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Partielle Differentialgleichungen

Organised by
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Reiner Schätzle, Tübingen
Neil Trudinger, Canberra

July 22nd – July 28th, 2007

Abstract. The workshop dealt with partial differential equations in geometry and technical applications. The main topics were the combination of nonlinear partial differential equations and geometric problems, regularity of free boundaries, conformal invariance and the Willmore functional.

Mathematics Subject Classification (2000): 35 J 60, 35 J 35, 58 J 05, 53 A 30, 49 Q 15.

Introduction by the Organisers

The workshop *Partial differential equations*, organised by Tom Ilmanen (ETH Zürich), Reiner Schätzle (Universität Tübingen) and Neil Trudinger (Australian National University Canberra) was held July 22-28, 2007. This meeting was well attended by 50 participants, including 6 females, with broad geographic reprensentation. The program consisted of 19 talks and 3 shorter contributions and left sufficient time for discussions.

One focus was the combination of nonlinear partial differential equations and geometric problems. There were talks on monotonicity formulas in presence of degenerate singularities and on estimates of the singular set for branched minimal immersions. One striking new result was the proof of the diffeomorphic sphere theorem by the use of Ricci flow.

Many leading experts in the regularity of elliptic and variational problems and in fully nonlinear differential equations occuring in optimal transport and in Hessian equations attended the workshop. Several main contributions were given to this subject. Here, we mention the proof of interior estimates for quadratic Hessian equations in three dimensions, which has been an open problem for several years of similar relevance as the gradient estimates for the minimal surface equation.

New results were also presented for the Willmore functional. It was achieved to write the Euler-Lagrange equation in divergence form, and the existence of minimizers in fixed conformal classes with certain energy bounds were obtained.

The organisers and the participants are grateful to the Oberwolfach Institute for presenting the opportunity and the resources to arrange this interesting meeting.

Workshop: Partielle Differentialgleichungen

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Abstracts

Estimates of the Jacobian Determinant in Terms of Subdeterminants Hardy & Littlewood unite with Fefferman & Stein in maximal inequalities

Tadeusz Iwaniec
(joint work with Flavia Giannetti, Jani Onninen and Anna Verde)

Recent advances in the Calculus of Variations (polyconvex energy integrals), Geometric Function Theory and Nonlinear Elasticity rely on various estimates of the Jacobian determinants of weakly differentiable mappings. This is where both harmonic analysis and geometric PDEs come in extremely useful. Two results will be presented. First we discuss local integrability property of nonnegative Jacobians under minimal integrability hypotheses on the differential and its subdeterminants. Then we shall allow the Jacobian to change sign, and show that it lies in the Hardy space. These questions arise naturally in the study of mappings of finite distortion and, in particular, deformations of hyperelastic materials. Some new cancellation phenomena, which we shall observe in commutators of singular integrals, are the key elements of the proofs. However, the true novelty of our approach are maximal operators which we obtain by interpolating between the classical Hardy-Littlewood and the spherical Fefferman-Stein maximal operators.
This lecture is based on two joint publications with Flavia Giannetti, Jani Onninen and Anna Verde.

A Hilbertian approach to some hyperbolic conservation laws

Yann Brenier

First order systems of conservation laws read: $\partial_t u + \nabla_x \cdot (Q(u)) = 0$, where $u = u(t,x) \in R^m$ depends on $t \geq 0$, $x \in R^d$, and $\cdot$ denotes the inner product in R^d. The Q_i (for $i = 1, \cdots, d$) are given smooth functions from an open subset U of R^m into R^m. The system is called hyperbolic when, for each $\tau \in R^d$ and each $w \in U$, the $m \times m$ matrix $\sum_{i=1,d} \tau_i Q_i'(w)$ can be put in diagonal form with real eigenvalues. There is no general theory to solve globally in time the initial value problem for such systems of PDEs. (See [Da] for a recent general description of the field.) In the non-linear case, local smooth solutions usually blow up in finite time (shock waves). The only theories available providing existence and uniqueness of global solutions (beyond singularity formation) are limited to the scalar case (Kruzhkov's theory: $m = 1$ [Kr]) or to one space dimension (Glimm-Bressan theory: $d = 1$, for small initial conditions [BB]). In both cases the functional spaces L^1 and BV play a crucial role while Hilbert spaces are of no use. Unfortunately, most linear multidimensional systems (such as linear acoustics, or Maxwell's equations) can be shown to be well posed only on Hilbert spaces such as L^2 [Brn].
The main point of our discussion is to revisit the Kruzhkov theory by showing that, unexpectedly, there is a Hilbertian structure behind it. Roughly speaking,

any Kruzhkov 'entropy' solutions $u(t,x)$, $t \geq 0$, $x \in R^d/Z^d$, assumed to be valued in $[0,1]$ for simplicity, can be written:

$$u(t,x) = \int_0^1 1_{\{Y(t,a,x) \leq 0\}} da,$$

where $Y = Y(t,x,a)$, $x \in R^d/Z^d$, $a \in [0,1]$, solves the subdifferential equation

$$0 \in \partial_t Y + Q'(a) \cdot \nabla_x Y + \partial K[Y],$$

where ∂K denotes the subdifferential of the convex lsc functional $Y \to K[Y]$ valued in $\{0, +\infty\}$, with value 0 whenever $\partial_a Y \geq 0$ and $+\infty$ otherwise. According to maximal monotone operator theory, this subdifferential equation is well posed in the Hilbert space $H = L^2(R^d/Z^d \times [0,1])$ and enjoys the non-expansive property:

$$\int |Y(t,x,a) - \tilde{Y}(t,x,a)|^p dadx \ \leq \ \int |Y(0,x,a) - \tilde{Y}(0,x,a)|^p dadx,$$

for all pairs of solutions $Y, \tilde{Y}$ and all $p \in [0, +\infty]$. The proof [Br5] is based on a combination of level-set, kinetic and transport-collapse approximations, in the spirit of previous works by Giga, Miyakawa, Osher, Tsai and the author [Br2, GM, TGO]. Let us mention that the indicator function $1_{\{Y(t,a,x) \leq 0\}}$ coincides with the one involved in the Lions-Perthame-Tadmor kinetic formulation [LPT]. Also notice recent related works [BBL, CFL].

Of course, dealing with scalar conservation laws is of limited interest. (However the regularizing effect of shock waves in that case is not yet fully understood, in spite of remarkable results by Lions-Perthame-Tadmor, Ambrosio-Kirchheim-Lecumberry-Rivière and De Lellis-Otto-Westdickenberg. Maybe our approach could provide new insights.) There is a very limited hope that our approach can be generalized to interesting multidimensional systems of conservation laws. Let us mention one positive example we treated earlier (which was crucial for our understanding). The so-called 'Born-Infeld-Chaplygin' system considered in [Br4], and the related concept of 'order-preserving strings'. This system reads:

$$\partial_t(hv) + \partial_y(hv^2 - hb^2) - \partial_x(hb) = 0, \tag{1}$$

$$\partial_t h + \partial_y(hv) = 0, \quad \partial_t(hb) - \partial_x(hv) = 0,$$

where h, b, v are real valued functions of time t and two space variables x, y. In [Br4], this system is related to the following subdifferential system:

$$0 \in \partial_t Y - \partial_x W + \partial K[Y], \quad \partial_t W = \partial_x Y, \tag{2}$$

where (Y, W) are real valued functions of (t, a, x) and K is as above. Unfortunately, this system is very special (its smooth solutions are easily integrable).

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Regularity in gradient constrained problems and thin-film micromagnetics

Stefan Müller

(joint work with John Andersson and Bisjawit Karmakar)

Let $S \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and let h_{ext} be a vector in $\mathbb{R}^2$. For a magnetization

$$m : S \to \mathbb{R}^2 \tag{1}$$

consider the energy

$$E(m) = -\int_S h_{\text{ext}} \cdot m \, dx + \frac{1}{4} ||\text{div}\,(m\chi_S)||^2_{H^{-1/2}}. \tag{2}$$

Here the notation $m\chi_S$ indicates that m is extended by zero outside S and the divergence is understood in the sense of distributions. The homogeneous $H^{-1/2}$ norm is defined by Fourier transform: $||f||^2_{H^{-1/2}} = \int |\xi|^{-1} |\hat{f}(\xi)|^2 \, d\xi$. Alternatively this norm can be expressed as the electrostatic energy of a potential $u : \mathbb{R}^3 \to \mathbb{R}$ generated by the surface charge f, i.e.,

$$\frac{1}{4}||f||^2_{H^{-1/2}} = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2, \tag{3}$$

where

$$-\Delta u = f(x_1, x_2)\delta_0(x_3) \tag{4}$$

in the sense of distributions.

We consider the minimization problem

$$\text{Minimize} \quad E(m) \qquad \text{subject to } |m| \le 1 \quad (P). \tag{5}$$

This problem arises as a Γ-limit of the full three-dimensional micromagnetic energy for films which are thin (small aspect ratio) and whose lateral dimension is large compared to a material parameter, the exchange length (that length is typically of the order of a few nanometers), see [3]. We are interested in the regularity of minimizers of (P).

Note first that (P) is convex and hence existence of minimizers is easy. On the other hand (P) is strongly degenerate, the energy $E(m)$ depends only on m only through $\operatorname{div} m$. More precisely

$$E(m + \nabla^\perp \psi) = E(m) \quad \forall \psi \in W_0^{1,\infty}(S), \tag{6}$$

where $\nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$. Hence we have neither uniqueness nor regularity of all minimizers, not even for $h_{\text{ext}} = 0$ (take, e.g., $m(x) = x^\perp/|x|$).

C. Melcher [6] has shown that one has at least higher regularity for $\operatorname{div} m$ (see also [3] for simply connected domains).

Theorem 1. *Let m be a solution of (P). Then $\operatorname{div} m \in H^{1/2}_{\text{loc}}(S)$.*

Our first result is

Theorem 2. *For each $h_{\text{ext}} \in \mathbb{R}^2$ there exists a solution $\bar{m}$ of (P) with*

$$\bar{m} \in C^{0,\beta}_{\text{loc}}(S), \tag{7}$$

for some $\beta > 0$.

To prove this we first note that Melcher's result and Hodge decomposition show that any minimizer is of the form $m_0 + \nabla^\perp \psi$ with $m_0 \in H^{3/2}_{\text{loc}}(S) \hookrightarrow C^{0,1/2}_{\text{loc}}(S)$. It just remains to make a good choice of ψ, keeping in mind the constraint $|m_0 + \nabla^\perp \psi| \le 1$. A natural attempt is to choose ψ in such a way that the L^∞ norm of $|m_0 + \nabla^\perp \psi|$ is minimal. This leads to an interesting regularity problem for a perturbation of the infinity Laplacian, which to the best of our knowledge is unsolved (see [7, 5] for striking recent progress on infinity harmonic functions). We will instead minimize the Dirichlet integral of ψ subject to the constraint $|m_0 + \nabla^\perp \psi| \le 1$. Then the theorem follows from the following result of independent interest.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Let $\alpha > 0$, $a \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$ and consider the admissible set*

$$K = \{w \in H_0^1(\Omega) : |\nabla w - a| \le 1 \ a.e.\}. \tag{8}$$

If K is nonempty then

$$I(w) = \int_\Omega |\nabla w|^2 \, dx \tag{9}$$

has a unique minimizer $\bar{w}$ in K and $\bar{w} \in C^{1,\beta}_{\text{loc}}(\Omega)$, for some $\beta > 0$.

If $a = 0$ then the problem turns out to be equivalent to a double obstacle problem, i.e., the minimization of $I(w)$ subject to $w^- \leq w \leq w^+$, and classical results [1, 4] give $\bar{w} \in C^{1,1}$. There is by now a large literature on elliptic problems with constraints on the gradient (see, e.g., [2] for more general homogeneous constraints and a review of earlier work), but we are not aware of other results where the constraint depends on x with relatively low regularity (Evans [4] treats $|\nabla w| \leq g$ with $g^2 \in C^2$). The main ingredient in the proof is a comparison between the homogeneous and the x-dependent problem. From this the assertion follows by the $C^{1,1}$ estimate for the homogeneous problem and the usual application of Morrey's decay lemma.

Lemma 1. *Let $0 < \varepsilon \leq 1$ and suppose that $||a||_\infty \leq \varepsilon$. Let B_1 denote the unit ball and consider the admissible sets*

$$\begin{aligned} K_a &= \{w \in H^1(B_1) : |\nabla w - a| \leq 1 \text{ a.e.}\}, \\ K_\varepsilon &= \{w \in H^1(B_1) : |\nabla w| \leq 1 + \varepsilon \text{ a.e.}\}. \end{aligned}$$

Let $w_0 \in K_a$ and let u and v be the minimizers of $I(w)$ in $(w_0 + H_0^1(B_1)) \cap K_a$ and $(w_0 + H_0^1(B_1)) \cap K_\varepsilon$, respectively. Then

$$I(u) - I(v) \leq C_\gamma \varepsilon^\gamma, \quad \text{for all } \gamma < 1/3. \tag{10}$$

Idea of proof. Clearly $K_a \subset K_\varepsilon$. We construct a test function in K_a which agrees with v if ∇v is small. More precisely set

$$A = \{x \in B_{1-\varepsilon^\gamma} : |\nabla v| \leq 1 - 2\varepsilon^\beta\}, \tag{11}$$

$$\tilde{K} = \{w \in K_a \cap (w_0 + H_0^1) : w = v \text{ in } A\} \tag{12}$$

(we can take later $\beta = \gamma \in (0, 1/3)$). If we can show that $\tilde{K}$ is nonempty the estimate for $I(u) - I(v)$ follows easily. To see that $\tilde{K}$ is nonempty we adapt the argument that Lipschitz functions on a subset can be extended to globally Lipschitz functions with the same Lipschitz constant. We first use the estimate $|D^2 v| \leq C\varepsilon^{-\gamma}$ in $B_{1-\varepsilon^\gamma}$ to show that

$$\frac{|v(x) - v(y)|}{|x - y|} \leq 1 - \varepsilon \quad \forall x \in A, \ \forall y \in A \cup \partial B_1. \tag{13}$$

Using this and the fact that $w_0 \in K_a$ we easily see that for each $x_0 \in A \cup \partial B_1$ the barriers

$$w_{x_0} = \inf\{w \in K_a : w(x_0) = v(x_0)\} \tag{14}$$

satisfy $w_{x_0} \leq v$ on $A \cup \partial B_1$. Thus $\tilde{u} = \sup_{x_0 \in A \cup \partial B_1} w_{x_0}$ belongs to $\tilde{K}$, as desired.
We are grateful to R. Mingione for bringing the work of Choe and Shim [2] to our attention during this workshop.

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Heat flow of harmonic maps whose gradients belong to Serrin class

Changyou Wang

For two smooth compact Riemannian manifolds (M, g) and (N, h) without boundaries and (N, h) isometrically embedded into an Euclidean space R^L, we consider the heat flow of harmonic maps from M to N, that is a map $u : M \times R_+ \to N$ solving:

$$\partial_t u - \Delta_M u = A(u)(\nabla u, \nabla u)\ x \in M,\ t > 0 \tag{1}$$

$$u|_{t=0} = \phi \in C^\infty(M, N). \tag{2}$$

It is a well known fact that there exists a maximal time interval $0 < T \leq +\infty$ depending on ϕ, M, N such that there is a unique smooth solution $u \in C^\infty(M \times [0, T), N)$ to (1) and (2). Moreover, if such a T is finite then

$$\limsup_{t \uparrow T} \|\nabla u(t)\|_{C^0(M)} = \infty. \tag{3}$$

In this talk, we are interested in finding a scaling invariant integral characterization for the finite, maximal time interval T. We establish the following result

Theorem 1. *For $n = dim(M) \geq 3$, $0 < T < +\infty$ is a maximal time interval of (1) and (2) iff*

$$\limsup_{t \uparrow T} \|\nabla u(t)\|_{L^n(M)} = \infty. \tag{4}$$

This theorem is a consequence of the following regularity theorem for weak solutions to (1) and (2) whose gradients belong to the *Serrin* class $L^n_x L^\infty_t$.

Theorem 2. *For $n \geq 4$ and $0 < T \leq \infty$, if $u : M \times (0, T) \to N$ is a weak solution to (1) and (2) such that $\nabla u \in L^\infty([0, T], L^n(M))$, then $u \in C^\infty(M \times (0, T], N)$.*

The ideas to prove both theorems involve: (1) an improved ϵ_0-regularity theorem, and (2) exclusion of self-similar solutions to the heat equation of harmonic maps in the Serrin class. A crucial step to prove (2) is the application of an unique continuation theorem of the backward heat equation due to Escauriaza-Seregin-Sverák.

Continuity of maps of optimal transportation

Gregoire Loeper

We give a necessary and sufficient condition on the cost function so that the map solution of Monge's optimal transportation problem is continuous for arbitrary smooth positive data. This condition was first introduced by Ma, Trudinger and Wang for a priori estimates of the corresponding Monge-Ampère equation. It is expressed by a so-called cost-sectional curvature being non-negative. We show that when the cost function is the squared distance of a Riemannian manifold, the cost-sectional curvature yields the sectional curvature. As a consequence, if the manifold does not have non-negative sectional curvature everywhere, the optimal transport map can not be continuous for arbitrary smooth positive data. The non-negativity of the cost-sectional curvature is shown to be equivalent to the connectedness of the contact set between any cost-convex function (the proper generalization of a convex function) and any of its supporting functions. When the cost-sectional curvature is uniformly positive, we obtain that optimal maps are continuous or Hölder continuous under quite weak assumptions on the data, compared to what is needed in the Euclidean case. This case includes the reflector antenna problem and the squared Riemannian distance on the sphere.

A new frequency formula and the singular set of a free boundary problem

Georg S. Weiss
(joint work with David Jerison)

Consider the parabolic free boundary problem

$$\Delta u - \partial_t u = 0 \text{ in } \{u > 0\}\ ,\ |\nabla u| = 1 \text{ on } \partial\{u > 0\}\ .$$

Originally it has been derived by J.D. Buckmaster (formally) as singular limit from the following model for the propagation of equidiffusional premixed flames as $\varepsilon \to 0$, i.e. the as the activation energy goes to infinity:

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$$

Here $\beta_\varepsilon(z) = \frac{1}{\varepsilon}\beta(\frac{z}{\varepsilon})$, $\beta \in C^1_0([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$. In the model $u_\varepsilon = \lambda(T_c - T)$, T_c is the flame temperature, which is assumed to be constant, T is the temperature outside the flame and λ is a normalization factor.
By [1], minimizers of the energy $\int(|\nabla u|^2 + \chi_{\{u>0\}})$ are solutions in the sense of distributions, whose free boundary $\partial\{u > 0\}$ can be decomposed into a relatively open regular part and a singular part of vanishing $\mathcal{H}^{n-1}$-measure.
For non-minimizers cones are not the only possible singularities: there are *cusps, crosses* and not much is known on the dimension of the singular set.

In the time-dependent case it is known that for initial data u^0 that are strictly mean concave in the interior of their support, a sequence of ε-solutions converges to a solution of the free boundary problem in the sense of distributions (cf. [2]). Moreover, for general initial data a sequence of ε-solutions converges to a solution of the free boundary problem in the sense of domain variations ([4]).
The questions of how big the singular set can be and whether the ε-limit is in general a solution in the sense of distributions are related to the harmonic/caloric measure of the free boundary (cf. [3]).
Here we describe progress in a project with David Jerison. Using a new frequency formula, we obtain in the stationary case results on the structure of singularities. We also prove that the *Hausdorff dimension* of the topological free boundary of each ε-limit is $\leq n-1$.
We point out possible generalizations of the frequency formula to more general nonlinearities.

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Regularity of measures with Hölder density ratio

Tatiana Toro
(joint work with G. David - C. Kenig & D. Preiss - X. Tolsa)

We study the following question: to what extend does the behavior of the density ratio of a measure determine the regularity of its support?
Let μ be a Borel locally finite measure on $\mathbb{R}^m$. For $0 \leq n \leq m$, the n-density ratio of μ is defined by

$$\theta(x,r) = \frac{\mu(B(x,r))}{\omega_n r^n} \text{ for } x \in \mathbb{R}^m,\ r > 0, \tag{1}$$

where ω_n denotes the Lebesgue measure of a ball of radius 1 in $\mathbb{R}^n$. The n-density of μ at x exists if

$$0 < \lim_{r\to 0} \theta(x,r) < \infty.$$

In the 1920's, Besicovitch proved that if $m = 2$, $n = 1$ and the 1-density of μ exits μ-a.e then μ is 1-rectifiable ([1]). In the 1950's Marstrand showed that if the density exists on a set of μ positive measure then n is an integer ([4]). In 1986 Preiss proved that if the n-density of μ exits μ-a.e then μ is n-rectifiable ([5]).

Preiss' argument could be outlined as follows:

- A blow up procedure shows that if the n-density of μ exists then for μ-a.e. $x \in \mathbb{R}^m$ all tangent measures of μ at x are n-uniform. The set of tangent measures of μ at x, $\mathrm{Tan}\,(\mu, x)$, consists of all non-zero scaled blow-ups of the measure μ at x. A measure ν is n-uniform if there is a constant $C > 0$ so that for all $r > 0$ and $x \in \operatorname{spt} \nu$, $\nu(B(x, r)) = Cr^n$.
- Let ν be an n-uniform tangent measure then ν is either a flat measure (i.e a multiple of the n-dimensional Hausdorff measure restricted to an n-plane) or its support is very far away from any n-plane.
- For μ a.e. $x \in \operatorname{spt} \mu$, $\mathrm{Tan}\,(\mu, x)$ contains a flat measure.
- By connectivity for μ a.e $x \in \operatorname{spt} \mathbb{R}^m$ all measures in $\mathrm{Tan}\,(\mu, x)$ are flat.

Consider the following two examples of n-inform measures in $\mathbb{R}^m$:

- Let L be an n-plane in $\mathbb{R}^m$ and $\nu = \mathcal{H}^n \llcorner L$ be the n-dimensional Hausdorff measure on an L.
- Kowalski-Preiss cone: let $\Sigma = \{(x_1, x_2, x_3, x_4, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$ then $\nu = \mathcal{H}^n \llcorner \Sigma$ satisfies $\nu(B(x, r)) = \omega_n r^n$ for $x \in \Sigma$ and $r > 0$. Note that here $m = n + 1$.

What distinguishes these two n-uniform measures is the degree of flatness of their support. Thus we introduce two quantities which measure flatness. Let $\Sigma \subset \mathbb{R}^m$, for $x \in \Sigma$ and $r > 0$ define

$$\beta_2(x, r) = \min_{L \in G(n,m)} \frac{1}{r^n} \int_{B(x,r)} \left(\frac{\mathrm{dist}\,(y, L)}{r} \right)^2 d\mu(y).$$

and

$$\beta_\infty(x, r) = \frac{1}{r} \inf_L D[\Sigma \cap B(x, r), L \cap B(x, r)],$$

where the infimun is taken over all n-planes L through x. D denotes the Hausdorff distance between 2 sets. Note that

$$\beta_2(x, r) \le C\beta_\infty(x, r). \tag{2}$$

Definition 1. *A point x is an L^∞ (resp. L^2) flat point of $\mathcal{S}$ if given $\epsilon > 0$ there is $r_0 > 0$ such that $\beta_\infty(x, r) < \epsilon$ (resp. $\beta_2(x, r) < \epsilon$) for $0 < r < r_0$. A set $\mathcal{S}$ is δ-Reifenberg flat (for some $\delta > 0$) if there exits $R > 0$ such that for all $x \in \mathcal{S}$ and $0 < r < R$, $\beta_\infty(x, r) \le \delta$.*

We now focus our attention in the following question:

Let μ be a Borel locally finite measure, and let

$$\mathcal{S} = \{x \in \mathbb{R}^m : \ \mu(B(x, r)) > 0, \ \forall r > 0\}.$$

Assume that there exist $\alpha \in (0, 1)$ and $C > 0$ such that

$$|\theta(x, r) - 1| \le Cr^\alpha \text{ for } x \in \Sigma \text{ and } 0 < r \le 1, \quad \text{how smooth is } \mathcal{S}? \tag{3}$$

First we note that $\mu = \mathcal{H}^n \llcorner \mathcal{S}$. Thus by Preiss' result $\mathcal{S}$ is n-rectifiable.

Theorem 1. *(***Local regularity** [2], [6]*) For $\alpha > 0$ there exists $\gamma > 0$ such that if $\Sigma \subset \mathbb{R}^m$, satisfies (3) and if $n \geq 3$, Σ is Reifenberg flat, then Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^m$.*

Theorem 2. *(***Global regularity** [6]*) For $\alpha > 0$ there exists $\gamma > 0$ such that if $\Sigma \subset \mathbb{R}^m$, satisfies (3) then if $n = 1, 2$, Σ is a $C^{1,\gamma}$ n-submanifold in $\mathbb{R}^m$. If $n \geq 3$, Σ is a $C^{1,\gamma}$ n-submanifold in $\mathbb{R}^m$ away from a closed set $\mathcal{S}$ such that $\mathcal{H}^n(\mathcal{S}) = 0$.*

We briefly sketch the proof of the global regularity theorem:

(1) (3) implies that all tangent and pseudo-tangent measures of μ are n-uniform (see [5] and [3]).
(2) Since $\mathcal{S}$ is rectifiable L^∞ flatness is a generic condition. Thus by (2), L^2 flatness is also a generic condition.
(3) L^2 flatness on $\mathcal{S}$ is an open condition (see[6]).
(4) L^2 flatness in a neighborhood of $x \in \mathcal{S}$ implies that there is a neighborhood of $x \in \mathcal{S}$ which is Reifenberg flat.
(5) The local regularity theorem ensures that a neighborhood of $x \in \mathcal{S}$ admits a $C^{1,\gamma}$ parameterization.

Corollary 3. *For each $\alpha > 0$ there is $\gamma > 0$ such that if $\Sigma \subset \mathbb{R}^{n+1}$ satisfies (3) then Σ is a $C^{1,\gamma}$ n-submanifold in $\mathbb{R}^{n+1}$ away from a closed set $\mathcal{S}$ of dimension $n-3$. If $n = 3$ $\mathcal{S}$, is discrete.*

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Hessian estimates for the sigma-2 equation in dimension three

MICAH WARREN
(joint work with Yu Yuan)

We derive an *interior a priori* Hessian estimate for the σ_2 equation

$$\sigma_2\left(D^2u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{1}$$

in dimension three, where λ_i are the eigenvalues of the Hessian D^2u.

Theorem 1. *Let u be a smooth solution to (1) on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\max_{B_R(0)}|Du|^3/R^3\right].$$

In the 1950's, Heinz [H] derived a Hessian bound for the two dimensional Monge-Ampère equation, $\sigma_2(D^2u) = \lambda_1\lambda_2 = \det(D^2u) = 1$, which is equivalent to (2) with $n = 2$ and $\Theta = \pm\pi/2$. In the 1970's Pogorelov [P] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$.

By Trudinger's [T] gradient estimates for σ_k equations, we can bound D^2u in terms of u as

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\max_{B_{2R}(0)}|u|^3/R^6\right].$$

We attack (1) via its special Lagrangian equation form

$$\sum_{i=1}^{n}\arctan\lambda_i = \Theta \tag{2}$$

with $n = 3$ and $\Theta = \pi/2$. Equation (2) stems from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is constant Θ, and it is special if and only if $(x, Du(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17].

In dimensions two and three, the special Lagrangian equations (2) can be expressed as

$$\cos\Theta(\sigma_1 - \sigma_3) + \sin\Theta(\sigma_2 - 1) = 0.$$

1. Outline of Proof

Heuristically, the proof breaks into the following three steps. We may assume by scaling that u is a solution on $B_4(0) \subset \mathbb{R}^3$.

Step 1)

Choose a function $b(D^2u(x))$ such that, with respect to the induced metric on the graph $(x, Du(x))$, b satisfies the (weak) Jacobi inequality

$$\triangle_g b \geq |\nabla_g b|_g^2.$$

We choose

$$b(D^2u) = \ln \sqrt{1 + \lambda_{\max}^2}$$

where $\lambda_{\max}$ is the largest eigenvalue of the Hessian D^2u.

Step 2)

By Michael and Simon's mean value and Sobolev inequalities for minimal surfaces, it follows from Step 1 (choosing appropriate exponents) that

$$b(0) \leq C(3) \left[\int_{B_2} \varphi^2 \left|\nabla_g b\right|^2 V dx + \int_{B_2} b \left|\nabla_g \varphi\right|^2 V dx \right]$$

where V is the volume element on the graph, and φ is an appropriately chosen test function.

Step 3)

Finally, we show that the integrals in Step 2 may be bounded by $\|Du\|_{L^\infty(B_4)}$. In fact, Step 1) implies that

$$\int_{B_2} \varphi^2 \left|\nabla_g b\right|^2 dv_g \leq C(3) \int_{B_2} \left|\nabla_g \varphi\right|^2 dv_g.$$

The identity

$$g^{ii} V = \sigma_1 - \lambda_i$$

yields an estimate

$$\int_{B_2} \left|\nabla_g \varphi\right|^2 V dx \leq C(3) \int_{B_2} \triangle u dx \leq \|Du\|_{L^\infty(B_2)}.$$

Further integration by parts, using the above identity, completes the estimate.

2. Questions

1) Estimates for $\sigma_2\left(D^2u\right) = 1$ when $n \geq 4$? The special Lagrangian structure is available only when $n = 2$ or 3, so a challenge is to replace the mean value and Sobolev inequalities in Step 2.

2) Estimates for $\sigma_2\left(D^2u\right) = f(x, u, Du)$? What conditions on $f(x, u, Du)$? The mean curvature is not given by a simple expression in terms of derivatives of f, so this generalization does not follow immediately.

3) Similar estimates for special Lagrangian equations (2) with larger phase $|\Theta| \geq \pi/2$ in dimension three are obtained in [WY]. The challenging problem is to derive estimates for (2) with lower phase, in particular the equation $\sigma_1 = \sigma_3$ corresponding to $\Theta = 0$.

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Boundary regularity for an elliptic system in geometry and an application

Sophie Szu-yu Chen

We consider in dimension four an elliptic system

$$\delta R_{ij} = (Rc * Rm)_{ij}$$

with boundary conditions $\nabla R_{\gamma\beta} = 0$ and $L_{\gamma\beta} = 0$, where Rc, Rm and L are Ricci tensor, Riemann tensor and the second fundamental form, respectively. The equation is motivated by a conformally invariant tensor (Bach tensor) and its matching conformally invariant tensor on the boundary. We will show a partial regularity result and an application to a compactness result in conformal geometry.

Calderón-Zygmund estimates around limit cases

Giuseppe Mingione

I am reporting on some results from my recent paper [8]. I will consider a Dirichlet problem of the type

$$\begin{cases} -\text{div } a(x, Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here and in the following I am assuming that $\Omega \subset \mathbb{R}^n$ is a bounded domain, μ is a signed Radon measure with finite total variation $|\mu|(\Omega) < \infty$, and $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathèodory vector field satisfying the following standard monotonicity and Lipschitz assumptions:

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_2 - z_1|^2 \le \langle a(x, z_2) - a(x, z_1), z_2 - z_1 \rangle \\ |a(x, z_2) - a(x, z_1)| \le L(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_2 - z_1| \\ |a(x, 0)| \le L s^{p-1} , \end{cases} \tag{2}$$

for every $z_1, z_2 \in \mathbb{R}^n$, $x \in \Omega$. When referring to the structural properties of a, and in particular to (2), I do always assume

$$2 \le p \le n, \qquad 0 < \nu \le 1 \le L, \qquad s \ge 0\, . \tag{3}$$

The measure μ will be considered as defined on the whole $\mathbb{R}^n$ by simply letting $|\mu|(\mathbb{R}^n \setminus \Omega) = 0$. At certain stages, I shall also require the following Lipschitz continuity assumption on the map $x \mapsto a(x,z)$:

$$(4) \qquad |a(x,z) - a(x_0,z)| \leq L|x-x_0|(s^2+|z|^2)^{\frac{p-1}{2}}, \qquad \forall\, x, x_0 \in \Omega,\ z \in \mathbb{R}^n .$$

Assumptions (2) are modeled on the basic example

$$(5) \qquad -\text{div}[c(x)(s^2+|Du|^2)^{\frac{p-2}{2}}Du] = \mu, \qquad \nu \leq c(x) \leq L, \qquad [c(\cdot)]_{0,1} \leq L ,$$

which is indeed covered here. When $s=0$ and $c(x) \equiv 1$ the equation in (5) has on the left hand side the familiar p-Laplacean operator

$$(6) \qquad -\triangle_p u = -\text{div}(|Du|^{p-2}Du) = \mu .$$

The basic result, see [4], about the problem (1) is the existence of a distributional solution $u \in W^{1,1}_0(\Omega)$, obtained by means of an approximations argument, i.e. uniform a priori estimates for problems with smooth right hand side + passage to the limit, such that

$$(7) \qquad Du \in L^q(\Omega,\mathbb{R}^n), \qquad \text{for every} \quad q < \frac{n(p-1)}{n-1} \quad \text{when} \quad p \leq n .$$

The inclusion in (7) is in a certain sense sharp in the scale of Lebesgue's spaces (a discussion about sharpness in connection to the definition of solution is hereby omitted for the sake of brevity, since it would be definitely a long story...). This is clear when considering the so called fundamental solution to the p-Laplacean equation, that is, up to multiplicative constant, $|x|^{\frac{p-n}{p-1}}$ when $p<n$, otherwise it is $\log|x|$. Anyway a significant sharpening of (7) can be obtained by considering so called weak-L^q spaces, often called Marcinkiewicz spaces; in fact, see [3, 6], it holds that

$$(8) \qquad Du \in \mathcal{M}^{\frac{n(p-1)}{n-1}}(\Omega,\mathbb{R}^n) .$$

as proved in [3, 6]. This means that

$$\sup_{\lambda\geq 0} \lambda^{\frac{n(p-1)}{n-1}}|\{x \in \Omega \ : \ |Du(x)| \geq \lambda\}| < \infty .$$

The results (7)-(8) are more or less all is known about the regularity of solutions to general measure data problems as (1). My aim here is to show how these integrability properties are connected to, and actually are, a consequence of a much deeper regularity property of Du, at least in a local fashion - in [8] I am concentrating on local estimates for the sake of brevity. In turn, this represents *the maximal regularity of solutions* which is expectable under the general assumptions (2)-(4). In some sense I will show an extension of the classical Calderón-Zygmund theory beyond those cases which are traditionally considered as limit ones.

Let me focus for simplicity on the case $p=2$, looking at (7) from a different viewpoint, considering $\triangle u = f$. In this case *the standard Calderón-Zygmund theory* asserts $f \in L^{1+\varepsilon}$ implies $Du \in W^{1,1+\varepsilon}$ for every $\varepsilon > 0$. Using Sobolev embedding theorem we have in particular $Du \in L^{n/(n-1)}$, that is, the "limit case" of (7) when $p=2$. This does not hold when $\varepsilon = 0$, since the inclusion $Du \in W^{1,1}$

generally fails, as is well-known. So, one could interpret (7) *as the trace of a potentially existent Calderón-Zygmund theory below the limit case* $W^{1,1}$. In this respect the first result I am presenting is

Theorem 1 (of Calderón-Zygmund type)**.** *Under the assumptions* (2)-(4) *there exists a solution* $u \in W_0^{1,1}(\Omega)$ *to the problem* (1) *such that*

$$Du \in W_{\mathrm{loc}}^{\frac{1-\varepsilon}{p-1},p-1}(\Omega,\mathbb{R}^n)\,, \tag{9}$$

holds for every $\varepsilon \in (0,1)$*. In particular*

$$Du \in W_{\mathrm{loc}}^{1-\varepsilon,1}(\Omega,\mathbb{R}^n)\,, \qquad \text{when} \quad p=2\,. \tag{10}$$

In other words, if we cannot have second differentiability of solutions, (10) tells us that we "almost have" second derivatives. Explicit local estimates is a sharp form are actually available, see [8, Theorem 1.3].
Let me recall here that the function spaces involved in Theorem 1 are the well-known "fractional Sobolev spaces", defined by saying that $w \in W^{\alpha,q}(A,\mathbb{R}^k)$ provided the following Gagliardo-type norm is finite:

$$\|w\|_{W^{\alpha,q}(A)} \;:=\; \left(\int_A |w(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\int_A\int_A \frac{|w(x)-w(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy\right)^{\frac{1}{q}}\,,$$

where $\alpha \in (0,1)$ and $q \geq 1$; the local variant $W_{\mathrm{loc}}^{\alpha,q}(A,\mathbb{R}^k)$ is obviously defined. Note that Sobolev embedding theorem in the fractional case ensures that

$$W^{\alpha,q} \hookrightarrow L^{\frac{nq}{n-\alpha q}} \qquad \text{provided} \qquad \alpha q < n\,. \tag{11}$$

Therefore, we observe that also (9) is sharp since, assuming (9) with $\varepsilon = 0$ would lead to $Du \in L_{\mathrm{loc}}^{\frac{n(p-1)}{n-1}}$, which is obviously false for the fundamental solution. The same argument then allows to recover, at least in a local fashion, the original result (7) from the one in (9), by means of (11).
A remarkable consequence of Theorem 1 is the following:

Corollary 1 (BV-type behavior)**.** *Let* Σ_u *denote the set of non-Lebesgue points of the solution found in Theorem 1, in the sense of*

$$\Sigma_u := \left\{ x \in \Omega : \liminf_{\rho\searrow 0} \fint_{B(x,\rho)} |Du(y)-(Du)_{x,\rho}|^{p-1}\,dy > 0 \quad \text{or} \quad \limsup_{\rho\searrow 0} |(Du)_{x,\rho}| = \infty \right\}. \tag{12}$$

Then its Hausdorff dimension $\dim(\Sigma_u)$ *satisfies* $\dim(\Sigma_u) \leq n-1$.

In fact, a classical result in potential theory asserts that the set of non-Lebesgue points of a $W^{\alpha,q}$-map, with $\alpha q < n$, has Hausdorff dimension less than or equal than $n - s\alpha$.
The optimality of the exponent $n(p-1)/(n-1)$ in (8) stems from considering measure data problems involving the Dirac mass, as $\triangle_p u = \delta$. The question is

now: What happens if the measure "diffuses"? We consider now a condition as

$$|\mu|(B_R) \leq MR^{n-\theta} \qquad 0 \leq \theta \leq n, \qquad M \geq 0, \tag{13}$$

satisfied for any ball $B_R \subset \mathbb{R}^n$ of radius R. Assuming (13) does not allow μ to concentrate on sets with Hausdorff dimension less than $n-\theta$, and indeed higher regularity of solutions can be obtained. Two cases must be anyway distinguished. When $\theta < p$ we have that a classical theorem of Adams [1] ensures that μ belongs to the dual of $W^{1,p}$, and therefore problem (1) can be uniquely solved in $W_0^{1,p}(\Omega)$ by standard monotonicity methods: this is in some sense the "uninteresting case". I shall concentrate on the case $\theta \in [p,n]$. An interesting phenomenon is coming up here: we shall see that the number θ replaces n everywhere, playing the role of a "new dimension". Indeed

Theorem 2 (Marcinkiewicz-Morrey regularity). *Under the assumptions* (2)-(3), *and* (13) *with $\theta \geq p$, there exists a solution $u \in W_0^{1,1}(\Omega)$ to the problem* (1) *such that*

$$Du \in \mathcal{M}_{\mathrm{loc}}^{\frac{\theta(p-1)}{\theta-1},\theta}(\Omega,\mathbb{R}^n) \subseteq \mathcal{M}_{\mathrm{loc}}^{\frac{\theta(p-1)}{\theta-1}}(\Omega,\mathbb{R}^n)\ . \tag{14}$$

In particular, in the limit case $\theta = p$ we have $Du \in \mathcal{M}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$.

The space $\mathcal{M}^{q,s}(A)$, with $q \geq 1$ and $s \in [0,n]$, is the Marcikiewicz-Morrey space defined by saying that $w \in \mathcal{M}^{q,s}(A)$ iff

$$\sup_{\lambda \geq 0} \lambda^q |\{x \in B_R \ :\ |w(x)| \geq \lambda\}| \leq cR^{n-s}\ ,$$

for every ball $B_R \subset A$. Observe how in the case $\theta = n$ inclusion (14) gives back the original one in (8). Moreover, in the limit case $p = \theta$, the gradient Du misses the L^p-integrability, which on the other hand holds as soon as $\theta < p$, just by a natural Marcinkiewicz factor. In this last case it is possible to prove that u is a Bounded Mean Oscillation Function in the sense of John-Nirenberg, and, under a slightly stronger assumption, that is even VMO in the sense of Sarason- see [8, Theorem 1.12]. When $p = n$, which forces $\theta = n$, Theorem 2 recovers in the case of scalar problems the results by Dolzmann & Hungerbühler & Müller [6]. Finally, the exponent $\theta(p-1)/(\theta-1)$ is sharp when $p = 2$, as shown by Adams [2]; it is also sharp when $p \neq 2$ and θ is an integer, as shown by Serrin [9], and it is safely conjectured to be sharp for every choice $p > 2$.
Finally, the density property (13) also reflects on the higher derivatives of Du:

Theorem 3 (Sobolev-Morrey regularity). *Under the assumptions* (2)-(4), *and* (13) *with $\theta \geq p$, let $u \in W_0^{1,1}(\Omega)$ be the solution found in Theorem 2. Then*

$$\int_{B_R}\int_{B_R} \frac{|Du(x)-Du(y)|^{p-1}}{|x-y|^{n+1-\varepsilon}}\ dx\ dy \leq cR^{n-\theta}\ , \tag{15}$$

holds for every $\varepsilon \in (0,1)$ and every ball $B_R \subset\subset \Omega$ of radius R, where c depends on ε, M, and on the distance between B_R and $\partial\Omega$.

The density property (15) actually means that Du belongs to the Sobolev-Morrey space $W^{\frac{1-\varepsilon}{p-1},p-1,\theta}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$, for every $\varepsilon \in (0,1)$, and for $p=2$, it is the natural borderline analog of the classical Calderón-Zygmund estimates in Morrey spaces derived by several authors like Campanato, Caffarelli, Liebermann, and so on. See for instance [5, 7, 10]. When $\theta = n$, Theorem 3 gives back Theorem 1.

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The role of integrability by compensation in the analysis of Willmore surfaces

Tristan Rivière

We present a new formulation to the Euler-Lagrange equation of the Willmore functional for immersed surfaces in $\mathbb{R}^m$. This new formulation of Willmore equation appears to be of divergence form, moreover, the non-linearities are made of jacobians. Additionally to that, if $\vec{H}$ denotes the mean curvature vector of the surface, this new form writes $\mathcal{L}\vec{H} = 0$ where $\mathcal{L}$ is a well defined locally invertible elliptic self-adjoint operator. These 3 facts have numerous consequences in the analysis of Willmore surfaces.

One first consequence is that the long standing open problem to give a meaning to the Willmore Euler-Lagrange equation for immersions having only L^2 bounded second fundamental form is now solved. We then establish the regularity of weak Willmore immersion with L^2 bounded second fundamental form. The proof of this result is based on the discovery of conservation laws for Willmore immersions which are preserved under weak convergences. These conservation laws are made of linear combinations of linear terms in one hand and non linear quantities which happen

to be quadratic and made of jacobians in the other hand. This last fact permits to make use of the integrability by compensation theory introduced originally by H. Wente in his work on prescribed mean curvature surfaces and systematized and extended in the works of L.Tartar and R.Coifman-P.L.Lions-Y.Meyer-S.Semmes. We shall explain how this use of integrability by compensation not only provides new results in border line situations but also permits to give a much simpler approach to existing analysis results for Willmore surfaces such as $\epsilon-$regularity results...etc.
In the last part of our presentation we make use of the previous analysis in order to establish a weak compactness result for Willmore surfaces of energy less than 8π (the Li-Yau condition which ensures the embeddedness of the surface). This theorem is based on a point removability result we prove for Willmore surfaces in $\mathbb{R}^m$. This result extends to arbitrary codimension a result previously established by E.Kuwert and R.Schätzle for surfaces in $\mathbb{R}^3$. Finally, we deduce from this point removability result the strong compactness, modulo the Möbius group action, of Willmore tori below the energy level 8π in dimensions 3 and 4. The dimension 3 case was already solved in the work of Kuwert and Schätzle.

Minimizers of the Willmore functional with prescribed conformal type

Ernst Kuwert
(joint work with Reiner Schätzle)

The Willmore integral of an immersed surface $f : \Sigma \to \mathbb{R}^n$ is given by

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |\vec{H}|^2\, d\mu_g,$$

where $g = f^* g_{euc}$ is the induced metric with area measure μ_g, and $\vec{H}$ is the mean curvature vector. An important property of the integral is its invariance under Möbius transformations of $\mathbb{R}^n$. By the work of L. Simon [8], the infimum β_p^n of the functional among all genus p immersions is attained (see also [1] for the case $p \geq 2$).

Willmore conjectured that $\mathcal{W}(f) \geq 2\pi^2$ for all tori in $\mathbb{R}^3$. By now, this inequality has been verified for several classes of tori, in particular Li and Yau proved it in [3] if the induced conformal type belongs to a certain compact subset of the moduli space, compare [4]. The inequality holds also for tori whose conformal type is sufficiently degenerate [2]. A proof of the full Willmore conjecture is claimed in [6].

Fixing a class τ in the Teichmüller space $\mathcal{T}(\Sigma)$ where Σ has genus $p \geq 1$, we put

$$\mu_p^n(\tau) = \inf\{\mathcal{W}(f) |\, f : \Sigma \to \mathbb{R}^n \text{ immersed}, \pi(f^* g_{euc}) = \tau\}.$$

Here π denotes the projection from the set of Riemannian metrics into $\mathcal{T}(\Sigma)$. We refer to [9] for the definition of the Teichmüller space. Our main result is as follows.

Theorem. *If $n \in \{3,4\}$ and if $\mu_p^n(\tau) < \omega_p^n$, then the infimum $\mu_p^n(\tau)$ is attained by a smooth embedding $f : \Sigma \to \mathbb{R}^n$, which satisfies the Euler-Lagrange equation*

$$\vec{W}(f) = g(A^\circ, q).$$

In the statement, $\vec{W}(f)$ is the L^2 gradient of the Willmore functional, A° is the vector-valued, traceless second fundamental form and q is a transverse traceless 2-tensor which comes up as a Lagrange multiplier. The condition on the infima is used to ensure the convergence of minimizing sequences using results from [2]. The constants ω_p^n are given by

$$\omega_p^n = \begin{cases} \min(8\pi, \tilde{\beta}_p^3) & \text{for } n = 3, \\ \min(8\pi, \tilde{\beta}_p^4, \beta_p^4 + \frac{8\pi}{3}) & \text{for } n = 4, \end{cases}$$

where for $p = 1$ we have $\tilde{\beta}_1^n = \infty$, and for $p \geq 2$

$$\tilde{\beta}_p^n = \min\Big\{4\pi + \sum_{i=1}^{k} (\beta_{p_i}^n - 4\pi) : 1 \leq p_i < p, \sum_{i=1}^{k} p_i = p\Big\}.$$

We have $2\pi^2 < 8\pi = \omega_1^3$, and in fact $\beta_p^n < \omega_p^n$ for all $p \geq 1$, $n \in \{3,4\}$. On the other hand, it is not known whether the assumption $\mu_p^n(\tau) < \omega_p^n$ is satisfied for all conformal classes τ.

Minimizers of prescribed conformal type are also considered in [7], without restrictions on the Willmore energy. The functional is extended to maps which are not necessarily immersed, and the minimization is approached in this generalized setting.

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Willmore blow-ups are never compact

Ralph Chill
(joint work with Eva Fašangová and Reiner Schätzle)

We study the qualitative behaviour near the maximal existence time of *Willmore flows* which are, by definition, solutions of the evolution equation

$$\partial_t f_t + \frac{1}{2}(\Delta^{\perp} H + Q(A^0)H) = 0, \quad t \geq 0. \tag{1}$$

Here, the $f_t : \Sigma \to \mathbb{R}^n$ are immersions of a closed surface Σ, H and A^0 are the corresponding mean curvature vector and the trace-free part of the second fundamental form, respectively, $\Delta^{\perp}$ is the Laplacian in the normal bundle along f_t, and $Q(A^0)H = A^0(e_i, e_j)\langle A^0(e_i, e_j), H\rangle$.
It is an important feature of this evolution equation that it is a gradient system for the *Willmore functional* given by

$$\mathcal{W}(f) := \frac{1}{4}\int_{\Sigma} |H|^2 \, d\mu_f;$$

see [3]. The Willmore functional is invariant under the full Möbius group of $\mathbb{R}^n$. Critical points of the Willmore functional are called *Willmore surfaces* or, more precisely, *Willmore immersions*.

Let $T \leq \infty$ be the maximal existence time of a Willmore flow (f_t). In [2] and [4], a blow-up procedure was described asserting that for each $t_j \nearrow T$, after passing to a subsequence, there exists $r_j > 0$ and $x_j \in \mathbb{R}^n$ such that

$$r_j^{-1}(f_{t_j + c_0 r_j^4} - x_j)$$

converges, for given $c_0 = c_0(n) > 0$ and after appropriate reparametrization, smoothly on compact subsets of $\mathbb{R}^n$ to a Willmore immersion $f_W : \hat{\Sigma} \to \mathbb{R}^n$, where $\hat{\Sigma} \neq \emptyset$ is a complete surface without boundary. Moreover, it is possible to select t_j in such a way that f_W is non-trivial in the sense $A_{f_W} \not\equiv 0$. After rescaling and passing to appropriate subsequences, one has either $r_j \to 0$, $r_j = 1$ or $r_j \to \infty$ (one always has $r_j \to 0$ if the maximal existence time is finite). We call the limit f_W *blow-up* if $r_j \to 0$ and *blow-down* if $r_j \to \infty$.

Our main result says that none of the components of blow-ups or blow-downs is compact.

The proof of this main result is based on the gradient structure of the Willmore flow, the fact that the Willmore functional satisfies the Lojasiewicz-Simon gradient inequality near every Willmore immersion (see [5] and [6] for first instances of this inequality in the literature), and the following global existence and convergence result for the Willmore flow.

Lemma. *For every* $k \in \mathbb{N}$, $\delta > 0$, *there exists* $\varepsilon > 0$ *such that the following is true: suppose that* $(f_t)_t$ *is a Willmore flow of* Σ *satisfying*

$$\|f_0 - f_W\|_{W^{2,2} \cap C^1} < \varepsilon \text{ and}$$

$$\mathcal{W}(f_t) \geq \mathcal{W}(f_W) \text{ whenever } \|f_t \circ \Phi_t - f_W\|_{C^k} \leq \delta, \tag{2}$$

for some appropriate diffeomorphisms $\Phi_t : \Sigma \xrightarrow{\approx} \Sigma$.
Then this Willmore flow exists globally, that is, $T = \infty$, *and converges, after reparametrization by appropriate diffeomorphisms* $\tilde{\Phi}_t : \Sigma \xrightarrow{\approx} \Sigma$, *smoothly to a Willmore immersion* f_∞, *that is,*

$$f_t \circ \tilde{\Phi}_t \to f_\infty \text{ as } t \to \infty.$$

Moreover, $\mathcal{W}(f_\infty) = \mathcal{W}(f_W)$ *and* $\|f_\infty - f_W\|_{C^k} \leq \delta$.

This global existence and convergence result also implies that if there is no blow-up or blow-down (i.e. $r_j = 1$ in the above blow-up procedure), then the Willmore flow necessarily exists globally and converges, after appropriate reparametrization, to a Willmore immersion.

In addition, the above lemma implies that if the Willmore flow starts near a local minimizer of the Willmore functional (near with respect to the topology in $W^{2,2} \cap C^1$), then again the flow exists globally and converges, after appropriate reparametrization, to a Willmore immersion which is itself a local minimizer of the Willmore functional. This corollary generalizes a result by Simonett [7].

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Some aspects and results for compressible and incompressible Navier-Stokes equation

Jens Frehse

(joint work with M.S. Neugebauer - M. Steinhauer & W. Weigant)

1. We present an example of a scalar uniformly elliptic equation with measurable complex valued coefficients having a complex valued discontinuous or unbounded solution in dimension $d > 2$. The case of $d = 3$ or 4 has been left open by Mazya-Nasarov-Plamenevsky.

2. (joint work with Mrs. M. Specovius Neugebauer). We present "re-normalized" estimates for the solution of the classical time dependent Navier-Stokes equations (incompressible case). Roughly spoken, the gradient of the solution divided by some power of the modulus of the solution is in $L^3(L^3)$.

3. (joint work with M.Steinhauer and Wladimir Weigant). We consider the classical Navier-Stokes equation for compressible fluids (stationary case) with some pressure law $p = c \cdot \rho^\gamma$. Depending on the boundary value problem in consideration we obtain existence of long-time-weak solutions for values of $\gamma < 3/2$ (up to 1 in special cases).

Curvature and continuity of optimal transport

Robert J. McCann

(joint work with Young-Heon Kim)

This abstract sketches a geometric framework proposed in [1] and its consequences concerning the general regularity theory for optimal mappings developed by Ma, Trudinger, Wang and Loeper, following pioneering work on special cost functions by (at least) Caffarelli, Delanoë, Huang, Guan, Gutierrez, Oliker, Urbas, and X-J Wang. Due to space limitations we do not attempt to cite the literature or give much historical context, referring the reader instead to our paper, except that we note a different approach to some of our results was discovered independently at about the same time by Trudinger & Wang in arXiv:math/0702807. For simplicity our assumptions here are more restrictive than required in [1].
Let M and $\bar{M}$ be domains with compact closure $\mathrm{cl}\, M \subset M'$ and $\mathrm{cl}\, \bar{M} \subset \bar{M}'$ in smooth manifolds M' and $\bar{M}'$. Suppose M and $\bar{M}$ are equipped with Borel probability measures ρ and $\bar{\rho}$, and let $s \in C^4(\Omega')$ be the *surplus* (= − transportation cost) defined on the product space $\Omega' = M' \times \bar{M}'$. The optimal transportation problem of Kantorovich is then to find the measure $\gamma \geq 0$ on $M \times \bar{M}$ which achieves the supremum

$$-W_{-s}(\rho, \bar{\rho}) := \max_{\gamma \in \Gamma(\rho, \bar{\rho})} \int_{M \times \bar{M}} s(x, \bar{x}) d\gamma(x, \bar{x}). \quad (1)$$

Here $\Gamma(\rho, \bar{\rho})$ denotes the set of joint probabilities having the same left and right marginals as $\rho \otimes \bar{\rho}$. It is not hard to check that this maximum is attained; any maximizing measure $\gamma \in \Gamma(\rho, \bar{\rho})$ is then called *optimal.* Each feasible $\gamma \in \Gamma(\rho, \bar{\rho})$ can be thought of as a weighted relation pairing points x distributed like ρ with points $\bar{x}$ distributed like $\bar{\rho}$; optimality implies this pairing also maximizes the average value of the specified surplus $s(x, \bar{x})$ for transporting each point x to its destination $\bar{x}$.

The optimal transportation problem of Monge amounts to finding a Borel map $F : M \longrightarrow \bar{M}$, and an optimal measure γ vanishing outside $\mathrm{Graph}(F) := \{(x, \bar{x}) \in M \times \bar{M} \mid \bar{x} = F(x)\}$. When such a map F exists, it is called an *optimal map* between ρ and $\bar{\rho}$; in this case, the relation γ is single-valued, so that ρ-almost every point x has a unique partner $\bar{x} = F(x)$, and optimality can be achieved in (1) without subdividing the mass at such points x between different destinations. Although Monge's problem is more subtle to solve than Kantorovich's, when M is a smooth manifold and ρ vanishes on every Lipschitz submanifold of lower dimension, a *twist* condition (see **(A1)** below) on the surplus function $s(x, \bar{x})$ guarantees existence and uniqueness of an optimal map F, as well as uniqueness of the optimal measure γ. One can then ask about the smoothness of the optimal map $F : M \longrightarrow \bar{M}$.

For $\rho, \bar{\rho}$ smooth and bounded away from zero on their respective domains, Ma, Trudinger & Wang gave hypotheses on Euclidean domains $M, \bar{M} \subset \mathbf{R}^n$ and $s \in C^4(\Omega')$ which ensure an affirmative answer. Their hypotheses may appear daunting, but inspired by Loeper's discoveries on Riemannian manifolds we recast them geometrically as follows. Use local coordinates $x^1, \dots, x^n$ on M' and $x^{\bar{1}}, \dots, x^{\bar{n}}$ on $\bar{M}'$ to define an inner product $d\ell^2 := (\partial^2 s/\partial x^i \partial x^{\bar{j}})(dx^i \otimes dx^{\bar{j}} + dx^{\bar{j}} \otimes dx^i)/2$ of indefinite sign and a symplectic form $\omega := (\partial^2 s/\partial x^i \partial x^{\bar{j}}) dx^i \wedge dx^{\bar{j}}$ on the tangent bundle $T\Omega'$ of the product space. Repeated indices are summed from $1, \dots, n$ or $n+1, \dots, n+\bar{n}$ according to whether they are barred or unbarred. Assume these bilinear forms are *non-degenerate* **(A2)** and $n = \bar{n}$. Then $d\ell^2$ defines a pseudo-Riemannian metric on Ω' with as many timelike as spacelike directions, i.e. signature (n, n). A vector $P \in T_{(x,\bar{x})}\Omega'$ is called *null* if it is self-orthogonal with respect to this metric. The canonical splitting of a vector in the tangent space $T_{(x,\bar{x})}\Omega' = T_x M' \oplus T_{\bar{x}} \bar{M}'$ is denoted by $P = p \oplus \bar{p}$. The metric $d\ell^2$ induces a pseudo-Riemannian curvature tensor $R_{i'j'k'l'}$ on Ω', which we use to define sectional curvature

$$\sec_{(x,\bar{x})} P \wedge Q := \sum_{i'=1}^{2n} \sum_{j'=1}^{2n} \sum_{k'=1}^{2n} \sum_{l'=1}^{2n} R_{i'j'k'l'} P^{i'} Q^{j'} P^{k'} Q^{l'} \tag{2}$$

in the standard way, except that we do not attempt to normalize it for fear of dividing by zero in the case of most interest to us, namely the null vectors $P = p \oplus \mathbf{0}$ and $Q = \mathbf{0} \oplus \bar{p}$ orthogonal to each other, or equivalently $p \oplus \bar{p}$ null.

The surplus function $s \in C^4(\Omega')$ is said to be *weakly regular* **(A3w)** if $d\ell^2$ is non-degenerate and

$$\sec_{(x,\bar{x})} (p \oplus \mathbf{0}) \wedge (\mathbf{0} \oplus \bar{p}) \geq 0 \tag{3}$$

for all $(x,\bar{x}) \in \Omega'$ and null-vectors $p \oplus \bar{p} \in T_{(x,\bar{x})}\Omega'$. It is said to be *strictly regular* **(A3s)** if, in addition, equality in (3) implies $p = \mathbf{0}$ or $\bar{p} = \mathbf{0}$. This terminology is motivated by the fact that weak regularity is known to be necessary [2] as well as sufficient for smoothness of optimal maps between nice probability measures. A set $\Lambda \subset \Omega'$ is *geodesically convex* if every pair of points $(x,\bar{x}), (y,\bar{y}) \in \Lambda$ is linked by a curve satisfying the geodesic equation for our pseudo-Riemannian metric. It is *vertically convex* if $\Lambda \cap (\{x\} \times \bar{M})$ is geodesically convex for each $x \in M$; *horizontally convex* if $\Lambda \cap (M \times \{\bar{x}\})$ is geodesically convex for each $\bar{x} \in \bar{M}$; and *bi-convex* if both hold. Our first main result is a maximum principle:

Theorem 1. *Let $s \in C^4(M' \times \bar{M}')$ be weakly regular. If $\Lambda \subset M' \times \bar{M}'$ is open, horizontally convex and $t \in [0,1] \longrightarrow (x,\bar{x}(t)) \in \Lambda$ is a geodesic then $\cup_{0 \le t \le 1}(y,\bar{x}(t)) \subset \Lambda$ implies $f(t,y) := s(y,\bar{x}(t)) - s(x,\bar{x}(t)) \le \max\{f(0,y), f(1,y)\}$.*

Idea of proof. Vanishing of $f'(t_0) = 0$ gives the null condition for weak regularity to imply $f''(t_0) \ge 0$, with strict inequality in the strictly regular case. This precludes a local maximum and is obtained using horizontal convexity to integrate the identity

$$2\frac{\partial^4}{\partial r^2 \partial t^2} s(y(r),\bar{x}(t)) = \sec_{(y(r),\bar{x}(t))}(y'(r) \oplus \mathbf{0}) \wedge (\mathbf{0} \oplus \bar{x}'(t)) \ge 0$$

along a geodesic from $(x,\bar{x}(t_0))$ (where all t derivatives of f vanish) to $(y,\bar{x}(t_0))$. □

For $\Lambda = M \times \bar{M}$, a version of this theorem was originally deduced [2] under additional hypotheses by relying on a sophisticated result of Trudinger & Wang. But our theorem requires no additional hypotheses, not even that the surplus $s \in C^4(M' \times \bar{M}')$ be *twisted*, meaning **(A1)**: for each $\bar{y}, \bar{z} \in \bar{M}'$ the function $x \in M' \longrightarrow s(x,\bar{y}) - s(x,\bar{z})$ has no critical points. If, in addition the reflected surplus $s^*(\bar{x},x) = s(x,\bar{x})$ is twisted on $\bar{M}' \times M'$, we say s is *bi-twisted*. Our theorem combines with a subtle yet elementary argument of Loeper to yield [2] [1]:

Theorem 2. *Let $s \in C^4(\Omega')$ be twisted and weakly regular on $\Omega' = \boldsymbol{R}^n \times \boldsymbol{R}^n$ and $M \times \bar{M} \subset \Omega'$ a bounded bi-convex domain. Suppose $u \in C(\mathrm{cl}\, M)$ and $\bar{u} \in C(\mathrm{cl}\, M)$ are continuous functions with $u(x) = \max_{\bar{x} \in \mathrm{cl}\, \bar{M}} s(x,\bar{x}) - \bar{u}(\bar{x})$ for each $x \in \mathrm{cl}\, M$. If there exist $(x,\bar{x}) \in M \times \mathrm{cl}\, \bar{M}$ such that $u(z) \ge u(x) + s(z,\bar{x}) - s(x,\bar{x})$ for all z close to x, then the same inequality holds for all $z \in \mathrm{cl}\, M$.*

For strongly regular, bi-twisted surpluses and probability densities $d\rho/d\,\mathrm{vol} \in L^\infty(M)$ and $d\,\mathrm{vol}/d\bar{\rho} \in L^\infty(\bar{M})$ on $M \times \bar{M} \subset \mathbf{R}^n \times \mathbf{R}^n$ bounded and bi-convex, powerful ideas of Loeper augmented by a few simplifications then yield a self-contained proof [1] of his Hölder continuity of optimal maps: $F \in C^{1,1/\max\{5,4n-1\}}_{loc}(M; \mathrm{cl}\, \bar{M})$. A future ambition is to extend his continuity result to more general geometries $M' \ne \mathbf{R}^n \ne \bar{M}'$. We must surrender smoothness of the cost to satisfy the twist condition as soon as the manifold M' is compact. Hölder continuity results from [2] for the restriction of $s(x,\bar{x}) = \log|x - \bar{x}|$ to the unit sphere $M = \bar{M} = \mathbf{S}^n$ in $\mathbf{R}^{n+1}$, and for the geodesic distance squared $s(x, \bar{x}) = -d^2(x, \bar{x})$ on the round sphere, are also recovered by our technique [1]. In our current work, they are extended to Riemannian submersions of geometries like the latter; (related work is in progress by Delanoë & Ge). We also explore products thereof.

Let us conclude by observing any s-optimal diffeomorphism $F : M \longrightarrow \bar{M}$ has a graph which is spacelike with respect to $d\ell^2$ and Lagrangian with respect to ω, and conversely, using results from Trudinger & Wang, that any diffeomorphism between suitable domains whose graph is $d\ell^2$-spacelike and ω-Lagrangian is in fact the s-optimal map between the measures $\rho := \pi_{\#}(\mathrm{vol}|_{\mathrm{Graph}(F)})$ and $\bar{\rho} := \bar{\pi}_{\#}(\mathrm{vol}|_{\mathrm{Graph}(F)})$ obtained by the canonical projections through $\pi(x, \bar{x}) = x$ and $\bar{\pi}(x, \bar{x}) = \bar{x}$ of the Riemannian volume vol induced by the pseudo-metric $d\ell^2$ on $\mathrm{Graph}(F) \subset \Omega'$. This reveals an unexpected connection between optimal transportation and symplectic or pseudo-Kähler geometry. There is related work of Wolfson and of Warren in the (pseudo-) Euclidean case with $s(x, \bar{x}) = x \cdot \bar{x}$.

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Euler–Poisson systems as energy minimizing paths in the Wasserstein space $\mathcal{P}_2(\mathbb{R})$

Wilfrid Gangbo

(joint work with T. Nguyen and A. Tudorascu)

This paper uses a variational approach to establish existence and uniqueness of solutions (σ_t, v_t) of the 1–d Euler-Poisson system, mimimizing an action. Here, σ_0, σ_T are prescribed and the time interval $[0, T]$ satisfies $T < \pi$. This extends the concept of the Euler-Poisson system to general measures. These solutions conserve the Hamiltonian, Legendre transform of the Lagrangian appearing in the action. They yield a path $t \to \sigma_t$ on $\mathcal{P}$ the set of probability measures on the real line and turn out to be characteristic of an infinite dimensional Hamilton-Jacobi equation on $\mathcal{P}$. The associated Hamiltonian system is nothing but the 1–d Vlasov-Poisson system. When $\sigma_t = \delta_{y(t)}$ is a dirac mass, the Euler-Poisson system reduces to $\ddot{y} + y = 0$, the Hamilton-Jacobi equation is merely finite dimensional and is given by $\partial_t u + 1/2(\partial_y u)^2 + 1/2y^2 + 1/24 = 0$.

Integral formulas for Levi curvature PDE's and applications to isoperimetric inequalities and to symmetry problems

Annamaria Montanari
(joint work with Vittorio Martino)

The study of surfaces in the Euclidean space with either constant Gauss curvature or constant mean curvature received in the past a great amount of attention. In 1899 Liebmann proved that the spheres are the only compact surfaces in $\mathbb{R}^3$ with constant Gauss curvature. In 1952 Süss extended the Liebmann result showing that a compact convex hypersurface in the Euclidean space must be a sphere, provided that for some j the $j-$th elementary symmetric polynomial in the principal curvatures is constant. In 1954 Hsiung proved that the "convexity" assumption can be relaxed to the "star-shapedness" one. The proofs of the above results are based on the Minkowski formulae. A breakthrough for this sort of problems was made by Alexandrov in 1956, who proved that the sphere is the only compact hypersurface embedded into the Euclidean space with constant mean curvature. Alexandrov method is completely different from the Liebmann-Süss method, and is based on the moving plane technique, on the interior maximum principle for elliptic equations and on the boundary maximum principle of Hopf type for uniformly elliptic equations. In 1978 Reilly obtained another proof of the Alexandrov theorem combining the Minkowski formulae with some new elegant arguments which are useful to study the Levi-curvature equations, too.
In [7] we introduced the j-th Levi curvature for real hypersurfaces in $\mathbb{C}^{n+1}$ in terms of elementary symmetric functions of the eigenvalues of the normalized Levi form and we proved that, if Ω is a bounded domain of $\mathbb{C}^{n+1}$ with boundary a real hypersurface of class C^2, then the j-th Levi curvature of $\partial\Omega$ at $z = (z_1, \dots, z_{n+1}) \in \partial\Omega$ writes in term of defining function f of $\Omega = \{f(z) < 0\}$ as

$$K^{(j)}_{\partial\Omega}(z) = -\frac{1}{\binom{n}{j}} \frac{1}{|\partial f|^{j+2}} \sum_{1 \le i_1 < \cdots < i_{j+1} \le n+1} \Delta_{(i_1, \cdots, i_{j+1})}(f) \tag{1}$$

for all $j = 1, \dots, n$, where $|\partial f| = \sqrt{\sum_{j=1}^{n+1} |f_j|^2}$,

$$\Delta_{(i_1,\cdots,i_{j+1})}(f) = \det \begin{pmatrix} 0 & f_{\bar{i}_1} & \cdots & f_{\bar{i}_{j+1}} \\ f_{i_1} & f_{i_1,\bar{i}_1} & \cdots & f_{i_1,\bar{i}_{j+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1}} & f_{i_{j+1},\bar{i}_1} & \cdots & f_{i_{j+1},\bar{i}_{j+1}} \end{pmatrix}$$

and $f_j = \frac{\partial f}{\partial z_j}$, $f_{\bar{j}} = \overline{f_j}$, $f_{j\bar{\ell}} = \frac{\partial^2 f}{\partial z_j \partial \bar{z}_\ell}$.
In [7] we then proved a strong comparison principle, leading to the following symmetry theorem for domains with constant curvatures.

Theorem 1. *Let $D \subseteq \mathbb{C}^{n+1}$ be a strictly j-pseudoconvex domain with connected boundary, $1 \le j \le n$. Let $B_R(z_0) \subseteq D$ be a ball of radius R and center at z_0. Assume that $B_R(z_0)$ is tangent to ∂D at some point $p \in \partial D$. If $K^{(j)}_{\partial D}(z)$ is the j-th Levi curvature of ∂D at $z \in \partial D$ and*

$$K^{(j)}_{\partial D}(z) \ge 1/R^j, \quad \forall z \in \partial D,$$

then D is a ball of radius R.

Theorem 1 suggested the following question: are spheres the unique compact hypersurfaces with constant Levi curvatures? Klingenberg in [4] gave a first positive answer to this problem by showing that a compact and strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^{n+1}$ is isometric to a sphere, provided that M has constant horizontal mean curvature and the CR structure $T_{1,0}(M)$ is parallel in $T^{1,0}(\mathbb{C}^{n+1})$. Later on in [5] we relaxed Klingerberg conditions and we proved that if the characteristic direction is a geodesic, then Alexandrov Theorem holds for hypersurfaces with positive constant Levi mean curvature.
The problem of characterizing hypersurfaces with constant Levi curvature has been recently studied by many authors. Hounie and Lanconelli in [3] showed Alexandrov for Reinhardt domain of $\mathbb{C}^2$, i.e. for domains D such that if $(z_1, z_2) \in D$, then $(e^{i\theta_1}z_1, e^{i\theta_2}z_2) \in D$ for all real θ_1, θ_2. Then in [8] Monti and Morbidelli proved a Darboux -type theorem for $n \ge 2$: the unique Levi umbelical hypersurfaces in $\mathbb{C}^{n+1}$ with all constant Levi curvatures are spheres or cylinders.
Our approach and our hypotheses to prove Alexandrov are different from the previous ones. Precisely in [6] we use the null Lagrangian property for elementary symmetric functions in the eigenvalues of the complex Hessian matrix and the classical divergence theorem to prove the following integral formula for a closed hypersurface in term of the j-th Levi curvature.

Theorem 2. *Let Ω be a bounded domain of $\mathbb{C}^{n+1}$ with boundary a real hypersurface of class C^2. For every defining function f of class C^2 of $\Omega = \{f(z) < 0\}$ and for every $j = 1, \dots, n$ we have*

$$\int_\Omega \sigma_{j+1}(\partial\bar\partial f)dx = \binom{n+1}{j+1}\frac{1}{2(n+1)}\int_{\partial\Omega} K^{(j)}_{\partial\Omega}(z)|\partial f|^{j+1}d\sigma(x),$$

where $K^{(j)}_{\partial\Omega}$ is the j-th Levi curvature of $\partial\Omega$ defined by (1) and $\sigma_j(\partial\bar\partial f)$ is the j-th elementary symmetric function in the eigenvalues of the complex Hessian matrix of f.

Then we follow the Reilly approach to prove the following isoperimetric estimate:

Theorem 3 (Isoperimetric estimate)**.** *Let Ω be a bounded domain of $\mathbb{C}^{n+1}$ with boundary a real hypersurface of class C^∞. If $K^{(j)}_{\partial\Omega}$ is non negative at every point of $\partial\Omega$ then*

$$\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}(x)}\right)^{1/j} d\sigma(x) \ge 2(n+1)|\Omega| \tag{2}$$

where $|\Omega|$ *is the Lebesgue measure of* Ω. *If* $K^{(j)}_{\partial\Omega}$ *is constant, then the equality holds in* (2) *if and only if* Ω *is a ball of radius* $\left(\frac{1}{K^{(j)}_{\partial\Omega}}\right)^{1/j}$.

Since there are non spherical sets which satisfy the equality in (2), then the class of sets which satisfy the equality in (2) is larger than the class of sets which satisfy the equality in the classical isoperimetric inequality and in the Alexandrov Fenchel inequalities for quermassintegrals (see [1], [2] and [10]).

On the other side, if $K^{(j)}_{\partial\Omega}$ is constant, then $\left(K^{(j)}_{\partial\Omega}\right)^{1/j} \leq \frac{\text{meas}\,(\partial\Omega)}{2(n+1)|\Omega|}$ and by the Minkowski formula for the classical mean curvature, together with Serrin theorem [9], in [6] we get the following Alexandrov type theorem for star-shaped domains whose classical mean curvature is bounded from above by a constant j-Levi curvature to $1/j$.

Theorem 4 (An Alexandrov type Theorem)**.** *Let* $\Omega \subset \mathbb{C}^{n+1}$ *be a bounded star-shaped domain with boundary a smooth real hypersurface. If the* j*-Levi curvature is a positive constant* K *at every point of* $\partial\Omega$ *and the maximum of the mean curvature of* $\partial\Omega$ *is bounded from above by* $K^{1/j}$, *then* $\partial\Omega$ *is a sphere and* Ω *is a ball.*

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Singular set bounds for branched minimal hypersurfaces

Neshan Wickramasekera
(joint work with Leon Simon)

A local $C^{1,\alpha}$ partial regularity theory for stable branched minimal hypersurfaces of multiplicity at most 2 has recently been established in [W].
One of the results obtained in [W] says that if V is the measure-theoretic (i.e. varifold) limit of a sequence of stable minimal hypersurfaces of an open set in $\mathbf{R}^{n+1}$ each of which is immersed away from a closed set (to be thought of as the singular set) of locally finite $(n-2)$-dimensional Hausdorff measure, and if at a point X the varifold V has a tangent cone equal to the multiplicity 2 varifold associated with a hyperplane, then, in a neighborhood of X, the support of V is a $C^{1,\alpha}$ 2-valued graph over the tangent plane for some $\alpha = \alpha(n) \in (0,1)$. It remained an open question how large the singular set of V could be near such a point. Also left open was the question of the optimal value of α.

One of the main results (Theorem 2 below) presented in this talk provides answers to these questions. It says that the varifold near the point is always a $C^{1,1/2}$ 2-valued graph, and that either it is regular there (meaning that its support decomposes as the union of two distinct smooth embedded intersecting minimal graphs or the varifold is equal to a multiplicity 2 copy of a single regular embedded minimal graph), or the set of its singularities (i.e. branch points) has codimension 2.

In fact these conclusions hold under much weaker assumptions, as asserted in Theorem 1 below. It suffices to assume that the graph is $C^{1,\alpha}$ for some $\alpha \in (0,1)$ and, viewed as a multiplicity 1 varifold, it is merely a stationary point (in the open cylinder over the domain of the graph) of the area functional. Thus, no stability hypothesis is necessary; nor is it necessary to impose *a priori* any control over the size of the set of branch points.

Before we can state the theorems, we need to establish some notation and definitions. Let $B_\rho(x_0)$ be the open unit ball of $\mathbf{R}^n$ of radius ρ and center x_0, and u denote a $C^{1,\alpha}(B_\rho(x_0))$ 2-valued function. Thus

$$u(x) = \{u_1(x), u_2(x)\} \tag{1}$$

(an unordered pair of real numbers) and

$$Du(x) = \{Du_1(x), Du_2(x)\} \tag{2}$$

(an unordered pair of vectors in $\mathbf{R}^n$) for each $x \in B_\rho(x_0)$, with

$$|u|_{C^{1,\alpha}(B_\rho(x_0))} \equiv \sup_{B_\rho(x_0)} \rho^{-1}|u| + [Du]_\alpha < \infty, \tag{3}$$

where $|u|$ and the Hölder coefficient $[Du]_\alpha$ are defined in the usual way for multiple valued functions; thus,

$$|u| = |u_1| + |u_2|, \tag{4}$$

$$[Du]_\alpha = \sup_{x_1,x_2\in B_\rho(x_0),x_1\neq x_2} |x_1 - x_2|^{-\alpha}|Du(x_1) - Du(x_2)|, \tag{5}$$

where

$$\begin{aligned}|Du(x_1) - Du(x_2)| = \min\{&|Du_1(x_1) - Du_1(x_2)| + |Du_2(x_1) - Du_2(x_2)|,\\ &|Du_1(x_1) - Du_2(x_2)| + |Du_2(x_1) - Du_1(x_2)|\}.\end{aligned} \tag{6}$$

Let us assume that

$$u(0) = \{0, 0\} \quad \text{and} \quad Du(0) = \{0, 0\}, \tag{7}$$

which we can always arrange by choosing coordinates appropriately. The area functional is given by

$$\mathcal{A}(u) = \int_{B_\rho(x_0)} \sqrt{1 + |Du_1|^2} + \sqrt{1 + |Du_2|^2}. \tag{8}$$

Note that this makes sense because the integrand is a well defined single-valued function on $B_\rho(x_0)$. We assume that u is a stationary point for this functional $\mathcal{A}$ in the sense that $G = \operatorname{graph} u$ (with multiplicity 1) is a stationary varifold. Thus we assume that

$$\int_G \nabla_j \zeta = 0, \qquad j = 1, ..., n + 1, \qquad \zeta \in C^1_c(B_\rho(x_0) \times \mathbf{R}), \tag{9}$$

where $\nabla\zeta$ denotes the gradient on G (i.e. $P_x(D\zeta)$, where P_x is the orthogonal projection of $\mathbf{R}^{n+1}$ onto the tangent space of G at any point $x \in G$) and $\nabla_j = e_j\cdot\nabla$. Let

$$\mathcal{K}_u = \{x \in B_\rho(x_0) : u_1(x) = u_2(x) \quad \text{and} \quad Du_1(x) = Du_2(x)\}. \tag{10}$$

Note that there is a well-defined single-valued ÒaverageÓ u_a given by

$$u_a = \frac{1}{2}(u_1 + u_2) \tag{11}$$

in $B_\rho(x_0)$. Denote by v the 2-valued difference. Thus

$$v = \{\pm(u_1 - u_2)\} \tag{12}$$

in $B_\rho(x_0)$. Since we are interested only in local results, we may assume (by rescaling) that

$$|u|_{C^{1,\alpha}(B_\rho(x_0))} < \epsilon \tag{13}$$

where ϵ is a small positive number. Our main result is the following:

Theorem 1 (SW). *There exists $\epsilon_0 = \epsilon_0(n) \in (0, 1/2)$ such that the following holds. Suppose u is a 2-valued $C^{1,\alpha}(B_\rho(x_0))$ function for some $\alpha \in (0,1)$ (in the sense described above in (1)—(6)) such that $G = \operatorname{graph} u$, taken as a multiplicity 1 varifold, is a stationary point of the area functional $\mathcal{A}(\cdot)$ (notation as in (8)) in the cylinder $B_\rho(x_0) \times \mathbf{R}$ in the sense that (9) holds. Suppose $\epsilon \le \epsilon_0$ and that (7) and (13) hold. Then*

(1) *u is in $C^{1,1/2}(B_{\rho/4}(x_0))$, with $|u|_{C^{1,1/2}(B_{\rho/4}(x_0))} \le C\epsilon$. In fact, we have that the 2-valued difference $v \in C^{1,1/2}(B_{\rho/4}(x_0))$ with $|v|_{C^{1,1/2}(B_{\rho/4}(x_0))} \le C\epsilon$ and the single valued average $u_a \in C^{1,1}(B_{\rho/4}(x_0))$ with $|u_a|_{C^{1,1}(B_{\rho/4}(x_0))} \le C\epsilon$. Here the notation is as in (11) and (12) and $C = C(n) \in (0, \infty)$.*

(2) *Either $u_1 \equiv u_2$ in $B_\rho(x_0)$, or the Hausdorff dimension of $\mathcal{K}_u$ (notation as in (10)) is at most $(n-2)$ and in case $n = 2$ $\mathcal{K}_u$ is locally finite. In particular, the branching set $\mathcal{B}_u$ of u (i.e. the set of points $z \in B_\rho(x_0)$ such that there exists no $\sigma > 0$ with the property that $G \cap (B_\sigma(z) \times \mathbf{R})$ is equal to the union of two embedded, possibly intersecting, smooth minimal disks) is either empty or has Hausdorff dimension $(n-2)$, and in case $n = 2$, $\mathcal{B}_u$ is locally finite.*

As discussed above, in view of the results of [W], this theorem immediately implies the following result concerning the size of the singular sets of certain stable, stationary integral varifolds.

Theorem 2 (SW). *If V is an n-dimensional stationary integral varifold of an open set U in $\mathbf{R}^{n+1}$ arising as the weak limit of a sequence of stable minimal hypersurfaces M_k, $k = 1, 2, \dots,$ of U, and if for each k, M_k is immersed away from a closed set of locally finite $(n-2)$-dimensional Hausdorff measure, then the set of points $z \in \operatorname{spt}\|V\|$ where V has a multiplicity 2 tangent plane but $\operatorname{spt}\|V\|$ is not a smooth embbeded minimal hypersurface in any neighborhood of z has Hausdorff dimension at most $(n-2)$, and is locally finite if $n = 2$. In particular, the set of multiplicity 2 branch points of V is either empty or has positive $(n-2)$-dimensional Hausdorff measure and Hausdorff dimension $(n-2)$, and is locally finite if $n = 2$. Furthermore, near each point $z \in \operatorname{spt}\|V\|$ where V has a multiplicity 2 tangent plane, $\operatorname{spt}\|V\|$ is a 2-valued $C^{1,1/2}$ graph over the tangent plane, with a local estimate for the $C^{1,1/2}$ norm (over a sufficiently small ball, in terms of the supremum of the height of the graph over the tangent plane over a larger ball).*

The main tool which we use to bound the size of of the branch set is a monotone frequency function for the 2-valued difference v. The frequency function allows us to produce non-trivial, homogeneous 2-valued stationary harmonic blow-ups at branch points. We then use the "dimension reducing" arguments in a standard way to estimate the size of the branch set. F.J. Almgren, Jr [A] introduced the notion of frequency function first and used this method in the setting of area minimizing currents (of arbitrary dimension and codimension) and energy minimizing multiple-valued harmonic functions.

Establishing monotonicity properties of the frequency function in the present PDE setting depends crucially on knowing the $C^{1,1/2}$ regularity of v and $C^{1,1}$ regularity of u_a. Once these regularity results are established, it is straightforward to prove that the 2-valued function v satisfies a divergence form elliptic equation with Lipschitz coefficients, and it can then be checked that the work of N. Garofalo and F.-H. Lin [GL] (which establishes monotonicity of the frequency function for single valued solutions to divergence form elliptic equations with Lipschitz coefficients) applies in our setting.

Finally, we mention that using the same procedure, but in a much more straightforward manner, one can obtain entirely analogous results for the $C^{1,\alpha}$ solutions to the corresponding "linear problem"; i.e. for the 2-valued $C^{1,\alpha}$ stationary points of the Dirichlet energy.

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Quantization for a fourth order equation with critical exponential growth

Michael Struwe

We describe results from our recent work [11]. Let Ω be a bounded domain of $\mathbb{R}^4$ and let $u_k > 0$ be solutions to the equation

$$\Delta^2 u_k = \lambda_k u_k e^{2u_k^2} \text{ in } \Omega \tag{1}$$

with constants $\lambda_k > 0$, where we prescribe Navier boundary conditions

$$u_k = \Delta u_k = 0 \text{ on } \partial\Omega. \tag{2}$$

We assume that $\lambda_k \to 0$ and

$$\int_\Omega |\Delta u_k|^2\,dx = \int_\Omega u_k \Delta^2 u_k\,dx = \lambda_k \int_\Omega u_k^2 e^{2u_k^2}\,dx \to \Lambda > 0 \tag{3}$$

as $k \to \infty$. In view of the boundary condition $u_k = 0$ on $\partial\Omega$, then by standard elliptic estimates we also have the uniform estimate

$$\int_\Omega |\nabla^2 u_k|^2\,dx \le C \int_\Omega |\Delta u_k|^2\,dx \le C \tag{4}$$

for all k. Since $\lambda_k \to 0$, from (3), (4) we conclude that $\Delta^2 u_k \to 0$ in $L^1(\Omega)$ and $u_k \rightharpoonup 0$ weakly in $H^2(\Omega)$ as $k \to \infty$, but not strongly. In fact, as shown in [10], the sequence (u_k) blows up in a finite number of points where after rescaling spherical bubbles form in the following sense.

Theorem 1. *Let Ω be a bounded domain of $\mathbb{R}^4$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of positive solutions to problem (1), (2), satisfying (3) for some $\Lambda > 0$ as above. Then there exist a subsequence (u_k) and finitely many points $x^{(i)} \in \Omega$, $1 \le i \le I \le C\Lambda$, such that for each i with suitable points $x_k = x_k^{(i)} \to x^{(i)}$ and scale factors $0 < r_k = r_k^{(i)} \to 0$ satisfying*

$$\lambda_k r_k^4 u_k^2(x_k) e^{2u_k^2(x_k)} = 96 \tag{5}$$

we have

$$\eta_k(x) = \eta_k^{(i)}(x) := u_k(x_k)(u_k(x_k + r_k x) - u_k(x_k)) \to \eta_0 = \log\left(\frac{1}{1+|x|^2}\right) \tag{6}$$

locally C^3-uniformly on $\mathbb{R}^4$ as $k \to \infty$, where $\eta = \eta_0 + \log 2 = \log\left(\frac{2}{1+|x|^2}\right)$ solves the fourth order analogue of Liouville's equation

$$\Delta^2 \eta = \Delta^2 \eta_0 = 96 e^{4\eta_0} = 6 e^{4\eta} \text{ on } \mathbb{R}^4. \tag{7}$$

In addition we have

$$\frac{|x_k^{(i)} - x_k^{(j)}|}{r_k^{(i)}} \to \infty \text{ for all } 1 \le i \ne j \le I, \tag{8}$$

and there holds the pointwise estimate

$$\lambda_k \inf_i |x - x_k^{(i)}|^4 u_k^2(x) e^{2u_k^2(x)} \le C, \tag{9}$$

uniformly for all $x \in \Omega$ and all k.

Geometrically, the solutions η to the limit equation (7) correspond to conformal metrics $g = e^{2\eta} g_{\mathbb{R}^4}$ on $\mathbb{R}^4$ of constant Q-curvature $Q = \frac{1}{2} e^{-4\eta} \Delta^2 \eta = 3 = Q_{S^4}$, which are obtained by pull-back of the spherical metric on S^4 under stereographic projection and with total Q-curvature

$$2 \int_{\mathbb{R}^4} Q \, d\mu_g = \int_{\mathbb{R}^4} 6e^{4\eta} \, dx = 2 \int_{S^4} Q_{S^4} \, d\mu_{g_{S^4}} = 16\pi^2 =: \Lambda_1. \tag{10}$$

This geometric interpretation of η is the reason why we prefer to state the preceding result in the present form rather than choosing scaling constants as in [10].

Continuing our previous work, in [11] we establish the following result, showing that the concentration energy Λ is quantized in multiples of Λ_1.

Theorem 2. *In the context of Theorem 1 we have* $\Lambda = L\Lambda_1$ *for some* $L \in \mathbb{N}$.

Theorem 2 is the four-dimensional analogue of a recent result by Druet [4] for the corresponding 2-dimensional equation

$$-\Delta u_k = \lambda_k u_k e^{2u_k^2} \text{ in } \Omega \subset \mathbb{R}^2, \tag{11}$$

which refines our previous result with Adimurthi in [2], characterizing only the first blow-up energy level.

A similar quantization has been observed by Wei [12] for the fourth order analogue of Liouville's equation

$$\Delta^2 u_k = \lambda_k e^{4u_k} \text{ in } \Omega \subset \mathbb{R}^4, \tag{12}$$

with Navier boundary conditions (2), assuming the uniform L^1-bound

$$\int_\Omega \lambda_k e^{4u_k} \, dx \le \Lambda \tag{13}$$

and with $\lambda_k \to 0$ as $k \to \infty$. Quite remarkably, Wei is able to show that for (12) each blow-up point is simple in the sense that $L = I$. In the geometric context of the problem of prescribed Q-curvature on S^4, an analogous result was obtained by Malchiodi and this author [8]. It is an interesting open question whether the same strong quantization property holds true for equation (1) as well.

Related results on compactness issues for fourth order equations were obtained by Adimurthi-Robert-Struwe [1], Hebey-Robert-Wen [5], Robert [9], or Malchiodi [7].

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Ricci flow on manifolds with 1/4-pinched sectional curvature

Simon Brendle
(joint work with Richard Schoen)

Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$, and let $g(t)$ be the solution to the normalized Ricci flow with initial metric g_0. We provide a sufficient condition for $g(t)$ to converge to a constant curvature metric as $t \to \infty$. This condition is closely related to the concept of positive isotropic curvature. The class of manifolds satisfying this condition includes all manifolds with 1/4-pinched sectional curvatures, as well as those with positive curvature operator.

Isoperimetric Inequalities in 3-manifolds of non-negative Ricci curvature

Gerhard Huisken

We use Mean Curvature Flow and Inverse Mean Curvature Flow to prove sharp isoperimetric inequalities for 3-manifolds of non-negative Ricci curvature.

In particular we show that

$$\begin{aligned} & \inf \{ \frac{1}{2}(\int_{\Sigma^2} H^2 d\mu)^{1/2} \mid \Sigma^2 = \partial\Omega,\ \Omega \subset (M^3, g) \text{ outward minimizing}\} \\ = \quad & \inf \{ \frac{|\Sigma^2|^{3/2}}{3\,\mathrm{Vol}(\Omega)} \mid \Sigma^2 = \partial\Omega,\ \Omega \subset (M^3, g)\}. \end{aligned}$$

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