi_{G/P} = p_*(\pi_G)$ is a well defined Poisson structure on G/P and the map $p \colon (G, \pi_G) \to (G/P, \pi_{G/P})$ is a surjective Poisson submersion. We study the Poisson structure $\pi_{G/P}$ on G/P by means of a certain weak splitting of the surjective Poisson submersion p and then use it for applications to Lie theory, combinatorics, and dynamical systems.

First recall:

Definition 1. *(*Crainic, Fernandes, [3]*)* Assume that (M, Π) is a smooth Poisson manifold. A submanifold X of M is called a *Poisson–Dirac submanifold* if the following two conditions are satisfied:

(i) For each symplectic leaf S of (M, Π), the intersection $S \cap X$ is clean *(*i.e., it is smooth and $T_x(S \cap X) = T_x S \cap T_x X$ for all $x \in S \cap X$*)* and $S \cap X$ is a symplectic submanifold of $(S, (\Pi|_S)^{-1})$.

(ii) The family of symplectic structures $(\Pi|_S)^{-1}|_{S \cap X}$ is induced by a smooth Poisson structure π on X.

One has the following criterion:

Proposition 2. (*Xu, Crainic, Fernandes,* [9, 1]) *Assume that* (M, Π) *is a Poisson manifold and that* X *is a submanifold for which there exists a subbundle* E *of* $T_X M$ *such that*
(i) $T_X M = TX \oplus E$ *and*
(ii) *the restriction of the Poisson tensor* Π *to* X *splits as*

$$\Pi|_X = \pi + \pi_E$$

for some smooth bivector fields $\pi \in \Gamma(X, \wedge^2 TX)$ *and* $\pi_E \in \Gamma(X, \wedge^2 E)$.

Then X *is a Poisson–Dirac submanifold of* (M, Π) *and the induced Poisson structure on it coincides with* π.

Now we define the notion of weak splitting of a surjective Poisson submersion:

Definition 3. *(*[5]*)* Assume that (M, Π) and (N, π) are Poisson manifolds and that $p\colon (M, \Pi) \to (N, \pi)$ is a surjective Poisson submersion. A *weak splitting* of p is a partition

$$N = \bigsqcup_{a \in A} N_\alpha$$

of (N, π) into Poisson submanifolds such that for each $a \in A$, there exists a smooth lifting $i_a\colon N_\alpha \rangle M$ *(*of $p|_{p^{-1}(N_a)}\colon p^{-1}(N_a) \rangle N_\alpha$*)* with the properties:
(i) $i_a(N_a)$ is a Poisson–Dirac submanifold of (M, Π) and
(ii) the induced Poisson structure on $i_\alpha(N_a)$ is $i_{a*}(\pi|_{N_a})$.

This notion can be viewed as a generalization of the quasiclassical analog of the notion of *conditional expectation* in operator algebras.

The Poisson structure π_G is invariant under the left and right actions of the maximal torus T on G. Denote the corresponding induced structure on G/T by $\pi_{G/T}$. Let $p'\colon (G/T, \pi_{G/T}) \to (G/P, \pi_{G/P})$ be the surjective Poisson submersion induced by p.

Denote by W the Weyl group of G and by W_P the parabolic subgroup of W corresponding to P. Let $W^{W_P}_{\max}$ be the set of (unique) maximal length representatives of the cosets from W/W_P. Recall that one has the decomposition of the flag variety G/P into Schubert cells:

$$G/P = \bigsqcup_{w \in W^{W_P}_{\max}} B^- \cdot wP.$$

Now we have:

Theorem 4. ([5]) *The partition into Schubert cells*

$$G/P = \bigsqcup_{w \in W^{W_P}_{\max}} B^- \cdot wP$$

and the morphisms

$$i_w : B^- \cdot wP \to G/T, \quad \text{given by} \quad i_w(uwP) \to uwT, \text{ for } u \in U^- \cap wB^-w^{-1},$$

where U^- is the unipotent radical of B^-, provide a weak splitting of the surjective Poisson submersion $p' \colon (G/T, \pi_{G/T}) \to (G/P, \pi_{G/P})$. In addition, the images of i_w satisfy the condition in Proposition 2.

As a consequence of this, all symplectic leaves of $(G/P, \pi_{G/P})$ are symplectic submanifolds of symplectic leaves of $(G/T, \pi_{G/T})$. It also implies the following result on T-orbits of symplectic leaves of $(G/P, \pi_{G/P})$.

Theorem 5. ([2, 5]) *There are only finitely many T-orbits of symplectic leaves on $(G/P, \pi_{G/P})$, parametrized by pairs $(w_1, w_2) \in W^{W_P}_{\max} \times W$ such that $w_1 \leq w_2$ in the Bruhat order. The torus orbit corresponding to the pair (w_1, w_2) is given by*

$$\mathcal{S}_{w_1,w_2} = (U^-_{w_1} \dot{w}_1 \cap B^+ w_2 B^+) \cdot P,$$

(where $\dot{w}_1$ is a representative of w_1 in the normalizer of T in G) and is biregularly isomorphic to the intersection $B^- \cdot w_1 B \cap B \cdot w_2 B$ of opposite Schubert cells in the full flag variety G/B. Thus, the T-orbits of symplectic leaves on $(G/B, \pi_{G/B})$ are exactly the intersections of opposite Schubert cells in G/B.

The above partition of G/P is exactly Lusztig's partition [6], defined for the purposes of studying total positivity in partial flag varieties.

The Poisson structures $\pi_{G/P}$ have interesting applications to Lie theory, combinatorics, and dynamical systems. Consider the special case when P has abelian unipotent radical. Then G/P is an example of a Hermitian symmetric space of compact type. The orbit structure of the action of the standard Levi factor L of P on G/P was described in [7]. It is shown in [5, §4] that all L-orbits on G/P are Poisson submanifolds and that the Poisson structure $\pi_{G/P}$ vanishes at all base points of Richardson, Röhrle, and Steinberg [7] and conjecturally only there. In other words the Poisson structure $\pi_{G/P}$ "sees" these special base points. Now consider the case of the full flag varieties G/B. In relation to combinatorics, it is proved in [8], that all strata of the Deodhar stratifications [4] of intersections of dual Schubert cells in G/B are coisotropic with respect to $\pi_{G/B}$. The coordinate rings of these intersections are also natural candidates for "upper cluster algebras" [1].

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Morita equivalence of Poisson manifolds via stacky groupoids

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(joint work with Henrique Bursztyn)

The aim of this talk is to present our program to define Morita equivalence in the category of all Poisson manifolds via Morita equivalence of their stacky symplectic groupoids. The talk is based on [3]. Early in [7], Xu invented Morita equivalence of Poisson manifolds with the inspiration from Rieffel's Morita equivalence of $\mathbb{C}^*$-algebras. However it works only for integrable Poisson manifolds, i.e. those Poisson manifolds who process symplectic groupoids. A symplectic groupoid [6] is a Lie groupoid $S \Rightarrow P$ with a symplectic form ω on S satisfying

$$pr_1^*\omega + pr_2^*\omega = m^*\omega, \tag{1}$$

on the set of composable arrows $S \times_P S$ (m is the multiplication on S). Then the base P of the symplectic groupoid $S \Rightarrow P$ has an induced Poisson structure such that the source map $\mathbf{s} : S \to P$ is a Poisson map and the target $\mathbf{t} : S \to P$ is anti-Poisson. In fact there is a one-to-one correspondence between integrable Poisson manifolds and source-simply connected symplectic groupoids.

Morita equivalence of Lie groupoids is well-studied and now widely used in the theory of differentiable stacks. Roughly speaking, differentiable stacks can be viewed as Lie groupoids up to Morita equivalence (see for example [1]). Adding compatible symplectic structure inside, [7] established Morita equivalence of symplectic groupoids and proved further that Poisson manifolds P_1 and P_2 are Morita equivalent if and only if their symplectic groupoids are Morita equivalent.

Now [4] [5] show that even a non-integrable Poisson manifold processes a sort of symplectic groupoid $\mathcal{S} \Rightarrow P$, but $\mathcal{S}$ is not anymore a manifold but an étale differentiable stack[1] which processes a compatible symplectic form as in (1). Then

[1]An étale differentiable stack is a differentiable stack presented by an étale Lie groupoid. Careful readers find out that $\mathcal{S}$ is presented by a groupoid and itself again is a groupoid over a manifold P. But these two groupoids are two different ones. In fact putting them together we have a Lie 2-groupoid [8].

the one-to-one correspondence is extended to the set of all Poisson manifolds and that of source-2-connected symplectic stacky groupoids (see Theorem 8).

In our program, we first build Morita equivalence for stacky groupoids, then we add compatible symplectic forms inside and build Morita equivalence for symplectic stacky groupoids and hence for the base Poisson manifolds.

1. Stacky groupoids and their principal bundles

We first say a few more words on the stacky groupoid $\mathcal{G} \rightrightarrows M$ we use. For an exact definition, we refer the reader to [8]. The space of arrows $\mathcal{G}$ is a differentiable stack, and the space of objects M is a manifold. It has $\mathbf{s}$, $\mathbf{t}$, m, e, i as source, target, multiplication, identity, and inverse map respectively, just as in the case of Lie groupoids. The only difference now is that the multiplication is not strictly associative but associative up to a 2-morphism α which satisfies a pentagon condition. The same happens to all the other identities we had before for Lie groupoids. Namely all these identities such as $(gh)k = g(hk)$, $1g = g$, etc., do not hold strictly, but still hold up to something in a controlled way. This '2'-phenomenon is new when we step into the world of stacks. It will come back to haunt us all the time (for example Definition 1). The alternative way is to work with Lie 2-groupoids which are essentially equivalent to SLie groupoids [8]. We established Morita equivalence of Lie 2-groupoids there.

To shorten the notation, we call these stacky groupoids *SLie groupoids*, and when $\mathcal{G}$ is further an étale differentiable stack, a *W-groupoid*[2]. A *symplectic W-groupoid* is a W-groupoid which has a compatible symplectic form as in (1).

To build Morita equivalence, we first need the notion of principal bundles of stacky groupoids.

Definition 1 (SLie (W-)groupoid actions)**.** Let $\mathcal{G}$ be an SLie (W-)groupoid over M, $\mathcal{X}$ differentiable stack and $J : \mathcal{X} \to M$ a smooth morphism. A right $\mathcal{G}$-action on $\mathcal{X}$ is a smooth morphism

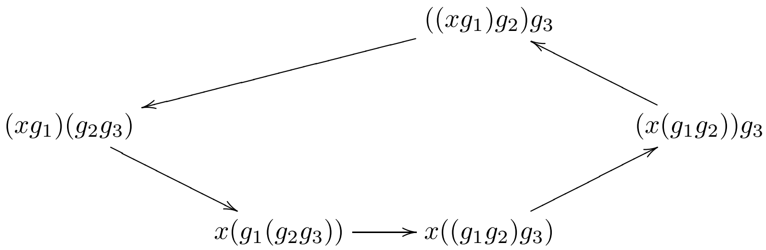
$$\Phi : \mathcal{X} \times_M \mathcal{G} \to \mathcal{X},$$

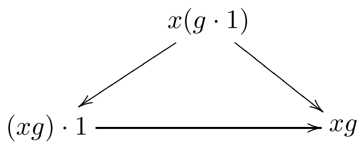
satisfies the following properties:

(1) $\Phi \circ (\Phi \times id) = \Phi \circ (id \times m)$ holds up to a 2-morphism a;
(2) $J \circ \Phi = \mathbf{s} \circ pr_2$, where $pr_2 : \mathcal{X} \times_M \mathcal{G} \to \mathcal{G}$;
(3) $\Phi \circ (id \times (e \circ J)) = id$ holds up to a 2-morphism b.

[2]The 'W' comes from Alan Weinstein, who suggested this stacky approach to one of the authors.

The 2-morphisms satisfy higher coherences, which roughly says that the following diagrams commute:





Given such an action, we can form a quotient stack $\mathcal{X}/\mathcal{G}$ as in [2]. Unfortunately, the quotient stack is not always a differentiable stack again. For this, we need principality of the action.

Recall that an action $\Phi : X \times_M G \to X$ of a Lie groupoid $G \rightrightarrows M$ on a manifold X is principal if and only if X/G is a manifold and $pr_1 \times \Phi : X \times_M G \longrightarrow X \times_{X/G} X$ is an isomorphism. We have the following definition:

Definition 2 (**Principal SLie (W-) groupoid bundles).** Let $\mathcal{G} \Rightarrow M$ be an SLie (-W) groupoid. A left *$\mathcal{G}$-bundle over a differentiable stack* $\mathcal{X}$ is a differentiable stack $\mathcal{X}$ together with a smooth morphism $\pi : \mathcal{X} \to \mathcal{S}$ and a right action Φ satisfying

$$\pi \circ \Phi = \pi \circ pr_2 \tag{2}$$

up to a 2-isomorphism $\alpha : \pi \circ pr_2 \to \pi \circ \Phi$. (Here $pr_2 : \mathcal{G} \times_M \mathcal{X} \to \mathcal{X}$ is the natural projection.) The 2-isomorphism α satisfies a further coherence condition.

The bundle is *principal* if π is a surjective submersion and

$$pr_1 \times \Phi : \mathcal{X} \times_M \mathcal{G} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{X}$$

is an isomorphism. Then the action Φ is also called *principal*.

Example 3 (A point as a principal $\mathbb{Z}$ bundle). A point pt is a principal $\mathbb{Z}$ bundle over the stack $B\mathbb{Z}$. The action of $\mathbb{Z}$ on pt is trivial, so it is not principal in the classical sense. However, pt is a principal $\mathbb{Z}$ bundle as in Definition 2 because $pt \times_{B\mathbb{Z}} pt = \mathbb{Z}$ (see [1] for the definition of fibre product of differentiable stacks) and

$$pt \times \mathbb{Z} \to pt \times_{B\mathbb{Z}} pt,$$

is an isomorphism of stacks.

Theorem 4. *Let $\mathcal{G}$ be an SLie (W-) groupoid. If $\pi : \mathcal{X} \to \mathcal{S}$ is a $\mathcal{G}$-principal bundle over $\mathcal{S}$, then $\mathcal{X}/\mathcal{G}$ is a differentiable stack and is isomorphic to the base $\mathcal{S}$. Moreover $\mathcal{X}/\mathcal{G}$ is presented by a Lie groupoid whose space of arrows is E_Φ/G_1 and whose space of objects is X_0. Here $X_1 \Rightarrow X_0$ is a Lie groupoid presentation of $\mathcal{X}$, $G_1 \Rightarrow G_0$ is that of $\mathcal{G}$ and E_Φ is the H-S bibundle of the $\mathcal{G}$-action Φ.*

2. Morita equivalence of SLie groupoids

Definition 5 (Morita equivalence of SLie groupoids)**.** Two SLie groupoids $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ are *Morita equivalent* if there is a differentiable stack $\mathcal{X}$ and two smooth morphisms $J_i : \mathcal{X} \to \mathcal{G}_i$ (moment maps) such that

(1) $J_1 : \mathcal{X} \to M_1$ is a right principal $\mathcal{G}_2$-bundle;
(2) $J_2 : \mathcal{X} \to M_2$ is a left principal $\mathcal{G}_1$-bundle;
(3) $\Phi_2 \circ (\Phi_1 \times id) = \Phi_1 \circ (id \times \Phi_2)$ holds up to a 2-isomorphism a which satisfies six higher coherence conditions.

In this case we call $\mathcal{X}$ a $(\mathcal{G}_1, \mathcal{G}_2)$-*Morita bibundle.*

It is simple to check that Morita equivalence is reflexive ($\mathcal{G}$ itself is a $(\mathcal{G}, \mathcal{G})$-Morita equivalence) and symmetric (use inverses to make right actions into left and vice-versa). However transitivity is nontrivial and we need to use Theorem 4.

Moreover we also have,

Proposition 6. *If two W-groupoids are Morita equivalent via Morita bibundle $\mathcal{X}$, then $\mathcal{X}$ is an étale differentiable stack.*

Proposition 7. *Two W-groupoids $\mathcal{G}_i \Rightarrow M_i$ are Morita equivalent via Morita bibundle $\mathcal{X}$. If $\mathcal{G}_1 \Rightarrow M_1$ is a Lie groupoid, then $\mathcal{X}$ is a manifold and $\mathcal{G}_2 \Rightarrow M_2$ is also a Lie groupoid.*

Finally, two symplectic W-groupoids $(\mathcal{G}_1, \omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2, \omega_2) \rightrightarrows M_2$ are *Morita equivalent* if they are Morita equivalent as SLie groupoids via a symplectic étale stack $(\mathcal{X}, \omega)$ satisfying

$$pr_1^* \omega_1 + pr_2^* \omega = \Phi_1^* \omega, \quad \text{on } \mathcal{G}_1 \times_{M_1} \mathcal{X},$$

where Φ_1 is the action of $\mathcal{G}_1$ on $\mathcal{X}$, and the same for ω and ω_2.

Theorem 8. [5] *For any symplectic W-groupoid $\mathcal{G} \rightrightarrows M$, the base manifold M has a unique Poisson structure such that the source map* **s** *is Poisson. In this case, we call $\mathcal{G}$ a symplectic W-groupoid of the Poisson manifold M.*

On the other hand, for any Poisson manifold M, there are two symplectic groupoids $\mathcal{G}(M)$ and $\mathcal{H}(M)$ of M. $\mathcal{G}(M)$ has 2-connected source fibre and $\mathcal{H}(M)$ has only 1-connected source fibre.

Definition 9. Two Poisson manifolds M_1 and M_2 are called *strongly Morita equivalent* if $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$ are Morita equivalent as symplectic W-groupoids. Respectively, they are called *weakly Morita equivalent* if $\mathcal{H}(M_1)$ and $\mathcal{H}(M_2)$ are Morita equivalent as symplectic W-groupoids.

Strong Morita equivalence implies the weak one, and weak Morita equivalence coincides with the classical one in [7] when applied to integrable Poisson manifolds. But strong Morita equivalence is something new. For example, in [7], with their usual symplectic forms, $\mathbb{R}^2$ and the 2-sphere S^2 are Morita equivalent since all the simply connected symplectic manifolds are Morita equivalent in the classical sense. But they are *not* strongly Morita equivalent because they have different π_2 groups. In fact, only 2-connected symplectic manifolds are strongly Morita equivalent to each other. We hope this π_2-phenomenon will help in symplectic geometry, for example, in the aspect of preservation of prequantization.

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