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Analysis and Geometric Singularities

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Abstract. This workshop focused on several main areas of current research concerning analysis on singular and noncompact spaces. Topics included classical areas like spectral asymptotics, propagation of singularities and scattering theory, index theory on singular spaces, exotic characteristic classes, boundary value problems, surgery formulae adiabatic limits, and singularities in mathematical relativity.

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Introduction by the Organisers

The conference on “Analysis and Geometric Singularities” took place from August 19 to August 25, 2007. The meeting was a popular one, with all spots taken, and its atmosphere was lively and full of activity. Participants ranged from many experts and old hands in the field to many young postdocs and graduate students, and the interaction between researchers at all levels was clearly productive for everyone. The talks were notably very well attended throughout the week, and many were accompanied by lively discussions.

Each day was structured thematically: every day except Wednesday began with a longer (70 minute) survey talk on some general theme in the subject; the other talks each day were shorter (55 minutes) and focused on that theme. Wednesday morning, however, was devoted to five half-hour lectures by some of the younger participants. The first day’s theme was spectral geometry; the lecture by Iosif Polterovich surveyed many recent results connecting spectral asymptotics and dynamics; later talks that day were given by Mueller, Kordyukov, Paycha and Grieser. Tuesday’s survey talk by Mihalis Dafermos on the mathematics of black holes was followed by the lectures of Rodnianski, Vasy, Baer and Perry; the first two of these especially focusing on mathematical relativity. Thursday's survey by Jared Wunsch on propagation of singularities for the Schroedinger equation was followed by talks by Lesch, Ammann, Moscovici and Dai. Friday's theme was index theory, with a survey by Richard Melrose and other lectures by Carron, Albin, Bunke and Richardson. The young researchers who spoke on Wednesday were Azzali, Degeratu, Mazzieri, J. Mueller and Rochon.

The scientific level of the talks was uniformly high, as will be seen from the abstracts which follow; many interesting results were announced, and the survey talks helped provide a focus which in turn facilitated communication between researchers in the various different fields represented. Overall, the conference gave clear evidence that the field of analysis on singular spaces is very active, and indeed is extending in many new directions. The many strong young researchers here indicate that the field has a bright future!

Workshop: Analysis and Geometric Singularities

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Abstracts

Spectral asymptotics and dynamics on Riemannian manifolds

Iosif Polterovich

According to Bohr's correspondence principle in quantum mechanics, the asymptotic properties of the eigenvalues and the eigenfunctions of the Laplacian on a Riemannian manifold are linked to the behavior of the geodesic flow. In particular, the growth of various spectral quantities (such as the spectral function, the error term in Weyl's law, eigenfunctions) depends on the underlying dynamics.

Let M be a compact Riemannian manifold of dimension n without boundary. Consider the Laplacian Δ on M with the eigenvalues $0 < \lambda_1^2 \le \lambda_2^2 \le \dots$, and the corresponding orthonormal basis $\{\phi_i\}$ of eigenfunctions: $\Delta\phi_i = \lambda_i^2\phi_i$. Let $N_{x,y}(\lambda) = \sum_{\lambda_i<\lambda}\phi_i(x)\phi_i(y)$, where $x, y \in M$, be the spectral function of the Laplacian. As $\lambda \to \infty$, it satisfies

$$N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y. \tag{1}$$

The asymptotics of the spectral function on the diagonal is given by the pointwise Weyl's law:

$$N_{x,x}(\lambda) = \frac{\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R_x(\lambda), \;\; R_x(\lambda) = O(\lambda^{n-1}). \tag{2}$$

Integrating (2) over M one gets the well-known Weyl's law for the eigenvalue counting function $N(\lambda) = \#\{\lambda_i < \lambda\}$:

$$N(\lambda) = \int_M N_{x,x}(\lambda) = \frac{\mathrm{Vol(M)}\,\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R(\lambda), \;\; R(\lambda) = O(\lambda^{n-1}). \tag{3}$$

The upper bounds on the spectral function and the error terms (due to Avakumovič and Levitan) are sharp and attained on a round sphere. At the same time, the upper bound on $R(\lambda)$ (respectively, $R_x(\lambda)$ and $N_{x,y}(\lambda)$) can be improved to $o(\lambda^{n-1})$ under the condition that the measure of directions corresponding to closed geodesics (respectively, geodesic loops at x and geodesic segments from x to y) is zero (see [DG], [Saf]).

Further improvements of estimates (1-3) are expected on manifolds whose geodesic flow is either completely integrable or ergodic (cf. [St, Conjecture I]). For the sake of simplicity, we focus here on manifolds of dimension two. Consider the integrable case first.

Conjecture 1. *Let M be a sphere with a generic (cf.* [CdV, section 6]*) metric of revolution. Then $R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$. Moreover, if x is not a pole, then $R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$. Also, if at least one of the points $x \neq y \in M$ is not a pole, then $N_{x,y}(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$.*

It was shown in [CdV] that on a generic surface of revolution $R(\lambda) = O(\lambda^{2/3})$. The genericity condition is important, since one has to eliminate such " degenerate" cases as a round sphere or a Zoll surface, for which the remainder estimate

(3) is attained. Note also that at poles of a surface of revolution the pointwise estimate (2) is attained (see [Saf]), and, most likely, (1) is attained if x, y are both poles. A dynamical explanation of this phenomenon is that all geodesics starting from one pole pass through another pole and return back.

Conjecture 2. *Let M be a torus with a Liouville metric. Then $R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, $R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, $N_{x,y}(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, for any $\varepsilon > 0$ and for all $x \neq y \in M$.*

It is shown in [La] that for a large class of Liouville tori $R(\lambda) = O(\lambda^{2/3})$. As in [CdV], the main idea is to approximate the eigenvalue count by a lattice count (in the integrable case this is possible, as follows from the EBK quantization conditions, see [St] for a discussion). Then one can apply the van der Corput method of exponential sums and obtain a $O(\lambda^{2/3})$ remainder estimate.

For the flat square torus, counting eigenvalues is equivalent to counting integer points inside a circle (the latter is known as the Gauss's circle problem). In this case, a slightly better estimate on $R(\lambda)$ was proved by Huxley. The celebrated Hardy's conjecture about the error term in the Gauss' circle problem is an important motivation for Conjectures 1 and 2 (in fact, it is a special case of Conjecture 2 when M is a flat square torus).

The estimates in Conjectures 1 and 2 can not be significantly improved. On a flat torus, the Hardy-Landau lower bound yields $R(\lambda) \equiv R_x(\lambda) \neq O(\sqrt{\lambda})$, and using the methods of [JP] one can also show that in this case $N_{x,y}(\lambda) \neq O(\sqrt{\lambda})$. It is not known whether ε can be removed from any of the estimates in Conjecture 1, but it seems unlikely. It was proved in [Sar] that $R(\lambda) \neq o(\sqrt{\lambda})$ on any surface with integrable geodesic flow. For $R_x(\lambda)$ and $N_{x,y}(\lambda)$ such a lower bound was proved in a much higher generality in [JP, LPS].

Consider now the negatively curved case. As was proved by Anosov, the geodesic flow on a surface of negative curvature is ergodic.

Conjecture 3. *Let M be a generic negatively curved surface. Then $R(\lambda) = O(\lambda^{\varepsilon})$, $R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, $N_{x,y}(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$ and for all $x \neq y \in M$.*

Note that we expect $R(\lambda)$ to grow much slower on a generic negatively curved surface than in the integrable case (cf. [St, Conjecture I]). Moreover, lower bounds on $R_x(\lambda)$ proved in [JP] indicate that substantial cancelations occur when the pointwise remainder is integrated over a negatively curved surface. In particular, results of [JP] imply that $N_{x,y}(\lambda) \neq O(\sqrt{\lambda})$ and $R_x(\lambda) \neq O(\sqrt{\lambda})$ for all points on a negatively curved surface.

The genericity assumption in Conjecture 3 can not be removed: indeed, as shown by Hejhal and Selberg [Hej], on arithmetic surfaces of constant negative curvature $R(\lambda) \neq o\left(\frac{\sqrt{\lambda}}{\log \lambda}\right)$. Note that arithmetic surfaces have exceptionally high multiplicities in the length spectrum. As mentioned in [Sar], this gives a dynamical explanation for the faster growth of the remainder. Randol (see [Ran]) conjectured that for arithmetic surfaces (and, in general, for all surfaces of constant negative curvature) $R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$ — similarly to the integrable case.

It is shown in [JPT] that on any negatively curved surface $R(\lambda) \neq o\left((\log\lambda)^{\alpha}\right)$ for some positive constant α that can be expressed in terms of certain dynamical characteristics of the geodesic flow. This lower bound is consistent with Conjecture 3. For surfaces of constant negative curvature (in this case one can take any $\alpha < 1/2$) it was proved by Randol and Hejhal using the Selberg zeta function techniques.

Conjecture 3 is very far from being proved. The best result up-to-date is $R_x(\lambda) = O\left(\frac{\lambda}{\log\lambda}\right)$ and $R(\lambda) = O\left(\frac{\lambda}{\log\lambda}\right)$ obtained in [Ber] (see also [Vol]).

Let us conclude by a recent result showing that the off-diagonal bounds on $N_{x,y}(\lambda)$ in Conjectures 1–3 hold *on average* in a much higher generality.

Theorem 1. [LPS] *Let M be an arbitrary compact n-dimensional Riemannian manifold. For any finite measure ν on $\mathbb{R}$ and any $x \in M$ there exists a subset $M_{x,\nu} \subset M$ of full measure such that*

$$\frac{N_{x,y}(\lambda)}{1+\lambda^{\frac{n-1}{2}}} \in L^2(\mathbb{R},\nu), \quad \forall y \in M_{x,\nu}. \tag{4}$$

Theorem 1 is inspired by [Ran] where similar averaging was considered. We prove it using purely analytic methods. It would be interesting to find a dynamical interpretation of this result — in particular, to check whether (4) is true if x is not conjugate to y along any geodesic.

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Spectral asymptotics for arithmetic quotients of symmetric spaces

Werner Müller
(joint work with Erez Lapid)

Let $S = G/K$ be a Riemannian symmetric space of non-compact type, where G is a real semi-simple Lie group of non-compact type with finite center, and K is a maximal compact subgroup of G. Let Γ be a non-uniform lattice in G, that is a discrete subgroup of G whose quotient $\Gamma\backslash G$ is of finite volume, but not compact. Then Γ acts properly discontinuously on S and $\Gamma\backslash S$ is a non-compact locally symmetric space of finite volume. Of particular interest are arithmetic subgroups Γ. This means that $G = \mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a semi-simple algebraic group defined over $\mathbb{Q}$, and Γ is a subgroup of $\mathbf{G}(\mathbb{Q})$ which is commensurable with $\mathbf{G}(\mathbb{Z})$. Here $\mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}(n,\mathbb{Z})$ with respect to some embedding $\mathbf{G} \subset \mathrm{GL}(n)$. A basic example is the principal congruence subgroup $\Gamma(N) \subset \mathrm{SL}(n,\mathbb{Z})$ of level N. An arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called congruence subgroup, if it contains a principal congruence subgroup.

Spectral theory on $\Gamma\backslash S$ is intimately connected with the theory of automorphic forms. For simplicity, assume that Γ is tosion free. Then $\Gamma\backslash S$ is a complete Riemannian manifold. Let Δ be the Laplacian of $\Gamma\backslash S$. Then Δ is essentially self-adjoint in $L^2(\Gamma\backslash S)$. Denote its unique self-adjoint extension by $\bar{\Delta}$. It is well known that the spectrum of $\bar{\Delta}$ is the union of a pure point spectrum and an absolutely continuous spectrum. The point spectrum consists of a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ of finite multiplicity. The absolutely continuous spectrum is described in terms of Eisenstein series. Espcially, it follows that $\sigma_{\mathrm{ac}}(\bar{\Delta}) = [c,\infty)$ for some $c > 0$. One of the main problems in theory of automorphic forms is to study the point spectrum. The presence of the continuous spectrum makes this very difficult. The basic tool to study the discrete spectrum is the Arthur-Selberg trace formula.

Let $L^2_{\mathrm{dis}}(\Gamma\backslash S)$ be the span of all eigenfunctions of $\bar{\Delta}$. It contains the space of cusp forms, which is defined as follows. Let $\mathcal{D}(S)$ be the algebra of invariant differential operators on S. It is well known that $\mathcal{D}(S)$ is a finitely generated commutative algebra. The minimal number of generators of $\mathcal{D}(S)$ equals the rank of S. Let P be a parabolic subgroup of G with Langlands decomposition $P = M_P A_P N_P$. Then P is called a Γ-cuspidal parabolic subgroup, if $(N_P \cap \Gamma)\backslash N_P$ is compact. A function $\phi \in C^\infty(\Gamma\backslash S)$ is called cusp form, if ϕ is a simultaneous eigenfunction of all $D \in \mathcal{D}(S)$ and which also satisfies

$$\int_{N_P\cap\Gamma\backslash N_P} \phi(nx)\, dn = 0$$

for all Γ-cuspidal parabolic subgroups $P \neq G$. Any cusp form is in particular a square integrable eigenfunction of Δ. The subspace of $L^2_{\mathrm{dis}}(\Gamma\backslash S)$ spanned by all cusp forms is denoted by $L^2_{\mathrm{cus}}(\Gamma\backslash S)$. The orthogonal complement of $L^2_{\mathrm{cus}}(\Gamma\backslash S)$ in $L^2_{\mathrm{dis}}(\Gamma\backslash S)$ is the residual subspace $L^2_{\mathrm{res}}(\Gamma\backslash S)$. By Langlands' theory of Eisenstein series, $L^2_{\mathrm{res}}(\Gamma\backslash S)$ is spanned by iterated residues of cuspidal Eisenstein series. So cusp forms are the building blocks in the theory of automophic forms. A basic problem is the question of existence and construction of cusp form. For a non-uniform lattice it is not clear that any cusp forms exist. For $\mathrm{SL}(2,\mathbb{R})$ conjectures of Phillips and Sarnak state that a generic non-uniform lattice in $\mathrm{SL}(2,\mathbb{R})$ has no cusp forms. It is believed that the existence of cusp forms is related to the arithmeticity of the lattice. Note that if $\mathrm{rank}(S) > 1$, then by the Margulis arithmeticity theorem, every irreducible non-uniform lattice is arithmetic in an appropriate sense.

A convenient way of counting the number of cusp forms for Γ is to use their eigenvalues of the Laplace operator. Let $N^{\Gamma}_{\mathrm{cus}}(\lambda)$ be the number of linearly independent cusp forms whose eigenvalues satisfy $\sqrt{\lambda_j} \le \lambda$. Recently, the following theorem has been proved by Lindenstrauss and Venkatesh [3].

Theorem 1. *Let* $\mathbf{G}$ *be a split adjoint semi-simple algebraic group over* $\mathbb{Q}$*,* $G = \mathbf{G}(\mathbb{R})$*,* $K \subset G$ *a maximal compact subgroup, and* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *a congrunece subgroup. Let* $S = G/K$ *and* $d = \dim S$*. Then*

$$N^{\Gamma}_{\mathrm{cus}}(\lambda) \sim \frac{\mathrm{vol}(\Gamma\backslash S)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)}\lambda^d, \quad \lambda \to \infty. \tag{1}$$

This establishes Weyl's law for the cuspidal spectrum of such lattices. This had been conjectured by Sarnak [5]. For congrunece subgroups of $\mathrm{SL}(2,\mathbb{Z})$, Weyl's law was first proved by Selberg [7], using his trace formula. For congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$, Theorem 1 was proved by the author in [4]. Especially, (1) implies that for the congruence subgroups as above, cusp forms exist in abundance.

Formula (1) does not say much about the finer structure of the eigenvalue distribution. One of the basic problems is the estimation of the remainder term

$$R(\lambda) = N^{\Gamma}_{\mathrm{cus}}(\lambda) - \frac{\mathrm{vol}(\Gamma\backslash S)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)}\lambda^d.$$

In joint work with E. Lapid [2] we have established the following estimation of the remainder term in the case of $G = \mathrm{SL}(n,\mathbb{R})$.

Theorem 2. *Let* $G = \mathrm{SL}(n,\mathbb{R})$ *and let* $\Gamma(N)$ *be the principal congruence subgroup of* $\mathrm{SL}(n,\mathbb{Z})$ *of level* N*. Let* $S = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ *and* $d = \dim S$*. Then for* $N \ge 3$ *we have*

$$N^{\Gamma(N)}_{\mathrm{cus}}(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S)}{(4\pi)^{d/2}\mathbf{\Gamma}\left(\frac{d}{2}+1\right)}\lambda^d + O\left(\lambda^{d-1}(\log\lambda)^{\max(n,3)}\right), \quad \lambda \to \infty.$$

The method to prove Theorem 2 is a combination of Hörmander's method to estimate the remainder term in Weyl's law for an elliptic operator on a compact manifold and the Arthur trace formula. For a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, Theorem 2 is due to Selberg with an error term of order $O(\lambda\log\lambda)$.

Theorem 2 is a special case of a more general result which deals with the joint spectrum of the algebra $\mathcal{D}(S)$ of invariant differntial operators acting in $L^2_{\mathrm{cus}}(\Gamma\backslash S)$. Let $G = KAN$ be the Iwasawa decomposition of G, W the Weyl group of (G, A) and $\mathfrak{a}$ the Lie algebra of A.

Let $S(\mathfrak{a}_\mathbb{C})$ be the symmetric algebra of the complexification $\mathfrak{a}_\mathbb{C}$ of $\mathfrak{a}$ and let $S(\mathfrak{a}_\mathbb{C})^W$ be the subspace of Weyl group invariants in $S(\mathfrak{a}_\mathbb{C})$. Then by a theorem of Harish-Chandra there is a canonical isomorphism $\mathcal{D}(S) \cong S(\mathfrak{a}_\mathbb{C})^W$. This implies that the characters of $\mathcal{D}(S)$ are parametrized by $\mathfrak{a}_\mathbb{C}^*/W$. The space of cusps forms $L^2_{\mathrm{cus}}(\Gamma\backslash S)$ is invariant under $\mathcal{D}(S)$ and therefore, can be decomposed into joint eigenspaces of $\mathcal{D}(S)$. Let $\Lambda_{\mathrm{cus}}(\Gamma) \subset \mathfrak{a}_\mathbb{C}^*/W$ be the corresponding characters. We call $\Lambda_{\mathrm{cus}}(\Gamma)$ the *cuspidal spectrum* of Γ. Given $\lambda \in \mathfrak{a}_\mathbb{C}^*$, let $m(\lambda)$ be the dimension of the associated eigenspace.

Given an open subset $\Omega \subset \mathfrak{a}^*$ and $t > 0$, let $t\Omega = \{t\mu \colon \mu \in \omega\}$. Furthermore, for $t > 0$ let $B_t(0) \subset \mathfrak{a}_\mathbb{C}^*$ be the ball of radius $t > 0$ around the origin. Let $\beta(i\lambda)$ be the Plancherel measure for spherical functions on $\mathrm{SL}(n,\mathbb{R})$, and let the notation be as in Theorem 2. In [2] we established the following result about the distribution of the cuspidal spectrum.

Theorem 3. *Let $\Omega \subset \mathfrak{a}^*$ be a bounded domain with piecewise smooth boundary. Then for $N \geq 3$ we have*
(2)
$$\sum_{\lambda\in\Lambda_{\mathrm{cus}}(\Gamma(N)),\lambda\in it\Omega} m(\lambda) = \frac{\mathrm{vol}(\Gamma(N)\backslash S)}{|W|}\int_{t\Omega}\beta(i\lambda)\,d\lambda + O\left(t^{d-1}(\log t)^{\max(n,3)}\right),$$
as $t \to \infty$, and

(3)
$$\sum_{\substack{\lambda\in\Lambda_{\mathrm{cus}}(\Gamma(N))\\ \lambda\in B_t(0)\backslash i\mathfrak{a}^*}} m(\lambda) = O\left(t^{d-2}\right), \quad t\to\infty.$$

For compact locally symmetric manifolds $\Gamma\backslash S$, the corrsponding result was proved by Duistermaat, Kolk and Varadarajan [1]. Note that $\Lambda_{\mathrm{cus}}(\Gamma(N)) \cap i\mathfrak{a}^*$ is the cuspidal tempered spherical spectrum. The Ramanujan conjecture [6] for $\mathrm{GL}(n)$ at the Archimedean place states that $\Lambda_{\mathrm{cus}}(\Gamma(N)) \subset i\mathfrak{a}^*$ so that (3) is empty, if the Ramanujan conjecture is true. However, the Ramanujan conjecture is far from being proved. Moreover, it is known to be false for other groups $\mathbf{G}$ and (3) is what one can expect in general.

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The periodic magnetic Schrödinger operators: spectral gaps and tunneling effect

Yuri A. Kordyukov
(joint work with Bernard Helffer)

Let M be a noncompact oriented manifold of dimension $n \geq 2$ equipped with a properly discontinuous action of a finitely generated, discrete group Γ such that M/Γ is compact. Suppose that $H^1(M, \mathbb{R}) = 0$. Let g be a Γ-invariant Riemannian metric and $\mathbf{B}$ a real-valued Γ-invariant closed 2-form on M such that $\mathbf{B} = d\mathbf{A}$ for some real-valued 1-form $\mathbf{A}$ on M. Consider the magnetic Schrödinger operator

$$H^h = (ih\, d + \mathbf{A})^*(ih\, d + \mathbf{A})$$

as a unbounded self-adjoint operator in the Hilbert space $L^2(M)$ (here $h > 0$ is a semiclassical parameter).

We discuss sufficient conditions on the magnetic field $\mathbf{B}$, which ensure the existence of a gap (or, even more, an arbitrarily large number of gaps) in the spectrum of H^h on some interval for any $h > 0$ small enough. By a gap in the spectrum $\sigma(T)$ of a self-adjoint operator T we mean a maximal interval (a, b) such that $(a, b) \cap \sigma(T) = \emptyset$.

For any $x \in M$, denote by $B(x)$ the anti-symmetric linear operator on the tangent space T_xM associated with the 2-form $\mathbf{B}$:

$$g_x(B(x)u, v) = \mathbf{B}_x(u, v), \quad u, v \in T_xM.$$

Consider the function $\operatorname{Tr}^+ B$ on M defined as

$$\operatorname{Tr}^+(B(x)) = \frac{1}{2} \operatorname{Tr}([B^*(x) \cdot B(x)]^{1/2}), \quad x \in M.$$

Put

$$b_0 = \min\{\operatorname{Tr}^+(B(x)) \,:\, x \in M\}.$$

We always assume that there exist a (connected) fundamental domain $\mathcal{F}$ and $\epsilon_0 > 0$ such that

$$\operatorname{Tr}^+(B(x)) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}. \tag{1}$$

First, we show that, under this assumption, there exists an arbitrarily large number of gaps in the spectrum of H^h on the interval $[0, h(b_0 + \epsilon_0)]$ for any sufficiently small $h > 0$.

Theorem 1 ([2])**.** *For any natural N, there exists $h_0 > 0$ such that the spectrum of H^h on the interval $[0, h(b_0 + \epsilon_0)]$ has at least N gaps for any $h \in (0, h_0)$.*

If, in addition, we suppose that $b_0 = 0$ (i.e., there exists at least one zero of B) and impose some additional assumptions on the zero set of B, we state the existence of an arbitrarily large number of gaps in the spectrum of H^h on some intervals of size $O(h^\alpha)$ with some $\alpha > 1$ for any sufficiently small $h > 0$.

We consider two cases. The first case deals with non-degenerate (in a weak sense) isolated zeroes of B.

Theorem 2 ([2]). *Suppose that there exists a zero $\bar{x}_0$ of B, $B(\bar{x}_0) = 0$, such that*

$$c^{-1}d(x,\bar{x}_0)^k \le \mathrm{Tr}^+(B(x)) \le cd(x,\bar{x}_0)^k$$

for all x in some neighborhood of $\bar{x}_0$ with some $c > 0$ and some integer $k > 0$ (here d denotes the geodesic distance on M). Then, for any natural N, there exist $C > 0$ and $h_0 > 0$ such that the spectrum of H^h on the interval $[0, Ch^{\frac{2k+2}{k+2}}]$ has at least N gaps for any $h \in (0, h_0)$.

In the second case, the zero set of B contains a non-degenerate one-dimensional component. More precisely, we assume that M is an oriented two-dimensional Riemannian manifold. We also suppose that:

- the zero set of B has a component γ, which is a bounded smooth curve;
- there are constants $k \in \mathbb{N}$ and $C > 0$ such that for all x in some neighborhood of γ the estimates hold:

$$C^{-1}d(x,\gamma)^k \le |B(x)| \le Cd(x,\gamma)^k \,.$$

In particular, for $k = 1$ the last condition means that ∇B does not vanish on γ.

Finally, we assume that the leading term of the Taylor expansion of B at γ is constant along γ. More precisely, write the volume 2-form $\mathbf{B}$ as

$$\mathbf{B}_x = B(x)\omega, \quad x \in M,$$

where ω is the Riemannian volume form on M. Denote by N the external unit normal vector to γ. Let $\tilde{N}$ denote an arbitrary extension of N to a smooth vector field on M. Consider the function β_1 on M given by the formula

$$\beta_1(x) = \tilde{N}^k B(x), \quad x \in M.$$

By hypothesis, $\beta_1(x) \neq 0$ for any $x \in \gamma$. Our assumption is that the restriction of β_1 to γ (which is independent of the choice of a smooth extension $\tilde{N}$) is constant along γ:

$$\beta_1(x) \equiv \beta_1 = \mathrm{const}, \quad x \in \gamma.$$

For $k = 1$, this means that the length of the gradient $|\nabla B|$ is constant along γ.

Theorem 3 ([2]). *Under the current assumptions, for any natural N, there exist constants $C > 0$ and $h_0 > 0$ such that the spectrum of H^h on the interval $[0, Ch^{\frac{2k+2}{k+2}}]$ has at least N gaps for any $h \in (0, h_0)$.*

The proofs are based on the semiclassical analysis of the tunneling effect for the corresponding quantum system. The key result is the following localization theorem for the spectrum of the magnetic Schrödinger operator H^h stated in [1].

Assume that the operator H^h satisfies the condition (1). Fix $\epsilon_1 > 0$ such that $\epsilon_1 < \epsilon_0$. The set

$$U_{\epsilon_1} = \{x \in \mathcal{F} \,:\, \mathrm{Tr}^+(B(x)) < b_0 + \epsilon_1\}$$

is an open subset of $\mathcal{F}$ such that $U_{\epsilon_1} \cap \partial\mathcal{F} = \emptyset$. Its closure $\overline{U_{\epsilon_1}}$ is compact and included in the interior of $\mathcal{F}$. Any connected component of $\overline{U_{\epsilon_1}}$ can be understood as a magnetic well (attached to the effective potential $h \cdot \mathrm{Tr}^+(B(x))$).

Take any $\epsilon_2 > 0$ such that $\epsilon_1 < \epsilon_2 < \epsilon_0$. Let $D = \overline{U_{\epsilon_2}}$. Denote by H^h_D the unbounded self-adjoint operator in the Hilbert space $L^2(D)$ defined by the operator H^h in D with the Dirichlet boundary conditions. The operator H^h_D has discrete spectrum.

Theorem 4 ([1])**.** *Under the assumption* (1)*, for any* $\epsilon_1 < \epsilon_2 < \epsilon_0$*, there exist* $C, c, h_0 > 0$ *such that, for any* $h \in (0, h_0]$

$$\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \mathrm{dist}(\lambda, \sigma(H^h_D)) < Ce^{-c/\sqrt{h}}\},$$

$$\sigma(H^h_D) \cap [0, h(b_0 + \epsilon_1)] \subset \{\lambda \in [0, h(b_0 + \epsilon_2)] : \mathrm{dist}(\lambda, \sigma(H^h)) < Ce^{-c/\sqrt{h}}\}.$$

Theorem 4 allows us to reduce the investigation of gaps in the spectrum of the operator H^h to the study of eigenvalues of the operator H^h_D.

Theorem 5. *Let* $N \geq 1$*. Suppose that there is a subset* $\mu^h_0 < \mu^h_1 < \ldots < \mu^h_N$ *of an interval* $I(h) \subset [0, h(b_0 + \epsilon_1))$ *such that*

(1) *There exist constants* $c > 0$ *and* $M \geq 1$ *such that*

$$\mu^h_j - \mu^h_{j-1} > ch^M, \quad j = 1, \ldots, N,$$
$$\mathrm{dist}(\mu^h_0, \partial I(h)) > ch^M, \quad \mathrm{dist}(\mu^h_N, \partial I(h)) > ch^M,$$

for any $h > 0$ *small enough;*

(2) *Each* $\mu^h_j, j = 0, 1, \ldots, N$*, is an approximate eigenvalue of the operator* H^h_D*: for some* $v^h_j \in C^\infty_c(D)$ *we have*

$$\|H^h_D v^h_j - \mu^h_j v^h_j\| = \alpha_j(h)\|v^h_j\|,$$

where $\alpha_j(h) = o(h^M)$ *as* $h \to 0$*.*

Then the spectrum of H^h *on the interval* $I(h)$ *has at least* N *gaps for any sufficiently small* $h > 0$*.*

To complete the proof in each case under consideration, it remains to construct arbitrarily long sequences of approximate eigenvalues of the operator H^h_D as requested in Theorem 5. For this purpose, we use constructions of approximate eigenvalues given in [3, 4].

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Chern-Weil forms in infinite dimensions

Sylvie Paycha

This is based on joint work with Steven Rosenberg [PR1], [PR2], with Simon Scott [PS] and with Jouko Mickelsson [MP].
Classical Chern-Weil formalism relates geometry to topology, assigning to the curvature of a connection, de Rham cohomology classes of the underlying manifold. This theory developped in the 40's by Shing-Shen Chern [C2] and André Weil[1] which can be seen as a generalisation of the Chern-Gauss-Bonnet theorem [C1], was an important step in the theory of characteristic classes.
Let G be a lie group with Lie algebra Lie(G). The Chern-Weil homomorphism assigns to an Ad(G)-invariant polynomial on Lie(G) a de Rham cohomology class defined as follows.
When G is a matrix group, Ad(G)-invariant monomials on Lie(G) can be built from the trace on matrices and invariant polynomials are actually generated by the monomials $X \mapsto \mathrm{tr}(X^j)$.
Let $P \to M$ be a G-principal bundle equipped with a connection ∇, since the curvature ∇^2 is a Lie(G)-valued two-form on P, to an analytic map f corresponds a form $f(\nabla^2)$:

$$\begin{array}{rcl} f : \mathcal{C}(P) & \to & \Omega(X, \mathcal{A}) \\ \nabla & \mapsto & f(\nabla^2) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} \nabla^{2i}, \end{array}$$

where $\mathcal{C}(P)$ is the space of connections on P, $\mathcal{A} = \mathrm{Ad}\, P$ the adjoint bundle and $\Omega(X, \mathcal{A})$ the algebra of $\mathcal{A}$-valued forms on X. The form

$$\mathrm{tr}\left(f(\nabla^2)\right) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} \mathrm{tr}\left(\nabla^{2i}\right)$$

turns out to be closed and its de Rham class independent of the choice of connection ∇.

The closedness of the forms and the independence of the corresponding de Rham classes on the choice of the connection which follows from the invariance of the polynomials $\mathrm{tr}(X^i)$, can be inferred from the following properties of the trace:

$$(1) \qquad [d, \mathrm{tr}]\,(\alpha) := d\,\mathrm{tr}(\alpha) - \mathrm{tr}(d\,\alpha) = 0 \quad \forall \alpha \in \Omega\,(X, \mathcal{A})$$

[1] In an unpublished paper

and

$$(2) \qquad \partial\mathrm{tr}(\alpha,\beta) := \mathrm{tr}\left(\alpha\wedge\beta + (-1)^{|\alpha|\,|\beta|}\beta\wedge\alpha\right) = 0 \quad \forall \alpha,\beta\in\Omega\,(X,\mathcal{A})\,,$$

When G is an infinite dimensional Lie group, there is a priori a problem to define a trace and therefore to get invariant polynomials on Lie(G).
We are concerned here with the Fréchet Lie group $C\ell^{0,*}(M,E)$ (and its subgroups) of invertible zero order classical pseudodifferential operators (see e.g. [Sh], [T],[Tr]) acting on smooth sections of some vector bundle $E\to M$ over a closed Riemannian manifold M. Its Lie algebra is the Fréchet algebra $C\ell^0(M,E)$ of zero order classical pseudodifferential operators acting on smooth sections $E\to M$. The algebra $C\ell^0(M,E)$ carries two types of traces [LP] together with their linear combinations, the noncommutative residue introduced by Adler, Manin, generalised by Wodzicki [W1],[W2] (see [K] for a survey) and leading symbol traces used in [PR1, PR2].
An explicit example of an infinite rank bundle with non vanishing first Chern class is built in [RT] using the noncommutative residue on classical pseudodifferential operators as an Ersatz for the trace on matrices. However, generally speaking, Chern classes built from the noncommutative residue or leading symbol traces seem too coarse to capture non trivial cohomology classes [2] so that we turn to mere linear extensions to the algebra $C\ell^0(M,E)$ of the ordinary trace on smoothing operators. The latter might not be traces since they are not expected to vanish on brackets. We refer all the same to these as regularised traces.
In constrast with the noncommutative residue and leading symbol traces which vanish on smoothing pseudodifferential operators, regularised traces coincide with the usual trace on smoothing operators. The price to pay for choosing regularised traces instead of genuine traces is that analogues of Chern-Weil invariant polynomials do not give rise to closed forms because the two fundamental properties (1) and (2) fail to hold. Implementing techniques borrowed from the theory of classical pseudodifferential calculus, one measures the obstructions to the closedness in terms of noncommutative residues [PR1], [PR2].
In specific situations such as in hamiltonian gauge theory where we need to build Chern classes on pseudodifferential Grassmannians, the very locality of the noncommutative residue can provide a way to build counterterms, and thereby to renormalise the original non closed forms in order to turn them into closed ones. Loop groups [F] also provide an interesting geometric setup since obstructions to the closedness can vanish, thus leading to closed forms [MP].
On infinite rank vector bundles associated with a family of Dirac operators on even dimensional closed spin manifolds (see e.g. [BGV]), these obstructions can be circumvented by an appropriate choice of regularised trace [PS] involving the very superconnection [Q] which gives rise to the curvature. Choosing the Bismut

[2] Interestingly, the very fact that the class vanishes can be used as a starting point to define Chern-Simons classes as in [MRT] where the authors build non trivial Wodzicki-Chern-Simons classes via a transgression of Wodzicki-Chern forms.

superconnection [B] associated with the family of Dirac operators yields explicit formulae for the corresponding Chern-Weil forms, which are reminiscent of the well-known formula for the corresponding Chern character involving the $\hat{A}$-genus of the total manifold [MP].

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Geodesics on singular surfaces

Daniel Grieser
(joint work with Vincent Grandjean)

Let $X \subset \mathbb{R}^n$ be a real algebraic surface. Let $p \in X$ be an isolated singularity of X, and let X_{reg} be the regular part of X. Assume p is not an isolated point of X. By considering shortest curves from p to nearby points, it is easy to see that there are many geodesics on X_{reg} ending at p, see for example [BL04]. There it is also shown that any such geodesic has a limit direction at p. We study the higher regularity properties of these geodesics; moreover, we study the exponential map based at p, that is, the whole family of geodesics ending at p. Simple examples show that such a geodesic may have no second derivative at p. However, one may hope for a complete asymptotic expansion in which fractional powers and possibly logarithmic terms occur. We prove that for a certain class of singular surfaces such complete expansions exist, in a suitable sense, for the exponential map.

Applications of such a detailed description of the exponential map are, among others:

- A notion of normal coordinates based at the singularity, with all the uses that normal coordinates have, for example in the analysis of the geometric partial differential operators on X_{reg} and of the diffraction of waves by the singularity.
- Complete asymptotic expansions of the volume of small balls, $B_r(p) = \{q : \text{dist}(p,q) < r\}$, for $r \to 0$. Here, dist is the intrinsic distance on X. Once the existence of such an expansion is established, an interesting question would be to relate the coefficients of such an expansion to generalized notions of curvature of X at p, as is well-known in the case of a smooth point p.

Apart from trivial cases, a precise description of the local Riemannian geometry near a singularity is, to the best of our knowledge, only known in the case of asymptotically conical singularities: For this case, the exponential map is analyzed completely in [MW04], and this is then used in the analysis of the propagation of singularities for the wave equation on manifolds with conical singularities.

In our approach we analyze directly the system of ordinary differential equations (ODEs) describing the geodesics. To do this, one needs to use suitable coordinate systems. In the case of an isolated singularity it is natural to use polar coordinates centered at the singularity. Geometrically, introducing polar coordinates corresponds to blowing up the space X at the point p. It is known that by repeated blow-ups, a smooth space X' can be obtained (resolution of singularities); however, the metric on X' corresponding to the metric on X is degenerate at the exponential divisor (the preimage of p), and hence the coefficients of the ODEs blow up there. The problem is to deal with these singularities.

Our guiding idea is that geodesic flow should behave rather regularly when considered on a suitably blown-up space and with a suitable rescaling of time.

The existence of such a blown-up space is by far not obvious. Roughly speaking, the problem is to find a path between two opposing forces:

- with each blow-up the metric becomes more degenerate, hence the geodesic equations become more singular
- one needs to make sufficiently many blow-ups to resolve the singularity as well as the metric (here, 'resolution of a metric' needs to be defined; essentially, this means that the degeneration of the metric tensor at the exceptional divisor has a monomial normal form; for surfaces, this may be taken to be the normal form derived by Hsiang and Pati [HP85])

We are able to carry out this program for the following class of surfaces, modelled on a $(1,2,1)$-quasihomogeneous algebraic singularity: Let $C \subset \mathbb{R}^2$ be a smooth simple closed curve and set $\tilde{X} = C \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$. Let

$$\beta : \mathbb{R}^3 \to \mathbb{R}^3, \quad (u,v,z) \mapsto (uz, vz^2, z), \qquad X := \beta(\tilde{X}) \subset \mathbb{R}^3. \tag{1}$$

That is, the singularity of X is at the origin and quasihomogeneous of type $(1,2,1)$. $\tilde{X}$ is the blow-up (resolution) of X by the quasihomogeneous blow-down map β. Note that X is not necessarily algebraic. We can also allow higher order perturbations of X.

We make the following assumptions on C. Let

$$C_0 = \{q \in C : \ i^* du = 0 \text{ at } q\}, \quad \text{where } i : C \mapsto \mathbb{R}^2 \text{ is inclusion.}$$

We assume

$$(u,v) \in C_0 \Longrightarrow v = 0 \text{ and } C \text{ is nondegenerate at } (u,v). \tag{2}$$

Thus, the only points where the tangent to C is parallel to the v-axis lie on the u-axis, and C has only first order contact with its tangent there.

An example is the surface

$$X = \{(x,y,z) \in \mathbb{R}^3 : \ \left(\frac{x}{z}\right) + \left(\frac{y}{z^2}\right) = 1, \ z > 0\} \cup \{(0,0,0)\}$$

for which $C = \{u^2 + v^2 = 1\}$, the circle. Here $C_0 = \{(\pm 1, 0)\}$.

We consider the metric on X induced by the Euclidean metric on $\mathbb{R}^3$. This induces a smooth semi-Riemannian metric g on $\tilde{X}$ which is Riemannian in the interior of $\tilde{X}$.

Our first theorem shows the existence of the exponential map based at the singular point 0. This will be expressed in terms of geodesics on $\tilde{X}$ rather than X. We call a curve $\gamma : [0,T) \to \tilde{X}$ with $\gamma(0) \in \partial\tilde{X}$ and $\gamma(t)$ in the interior of $\tilde{X}$ for $t > 0$ a *geodesic* if $\gamma_{|(0,T)}$ is a geodesic with respect to the Riemannian metric g and γ is continuous at $t = 0$. We say γ starts at $\gamma(0)$.

Theorem 1. *Let C and $\tilde{X}$ be given as above.*

(a) *For each $q \in C$ there is a unique geodesic $\gamma_q : \mathbb{R}_+ \to \tilde{X}$ starting at $(q,0)$.*
(b) *The map $\exp : C \times \mathbb{R}_+ \to \tilde{X}$, $(q,t) \mapsto \gamma_q(t)$ is a homeomorphism.*

Our main result is a precise description of the analytic properties of exp. To state this, it is more convenient to consider the inverse map

$$\exp^{-1} = (I, d) : \tilde{X} \to C \times \mathbb{R}_+.$$

Here, if $p \in \tilde{X}$ then $I(p)$ is the unique point $q \in C$ for which there is a geodesic from q to p, and $d(p)$ is the time it takes to get from q to p. In terms of X, $I(p)$ corresponds to the direction in which the unique geodesic from $\beta(p)$ to the singular point 0 arrives at 0, and $d(p)$ is the distance from $\beta(p)$ to 0.

Let

$$\pi : X' = [\tilde{X}, C_0 \times \{0\}, C_0 \times \mathbb{R}_+] \to \tilde{X}$$

be the iterated blow-up of $\tilde{X}$ in the (isolated) points of $C_0 \times \{0\}$, followed by a blow-up in the (preimage of) the lines $C_0 \times \mathbb{R}_+$. While the blow-up $\beta : \tilde{X} \to X$ resolves X in the sense of manifolds, the blow-up $[\tilde{X}, C_0 \times \{0\}] \to \tilde{X}$ resolves the metric (for example, in the sense of Hsiang-Pati). The last blow-up, of the lines $C_0 \times \mathbb{R}_+$, is needed for the analytic description of the exponential map below: It replaces each of these lines by two copies of itself (making them boundary lines) and has only the effect that non-smooth behavior is permitted transversal to these lines.

We denote the faces of X' as follows: C_{conic} is the preimage of $(C \setminus C_0) \times \{0\}$, ff is the front face of the first blow-up, that is, the preimage of $C_0 \times \{0\}$, and Z is the preimage of $C_0 \times (0, \infty)$.

Theorem 2. *$\pi^* I$ and $\pi^* d$ are polyhomogeneous conormal functions on X'. They are smooth at C_{conic} and* ff *but their expansion at Z contains half integral powers and may contain logarithms.*

Whether or not the logarithmic terms actually appear remains to be checked, but at present seems very likely. The question of their presence (for the case of semi-algebraic X) is interesting since an affirmative answer would imply that *Hardt's conjecture*, stating that the distance function on a semialgebraic set be subanalytic, is false. In fact, Theorem 2 suggests a *replacement for Hardt's conjecture*, namely that the distance function on a semi-algebraic set is conormal on a suitably blown-up space.

Outline of the proof:

We analyze directly the Hamiltonian flow describing the geodesics in the smooth part of X, that is, the interior of $\tilde{X}$, uniformly up to $\partial\tilde{X}$. On $\partial\tilde{X}$, the Hamilton vector field W is singular (i.e. blows up), and even more so at the points in $C_0 \times \{0\} \subset \partial\tilde{X}$. This singularity is resolved in three ways:

- rescaling the cotangent bundle of $\tilde{X}$, to the conic cotangent bundle ${}^cT^*\tilde{X}$
- rescaling time, by considering $V = zW$ instead of the Hamilton field W, where z is as in (1).
- blowing up certain points in ${}^cS^*\tilde{X}$ (the cosphere bundle corresponding to ${}^cT^*\tilde{X}$), lying over $C_0 \times \{0\}$, with suitable quasi-homogeneity.

The first two steps were already used by Melrose and Wunsch in the conic case, where they showed that (the pull-back of) V is a smooth vector field on ${}^cS^*\tilde{X}$ with non-degenerate hyperbolic critical points along a curve L which projects diffeomorphically to $\partial\tilde{X}$, under the natural projection $\pi_{\tilde{X}} : {}^cS^*\tilde{X} \to \tilde{X}$. The invariant manifold theorem can then be used to deduce the existence and smoothness of the exponential map in the conic case.

In our quasihomogenous situation, the metric on $\tilde{X}$ is conical except at $C_0 \times \{0\}$, and this yields additional singularities of V there. Therefore, an additional blow-up is needed. The main (and highly non-obvious) point of the proof is that these singularities can be resolved by blowing up the points in $(\pi_{\tilde{X}|L})^{-1}(C_0 \times \{0\})$ with a $1, 1, 3$-quasihomogeneity, and that the pull-back of V under that blow-up is not too degenerate. More precisely, one obtains a smooth vector field tangent to the boundary which has only hyperbolic critical points (at least in the regions that matter for the exponential map). The flow near these critical points, and hence everywhere, can then be analyzed.

The emergence of logarithms is related to the fact that the hyperbolic critical points are resonant and therefore (most likely) do not have smooth linearizations (but rather 'log-smooth' linearizations).

While our result is restricted to a special class of singularities, it is the first detailed (that is, to higher order) investigation of the inner geometry of algebraic sets beyond the conic case. Since for general algebraic surfaces a normal form of the metric is known (see [HP85]), we conjecture that similar results will be true in this more general context.

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Stability of black holes and singularities in general relativity

Mihalis Dafermos

This talk surveyed results and open problems in general relativity pertaining to issues surrounding black holes and singularities. The aspects concerning stability of black holes are reviewed in this extended abstract.

General relativity is the study of four-dimensional Lorentzian manifolds $(\mathcal{M}, g)$ satisfying the *Einstein equations*

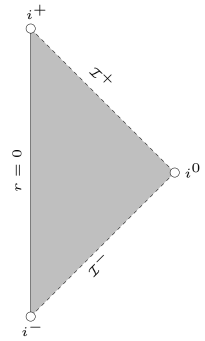
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}, \qquad (1)$$

where the tensor $R_{\mu\nu}$ denotes the Ricci curvature of the metric g, and Λ is the so-called *cosmological constant.*

If we set $T_{\mu\nu} = 0$, then the equations (1) describe the vacuum and constitute a closed system of non-linear hyperbolic partial differential equations for the metric components $g_{\mu\nu}$. See [15] for a general introduction.

1. Penrose diagrams

Many interesting questions concerning (1) refer to the global causal structure of the solution manifold $(\mathcal{M}, g)$. The very notion of a black hole is defined with respect to this. An important object for understanding the global causal structure of Lorentzian manifolds is that of a *Penrose diagram* [5]. In the case of spherically symmetric spacetimes, Penrose diagrams are just the image of a time-orientation-preserving conformal map of the Lorentzian 2-manifold $\mathcal{Q} \doteq \mathcal{M}/SO(3)$ into a bounded subset of $\mathbb{R}^{1+1}$. In the case of Minkowski space, $\mathbb{R}^{3+1}$, a *Penrose diagram* is given below:



From the diagram, one can read off by sight the causal structure of $\mathcal{Q}$. Moreover, $\mathcal{Q}$ inherits an additional boundary induced by the embedding into $\mathbb{R}^{1+1}$, to which one can also apply causal relations. The set of boundary points which are limit points of future directed null rays in $\mathcal{Q}$ for which $r \to \infty$ is denoted $\mathcal{I}^+$ and termed *future null infinity*. Past null infinity $\mathcal{I}^-$ is defined similarly.
Exercise: Show $\mathcal{Q} = J^-(\mathcal{I}^+)$, i.e., the complement of $J^-(\mathcal{I}^+)$ is empty.[1]

2. Black holes

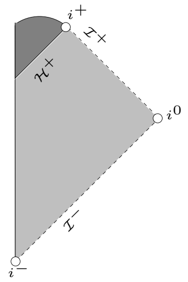
When $\mathcal{I}^+$ is "complete" yet $J^-(\mathcal{I}^+)$ has a nontrivial complement[2], we say that our spacetime $\mathcal{M}$ has a black hole region, which is defined to be precisely this complement

$$\text{Black Hole} = \mathcal{M} \setminus J^-(\mathcal{I}^+).$$

[1] J^- here denotes causal past. See [15] for the basic notions of Lorentzian causality.

[2] We have defined these notions only for spherically symmetric spacetimes. For non-spherically symmetric spacetimes, one cannot identify $\mathcal{I}^+$ and $J^-(\mathcal{I}^+)$ *a priori* in terms of a diagram as discussed above. One can, however, hope to reconstruct the Penrose picture using optical foliations as in Christodoulou-Klainerman [7]. It must be remembered that the construction itself will be coupled with the global analysis of the solution spacetime.

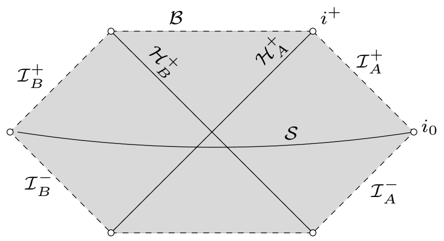
Such a situtation is illustrated in the Penrose diagram below:



We call the future boundary $\mathcal{H}^+$ of $J^-(\mathcal{I}^+)$ in $\mathcal{M}$ the *event horizon*. By our previous "exercise", it follows that Minkowski space does not contain a black hole. As proven by Christodoulou and Klainerman [7], this property of Minkowski space is *stable*. Stability here refers to perturbation of initial data for the Einstein vacuum equations with $\Lambda = 0$.

3. Schwarzschild

Minkowski space and its perturbations do not contain black holes, but spacetimes with black holes solving (1) do exist. The most famous such example is the *Schwarzschild solution*: this was in fact the second solution of the vacuum, $\Lambda = 0$ Einstein equations (1) to have been written down.[3] See [15]. A Penrose diagram is given below:

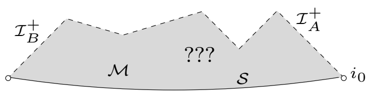


$\mathcal{S}$ denotes a Cauchy surface. An explicit (defined almost everywhere) form of the metric g in coordinates is given by:

$$-(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\sigma_{\mathbb{S}^2}$$

These coordinates degenerate on the horizons $\mathcal{H}_A^+ \cup \mathcal{H}_B^+$ where $r = 2M$.

The problem of nonlinear stability is as follows. Perturb (in a sufficiently small way) Schwarzschild initial data on a Cauchy surface $\mathcal{S}$.



Question: *Does the (say vacuum) maximum development $(\mathcal{M}, g)$ that arises from data still contain a black hole, i.e. is $\mathcal{I}^+$ complete with $\mathcal{M} \setminus J^-(\mathcal{I}^+) \neq \emptyset$?*

[3]Already in 1916. The first is Minkowski space.

It is unlikely that this problem can be resolved without showing at the same time that in the exterior regions (that is to say, $J^-(\mathcal{I}^+)$) the solution approaches a so-called *Kerr solution*. That is to say, it is unlikely that orbital stability can be shown without showing that the Kerr family is *asymptotically stable* around Schwarzschild.

4. Orbital and asymptotic stability in symmetry classes

The reason why asymptotic stability is a prerequisite for orbital stability is connected to the supercriticality of the Einstein equations. Let us compare with the situation under symmetry assumptions.

It turns out that in spherical symmetry, orbital stability in the sense described previously can be proven *without* asymptotic stability for the Einstein equations coupled to a very wide class of matter [8, 11]. In $4+1$ dimensions, orbital stability of *vacuum* black holes has been proven in the class of triaxial Bianchi IX symmetric spacetimes [9]. This gives the first example of vacuum black hole spacetimes which are not identically static or stationary solutions. The above problems all reduce to $1+1$-dimensional p.d.e.'s.

The results described above are not too difficult for the following reasons: (1) the presence of the black hole in conjunction with the symmetry breaks the supercriticality of the system from the point of view of the exterior (there is no $r = 0$!), (2) these systems come with a conservation law provided by a quantity known as the *Hawking mass*. One can thus compare these results with "global existence" results for subcritical nonlinear equations, which do not always retrieve control on asymptotic behaviour.

Asymptotic stability, on the other hand, is much more difficult, even under symmetry assumptions. Asymptotic stability of Schwarzschild and Reissner-Nordström has been proven [12] for the Einstein-(Maxwell)-scalar field system in spherical symmetry.

5. The linearised problem

A prerequisite to begin thinking about this problem is to be able to show decay for an associated linearised problem. Let $(\mathcal{M}, g)$ be Schwarzschild, let $\mathcal{S}$ be an arbitrary Cauchy surface for $(\mathcal{M}, g)$, and let ϕ be the unique solution to the wave equation

$$\Box_g \phi = 0$$

on $(\mathcal{M}, g)$, with sufficiently regular compactly supported initial data prescribed on $\mathcal{S}$. No symmetry is to be assumed on ϕ. The problem then is to understand the decay properties of ϕ in $\mathcal{R} = J^+(I_A^-) \cap J^-(\mathcal{I}_A^+)$. The study of this problem in the physics literature goes back to work of Price [17] in 1972. The first rigorous theorem is

Theorem. *(Kay-Wald* [16]*, 1987) There exists a constant C depending on a suitable norm of initial data such that $|\phi| \leq C$.*

(Note that for the Kerr solution, the validity of the analogue of the above theorem is not known even today). The question of uniform decay is addressed in

Theorem 1. *(M.D.-I. Rodnianski* [13]*) With suitable null coordinates u, v (which generalise $v = t + r$, $u = t - r$ of Minkowski space) mapping $\mathcal{R}$ to $(-\infty,\infty)\times(-\infty,\infty)$, we have*

$$|\phi| \leq C v_+^{-1},$$

and $|r\phi| \leq C u_+^{-1/2}$ along $\mathcal{I}^+$, where C depends on an appropriate norm of the data, and $v_+ = \max\{v, 1\}$, $u_+ = \max\{u, 1\}$.

There are also related weaker results of Blue-Soffer [1, 2] and Blue-Sterbenz [3].

6. The method of energy currents

The proof of Theorem 1 uses energy currents constructed from vector fields. That is to say, recalling the so-called energy momentum tensor

$$T_{\mu\nu} = \phi_\mu\phi_\nu - \frac{1}{2}g_{\mu\nu}\phi^\alpha\phi_\alpha$$

and defining $P_\mu = T_{\mu\nu}V^\nu$ and $\pi^{\mu\nu} = \frac{1}{2}V^{(\mu;\nu)}$, for an arbitrary vector field V, one exploits identities

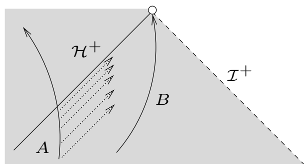
$$-\int_{\Sigma_2} P_\mu N^\mu + \int_{\mathcal{B}} \pi^{\mu\nu}T_{\mu\nu} = -\int_{\Sigma_1} P_\mu N^\mu \qquad (2)$$

where Σ_1 is homologous to Σ_2, bounding some region $\mathcal{B}$. The case where V is Killing and timelike is particularly useful as the bulk term vanishes and the boundary terms are positive definite. See [6] for a general discussion of such identities.

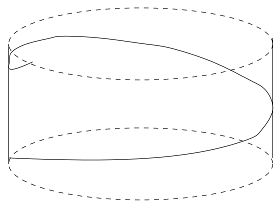
The vector fields applied as multipliers in the proof of Theorem 1 are:

(1) the Killing vector field $T = \partial_t = \partial_u + \partial_v$
(2) a Morawetz-type vector field $K = u^2\partial_u + v^2\partial_v$,
(3) a momentum vector field $X \sim f(u-v)(\partial_u - \partial_v) = f(r^*)\partial_{r^*}$, for a well-chosen function f,
(4) a "local observer" vector field $Y \sim \left(1 - \frac{2M}{r}\right)^{-1}\partial_u$, cut off far from $\mathcal{H}^+$

The vector field Y allows us to understand the red-shift effect of geometric optics from the point of view of energy estimates. In textbooks, this effect is typically described in terms of the frequency shift of a signal emitted by an observer crossing the horizon (A) as measured by observer B. See:



The vector field X, on the other hand, captures the obstruction to decay caused by the presence of the so-called photon sphere at $r = 3M$.



The timelike hypersurface $r = 3M$ contains null geodesics and they neither escape to infinity nor enter the black hole. The bulk integral corresponding to X in identity (2) degenerates precisely at the photon sphere. This is the origin of the loss of derivatives in the decay estimates for the energy flux.

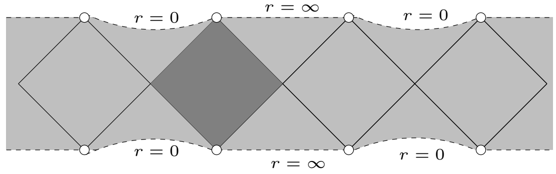
The fact that the fundamental aspects of this problem can be captured by vector field multipliers is a hopeful sign that the non-linear stability problem will be eventually tractable, as the use of vector field multipliers has indeed proven robust in the study of non-linear problems (cf. [7]). On the other hand, the geometry of trapped null geodesics is more complicated even in the special case of the Kerr solutions, and this poses a fundamental challenge for applying these techniques to general perturbations of Schwarzschild.

7. The case $\Lambda > 0$: Schwarzschild-de Sitter

In recent years there has been intense interest in solutions of (1) with positive cosmological constant $\Lambda > 0$. There is a version of the Schwarzschild solution in this case with metric element in local coordinates given by the expression

$$-\left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2}. \tag{3}$$

This spacetime is conventionally called *Schwarzschild-de Sitter*. A Penrose diagram for this solution is given below



The darker shaded region $\mathcal{R}$ is the analogue of the black hole exterior region in the Schwarzschild case.

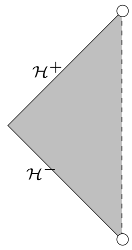
Theorem 2. *(M.D.-I. Rodnianski* [14]*) Let ϕ be an arbitrary smooth solution of the wave equation $\Box\phi = 0$ on Schwarzschild-de Sitter. Then ϕ in $\mathcal{R}$ decays to a constant faster than any poloynomial rate.*

The faster the decay rate desired, the more angular derivatives required on initial data. (For any fixed spherical harmonic, one obtains in fact exponential decay.)

Using the technology of time-independent scattering theory and work of Sá Barreto-Zworski [18], exponential decay away from the horizons has been shown [4] for solutions of the wave equation arising from a restricted class of initial data, in particular, not supported near the bifurcate horizons. It would be interesting to explore whether the restrictive hypothesis on the initial data can be removed, and moreover, whether these methods can in principle be applied to the non-linear problem.

8. The case $\Lambda < 0$: Schwarzschild-anti de Sitter

In the case $M > 0$, $\Lambda < 0$, the expression (3) is known as the Schwarzschild-anti de Sitter solution. The motivation for studying this arises in particular from speculations in high energy physics. In the case where $M = 0$, one obtains the anti de-Sitter solution, denoted as AdS. The infinity of AdS is in fact timelike, and thus, dynamics of (1) for $\Lambda < 0$ near the AdS solution must be studied in the context of a suitable initial-boundary value problem. The boundary condition typically imposed is that the mass at infinity be preserved. This precludes asymptotic stability of AdS spacetime. It is completely unclear what the physical mechanism would be that ensures that AdS is orbitally stable. Remarkably, the situtation appears much more favourable for Schwarzschild-AdS. This is immediately evident from its Penrose diagram



The horizon provides an escape mechanism for radiation. In work in progress [13], Schwarzschild-AdS is proven asymptotically stable in the context of spherically symmetric solutions to (1) coupled with a scalar field.

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On Formation of Singularities in the Critical Wave Map Problem

Igor Rodnianski

This abstract is a short report on the results[1] obtained jointly with Jacob Sterbenz (UCSD). In this work we studied the phenomena of energy concentration for the critical $O(3)$ sigma model, also known as the wave map flow from $\mathbb{R}^{2+1}$ Minkowski space into the sphere $\mathbb{S}^2$. The Lagrangian for this model is given by the expression

$$\mathcal{L}[\Phi] = \frac{1}{2}\partial_\alpha\Phi \cdot \partial_\beta\Phi \ m^{\alpha\beta} \ , \tag{1}$$

where $m_{\alpha\beta}$ is the Minkowski metric on $\mathbb{R}^{2+1}$, while the evolution of the nonlinear field $\Phi : \mathbb{R}^{2+1} \to \mathbb{S}^2$ is described by the Euler-Lagrange equations:

$$\Box\Phi = -\Phi\left(\partial^\alpha\Phi \cdot \partial_\alpha\Phi\right) \ . \tag{2}$$

We established constructively existence of a set of smooth initial data resulting in a dynamic finite time formation of singularities. Our basic result is contained in the following theorem.

[1]The results are contained in the paper: I. Rodnianski, J. Sterbenz *On the formation of singularities in the critical $O(3)$ σ-model*, to appear in Ann. Math.

Theorem 1. *There exists a set of smooth initial data* $(\Phi_0, \dot{\Phi}_0) \in (\mathbb{S}^2, T\mathbb{S}^2)$*, and a finite time* $T = T(\Phi_0, \dot{\Phi}_0)$*, such that the corresponding solution* $\Phi(t,x)$ *of problem* (2) *remains smooth on the interval* $[0,T)$ *and develops a singularity at* T*. Moreover, as* $t \to T$ *we have that for some constant* $C > 0$*:*

$$C^{-1} \frac{\sqrt{\ln(T-t)}}{T-t} \leqslant \sup_{x\in\mathbb{R}^2} |\nabla_x \Phi(t,x)| \ . \tag{3}$$

Furthermore, sufficiently small perturbations of $(\Phi_0, \dot{\Phi}_0)$ *also lead to blowup with bound* (3)*.*

The construction and analysis is done in the context of the k-equivariant symmetry reduction, and we restrict to maps with homotopy class $k \geqslant 4$. Under this symmetry assumption the wave map equation takes the form

$$-\partial_t^2\phi + (\partial_r^2 + \frac{1}{r}\partial_r)\phi \ = \ k^2 \, \frac{\sin(2\phi)}{2r^2} \ , \qquad\qquad k \ \in \ \mathbb{N}^+ \ , \tag{4}$$

for a scalar valued function satisfying boundary conditions $\phi(0) = 0$ and $\phi(\infty) = \pi$. The concentration mechanism we uncovered is essentially due to a resonant self-focusing (shrinking) of a corresponding static solution of (4) – harmonic map $I^k(r) = 2\tan^{-1} r^k$. We showed that the phenomenon is generic (e.g. in certain Sobolev spaces) and persists under small perturbations of initial data, while the resulting blowup obeys an essentially universal log-modified self-similar asymptotic. The dynamics of our blowing up solutions remains close to that of a dynamically collapsing profile $I^k(\lambda(t)r)$ with

$$\lambda(t) \ = \ \left(\frac{C_0}{2C_*}\right)^{\frac{1}{2}} \big(1 + O(\epsilon^{\frac{1}{2}})\big) \frac{\sqrt{\ln(T-t)}}{(T-t)} \ . \tag{5}$$

where the constants C_0 and C_* in the above formula depend on k and are explicit, and small constant ϵ reflects dependence on the initial data.

The logarithmic acceleration of the linear collapse rate is a correction to the previously proposed geodesic behavior, which conjectures a connection between the behavior of low energy solutions of certain infinite dimensional PDE's, in particular (2), and geodesic motion on the finite dimensional moduli space of static solutions.

The wave equation on asymptotically de Sitter-like spaces

András Vasy

(joint work with Antônio Sá Barreto, Richard Melrose)

Consider a de Sitter-like pseudo-Riemannian metric g of signature $(1, n-1)$ on an n-dimensional $(n \geq 2)$ manifold with boundary X, with boundary Y, which near Y is of the form

$$g = \frac{dx^2 - h}{x^2},$$

h a smooth symmetric 2-cotensor on X such that with respect to some product decomposition of X near Y, $X = Y \times [0,\epsilon)_x$, $h|_Y$ is a section of $T^*Y \otimes T^*Y$ (rather than merely $T^*_Y X \otimes T^*_Y X$) and is a Riemannian metric on Y. Such manifolds are Lorentzian analogues of the so-called Riemannian conformally compact (or asymptotically hyperbolic) spaces, see [6].

It is easy to see that de Sitter space, given by the hyperboloid $z_1^2 + \ldots + z_n^2 = z_{n+1}^2 + 1$ in $\mathbb{R}^{n+1}$ equipped with the pull-back of the Lorentzian metric $dz_{n+1}^2 - dz_1^2 - \ldots - dz_n^2$, fits into this framework. Indeed, introducing polar coordinates (r,θ) in the first n variables and writing $t = z_{n+1}$, the hyperboloid can be identified with $\mathbb{R}_t \times \mathbb{S}^{n-1}_\theta$; letting $x = t^{-1}$ for $t > 1$ (with analogous constructions in $t < -1$) compactifies the real line and gives a metric of the desired form.

Let the wave operator $\Box$ be the Laplace-Beltrami operator associated to this metric, and let $P = P(\lambda) = \Box - \lambda$ be the Klein-Gordon operator, $\lambda \in \mathbb{R}$. Below we consider solutions of $Pu = 0$. The *bicharacteristics* of P over X° are the integral curves of the Hamilton vector field of the principal symbol $\sigma_2(P)$ (given by the dual metric function) *inside the characteristic set* of P. As g is conformal to $dx^2 - h$, bicharacteristics of P are reparameterizations of bicharacteristics of $dx^2 - h$ (near Y, that is). Since g is complete, this means that the bicharacteristics γ of P have limits $\lim_{t\to\pm\infty}\gamma(t)$ in $S^*_Y X$, provided that they approach Y. While many of the results below are local in character, it is simpler to state a global result, for which we need to assume that

(A1) $Y = Y_+ \cup Y_-$ with Y_+ and Y_- a union of connected components of Y
(A2) each bicharacteristic γ of P converges to Y_+ as $t \to +\infty$ and to Y_- as $t \to -\infty$, or vice versa

Due to the conformality, the characteristic set $\Sigma(P)$ of P can be identified with a smooth submanifold of S^*X, transversal to ∂X, so $S^*_Y X \cap \Sigma(P)$ can be identified with two copies $S^*_\pm Y$ of S^*Y, one for each sign of the dual variable of x. Under our assumptions we thus have a classical scattering map $\mathcal{S}_{\mathrm{cl}} : S^*_+ Y_+ \to S^*_- Y_-$.

It is well-known, cf. [4], that (A1) and (A2) imply the existence of a global compactified 'time' function T, with $T \in \mathcal{C}^\infty(X)$, $T|_{Y_\pm} = \pm 1$, and the pullback of T to S^*X having positive/negative derivative along the Hamilton vector field inside the characteristic set $\Sigma(p)$ depending on whether the corresponding bicharacterstics tend to Y_+ or Y_-. Notice that $1 - x$ resp. $x - 1$ has the desired properties near Y_+ resp. Y_-, so the point is that a function like these can be extended to all of X. Moreover, such a function gives a fibration $T : X \to [-1,1]$, hence X is in fact diffeomorphic to $[-1,1] \times S$ for a compact manifold S. In particular, Y_+ and Y_- are both diffeomorphic to S. Denote the level set $T = t_0$ by S_{t_0}. With any choice of such a function T, a constant $t_0 \in (-1,1)$, and a vector field V transversal to S_{t_0} (e.g. take the vector field corresponding to dT under the metric identification of TX° and T^*X°), P is strictly hyperbolic, and the Cauchy problem $Pu = 0$ in X°, $u|_{S_{t_0}} = \psi_0$, $Vu|_{S_{t_0}} = \psi_1$, $\psi_0, \psi_1 \in \mathcal{C}^\infty(S_{t_0})$ is well posed.

Theorem 1. *Let* $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} - \lambda}$. *Assuming (A1) and (A2), the solution* u *of the Cauchy problem has the form*

$$u = x^{s_+(\lambda)} v_+ + x^{s_-(\lambda)} v_-, \ v_\pm \in \mathcal{C}^\infty(X), \tag{1}$$

if $s_+(\lambda) - s_-(\lambda) = 2\sqrt{\frac{(n-1)^2}{4} - \lambda}$ *is not an integer. If* $s_+(\lambda) - s_-(\lambda)$ *is an integer, the same conclusion holds if we replace* $v_- \in \mathcal{C}^\infty(X)$ *by* $v_- = \mathcal{C}^\infty(X) + x^{s_+(\lambda)-s_-(\lambda)} \log x\, \mathcal{C}^\infty(X)$.

Conversely, the asymptotic behavior of $v_\pm$ either at Y_+ or at Y_- can be prescribed arbitrarily. Thus, assuming A1 and A2, if $s_+(\lambda) - s_-(\lambda)$ is not an integer, we show that given $g_\pm \in \mathcal{C}^\infty(Y_+)$ there exists a unique $u \in \mathcal{C}^\infty(X^\circ)$ such that $Pu = 0$ and which is of the form (1) and such that

$$v_+|_{Y_+} = g_+, \ v_-|_{Y_+} = g_-. \tag{2}$$

If $s_+(\lambda) - s_-(\lambda)$ is a non-zero integer, the same conclusion holds after modifying the asymptotics (with log terms) as in the statement of Theorem 1.

That is, for all $\lambda \in \mathbb{R}$, there is a unique solution of $Pu = 0$ with two pieces of 'Cauchy data' specified at Y_+. Note the contrast with the elliptic asymptotically hyperbolic problem (conformally compact Riemannian metrics): there one specifies one of the two pieces of the Cauchy data, but over all of Y (not only at Y_+), see [6]. The quantum scattering map is the map:

$$\mathcal{S} : \mathcal{C}^\infty(Y_+) \oplus \mathcal{C}^\infty(Y_+) \to \mathcal{C}^\infty(Y_-) \oplus \mathcal{C}^\infty(Y_-), \ \mathcal{S}(g_+, g_-) = (v_+|_{Y_-}, v_-|_{Y_-}).$$

Of course, the labelling of Y_+ and Y_- can be reversed, so $\mathcal{S}$ is invertible.

Theorem 2. *Suppose that* $s_+(\lambda) - s_-(\lambda)$ *is not an integer, i.e.* $\lambda \neq \frac{(n-1)^2 - m^2}{4}$, $m \in \mathbb{N}$. *Then* $\mathcal{S} = \mathcal{S}(\lambda)$ *is a Fourier integral operator with canonical relation given by* $\mathcal{S}_{\mathrm{cl}}$, *and its suitable renormalization* $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}(\lambda)$ *(see [9]) is an invertible elliptic* 0*th order Fourier integral operator with the same canonical relation.*

It is expected that with more detailed analysis (changing the ansatz slightly to allow logarithmic terms in x) one can prove the theorem even if $s_+(\lambda) - s_-(\lambda)$ is an integer. In addition, if g is *even*, i.e. there is a boundary defining function x such that only even powers of x appear in the Taylor series of g at ∂X expressed in geodesic normal coordinates, see [5] for the Riemannian case, then the $\log x$ terms in v_- disappear and our parametrix construction for $\mathcal{S}(\lambda)$ goes through provided that $s_+(\lambda) - s_-(\lambda)$ is odd. In particular, this covers the case $\lambda = 0$ if n is even.

The de Sitter-Schwarzschild metric, which we do not describe here for the sake of brevity, has many similar features to our class of metrics, at least if we blow up an arbitrary point $q_0 \in Y_+$ in our setting. This blow-up arises in this work as it desingularizes the backward light cone from q_0: the lift of the latter intersects the front face of the blow-up transversally. The interior of the light cone in the front face contains the so-called static model of de Sitter space. If $v \in \mathcal{C}^\infty(X)$, then its lift to $[X; \{q_0\}]$ is constant along the front face, so as long as one restricts one's attention to the interior of this front face, for the actual wave equation ($\lambda = 0$), one observes the convergence of solutions to a constant, $v(q_0)$. This convergence is exponentially fast in terms of the distance function.

The analogous result for de Sitter-Schwarzschild, but with only a polynomial rate of convergence, was recently proved with Antônio Sá Barreto and Richard Melrose, and it was also announced by Dafermos and Rodnianski at this meeting. Weaker results on the asymptotics in that case are contained in the part of works of Dafermos and Rodnianski [2, 3] (they also study a non-linear problem), and local energy decay was studied by Bony and Häfner [1], in part based on the stationary resonance analysis of Sá Barreto and Zworski [8] (which corresponds to the 'static model').

We also note that on de Sitter space itself, one can solve the wave equation explicitly, see [7], but even the 'smooth asymptotics' result, Theorem 1, is not apparent from such a solution.

There are two rather different techniques used to prove these results in [9]. The 'rough' results yielding the existence of the asymptotics, such as Theorem 1, are proved using positive commutator estimates, which roughly speaking describe the microlocal (i.e. phase space) propagation of L^2 (or Sobolev) mass ('energy'). Such methods are very robust, but (unless they are used in a much more sophisticated form) give less precise results. The Fourier integral operator results are proved by a more delicate parametrix construction.

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Path integrals on manifolds by finite dimensional approximation

Christian Bär
(joint work with Frank Pfäffle)

This talk is based on the results in [2]. Let M be a compact Riemannian manifold without boundary, let $E \to M$ be a Riemannian or Hermitian vector bundle with compatible connection ∇. We study selfadjoint generalized Laplace operators, i. e. operators of the form $H = \nabla^*\nabla + V$ where V is a potential (symmetric endomorphism field on E). For simplicity of notation we restrict ourselves to scalar potentials even though this is not necessary.

Our main result can formally be stated as follows: The solution to the heat equation

$$\frac{\partial U}{\partial t} + HU = 0$$

with initial condition

$$U(0,x) = u(x)$$

is given by the path integral

$$U(t,x) = \frac{1}{Z}\int_{\mathfrak{C}_x(M,t)} \exp\left(-\frac{1}{2}\mathrm{E}(\gamma) + \int_0^t \left(\frac{1}{3}\mathrm{scal}(\gamma(s)) - V(\gamma(s))\right) ds\right) \cdot \tau(\gamma)_t^0 \cdot u(\gamma(t))\,\mathcal{D}\gamma.$$

Here $\mathfrak{C}_x(M,t)$ is the space of all continuous paths $\gamma : [0,t] \to M$ emanating from x, $\mathrm{E}(\gamma)$ denotes the energy of the path γ, $\tau(\gamma)$ is parallel translation along γ, $\mathcal{D}\gamma$ is a formal measure on $\mathfrak{C}_x(M,t)$ and Z is a normalizing constant.

Such formulas are very common in the physics literature but there are various problems with a rigorous mathematical interpretation:

- $\mathfrak{C}_x(M,t)$ is an infinite dimensional space and the meaning of the measure $\mathcal{D}\gamma$ is unclear,
- $\mathrm{E}(\gamma)$ and $\tau(\gamma)$ are not defined for continuous paths without differentiability properties,
- Z is infinite.

It is well-known that $\frac{1}{Z}\exp\left(-\frac{1}{2}\mathrm{E}(\gamma)\right)\mathcal{D}\gamma$ yields a well-defined measure on path space $\mathfrak{C}_x(M,t)$, the *Wiener measure*. Parallel transport $\tau(\gamma)$ can be treated using stochastic differential equations. This then generalizes the *Feynman-Kac formula*, see e. g. [4].

We follow a different approach. We approximate $\mathfrak{C}_x(M,t)$ by finite dimensional spaces of geodesic polygons. It turns out that the formally identical integrals over these finite dimensional space approximate the solution to the heat equation. The necessary analysis can be organized nicely using a classical theorem of Chernoff's [3]. The short time asymptotics of the heat kernel also play an important role.

Our technique allows us to derive different versions of the path integral formula. For example, one can remove the scalar curvature term if one uses another measure on the approximating spaces of geodesic polygons. This clearifies a discussion in [1] where our main result has been proved by different methods in the special case of the Laplace-Beltrami operator acting on functions.

As an application we find a very simple and natural proof of the Hess-Schrader-Uhlenbrock estimate for the heat kernel by the kernel of a scalar comparison operator, see [5]. Moreover, we can express the trace of the heat operators by a path integral. Formally,

$$\mathrm{Tr}(e^{-tH}) = \frac{1}{Z}\int_{\mathfrak{C}_{\mathfrak{cl}}(M,t)} \exp\left(-\frac{1}{2}\mathrm{E}(\gamma) + \int_0^t \left(\frac{1}{3}\mathrm{scal}(\gamma(s)) - V(\gamma(s))\right) ds\right) \mathrm{tr}(\mathrm{hol}(\gamma))\,\mathcal{D}\gamma.$$

Here $\mathfrak{C}_{\mathfrak{cl}}(M,t)$ denotes the space of closed continuous loops in M, parametrized on $[0,t]$, and $\mathrm{hol}(\gamma)$ is the holonomy of such a loop γ.

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Scattering Theory for Complex Manifolds and CR-Invariant Differential Operators

Peter A. Perry

(joint work with Peter D. Hislop, Siu-Hung Tang)

In this talk, we describe the relationship between scattering theory for the Laplacian on a complex manifold X and CR-invariants of its strictly pseudoconvex boundary M. Our work is inspired by earlier work of Robin Graham and Maciej Zworski [7], who studied scattering theory for the Laplacian on Poincaré-Einstein manifolds X. In their case, X compactifies to a manifold with boundary M which is the "conformal infinity" of X, and the conformal structure on M is determined by the Poincaré-Einstein metric on M. They show that the conformally covariant powers of the Laplacian on the boundary M (the GJMS operators, see [5]) and Branson's Q-curvature (an analog in conformal geometry of the scalar curvature, see for example [3] for discussion and references to the literature) can be recovered from the scattering operator.

In an analogous way, we show that CR-invariants of the strictly pseudoconvex boundary M of our complex manifolds X can be recovered from the scattering operator of the Laplacian associated to a Kähler metric on X. We require that the Kähler metric on the interior is an approximate Kähler-Einstein metric, i.e.,

$$\mathrm{Ric}(g) + (m+1)g = \mathcal{O}(\varphi^{m-1})$$

where m is the complex dimension of X and φ is a defining function for M. In order to obtain such a metric, we construct approximate solutions of the complex Monge-Ampère equation which are, in an appropriate sense, globally defined near the boundary.

A motivating example for the class of complex manifolds we consider is the complex unit ball or, more generally, a strictly pseudoconvex domain X in $\mathbb{C}^m$ (see Melrose [13] for fundamental results and discussion of scattering theory in this setting). For a strictly pseudoconvex domain in $\mathbb{C}^n$, there is a defining function φ for the boundary of X so that the Kähler form

$$\omega = -\partial\overline{\partial}\log(-\varphi)$$

defines a complete Hermitian metric on X. Moreover, if φ is Fefferman's [2] approximate solution φ of the complex Monge-Ampère equation, the metric is approximately Kähler-Einstein.

Epstein, Melrose, and Mendoza [1] constructed the resolvent and studied the Poisson map for the Laplacian on a class of manifolds that includes those studied here; more recently, Guillarmou and Sa Barreto [8] have studied the resolvent, Poisson map, and scattering operator for the same class in greater depth. In particular, it is known that the scattering operator $S_X(s)$ for the Laplacian is a Heisenberg pseudodifferential operator on $C^\infty(M)$ and is meromorphic as an operator-valued function of s. We study the scattering operator using on the mapping properties of the resolvent established in [1] and the methods of [7].

If X is a complex manifold of complex dimension m with strictly pseudoconvex boundary, and X admits an approximately Kähler-Einstein metric g, we can identify the poles of the scattering operator $S_X(s)$ at $s = m/2 + k/2$, $1 \leq k \leq m$, with the CR-covariant differential operators on M (covariant powers of the sub-Laplacian on M) studied by Fefferman and Hirachi [3] (see also Gover and Graham [4]. We also recover a formula for the renormalized volume analagous to similar formulae obtained for the Poincaré-Einstein manifolds considered by Graham and Zworski.

We also obtain a necessary and sufficient condition on M for X to admit an approximate Kähler-Einstein metric: M must admit a contact form which is pseudo-Einstein as defined by Lee in [11]. The pseudo-Einstein condition is necessary and sufficient to patch together local approximate solutions of the complex Monge-Ampère equation obtained by Fefferman's method and obtain a global defining function that gives an approximate Kähler-Einstein metric on X.

An important ingredient in our analysis is the study of asymptotic geometry of a complex manifold near its strictly pseudoconvex boundary carried out by Lee and Melrose [12] and refined by Graham and Lee in [6].

This work was announced in [9].

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Two spectral invariants of type Rho

Sara Azzali

Rho invariants are spectral invariants which naturally arise in the index theory of elliptic operators treated with heat-kernel techniques. Their geometric properties are very interesting and show that they behave as secondary invariants with respect to the index. The classical invariants invariants of type rho are the Atiyah-Patodi-Singer rho invariant $\rho_{\alpha-\beta}(D)$, defined in [1] and the Cheeger-Gromov L^2 rho invariant $\rho_{(2)}(D))$ ([5]).

In this talk we first recall examples of the use of rho-invariants in geometric issues, and then we will define two new quantities of type rho.

1. Introduction

Let M be a closed compact odd dimensional oriented riemannian manifold and let D be a Dirac-type operator on the sections of a Clifford modules bundle $E \to M$. In fact D will be either the signature operator on M, or, in a parallel case, the spin Dirac operator on a spin manifold.

If D is the signature operator, $\rho_{\alpha-\beta}(D)$ and $\rho_{(2)}(D)$ are independent of the metric on M ([5]). A parallel situation occurs when $\not{D}$ is the Dirac operator on a spin manifold M: then $\rho_{\alpha-\beta}(\not{D})$ and $\rho_{(2)}(\not{D})$ are constant on the connected components of the space $\mathcal{R}^+(M)$ of metrics on M which are positive scalar curvature.

A wider class of results were more recently obtained, by getting into higher index theory: let's recall here some of them. Let $D = D^{sign}$ be the signature operator on a oriented M, let $\Gamma = \pi_1(M)$

- **(b1)** *If Γ is torsion-free and satisfies the Baum-Connes conjecture for the maximal C^*-algebra, then $\rho_{(2)}(D)$ depends only on the oriented Γ-homotopy type of $(M, r : M \to B\Gamma)$* ([Ke], see also [8]).
- **(b2)** *If Γ has torsion and M is $4k+3$-dimensional, $k > 0$, then there are infinitely many manifolds that are homotopy equivalent to M but not homeomorphic to M: they are distinguished by $\rho_{(2)}(D^{sign})$* (Chang and Wienberger [4]).

Let $\not{D}$ be the Dirac operator on a spin manifold M

- **(a1)** *If Γ is torsion-free and satisfies the Baum-Connes conjecture for the maximal C^*-algebra, then $\rho_{(2)}(\not{D}_g)$=0 if $g \in \mathcal{R}^+(Z)$* (Piazza-Schick [8]).
- **(a2)** *If Γ has torsion, M is $4k+3$-dimensional, $k > 0$, and if $\mathcal{R}^+(M)$ is non void, then M has infinitely many different Γ-bordism-classes of metrics with positive scalar curvature: they are distinguished by $\rho_{(2)}(\not{D})$* ([9]).

2. Rho-form for fibrations

We construct a natural generalization of the Cheeger-Gromov L^2-rho-invariant in the case of families of Dirac operators: it is in fact a differential form, which we call rho-form.

Consider a family $\mathcal{D} = (D_b)_{b\in B}$ of Dirac operators along the fibres of $\pi : M \to B$. Suppose that $\dim \operatorname{Ker} D_z$ is constant in z: then one of the results of the heat-kernel proof by Bismut of the family index theorem is that the so-called *eta-form* of the family is well defined (see [2]): it is given in the case of even dimensional fibre by

$$\hat{\eta}(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Str}\left(\frac{d\mathbb{B}_s}{ds} e^{-\mathbb{B}_s^2}\right) ds \quad \in \mathcal{C}^\infty(B, \Lambda^{even} T^*B) \tag{1}$$

where $\mathbb{B}$ is the Bismut superconnection.

Suppose now $\tilde{M} \to M$ is a normal Γ-covering of M. Let $\tilde{\mathcal{D}}$ be the family of Dirac operators lifted on the fibres of the coverings fibration. Our goal is first to construct an L^2 eta form $\hat{\eta}_{(2)}(\tilde{\mathcal{D}}) \in \mathcal{C}^\infty(B, \Lambda T^*B)$ and then to consider the rho-type difference $\hat{\eta}_{(2)}(\tilde{\mathcal{D}}) - \hat{\eta}$. The candidate for $\eta_{(2)}(\tilde{\mathcal{D}})$ is taking the two limits for $t \to 0$ and $T \to \infty$ of

$$\int_t^T \operatorname{Str}_\Gamma\left(\frac{d\tilde{\mathbb{B}}_s}{ds} e^{-\tilde{\mathbb{B}}_s^2}\right) ds \qquad \dim Z = 2l$$

Some hypothesis on the spectrum of the family of operators on the covering fibres must be made. The delicate point is in fact in the $s \to \infty$-asymptotic of the heat operator for the Bismut superconnection on the covering.

Following some ideas of [6], we can give the following sufficient conditions:

Proposition 1. *Let* $\tilde{P} = (P_b)_{b\in B}$ *the family of projections onto the kernel* $\ker \tilde{\mathcal{D}} = \tilde{\mathcal{D}}^2$. *Let* $\tilde{P}_\epsilon$ *be the family of spectral projections of* $\tilde{\mathcal{D}}$ *relative to the interval* $(0, \epsilon)$ *and* $\tilde{Q}_\epsilon = 1 - \tilde{P}_\epsilon - \tilde{P}$ *relative to* $[\epsilon, \infty)$. *If the following regularity hypothesis are satisfied*

(1) $\tilde{P}$, $\tilde{P}_\epsilon$ *are smooth in* z
(2) $\mathrm{tr}_\Gamma(\tilde{P}_\epsilon) = \mathcal{O}(\epsilon^\beta)$ *with* $\beta > 3(\dim B + 1)$

then the form $\hat{\rho}(M, \tilde{M}, \mathcal{D}) := \hat{\eta}(\mathcal{D}) - \hat{\eta}_{(2)}(\tilde{\mathcal{D}}) \in \mathcal{C}^\infty(B, \Lambda T^*B)$ *is well defined.*

A simple observation is the following

Proposition 2. *Let* $\pi : M \to B$ *a fibration of compact spin manifolds. Let* $\hat{g}$ *be a metric on* $T(M/B)$ *s.t.* $\mathrm{scal}\,\hat{g}_b > 0\ \forall b \in B$: *then* $\hat{\rho}_{(2)}$ *is a closed form, hence giving a class* $[\hat{\rho}_{(2)}] \in H^*(B, \mathbb{R})$.

We can relate $\hat{\rho}_{(2)}$ with the space of metrics of positive scalar curvature along the fibres of a fibration $M \to B$ of spin manifolds. We prove that

Proposition 3. *Suppose* B *is compact.* $[\hat{\rho}_{(2)}]$ *is constant on the connected components of* $\mathcal{R}^+(M/B)$, *the space of metrics on the vertical tangent, which are positive scalar curvature along the fibres.*

3. A rho invariant associated to $[c] \in H^2(B\Gamma, \mathbb{R})$

We come back to a single Dirac operator on a compact manifold M and follow the idea of constructing a rho invariant for D which takes into account also a fixed class $[c] \in H^2(B\pi_1(M); \mathbb{R})$. In this sense this new rho quantity would like to be the rho-analogue of the higher indexes of Connes-Moscovici. Here we follow some ideas and of Mathai's works [7], where he proves the Novikov conjecture for higher signatures associated with low-degree classes.[1] A basic idea which we extract from Mathai's is that of twisting the bundle E by a bundle of small curvature. We will fix a 2-cohomology class $[c] \in H^2(B\Gamma, \mathbb{R})$ in the cohomology of the classifying space $B\Gamma$, and a couple (α, β) of *projective* $(\Gamma, \bar{\sigma})$-representations of the group Γ, where σ is the multiplier naturally associated to $[c]$.

By Mathai constructions we have a projective (Γ, σ)-action on every Γ-invariant bundle on the universal covering $\tilde{M}$. Due to this fact, for each *projective* (Γ, σ)-*representation* α of the group Γ we are able to define a vector bundle F_α associated with α and $[c]$, endowed with a connection ∇^α, whose curvature is explicitely written. If α and β are a couple of projective (Γ, σ)-representations of Γ, let D_α and D_β the operators obtained twisting D by the two connections ∇^α and ∇^β respectively. We define

$$\rho^{[c]}_{\alpha-\beta}(D) := \eta(D_\alpha) - \eta(D_\beta)$$

[1]Those which belong to the subring of $H^*(B\Gamma; \mathbb{R})$ generated by classes of degree less than or equal to 2.

Then we concentrate on the case of the Dirac operator on a spin manifold and give the following geometric property.

Proposition 4. *Let M be a closed spin manifold. Then $\rho^{[c]}_{\alpha-\beta}(\not{D})$ has the same value for metrics in $\mathcal{R}^+(M)$ which belong to the same concordance class.*

An open question is whether it possible to define an L^2 invariant $\rho^{[c]}_{(2)}(D)$.

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Fredholm theory on quasi-asymptotically conical manifolds

Anda Degeratu

(joint work with Rafe Mazzeo)

Recall that an operator between Hilbert spaces is called Fredholm if it has finite dimensional kernel, closed range, and finite dimensional cokernel. To it one associates the index, which is the dimension of the kernel minus the dimension of the cokernel.

On a compact Riemannian manifold (X, g), an elliptic operator is always Fredholm as an operator acting between the L^2-spaces. Moreover, the index is a topological object, and it is given by the Atiyah-Singer index theorem. For example, the Dirac operator

$$\not{D}^+_E : H^1(X, S^+ \otimes E) \to L^2(X, S^- \otimes E) \tag{1}$$

twisted by some Hermitian bundle E is Fredholm, and its index is

$$\operatorname{index} \not{D}^+_E = \int_X \hat{A}(g)\mathrm{ch}(E). \tag{2}$$

When we have a non-compact manifold, ellipticity is not enough to guarantee Fredholmness, as usually the range of the operator is not closed. However, choosing the right weights for the Sobolev spaces, usually allows to conclude that the operator is Fredholm. For example, in the case when (X, g) is an asymptotically conical manifold – a non-compact manifold with the infinite end being asymptotically a cone – one introduces a weight which has to do with the distance ρ on the infinite end of X, and then the Laplace operator (twisted by some bundle) is Fredholm as an operator

$$\Delta_E : \rho^\delta H^2(X, E) \to \rho^{\delta-2} L^2(X, E) \tag{3}$$

if and only if δ is not an indicial root of Δ_E. For the Dirac operator one has

$$\not{D}_E^+ : \rho^\delta H^1(X, S^+ \otimes E) \to \rho^{\delta-1} L^2(X, S^- \otimes E), \tag{4}$$

and this is Fredholm if and only if again δ is not an indicial root of $\not{D}_E^+$. The Atiyah-Patodi-Singer index theorem gives the index of this operator, as the integral contribution of the compact version, plus a contribution of the boundary at infinity, contribution which is given by the eta-invariant,

$$\text{index}\, \not{D}_E^+ = \int_X \hat{A}(g)\text{ch}(E) - \frac{\eta_E}{2}. \tag{5}$$

In this talk we are concerned with a new class of non-compact manifolds which we call "quasi-asymptotically conical", or QAC for short. Our ultimate goal is to generalize the formulas (2) and (5) to the class of these manifolds. For now, we are concerned with the more modest goal of figuring out the spaces of functions for which the Laplace operator (and other geometrical operators) are Fredholm, thus generalizing (3) to this new class of manifolds.

Before diving into the technical definition of the QAC spaces, let us first present a bit of motivation, and show why these spaces deserve to be looked at.

The QAC spaces arise naturally as resolutions of singularities in algebraic geometry. Locally, a complex orbifold is modeled on $\mathbb{C}^n/G$, with G a finite subgroup of $U(n)$. Note that the origin of $\mathbb{C}^n$ always gives a singular point in $\mathbb{C}^n/G$. Depending on the way G acts on $\mathbb{C}^n$, we might have some other singular points or not. A resolution of singularities of $\mathbb{C}^n/G$ is a pair (X, π), with X a smooth complex manifold of dimension n, and $\pi : X \to \mathbb{C}^n/G$ a proper surjective map that is a biholomorphism between dense open sets. If the origin gives the only singular point of $\mathbb{C}^n/G$, then X is a non-compact manifold whose geometry is $(\mathbb{C}^n \setminus B_R(0))/G$ outside a compact set. Such a geometry is an example of asymptotically conical manifold. On the other hand, the action of G on $\mathbb{C}^n$ might have more singular points, and then the singular set is non-compact (it arises from subspaces of $\mathbb{C}^n$ with non-trivial stabilizers under the action of G). By resolving the singularities of such manifolds, one is lead to consider the notion of "quasi-asymptotically conical manifolds", geometries which outside a compact set are composed of pieces which are either cones over (possibly) singular spaces, or products between such cones and euclidean spaces.

We introduce three types of spaces which are closely related to each other: (1) the class $\mathcal{I}$ of *iterated cone-edge spaces*, singular spaces obtained via an iterated coning procedure; (2) the class $\mathcal{D}$ of *resolution blowup spaces*, a class of smooth spaces which arise as smoothings of spaces in $\mathcal{I}$; and (3) the class $\mathcal{Q}$ of *quasi-asymptotically conical spaces*, noncompact spaces which on the infinite end have as link an element in $\mathcal{D}$. Basically, if $(Y_0, h_0) \in \mathcal{I}$ is an iterated-cone edge singular space, then a smooth compact manifold Y is in $\mathcal{D}$, if there exists a family of metrics $\{h_\epsilon\}$ on Y so that $(Y, h_\epsilon) \to (Y_0, h_0)$ in Gromov-Hausdorff sense. We call (Y, h_ϵ) a resolution blowup space associated to (Y_0, h_0). Such a space comes with a radius function w_ϵ which converges to s, the distance to the singular stratum of (Y_0, h_0), as $\epsilon \to 0$. On the other hand, a QAC space is a smooth manifold (X, g) with the metric outside a compact set asymptotic to

$$d\rho^2 + \rho^2 h_{1/\rho},$$

meaning that the link at radius ρ is the resolution blowup space $(Y, h_{1/\rho})$. It comes with a pair of two radius functions (ρ, w), with $\rho : X \to [1, +\infty)$ the distance on the quasi-asymptotically conical end, and with w at radius ρ being the radius function $w_{1/\rho}$ on the resolution blowup space $(Y, h_{1/\rho})$.

Note that the construction of resolution blowup and QAC spaces is an inductive one. Once we constructed a QAC space, we can go on and construct a resolution blowup space for an iterated cone-edge space with higher depth singularities – thus a *depth* induction. As such, to prove a Fredholmness result on a QAC space (X, g), one first need to show that the restriction of the operators on each slice behaves well in the limit, meaning one needs to prove a spectral convergence result for the resolution blowup spaces (Y, h_ϵ) which come with the QAC package. This spectral result is based on the Fredholmness of the corresponding operator on the lower depth QAC spaces used to construct (Y, h_ϵ), and this is the inductive step for its proof.

Theorem 1 (Spectral Convergence)**.** *Let $(Y, h_\epsilon) \in \mathcal{D}$ be a resolution blowup space. Let L_ϵ be a generalized Laplace operator acting on the sections of a geometric vector bundle. Assume that each model operator on the lower depth QAC spaces appearing in the deconstruction of Y is positive, and that 0 is not a L^2-eigenvalue. Then the spectrum of L_ϵ converges to the spectrum of the Friedrichs extension of the limiting operator L_0 on (Y_0, h_0).*

The analogue of (3) in the QAC context is the following:

Theorem 2. *Let (X, g) be a QAC manifold with radius functions (ρ, w). Let $\mathcal{L}$ be a generalized Laplace operator on X twisted by some geometrical vector bundle E. Then*

$$\mathcal{L} : \rho^\delta w^\tau H^2(X, E) \to \rho^{\delta-2} w^{\tau-2} L^2(X, E) \qquad (6)$$

is a Fredholm operator provided δ is not an indicial root for $\mathcal{L}$, and τ is so that the lower depth operators on the corresponding QAC spaces are positive and do not have 0 as a L^2-eigenvalue.

The proofs of these two theorems are interlinked, and they go inductively. The first inductive step in the proof of Theorem 1 – the case of a resolution blowup space corresponding to a space (Y_0, h_0) with isolated conical singularities – was proved in the PhD thesis of Rowlett [6]. Then the general case (assuming Theorem 2) was presented in [5]. Note that in the context of the scalar Laplacian acting on QALE manifolds (a special class of QAC manifolds), the proof of Theorem 2 appears in Joyce [4]. Since it is based on the maximum principle, his proof cannot be generalized to the case of systems. In the process of proving Theorem 2 we also prove a similar result for weighted Hölder spaces.

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Generalized gluing for Einstein constraint equations

Lorenzo Mazzieri

CMC solutions and conformal method. It is well known [2] that short time vacuum solutions for the Einstein hyperbolic system on a Lorentzian $(m+1)$-dimensional manifold (Z, γ) may be obtained from solutions of the Einstein constraint equations on a m-dimensional space-like Riemannian submanifold (M, g) of Z. In fact the solutions to the constraints form a suitable set of vacuum initial data for the hyperbolic Cauchy problem (for further details see [1]). More precisely, when we are talking about a solution of the constraints we refer to a triple (M, g, Π), where M is a smooth manifold and g and Π are symmetric $(2,0)$ tensors (respectively the induced Riemannian metric and the second fundamental form), verifying the relationships

$$\mathrm{div}_g \, \Pi - \mathrm{d}\,(\mathrm{tr}_g \, \Pi) \quad = \quad 0\,, \tag{1}$$

$$R_g - |\Pi|_g^2 + (\mathrm{tr}_g \, \Pi)^2 \quad = \quad 0\,. \tag{2}$$

where div_g and tr_g are respectively the divergence operator and the trace operator computed with respect to the metric g and R_g is the scalar curvature of the metric g. In the case we are looking for constant mean curvature (CMC) solutions of the constraints (i.e. when $\tau := \mathrm{tr}_g \, \Pi$ is a constant) the system above becomes equivalent to an uncoupled system by means of the so called conformal method. Following [2], and [3], one can split the second fundamental form Π into trace free and pure trace part $\Pi \; = \; \mu + \frac{\tau}{m}\, g$, where μ is a symmetric $(2,0)$-tensor such that $\mathrm{tr}_g \mu = 0$. Then it is convenient to consider the double conformal change $g = u^{4/(m-2)}\,\bar g$ and $\mu = u^{-2}\,\bar\mu$, where the conformal factor u is a positive smooth function on M. It is now straightforward to see that g and Π verify the Einstein constraint equations (1) and (2) if and only if the following holds for $\bar g, \bar\mu$ and u

$$\mathrm{tr}_{\bar g}\,\bar\mu = 0 \tag{3}$$

$$\mathrm{div}_{\bar g}\,\bar\mu = 0 \tag{4}$$

$$\mathrm{Lic}_{\bar g}(u) = 0\,, \tag{5}$$

where Lic is the semi-linear elliptic operator given by

$$\mathrm{Lic}_{\bar g}(u) \;=\; \Delta_{\bar g}\, u + c_m\, R_{\bar g}\, u - c_m\,|\,\mu_\varepsilon\,|^2_{\bar g}\, u^{-\frac{3m-2}{m-2}} + c_m\,\frac{m-1}{m}\,\tau^2\, u^{\frac{m+2}{m-2}} \tag{6}$$

with $c_m = (m-2)/[4\,(m-1)]$. Therefore, in order to produce a CMC solution for the Einstein constraints it is sufficient to provide a symmetric transverse (4) traceless (3) tensor (briefly TT-tensor) and the right conformal factor, it is to say a solution of the Lichnerowicz equation (5). In this context and because of their physical meaning [1] the equations (1) (or equivalently (4)) and (2) (or equivalently (5)) are known as the momentum constraint and the Hamiltonian or energy constraint respectively.

Strategy of the gluing and statement of the main result. In the spirit of [3] suppose now to have two set of solutions $(M_i,\, g_i,\, \mu_i,\, u_i)$, $i = 1, 2$ to equations (3), (4) and (5) with the same τ and suppose that we construct the generalized connected sum of the compact m-dimensional manifolds M_1 and M_2 along a common isometrically embedded k-dimensional Riemannian submanifold (K, g_K). This construction consists in excising a small ε-tubular neighborhood (i.e. a tubular neighborhood of size $\varepsilon \in (0, 1)$) of K in both the starting manifolds and in identifying the differentiable structures along the left over boundaries as explained in [4]; also notice that in order to perform the generalized connected sum we have to require that the normal bundles of K in M_1 and in M_2 are diffeomorphic and that the injectivity radius of K in M_1 and M_2 is grater than a positive constant. The purpose is then to endow - in correspondence to each value of ε - the new manifold $M_\varepsilon = M_1 \sharp_{K,\varepsilon} M_2$ with a Riemannian structure g_ε and a symmetric TT-tensor μ_ε such that a solution u_ε to equation (5) can be found. As is typical of the gluing results, the new solution has to preserve the information about the starting solutions insofar as is possible. In our case the metric g_ε will coincide by construction with the metrics g_1 and g_2 away from the gluing locus. Moreover, as the geometric parameter ε tends to zero, the metric g_ε tends to the metric g_i on the compact sets of $M_i \setminus K$, with respect to the $\mathcal{C}^2$ topology, for $i = 1, 2$. The TT-tensor μ_ε too tends to μ_i away from the gluing locus, as ε goes to zero. In addition, we can make the conformal factor u_ε as close to the constant 1 as we want, by choosing ε to be small. In this sense, we are allowed to look at the metric g_ε and at the TT-tensor μ_ε as an approximate solution of the system (1)-(2), which can be made exact by means of a small conformal perturbation $u_\varepsilon \simeq 1$

$$\tilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon\,, \tag{7}$$

$$\tilde{\Pi}_\varepsilon = u_\varepsilon^{-2}\,\mu_\varepsilon + \frac{\tau}{m}\, u_\varepsilon^{\frac{4}{m-2}}\, g_\varepsilon\,. \tag{8}$$

As already explained, the real advantage in considering CMC solutions is that one has an uncoupled system to solve (3)-(4)-(5) instead of the system (1)-(2). In particular, once an approximate solution metric g_ε is available, the natural way to proceed is to solve first the equations (3) and (4), and then to put the solution μ_ε in the equation (5) and solve this one for u_ε. Since the latter equation is nonlinear and we wish to solve it by means of a perturbation argument which allows us to choose $u_\varepsilon \simeq 1$, we are led to linearize the Lichnerowicz operator around the constant 1 and to consider the left terms as error terms. Among these error terms, the squared norm of μ_ε plays a significant role. As a consequence of this special role of μ_ε it is crucial to get an ε-uniform bound for solutions of the equation (4). In this form the momentum constraint is a linear system of partial differential equations and there is a standard two step procedure to produce trace free solutions of it [5]. In our case we will proceed as follows. Starting with μ_1 and μ_2 and using suitable cut-off functions, we produce a g_ε-trace free symmetric $(2,0)$-tensor μ, which in general is not a solution to (4) (notice that μ actually depends on ε as long as it has to be trace free with respect to the metric g_ε). The second step consists then in finding a correction term σ_ε which repairs the momentum constraint (i.e., $\mathrm{div}_{g_\varepsilon}(\mu + \sigma_\varepsilon) = 0$). Since the system is largely underdetermined, we may force the solution to have a special shape. In particular we look for a solution of the form $\sigma_\varepsilon = D_{g_\varepsilon} X$, where X is a vector field on M and D_{g_ε} is the so called conformal Killing operator for the metric g_ε. This operator enjoys a nice algebraic property: it is the negative of the formal adjoint of the divergence, more precisely $D_g = -\,(\sharp\,\mathrm{div}_g)^*$. As a consequence when we perform the repair of the momentum constraint, we are induced to consider the elliptic self-adjoint operator $L_g := -\sharp\,\mathrm{div}_g \circ D_g = D_g^* \circ D_g$, known as the vector Laplacian, and to solve the equation

$$L_g\, X = \sharp\,\mathrm{div}_g\,\mu \tag{9}$$

with respect to each g_ε metric, therefore providing the solutions with an *a priori* ε-uniform bound. The vector fields in the kernel of D_g are called conformal Killing vector fields, in fact their flow leaves the metric invariant up to conformal changes. For technical reasons, in order to deduce the ε-uniform estimate, we have to require a non-degeneracy condition about the conformal Killing vector fields of the starting manifolds. The hypothesis we need is that there are no nontrivial conformal Killing vector fields on either (M_1, g_1) or (M_2, g_2). Following the strategy summarized above, we can prove:

Theorem 1. *Let* $(M_1, \tilde{g}_1, \tilde{\Pi}_1)$ *and* $(M_2, \tilde{g}_2, \tilde{\Pi}_2)$ *be two compact* m*-dimensional CMC solutions to the Einstein constraint equations (1)-(2) having the same constant mean curvature* τ *and verifying the non-degeneracy condition and let* $(K, \tilde{g}_K)$ *be a common isometrically embedded* k*-dimensional sub-manifold with codimension* $n := m - k \geq 3$. *Then there exists a real value* $\varepsilon_0 \in (0, 1)$ *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *it is possible to endow the* ε*-generalized connected sum* $M_\varepsilon = M_1 \sharp_{K,\varepsilon} M_2$ *of* M_1 *and* M_2 *along* K *with a metric* $\tilde{g}_\varepsilon$ *and a second fundamental form* $\tilde{\Pi}_\varepsilon$ *such that the triple* $(M_\varepsilon, \tilde{g}_\varepsilon, \tilde{\Pi}_\varepsilon)$ *is still a* τ*-CMC solution to the Einstein constraint equations. Moreover the new metric* $\tilde{g}_\varepsilon$ *tends to the starting metric* $\tilde{g}_i$ *in the* $\mathcal{C}^2$ *topology on the compact sets of* $M_i \setminus K$, *for* $i = 1, 2$, *as the geometric parameter* ε *tends to zero. The symmetric TT-tensor* $\tilde{\mu}_\varepsilon$ *does the same away from a fixed tubular neighborhood of* K *(gluing locus) whose radius can be chosen to be arbitrarily small.*

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Spectral theory for Laplace-Operators on manifolds with fibred cusp metrics

Jörn Müller

In this talk I am presenting the current state and goals of my PhD thesis.

A Riemannian manifold with a metric of fibred cusp type can be considered as a generalization of the geometric structure of a locally symmetric space of $\mathbb{Q}$-rank one. The structure at infinity of such a manifold X is determined by a Riemannian submersion $\pi\colon M \to B$ with $q-$dimensional fibres F as follows.

Let $Z = \mathbb{R}^+ \times M$ and equip Z with the metric $g^Z = du^2 + \pi^* g^B + e^{-2u} g^{F_b}$, where g^B and g^{F_b} are the Riemannian metrics on B and on the vertical tangent bundle, respectively. Then outside a compact set, X is isometric to (Z, g^Z).

In my thesis I examine the spectral theory of the Laplace-operator on differential forms $\Omega^p(X)$. The results then will be used to proof a Hodge-type theorem, i.e. to identify certain subspaces of L^2-harmonic forms on X, and to represent cohomology classes by singular values of generalized eigenforms.

The work bases mainly on methods developed and used by my advisor Werner Müller for example in [3] and [4]. The case of locally symmetric spaces has been studied extensively by Günter Harder, e.g. in [1].

A different approach to treat the spectral theory of manifolds with fibred cusp metrics, the $b-$ and $\phi-$calculus by Melrose, is used in the PhD-thesis [5] of Boris Vaillant. In [2] some Hodge-type theorems are proved, also using methods from $\phi-$calculus.

In my talk I first restrict myself to the case of the Laplacian Δ on functions to illustrate the approach. For example, using methods from scattering theory, it is shown that the essential spectrum of Δ is determined by the restriction of Δ to fibre-wise constant functions. The spectral resolution of Δ can be given explicitely in terms of generalized eigenfunctions and L^2-eigenfunctions. Many of the results for functions can be generalized to the setting of differential forms. In particular, fibre-wise constant functions have to be replaced by fibre-wise harmonic forms. Forms in their L^2-orthogonal complement are called "cusp forms".

At the end of the talk I explain the following result due to Werner Müller, which I want to generalize to manifolds with fibred cusp metrics:
Let X be a manifold with cusp ends, $\mathscr{H}^p(X)$ harmonic L^2-forms on X. The space of harmonic cusp forms is naturally isomorphic to the image $H^p_!(X)$ of the cohomology with compact support in the usual (smooth) cohomology $H^p(X)$ under the map $\mathscr{H}^p(X) \to H^p(X)$. Classes in the complement of $H^p_!(X)$ in $H^p(X)$ can be represented by certain singular values (i.e. special values or residues) of generalized eigenforms.

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On the uniqueness of certain families of holomorphic disks

Frédéric Rochon

Since their introduction by Roger Penrose [6] (see also [1]), twistor spaces have proven to be very successful in the study of Riemannian manifolds with special properties. They allow one to encode subtle geometric information into holomorphic objects. In this way, powerful tools of complex and algebraic geometry can come into play and reveal in a rather unexpected way new properties of the original geometric situation. A recent example of such a phenomenon was discovered by LeBrun and Mason [5] in their study of Zoll metrics on a compact surface.

In general, a Zoll metric on a manifold is a metric whose geodesics are simple closed curves of equal length. The terminology is in honor of Otto Zoll [7], who discovered that the 2-dimensional sphere $\mathbb{S}^2$ has many such metrics beside the standard one. Later on, it was speculated by Funk [2] and proven by Guillemin [4] that the tangent space of the moduli space of Zoll metrics on $\mathbb{S}^2$ (modulo isometries and rescaling) at the standard round metric is isomorphic to the space of odd functions $f : \mathbb{S}^2 \to \mathbb{R}$. In particular, this indicates that the moduli space of Zoll metrics on $\mathbb{S}^2$ is infinite dimensional. The only other compact surface admitting a Zoll metric, $\mathbb{RP}^2$, has a very different behavior in that respect since, as was conjectured by Blasche and proved by Leon Green [3], it admits only one Zoll metric modulo rescaling and isometries.

Via the use of twistor theory, LeBrun and Mason were able in [5] to recover all these results in a very elegant and novel way. In the case of the sphere $\mathbb{S}^2$, their twistor correspondence associates to a Zoll metric on $\mathbb{S}^2$ a family of holomorphic disks in $\mathbb{CP}_2$ with boundary in a totally real submanifold $P \subset \mathbb{CP}_2$ such that there exists a diffeomorphism $\varphi : \mathbb{CP}_2 \to \mathbb{CP}_2$ identifying P with the standard $\mathbb{RP}^2 \subset \mathbb{CP}_2$. Moreover, it is such that each holomorphic disk of the family represents a generator of $\mathrm{H}_2(\mathbb{CP}_2, P) \cong \mathbb{Z}$. For instance, the standard round metric on $\mathbb{S}^2$ corresponds to a family of holomorphic disks with boundary lying on the standard $\mathbb{RP}^2 \subset \mathbb{CP}_2$.

In the case of the standard $\mathbb{RP}^2 \subset \mathbb{CP}_2$, one can check rather directly that there is a unique family of holomorphic disks associated to it. This is because considering the involution

$$\begin{array}{cccc} \rho : & \mathbb{CP}_2 & \to & \mathbb{CP}_2 \\ & [x_0 : x_1 : x_2] & \mapsto & [\overline{x}_0 : \overline{x}_1 : \overline{x}_2] \end{array}$$

having the standard $\mathbb{RP}^2$ as a fixed set, we can double any holomorphic disk $D \subset \mathbb{CP}_2$ with $\partial D \subset \mathbb{RP}^2$ and $D \setminus \partial D \subset \mathbb{CP}_2 \setminus \mathbb{RP}^2$ to obtain a holomorphic curve

$$\Sigma = D \cup \rho(D) \subset \mathbb{CP}_2.$$

In the case of a Zoll family, D has to represent a generator of $\mathrm{H}_2(\mathbb{CP}_2, \mathbb{RP}^2)$, which means $\Sigma \cong \mathbb{CP}_1$ is a curve of degree 1 in $\mathbb{CP}_2$. Thus, Σ is the zero locus of a homogeneous polynomial of degree 1. Furthermore, the coefficients of this polynomial can be chosen to be real since $\rho(\Sigma) = \Sigma$. This indicates that D has to be a member of the family of holomorphic disks corresponding to the standard Zoll metric on $\mathbb{S}^2$.

However, in the more general situation where $P \subset \mathbb{CP}_2$ does not come from the fixed set of an involution, the above argument has no obvious generalization. In principle, one could therefore imagine situations where two distincts Zoll metrics on $\mathbb{S}^2$ would lead to two distinct families of holomorphic disks with boundary on the same totally real submanifold $P \subset \mathbb{CP}_2$. This would mean the totally real submanifold $P \subset \mathbb{CP}_2$ would not be sufficient to recover the Zoll metric from which it comes from.

In this work, we rule out this possibility by showing that for a totally real submanifold $P \subset \mathbb{CP}_2$ such that there exists a diffeomorphism $\varphi : \mathbb{CP}_2 \to \mathbb{CP}_2$ identifying P with the standard $\mathbb{RP}^2$, there is at most one family of holomorphic disks coming from a Zoll metric on $\mathbb{S}^2$. In particular, the twistor correspondence of LeBrun and Mason, which to a Zoll metric on $\mathbb{S}^2$ associate a totally real submanifold $P \subset \mathbb{CP}_2$, is in some sense injective. In fact, we show that given such a family, any holomorphic disk $D \subset \mathbb{CP}_2$ with $\partial D \subset P$, $D \setminus \partial D \subset \mathbb{CP}_2 \setminus P$ and such that it is a generator of $\mathrm{H}_2(\mathbb{CP}_2, P)$ necessarily has to be a member of the family.

Our proof is by contradiction. We suppose that there is such a holomorphic disk $D \subset \mathbb{CP}_2$ which is not in the family. Then we show that there exists a holomorphic disk D_b in the family such that D and D_b intersect transversally and have a non-empty intersection in $\mathbb{CP}_2 \setminus P$. Since D and D_b have boundaries, the intersection number one gets is not a homotopy invariant. Our strategy is to look at the lifts $\widetilde{D}$ and $\widetilde{D}_b$ of D and D_b in the blow-up (in the sense of Melrose)

$$\widetilde{\mathbb{CP}}_2 := [\mathbb{CP}_2; P].$$

The family of holomorphic disks defining a circle fibration on $\partial\widetilde{\mathbb{CP}}_2$, we have an alternative way of blowing down $\widetilde{\mathbb{CP}}_2$. Let

$$\alpha : \widetilde{\mathbb{CP}}_2 \to Y$$

denote the corresponding blow-down map. In this blow-down procedure, each disk becomes a sphere, so eliminating the boundary. We can also get a sphere out of D by deforming it before blowing down with α. Doing it carefully, we can insure that the intersection number between $\widetilde{D}$ and $\widetilde{D}_b$ remains positive. One can also check that the corresponding spheres $\alpha(\widetilde{D})$ and $\alpha(\widetilde{D}_b)$ in the blow-down picture are homologous. By construction, the oriented intersection number of these two spheres will be positive, which leads to a contradiction, since the self-intersection number of the corresponding homology class is shown to be equal to zero.

Using this uniqueness result, we are able to use the continuity method to obtain a partial result concerning the existence of families of holomorphic disks arising from a projective Zoll structure. This includes some situations where the totally real submanifold $P \subset \mathbb{CP}_2$ is not necessarily $\mathcal{C}^\infty$-close to the standard $\mathbb{RP}^2 \subset \mathbb{CP}_2$.

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Schrödinger evolution and geometry

Jared Wunsch

Consider the initial value problem for the time-dependent Schrödinger equation

$$(-i\partial_t + (1/2)\Delta + V)\psi = 0$$
$$\psi|_{t=0} = \psi_0 \in L^2(X)$$

on $\mathbb{R} \times X$, where (X, g) is a Riemannian manifold, V is real-valued, and

$$\Delta = \Delta_g = d^* d$$

is the (nonnnegative) Laplace-Beltrami operator on X. This equation describes the (nonrelativistic) quantum evolution of a particle moving on X.

The effects of the geometry of the underlying manifold X on the qualitative nature of the solutions ψ can be subtle. On $\mathbb{R}^n$ (with $V = 0$) we have the "local smoothing estimate"

$$\|\psi\|_{L^2([0,T];H^{1/2}_{\text{loc}}(\mathbb{R}^n))} \leq C\|\psi_0\|. \tag{1}$$

This estimate tells us that *on average* in time, ψ is half a derivative smoother than its initial data [10, 19, 22]. Likewise, if we measure regularity in L^p spaces, we have the *Strichartz estimates*

$$\|\psi\|_{L^q([0,T];L^p(\mathbb{R}^n))} \leq C\|\psi_0\|_{L^2}. \tag{2}$$

for certain values of p, q [21, 13]

That these estimates may fail on manifolds is easy to see. For instance, if X is compact, the H^s norm is conserved under the evolution, so (1) cannot possibly be valid. More precisely, a theorem of Doi [11] states, loosely speaking, that it cannot hold in a noncompact manifold with trapped geodesics. By contrast, both estimates (1), (2) continue to hold with only an epsilon derivative loss in various settings with *mild* trapping, including the exterior problem with Dirichlet conditions outside two strictly convex obstacles [4], asymptotically Euclidean manifolds with a single hyperbolic closed geodesic [8], and asymptotically Euclidean manifolds with hyperbolic dynamics on a trapped set, subject to a condition on the topological pressure [9, 16]. Estimates (1) and (2) are also known to hold without loss in a wide variety of geometric situations, including on asymptotically Euclidean spaces and even asymptotically conic spaces [20, 5, 15, 18, 3], and on Euclidean space with a potential having one or more poles of order zero [7, 12]. Results on hyperbolic space have recently been obtained by several authors [1, 2, 17]. By contrast, the role of trapping in Strichartz estimates remains somewhat obscure, especially given that Burq-Gérard-Tzvetkov [6] have obtained an estimate with derivative loss that holds even when X is compact.

The propagation of singularities for solutions to the Schrödinger equation involves more than merely the wavefront set, as can easily seen from the fact that the fundamental solution is smooth for $t \neq 0$. It turns out to be the asymptotic behavior at infinity (in particular, the oscillation) of the initial data that determines the wavefront set of the solution at time t. In particular, Hassell-Wunsch [14] showed that if we take a solution to the Schrödinger equation on an asymptotically conic manifold with no trapped geodesics, then it is $\mathrm{WF}_{\mathrm{sc}}(e^{ir^2/2T}\psi_0)$ that determines the wavefront set of ψ at time T. Here, $\mathrm{WF}_{\mathrm{sc}}$ denotes Melrose's "scattering wavefront set," and the connection to $\mathrm{WF}(\psi(T))$ is via a "sojourn relation" relating the cotangent bundle of X and the (rescaled) cotangent bundle at infinity.

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The invertible double construction for general first order elliptic differential operators

Matthias Lesch
(joint work with Bernhelm Booß–Bavnbek, Chaofeng Zhu)

1. The invertible double of a Dirac operator revisited

Let M be a smooth compact connected riemannian manifold with boundary and let D be a Dirac type operator on M acting between sections of the hermitian vector bundle E. A natural question is whether D has an extension to an elliptic or even invertible operator on the closed double $\widetilde{M} = M \cup_{\partial M} M$ obtained by gluing two copies of M along their common boundary. In fact, if all structures are product near the boundary there is a nice answer to the problem which can be found, e.g., in the book by Booß and Wojciechowski [3].

Proposition 1. *Assume that all structures are product near the boundary. Then D and $-D$ can be glued together to obtain an* invertible *elliptic operator $\widetilde{D} = D \cup_{\partial M} (-D)$ on the closed double $\widetilde{M}$.*

Sketch of Proof. Since all structures are product near the boundary there is a collar $[0, \varepsilon) \times \partial M$ on which D takes the form (is unitarily equivalent to)

$$D_{|\text{collar}} = J\big(\frac{d}{dx} + B\big), \tag{1}$$

with J, B independent of the normal variable. The normal symbol J is a unitary bundle endomorphism with $J^2 = -I$ and the tangential operator B is a first order self–adjoint elliptic differential operator on ∂M. Furthermore, the (formal) self–adjointness of D and B imply $JB = -BJ$.

The unitary map sending a section f over $[0, \varepsilon)\times\partial M$ to $Jf(-.)$ over $(-\varepsilon, 0]\times\partial M$ conjugates D and $-D$. Hence if we use J as a clutching function for the bundle E then indeed $\widetilde{D} = D \cup_{\partial M} (-D)$ is an elliptic operator with smooth coefficients on the double.

To prove that $\widetilde{D}$ is invertible assume that $\widetilde{D}f = 0$. Denoting the two different copies of M in $\widetilde{M}$ by $M_\pm$ we then have $D(f_{|M_\pm}) = 0$ and Green's formula gives

$$\begin{aligned} \langle f_{\partial M}, f_{\partial M}\rangle &= \langle J(f_{|M_-})_{|\partial M}, (f_{|M_+})_{|\partial M}\rangle \\ &= \langle Df_{|M_-}, f_{|M_+}\rangle - \langle f_{|M_-}, Df_{|M_+}\rangle = 0. \end{aligned} \tag{2}$$

Thus $f_{|\partial M} = 0$. The weak unique continuation property of Dirac operators then implies $f = 0$. □

The weak unique continuation property exactly means that on the connected manifold M there are no nontrivial solutions of $Df = 0$ which vanish on the boundary, cf. e.g. [2] and the references therein.

For general Dirac operators in the non–product case one can still extend the collar a bit and deform to the product situation. The resulting operator will still be invertible; however it will neither be an exact double nor will it be canonical.

The invertible double is used for the construction of the Calderón projector. This is a pseudo–differential idempotent (R.T. Seeley [9, 10]) with range being the space of Cauchy data for D. "Construction of the Calderón projector" here means to provide a formula in terms of D such that mapping properties and dependencies on the data become transparent. See Theorem 2 below.

For more general elliptic operators, in the literature an invertible elliptic extension is often assumed "for convenience". Although a geometric invertible double is not available in general, we will see below that there is always a nice boundary value problem which provides an "invertible" double.

2. General first order elliptic operators (no product assumption)

Let now M be a smooth compact connected riemannian manifold with boundary and let $A : \Gamma^\infty(M, E) \longrightarrow \Gamma^\infty(M, F)$ be a first order elliptic differential operator acting between sections of the hermitian vector bundles E, F.

The goals of the current project are

- A canonical "invertible double" construction. The quotation marks are used since in general we cannot expect weak UCP to hold. For details see Theorem 1 below.
- A canonical (and in a sense optimal) construction of the Calderón projector.

At the end we will point out a couple of applications.

As above we separate variables in a collar of the boundary and write

$$A_{|\text{collar}} = J(x)\big(\frac{d}{dx} + B(x)\big), \quad J_0 := J(0), B_0 := B(0), \tag{3}$$

where $J(x)$ is a smooth family of bundle morphisms and $B(x)$ is a smooth family of first order elliptic differential operators on ∂M.

Put

$$\widetilde{A} := A \oplus (-A^t), \tag{4}$$

acting on sections of $E \oplus F$. A^t denotes the formal adjoint of A. We choose a bundle morphism $\Lambda \in \operatorname{Hom}(E_{|\partial M}, F_{|\partial M})$ and impose the following boundary condition of $\widetilde{A}$:

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \operatorname{dom}(\widetilde{A}_\Lambda) :\Leftrightarrow f_{-|\partial M} = \Lambda f_{+|\partial M}. \tag{5}$$

The two most important cases are $\Lambda = (J_0^*)^{-1}$ and, if $J_0 = -J_0^*$, $\Lambda = J_0|J_0|^{-1}$. In both cases the endomorphism $J_0^*\Lambda$ is positive definite.

Theorem 1. *Assume that $J_0^*\Lambda$ is positive definite. Then*

1. $\widetilde{A}_\Lambda$ *is a realization of a local elliptic boundary condition (in the classical sense of Shapiro-Lopatinski), hence* $\widetilde{A}_\Lambda$ *is a Fredholm operator with compact resolvent.*
2. $\ker \widetilde{A}_\Lambda = Z_0(A) \oplus Z_0(A^t), \operatorname{coker} \widetilde{A}_\Lambda \cong \ker A^*_\Lambda = Z_0(A^t) \oplus Z_0(A).$

Here $Z_0(A) = \{u \in L^2(M,E) \,|\, Au = 0, u_{|\partial M} = 0\}$ is the space of "ghost solutions". By elliptic regularity it is easy to see that $Z_0(A)$ is a finite–dimensional subspace of $\Gamma^\infty(M,E)$ and hence does not depend on the choice of a Sobolev regularity for u. $Z_0(A) = \{0\}$ if and only if weak UCP holds for A. We note in passing that, although it is generally believed that weak UCP does not hold in general, we do not know of an example of a first order elliptic differential operator A with $Z_0(A) \neq \{0\}$. The known counterexamples to UCP, cf. e.g. [7], do not apply in an obvious way.

If the operator A is formally self–adjoint then for $\Lambda := J_0|J_0|^{-1}$ the double $\widetilde{A}_\Lambda$ is self–adjoint.

From Theorem 1 the Calderón projector may be constructed in the usual way. Let

$$r_\pm \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := f_\pm, \quad \varrho_\pm \begin{pmatrix} f_+ \\ f_- \end{pmatrix} := f_{\pm|\partial M}, \tag{6}$$

and denote by ϱ^* the L^2–dual of ϱ_+. It is well–known that $\varrho_\pm$ maps the Sobolev space $L^2_s(M,\dots)$ continuously into $L^2_{s-1/2}(\partial M,\dots)$ for $s > 1/2$ and consequently ϱ^* maps $L^2_s(\partial M,\dots)$ continuously into $L^2_{s-1/2}(M,\dots)$ for $s < 0$. These constraints on s cause some technical difficulties.

Definition 1. *Let A_Λ^{-1} be the pseudoinverse of the operator A_Λ. Put $K_\pm := \pm r_\pm \widetilde{A}_\Lambda^{-1} \varrho^* J_0$, $C_+ := \varrho_+ K_+$, $C_- := \Lambda^{-1}\varrho_- K_-$.*

Theorem 2. *Under the assumptions of Theorem 1 and under the additional technical assumption that the commutator $[B_0^t, J_0^*\Lambda]$ is of order 0 we have:*

1. *For $s \geq -1/2$ the operator K_+ maps $L^2_s(\partial M, E)$ continuously into $L^2_{s+1/2}(M,E) \cap \ker A$.*
2. *$C_\pm$ are complementary idempotents with*

$$\begin{aligned} \operatorname{range} C_+ &= \{u_{|\partial M} \in L^2 \,|\, Au = 0\}, \\ \operatorname{range} C_- &= \Lambda^{-1}\{u_{|\partial M} \in L^2 \,|\, A^t u = 0\}. \end{aligned} \tag{7}$$

If $\Lambda = (J_0^*)^{-1}$ then $C_\pm$ are orthogonal projections.

More interesting than this result, which looks pretty similar to what one gets from geometric invertible double constructions respectively invertible non–canonical closed extensions, is the method. Namely, an approximation to K_+ near the boundary is constructed from the operator $Q_+(x) := \frac{1}{2\pi i}\int_\Gamma e^{-x\lambda}(\lambda - B_0)^{-1} d\lambda$. Here Γ is a contour which encircles the eigenvalues of B_0 in the right half plane and such that $\Re z_n \to \infty$ if z_n is on Γ with $|z_n| \to \infty$. $Q_+(0)$ is the positive sectorial spectral projection of B_0 which has recently been discussed by Ponge [8]. A crucial observation is that the tangential operator B_0 is not an arbitrary elliptic operator. Rather the ellipticity of A implies that $B_0 - it, t \in \mathbb{R}$, is elliptic in the parametric sense. This is much stronger than ellipticity.

The approximation to K_+ constructed from Q_+ allows to control the error when replacing C_+ by the sectorial projection of B_0.

Applications of our results are:

- Proof of the Lagrangian property of the Cauchy data space in great generality.
- The continuous dependence of the Calderón projector on the input data. This has consequences for the so–called Spectral Flow Theorem, cf. [1, 4, 5, 6].
- A generalization of the Cobordism Theorem to first order formally self–adjoint elliptic operators.

Details will appear soon.

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A surgery formula for the smooth Yamabe invariant

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(joint work with M. Dahl, E. Humbert)

1. Historical overview

For a compact Riemannian manifold (M, g_0) of dimension $n \geq 3$ we define the conformal Yamabe invariant to be

$$Y(M, [g_0]) := \inf_{g \in [g_0]} \int_M \mathrm{Scal}_g \, dv^g$$

where $[g_0]$ denotes the set of all metrics conformal to g and of volume 1. The solution of the Yamabe problem (see e.g. [10] for a good overview) states that this infimum is always attained. The scalar curvature of the mimimzing metric satisfies $\mathrm{Scal}_g \equiv Y(M, [g_0])$. It is not difficult to show that

$$Y(M, [g_0]) \leq Y(\mathbb{S}^n) = n(n-1)\omega_n$$

where $\mathbb{S}^n$ denotes the sphere with the standard metric, and where $\omega_n = \mathrm{Vol}(\mathbb{S}^n)$.

The smooth Yamabe invariant, also called Yamabe number or Schoen's σ-invariant is then defined as

$$\sigma(M) := \sup_{[g_0]} Y(M, [g_0]) \in (-\infty, Y(\mathbb{S}^n)]$$

where the supremum runs over all conformal classes $[g_0]$ on M.

The σ-invariant is positive if and only if M carries a metric of positive scalar curvature. Calculating σ-invariants in the positive case is also difficult. The first examples have been calculated by Schoen [12, Section 2] where one sees that $\sigma(S^{n-1} \times S^1) = Y(\mathbb{S}^n)$, and the same method yields for $S^{n-1}\tilde{\times}S^1$, the non-orientable S^{n-1} bundle over S^1, that $\sigma(S^{n-1}\tilde{\times}S^1) = Y(\mathbb{S}^n)$. By Kobayashi's surgery formula [8] it is also clear that the σ-invariant of connected sums of above manifolds have the value $Y(\mathbb{S}^n)$. It is conjectured [12, Page 10, lines 6–11] that all quotient of round spheres satisfy $\sigma(S^n/\Gamma) = (\#\Gamma)^{-2/n} Y(\mathbb{S}^n)$, but this conjecture is only verified for $\mathbb{R}P^3$ [5]. An extended Aubin's Lemma enabled the complete characterization of all 3-dimensional manifolds with $\sigma(M) > \sigma(\mathbb{R}P^3)$, see [1]. The smooth Yamabe invariant of $\sigma(\mathbb{C}P^2)$ was calculated in [9] and [7]. With similar methods the σ-invariants of some several other Kähler manifolds could be determined.

As smooth Yamabe invariants are difficult to calculate explicitly, estimates are very helpful. An important class of estimates can be proven with the help of surgery formulas. Let $\Phi : S^k \times \overline{B^{n-k}} \to M$ be an embedding. We define

$$M_k^\Phi := M \setminus \Phi(S^k \times B^{n-k}) \cup_\sim \overline{B^{k+1}} \times S^{n-k-1}$$

and we say that M_k^Φ *is obtained from* M *by* k*-dimensional surgery along* Φ. Connected sums are examples for 0-dimensional surgery along non-connected manifolds.

Theorem 1 ([8]). *If* M_0^Φ *is obtained from* M *by* 0*-dimensional surgery, then*

$$\sigma(M_0^\Phi) \geq \sigma(M).$$

The next theorem states that monotonicity still holds in the case $\sigma(M) \leq 0$ for $k \in \{1, \ldots, n-3\}$.

Theorem 2 ([11]). *If* M_k^Φ *is obtained from* M *by* k*-dimensional surgery,* $1 \leq k \leq n-3$*, then*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), 0\}.$$

Using results from bordism theory and from [14] this implies that all simply connected manifolds of dimension ≥ 5 have $\sigma(M) \geq 0$. On the other hand positivity of $\sigma(M)$ is preserved under surgeries of dimension $k \in \{1, \ldots, n-3\}$.

Theorem 3 ([6],[13]). *If* M_k^Φ *is obtained from* M *by* k*-dimensional surgery,* $1 \leq k \leq n-3$*, and* $\sigma(M) > 0$*, then we also have* $\sigma(M_k^\Phi) > 0$.

2. New surgery formula for σ

In [4] we prove a surgery formula for the σ-invariant that is stronger than Theorems 1, 2 and 3 together.

Theorem 4. *Let $0 \leq k \leq n-3$. There is a positive constant $\Lambda_{n,k}$ depending only on n and k such that*

$$\sigma(M_k^\Phi) \geq \min\{\sigma(M), \Lambda_{n,k}\}$$

if M_k^Φ is obtained from M by k-dimensional surgery. Furthermore $\Lambda_{n,0} = Y(\mathbb{S}^n)$.

We will not give a detailed proof in this short announcement, but restrict to a short sketch of the method used herein. Details will be published in [4]. The theorem directly follows from the following:

Theorem 5. *Suppose that M_k^Φ is obtained from M by k-dimensional surgery. Then for any metric g on M there is a family of metrics g_θ, $\theta \in [0,1]$, on M_k^Φ such that*

$$\min\{Y(M,[g]), \Lambda_{n,k}\} \leq \liminf_{\theta\to 0} Y(M_k^\Phi, [g_\theta]) \leq \limsup_{\theta\to 0} Y(M_k^\Phi, [g_\theta]) \leq Y(M,[g]).$$

These metrics are very similar to the metrics g_ρ described in [2] and the metrics g_τ described in [3]. The inequality

$$\limsup_{\theta\to 0} Y(M_k^\Phi, [g_\theta]) \leq Y(M,[g])$$

in Theorem 5 can be shown by standard cut-off arguments. The other inequality, namely

$$\min\{Y(M,[g]), \Lambda_{n,k}\} \leq \liminf_{\theta\to 0} Y(M_k^\Phi, [g_\theta])$$

is much harder to show, but it is the important one for proving Theorem 4.

For showing this inequality we study minimzers of the Yamabe functional of $(M_k^\Phi, [g_\theta])$ for a sequence of $\theta \to 0$. A sophisticated blow-up analysis for such minimizers is worked out. The arguments of [3] can only be applied for $2k < n-2$, hence completely new tools have to be developped to treat the general case. We see that the sequence either concentrates in the part $(M_k^\Phi, [g_\theta])$ that is locally conformal to $(M \setminus \Phi(S^k \times \{0\}), g)$. In this case we obtain convergence to a solution of the Yamabe equation on $(M \setminus \Phi(S^k \times \{0\}), g)$ that can be extended to a solution on (M,g) and we derive

$$Y(M,[g]) \leq \liminf_{\theta\to 0} Y(M_k^\Phi, [g_\theta]).$$

Or the minimizers concentrate on the new "handle" and then blowup analysis provides a solution of

$$Lu = \lambda |u|^{\frac{2}{n-2}} u \tag{1}$$

on $(P(n,k), H_K) := \mathbb{S}^{n-k-1} \times (B^{k+1}, h_K)$ for a suitable $K \in [-1,0]$. Here h_K is "the" complete metric of sectional curvature $K \in [-1,0]$ on the ball B^{k+1}. For example (B^{k+1}, h_0) is isometric to euclidean space and (B^{k+1}, h_{-1}) is isometric to the hyperbolic space.

We define

$$Y(P(n,k),H_K) := \inf \lambda \in [0,\infty]$$

where λ runs over all values $\lambda \in (0,\infty)$ for which there is a positive function $u \in L^\infty(P(n,k)) \cap C^2_{\mathrm{loc}}(P(n,k))$, $\|u\|_{L^{\frac{2n}{n-2}}} = 1$ solving (1) on $(P(n,k),H_K)$. and we obtain

$$Y(P(n,k),H_K) \leq \liminf_{\theta\to 0} Y(M_k^\Phi,[g_\theta]).$$

Other blowup arguments show

$$\Lambda_{n,k} := \inf_{K\in[-1,0]} Y(P(n,k),H_K) > 0$$

and Theorem 5 follows.

As $(P(n,k),H_K)$ is conformal to $\mathbb{S}^n \setminus \{p_1,p_2\}$, one can see that $\Lambda_{n,0} = Y(\mathbb{S}^n)$.

3. Topological applications

We consider pairs (M,f) where M is a compact spin manifold and where $f : M \to B\Gamma$ is continuous. Two such pairs (M_1,f_1) and (M_2,f_2) are spin bordant if there exists an $n+1$-dimensional manifold W with boundary $-M_1 \cup M_2$ with a map $F : W \to B\Gamma$ such that the restriction of F to the boundary yields f_1 and f_2. It is implicitly required that the boundary carries the induced orientation and spin structure and $-M_1$ denotes M_1 with reversed orientation. Bordance is an equivalence relation, the corresponding equivalence classes are denoted by $[(M,f)]$ and the set of equivalence classes is $\Omega_n(B\Gamma)$. Disjoint union defines a sum on $\Omega_n(B\Gamma)$ which turns it into a group.

We say that a tuple (M,f) with $f : M \to B\Gamma$ is a *π_1-bijective representative of* $[(M,f)]$ if M is connected and if the induced map $f_* : \pi_1(M) \to \Gamma$ is a bijection. Any equivalence class in $\Omega_n(B\Gamma)$ has a π_1-bijective representative.

Now we define

$$\Lambda_n := \min\{\Lambda_{n,1},\dots \Lambda_{n,n-3}\} > 0,$$
$$\bar\sigma(M) := \min\{\sigma(M),\Lambda_n\}.$$

Proposition 1. *Let $n \geq 5$. Let (M_1,f_1) and (M_2,f_2) be compact spin manifolds with maps $f_i : M_i \to B\Gamma$. If (M_1,f_1) and (M_2,f_2) are spin bordant and if f_2 is π_1-bijective, then*

$$\bar\sigma(M_1) \leq \bar\sigma(M_2).$$

Hence $\bar\sigma$ defines a well-defined map $\Omega_n(B\Gamma) \to \mathbb{R}$. We conclude from the surgery formula.

Theorem 6. *Let $t \geq 0$. Then the sets*

$$G(t) := \{[x] \in \Omega_n(B\Gamma) \,|\, \bar\sigma(x) > t\}$$

and

$$\overline{G}(t) := \{[x] \in \Omega_n(B\Gamma) \,|\, \bar\sigma(x) \geq t\}$$

are subgroups of $\Omega_n(B\Gamma)$.

The theorem admits — among other interesting conclusions — the following application. For $p \in \mathbb{N}^*$ we write $p\#M$ for $M\# \cdots \#M$ where M appears p times. We already know $\bar{\sigma}(p\#M) \geq \bar{\sigma}(M)$ if $\bar{\sigma}(M) \geq 0$.

Corollary 7. *Suppose M is a compact spin manifold with $\sigma(M) \in (0, \Lambda_n)$. Let p and q be two relatively prime positive integers. If $\sigma(p\#M) > \sigma(M)$, then $\sigma(q\#M) = \sigma(M)$.*

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Computing characteristic classes of noncommutative spaces

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(joint work with Alain Connes)

The Chern character for K-homology cycles over C^*-algebras was introduced by Connes [1], by a formula which essentially gave birth to *cyclic cohomology.* In particular, the Chern-Connes character provides the total characteristic class of a noncommutative space (*cf.* [2]), encoded by the datum of a *spectral triple.* In the case of spectral triples with the meromorphic continuation property, this total characteristic class was given a *local* (in the sense of Bohr's correspondence principle) formulation in [3]. Moreover, for the *hypoelliptic* spectral triple representing the transverse geometry of a general foliation, the local formula has been used to express the Chern-Connes character in terms of classical geometric characteristic classes of foliations (*cf.* [4]).

After briefly reviewing these basic facts, the talk focused on the appropriate definitions and means to compute the characteristic classes of noncommutative spaces described by *twisted spectral triples*. The latter notion, recently introduced in [5], is a variant of the usual one which allows to treat type III situations, when the underlying algebra of noncommuting 'coordinates' admits only a KMS-state instead of a trace.

More precisely, with σ denoting an automorphism of $\mathcal{A}$, an *ungraded σ-spectral triple* $(\mathcal{A}, \mathfrak{H}, D)$ is given by an action of $\mathcal{A}$ in the Hilbert space $\mathfrak{H}$, while D is a self-adjoint operator with compact resolvent and such that the twisted commutators $d_\sigma a := D\,a - \sigma(a)\,D$ are bounded for all $a \in \mathcal{A}$.

A *graded* σ-spectral triple is similarly defined, with the additional datum of a grading operator $\gamma = \gamma^* \in \mathcal{L}(\mathfrak{H})$, $\gamma^2 = I$, which commutes with the action of $\mathcal{A}$ and anticommutes with D, *i.e.* $a\,\gamma = \gamma\,a$, $\forall\, a \in \mathcal{A}$, $D\,\gamma = -\gamma\,D$. A *Lipschitz-regular* σ-spectral triple is one that satisfies the additional condition $|D|\,a - \sigma(a)\,|D|$ are bounded for all $a \in \mathcal{A}$.

Example: codimension 1 foliations. To simplify the presentation, we restrict to a complete transversal and take for the algebra $\mathcal{A}$ of 'transverse coordinates' the crossed product of the algebra $C^\infty(S^1)$ by a group Γ of orientation preserving diffeomorphisms, whose elements can be is represented as finite sums of the form $a = \sum_\Gamma a_\phi\, U_\phi^*$, with the product rule $U_\phi^* f = (f \circ \phi)\, U_\phi^*$, $U_\phi^*\, U_\psi^* = U_{\psi\,\phi}^*$. Then $\mathcal{A} = C^\infty(S^1) \rtimes \Gamma$ acts on the Hilbert space $\mathfrak{H} = L^2(S^1)$ by the $*$-representation

$$(\pi(f\, U_\phi^*)\,\xi)(x) = f(x)\,\phi'(x)^{\frac{1}{2}}\,\xi(\phi(x))\,, \quad \forall \xi \in \mathfrak{H}\,, \quad x \in \mathbb{R}/\mathbb{Z},$$

in the role of D one take the operator $\frac{1}{i}\frac{d}{dx}$, and the automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$ is defined on the monomials generating $\mathcal{A}$ by $\sigma(g\, U_\phi^*) = \frac{d\phi(x)}{dx}\, g\, U_\phi^*$. Both boundedness properties listed above are easily seen to hold: the twisted commutators

$$D \circ \pi(a) - \pi(\sigma(a)) \circ D \qquad \text{and} \qquad |D| \circ \pi(a) - \pi(\sigma(a)) \circ |D|$$

are bounded for all $a \in \mathcal{A}$. Furthermore, the canonical state φ on $\mathcal{A}$,

$$\varphi(f\, U_\phi^*) = 0 \;\; \textit{if} \;\; \phi \neq 1\,, \quad \text{and} \quad \varphi(f) = \int_{\mathbb{R}/\mathbb{Z}} f(x)\, dx$$

is a σ^{-1}-*trace*, *i.e.* satisfies $\varphi(a\,b) = \varphi(b\,\sigma^{-1}(a))$, $\forall\, a, b \in \mathcal{A}$.

Quite surprisingly, in spite of the twisting, the Chern-Connes character of a general σ-spectral triple is still given by a "straight" cocycle in cyclic cohomology.

Theorem 1. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$. Then the $(n+1)$-linear forms on $\mathcal{A}$*

$$\Phi^\pm_{D,\sigma}(a^0, \ldots, a^n) := \mathrm{Tr}\left(D_\pm^{-1}(D_\pm\, a^0_\pm - \sigma(a^0_\mp)\, D_\pm) \cdots D_\pm^{-1}(D_\pm\, a^n_\pm - \sigma(a^n_\mp)\, D_\pm)\right)$$

are cyclic cocycles in $Z^n_\lambda(\mathcal{A})$, and for any idempotent $e^2 = e \in M_N(\mathcal{A})$ one has

$$\mathrm{Index}\big(D_\pm^{-1}\sigma(e_\mp)D_\pm\big) = \Phi^\pm_{D,\sigma}(e, \ldots, e).$$

In the Lipschitz-regular case the twisted spectral triple can be effectively "untwisted" by simply passing to its "phase" $F = D\,|D|^{-1}$. Indeed, in that case the commutators

$$[F,a] \,=\, |D|^{-1}\left((D\,a \,-\, \sigma(a)\,D) \,-\, (|D|\,a - \,\sigma(a)\,|D|)\,F\right)$$

are compact operators, for all $a \in \mathcal{A}$.

Theorem 2. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a σ-spectral triple with $D^{-1} \in \mathcal{L}^{n,\infty}$. Then the linear functional defined by a Dixmier trace* $\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty} \to \mathbb{C}$*,*

$$a \in \mathcal{A} \mapsto \varphi(a) = ⨍ a\,D^{-n} \;:= \mathrm{Tr}_\omega(a\,D^{-n}),$$

is a σ^{-n}-trace on $\mathcal{A}$: $\quad \varphi(a\,b) \,=\, \varphi(b\,\sigma^{-n}(a)), \quad \forall\, a,b \in \mathcal{A}$. More generally, for any bounded operator $T \in \mathcal{L}(\mathfrak{H})$,

$$Tr_\omega(T\,\sigma^{-n}(a)\,D^{-n}) \,=\, Tr_\omega(a\,T\,D^{-n})\,, \qquad \forall\, a \in \mathcal{A}\,.$$

When the σ-spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is Lipschitz-regular, the same hold true when D^{-n} is replaced by $|D|^{-n}$.

As a step in the direction of extending the local index formula to twisted spectral triple, we can construct the analogue of the local Hochschild cocycle that gives the Hochschild class of the Chern character in the untwisted case.

Theorem 3. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a graded σ-spectral triple such that $D^{-1} \in \mathcal{L}^{n,\infty}$ for some even $n \in \mathbb{N}$. Then the $(n+1)$-linear form on $\mathcal{A}$*

$$\Psi_{D,\sigma}(a^0, a^1, \ldots, a^n) \,:=\, ⨍ \gamma\, a^0\, d_\sigma(\sigma^{-1}(a^1)) \,\cdots\, d_\sigma(\sigma^{-n}(a^n))\,D^{-n}$$

is a Hochschild cocycle in $Z^n(\mathcal{A}, \mathcal{A}^*)$. In the ungraded case, for a σ-spectral triple of odd summability degree which is Lipschitz-regular, the corresponding Hochschild cocycle is defined by the expression

$$\Psi_{D,\sigma}(a^0, a^1, \ldots, a^n) \,:=\, ⨍ a^0\, d_\sigma(\sigma^{-1}(a^1)) \,\cdots\, d_\sigma(\sigma^{-n}(a^n))\,|D|^{-n}\,.$$

To illustrate this result, it is instructive to compute the above Hochschild cocycle for the Dirac spectral triple over $\mathcal{A} = C^\infty(P^1(\mathbb{R})) \rtimes \mathrm{PSL}(2,\mathbb{R})$. The Dixmier trace, which is given by the Wodzicki-Guillemin residue, can be explicitly evaluated in this case. One thus obtains that the Hochschild cocycle $\Psi_{D,\sigma}$ is actually *cyclic* and, up to a coboundary, is proportional to the *transverse fundamental cocycle* $\tau_\mathcal{A} \in ZC^1_\lambda(\mathcal{A})$,

$$\tau_\mathcal{A}(f\,U^*_\phi,\, g\,U^*_{\phi^{-1}}) \quad = \quad \int_{\mathbb{R}/\mathbb{Z}} f\,\phi^*(dg)\,,$$

$$\text{resp.} \qquad \tau_\mathcal{A}(f\,U^*_\phi,\, g\,U^*_\psi) \quad = \quad 0\,, \qquad \text{if} \quad \psi \neq \phi^{-1}.$$

The facility of the above mentioned exercise stands in sharp contrast to the rather forbidding amount of computations required to get the explicit form of the local index cocycle for the (type II and 4-dimensional) lift of the above spectral triple to the frame bundle (*cf.* [4, Appendix]).

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Adiabatic Limit, Bismut-Freed Connection and Real Analytic Torsion Form

Xianzhe Dai

(joint work with Weiping Zhang)

For a complex flat vector bundle over a fibered manifold, we consider the 1-parameter family of certain deformed sub-signature operators introduced by Ma-Zhang in [12]. We compute the adiabatic limit of the Bismut-Freed connection [5] associated to this family and show that the Bismut-Lott analytic torsion form [7] shows up naturally under this procedue.

1. Adiabatic limit, higher invariants and transgression

Adiabatic limit refers to a geometric degeneration when metric in certain directions are blown up, while the remaining directions are kept fixed. Typically, the underlying manifold has a so called fibration structure (or fiber bundle structure). That is

$$Z \longrightarrow M \xrightarrow{\pi} B,$$

where π is a submersion and the typical fiber $Z \simeq Z_b = \pi^{-1}(b)$, for $b \in B$. Given a submersion metric on M:

$$g = \pi^* g_B + g_Z,$$

the adiabatic limit refers to the limit as $\epsilon \to 0$ of

$$g_\epsilon = \epsilon^{-2} \pi^* g_B + g_Z.$$

This is first introduced by Witten in his famous work on global gravitational anomalies [15].

Witten considered the adiabatic limit of the eta invariant of Atiyah-Patodi-Singer. Full mathematical treatment and generalizations are given by Bismut-Freed [5], Cheeger [8], Bismut-Cheeger [2], Dai [9] among others. The adiabatic limit of the eta invariant gives rise to the Bismut-Cheeger eta form, a canonically defined differential form on the base B. The eta form is a higher dimensional generalization of the eta invariant as it gives the boundary contribution of the family index theorem for manifolds with boundaries [3, 4], [13, 14]. The degree zero component of the eta form here is exactly given by the eta invariants of the fibers. The nonzero degree components therefore contains new geometric information about the fibration.

Another important geometric invariant is the analytic torsion. The adiabatic limit of the analytic torsion has been considered by Dai-Melrose [10]. In contrast to the case of the eta invariant, the adiabatic limit here does not lead to a higher invariant. This is because the associated characteristic class is the Pfaffian, a top form which kills any possible higher degree components arising form the adiabatic limit.

It should be noted that there is a complex analogue of the analytic torsion for complex manifolds called the holomorphic torsion. Its adiabatic limit has been considered by Bismut-Berthomieu [1]. It does give rise to the holomorphic torsion form of Bismut-Köhler [6]. The difference with the real case comes from the fact that the characteristic class here is the Todd class—a stable class (which then contains differential forms of lower degree).

There is another way to view the higher invariants, namely via transgression. The eta form transgresses between the Chern-Weil representative of the family index and its Atiyah-Singer representative. Similarly, the holomorphic torsion form is the double transgression of the family index in the complex setting. This is precisely the view point used by Bismut-Lott [7] to define the real analytic torsion form, a higher dimensional generalization of the analytic torsion. It is a canonical transgression of some odd cohomology classes.

2. Adiabatic limit, Bismut-Freed connection and real analytic torsion form

Let $\pi : M \to B$ be a smooth fiber bundle with compact fiber Z of dimension n. We denote by $m = \dim M,\ p = \dim B$. Denote by TZ the vertical tangent bundle of the fiber bundle, and choose a splitting of TM: $TM = T^H M \oplus TZ$. Let F be a flat complex vector bundle with the flat connection ∇^F and an hermitian metric h^F on M. Denote by

$$\omega(F, h^F) = (h^F)^{-1}(\nabla^F h^F)$$

the endomorphism valued 1-form.

Let $E = \oplus_{i=0}^n E^i$ be the smooth infinite-dimensional $\mathbb{Z}$-graded vector bundle over B whose fiber over $b \in B$ is $C^\infty(Z_b, (\Lambda(T^*Z) \otimes F)_{|Z_b})$. Then exterior differentiation d^M on M is a flat superconnection of total degree 1 on E:

$$d^M = d^Z + \nabla^E + i_T,$$

and

$$\left(d^Z\right)^2 = 0, \quad \left[\nabla^E, d^Z\right] = 0.$$

Here T is a TZ-valued horizontal 2-form

Let N_Z be the number operator of E, i.e. N_Z acts by multiplication by k on $C^\infty(M, \Lambda^k(T^*Z) \otimes F)$. For $u > 0$, set

$$C'_u = u^{N_Z/2} d^M u^{-N_Z/2}, \quad C''_u = u^{-N_Z/2} \left(d^M\right)^* u^{N_Z/2},$$
$$C_u = \frac{1}{2}\left(C'_u + C''_u\right), \quad D_u = \frac{1}{2}\left(C''_u - C'_u\right).$$

Then

$$C_u = \frac{\sqrt{u}}{2} D^Z + \nabla^{E,e} - \frac{1}{2\sqrt{u}} c(T), \quad D^Z = d^Z + d^{Z*}$$
$$D_u = \frac{\sqrt{u}}{2}\left(d^{Z*} - d^Z\right) + \frac{1}{2}\omega\left(E, h^E\right) - \frac{1}{2\sqrt{u}}\widehat{c}(T).$$

Let (μ, h^μ) be a Hermitian complex vector bundle over B. Consider

$$D^{\pi^*\mu\otimes F} = \sum_{a=1}^m c(e_a)\widetilde{\nabla}^e_{e_a} - \frac{1}{2}\sum_{i=1}^n \widehat{c}(e_i)\omega\left(F, h^F\right)(e_i),$$
$$\widehat{D}^{\pi^*\mu\otimes F} = -\sum_{i=1}^n \widehat{c}(e_i)\widetilde{\nabla}^e_{e_i} + \frac{1}{2}\sum_{a=1}^m c(e_a)\omega\left(F, h^F\right)(e_a)$$
$$- \frac{1}{4}\sum_{\alpha,\beta=1}^p \widehat{c}\left(T\left(f_\alpha, f_\beta\right)\right)\widehat{c}(f_\alpha)\widehat{c}\left(f_\beta\right).$$

These two operators are closely related to C_u and D_u via quantization.

For any real number r, define

$$D^{\pi^*\mu\otimes F}(r) = D^{\pi^*\mu\otimes F} + \sqrt{-1} r \widehat{D}^{\pi^*\mu\otimes F}.$$

This family of operators was introduced by Ma-Zhang [12] in their new proof of Bismut-Lott's result. Now we define

$$\delta(F)(r) = -\frac{1}{2}\int_0^{+\infty} \mathrm{Tr}_s\left[\widehat{D}^{\pi^*\mu\otimes F} D^{\pi^*\mu\otimes F}(r) e^{-t\left(D^{\pi^*\mu\otimes F}(r)\right)^2}\right] dt.$$

This is well defined, and, when the flat vector bundle F over M is fiberwise acyclic, that is $H^*(Z_b, F|_{Z_b}) = \{0\}$ on each fiber Z_b, $b \in B$, $\delta(F)(r)$ is just the imaginary part of the Bismut-Freed connection for the determinant line bundle of $D^{\pi^*\mu\otimes F}(r)$.

Finally, we denote by $\delta(F)_\epsilon(r)$ the invariant when the metric is the adiabatic metric g_ϵ.

Theorem 1. *Under the assumption that the flat vector bundle F over M is fiberwise acyclic, the following identity holds,*

$$\frac{1}{2}\lim_{\epsilon\to 0} \delta_\epsilon(F)(r) = \int_B L\left(TB, \nabla^{TB}\right) \mathrm{ch}\left(\mu, \nabla^\mu\right) \mathcal{T}_r,$$

where

$$\mathcal{T}_r = -\int_0^\infty \varphi\, \mathrm{Tr}_s\left[N D_t^2 e^{\left(1+r^2\right)D_t^2}\right] \frac{dt}{t}.$$

Up to scaling, $\mathcal{T}_r$ is the positive degree component of Bismut-Lott's real analytic torsion form.

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Recent advances in Index formulæ

Richard B. Melrose

I was asked to present an overview of index theorems and formulæ in this talk. This is rather a daunting task since there are so many variants – including at least the following

- Single/Family
- Dirac/pseudodifferential/product-type
- Elliptic/subelliptic/Heisenberg
- K-theory/cohomology
- Equivariant, transversally elliptic
- Higher
- Boundary/Corner, Complete/Incomplete
- Algebraic
- Twisted by Azumaya

- C^*

and combinations thereof! In fact there are a lot of things it would be desirable to report on. For instance, it would be nice to discuss the relationship between the various proofs of the index theorems and formulæ. These fall into classes including the Local index for Dirac (. . . , Getzler, Quillen, . . .), Embedding (Atiyah-Singer, . . .), K-homology (Kasparov, . . .), Asymptotic morphism (Connes, Higson, Trout), Homological (Fedosov, . . .) and it is now possible to describe the relationship between them; however I will not do so!

Rather, from personal interest, I decided to concentrate on one thing that is still not well understood. Namely, the general (and in fact very general) index formula for pseudodifferential cases rather than the better-understood Dirac setting. The main point I emphasized was that there is now the possibility of rather explicit formulæ, following ongoing work with Pierre Albin and Frédéric Rochon, which is closely related to theorems on the index in K-theory [1] and [4]. This work is in the context of manifolds with boundary. Rather than talk once again about the various algebras of pseudodifferential operators which are known on a manifold with boundary, I chose instead to limit attention to a less well-known, but very similar, setting of product-type pseudodifferential operators associated to a fibration. The sorts of questions I addressed were first attacked (for the circle) by Nicola and Rodino, [5], although in a somewhat different context. The similarity of the results in these two apparently quite different settings can be attributed to the fact that both correspond to 'second level' algebras in which part of the 'symbol' is a pseudodifferential operator.

First, consider the setting of the Atiyah-Singer index theorem. Let $\phi : M \longrightarrow B$ be a fibration of compact manifolds, $\mathbb{E} = (E^+, E^-) \longrightarrow M$ a $\mathbb{Z}_2$-graded bundle, $P \in \Psi^m(M/B; \mathbb{E})$ a family of elliptic operators on the fibres and $\sigma(P) \in \mathcal{C}^\infty(S^*(M/B); \hom(\mathbb{E}) \otimes N_m)$ the principal symbol – invertible by assumption. Then $R \in \Psi^{-\infty}(M/B; \mathbb{E})$ exists such that $\operatorname{null}(P + R) \subset \mathcal{C}^\infty(B; \mathcal{C}^\infty(M/B; E^+))$ is a bundle and the cohomological (so not complete, like the index in K-theory) obstruction to existence of $R' \in \Psi^{-\infty}(M/B; \mathbb{E})$ such that $P + R'$ is invertible is the Chern character of the index superbundle

$$\begin{gathered}\operatorname{ind}_{\mathrm{a,hom}}(P) = \operatorname{Ch}(\operatorname{null}(P + R)) - \operatorname{Ch}(\operatorname{null}(P^* + R^*)) = \\ \operatorname{ind}_{\mathrm{t,hom}}(P) = (\phi \circ \pi)_*(\operatorname{Ch}(\sigma(P)) \wedge \operatorname{Td}(M/B)) \in H^{\mathrm{even}}(B).\end{gathered}$$

Following Quillen, but with modifications by Pierre Albin and myself, for a real vector bundle $W \longrightarrow X$ take the chain space $\mathcal{C}^\infty(X; \Lambda^*) \oplus \mathcal{C}^\infty(\mathbb{S}W; \Lambda^{*-1})$, $\mathbb{S}W = (W \setminus 0_M)/\mathbb{R}^+$ and differential $D = \begin{pmatrix} d & 0 \\ \pi^* & -d \end{pmatrix}$. The cohomology is then naturally isomorphic to $H^*_{\mathrm{c}}(W)$. The Chern-Weil character of the K-class of the symbol, essentially as described by Fedosov [2], fits very nicely into this formalism. Be aware that I have set $2\pi i = 1$ and $m = 0$ here. Take a graded connection ∇

with curvature $\omega = \nabla^2$. The base and (regularized) 'odd' Chern characters are

$$\alpha = \mathrm{STr}\left(\exp(\omega)\right) \in \mathcal{C}^\infty(M;\Lambda^{\mathrm{even}})$$

$$\beta = -\int_0^1 \mathrm{tr}\left(\sigma^{-1}(\nabla\sigma)e^{w(t)}\right)dt \in \mathcal{C}^\infty(S^*(M/B);\Lambda^{\mathrm{odd}}),$$

$$\text{where } w(t) = (1-t)\omega_+ + t\sigma^{-1}\omega_-\sigma + t(1-t)(\sigma^{-1}\nabla\sigma)^2.$$

Then, in terms of the chain complex above, $\mathrm{Ch}(\sigma,\mathbb{E}) = [(\alpha,\beta)] \in H_{\mathrm{c}}^{\mathrm{even}}(T^*(M/B)$.

To get an explicit Atiyah-Singer formula represent the Todd class $\mathrm{Td}(M/B)$ in the same way as the Chern character of the creation complex on $NM \otimes N^*M$ for the normal bundle of an embedding $M \hookrightarrow B \times \mathbb{R}^N$ consistent with the fibration. Then, essentially as in [2]

$$\mathrm{ind}_{\mathrm{t,hom}}(P) = \int_{S^*(M/B)} \beta \wedge \pi^*\,\mathrm{Td}(M/B).$$

For a nested pair $Y_1 \longrightarrow Y_2 \longrightarrow X$ of embeddings the spaces of 'product-type' or 'marked' conormal distributions, $I^{m_2,m_1}(X,Y_1,Y_2;E)$, are well-defined. These are completely coordinate independent and have a long, and somewhat tortuous, history into which I will not go. Product-type pseudodifferential operators have kernels of this type. For a fibration $\phi : M \longrightarrow B$ set

$$\Psi^{m,m'}_{\mathrm{pt}-\phi}(M;\mathbb{E}) = I^{m',m}(M^2, M^2_\phi; \mathrm{Diag}(M); \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R)$$

where M^2_ϕ is the fibre diagonal.

These have composition and boundedness properties similar to the usual pseudodifferential operators on appropriate Sobolev spaces. There are two symbols homomorphisms with values $a \in \mathcal{C}^\infty([S^*M, \phi^*S^*B]; R^m_B R^{m'} \otimes \mathrm{hom}(\mathbb{E}))$ and $b \in \mathcal{C}^\infty(S^*B; \phi^*\Psi^m(M/B;\mathbb{E}) \otimes R^{m'})$, respectively the usual and the base symbols. They are restricted by the condition $\sigma(b) = a_{|\partial}$. If both are invertible then

$$\text{(APS)} \qquad \mathrm{ind}(A) = \int_{S^*M} \mathrm{Ch}(a) \wedge \mathrm{Td}(M) - \frac{1}{2}\int_{S^*B}(\eta(b) + \tau)$$

which I am thinking of as a version of the Atiyah-Patodi-Singer formula.

The eta form here is a regularized version of the form β considered earlier, obtained by replacing the bundles by the infinite-dimensional bundles over B (and hence S^*B) of sections

$$\eta(b) = -\int_0^1 \overline{\mathrm{Tr}}\left(b^{-1}(\nabla b)e^{w(t)}\right)dt \in \mathcal{C}^\infty(S^*(B);\Lambda^{\mathrm{odd}}).$$

The trace functional has been replaced here by a regularized trace functional, which means that η need not be closed.

The form τ is a *transgression* form, which depends on more than top symbol data. Even its existence is not obvious, but follows from a refinement of the proof of the Atiyah-Singer index theorem itself. To do this the standard arguments need to be carried through smoothly and in one step, the product formula for the index, this is not done classically. Using product-type operators this can be done smoothly (see [3]). Still a better understanding of τ would be helpful! Even without this better understanding, the 'APS' formula can be interpreted as a formula in relative cohomology much as before.

Consider an iterated fibration $\phi : M \longrightarrow B$, $\phi : B \longrightarrow D$. A 'fully elliptic' family $P \in \Psi^{m,m'}_{\text{pt-}\phi}(M/D;\mathbb{E})$ can be perturbed by a family of smoothing operators so the null spaces form a bundle and then $\operatorname{ind}_{a,K}(P)$ is again the full obstruction to the existence of such a perturbation to an invertible family. The standard Atiyah-Singer theorem, asserting the equality of this 'analytic' index and a topological index defined through embedding, does not apply but the symbol pair (a, b) defines a class $\delta(a, b) \in K(T^*(B/D))$ and

$$\operatorname{ind}_{a,K}(P) = \operatorname{ind}_{a,K} \circ \delta.$$

This is a form of 'quantization commutes with *almost everything*.

My claim here is that the formula above is a model for several theorems for boundary calculi which should soon be forthcoming. Namely for Boutet de Monvel's transmission calculus (work with Pierre Albin, following results of Fedosov), for the scattering calculus (relatively easy), for the zero calculus (corresponding to conformally compact geometry) with Pierre Albin, for the cusp and fibred cusp calculi – the hardest case – in work with Frédéric Rochon. This should all extend to manifolds with corners, although less is known for the moment.

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Index theory for manifolds with cusps

Gilles Carron

(joint work with Werner Ballmann, Jochem Brüning)

The geometric set-up. We consider a complete non-compact Riemannian manifold (M, g) with finite volume and pinched negative curvature :

$$\operatorname{vol}_g(M) < \infty \text{ and } -b^2 \leq K \leq -a^2 < 0.$$

Since the work of Eberlein, Heintze & Im-Hof [6],[7], the geometry at infinity of such manifold is well understood : outside some compact set $K \subset M$, $U := M \setminus K$ is diffeomorphic to $)0, \infty(\times N$ where N is a compact manifold equipped with the metric

$$g = (dt)^2 + h_t$$

where $(h_t)_t$ is a smooth family of C^1 metric on N. We moreover know that $h_t^{-1}\dot{h}_t \in L^\infty$ and that

$$e^{-bt}h_0 \leq h_t \leq e^{-at}h_0.$$

For sake of simplicity, we will assume that N is connected, that is M has only one end.

The analytic set-up. We consider on M a geometric Dirac operator $D : C^\infty(M,E) \to C^\infty(M,E)$ where $E \to M$ is a Clifford bundle. On U, we have $L^2(U,E) = \int^{\oplus} L^2(E_t)dt$ where $E_t = E|_{\{t\}\times N}$.

In [1], Ballmann and Brüning had introduced a unitary flat connexion $\bar{\nabla}$ on $E|_U$ such that

$$\nabla^{\text{Levi-Civita}} = \bar{\nabla} + S$$

where S is uniformly bounded : $\|S\|_{L^\infty} < \infty$. This connexion has only C^0 coefficient and the flatness of $\bar{\nabla}$ has the following meaning : "*$\bar{\nabla}$-parallel transport depends only on homotopy class of curves.*"

Let $T := \partial_t$ be the radial vector field on U, we can decompose

$$D = T.\left(\nabla_T - \frac{\kappa}{2} + A_t\right)$$

where $\kappa = \operatorname{tr}\left(h_t^{-1}\dot{h}_t\right)$ and $A_t : C^\infty(E_t) \to C^\infty(E_t)$ is a Dirac type operator with only C^0 coefficients. It is easy to check that the operator $T.\left(\nabla_T - \frac{\kappa}{2}\right)$ is symmetric on $C^1_0(U,E)$. Using the connexion $\bar{\nabla}$ we have another decomposition :

$$D = T.\left(\nabla_T - \frac{\kappa}{2} + \bar{A}_t + B_t\right)$$

where

$$\bar{A}_t = -T.\sum_i E_i\bar{\nabla}_{E_i}$$

where $(E_i)_i$ a local orthonormal frame of (N,h_t). $\bar{A}_t$ is a kind of Dirac operator but is not necessary symmetric. Moreover B_t is a zero order operator which is uniformly bounded.

Some spectral theory. In [2], we obtain a result which implies the following

Proposition 1. *Let $k = \dim\ker\bar{\nabla}$ and denote*

$$\lambda_0(t) \leq \lambda_1(t) \leq ... \leq \lambda_l(t) \leq ...$$

be the spectrum of the operator $A_t^*A_t$. then $\lambda_0(t) = \lambda_1(t) = ... = \lambda_{k-1}(t)$ and $\lim_{t\to\infty}\lambda_k(t) = \infty$.

Since $A_t = \bar{A}_t + B_t$ and B_t is uniformly bounded, we can decompose $L^2(E_t)$ in a low energy part and a high energy part :

$$L^2(E_t) = H_t^l + H_t^h$$

where $\dim H_t^l = k$ and in this decomposition A_t is diagonal :

$$A_t = \begin{pmatrix} A_t^l & 0 \\ 0 & A_t^h \end{pmatrix}$$

where A_t^l is uniformly bounded and A_t^h is invertible with $\lim_{t\to\infty} |A_t|^{-1} = 0$. Let P_t be the orthogonal projection onto H_t^l and let

$$D^l := P_t D P_t = P_t \left(T. \left(\nabla_T - \frac{\kappa}{2} \right) \right) + A_t^l$$

is a Dirac type operator acting on $C^1(\mathbb{R}_+, \mathbb{C}^k)$. We can recover a result of J. Lott [10]

Theorem 1. *Let* $\mathrm{spec}_\infty D^l$ *be the subset of real number* $\lambda \in \mathbb{R}$ *such that there is a sequence* $\sigma_p \in C_0^1(\mathbb{R}_+, \mathbb{C}^k)$ *with* $\mathrm{supp}\sigma_p \subset [p, \infty[$, $\|\sigma_p\| = 1$ *and* $\lim_{p\to\infty} \|D\sigma_p - \lambda\sigma_p\| = 0$. *Then we have the equality :*

$$\mathrm{spec}_\infty D^l = \mathrm{spec}_{ess} D.$$

Finiteness of the L^2 kernel : we also obtain a generalisation of a result of J. Lott [9]

Theorem 2. *For any* $w \in \mathbb{R}$, *the space*

$$\{\sigma \in L^2_{loc}(M, E), D\sigma = 0 \ \& \ \int_U |\sigma|^2 e^{-wt} d\mathrm{vol}_g < \infty\}$$

has finite dimensional. In particular, the L^2 *kernel of* D *is finite dimensional.*

The result of J. Lott was that the space of L^2 harmonic forms has finite dimension. In fact we show a Poincaré inequality : for $w > 0$ large enough : $\forall \sigma \in C_0^1(U, E)$

$$\int_U |\sigma|^2 e^{-wt} d\mathrm{vol}_g \leq \int_U |D\sigma|^2 e^{-wt} d\mathrm{vol}_g \leq \int_U |D\sigma|^2 d\mathrm{vol}_g$$

in particular D satisfies the non parabolic at infinity assumption that was introduced in [5].

Some index formulae. We turn to index computation, we assume that E and D are $\mathbb{Z}_2$ graded : $E = E^+ \oplus E^-$ and

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : C^\infty(M, E^+) \oplus C^\infty(M, E^-) \to C^\infty(M, E^+) \oplus C^\infty(M, E^-).$$

We assume that $0 \notin \mathrm{spec}_{ess} D$, then the operator $D^+ : \mathcal{D}(D^+) \to L^2(E^-)$ is Fredholm and his index is

$$\mathrm{ind}\, D^+ = \dim\ker D^+ \cap L^2(E^+) - \dim\ker D^- \cap L^2(E^-).$$

Our goal is to express the index of D^+ as a local contribution coming from the index density of D^+ and a contribution of the end. An important tool to do this is the splitting formula :

$$\mathrm{ind}\, D^+ = \mathrm{ind}\, D_K^+ + \mathrm{ind}\, D_U^+$$

where D_K^+ is the operator D^+ with domain given a the Atiyah-Patodi-Singer boundary condition :

$$\mathcal{D}(D_K^+) = \{\sigma \in H^1(K, E^+),\ Q_{\leq 0}\left(\sigma|_{\partial K}\right) = 0\}$$

where $Q_{\leq 0}$ is orthogonal projection onto the non positive eigenspace of A_0. And similarly for D_U^+ :

$$\mathcal{D}(D_U^+) = \{\sigma \in H^1(U, E^+),\ Q_{>0}(\sigma|_{\partial K}) = 0\}$$

where $Q_{>0} = \mathrm{Id} - Q_{\leq 0}$ is orthogonal projection onto the positive eigenspace of A_0. There is a problem in our setting, this splitting formula is known only in the smooth realm and the operators A_t have only C^0 coefficients. An important tool to show such a spitting formula for the index is a result of regularity for the Calderòn projector, see for instance [4]. In [3], we have obtain

Theorem 3. *Let $\mathcal{C} = \{\sigma|_{\partial K}, \sigma \in L^2(U, E) \& D\sigma = 0\}$, then we have $\mathcal{C} \subset H^{-1/2}$. and let C be the orthogonal projection onto the closure of $\mathcal{C} \cap L^2(E_0)$, then $C = Q_{>0} + E$ where $E : H^{-1/2} \to H^{1/2}$.*

This result (and a similar result concerning the regularity of the Calderon projector associate to solution of $D\sigma = 0$ on K is then used to show that the splitting formula for the index is true in our case. And we obtain several results

Theorem 4. *For $w > 0$ large enough and without any assumption on $\mathrm{spec}_{ess} D$, then*

$$\frac{\dim \ker \bar{\nabla}}{2} = \dim\{\sigma \in L^2_{loc}(M, E), D\sigma = 0 \ \& \int_U |\sigma|^2 e^{-wt} d\mathrm{vol}_g < \infty\}$$
$$- \dim\{\sigma \in L^2_{loc}(M, E), D\sigma = 0 \ \& \int_U |\sigma|^2 e^{wt} d\mathrm{vol}_g < \infty\}$$

Theorem 5. *With the assumption that $0 \notin \mathrm{spec}_{ess} D$, we get*

$$\mathrm{ind}\, D^+ = \int_M \omega_{D^+} + \frac{1}{2} \lim_{t\to\infty} \eta^{he}(\tilde{A}_t^+)$$
$$+ \frac{1}{2} \left(\dim \ker_{L^2} D^{l,+} - \dim \ker_{L^2} D^{l,-}\right),$$

where $D^{l,\pm}$ are the $\pm$ part of the operator D^l and $\ker_{L^2} D^{l,\pm}$ are the set of L^2 solution of the ODE $D^{l,\pm}\sigma = 0$. Moreover $\eta^{he}(\tilde{A}_t^+)$ is a formal quantity which is the eta invariant of high energy part the geometrical operator $\tilde{A}_t$ induced on $\{t\} \times N$ when submanifold $\{t\} \times N \subset M$ is smooth ; in general the definition of this quantity is the same as definition of Hilsum concerning the eta invariant of Lipschitz manifold [8]

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Index theory on conformally compact manifolds

Pierre Albin
(joint work with Richard Melrose)

Conformally compact metrics have been involved in many recent studies in conformal geometry and the AdS/CFT correspondence in physics. This talk was a report on recent work on the index of pseudo-differential operators naturally associated to these metrics.

Let X be the interior of a manifold with boundary and let x be a smooth non-negative function on $\overline{X}$ that is equal to zero precisely at the boundary with only a simple zero there (i.e., a boundary defining function). A metric g on X is a conformally compact metric if $x^2 g$ extends to a smooth metric on $\overline{X}$. Infinite volume hyperbolic metrics without cusp singularities are examples of conformally compact metrics.

Natural differential operators associated to these metrics are examples of zero pseudo-differential operators (so called because they are associated to the vector fields that vanish at the boundary). This calculus was introduced in [7] and [8] where it was shown that the essential behavior of these operators is controlled by two model operators, the princial symbol and the normal operator. For an operator A of order zero acting on sections of a bundle E, the first is a straightforward extension of the classical principal symbol,

$$\sigma(A) \in \mathcal{C}^\infty\left(S^*X, \mathrm{End}(E)\right),$$

while the second is a smooth family of pseudo-differential operators (in the b, c calculus [6], [1]) acting on an interval bundle over the boundary,

$$\mathcal{N}(A) \in \mathcal{C}^\infty\left(S^*\partial X, \Psi^0_{b,c}(\mathcal{I}, E)\right).$$

As advertised these determine whether or not A is compact (if and only if they both vanish) or Fredholm (if and only if they are both invertible).

Consider a fibration $M \xrightarrow{\phi} B$ whose fibers are manifolds with boundary. In the paper [1] we investigated the group of stable homotopy classes of families of Fredholm zero operators on the fibers of ϕ, $\mathcal{K}(\phi)$. We were able to identify this group with the relative topological K-theory group of the vertical cotangent bundle,

$$\mathcal{K}(\phi) \cong K^0_c\left(T^*M^\circ/B\right),$$

by showing that every class $[A] \in \mathcal{K}(\phi)$ could be represented by a family of zero pseudo-differential operators $\widetilde{A}$ with $\mathcal{N}(\widetilde{A}) = \mathrm{Id}$ and $\sigma(\widetilde{A}) = \mathrm{Id}$ near the boundary. One interesting consequence of our results in [1] is that in order for a family of zero pseudo-differential operators A to be Fredholm it is necessary that the Atiyah-Bott obstruction of its principal symbol vanish (just as in the study of local elliptic boundary problems). A second interesting consequence is the families index theorem for the zero calculus which we obtain by doubling the fibers across their boundaries, extending $\widetilde{A}$ as the identity to the double, and then applying the usual Atiyah-Singer families index theorem.

This is great as far as it goes, but if now we want a formula for the Chern character of the index bundle it would be in terms of $\sigma(\widetilde{A})$ and we would much rather have one in terms of $(\sigma(A), \mathcal{N}(A))$. We find this improved formula in [2].

First recall the situation for a fibration of *closed* manifolds $M' \xrightarrow{\phi'} B$ and a family of Fredholm pseudo-differential operators P acting on a graded vector bundle $\mathbb{E} = E^+ \oplus E^-$. The symbol of P determines a class $[\sigma(P)] \in K^0_c(T^*M'/B)$ and hence a class $\mathrm{Ch}(\sigma(P)) \in H^{\mathrm{even}}_c(T^*M'/B)$. To write down a formula for this class, notice that the complex

$$\mathcal{C}^k := \Omega^{k+1}(M'/B) \oplus \Omega^k(S^*M'/B), \quad D := \begin{pmatrix} d & 0 \\ \pi^* & -d \end{pmatrix}$$

computes $H^*_c(T^*M'/B)$ (cf. the definition of relative cohomology in [3]). Then for a choice of graded connection ∇ on $\mathbb{E}$ we obtain a representative of $\mathrm{Ch}(\sigma(P))$ as

$$\left(\mathrm{Ch}(\mathbb{E}), \widetilde{\mathrm{Ch}}(\sigma(P))\right) \in \mathcal{C}^{\mathrm{even}}, \quad \mathrm{Ch}(\mathbb{E}) = \mathrm{tr}(e^{\omega_+}) - \mathrm{tr}(e^{\omega_-}),$$

$$\widetilde{\mathrm{Ch}}(\sigma(P)) = \int_0^1 \mathrm{tr}\left(\left(\sigma^{-1}\nabla\sigma\right) e^{\omega(t)}\right) \, dt,$$

$$\omega(t) = (1-t)\omega_+ + t\sigma^{-1}\omega_+\sigma + t(1-t)\left(\sigma^{-1}\nabla\sigma\right)^2$$

where $\omega_\pm$ are the curvature forms for ∇ restricted to $E^\pm$ (and we have left out all factors of $2\pi i$). This formula is a slight reinterpration of a formula of Fedosov [5] that, for instance, allows us to dispense with cut-off functions. It follows that a representative of $\mathrm{Ch}(\mathrm{Ind}P) \in H^{\mathrm{even}}(B)$ is given by

$$\mathrm{Ch}\left(\mathrm{Ind}P\right) = \int_{S^*M'/B} \widetilde{\mathrm{Ch}}(\sigma(P))\mathrm{Todd}(M'/B).$$

Returning to the fibration $M \xrightarrow{\phi} B$ of manifolds with boundary and the family A of Fredholm zero pseudo-differential operators acting on a graded bundle $\mathbb{E}$ over M, we know that the joint symbol $(\sigma(A), \mathcal{N}(A))$ determines a class in $K^0_c(T^*M^\circ/B)$ and hence its Chern character a class in $H^{\mathrm{even}}_c(T^*M^\circ/B)$ which we will describe using only $(\sigma(A), \mathcal{N}(A))$ and a choice of graded connection ∇ on $\mathbb{E}$. This data allows us to define $\mathrm{Ch}(\mathbb{E})$ and $\widetilde{\mathrm{Ch}}(\sigma(A))$ as before, we will also need the even Chern character of the indicial operator of A, $I_b(A)$ (the normal operator of $\mathcal{N}(A)$ as an element of the b-calculus),

$$\widetilde{\mathrm{Ch}}^{\mathrm{even}}(I_b(A)) = \int_{\mathbb{R}} \mathrm{i}_{\partial_\xi} \widetilde{\mathrm{Ch}}_{\partial M \times \mathbb{R}^+_\xi}(I_b(A)) \in \Omega^*(\mathcal{I}),$$

and an eta-form (in $\Omega^{\mathrm{odd}}(S^*\partial M/B)$)

$$\eta(\mathcal{N}(A)) = \int_0^1 (1-t)^R \mathrm{Tr}\left(\left(\mathcal{N}^{-1}\nabla\mathcal{N}\right) e^{\omega_N(t)} \right) + t^R \mathrm{Tr}\left(\left(\nabla\mathcal{N}\right) e^{\omega_N(t)} \mathcal{N}^{-1} \right) \, dt$$

which involves (a slight variation of) the renormalized trace introduced by Melrose and Nistor [9].

Viewed as a whole, these Chern forms represent a class in $H^{\mathrm{even}}_c(T^*M^\circ/B)$ once this group is written down in an appropriate way. Indeed, we define

$$\mathcal{C}^k = \Omega^k X \oplus \Omega^{k-1}(S^*M/B) \oplus \Omega^{k-1}(\mathcal{I}) \oplus \Omega^{k-3}(S^*\partial M/B)$$

$$D = \begin{pmatrix} d & & & \\ -\pi^*_{S^*M/B} & -d & & \\ -\pi^*_{\mathcal{I}} i^*_\partial & & -d & \\ & \nu^{\mathcal{I}}_* & \pi^*_\partial \nu^{\mathcal{I}}_* & d \end{pmatrix}$$

(with a compatibility condition at the boundary) where $\nu^{\mathcal{I}}_*$ is a certain push-forward map from $S^*_{\partial M}M/B$ to $S^*\partial M/B$ and we show that this complex computes the cohomology groups $H^*_c(T^*M^\circ/B)$. Then we show that a representative of $\mathrm{Ch}(\sigma(A), \mathcal{N}(A))$ in $\mathcal{C}^{\mathrm{even}}$ is given by

$$\mathrm{Ch}(\sigma(A), \mathcal{N}(A)) = \left(\mathrm{Ch}(\mathbb{E}), \widetilde{\mathrm{Ch}}(\sigma(A)), \widetilde{\mathrm{Ch}}^{\mathrm{even}}(I_b(A)), \eta(\mathcal{N}(A)) \right).$$

Finally we use this representation together with a natural push-forward map to get a formula for the Chern character of the index bundle of A as an element in $\Omega^{\mathrm{even}}(B)$

$$\mathrm{Ch}\,(\mathrm{Ind}A) = \int_{S^*M/B} \widetilde{\mathrm{Ch}}(\sigma(A))\mathrm{Todd}(M/B) + \int_{S^*\partial M/B} \eta(\mathcal{N}(A))\mathrm{Todd}(\partial M/B).$$

In closing we mention that [2] contains a similar treatment of the scattering calculus which models asymptotically locally Euclidean manifolds (where the K-theory computations were carried out in [10]) and the Boutet de Monvel calculus (where the K-theory computations were carried out in [4] and the index formula is due to Fedosov [5]).

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Smooth K-theory

Ulrich Bunke
(joint work with Th. Schick)

The main objective of the talk was to report on [BS07]. Smooth extensions of generalized cohomology theories recently became an important tool for setting up field theories with differential form field strength on the one hand [Fre00], [FH00], and for capturing secondary geometric and analytic data in problems of global analysis on the other hand [CS85], [Lot02], [Bun].

Let N be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$. We consider a generalized cohomology theory h with a natural transformation of cohomology theories $c : h(B) \to H(B,N)$. Let $\Omega(B,N) := \Omega(B) \otimes_{\mathbb{R}} N$.

Definition 1. *A smooth extension of the pair (h,c) is a functor $B \to \hat{h}(B)$ from the category of compact smooth manifolds*[1] *to $\mathbb{Z}$-graded groups together with natural transformations*

a) $R : \hat{h}(B) \to \Omega_{d=0}(B,N)$ *(curvature)*
b) $I : \hat{h}(B) \to h(B)$ *(forget smooth data)*
c) $a : \Omega(B,N)/\mathrm{im}(d) \to \hat{h}(B)$ *(action of forms)* .

We require that these transformations satisfy the following axioms:

i) *The following diagram commutes*

$$\begin{array}{ccc} \hat{h}(B) & \xrightarrow{I} & h(B) \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle c} \\ \Omega_{d=0}(B,N) & \xrightarrow{dR} & H(B,N) \end{array} .$$

[1] possibly with boundary

ii)

(1) $$R \circ a = d \ .$$

iii) *a is of degree* 1.
iv) *The sequence*

(2) $$h(B) \stackrel{c}{\to} \Omega(B,N)/\mathrm{im}(d) \stackrel{a}{\to} \hat{h}(B) \stackrel{I}{\to} h(B) \to 0 \ .$$

is exact.

The smooth integral cohomology $B \mapsto \hat{H}(B,\mathbb{Z})$ first introduced in [CS85] is the smooth extension of the pair $(H(\ldots,\mathbb{Z}), i)$, with the canonical $i : H(B,\mathbb{Z}) \to H(B,\mathbb{R})$. In general, the existence of smooth extensions has been shown in [HS05]. In the special case of ordinary cohomology uniqueness is known [SS], [Wie]. In other examples, e.g. complex K-theory, there exist non-isomorphic smooth extensions, see [Wie].

If h is a multiplicative cohomology theory, then one can consider a $\mathbb{Z}$-graded ring N over $\mathbb{R}$ and a multiplicative transformation $c : h(B) \to H(B,N)$.

Definition 2. *A smooth extension $\hat{h}$ of (h,c) is called multiplicative, if $\hat{h}$ together with the transformations R, I, a is a smooth extension of (h,c), and in addition*

(1) *$\hat{h}$ is a functor to $\mathbb{Z}$-graded rings,*
(2) *R and I are multiplicative,*
(3) *$a(\omega) \cup x = a(\omega \wedge R(x))$ for $x \in \hat{h}(B)$ and $\omega \in \Omega(B,N)/\mathrm{im}(d)$.*

The smooth extension $\hat{H}(\ldots,R)$ of ordinary cohomology $H(\ldots,R)$ with coefficients in a subring $R \subset \mathbb{R}$ is multiplicative in a unique way.

In [BS07] we introduce a geometric/analytic model of a multiplicative smooth extension of the pair $(K, \mathbf{ch})$, where K is complex K-theory, and $\mathbf{ch}$ is the Chern character. A cycle for a smooth K-theory class over B is a pair $(\mathcal{E},\alpha)$, where $\mathcal{E}$ is a geometric family (see [Bun]), and $\alpha \in \Omega(B)/\mathrm{im}(d)$. Addition of cycles is defined in the obvious way by the disjoint union of families and the sum of forms. For the relations: the pair $(\mathcal{E},\alpha)$ represents the trivial smooth K-theory class if $\mathcal{E}$ admits a taming $\mathcal{E}_t$ such that $\eta(\mathcal{E}_t) = \alpha$, where $\eta(\mathcal{E}_t)$ is the eta form, see [Bun] for the notion of a taming.

The most interesting operation with smooth K-theory is integration. In [BS07] we introduce the notion of a smooth K-orientation of a proper submersion $\pi : W \to B$. Then we construct the push-forward $\pi_! : \hat{K}(W) \to \hat{K}(B)$ which has all expected properties:

(1) it is compatible with the push-forward on the level of topological K-theory and on forms,
(2) it satisfies a projection formula and commutes with pull-backs
(3) it is functorial with respect to the composition of smoothly K-oriented maps.

It was shown in [Wie] that the pair $(K, \mathbf{ch})$ has a unique extension, if in addition to the structures discussed above one requires a push-forward along the bundles

$S^1 \times B \to B$ with the corresponding subset of the properties above. This implies in particular that our model of $\hat{K}$ is canonically isomorphic to that of [HS05] (here we disregard the multiplicative structure).

The second main construction of [BS07] is a lift of the Chern character to a multiplicative transformation

$$\hat{\mathbf{ch}} : \hat{K}(B) \to \hat{H}(B, \mathbb{Q})$$

which induces a rational isomorphism. The main theorem is the following index theorem:

Theorem 1. *If $\pi : W \to B$ is a smoothly K-oriented proper submersion, then the following diagram commutes*

$$\begin{array}{ccc} \hat{K}(W) & \xrightarrow{\hat{\mathbf{ch}}} & \hat{H}(W, \mathbb{Q}) \\ \downarrow{\scriptstyle \pi_!} & & \downarrow{\scriptstyle \pi_!^K} \\ \hat{K}(B) & \xrightarrow{\hat{\mathbf{ch}}} & \hat{H}(B, \mathbb{Q}) \end{array} ,$$

where $\pi_!^K(x) := \int_{W/B} \hat{\hat{\mathbf{A}}} \cup x$, $\int_{W/B}$ denotes the integration over the fibre in rational smooth cohomology, and $\hat{\hat{\mathbf{A}}} \in \hat{H}(W, \mathbb{Q})$ is a lift of the $\hat{\mathbf{A}}$-class of the vertical bundle of π determined by the smooth K-orientation.

This theorem subsumes and generalizes previous results of [Lot94], [Bera], [Berb], [Bis05].

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The eta invariant and the equivariant index theorem

Ken Richardson
(joint work with Jochen Brüning, Franz W. Kamber)

Let E be a Hermitian vector bundle over a closed, Riemannian manifold M, such that a compact Lie group G acts on (M,E) by isometries. Let $\rho : G \to U(V_\rho)$ be an irreducible unitary representation. Let $D : \Gamma(M,E^+) \to \Gamma(M,E^-)$ be a first order, transversally elliptic, G-equivariant differential operator. We assume that near each singular stratum Σ_j, D can be written as the product

$$D = \left\{ Z_j \left(\nabla^E_{\partial_r} + \frac{1}{r} D^S_j \right) \right\} * D^{\Sigma_j},$$

where r is the distance from Σ_j, where Z_j is a local bundle isomorphism, the map D^S_j is a purely first order operator that differentiates in the unit normal bundle directions tangent to $S_x\Sigma_j$, and D^{Σ_j} is a global transversally elliptic, G-equivariant, first order operator on the stratum Σ_j.
Following is the main result ([2]).

Theorem 1. *(Equivariant Index Theorem) The $[\rho]$ part of the equivariant index of D is*

$$\begin{aligned} \operatorname{ind}^\rho(D) &= \int_{\widetilde{M_0}/G} A^\rho_0(x)\ \widetilde{|dx|} \\ &\quad + \frac{1}{2} \sum_{i,j,a,b} \frac{n^\rho_j \gamma^j_{ab}}{m^{\Sigma_i}_b} \left(-\eta\left(D^{S+,\sigma_a}_i\right) + h\left(D^{S+,\sigma_a}_i\right)\right) \int_{\widetilde{\Sigma_i}/G} A^{\rho_0}_{i,\sigma^*_b}(x)\ \widetilde{|dx|}, \end{aligned}$$

where

1. *$\widetilde{M_0}$ is the equivariant blowup of the principal stratum of M. That is, $\widetilde{M_0}$ is the fundamental domain of the principal stratum of M after blowing up and doubling along all singular strata in its closure to produce the G-manifold M_0', whose orbits $\{\mathcal{O}_x : x \in M_0'\}$ form a fiber bundle;*
2. *$\widetilde{\Sigma_i}$ is the equivariant blowup of the singular stratum Σ_i;*
3. *$A^\rho_0(x)$ is the Atiyah-Singer integrand, the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from D' (blown-up and doubled from D) on the quotient M_0'/G, where the bundle E is replaced by the bundle $\mathcal{E}_\rho$ defined as $\mathcal{E}_\rho(\mathcal{O}_x) = \operatorname{Hom}_G\left(V_\rho, L^2(\mathcal{O}_x, E)\right)$.*
4. *For an irreducible representation $\sigma_b : H_i \to U(W_{\sigma_b})$ of H_i, $A^{\rho_0}_{i,\sigma^*_b}$ is similarly the local supertrace of the ordinary heat kernel associated to the elliptic operator induced from $D^{\Sigma_i\prime} \otimes \mathbf{1}$ (blown-up and doubled from D^{Σ_i}, where $[H_i]$ is the isotropy type associated to the stratum Σ_i) on the quotient Σ_i'/G,*

where the bundle $E_{\Sigma_i} \otimes \widetilde{E_{\sigma_b}}^*$ *is replaced by the space of invariant sections over each orbit;*

(5) $\eta\left(D_i^{S+,\beta}\right)$ *is the eta invariant of the operator* D_i^{S+} *induced on any unit normal sphere* $S_x\Sigma_i$*, restricted to sections of type* β.

(6) $h\left(D_i^{S+,\beta}\right)$ *is the dimension of the kernel of* D_i^{S+}*, restricted to sections of type* β.

(7) n_j^ρ, $m_b^{\Sigma_i}$ *and* γ_{ab}^j *are defined by the equations*

$$\operatorname{Res}(\rho) = \bigoplus_j n_j^\rho \sigma_j;\ \gamma_{ab}^j = \textit{multiplicity of } \sigma_j \textit{ in } \sigma_a \otimes \sigma_b;$$
$$m_b^{\Sigma_i} = \textit{multiplicity of } \sigma_b \textit{ in } \left.E_{\Sigma_i}\right|_{\mathrm{pt}},$$

with the σ_j,σ_a,σ_b *being irreducible* H_i *representations.*

(8) $\widetilde{V_{\sigma_b}}^*$ *is the bundle over* Σ_i *induced from the* H_i *representation that is a direct sum of* $n_{\sigma_b}^\rho$ *copies of* $W_{\sigma_b}^*$.

A consequence of the theorem above is the following formula (from [3]) for the index of a foliation-equivariant transverse Dirac operator D_b restricted to basic sections on a Riemannian foliation $(M, \mathcal{F})$. See [5] and [4] for definitions.

Theorem 2. *(Basic Index Theorem for Riemannian foliations) We have*

$$\begin{aligned} \operatorname{ind}_b(D_b) &= \int_{\widetilde{M_0}/\overline{\mathcal{F}}} A_{0,b}(x)\ \widetilde{|dx|} \\ &\quad + \frac{1}{2}\sum_{i,\sigma} \frac{1}{m_\sigma^{M_i}} \left(-\eta\left(D_i^{S+,\sigma}\right) + h\left(D_i^{S+,\sigma}\right)\right) \int_{\widetilde{M_i}\overline{\mathcal{F}}} A_{i,b}^\sigma(x)\ \widetilde{|dx|}, \end{aligned}$$

where the sum is over all strata M_i *of* $\left(M, \overline{\mathcal{F}}\right)$ *and over all irreducible representations* $\rho : O(q) \to U(V_\rho)$ *(only a finite number occur), where*

(1) $\widetilde{M_0}$ *is the fundamental domain of the principal stratum of the foliation after blowing up and doubling along all singular strata;*

(2) $\widetilde{M_i}$ *is the fundamental domain of the singular stratum* M_i *after blowing up and doubling* M_i *along all singular strata properly contained in the closure of* M_i*;*

(3) *The other notation is similar to that in the last theorem.*

In the following section, we explain the meaning of the equivariant index multiplicity $\operatorname{ind}^\rho(D)$.

1. What is the equivariant index?

As in the last section, suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold M, and let $E = E^+ \oplus E^-$ be a graded, G-equivariant Hermitian vector bundle over M. We consider a first order G-equivariant differential operator $D = D^+ : \Gamma(M, E^+) \to \Gamma(M, E^-)$ which is transversally elliptic, and let D^- be the formal adjoint of D^+.

The group G acts on $\Gamma(M, E^{\pm})$ by $(gs)(x) = g \cdot s\left(g^{-1}x\right)$, and the (possibly infinite-dimensional) subspaces $\ker(D)$ and $\ker(D^*)$ are G-invariant subspaces. Let $\rho : G \to U(V_\rho)$ be an irreducible unitary representation of G, and let $\chi_\rho = \operatorname{tr}(\rho)$ denote its character. Let $\Gamma(M, E^{\pm})^\rho$ be the subspace of sections that is the direct sum of the irreducible G-representation subspaces of $\Gamma(M, E^{\pm})$ that are unitarily equivalent to the ρ representation. It can be shown that the extended operators

$$\overline{D}_{\rho,s} : H^s\left(\Gamma\left(M, E^+\right)^\rho\right) \to H^{s-1}\left(\Gamma\left(M, E^-\right)^\rho\right)$$

are Fredholm and independent of s, so that each irreducible representation of G appears with finite multiplicity in $\ker D^{\pm}$. Let $a_\rho^{\pm} \in \mathbb{Z}^+$ be the multiplicity of ρ in $\ker(D^{\pm})$.

We define the virtual representation-valued index of D as in [1], as

$$\operatorname{ind}^G(D) := \sum_\rho \left(a_\rho^+ - a_\rho^-\right)[\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ. The index multiplicity is

$$\operatorname{ind}^\rho(D) := a_\rho^+ - a_\rho^- = \frac{1}{\dim V_\rho} \operatorname{ind}\left(D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho}\right).$$

In particular, if ρ_0 is the trivial representation of G, then

$$\operatorname{ind}^{\rho_0}(D) = \operatorname{ind}\left(D|_{\Gamma(M,E^+)^G \to \Gamma(M,E^-)^G}\right),$$

where the superscript G implies restriction to G-invariant sections.

There is a clear relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\operatorname{ind}_g(D)$ (see [1]); the multiplicities determine the distributional index, and vice versa. Because the operator $D|_{\Gamma(M,E^+)^\rho \to \Gamma(M,E^-)^\rho}$ is Fredholm, all of the indices $\operatorname{ind}^G(D)$, $\operatorname{ind}_g(D)$, and $\operatorname{ind}^\rho(D)$ depend only on the homotopy class of the principal transverse symbol of D.

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