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Komplexe Algebraische Geometrie

Organised by
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Abstract. The Conference focused on several classical theories in the realm of complex algebraic geometry, such as Abelian Varieties, Jacobians and Pryms, Moduli spaces, Variation of Hodge structures and Algebraic surfaces. New inputs concerned the minimal model program, resp. the Hodge conjecture, and algebraic fundamental groups. New insights relate to arithmetic (integrality, hyperbolicity) and physics (Mirror Symmetry, quantization).

Mathematics Subject Classification (2000): 14xx, 11xx, 32xx, 53xx.

Introduction by the Organisers

The Workshop *Komplexe algebraische Geometrie*, organized by Fabrizio Catanese (Bayreuth), Yujiro Kawamata (Tokyo), Gang Tian (Princeton), and Eckart Viehweg (Essen), drew together 50 participants. There were several young PhD students and other PostDocs in their 20's and early 30's, together with established leaders of the fields related to the thematic title of the workshop. There were 23 talks, each lasting 55 minutes or one hour, and each followed by a lively 10 minutes discussion.

As usual at an Oberwolfach Meeting, the mathematical discussions continued outside the lecture room throughout the day and the night. The Conference fully fulfilled its purported aim, of setting in contact mathematicians with different specializations and non uniform background, of presenting new fashionable topics alongside with new insights on long standing classical open problems, and also cross-fertilizations with other research topics as arithmetic and physics. For the latter, cf. the talk by Bernd Siebert on the new approach to Mirror Symmetry through logarithmic geometry and toric affine Calabi Yau varieties and the one by van Straten on quantization of completely integrable Hamiltonian Systems. For the former, cf. the talks by Winkelmann on integral sets and by McQuillan on the Bloch principle.

A central role was occupied by the new results around the Hodge conjecture presented by Voisin, and the new results by Hacon, M^{c}Kernan et al which give an essential step towards the final solution of the Minimal Model Program, and were presented here by M^{c}Kernan with a proposed approach to the Sarkisov program.

There were many expositions dealing with several classical problems and classical and modern theories. It would take too long to dwell on each of the outstanding contributions presented at the Conference. For some topics there were several interrelated talks, for instance we could mention the following classical themes:

(1) Abelian Varieties, Jacobian and Prym Varieties and their Moduli spaces (van der Geer, Farkas, Lange, Hulek)
(2) Hodge Theory and Variation of Hodge structures (Voisin, Möller, Barja)
(3) Fibred varieties (Oguiso, Moeller)
(4) Algebraic Surfaces (Mukai, Pardini)
(5) Fundamental groups and algebraic fundamental groups (Esnault, Bauer, Pardini).

There were also expositions on many other beautiful topics:

(1) Moduli spaces of sheaves on higher dimensional varieties (Lehn)
(2) Deformations of special complex maifolds (Rollenske)
(3) Varieties of power sums (Takagi)
(4) Ball quotients (Müller-Stach)
(5) Stacks and Azumaya algebras (Schröer)
(6) nefness and vector bundles (Peternell)

The variety of striking results and the very interesting and challenging proposals made the participation in the workshop rather strenuous but certainly highly rewarding. We hope that the quality of the expositions in these abstracts will make them quite useful to the mathematical community.

Workshop: Komplexe Algebraische Geometrie

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Abstracts

Hodge loci and absolute Hodge classes

Claire Voisin

Let X be a smooth complex algebraic variety, and denote by X^{an} the corresponding complex manifold. A Hodge class α on X is a class $\alpha \in (2\iota\pi)^k H^{2k}(X^{an}, \mathbb{Q}) \cap F^k H^{2k}(X^{an}, \mathbb{C})$ where $F^{\cdot}$ stands for the Hodge filtration. According to [1], α is said to be absolute Hodge if for any $\tau \in Aut\,\mathbb{C}$, the class $\alpha_\tau \in F^k H^{2k}(X_\tau^{an}, \mathbb{C})$ is again a Hodge class, that is belongs to $(2\iota\pi)^k H^{2k}(X_\tau^{an}, \mathbb{Q})$. Here α_τ is obtained by using the isomorphism $F^k H^{2k}(X^{an}, \mathbb{C}) \cong \mathbb{H}^{2k}(X^{an}, \Omega_{X^{an}}^{\cdot \geq k})$ which gives by GAGA

$$F^k H^{2k}(X^{an}, \mathbb{C}) \cong \mathbb{H}^{2k}(X, \Omega_{X/\mathbb{C}}^{\cdot \geq k}).$$

We first reinterpret this notion in terms of the locus of Hodge classes (cf [2]) : X is a complex fiber of a smooth quasi-projective family $\pi : \mathcal{X} \to B$ defined over $\mathbb{Q}$. There is an algebraic vector bundle $F^k H^{2k}$ over B which is defined over $\mathbb{Q}$ and whose analytisation is the Hodge bundle with fiber $F^k H^{2k}(X_t^{an}, \mathbb{C})$ over $t \in B(\mathbb{C})$. Inside $F^k H^{2k}(\mathbb{C})$, let Z be the set of all Hodge classes in fibers of π. Let Z_α be the connected component of Z passing through α. It is proved in [2] that Z_α is closed algebraic. We show that α is absolute Hodge iff Z_α is defined over $\overline{\mathbb{Q}}$ and its Galois transforms under $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$ are again Z_β's. This leads to a weakening a the notion of absolute Hodge classes. Namely consider the projection B_α of Z_α to B. Then we can study whether B_α is defined over $\overline{\mathbb{Q}}$ and its Galois transforms under $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$ are again B_β's.

This last property is enough to guarantee that the Hodge conjecture for α is implied by the Hodge conjecture for Hodge classes on varieties defined over $\overline{\mathbb{Q}}$ (a question asked by Maillot and Soulé). On the other hand, it is much easier to address. We prove the following criterion.

Theorem. *Suppose that the only locally constant sub-Hodge structure* $L \subset H^{2k}(X_t, \mathbb{Q})$, $t \in B_\alpha$, *is trivial, that is of type* (k, k). *Then* B_α *is defined over* $\overline{\mathbb{Q}}$ *and its Galois transforms under* $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$ *are again* B_β*'s.*

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Cycle Relations on Jacobians

Gerard van der Geer

This is a report on joint work with Alexis Kouvidakis. The Chow ring $CH^*_{\mathbb{Q}}(X)$ with rational coefficients of a principally polarized abelian variety X over an algebraically closed field k comes with a rich structure. It carries a grading $CH^*_{\mathbb{Q}}(X) = \oplus_i CH^i_{\mathbb{Q}}(X)$ by codimension and an intersection product $(x, y) \mapsto x \cdot y$ making it into a commutative ring. But there is also the Pontryagin product $(x, y) \mapsto x * y$ which provides $CH^*_{\mathbb{Q}}(X)$ with a second structure of commutative ring and a Fourier-Mukai transform $F : CH^*_{\mathbb{Q}}(X) \to CH^*_{\mathbb{Q}}(X^t)$, where $X^t \cong X$ is the dual abelian variety. This transform F gives an isomorphism of $(CH^*_{\mathbb{Q}}(X), \cdot)$ with $(CH^*_{\mathbb{Q}}(X), *)$ interchanging the intersection product and the Pontryagin product. Furthermore we have the action of the integers $\mathbb{Z} \subset \mathrm{End}(X)$ on $CH^*_{\mathbb{Q}}(X)$. Following Beauville (cf. [1]) we can put

$$CH^i_{(j)}(X) = \{x \in CH^i_{\mathbb{Q}}(X) : n^* x = n^{2i-j} x \text{ for all } n \in \mathbb{Z}\}.$$

Then $F : CH^i_{(j)}(X) \cong CH^{g-i+j}_{(j)}(X)$ which implies that $i - g \leq j \leq i$; but for $i = 1$ and $g - 1$ we also know that $j \geq 0$. The Chow ring modulo algebraic equivalence $A(X) = CH^*_{\mathbb{Q}}(X)/\sim_{\mathrm{alg}}$ inherits this rich structure and we can write $A(X) = \oplus A^i_{(j)}$.

Let now C be a smooth irreducible algebraic curve of genus g over an algebraically closed field k and embed C in its Jacobian J via $p \mapsto (p - p_0)$ for some point p_0. Then class $[C]$ of the image in $A(J)$ is well-defined and can be decomposed as

$$[C] = C_0 + C_1 + \cdots + C_{g-1} \qquad \text{with } C_j \in A^{g-1}_{(j)}.$$

Note that the classes C_j are homologically trivial for $j > 0$ because $n \in \mathbb{Z}$ does not act with the right power. We put

$$p_j := F(C_{j-1}) \in A^j_{(j-1)} \qquad \text{for } j = 1, \ldots, g.$$

Let now R be the smallest subring of $A(J)$ containing the class $[C]$ which is stable under intersection and Pontryagin product, the action of $\mathbb{Z}$ and under F. It is a theorem of Beauville ([2]) that R is generated by the classes $p_1, \ldots, p_g$. In particular the ring R is finite-dimensional. It is called the *tautological ring* of C. The basic question is: what is the structure of R? It is a very subtle invariant of C. It encodes both geometric information about the curve, but also arithmetic information. A theorem of Colombo and van Geemen ([4]) says that if C possesses a base-point free linear system g^1_d then $C_{(j)} = 0$ for $j \geq d - 1$. But there is also the celebrated theorem of Ceresa ([3]) that says that for a general curve of genus $g \geq 3$ the class p_2 does not vanish and $C - C^- \not\sim_{\mathrm{alg}} 0$.

In 2006 Herbaut found a generalization of the Colombo-van Geemen theorem.

Theorem 1. *(Herbaut, [7]) If C has a base-point-free g_d^r then*

$$\sum_{a_1+\dots a_r=N} B_d(a_1,\dots,a_r)\, C_{a_1} * \cdots * C_{a_r} = 0$$

for all $N \geq 0$ with

$$B_d(a_1,\dots,a_r) = \sum_{n_1,\dots,n_r\geq 1} (-1)^{d-\sum n_j} \binom{d}{\sum n_j} n_1^{a_1}\cdots n_r^{a_r}.$$

A little later Kouvidakis and I found the following result.

Theorem 2. *(van der Geer–Kouvidakis, [6]) If C has a base-point-free g_d^r then*

$$\sum_{a_1+\dots a_r=N} (a_1+1)!\cdots(a_r+1)!\, C_{a_1} * \cdots * C_{a_r} = 0$$

for all $N \geq d-2r+1$.

It turns out the Herbauts relations are vacuous for $N \leq 2d-r$, but for $N = d-2r+1$ one finds the first new relation beyond the Colombo-van Geemen relation. The relations in our theorem are much simpler. However, Zagier proved that the set of relations of Herbaut is equivalent to that of our theorem, cf. [6]. But although the theorems amount to the same the proofs are rather different. Herbaut works on symmetric powers of C and calculates there cycle classes of loci that are blown down under the map to the Jacobian. We use Grothendieck-Riemann-Roch to deduce the relations.

The structure of R for a given curve is a difficult problem. Note that the general curve of genus g has gonality $[(g+3)/2]$, hence $p_j = 0$ for $j \geq g/2+1$.

Observe that a (weighted) monomial of degree g and positive weight in the p_i has zero cohomology class, where we consider p_j to be of degree j and weight $j-1$, because a zero cycle which is homologically zero is algebraically equivalent to 0.

Polishchuk has studied in [9, 10] the operator $x \mapsto x * \theta^{g-1}/(g-1)!$, where θ is the class of the theta divisor. The effect of this operator on R is given by the differential operator

$$D = -g\partial_1 + \sum_{m,n\geq 1} \binom{m+n}{n} p_{m+n-1}\partial_m\partial_n,$$

where $\partial_i = \partial/\partial p_i$. This gives a way of creating new relations from the positive weight degree g monomials in the p_j.

The ring R gets the structure of an sl_2-module via

$$e(x) = p_1\cdot x,\ h(x) = -g + \sum_{n\geq 1}(n+1)p_n\partial_n(x),\ f(x) = -D(x),$$

as Polishchuk observed (cf. [9, 10], also [8]).

Consider now the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$ and let I be the smallest ideal containing all monomials in the x_i of degree $> g$ and all monomials of degree g and weight > 0 where the degree (resp. weight) of x_i is i (resp $i-1$) and stable under the operator $D = -g\partial_1 + \sum_{m,n\geq 1}\binom{m+n}{n} x_{m+n-1}\partial_m\partial_n$ with $\partial_i = \partial/\partial x_i$.

The quotient $S := \mathbb{Q}[x_1, \ldots]/I$ maps surjectively onto R via $x_i \mapsto p_i$. Polishchuk conjectures (cf. [9]) that for a general curve this is an isomorphism $S \cong R$.

The structure of S is a combinatorial problem. We conjecture the following for its structure, and then assuming Polishchuk's conjecture also for the structure of R.

Conjecture 1. *For a general curve C of genus g the ring R satisfies*

(1) $\dim_{\mathbb{Q}} R = p(g+1)$*, the number of partitions of* $g+1$*.*
(2) $\dim R^i_{(j)} = p(i, g+1-i, j)$*, the number of partitions of i with $i-j$ parts and with all parts* $\leq g+1-i$*.*

Note that this conjecture is compatible with the duality between $R^i_{(j)}$ and $R^{g-i+j}_{(j)}$. It also connects well with Brill-Noether theory. Let $d = d(g,r)$ be the smallest d such that the general curve of genus g has a g^r_d. We have $d(g,r) = g+r-[g/(r+1)]$. Then our conjecture predicts that $R^{j+r}_{(j)} = (0)$ if $j \geq d(g,r) - 2r + 1$. This is true for $r = 1$ since by Colombo-van Geemen we have that $p_j = 0$ for $j \geq g/2 + 1$. It has been checked by Moonen (cf. [8]) for $r = 2$ and $r = 3$. Some of the results on R can be lifted to the level of CH^* instead of A, cf. [5, 8].

As a final remark, note that R also seems to carry subtle arithmetic information. For example, if C is defined over a number field then one expects that $p_j = 0$ for all $j > 2$.

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Mordell-Weil group of an abelian fibered variety and its application to hyperkähler manifolds

Keiji Oguiso

We work over $\mathbf{C}$. In 70's, Shioda [Sh] proved the following important:

Theorem 1. *Let $f : S \longrightarrow C$ be a relatively minimal Jacobian fibration, i.e., a relatively minimal elliptic fibration with a section O, having at least one singular fibers, say, S_{t_i} $(1 \le i \le k)$. Then, the Mordell-Weil group* $\mathrm{MW}(f)$ *is a finitely generated abelian group of rank*

$$\mathrm{mw}\,(f) = \rho(S) - 2 - \sum_{i=1}^{k}(m_i - 1)\ .$$

Here $\rho(S)$ is the Picard number of S and m_i is the number of irreducible components of S_{t_i}. In particular, $\rho(S) \ge 2$ and $\mathrm{mw}\,(f) \le \rho(S) - 2$.

It is natural to ask the optimality of the last estimate. In this direction, the following result was shown by [O1] (note that $\rho(S) \le 20$ for a K3 surface S):

Theorem 2. *Let ρ be an integer s.t. $2 \le \rho \le 20$. Then, for each such ρ, there is a Jacobian K3 surface $f : S \longrightarrow \mathbf{P}^1$ s.t. $\rho(S) = \rho$ and $\mathrm{mw}\,(f) = \rho - 2$.*

In the talk, I explained possible generalizations of these two theorems.

Definition 3. *Let $f : X \longrightarrow Y$ be a surjective morphism between normal projective varieties. We call f an abelian fibration if f has a rational section O and the generic fiber (in the sense of scheme) $A_K := X_\eta$ is an abelian variety defined over $K := \mathbf{C}(Y)$ with origin $O \in A_K(K)$. The Mordell-Weil group $\mathrm{MW}\,(f)$ of f is the set of K-rational points $A_K(K)$, or more geometrically, the set of rational sections of f.*

$\mathrm{MW}\,(f)$ forms an abelian group and acts faithfully on X as birational automorphisms of X. We assume the following:
(i) X and Y have only $\mathbf{Q}$-factorial rational singularities;
(ii) there is no prime divisor D on X s.t. $f(D)$ is of codimension ≥ 2 on Y;
(iii) $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y)$.

The conditions (i) and (ii) are natural in the view of flattening theorem and probably the minimal model theory for higher dimensional varieties. Some condition like (iii) is necessary for the finite generation of $MW(f)$. For instance, $MW(p_2)$ is far from being finitely generated for the product manifold $p_2 : A \times Y \longrightarrow Y$ where A is a positive dimensional complex abelian variety. We have $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_C)$ in Theorem 1, as f has a singular fiber, and also $\rho(C) = \rho(E) = 1$, where $E = S_\eta$.

Definition 4. *Let X be a compact Kähler manifold. We call X a hyperkähler manifold (HK manifold, for short) if X is simply connected and X has an everywhere non-degenerate global holomorphic 2-form σ_X s.t. $H^0(\Omega_X^2) = \mathbf{C}\sigma_X$.*

Typical examples are the Hilbert schemes $S^{[n]}$ of n points on K3 surfaces S and their small deformations [Be]. When $n \geq 2$, $\rho(S^{[n]}) = \rho(S) + 1$ and $\rho(X) \leq 21$ for any small deformation of $S^{[n]}$. Note that a HK manifolds is even dimensional and both projective HK manifolds and non-projective HK manifolds are dense in the Kuranishi space.

The following theorem is due to Matsushita [M1], [M2]:

Theorem 5. *Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibers from a HK manifold of dimension $2n$ to a normal projective variety Y s.t. $0 < \dim Y < 2n$. Then, any irreducible component of the fiber is Lagrangian. In particular, any smooth fiber is a complex torus of dimension n and f is equi-dimensional. Moreover, if X is projective, then Y is a $\mathbf{Q}$-Fano variety with $\mathbf{Q}$-factorial klt singularities and $\rho(Y) = 1$. In particular, (i), (ii) as well as (iii) (as $h^1(\mathcal{O}_X) = 0$) are satisfied for an abelian fibered HK manifold.*

It is conjectured that the base space Y is always isomorphic to $\mathbf{P}^n$.

The following is one of possible generalizations of Theorem 1 [O2]:

Theorem 6. *Let $f : X \longrightarrow Y$ be an abelian fibration with properties (i), (ii), (iii). Let $\Delta = \cup_{i=1}^{k} \Delta_i \subset Y$ be the irreducible decomposition of the codimension 1 locus of the critical loci of f and let m_i be the number of prime divisors lying over Δ_i. Then $\mathrm{MW}\,(f)$ is a finitely generated abelian group of rank*

$$\mathrm{mw}\,(f) = \rho(X) - \rho(Y) - \rho(A_K) - \sum_{i=1}^{k}(m_i - 1)\ .$$

Here $\rho(A_K)$ is the rank of the Néron-Severi group of A_K, i.e., the rank of group of algebraically equivalent classes of divisors on A_K defined over K. In particular, $\rho(X) \geq 2$ and $\mathrm{mw}\,(f) \leq \rho(X) - 2$.

A similar result is also obtained independently by [Kh]. As the dual abelian variety $\hat{A}$ of A is defined over K and is isogenous to A over K, the two groups $\mathrm{MW}(f) = A(K)$ and $\mathrm{Pic}^0 A_K(K) = \hat{A}(K)$ are isomorphic modulo finite groups. This is the essential part of the proof, as it reduces the problem to the one on divisor classes on X, Y, and A_K. The rest of the proof is quite close to the proof of Theorem 1 [Sh] and an argument of [Ka] for certain Calabi-Yau fiber spaces. See [O2] for a complete proof.

The following is a partial generalization of Theorem 2:

Theorem 7. *For each integers $n \geq 2$ and $2 \leq \rho \leq 21$, there is an abelian fibered HK manifold $f : X \longrightarrow \mathbf{P}^n$ s.t. X is a small deformation of $S^{[n]}$ of a K3 surface S, $\rho(X) = \rho$ and $\mathrm{mw}\,(f) = \rho - 2$.*

Example 1. *A Jacobian K3 surface $f : S \longrightarrow \mathbf{P}^1$ of Mordell-Weil rank $\rho(S) - 2$ induces an abelian fibration $f_n : S^{[n]} \longrightarrow \mathbf{P}^n$ of Mordell-Weil rank $\geq \rho(S) - 2$. For f_n, the exceptional divisor of the Hilbert-Chow morphism becomes one of two irreducible components over some critical prime divisor. Thus, from Theorem 6, we have $\mathrm{mw}\,(f) = \rho(S) - 2 = \rho(S^{[n]}) - 3$ and $\rho(A_K) = 1$. Note that any* *smooth closed fiber X_t of f_n is the product of elliptic curves. Thus $\rho(X_t) \geq 2$. In particular, $\rho(A_K) \neq \rho(X_t)$.*

The crucial part of Theorem 7 is to compute somewhat mysterious $\rho(A_K)$:

Theorem 8. *Let $f : X \longrightarrow \mathbf{P}^n$ be an abelian fibered HK manifold with generic fiber A_K. Then $\rho(A_K) = 1$. In particular,* $\mathrm{mw}\,(f) = \rho(X) - 2 - \sum_{i=1}^{k}(m_i - 1)$.

For the proof, we use deformation theory. Let F be a general closed fiber of f and let $\iota : F \longrightarrow X$ be the inclusion map. As f is fibered over $\mathbf{P}^n$, by Matsushita [M3] (see also [Sa]), deformation of X that keeps fibration is of codimension 1 in the Kuranishi space. This deformation is a (part of) deformation of X that keeps F Lagrangian. Therefore, by Voisin [Vo] (an easier direction), it is of codimension $\mathrm{rank}\,\mathrm{Im}(\iota^* : H^2(X, \mathbf{Z}) \longrightarrow H^2(F, \mathbf{Z}))$. Thus $\mathrm{rank}\,\mathrm{Im}\,\iota^* = 1$. If $\rho(A_K) \geq 2$, then the specialization of divisors D_1 and D_2 on X corresponding to independent elements of $\mathrm{NS}\,(A_K)$ would yield independent elements of $\mathrm{NS}\,(F)$, a contradiction. In this way, Theorem 8 can be proved.

Now one can show Theorem 7 by starting from $f_n : S^{[n]} \longrightarrow \mathbf{P}^n$ in Example 1 and deforming it as in the proof for the K3 case. The argument is based on the jumping of Picard numbers under deformation [O1], again Voisin's deformation theory of Lagrangian submanifolds [Vo] (harder part), and the fact that fibered HK manifold with a bimeromorphic section over a projective base space is projective [O2]. See [O3] (which will be avaiable when this report will be published) for details.

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The Sarkisov Program

James McKernan
(joint work with Christopher Hacon)

Recall the following well known conjecture of higher dimensional geometry:

Conjecture 1. *Let X be a smooth projective variety.*

Then there is a K_X-negative birational map $f\colon X \dashrightarrow Y$, whose inverse does not contract any divisors, where Y has $\mathbb{Q}$-factorial terminal singularities and either

(1) *Y is a **minimal model**, so that K_Y is nef (that is $K_Y \cdot \Sigma \geq 0$ for every curve C in Y), or*
(2) *Y is a **Mori fibre space**, so that there is a contraction morphism $\psi\colon Y \to V$ of relative Picard number one, $\dim V < \dim Y$ and $-K_Y$ is relatively ample.*

Negativity means that that the difference between the pullbacks of K_X and K_Y to a common resolution is effective and exceptional. The key point is that then X and Y have the same pluricanonical forms:

$$\forall m \geq 0 \qquad H^0(X, \mathcal{O}_X(mK_X)) \simeq H^0(Y, \mathcal{O}_Y(mK_Y)).$$

Note that we do know some cases of Conjecture 1:

Theorem 1 (Birkar, Cascini, Hacon, McKernan, [1])**.** *Let X be a smooth projective variety.*

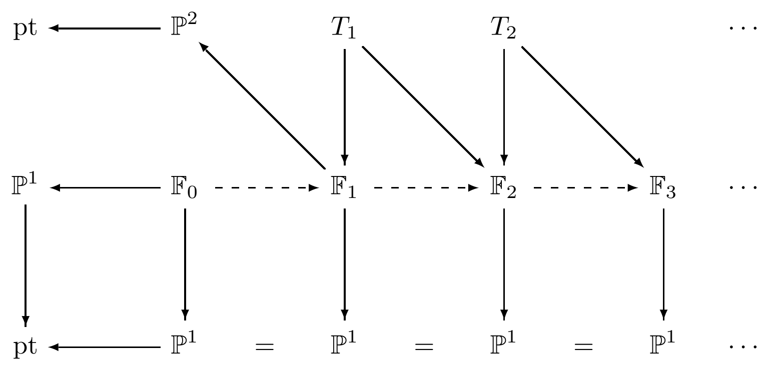
(1) *If X is of general type then X has a minimal model.*
(2) *If $-K_X$ is not pseudo-effective (ie K_X is not a limit of big divisors) then X has a Mori fibre space.*

However in this talk I am much more interested in the fact that the output of the MMP is not unique in either case. Fortunately we do have a satisfactory understanding of what happens in the case of minimal models:

Theorem 2 (Kawamata, [3])**.** *If $f\colon X_1 \dashrightarrow X_2$ is a birational map between two minimal models then f is a composition of flops.*

We also know that if X is of general type, then X has only finitely many minimal models (in fact see below for a much sharper statement).

However the situation for Mori fibre spaces is much more complicated. To understand the situation better, consider the case of surfaces. In this case, a Mori fibre space is a contraction morphism $\phi\colon X \longrightarrow U$, where X is a smooth surface and ϕ is a $\mathbb{P}^1$-bundle, unless U is a point, in which case $X = \mathbb{P}^2$. The problem is that rational surfaces have infinitely many Mori fibre spaces. Fortunately however they are arranged in an appealing fashion:



The morphism $\mathbb{F}_1 \longrightarrow \mathbb{P}^1$ is simply the blow up of a point and the birational map $\mathbb{F}_i \dashrightarrow \mathbb{F}_{i+1}$ is an elementary transformation, which is given by the morphism $T_i \longrightarrow \mathbb{F}_i$ which blows up a point of a fibre of the $\mathbb{P}^1$-fibration where it meets a section of minimal self-intersection and then the morphism $T_i \longrightarrow \mathbb{F}_{i+1}$ which contracts the old fibre (or the inverse of such a map). We then have the following classical result, whose modern formulation is due to Iskovskikh:

Theorem 3. *Let $f\colon X \dashrightarrow Y$ be a birational map between any two Mori fibre spaces $\phi\colon X \longrightarrow U$ and $\psi\colon Y \longrightarrow V$.*

Then f is a composition of elementary links.

An elementary link is one of an elementary tranformation, blowing up a point of $\mathbb{P}^2$ and switching which factor of $\mathbb{P}^1 \times \mathbb{P}^1$ we project down to. The key point about Theorem 3 is that the intermediary links of the factorisation of f are all Mori fibre spaces themselves. It is quite instructive to factor the Cremona transformation

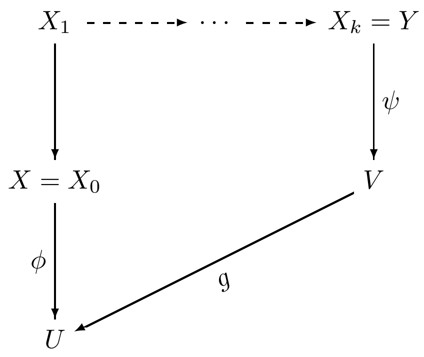
$$f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \qquad \text{where} \qquad [X : Y : Z] \longrightarrow [X^{-1} : Y^{-1} : Z^{-1}],$$

into a sequence of elementary transformations. Perhaps one of the most interesting applications of Theorem 3 is to a proof of the following classical result:

Theorem 4. *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *of birational automorphisms of* $\mathbb{P}^2$ *is generated by the Cremona transformation and* $\mathrm{PGL}(3)$.

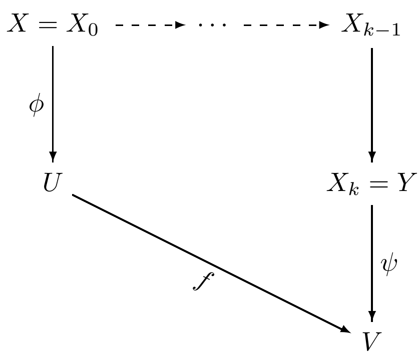
Sarkisov was the first to realise that such a result should hold in all dimensions. We recall the definition of the Sarkisov links I-IV:

(I)



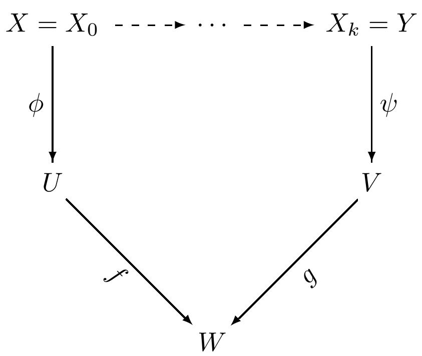
where $X_1 \longrightarrow X$ is an extremal divisorial contraction, $X_1 \dashrightarrow X_k = Y$ is a sequence of flops and $\rho(V/U) = 1$. A Sarkisov link of type II is the mirror reflection of this diagram in a central vertical line.

(III)



where $X = X_0 \dashrightarrow X_{k-1}$ is a sequence of flips, $X_{k-1} \longrightarrow X_k = Y$ is an extremal divisorial contraction and $\rho(U/V) = 1$.

(IV)



Note that the blow up of a point of $\mathbb{P}^2$ is a link of type I, the blow down is a link of type II, an elementary tranformation is a link of type III, and switching the factors is a link of type IV.

Theorem 5 (Corti, Hacon, Iskovskikh, M^cKernan, Sarkisov, Shokurov)**.** *Let $f\colon X \dashrightarrow Y$ be a birational map between two Mori fibre spaces.*

Then f is a composition of Sarkisov links.

This result was proved by Corti [2] in dimension three, and some special cases were proved by Sarkisov in all dimensions and independently by Iskovskikh and Shokurov. The key trick to prove Theorem 5 is to realise the intermediary links as log terminal models, for an appropriate choice of divisors on some common resolution W of X and Y. The result then follows by finiteness of these models, which is proved in [1]. It seems worth pointing out though that we do not even have a putative set of generators of $\mathrm{Bir}(\mathbb{P}^3)$.

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The Kodaira dimension of the moduli space of Prym varieties

Gavril Farkas

We consider the moduli stack $\mathcal{R}_g$ parametrizing pairs (C,η) where $[C]\in\mathcal{M}_g$ is a smooth curve and $\eta\in \mathrm{Pic}^0(C)[2]$ is a torsion point of order 2 giving rise to an étale double cover of C. We denote by $\pi:\mathcal{R}_g\to\mathcal{M}_g$ the natural projection forgetting the point of order 2 and by $P:\mathcal{R}_g\to\mathcal{A}_{g-1}$ the Prym map given by

$$P(C,\eta):=\mathrm{Ker}\{f_*:\mathrm{Pic}^0(\widetilde{C})\to\mathrm{Pic}^0(C)\}^0,$$

where $f:\widetilde{C}\to C$ is the étale double covering determined by η. It is known that P is generically injective for $g\geq 7$ (cf. [FS]), hence one can view $\mathcal{R}_g$ as a birational model for the moduli stack of Prym varieties of dimension $g-1$. If $\overline{\mathcal{R}}_g$ the normalization of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_g$, then it is known that $\overline{\mathcal{R}}_g$ is isomorphic to the stack of Beauville admissible double covers (cf. [B]), and also to the stack of Prym curves in the sense of [BCF]. It is known that the space $\mathcal{R}_g$ is unirational for $g\leq 6$ (cf. [D]) and the main result of this paper is the following:

Theorem 1. *The moduli space $\overline{\mathcal{R}}_g$ is of general type for all $g>13$.*

The strategy of the proof is similar to the one used by Harris and Mumford for proving that $\overline{\mathcal{M}}_g$ is of general type for large g (cf. [HM]). One first computes the canonical class $K_{\overline{\mathcal{R}}_g}$ in terms of the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)$ and then shows that $K_{\overline{\mathcal{R}}_g}$ is effective for $g>13$ by explicitly computing the class of a specific effective divisor on $\overline{\mathcal{R}}_g$ and comparing it to $K_{\overline{\mathcal{R}}_g}$. The divisors we construct are of two types, dpending on whether g is even or odd. In an appendix, K. Ludwig will show that for $g\geq 4$ any pluricanonical form on $\overline{\mathcal{R}}_{\mathrm{g,reg}}$ automatically extends to any desingularization. This is a key ingredient in carrying out the program of computing the Kodaira dimension of $\overline{\mathcal{R}}_g$.

In the odd genus case we we set $g = 2i+1$ and consider the vector bundle Q_C defined by the exact sequence

$$0 \longrightarrow Q_C^{\vee} \longrightarrow H^0(K_C) \otimes \mathcal{O}_C \to Q_C \longrightarrow 0.$$

(If other words, Q_C is the normal bundle of C embedded in its Jacobian). It is well-known that Q_C is a semi-stable vector bundle of rank $g-1$ on C of slope $\nu(Q_C) = 2 \in \mathbb{Z}$, so it makes sense to look at the theta divisors of its exterior powers. Recall that

$$\Theta_{\wedge^i Q_C} = \{\xi \in \mathrm{Pic}^{g-2i-1}(C) : h^0(C, \wedge^i Q_C \otimes \xi) \geq 1\},$$

and the main result from [FMP] identifies this locus with the difference variety $C_i - C_i \subset \mathrm{Pic}^0(C)$.

Theorem 2. *For $g+2i+1$, the locus E_i consisting of those points $[C,\eta] \in \mathcal{R}_{2i+1}$ such that $\eta \in \Theta_{\wedge^i Q_C}$, is an effective divisor on $\mathcal{R}_{2i+1}$. Its class on $\overline{\mathcal{R}}_{2i+1}$ is given by the formula*

$$E_i \equiv \frac{2}{i}\binom{2i-2}{i-1} \cdot \Big((3i+1)\lambda - \frac{i}{2}\delta_0^u - \frac{2i+1}{4}\delta_0^r - (\text{ higher boundary divisors})\Big).$$

This proves our main result in the odd genus case. The diviors we consider for even genus are of Koszul type in the sense of [F].

Theorem 3. *For $g = 2i+6$, the locus D_i of those $[C,\eta] \in \mathcal{R}_{2i+6}$ such that the Koszul cohomology group $K_{i,2}(C, K_C+\eta)$ does not vanish (or equivalently, $(C, K_C+\eta)$ fails the Green-Lazarsfeld property (N_i)), is a virtual divisor on $\mathcal{R}_{2i+6}$. Its class on $\overline{\mathcal{R}}_{2i+6}$ is given by the formula:*

$$D_i \equiv \frac{1}{2}\binom{2i+2}{i}\Big(\frac{6(2i+7)}{i+3}\lambda - 2\delta_0^u - 3\delta_0^r - \cdots\Big).$$

In both Theorems 2 and 3, $\lambda \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ denotes the Hodge class and $\pi^*(\delta_0) = \delta_0^u + 2\delta_0^r$ (that is δ_0^r is the ramification divisor of π whereas δ_0^u is the unramified part of the pull-back of the boundary divisor δ_0 from $\overline{\mathcal{M}}_g$.). The boundary divisors δ_0^u and δ_0^r have clear modular description in terms of Prym curves and the same holds for the higher boundary divisors.

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The computation of ...

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Canonical coordinates in mirror symmetry

Bernd Siebert
(joint work with Mark Gross)

Mirror symmetry is a statement relating the complex geometry of an algebraic variety to the symplectic geometry of a mirror partner and conversely. First limited to pairs of three-dimensional Calabi-Yau varieties, mirror phenomena have been observed in arbitrary dimensions and with varieties with only effective anticanonical class; in the latter case the mirror is a non-compact variety together with a holomorphic function.

Finer statements of this sort require to identify the moduli space of complex structures on one side with the (complexified) moduli space of deformations of the symplectic or rather the Kähler structure. Under this identification flat complexified Kähler parameters correspond to what are called *canonical coordinates* on the complex side, which are constructed by certain period integrals. For example, the celebrated computation in [2] of the numbers of rational holomorphic curves (genus zero Gromov-Witten invariants) on the quintic threefold works by expanding the so-called Yukawa-coupling on the family of mirror quintics with respect to canonical coordinates.

In [4] Mark Gross and myself laid the foundations for a program providing a general framework for the study of mirror phenomena. The basic idea is to use degenerations of the considered varieties into simpler (toric) pieces. For mirror symmetry for complete varieties with trivial canonical bundle, we look at so-called *toric degenerations*. These are degenerations with central fiber a union of toric varieties, glued torically along pairs of toric divisors, and such that the map to the base is toroidal near the zero-dimensional toric strata. The simplest interesting example is a sufficiently general degeneration of a quartic in $\mathbb{P}^3$ into a union of four hyperplanes. To such a degeneration we associate a combinatorial object, the *dual intersection complex* of the central fiber. This is a cell complex $\mathscr{P}$ of integral, convex polyhedra, together with a compatible structure of a complete fan at each vertex. The underlying topological space B is then a manifold, and it comes with a well-defined integral affine structure (transition functions in $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}(\mathbb{Z}^n)$) outside a closed, polyhedral subset $\Delta \subset B$ of codimension two.

In fashionable terms it is appropriate to call this data $(B, \mathscr{P})$ an *integral tropical manifold*.

To obtain mirror symmetry one needs a polarization on the degeneration. This leads to a (multi-valued) strictly convex, piecewise linear function φ on B. The basic duality between (I) integral, bounded convex polyhedra and (II) pairs consisting of a complete fan and an integral, convex, piecewise linear function on it, leads to a perfect Legendre-type duality on *polarized tropical manifolds*:

$$(B, \mathscr{P}, \varphi) \longleftrightarrow (\check{B}, \check{\mathscr{P}}, \check{\varphi}).$$

This provides the basic mirror mechanism: Toric degenerations with Legendre dual degeneration data are mirror dual. It can be viewed as an algebraic-geometric, limit version of the differential geometric SYZ-approach to mirror symmetry [8]. Mark Gross has shown that the largest class of known mirror pairs, complete intersections in toric varieties [1], fits into this framework [3]. Moreover, preliminary results on other cases (local mirror symmetry, Fano/Landau-Ginzburg duality, and even mirror phenomena of varieties of general type) suggest that this idea should work in complete generality.

In [5] we closed the main missing link in this picture by showing that, under certain natural conditions, any $(B, \mathscr{P}, \varphi)$ arises as the dual intersection of an *explicit, canonical* toric degeneration. This gives complete control of the complex side of mirror symmetry. Such canonical families were known from toric methods only for toric and abelian varieties (Mumford). These are, in a sense, linear cases, and a similarly direct method does certainly not work for proper Calabi-Yau varieties. This is directly related to the fact that the affine structure on $B \setminus \Delta$ has non-trivial local monodromy around $\Delta \subset B$. (The linear cases have $\Delta = \emptyset$.) Therefore, local models for the toric degeneration suggested by toric geometry do not patch. The insight in [5] is that tropical geometry on $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ provides a way to making the necessary adjustments canonically. Some inspiration for this came from the work of Kontsevich and Soibelman [7], where a rigid analytic $K3$-surface is constructed out of an affine structure on S^2 minus 24 singular points.

In the talk I argued that the canonical one-parameter families from our construction readily provide canonical coordinates. The main point is that it is easy to control the relevant period integrals over a large class of n-cycles throughout our algorithm. What is currently missing to make this a theorem is to check that the n-cycles of this form span the relevant subspace W_2 of the monodromy weight filtration on the middle homology. This will be addressed in the forthcoming paper [6].

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Mixed Tate motives and the fundamental group

Hélène Esnault
(joint work with Marc Levine)

Abstract: Let k be a number field, and let $S \subset \mathbb{P}^1(k)$ be a finite set of rational points. We relate the Deligne-Goncharov contruction of the motivic fundamental group of $X := \mathbb{P}^1 \setminus S$ to the Tannaka group scheme over $\mathbb{Q}$ of the category of mixed Tate motives over X.

More presicely, let $MT(k)$ be the full abelian subcategory of Voevodksy's category $DM_{gm}(k)$ of geometric motives over k. It has been defined by Marc Levine [6], based on Borel's theorem saying-in more modern language- that number fields satisfy the Beilinson-Soulé vanishing theorem. This is a $\mathbb{Q}$-linear, abelian, tensor rigid category, which is endowed with a natural neutral fiber functor gr_W associated to the weight filtration of a motive. Let $G(MT(k), gr_W)$ be its Tannaka group scheme over $\mathbb{Q}$. If X is as described, localization shows that it also satisfies the Beilinson-Soulé vanishing theorem. Cisinski-Déglise's definition of $DM_{gm}(X)$ allows to define $MT(X) \subset DM_{gm}(X)$ in the same way as over k. So one has its Tannaka group scheme $G(MT(X), gr_W)$ over $\mathbb{Q}$. The structure morphism $\epsilon : X \to \mathrm{Spec}(k)$ yields a surjective homomorphism $\epsilon_* : G(MT(X), gr_W) \to G(MT(k), gr_W)$. Thus any section s of ϵ_* defines $K := \mathrm{Ker}(\epsilon_*)$, which is a group scheme over $\mathbb{Q}$, as a representation of $G(MT(k), gr_W)$, thus $\mathbb{Q}[K]$ as a ind-representation of $G(MT(k), gr_W)$, and thus as an ind-object of $MT(k)$. Let K_s be the corresponding pro-groupscheme object in $GM(k)$. One shows

Theorem 1. *If s is the section associated to a rational point $a \in X(k)$, then K_s is isomorphic in $MT(k)$ to Deligne/Deligne-Goncharov motivic fundamental group scheme $\pi_1^{\mathrm{mot}}(X, a)$.*

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Enriques surfaces covered by Kummer's quartics

Shigeru Mukai
(joint work with H. Ohashi)

An Enriques surface is a quotient of a K3 surface by a (fixed point) free involution. It determines the K3 surface uniquely as its universal cover but not vice versa. In this talk we give an answer to the following problem in the case of a *very general* Jacobian Kummer surface.

Problem Given a K3 surface X, how many Enriques surfaces are obtained by taking quotient of X? Equivalently, how many conjugacy classes of free involutions are there in the automorphism group of X? Describe all Enriques quotients of X as explicit as possible.

Let $J(C)$ be the Jacobian of a (smooth projective) curve C of genus 2. Its image $\overline{Km}\, C \subset \mathbb{P}^3$ by the linear system $|2\Theta|$ is a quartic surface with 16 nodes and called *Kummer's quartic*. We denote the minimal resolution by $Km\, C$. It is known that $Km\, C$ is the intersection of three quadrics

$$(*) \qquad \sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} \lambda_i x_i^2 = \sum_{i=1}^{6} \lambda_i^2 x_i^2 = 0$$

in $\mathbb{P}^5$. (The coordinates x_i's correspond to the 6 Weierstrass points of C.)

'Theorem' *Assume that the Picard number of $J(C)$ is equal to 1. Then there are exactly* 31 *Enriques quotients of the Jacobian Kummer surface $Km\, C$. Moreover, they are $Km\, C/\varepsilon_G$, $Km\, C/Sw_\eta$ and $Km\, C/\varepsilon_W$ obtained from*

(1) *15 Göpel subgroups G of the 2-torsion group $J(C)_{(2)}$,*
(2) *10 even theta characteristics η of C, and*
(3) *6 cubic surfaces $S_W \subset \mathbb{P}^3$ whose Hesians $\tilde{H}(S_W)$ are isomorphic to $Km\, C$.*

Here we explain the 31 free involutions ε_G, Sw_η and ε_W briefly.

(1) A subgroup $G \subset J(C)_{(2)}$ of order 4 is called *Göpel* if the Weil pairing is identically zero on G. ε_G is induced from a *standard Cremona involution* of $\mathbb{P}^3$ with center the 4 nodes of $\overline{Km}\, C$ corresponding to G ([2], [3]).
(2) An even theta characteristic η corresponds to a partition of the 6 Weierstrass points into two parts of cardinality 3. The *switch* Sw_η in the theorem is the involution changing the three coordinates of x_i's in (*) belonging to one of two parts corresponding to η.

(3) A certain hexad of nodes of $\overline{Km}\, C$, called *a Weber hexad*, defines a birational embedding $Km\, C$ into $\mathbb{P}^3$ whose image is the Hessian quartic $H(S_W)$ of a cubic surface $S_W \subset \mathbb{P}^3$. $H(S_W)$ is defined by the two equations

$$\sum_1^5 x_i = \sum_1^5 \frac{a_i}{x_i} = 0$$

in $\mathbb{P}^4$ for nonzero constants $a_1, \ldots, a_5 \in \mathbb{C}$. The a free involution ε_W is induced from the standard Cremona involution $(x_i) \mapsto (a_i/x_i)$ of $\mathbb{P}^4$ (cf. [1]). There are 12 Weber hexads W modulo the translation by $J(C)_{(2)}$. These 12 hexads decomposes into six pairs such that two hexads in the same pair define the same Enriques quotients.

The theorem was conjectured and proved in the case where *the patching group* is of type $(2, 2)$ in my study of rank one involutions [4]. (Rank one involution is the next step of the numerically trivial, or rank zero, involution towards the classification of all involutions of Enriques surfaces.) The general case has been recently (almost) proved by Hisanori Ohashi.

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A characterizaion of Shimura varieties

Martin Möller

(joint work with Eckart Viehweg, Kang Zuo)

Let Y be a complex projective manifold of dimension n, and let U be the complement of a normal crossing divisor S. We are interested in families $f : A \to U$ of abelian varieties, up to isogeny, and we are looking for numerical invariants which take the minimal possible value if and only if U is a Shimura variety of certain type, or to be more precise, if $f : A \to U$ is a Kuga fibre space. Those invariants will be attached to $\mathbb{C}$-subvariations of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$. We will always assume that the family has semistable reduction in codimension one, hence that the local system $R^1 f_* \mathbb{C}_A$ has unipotent monodromy in the general points of the components of S.

The most important numerical invariant will be the slope of $\mathbb{V}$ or of the Higgs bundle (E, θ). Recall that the slope $\mu(\mathcal{F})$ of a torsion free coherent sheaf $\mathcal{F}$ on Y, is defined by

$$\Upsilon(\mathcal{F}) = \frac{\mathrm{c}_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \in H^2(Y, \mathbb{Q}) \quad \text{and} \quad \mu(\mathcal{F}) = \Upsilon(\mathcal{F}).\mathrm{c}_1(\omega_Y(S))^{\dim(Y)-1}.$$

We write

$$\mu(\mathbb{V}) := \mu(E^{1,0}) - \mu(E^{0,1}).$$

We require some positivity properties of the sheaf of differential forms on the compactification Y of U:

Assumptions 1. *Y is a connected projective manifold and U is the complement of a normal crossing divisor S such that:*

- *$\Omega^1_Y(\log S)$ is nef and $\omega_Y(S)$ is ample with respect to U.*

If the universal covering $\pi : \tilde{U} \to U$ is a bounded symmetric domain, hence isomorphic to $M_1 \times \cdots \times M_s$ for irreducible bounded symmetric domains M_i of dimension n_i, Mumford constructed in [Mu77, Section 4] a non-singular compactification satisfying the Assumption 1. We will call it the *Mumford compactification* in the sequel. The Mumford compactification has the following property:

Condition 2.

- *$\Omega^1_Y(\log S)$ is μ-polystable. If $\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_{s'}$ is the decomposition as a direct sum of stable direct factors, then $s = s'$ and for a suitable choice of the indices the pullback of $\Omega_i|_U$ to $\tilde{U}$ coincides with $\mathrm{pr}_i^* \Omega^1_{M_i}$.*

In particular the Mumford compactification exists for a Shimura variety of Hodge type or for the base of a Kuga fibre space. The following properties of Shimura varieties are presumably somehow known, they can serve as a “Leitfaden” for how Shimura varieties can be characterized.

Proposition 3. *Let $f : A \to U$ be a Kuga fibre space, such that $\mathbb{W} = R^1 f_* \mathbb{C}_A$ has unipotent local monodromies at infinity. Then there exists a compactification Y satisfying the Assumption 1 and the Condition 2 such that for all irreducible non-unitary $\mathbb{C}$ subvariation of Hodge structures $\mathbb{V}$ of $\mathbb{W}$ with Higgs bundle (E, θ) one has:*

i. *There exists some $i = i(\mathbb{V})$ such that the Higgs field θ factorizes through*

$$\theta : E^{1,0} \;>> E^{0,1} \otimes \Omega_i \;\subset>> E^{0,1} \otimes \Omega^1_Y(\log S).$$

(We say that Θ is pure of type i in this case*)*

ii. *The "Arakelov equality" $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$ holds.*

iii. *Assume for $i = i(\mathbb{V})$ that M_i is a complex ball of dimension $n_i \geq 1$. Then the length of the iterated Kodaira-Spencer map equals*

$$\varsigma(\mathbb{V}) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_i + 1)}{\mathrm{rk}(E) \cdot n_i}.$$

The "Arakelov Equality" in ii) will be our main condition. It is valid independently of the compactification. Assume that U has a compactification Y satisfying the Assumptions 1. This allows to apply Yau's Uniformization Theorem (as recalled in [VZ07, Theorem 1.4]). In particular the sheaf $\Omega^1_Y(\log S)$ is μ-polystable and the Condition 2 holds true. So one has again a direct sum decomposition

$$\Omega^1_Y(\log S) = \Omega_1 \oplus \cdots \oplus \Omega_s. \tag{1}$$

in stable sheaves of rank $n_i = \mathrm{rk}(\Omega_i)$. We say that Ω_i *is of type A*, if it is invertible, and *of type B*, if $n_i > 1$ and if for all $m > 0$ the sheaf $S^m(\Omega_i)$ is stable. Finally it is *of type C* in the remaining cases, i.e. if for some $m > 1$ the sheaf $S^m(\Omega_i)$ is unstable.

Let again $\pi : \tilde{U} \to U$ denote the universal covering with covering group Γ. The decomposition (1) of $\Omega^1_Y(\log S)$ gives rise to a product structure

$$\tilde{U} = M_1 \times \cdots \times M_s, \tag{2}$$

where $n_i = \dim(M_i)$. If $\tilde{U}$ is a bounded symmetric domain, the M_i in (2) are irreducible bounded symmetric domains, and on a Mumford compactification the decomposition (1) coincides with the one in Property 2.

Yau's Uniformization Theorem gives a criterion for the M_i to be bounded symmetric domains. In fact, if Ω_i is of type A, M_i is a one-dimensional complex ball, and it is a bounded symmetric domain of rank > 1, if Ω_i is of type C.

If Ω_i is of type B, then M_i is a n_i-dimensional complex ball if and only if

$$\big[2 \cdot (n_i + 1) \cdot \mathrm{c}_2(\Omega_i) - n_i \cdot \mathrm{c}_1(\Omega_i)^2\big].\mathrm{c}(\omega_Y(S))^{\dim(Y)-2} = 0. \tag{3}$$

Fix an irreducible polarized $\mathbb{C}$-variation of Hodge structures $\mathbb{V}$ on U of weight one and with Higgs bundle (E, θ). By [VZ07, Theorem 1] one has the Arakelov type inequality

$$\mu(\mathbb{V}) = \mu(E^{1,0}) - \mu(E^{0,1}) \leq \mu(\Omega^1_Y(\log S)). \tag{4}$$

We can now state a first part of a converse of Proposition 3.

Theorem 4. *Under the Assumptions 1 consider an irreducible polarized $\mathbb{C}$-variation of Hodge structures $\mathbb{V}$ of weight one with unipotent monodromy at infinity. If $\mathbb{V}$ satisfies the Arakelov equality then $\mathbb{V}$ is pure for some $i = i(\mathbb{V})$.*

The proof of Theorem 4 makes use of small twists of the slopes $\mu(\mathcal{F})$ and the behaviour of the Harder-Narasimhan filtration under such twists.

Finally we will obtain the numerical characterization of Kuga fibre spaces in the following form.

Theorem 5. *Let $f : A \to U$ be a smooth family of abelian varieties, such that the induced morphism $U \to \mathcal{A}_g$ is generically finite. Assume that U has a projective compactification Y satisfying the Assumptions 1.*

Then $f : A \to U$ is a Kuga fibre space if and only if for each irreducible subvariation of Hodge structures $\mathbb{V}$ of $R^1 f_* \mathbb{C}_A$ with Higgs bundle (E, θ) one has:

1. *If $\mathbb{V}$ is non-unitary, the Arakelov equality $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$ holds.*

2. *For each stable direct factor Ω_j of $\Omega^1_Y(\log S)$ of type B either the composition*

$$\theta_j : E^{1,0}\, \theta >> E^{0,1} \otimes \Omega^1_Y(\log S)\; \mathrm{pr} >> E^{0,1} \otimes \Omega_j$$

is zero, or

$$\varsigma((E, \theta_j)) = \frac{\mathrm{rk}(E^{1,0}) \cdot \mathrm{rk}(E^{0,1}) \cdot (n_j + 1)}{\mathrm{rk}(E) \cdot n_j}.$$

If in addition $f : A \to U$ is infinitesimally rigid U is a Shimura variety of Hodge type.

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Absolute Galois changes the fundamental group as much as possible

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(joint work with F. Catanese and F. Grunewald)

1. Motivation

The key slogan of the following is: *the absolute Galois group acts on the set of components of moduli spaces*, e.g., let $\mathfrak{M}_{x,y}$ be the moduli space of isomorphism classes of minimal complex surfaces S of general type with $K_S^2 = x$, $\chi(\mathcal{O}_S) = y$. It is wellknown that $\mathfrak{M}_{x,y}$ is defined over the integers and therefore the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of irreducible (or connected) components of $\mathfrak{M}_{x,y}$.

In particular, $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the 0-dimensional components of $\mathfrak{M}_{x,y}$, the *rigid* surfaces. Try to understand the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

Define:

$$\mathfrak{M} := \bigcup_{x,y} \mathfrak{M}_{x,y}.$$

There are the following natural

Question 1. *1) Given a variety defined over $\bar{\mathbb{Q}}$, which topological invariants of the corresponding complex space are preserved by the absolute Galois group?*
2) Is the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathfrak{M}$ faithful?
3) Is the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ faithful on rigid surfaces?

2. Some known results

1) If X is a nonsingular projective variety, then the Betti numbers are preserved by the absolute Galois group (Serre).

2)The profinite completion of the fundamental group of an algebraic variety is invariant under $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$; more generally, the profinite completion of the homotopy type of X is invariant under Galois conjugation (Artin-Mazur, [3]).

3) In the 60's J.P. Serre (cf. [8]) gave an elegant example of a smooth variety X (defined over $\bar{\mathbb{Q}}$) and a $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ such that the fundamental groups of the complex manifolds X and $(X)^\sigma$ are not isomorphic. In particular, X and $(X)^\sigma$ are not homeomorphic.

4) There are further recent examples of Galois conjugate non homeomorphic varieties, e.g., recently by Artal-Bartolo, Carmona Ruber, Cogolludo Augustin (cf. [2].

5) Abelson (cf. [1]) gave examples of Galois conjugate (nonsingular projective) varieties with the same fundamental group, yet of different homotopy type, and examples of conjugate (nonsingular quasiprojective) varieties which are homotopy equivalent, but not homeomorphic.

6) $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on coverings of the projective line branched only over $\{0, 1, \infty\}$; (*Grothendieck's dessins d'enfants*).

7) E. Girondo and G. Gonzalez-Diez: $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithful on dessins of any given genus g (cf. [7]).

8) R. Easton and R. Vakil show that the absolute Galois group acts faithfully on the set of irreducible components of $\mathfrak{M}$ (cf. [6]).

3. An explicit example

In this section we provide, an explicit example of surfaces with nonisomorphic fundamental groups which are conjugate under the absolute Galois group, hence with isomorphic profinite completions of their respective fundamental groups.

We consider (as in [4]) *normalized polynomials* $P(z) := z^n + a_{n-2}z^{n-2} + \dots a_0$ with only critical values $\{0, 1\}$. Once we choose the types of the respective cycle decompositions $(m_1, \dots, m_r)$ and $(n_1, \dots, n_s)$ of the respective local monodromies over 0 and 1, we can write our polynomial P in two ways, namely as: $P(z) = \prod_{i=1}^{r}(z - \beta_i)^{m_i}$, and $P(z) - 1 = \prod_{k=1}^{s}(z - \gamma_k)^{n_k}$.

Comparing variables we obtain a set $\mathbb{W}(n; (m_1, \dots, m_n), (n_1, \dots, n_s))$ in affine $(n-1)$ space, parametrizing these polynomials. This algebraic set is defined over $\mathbb{Q}$ since by Riemann's existence theorem they are either empty or have dimension 0 (we refer to [4] for more details).

Example 1. *We calculate (e.g., using MAGMA) that* $\mathbb{W}(7; (2, 2, 1, 1, 1); (3, 2, 2))$ *is irreducible over* $\mathbb{Q}$*, which implies that* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ *acts transitively on* $\mathbb{W}$*. Looking at the possible monodromies, one sees that there are exactly two real non equivalent polynomials (corresponding to the two orbits of the group of 7-th roots of unity). The two permutations of types* $(2, 2)$ *and* $(3, 2, 2)$ *are seen to generate* $\mathfrak{A}_7$ *and the* *respective normal closures of the two polynomial maps are easily seen to give (since the automorphism group of $\mathfrak{A}_7$ is $\mathfrak{S}_7$) nonequivalent triangle curves C_1, C_2.*

By Hurwitz's formula, we see that $g(C_i) = \frac{|\mathfrak{A}_7|}{2}(3 - \frac{1}{2} - \frac{1}{6} - \frac{1}{7}) + 1 = 241$.

We remark that $\mathfrak{A}_7$ has generators a_1, a_2 of order 5 such that their product has order five, yielding a triangle curve C (of genus 505). An easy MAGMA routine shows that there is exactly one Hurwitz class of triangle curves given by a spherical system of generators of type $(5,5,5)$ of $\mathfrak{A}_7$.

Obviously, $\mathfrak{A}_7$ acts freely on $C_1 \times C$ as well as on $C_2 \times C$ and we obtain two Beauville surfaces S_1, S_2, which are not diffeomorphic and therefore have non isomorphic fundamental groups by [5].

Proposition 1. *There is a field automorphism $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $S_2 = (S_1)^\sigma$.*

Proof. We know that $\sigma(S_1) = ((C_1)^\sigma \times (C)^\sigma)/G$. Since there is only one isomorphism class of triangle curves given by a spherical system of generators of type $(5,5,5)$ of $\mathfrak{A}_7$, we have $(C)^\sigma \cong C$. □

We give now explicitly the fundamental groups of S_1 and S_2.

We choose an arbitrary spherical system of generators of type $(5,5,5)$ of $\mathfrak{A}_7$, for instance $((1,7,6,5,4),(1,3,2,6,7),(2,3,4,5,6))$.

A MAGMA routine shows that

$$((1,2)(3,4),(1,5,7)(2,3)(4,6),(1,7,5,2,4,6,3))$$

and

$$((1,2)(3,4),(1,7,4)(2,5)(3,6),(1,3,6,4,7,2,5))$$

are two representatives of spherical generators of type $(2,6,7)$ yielding two non isomorphic triangle curves C_1 and C_2. The two corresponding homomorphisms $\Phi_1 : T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7$ and $\Phi_2 : T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7$ give two exact sequences ($i = 1,\ 2$)

$$1 \to \pi_1(C_i) \times \pi_1(C) \to T_{(2,6,7)} \times T_{(5,5,5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,$$

yielding two non isomorphic fundamental groups $\pi_1(S_1) = \Phi_1^{-1}(\Delta_{\mathfrak{A}_7})$ and $\pi_1(S_2) = \Phi_2^{-1}(\Delta_{\mathfrak{A}_7})$, where $\Delta_{\mathfrak{A}_7}$ is the diagonal of $\mathfrak{A}_7 \times \mathfrak{A}_7$ (cf. [5]), fitting both in an exact sequence of type

$$1 \to \Pi_{241} \times \Pi_{505} \to \pi_1(S_j) \to \Delta_{\mathfrak{A}_7} \cong \mathfrak{A}_7 \to 1,$$

where $\Pi_{241} \cong \pi_1(C_1) \cong \pi_1(C_2)$, $\Pi_{505} = \pi_1(C)$.

Remark 1. *1) Using a surjection of a group $\Pi_g \to \mathfrak{A}_7$ we get infinitely many examples of pairs of fundamental groups which are nonisomorphic, but which have isomorphic profinite completions. Each pair fits into an exact sequence*

$$1 \to \Pi_{241} \times \Pi_{g'} \to \pi_1(S_j) \to \mathfrak{A}_7 \to 1.$$

2) Many more explicit examples as the one above (but with cokernel group different from $\mathfrak{A}_7$) can be obtained using polynomials with two critical values.

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Intersection theory of divisors on compactifications of $\mathcal{A}_g$

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(joint work with Cord Erdenberger, Samuel Grushevsky)

The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is a quasi-projective variety. Several compactifications are known, notably the Satake (or minimal) compactification $\mathcal{A}_g^{\text{Sat}}$ and toroidal compactifications such as the second Voronoi compactification $\mathcal{A}_g^{\text{Vor}}$, the central cone compactification $\mathcal{A}_g^{\text{Centr}}$ or the perfect cone compactification $\mathcal{A}_g^{\text{Perf}}$. Alexeev [1], see also Olsson [6], showed that $\mathcal{A}_g^{\text{Vor}}$ represents a moduli functor. The central cone compactification $\mathcal{A}_g^{\text{Centr}}$ is known to coincide with the Igusa compactification, which is a partial desingularization of the Satake compactification. Finally, Shepherd-Barron [7] proved that $\mathcal{A}_g^{\text{Perf}}$ is a canonical model of $\mathcal{A}_g$ in the sense of Mori theory if $g \geq 12$.

The Picard group of $\mathcal{A}_g^{\text{Perf}}$ is very simple, namely

$$\text{Pic}(\mathcal{A}_g^{\text{Perf}}) \otimes \mathbb{Q} = \mathbb{Q}L + \mathbb{Q}D$$

where L is the Hodge line bundle and D is the boundary divisor. In view of this Shepherd-Barron [7, p. 41] posed the question to determine the intersection theory of divisors on $\mathcal{A}_g^{\text{Perf}}$. This amounts to computing the numbers

$$a_N^{(g)} = \langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\text{Perf}}}$$

where $G = g(g+1)/2 = \dim \mathcal{A}_g$. Our main result is

Theorem 1. *The only three intersection numbers with $N < 3g-3$ that are non-zero are those for $N = 0, g, 2g-1$ (and thus the power of L being equal to $\dim \mathcal{A}_g, \dim \mathcal{A}_{g-1}$, and $\dim \mathcal{A}_{g-2}$, respectively). The numbers are*

$$a_0^{(g)} = \langle L^{\frac{g(g+1)}{2}} \rangle_{\mathcal{A}_g^{\mathrm{Perf}}} = (-1)^G 2^{-g} G! \prod_{k=1}^{g} \frac{\zeta(1-2k)}{(2k-1)!!} \tag{1}$$

$$a_g^{(g)} = \langle L^{\frac{(g-1)g}{2}} D^g \rangle_{\mathcal{A}_g^{\mathrm{Perf}}} = \frac{1}{2}(-1)^{G-1}(g-1)!(G-g)! \prod_{k=1}^{g-1} \frac{\zeta(1-2k)}{(2k-1)!!} \tag{2}$$

and

$$a_{2g-1}^{(g)} = \langle L^{\frac{(g-2)(g-1)}{2}} D^{2g-1} \rangle_{\mathcal{A}_g^{\mathrm{Perf}}} = (I) + (II) + (III) \tag{3}$$

where the terms (I), (II) and (III) can be computed explicitly.

For $g = 2, 3$ van der Geer [4] has computed the Chow ring of $\mathcal{A}_g^{\mathrm{Perf}}$ which in these cases coincides with the other two toroidal compactifications. In [2] we determined the intersection theory of divisors not only for $\mathcal{A}_4^{\mathrm{Perf}} = \mathcal{A}_4^{\mathrm{Centr}}$, but also for $\mathcal{A}_4^{\mathrm{Vor}}$. It should also be noted that the number $a_0^{(g)} = \langle L^{\frac{g(g+1)}{2}} \rangle_{\mathcal{A}_g^{\mathrm{Perf}}}$ is essentially the Hirzebruch-Mumford volume of the symplectic group and has as such been known to Siegel [8]. The above result also holds, properly formulated, for all "reasonable" toroidal compactifications of $\mathcal{A}_g$.

The most striking result of our computations is that

$$\langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\mathrm{Perf}}} = 0 \quad \text{unless} \quad G - N = \dim \mathcal{A}_k \quad \text{for some } k \leq g \tag{4}$$

in the range $N < 3g-3$. Note that $\mathcal{A}_g^{\mathrm{Sat}}$ has the natural stratification

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_0.$$

This leads one naturally to

Conjecture 1. *The intersection numbers $a_N^{(g)}$ for any N vanish unless $G - N = k(k+1)/2$ for some $k \leq g$, i. e. unless $G - N$ equals the dimension of a stratum of the Satake compactification.*

One can also ask this question for other toroidal compactifications of $\mathcal{A}_g$, and it is tempting to conjecture that, if one interprets D as the closure of the boundary of the partial compactification, this still holds. Of course, one could even hope that such a vanishing result holds for (reasonable) toroidal compactifications of any quotient of a homogeneous domain by an arithmetic group. This is e. g. the case for the moduli space of polarized K3 surfaces. However, in this case the Baily-Borel (or minimal) compactification has only two boundary strata, which are of dimension 0 and 1 respectively and this easily implies the vanishing.

Our approach to computing intersection numbers is based on an analysis of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}$. Recall that every toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ admits a map $\pi : \mathcal{A}_g^{\mathrm{tor}} \to \mathcal{A}_g^{\mathrm{Sat}}$. Let $\beta_k = \pi^{-1}(\mathcal{A}_{g-k})$. The set $\mathcal{A}_g^{\mathrm{Part}} = \mathcal{A}_g^{\mathrm{tor}} \setminus \beta_2$ is Mumford's partial compactification and is independent of the chosen toroidal compactification. The boundary $D' = \mathcal{A}_g^{\text{tor}} \setminus \beta_2$ is the universal Kummer family over $\mathcal{A}_{g-1}$. More precisely, if $\pi : \mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ is the universal abelian variety (which exists in a stack sense) then there is a map $j : \mathcal{X}_{g-1} \to \mathcal{A}_g^{\text{Part}}$ such that $j_*([\mathcal{X}_{g-1}]) = 2D'$ as cycles. Note that we consider $\mathcal{A}_g$ as a stack with a non-trivial involution which comes from the fact that every abelian variety possesses an involution.

Our proof is based on two key observations: the first is that the universal family itself allows a partial compactification $\pi : \mathcal{X}_{g-1}^{\text{Part}} \to \mathcal{A}_g^{\text{Part}}$, which is obtained by adding corank 1 degenerations, such that there is a map $j : \mathcal{X}_{g-1}^{\text{Part}} \to \mathcal{A}_g^{\text{Perf}}$ with $j(\mathcal{X}_{g-1}^{\text{Part}}) = \mathcal{A}_g^{\text{Perf}} \setminus \beta_3$ (as sets). The second observation is that $L^M|_{\beta_k} = 0$ if $M > \dim \mathcal{A}_{g-k} = (g-k)(g-k+1)/2$. This follows easily from the fact that $L^{\otimes n}$ is free on $\mathcal{A}_g^{\text{Sat}}$ for $n >> 0$.

As an intermediate step we use the level covers $\mathcal{A}_g^{\text{Perf}}(\ell)$ where $\ell \geq 3$ is prime. This is a Galois cover $\sigma : \mathcal{A}_g^{\text{Perf}}(\ell) \to \mathcal{A}_g^{\text{Perf}}$ of degree $\nu_g(\ell) = |\text{Sp}\,(2g, \mathbb{Z}/\ell\mathbb{Z})|$ which is branched of order ℓ along the boundary. Hence $\sigma^*(D) = \ell \sum D_i$ where the number of the boundary components equals $d_g(\ell) = \frac{1}{2}\ell^{2g}(1-\ell^{-2g})$. We find that

$$a_N^{(g)} = \langle L^{G-N} D^N \rangle_{\mathcal{A}_g^{\text{Perf}}} = \frac{1}{\nu_g(\ell)} \langle \sigma^* L^{G-N} \sigma^* D^N \rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)}$$

$$= \frac{\ell^N}{\nu_g(\ell)} \left\langle \sigma^* L^{G-N} \left[\sum_i D_i^N + \sum_{i>j;\ a+b=N, a,b>0} \binom{N}{a} D_i^a D_j^b \right.\right.$$

$$\left.\left. + \sum_{i>j>k;\ a+b+c=N, a,b,c>0} \binom{N}{a,\ b,\ c} D_i^a D_j^b D_k^c \right] \right\rangle_{\mathcal{A}_g^{\text{Perf}}(\ell)}.$$

The intersection of four or more boundary components can be neglected since these cycles live in β_3 on which L^{G-N} vanishes if $N < 3g-3$. This explains the three summands in Theorem 1. The computation of these three summands can finally be reduced to intersection numbers on geometrically well understood varieties.

This talk is based on [3] where details can be found.

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On the variety of power sums of the Scorza quartics of trigonal curves

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(joint work with Francesco Zucconi)

The problem of representing a homogeneous form as a sum of powers of linear forms has been studied since the last decades of the 19th century. This is called the Waring problem for a homogeneous form. We are interested in the global structure of a suitable compactification of the variety parameterizing all such representations of a homogeneous form. Here is a precise definition of such a compactification:

Definition 1. *Let V be a $(v+1)$-dimensional vector space and $F \in S^m\check{V}$ be a homogeneous form of degree m on V, where $\check{V}$ is the dual vector space of V.*

$$\mathrm{VSP}\,(F,n) := \overline{\{([H_1],\dots,[H_n]) \mid H_1^m + \cdots + H_n^m = F\}} \subset \mathrm{Hilb}^n(\mathbb{P}_*\check{V}).$$

We sometimes denote $\mathbb{P}*\check{V}$ by $\check{\mathbb{P}}^v$.

We describe the varieties of power sums for some special quartic forms. Though we cannot fully describe such varieties, we can find some interesting subvarieties of the following type:

Definition 2. *For a subvariety S of $\check{\mathbb{P}}^v$,*

$$\mathrm{VSP}\,(F,n;S) := \overline{\{([H_1],\dots,[H_n]) \mid [H_i] \in S, H_1^m + \cdots + H_n^m = F\}} \subset \mathrm{VSP}\,(F,n).$$

We find some threefolds and study the geometry of some curves on them.

Let B be the smooth quintic del Pezzo 3-fold, and $f\colon A \to B$ the blow-up along a general smooth rational curve C of degree d on B, where d is an arbitrary integer greater than or equal to 5. Let E be the f-exceptional divisor.

The notions of lines and conics on A, and marked lines and marked conics on B are defined. For example, a conic on A is a reduced connected curve q such that $-K_A \cdot q = 2$, $E \cdot q = 2$ and $p_a(q) = 0$, and a marked conic is the pair (q,η) of a conic q on B and a length two subscheme $\eta \subset C \cap q$. There are natural one to one correspondences between lines on A and marked lines, and between conics on A and (a part of) marked conics. Marked lines and conics, hence lines and conics on A are parameterized nicely:

Proposition 3.

(1) *Marked lines are parameterized by a smooth trigonal canonical curve $\mathcal{H}_1$ of genus $d-2$ if $d \geq 5$, and*
(2) *(a part of) marked conics are parameterized by the surface $\mathcal{H}_2$ obtained by blowing up $S^2C \simeq \mathbb{P}^2$ at $\frac{(d-2)(d-3)}{2}$ points.*

For (1), recall that there are three lines (counted with multiplicities) through a point of B. This gives the triple cover $\mathcal{H}_1 \to C \simeq \mathbb{P}^1$.

For (2), the crucial point is that there exists a unique conic on B through two points t_1 and t_2 if there is no line on B through t_1 and t_2. Thus the natural morphism $\mathcal{H}_2 \to S^2C \simeq \mathbb{P}^2$ mapping a marked conic to its marking is birational. Let β_i be a bi-secant line of C. It is shown that there exist $\frac{(d-2)(d-3)}{2}$ bi-secant lines. Then for the length two subscheme $[\beta_i \cap C]$, there exist infinitely many marked conics $(\beta_i \cup \alpha, \beta_i \cap C)$, where α are lines intersecting β_i, and it is known that such α's are parameterized by $\mathbb{P}^1$. This explains why $\mathcal{H}_2 \to S^2C$ is the blow-up at $\frac{(d-2)(d-3)}{2}$ points, which are $[\beta_i \cap C]$.

To investigate $\mathcal{H}_2$ more, consider the locus $D_l \subset \mathcal{H}_2$ parameterizing conics which intersect a fixed line l. D_l turns out to be a divisor linearly equivalent to $(d-3)h - \sum e_i$, where h is the pull-back of a line, and e_i are the exceptional curves of $\mathcal{H}_2 \to S^2C$. It is shown that if $d \geq 6$, then $|D_l|$ is very ample and embeds $\mathcal{H}_2$ in $\check{\mathbb{P}}^{d-3}$, and if $d = 5$, $|D_l|$ defines a birational morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$. Here the dual notation is used for later convenience. If $d \geq 6$, then $\mathcal{H}_2$ is so called the *White surface*.

Assume that $d \geq 6$. Set $\mathcal{D}_2 := \{([q_1], [q_2]) \mid q_1 \cap q_2 \neq \emptyset\}$ and denote by D_q the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over a point $[q]$. By the seesaw theorem, it holds that $\mathcal{D}_2 \sim p_1^* D_q + p_2^* D_q$. Embed $\mathcal{H}_2 \times \mathcal{H}_2$ into $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$ by $|\mathcal{D}_2|$. By $H^0(\mathcal{H}_2 \times \mathcal{H}_2, \mathcal{D}_2) \simeq H^0(\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}, \mathcal{O}(2,2))$, $\mathcal{D}_2$ is the restriction of the unique $(2,2)$-divisor on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$, which is denoted by $\{\widetilde{\mathcal{D}}_2 = 0\}$. Since $\{\widetilde{\mathcal{D}}_2 = 0\}$ is also symmetric, the equation $\widetilde{\mathcal{D}}_2$ can be taken so that it is the bi-homogenization of an equation $\check{F}_4$ of a quartic in $\check{\mathbb{P}}^{d-3}$. It holds that $\check{F}_4$ is non-degenerate. Let F_4 be the quadratic form dual to $\check{F}_4$ (see [Dol04, §2.3]).

By the double projection $B \dashrightarrow \mathbb{P}^2$ from a general point b, we see that there are n conics through a general point of $a \in A$. It is crucial that the number n is equal to the dimension of the quadratic forms on $\check{\mathbb{P}}^{d-3}$.

Now we can state our main result:

Theorem 4. *Let $\rho\colon \widetilde{A} \to A$ be the blow-up of A along the strict transforms of bi-secant lines of C on B. There is an injection $\Phi\colon \widetilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$ mapping a point $\widetilde{a}$ of $\widetilde{A}$ to the point representing the n points in $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ corresponding to n conics on A 'attached' to a. Moreover $\mathrm{Im}\,\Phi$ is an irreducible component of* $\mathrm{VSP}\,(F_4, n; \mathcal{H}_2)$.

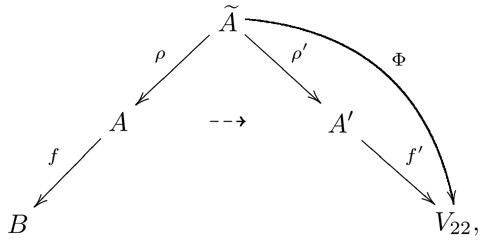
I will not explain the precise definition of attached conics. For a general point $\widetilde{a}$, they are just conics through $\rho(\widetilde{a})$.

For a point $\widetilde{a}$, let $H_1, \dots, H_n$ be the linear forms on $\mathbb{P}^{d-3}$ corresponding to n conics attached to $\widetilde{a}$. Then H_i gives a representation $\alpha_1 H_1^4 + \cdots + \alpha_n H_n^4 = F_4$ for some $\alpha_i \neq 0$.

Unfortunately, we did not succeed in proving Φ is an immersion or $\mathrm{Im}\,\Phi = \mathrm{VSP}\,(F_4, n; \mathcal{H}_2)$.

Even if $d = 5$, we have a similar result as follows: associated to the birational morphism $\Phi_{|D_l|}\colon \mathcal{H}_2 \to \check{\mathbb{P}}^2$, there exists a non-finite birational morphism $\Phi\colon \widetilde{A} \to \mathrm{VSP}\,(F_4, 6)$. Mukai showed that $\mathrm{VSP}\,(F_4, 6)$ is isomorphic to a smooth prime Fano

3-fold V_{22} of genus 12. Φ turns out to fits into the following diagram:



where $A \dashrightarrow A'$ is the flop of the strict transforms of bi-secant lines of C, $A' \to V_{22}$ is the blow-up along a general line m, and the rational map $V_{22} \dashrightarrow B$ is the famous double projection from m.

0.1. **Canonical curves and theta characteristics.** Finally, I explain some applications of our study of A for a pair of a canonical curve of any genus and a non-effective theta characteristic.

Using the incidence correspondence of intersections of lines on A

$$I := \{([l],[m] \mid l \cap m \neq \emptyset, l \neq m\} \subset \mathcal{H}_1 \times \mathcal{H}_1,$$

a non-effective theta characteristic θ on $\mathcal{H}_1$ can be defined such that

$$I = \{([l],[m]) \mid [m] \text{ is in the support of the unique member of } |\theta + [l]|\}.$$

We can define so called *the Scorza quartic* for a pair of a canonical curve of any genus and a non-effective theta characteristic (see [DK93, §9]). The Scorza quartic is not known to exist always. Dolgachev and Kanev proposed three conditions which guarantee the existence of the Scorza quartic. We prove that the pair $(\mathcal{H}_1, \theta)$ satisfies these conditions. By a standard deformation theoretic argument, we can verify these three conditions hold also for a general pair of a canonical curve and a non-effective theta characteristic, hence

Theorem 5. *The Scorza quartic exists for a general pair of a canonical curve and a non-effective theta characteristic.*

By the correspondence $[l] \mapsto D_l$, there is a natural identification $\mathbb{P}^{d-3} = \mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1})$, where $\mathbb{P}^{d-3}$ is the projective space dual to the ambient projective space $\check{\mathbb{P}}^{d-3}$ of $\mathcal{H}_2$. By definition, the Scorza quartic $\{F_4' = 0\}$ for $(\mathcal{H}_1, \theta)$ lives in $\mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1})$ but now it is possible to consider $\{F_4' = 0\} \subset \mathbb{P}^{d-3}$. We prove

Proposition 6. *The special quartic* $\{F_4 = 0\} \subset \mathbb{P}^{d-3}$ *in Theorem* 4 *coincides with the Scorza quartic* $\{F_4' = 0\}$.

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Abstract: Polarizations of Prym varieties via abelianization

Herbert Lange
(joint work with Christian Pauly)

1. Introduction

Let X be a smooth projective curve of genus g, G a simple, simply-connected complex Lie group, $\mathcal{M}_X(G)$ the moduli stack of principal G-bundles and and $\mathcal{L}$ the ample generator of $\mathrm{Pic}(\mathcal{M}_X(G))$. The Verlinde formula gives the numbers $N_{g,k} := \dim H^0(\mathcal{M}_X(G), \mathcal{L}^k)$. Some particular cases are

G	$SL(m)$	$Spin(2m)$	E_6	E_7	E_8
$N_{g,1}(G)$	m^g	4^g	3^g	2^g	1

The notion "Abelianization of principal G-bundles" goes back to Hitchin. Roughly speaking it means to give a map

$$\text{Prym variety} \to \text{Moduli space of principal bundles}$$

inducing an isomorphism between the Verlinde spaces and some spaces of Theta-functions. To give an example: In the case $G = SL(m)$, Beauville, Narasimhan and Ramanan showed in [1]: There is cover $Y \to X$ such that the direct image map $Prym(Y/X) \to \mathcal{M}_X(SL(m))$ induces by pull-back an isomorphism between the $SL(m)$-Verlinde space of level 1 and the space of abelian theta functions on $Prym(Y/X)$. Oxbury proved a similar result for the group $Spin(2m)$.

The main motivation for the paper [4] was to relate the Verlinde spaces for E_6, E_7 and E_8 to a space of theta functions.

In order to explain the types of theta functions we are looking at, we have to recall the definition of the Prym varieties we are considering. A polarized abelian variety (A, L) is called a *(generalized) Prym variety*, if there is a curve C and an embedding of A into its Jacobian JC such that the canonical principal polarization Θ of JC restricts to L. A principally polarized (A, Ξ) is called a *Prym-Tyurin variety of exponent* q if in addition $\Theta|A = q\Xi$. In [3] Kanev gave a construction of Prym-Tyurin varieties for the Weyl groups of type A_n, D_n, E_6 and E_7.

2. Results

Let W denote the Weyl group of G, $T \subset G$ a maximal torus, and $\mathbb{S}_w = Hom(T, \mathbb{C}^*)$ the weight lattice. Consider a commutative diagram

$$\begin{array}{ccc} Z & & \\ & \searrow & \\ \pi \downarrow & & Y = Z/Stab(\lambda) \\ & \swarrow \psi & \\ X & & \end{array}$$

where π is a Galois covering of smooth projective curves with group W and $\lambda \in \mathbb{S}_w$ dominant weight. Then Kanev's construction generalizes to give an abelian subvariety

$$P_\lambda \subset JY$$

which we call the *Prym variety* associated to λ.

The aim is to determine the type of the restriction of the canonical principal polarization to P_λ. Here the *type* of a polarization L is defined to be the finite group $K(L)$, the kernel of the induced isogeny of P_λ onto its dual abelian variety.

Recall that the Weyl groups of type E_i, $4 \le i \le 8$ are called *of del Pezzo type*, since a modified version of the weight lattice of $W(E_i)$ is isomorphic to the Picard lattice of a del Pezzo surface of degree $9 - i$. Consider the following table

Weyl group $W(E_i)$	$E_4 = A_4$	$E_5 = D_5$	E_6	E_7	E_8
weight λ	ϖ_2	ϖ_4, ϖ_5	ϖ_1, ϖ_6	ϖ_7	ϖ_8
del Pezzo S of deg. d	5	4	3	2	1
$\deg Y/X = \#$ lines $\subset S$	10	16	27	56	240

Here ϖ_i are the fundamental weights in the notation of Bourbaki [2]. In these cases we have the following theorem.

Theorem 1. *Suppose $\pi : Z \to X$ is étale. If P_λ denotes the Prym variety associated to one of the weights of the table, then*

$$\Theta_Y | P_\lambda \simeq M^{\otimes q_\lambda}$$

where $M \in Pic(P_\lambda)$ is of type $K(M) = (\mathbb{Z}/d\mathbb{Z})^{2g_X}$.

Remark. In the case of E_8 the line bundle M defines a principal polarization. Hence we obtain families of Prym-Tyurin varieties. These are different from Kanev's examples in [3], since π is étale and hence $X \neq \mathbb{P}^1$.

Theorem 1 is a consequence of the following more general result.

Theorem 2. *Assume*

- $\pi : Z \to X$ *étale, Galois with Galois group* W,
- $q_\lambda = d_\lambda$ *(= the Dynkin index of* λ*),*
- $\lambda\mathbb{Z}[W] = \mathbb{S}_\lambda$,
- λ *minuscule or quasiminuscule,*
- $\psi^* : JX \to JY$ *is injective.*

Then $\exists\, M \in Pic(P_\lambda)$ *with* $\Theta_Y | P_\lambda = M^{\otimes q_\lambda}$ *and*

$$K(M) = (\mathbb{Z}/m\mathbb{Z})^{2g_X}$$

where $m = \frac{\deg Y/X}{gcd(\deg K_\lambda - 1, \deg Y/X)}$.

In the talk a sketch of the proof of Theorem 2 was given. In particular the words of the title "via abelianization" were explained.

3. Applications and Problems

3.1. **Abelianization.** The Theorem implies $h^0(P_\lambda, M) = N_{g,1}(G)$. Hence our Prym varieties are candidates for the abelianization problem mentioned in the introduction. Moreover there exists a map between the corresponding spaces:

$$\gamma^* : H^0(\mathcal{M}_X(G), \mathcal{L}) \to H^0(P_\lambda, M).$$

The problem remains to show that γ^* is an isomorphism. It is in fact an isomorphism in the special case $G = SL(m)$. This is easily seen using the results of [1]. In all other cases we do not know the answer, mainly because an explicit description of special divisors in the linear system $|\mathcal{L}|$ seems to be missing. Particularly intriguing is the case of E_8, where both spaces are of dimension 1.

3.2. **The E_8-Prym-Tyurin varieties.** Study the families of Prym-Tyurin varieties associated to E_8 mentioned in the above remark. It is easy to see that they can be realised starting with an arbitrary curve X of genus ≥ 2. Moreover we have $\dim P_{\varpi_8} = 8(g_X - 1)$.

3.3. **Ramified coverings.** We expect similar results in the case of a ramified Galois covering $\pi : Z \to X$. There are however several problems in order to generalize our proof of Theorem 2.

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The fundamental group of surfaces with small K^2

Rita Pardini
(joint work with Ciro Ciliberto, Margarida Mendes Lopes)

Let S be a minimal complex surface of general type. It is well known that the numerical invariants of S satisfy the inequalities:

$$K_S^2 > 0, \chi(S) > 0,$$
$$2\chi(S) - 6 \leq K_S^2 \leq 9\chi(S)$$

It is expected that surfaces with K_S^2 small with respect to $\chi(S)$ have simpler fundamental group. For instance, it is known that surfaces on the Noether line $K^2 = 2\chi - 6$ are simply connected, while surfaces on the Bogomolov–Miyaoka–Yau line $K^2 = 9\chi$ have the unit ball in $\mathbb{C}^2$ as their universal cover.
The following conjecture of Miles Reid makes this expectation precise:
If $K_S^2 < 4\chi(S)$, then the algebraic fundamental group $\pi_1^{\rm alg}(S)$ of S is isomorphic, up to finite group extensions, to the fundamental group of a curve.

One possible approach to Reid's conjecture is to show the existence of a fibration $f \colon S \to B$ onto a smooth curve and then to prove that the kernel and cokernel of the induced map $\pi_1^{\rm alg}(S) \to \pi_1^{\rm alg}(B)$ are finite groups. Using this idea, Reid's conjecture has been verified in the following cases:

- when $K_S^2 < 3\chi(S)$ (work of Horikawa, Reid and other authors; cf. also [MP1]). In this case the fibration arises from the canonical map of étale covers of S;
- when S is irregular or has an irregular étale cover. In this case, by the the Severi inequality, proven in [Pa], the Albanese map of (an irregular étale cover of) S gives the required fibration.

To prove the conjecture in general, one should give a positive answer to the following:

Question 1. *If $K_S^2 < 4\chi(S)$ and S has no irregular cover, is $\pi_1^{\rm alg}(S)$ a finite group?*

The answer to Question 1 is known to be yes for $K^2 < 3\chi$, but it is unknown for $3\chi \leq K^2 < 4\chi$, even in the case $K^2 = 3$, $\chi = 1$ (the smallest possible invariants in this range).

A related simpler question is to give explicit bounds for the order of $\pi_1^{\rm alg}(S)$ when S is a surface with $K_S^2 < 3\chi(S)$ that has no irregular étale covers. Here one can give precise answers:

Theorem 1 ([MP1], [CMP])**.** *If S has no irregular finite étale cover and $K_S^2 < 3\chi(S)$, then:*

(1) $|\pi_1^{\rm alg}(S)| \leq 9$;
(2) *if $|\pi_1^{\rm alg}(S)| = 9$ or 8, then $K_S^2 = 2$ and $p_g(S) = 0$, namely S is a (numerical) Campedelli surface.*

Better bounds can be obtained if one assumes the stronger inequality $K^2 < 3\chi - 1$:

Theorem 2 ([MP2])**.** *If S has no irregular finite étale cover and $K_S^2 < 3\chi(S) - 1$, then:*

(1) $|\pi_1^{\text{alg}}(S)| \leq 5$;
(2) *if $|\pi_1^{\text{alg}}(S)| = 5$, then $K_S^2 = 1$ and $p_g(S) = 0$, namely S is a (numerical) Godeaux surface;*
(3) *if $|\pi_1^{\text{alg}}(S)| = 3$, then $K_S^2 = 3\chi(S) - 3$ and $2 \leq \chi(S) \leq 4$*

Theorem 1 and Theorem 2 are sharp. In fact, examples of the following are known:

- (numerical) Campedelli surfaces with fundamental group of order 8 and 9;
- (numerical) Godeaux surfaces with $\pi_1^{\text{alg}} = \mathbb{Z}_5$, e.g. the classical Godeaux surface;
- surfaces with $K^2 = 3\chi - 3$ and $\pi_1^{\text{alg}} = \mathbb{Z}_3$ for $\chi = 2, 3, 4$;
- infinitely many families of surfaces with $K^2 < 3\chi$ and $\pi_1^{\text{alg}} = \mathbb{Z}_2$, $\mathbb{Z}_2^2$.

The bounds given in Theorem 1 and Theorem 2, together with the above list of examples, suggest the following:

Question 2. *Let S be a surface with $K^2 < 3\chi$ having no irregular étale cover. Is it true that, up to a finite number of exceptions, $\pi_1^{\text{alg}}(S)$ is a subgroup of $\mathbb{Z}_2^2$?*

Finally, Campedelli surfaces with fundamental group of order 9 have been completely classified in [MP3]. They have some interesting properties:

Theorem 3. *Let $\mathcal{M}$ be the moduli space of (numerical) Campedelli surfaces with $|\pi_1^{\text{alg}}| = 9$. Then:*

(1) *$\mathcal{M}$ has two connected components: $\mathcal{M}_A$ (surfaces with $\pi_1^{\text{alg}} = \mathbb{Z}_9$) and $\mathcal{M}_B$ (surfaces with $\pi_1^{\text{alg}} = \mathbb{Z}_3^2$);*
(2) *$\mathcal{M}_A$ is irreducible of dimension 6 (= expected dimension) and $\mathcal{M}_B$ is irreducible of dimension 7 (= expected dimension+1);*
(3) *there is a codimension 1 subvariety $\mathcal{M}_{B2}$ of $\mathcal{M}_B$ such that for S in $\mathcal{M}_{B2} \cup \mathcal{M}_A$ the system $|2K_S|$ has two base points.*

Notice that the bicanonical system of a surface of general type with $K_S^2 > 1$ is known to be base point free, except possibly, for $2 \leq K_S^2 \leq 4$. The surfaces correspondings to points of $\mathcal{M}_{B2} \cup \mathcal{M}_A$ are at the moment the only known example of surfaces with $K_S^2 > 1$ whose bicanonical system has base points.

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Generic nefness

Thomas Peternell

We fix a projective manifold X of dimension n. Recall from [BDPP] that $\overline{ME}(X)$ is the closed cone generated by the classes of the following form:

$$\mu_*(H_1 \cdot \ldots \cdot H_{n-1}),$$

where $\mu : \tilde{X} \to X$ is a birational map from a projective manifold $\tilde{X}$ and H_i are very ample divisors on $\tilde{X}$. It is shown in [BDPP] that $\overline{ME}(X)$ is the dual cone to the cone of effective divisors; the pseudo-effecive cone of X. The question arises whether it is really necessary to take blow-ups or whether already the closed cone of complete intersection curves on X itself is dual to the pseudo-effective cone.

Definition. *A line bundle L on a projective manifold X_n is generically nef if*

$$L \cdot H_1 \cdot \ldots \cdot H_{n-1} \geq 0$$

for all ample line bundle H_i.

In this notation the above problem is equivalent to the question whether generically nef line bundle are already pseudo-effective. This is however in general not true (see the update of [BDPP]):

Example. Let E be the rank 3-vector bundle $E = \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3)$ over $\mathbb{P}_1$ or the rank 2-vector bundle on $\mathbb{P}_2$ given by a non-split extension

$$0 \to \mathcal{O} \to E \to \mathcal{I}_{\{p_1,p_2\}}(-2) \to 0,$$

where p_1, p_2 are two points in $\mathbb{P}_2$. Let $X = \mathbb{P}(E)$ and $L = \mathcal{O}_{\mathbb{P}(E)}(1)$. Then L is generically nef, but not pseudo-effective.

It is however an interesting open question whether a counterexample exists also for the canonical bundle:

Question 1. Suppose K_X generically nef. Is K_X pseudo-effective? In other words, if K_X is not generically nef, is X uniruled?

This question is a strong from of a reverse of Miyaoka's theorem, saying then Ω^1_X is generically nef unless X is uniruled. Here is the relevant definition:

Definition. *Let H_i be ample divisors on X. A vector bundle E is generically $(H_1, \ldots, H_{n-1})-$nef (ample), if $E|C$ is nef (ample) where C is "MR-general".*
E is generically nef if it is for all choices of H_i.

We say that C is MR-general, i.e. general in the sense of Mehta-Ramanathan, if C is cut by general elements of $|m_iH_i|$, $1 \leq n-1$ for $m_i \gg 0$. The importance of this notion comes in particular from the following fact:

if E is semi-stable w.r.t. $(H_1, \ldots, H_{n-1})$ *and if* $c_1(E) \cdot H_1, \ldots, H_{n-1}) \geq 0$, *then E is generically nef w.r.t.* $(H_1, \ldots, H_{n-1})$.

For applications the following would be useful:

Question 2. Let $\mu : \tilde{X} \to X$ be a modification from a projective manifold. Suppose X not uniruled. Is $\mu^*(\Omega^1_X)$ generically nef?

In [CP] the following wekaer version is proved:

Theorem 1. *If X is not uniruled and*

$$(\Omega^1_X)^{\otimes m} \to Q \to 0$$

a torsion free quotient, then $\det Q$ *is pseudo-effective.*

For applications we also refer to [CP].

A further strenghtening of Miyaoka's theorem would be

Question 3. Let $(C_t)_{t \in T}$ be a covering family of curves of X and suppose that the family is maximal, i.e. the parameter space T is an irreducible family of the Chow scheme. Suppose $\Omega^1_X|C_t$ is not nef for general $t \in T$. Is then X uniruled?

If one drops the assumption of maximality, the answer is "no": in [BDPP] it is shown that on a K3 surface or a Calabi-Yau threefold X there is a covering family (C_t) such that $\Omega^1_X|C_t$ is not nef for general t.

Turning sides, we now ask for which varieties X the tangent bundle might be generically nef (ample).

Theorem 2. *Assume T_X is generically nef w.r.t.* $(H_1, \ldots, H_{n-1})$. *Let $f : X \to Y$ be a surjective holomorphic map to the normal projective variety Y. Then either Y is uniruled or $\kappa(\hat{Y}) = 0$ for a desingularization $\hat{Y}$ of Y. In particular the Albanese map of X is surjective.*

For the proof we refer to [Pe2]. A theorem of Qi Zhang [Zh] says that projective manifolds with nef anti-canonical bundle have the same property. This leads to

Question 4. Let X be a projective manifold with $-K_X$ nef. Is T_X generically nef for some/all polarizations?

Before we discuss Question 4, let us mention that already the nefness/ampleness of T_X on *one* curve has strong consequences; in fact, in [Pe1] the following structure result is shown.

Theorem 3. *Let $C \subset X$ be an irreducible curve. If $T_X|C$ is nef, then $\kappa(X) < \dim X$. If $K_X \cdot C < 0$, then X is uniruled. If furthermore $T_X|C$ is ample, then X is rationally connected.*

As a corollary, if T_X is generically ample for $(H_1, \ldots, H_{n-1})$, e.g., T_X is semi-stable w.r.t $(H_1, \ldots, H_{n-1})$ and if $-K_X \cdot H_1 \cdot \ldots \cdot H_{n-1} > 0$, then X is rationally connected.

Concerning Question 4 the following holds:

Theorem 4. *Let X be a Fano manifold with $b_2(X) = 1$. Then T_X is generically ample.*

The proof uses essentially a theorem of Bogomolov-McQuillan and Kebekus-Sola Condé-Toma, see [KST], on foliations which are ample on a "sufficiently regular" curve.
We can prove Theorem 4 also for Fano manifolds with $b_2 > 1$, once the following cone theorem holds:

$\overline{ME}(X)$ is locally rationally polyhedral in $\{K_X < 0\}$. The extremal rays are represented by covering families of rational curves.

J. McKernan told me during the conference that he can prove this cone theorem, to be contained in a new version of [BCHM].
Question 4 has also a positive answer when $-K_X$ is nef admitting a K_X-trivial covering family of curves which is not "connecting".

We would like to use nefness properties of the tangent bundle to settle the following

Conjecture 1. Let X be a projective manifold with $-K_X$ nef. Then the Albanese is a (surjective) submersion.

Generic nefness is certainly not sufficient to prove Conjecture 1; one needs informations on every point of x. Therefore we propose

Definition. *A vector bundle E is sufficiently nef, if through every point x of X there is a covering family (C_t) of curves passing through x such that $E|C_t$ is nef for general t.*

Using the notation we state

Conjecture 2. If $-K_X$ is nef, then T_X is sufficiently nef.

Conjecture 2 can be shown for surfaces using a generalization of Bogomolov for surfaces of the theorem of Mehta-Ramanathan. It is also not difficult to see that Conjecture 2 implies Conjecture 1. In fact, Conjecture 1 is equivalent to saying that the holomorphic 1-forms on X do not have zeroes. This clearly holds when T_X is sufficiently nef. At the momemt we have some partial results supporting Conjecture 2.

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L^2–cohomology on ball quotients

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(joint work with Kang Zuo)

The complex ball $\mathbb{B}_n$ is a bounded Hermitian symmetric domain of type G/K with $G = SU(n,1)$ and $K = U(n)$. A ball quotient X is a quotient of $\mathbb{B}_n$ by a torsion–free discrete subgroup $\Gamma \subset SU(n,1)$. Such double quotients $\Gamma\backslash G/K$ are also called locally symmetric varieties. If Γ is an arithmetic subgroup, then X is quasi–projective and allows a natural normal projective compactification by adding a finite number of points (cusps) at infinity, the Baily–Borel–Satake compactification. Desingularizations $\overline{X}$ of this compactification are given by toroidal compactifications. If $X = \mathbb{B}_2/\Gamma$ is a compact ball quotient surface and $K_X = L^{\otimes 3}$ for some nef and big line bundle L, then Miyaoka [3] proved that $H^0(X, S^n\Omega^1_X \otimes L^{-m}) = 0$ for $m \geq n \geq 1$. Examples of compactified ball quotient surfaces are Picard modular surfaces X. Those are (components of) Shimura varieties which parametrize abelian 3–folds with given Mumford–Tate group. In this case Γ is a subgroup of $SU(2,1)$ with values in integers of an imaginary quadratic field E and X paramatrizes Jacobians of Picard curves of type $y^3 = P(x)$ with $\deg(P) = 4$.
Let us start with the following vanishing theorem of Ragunathan, which has later been generalized by Li–Schwermer [2] and Saper:

Theorem 1 (Ragunathan)**.** *Let $\mathbb{W}$ be an irreducible representation of Γ, i.e., a local system on X. If the highest weight of $\mathbb{W}$ is regular, then the intersection cohomology $IH^1(\overline{X}, \mathbb{W}) = 0$.*

Using the formalism of Higgs bundles and Higgs cohomology we show that this implies the following result [4]:

Theorem 2. *On has $H^0(\overline{X}, S^n\Omega^1_{\overline{X}}(\log D)(-D) \otimes L^{-m}) = 0$ for all $m \geq n \geq 3$.*

For the proof in the interesting case $m = n = a + 2$, consider the Higgs bundle associated to a regular representation with highest weight $(a, 1)$ with $a \geq 1$. In the first cohomology of the corresponding Higgs bundle $E_{a,1}$ only the term $H^0(\overline{X}, S^{a+2}\Omega^1_{\overline{X}}(\log D)(-D) \otimes L^{-a-2}) = 0$ survives and hence is zero.
The twist by $(-D)$ in the theorem is too strong and the proof gives a slightly better result. We also prove generalizations of such vanishing and related non–vanishing

results to higher–dimensional ball quotients in [4]. The symmetric powers S^n are then replaced by certain Schur functors of the type $\Gamma_{a_1,\ldots,a_{n-1}}$.
We also give applications to the intersection cohomology groups of universal families $f: A \to X$ of abelian varieties over Picard modular surfaces and threefolds. Standard methods from the theory of algebraic cycles imply vanishing and non–vanishing theorems for Chow groups as a consequence. For example we can show:

Theorem 3 ([4], Schoen)**.** *Let $f: A \to X$ be the universal familiy of abelian 3–folds over a Picard modular surface. Assume that the monodromy representation of $R^1 f_* \mathbb{C}$ has unipotent monodromy at infinity. Then a multiple of the normal function $AJ(C_t - C_t^-)$ associated to the Ceresa cycle is contained in the maximal abelian subvariety $J^2_{\rm ab}(JC_t)$ of the intermediate Jacobian $J^2(JC_t)$ for every t.*

This theorem gives some evidence for Clemens' conjecture saying that $C - C^-$ is never algebraically equivalent to zero unless C is hyperelliptic. It can be shown that the necessary multiple in the theorem is 3 (due to Chad Schoen, unpublished). Note that in this case C is hyperelliptic as a point on X only if C becomes singular as a curve (with smooth Jacobian).
There are also corresponding non–vanishing theorems of Kazdan which show that for $m < n$ the vanishing does not hold, if Γ is sufficiently small and $\mathbb{V}$ is not regular. For Picard modular 3–folds X with a universal family $f: A \to X$ (assuming unipotent monodromy) we get as a consequence:

Theorem 4. *(a) If Γ is sufficiently small, the general member A_t of the universal family $f: A \to X$ has non–trivial Griffiths group $\mathrm{Griff}^3(A_t)$.*
(b) If Γ is sufficiently small and $H^0(\overline{X}, \Omega^1_{\overline{X}})$ contains two linearly independent sections α, β with $\alpha \wedge \beta \neq 0$, then the group $Gr^3_F H^2_{L^2}(X, R^4 f_* \mathbb{C}_{\rm pr})$ does not vanish. In particular, assuming the Hodge conjecture, there are codimension 3–cycles in the kernel of the Abel–Jacobi map.

Case (b) seems to be a new phenomenon.

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A birational Local Torelli Theorem with respect to n- and 1- forms

Miguel A. Barja
(joint work with Francesco Zucconi)

A local family $f : \mathcal{X} \longrightarrow B$ is a proper flat family of smooth complex projective varieties of dimension n over a polydisk B. Set X_b for the fibre over b. In this context, local Torelli theorem asks whether the fibres are mutually isomorphic provided the Hodge structures of the fibres are constant.

Our main result is as follows: we give a set of properties for X_b in such a way that, if the global n-forms ant the global 1-forms of X_b are liftable to the family $\mathcal{X}$, then $var(f) = 0$ (see Theorem 3). When B is 1-dimensional this means that all the fibres are mutually birational. Liftability of forms is a direct consequence of having constant Hodge structure, and birationality (instead of biregularity) of the fibres can not be avoided since the global differential forms on the fibres are invariant under birational transformations.

As a byproduct we give a result which characterizes products of varieties as fibre spaces verifying Künneth formulas, when the general fibre verifies good properties (Theorem 4), as a generalization of a well known result of Beauville for surfaces.

We obtain the main results as a consequence of a generalization of the so called Adjoint Theorem in [PZ] and an inverse of it (Theorem 1). Let X be a smooth variety of dimension n and let $\mathcal{F}$ be a locally free sheaf of rank r. Fix an element of an extension class $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_X)$:

$$0 \to \mathcal{O}_X \stackrel{d\epsilon}{\to} \mathcal{E} \to \mathcal{F} \to 0$$

and assume that we have $\eta_1, ..., \eta_{r+1} \in H^0(X, \mathcal{F})$ which are liftable to $H^0(X, \mathcal{E})$. Let $\mathcal{L} = \det(\mathcal{F})$ and consider the linear system $|\bigwedge^r W|$ (where $W = \langle \eta_1, \dots, \eta_{r+1} \rangle$) inside $|\mathcal{L}|$ (assuming it is not empty). Call D the base divisor of that linear system. Choosing liftings $s_i \in H^0(X, \mathcal{E})$ of η_i we define its adjoint image ω as the image of $s_1 \wedge ... \wedge s_{r+1}$ through the chain of maps $\bigwedge^{r+1} H^0(X, \mathcal{E}) \to H^0(X, \det(\mathcal{E})) \cong \mathrm{H}^0(\mathrm{X}, \mathcal{L})$. Now, the Adjoint Theorem states (see [PZ]) that if $\omega \in |\bigwedge^r W|$ then $\xi \in Ker(H^1(X, \mathcal{F}^\vee) \to H^1(X, \mathcal{F}^\vee(D))$. The first result we have is

Theorem 1. *If* $h^0(X, \mathcal{O}_X(D)) = 1$ *then the inverse holds, i.e., if* $\xi \in Ker(H^1(X, \mathcal{F}^\vee) \to H^1(X, \mathcal{F}^\vee(D))$ *then* $\omega \in |\bigwedge^r W|$.

And as a consequence

Theorem 2. *Let* C *be a smooth curve and*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

an extension of a line bundle $\mathcal{L}$ *on* C*. Assume*

i) $H^0(C, \mathcal{E})$ *surjects onto* $H^0(C, \mathcal{L})$ *and*

ii) *the linear system* $|\mathcal{L}|$ *induces a base point free birational morphism on* C*.*

Then the extension splits.

Now we can state the main theorem. Let $f : \mathcal{X} \to B$ be a local family as above, such that $A = Alb(X_b)$ is constant (does not depend on b) and there exists a morphism of B-schemes $Alb : \mathcal{X} \to B \times A$, with $Alb_b = alb_{X_b}$.

Assume we have liftability of $n-$forms from the fibres to the family, i.e.

i) $H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^n_{X_b})$.

Observe that, as a consequence of the existence of the map Alb we can also lift 1-forms from the fibre to the family, i.e.

ii) $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}}) \twoheadrightarrow H^0(X_b, \Omega^1_{X_b})$.

The conditions we want to impose to the general fibre involves its Albanese map. If X_b is of Albanese general type (i.e., its Albanese map is generically finite over its image) we call D_b the ramification divisor of alb_{X_b} and denote $C_b := K_{X_b} - D_b$.

Theorem 3. *(Birational Local Torelli theorem) Assume that, for any $b \in B$, the fibre X_b verifies*

i) *X_b is of general type of dimension $n \geq 2$,*
ii) *$\deg(\mathrm{alb}_{X_b}) = 1$,*
iii) *$h^0(X_b, \mathcal{O}_{X_b}(D_b)) = 1$,*
iv) *$h^{n-1}(X_b, \Omega^1_{X_b}(2C_b)) = 0$.*

Then $var(f) = 0$.

Remark. The proof of the theorem is a consequence of the Adjoint theorem and Theorem 1 and a criterium of birational triviality given by the Volumetric Theorem in [PZ].

Remark.(i) It is easy to construct counterexamples when $\deg(\mathrm{alb}_{X_b}) \geq 2$.

(ii) The four conditions for X_b in the theorem are trivially verified by smooth general type subvarieties of abelian varieties, provided Ω^1_X is big and nef. By a result of Debarre (cf. [De]) this holds for any nondegenerate X with $\mathrm{dimX} \leq \frac{1}{2}\mathrm{dimA}$. Also, if X is a complete intersection of at least 2 divisors in A, an easy computation shows that $h^{n-1}(X, \Omega^1_X(2K_X)) = 0$.

Finally we can give a characterization of birationally trivial fibrations between those verifying Künneth formulas, under some conditions for the canonical or the Albanese map of the fibre.

Theorem 4. *Let $f : Z \longrightarrow Y$ be a fibration of complex projective varieties of relative dimension n. Let F be a general smooth fibre. Assume that F is of general type and verifies one of the following set of properties*

(i) *either the canonical map of F is birational or,*
(ii) *the albanese map of F is of degree 1, $h^0(F, \mathcal{O}_F(D_F)) = 1$ and $h^{n-1}(F, \Omega^1_F(2C_F)) = 0$.*

Assume that

$$\forall i = 1, ..., n \quad h^0(Z, \Omega^i_Z) = \sum_{j+k=i} h^0(Y, \Omega^j_Y) h^0(F, \Omega^k_F)$$

Then $var(f) = 0$. Moreover, if Y is a curve, then Z is birational to $Y \times F$.

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The Bloch principle

Michael McQuillan

In comparison with the purely qualitative theorem of Picard, there is the theorem of Montel that the space of maps, in the compact open topology, from the disc to $\mathbb{P}^1\backslash\{0,1,\infty\}$ is compact. The generalisation of Picard's theorem by E. Borel to a projective space complemented by planes is straightforward, but the corresponding generalisation by A. Bloch, [1], and H. Cartan [2], of Montel's theorem has resisted any substantive simplification for 80 years. Such discrepancy of difficulty is, as Bloch remarked, not easily explained since there can be little doubt that, *nihil est in infinito quod non fuerit prius in finito.* It is, however, already not a complete triviality to give a concrete mathematical formulation of Bloch's dictum, and many, supposed, counterexamples have been suggested. These supposed counterexamples are, however, completely explained by Gromov's theory of bubbling, [4], which in turn leads, [6], to counterexample free formulations of the principle such as,

Question Suppose for a quasi-projective variety X with boundary ∂ there is a Zariski subset Z of X such that every entire map $f : \mathbb{C} \to X$ factors through Z, then is a sequence of discs $f_n : \Delta \to X$ without a convergent subsequence in the sense of Gromov arbitrarily close to $Z \cup \partial$?

Which, in turn, for surfaces admits a certain refinement, cf. below & op. cit. §2, on taking, as we may, the boundary to be a stable curve. Nevertheless, there is some distance between a properly posed question, and a solution. A major step, however, has recently been taken by J. Duval, [3], who has established,

Fact Suppose that $f_n : \Delta \to X$ are a sequence of discs to a compact analytic space violating Gromov's isoperemetric inequality, or, equivalently, there is a subsequence such that the currents of integration,

$$\frac{1}{A_n}\int_{\Delta} f_n^*$$

converge to a closed current, T, where A_n is the area of the n^{th} disc, then if T has mass along a subvariety Z there is an entire mapping from $\mathbb{C}$ to Z.

Plainly, this reduces the study of the Bloch principle to that of appropriate T and Z as found in Duval's theorem. For surfaces, the requisite study, [5], had already been undertaken, and whence,

Conclusion Let S be a quasi-projective surface, or bi-dimensional Deligne-Mumford stack, with stable boundary ∂ such that there is a Zariski subset Z of S through which every entire map $f : \mathbb{C} \to X$ factors, then a sequence of discs

$f_n : \Delta \to X$ without a convergent subsequence in the sense of Gromov is arbitrarily close to Z.

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Entire curves, integral sets and fiber bundles

Jörg Winkelmann

Let X be a variety defined over a number field K. Conjecturely existence of entire curves (holomorphic maps from $\mathbb{C}$ to $X(\mathbb{C})$) is related to the existence of infinite integral point sets: More precisely: *Let W be a (irreducible) subvariety of X defined over a finite field extension K'/K. Then there should exist a holomorphic map $f : \mathbb{C} \to X$ with $\overline{f(\mathbb{C})}^{Zar} = W$ if and only if there is a finite field extension K''/K' for which there is a Zariski dense integral point set in $W(K'')$.*

This follows the philosophy proposed by Lang and Vojta ([1],[3]).

As evidence towards this conjecture we show that entire curves and integral point sets have similar functorial behaviour with regard to principal bundles:

Theorem 1. *Let G be a connected algebraic group (not necessarily linear) and let $p : E \to B$ be a G principal bundle which is locally trivial in the Zariski topology, all defined over some number field K.*

Then for every holomorphic map $f : \mathbb{C} \to B$ there exists a holomorphic map $F : \mathbb{C} \to E$ with $f = p \circ F$ and $\overline{F(\mathbb{C})}^{Zar} = p^{-1}\left(\overline{F(\mathbb{C})}^{Zar}\right)$.

For every integral point set $S \subset B(K)$ there is a finite field extension K' and an integral point set $R \subset E(K')$ such that $\overline{R}^{Zar} = p^{-1}\left(\overline{S}^{Zar}\right)$.

As a consequence one obtains:

Corollary 2. *Let X be a quasiprojective variety over a number field K. Then a subset $S \subset X(K)$ is integral if and only if there exists an affine variety Z with a closed embedding $i : Z \to \mathbb{A}^N$ and a morphism $\phi : Z \to X$ (all over K) such that $\phi(i^{-1}(\mathcal{O}_K{}^N) = S$.*

Additional evidence towards the conjecture is provided by the following result concerning ramified coverings over abelian varieties, which is based on recent results in Nevanlinna theory ([2]).

Theorem 3. *Let $\pi : X \to A$ be a finite morphism from a quasi-projective variety X to a semi-abelian variety A, all over some number field K.*

Then for every holomorphic map $f : \mathbb{C} \to X$ there is a finite field extension K'/K such that there exists an integral point set $S \subset X(K')$ for which $f(\mathbb{C}) \subset \overline{S}^{Zar}$.

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Singular Symplectic Moduli Space

Manfred Lehn

(joint work with D. Kaledin, Ch. Sorger)

A holomorphic symplectic manifold is a complex manifold X together with a global holomorphic form σ that is closed and non-degenerate in the sense that it induces an isomorphism $\sigma : T_X \to \Omega^1_X$. Such a manifold is called irreducible holomorphic symplectic if X is compact, simply connected and admits a Kähler structure and if σ spans the $\mathbb{C}$-vector space $H^{2,0}(X)$. There are only two known examples of irreducible holomorphic symplectic manifolds that are not deformation equivalent to K3-surfaces S and their Hilbert schemes or to generalised Kummer varieties associated to complex tori $A = \mathbb{C}^2/\Gamma$. These examples are due to O'Grady and arise as symplectic desingularisations of singular moduli spaces of semistable sheaves $M_S(2;0,4)$ or $M_A(2;0,2)$. We will show that the attempt to construct other examples of new topological types of irreducible holomorphic symplectic manifolds in this way from singular moduli spaces must fail.

For simplicity, let S be a K3-surface. Let $\langle -,-\rangle$ denote the Mukai pairing on $H^*(S,\mathbb{Z})$ and let $M = M(v)$ denote the moduli space of H-semistable sheaves on S with Mukai vector $v = v(F) := ch(F)\sqrt{td(T_S)}$, where the ample divisor H is assumed to be general with respect to v in the sense that for any $F \in M(v)$ and any destabilising subsheaf $F' \subset F$ one has $v(F') \in \mathbb{Q}v$. The virtual dimension of M is $2 + \langle v, v\rangle$. The Moduli spaces $M(v)$ are singular simyplectic varieties.

We may write $v = mv_0$ with $m \in \mathbb{N}$ and a uniquely defined primitive vector $v_0 \in H^*(S,\mathbb{Z})$. Now the singularity type of $M(v)$ is completely determined by the numbers m and $\langle v_0, v_0\rangle$. The first theorem extends part of O'Grady' result.

Theorem 1. (Lehn-Sorger [2]) — *Let $m = 2$ and $\langle v_0, v_0\rangle = 2$. Then blowing-up the singular locus of $M(v)$ provides a symplectic resolution $\tilde{M} \to M(v)$.*

The second theorem rules out the rest of possible candidates.

Theorem 2. (Kaledin-Lehn-Sorger [1]) — *If* $m, \langle v_0, v_0\rangle \geq 2$ *and* $m + \langle v_0, v_0\rangle \geq 5$, *then* $M(v)$ *is irreducible, l.c.i, and locally factorial, and the codimension of the singular locus is at least* 4. *In particular,* M *does not admit a symplectic resolution.*

A fundamental property of symplectic resolutions is that they are semismall. So if $M' \to M(v)$ were a symplectic resolution under the conditions of the last theorem, then any component of the exceptional locus would have to have codimension at least one half of the codimension of the singular locus in $M(v)$, i.e. 2. On the other hand, if $M(v)$ is locally factorial, any resolution is divisorial, a contradiction.

The method of proof for both theorems consists in a careful analysis of the local situation near a point $[F] \in M(v)$, represented by a polystable sheaf F. There is an $\mathrm{PAut}(F)$-equivariant germ of a map $\kappa : (\mathrm{Ext}^1(F,F),0) \to \mathrm{Ext}^2(F,F)_0$, the so-called Kuranishi map, with the property that $(M(v),[F]) \cong \kappa^{-1}(0)/\!\!/\mathrm{PAut}(F)$. Moreover, up to higher order terms, κ equals the momentum map $\mu : \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^2(F,F)_0 = \mathrm{Lie}(\mathrm{PAut}(V))^*$ for the action of $\mathrm{PAut}(V)$ on the representation $\mathrm{Ext}^1(F,F)$. We first show that the symplectic reduction $\mathrm{Ext}^1(F,F)/\!\!/\!\!/\mathrm{PAut}(V) := \mu^{-1}(0)/\!\!/\mathrm{PAut}(F)$ has the properties claimed about $(M(v),[F])$, and then extend these results to the moduli space itself.

This raises the more general question which symplectic singularities admit symplectic resolutions. In this talk we considered only quotients by finite groups, and omitted the discussion of symplectic reductions because of time constraints.

Let $G \subset \mathrm{Sp}(V)$ be a finite group. By a theorem of Verbitsky, if the quotient V/G admits a symplectic resolution then G is generated by symplectic reflections. Given a real reflection group $G \subset \mathrm{GL}(n,\mathbb{R})$, complexification yields a group generated by complex or pseudo-reflections, and the canonical embedding $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{Sp}(2n,\mathbb{C})$ turns any complex reflection group into a symplectic reflection group. All these types of reflection groups have been classified by Coxeter, Shephard and Todd, and A. Cohen, respectively. The theorem of Verbitsky limits the search for symplectically resolvable quotients to this range. A theorem of Kaledin and Ginzburg for real reflection groups and of Bellamy for complex reflection groups shows that only the following three representations of a group G on a vector space V_0 admit symplectic resolutions for their corresponding symplectic double $V_0 \oplus V_0^*/G$:

(1) The action of S_n on $\mathfrak{H} = \{z \in \mathbb{C}^n \mid z_1 + \ldots, z_n = 0\}$.
(2) The action of the wreath product $(\mathbb{Z}/2)^n \rtimes S_n$ on $\mathbb{C}^n$.
(3) The action of the binary tetrahedral group T on $\mathbb{C}^2$.

It is well-known that in the first cases of Coxeter type A and B resolutions are provided by the Hilbert scheme of points. For the last case we have the following resolution: T has three different 2-dimensional representations: the standard action S, which is in fact symplectic, the quotient being the E_6-singularity, and two representations S' and S'' that are dual to each other. S' contains a divisor C consisting of four lines that are made up by points with non-trivial stabiliser. Even though the divisor is invariant under T, its equation is not. Hence the quotient $W := (C \times S'')/T \subset Z := (S' \oplus S'')/T$ is a Weil divisor but not Cartier.

Theorem 3. (Lehn-Sorger [3]) — *Let $Z' \to Z$ be the blow-up of Z along W, and let $Z'' \to Z'$ be the blow-up along the singular locus of Z'. Then Z'' is smooth and $Z'' \to Z$ is semismall. In particular, $Z \to Z''$ is a symplectic resolution.*

As a by-product of the work on this example we find: The Nakamura Hilbert scheme $T - \mathrm{Hilb}(S' \oplus S'')$ is not irreducible. It consists of two smooth components, one of which lies dominantly over $(S' \oplus S'')/T$, whereas the other is isomorphic to $\mathbb{P}^2 \times \check{\mathbb{P}}^2$. They intersect transversely along the natural incidence variety. This seems to be first example of a reducible G-Hilbert scheme for an action of a finite group G on a smooth variety.

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Nilmanifolds with left-invariant complex structure and their deformations in the large

Sönke Rollenske

The aim of this work was to understand the deformations in the large of a certain class of compact, complex manifolds.

We say that two compact, complex manifolds X and X' are directly deformation equivalent $X \sim_{def} X'$ if there exists an irreducible, flat family $\pi : \mathcal{X} \to \mathcal{B}$ of compact, complex manifolds over an analytic space $\mathcal{B}$ such that $X \cong \pi^{-1}(b)$ and $X' \cong \pi^{-1}(b')$ for some points $b, b' \in \mathcal{B}$. The manifold X is said to be a deformation in the large of X' if both are in the same equivalence class with respect to the equivalence relation generated by $\sim_{def}$.

The problem of determining the deformations of a given complex manifold is very difficult in general; but while there is a general method due to Kuranishi, Kodaira and Spencer to tackle small deformations there is no general approach to deformations in the large.

From Tori to Nilmanifolds. Even the seemingly natural fact that any deformation in the large of a complex torus is again a complex torus has been fully proved only in 2002 by Catanese. In [Cat04] he studies more in general deformations in the large of principal holomorphic torus bundles, especially bundles of elliptic curves. This was the starting point for our research.

It turns out that the right context to generalise Catanese's results is the theory of left invariant complex structures on nilmanifolds, i.e., compact quotients of nilpotent real Lie groups [CF06].

Many (counter-)examples in complex differential geometry have been constructed from nilmanifolds:

- Thurston's example of a manifold which admits a complex structure and a symplectic structure but no Kähler structure.
- Guan's example of a simply connected, non-kählerian, holomorphic symplectic manifold.
- Manifolds with arbitrarily non degenerating Frölicher spectral sequence [Rol07]. This answers a question mentioned in the book of Griffith and Harris.

In fact, a nilmanifold M admits a Kähler structure if and only if it is a complex torus [BG88].

There are too many nilmanifolds. Even if every (iterated) principal holomorphic torus bundle can be regarded as a nilmanifold, the converse is far from true. Moreover it turns out that even a small deformation of an iterated principal holomorphic torus bundle may not admit such a structure.

A simple example showing this behaviour can already be found in complex dimension 3.

Addressed questions.

(1) What are the **small deformations** of nilmanifolds with left invariant complex structure.
(2) When has such a nilmanifold a **geometric description** as an (iterated) principal holomorphic torus bundle?
(3) Can we determine all **deformations in the large** of (iterated) principal holomorphic torus bundles?

Small deformations. A fairly complete answer to the first question is given by the following result:

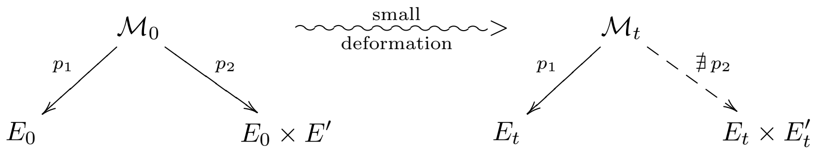
Theorem 1. *Let $M = \Gamma\backslash G$ be a nilmanifold with left-invariant complex structure J. If the Dolbeault cohomology $H^{p,q}(M, J)$ can be calculated using left-invariant differential forms then all small deformations of (M, J) are again nilmanifolds with left-invariant complex structure.*

The condition on Dolbeault cohomology is satisfied if (M, J) is an iterated principal holomorphic torus bundle or if J is generic (see [CF01, CFGU00]) and conjecturally holds true for all left-invariant complex structures.

The strategy of the proof is to show that the Kuranishi family can be described using only left-invariant differential forms generalising results of [CFP06].

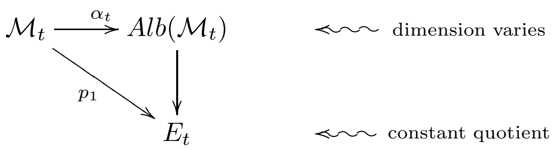
Stable geometries. In order to study deformations in the large we need more control over the geometry – the existence of a so-called stable torus bundle series.
Example: Let $\pi : S \to E_0$ be a Kodaira surface, i.e., a non-trivial principal bundle of elliptic curves over an elliptic curve, and let E' be an elliptic curve. We consider a family of nilmanifolds $\mathcal{M} \to \Delta$ such that $\mathcal{M}_0 = S \times E'$.

After a general small deformation the projection to the product of curves will vanish, while the projection induced by π always remains:



Hence the right approach is to study M_t as a principal 2-torus bundles over an elliptic curve.

Analysing the Albanese map it turns out that



Note that even the $\mathcal{C}^\infty$-map underlying p_1 does not change.

This is the simplest example of a stable torus bundle series and we determined several condition under which these exist.

Deformations in the large. Studying deformations in the large of such constant quotients (if they exist) instead of the whole Albanese variety we can generalise the results in [Cat04, CF06]. For technical reasons it is better to formulate the results in the language of Lie theory.

Theorem 2. *Let G be a simply connected nilpotent Lie group with Lie algebra* lg *and let $\Gamma \subset G$ be a lattice such that the following holds:*

(1) lg *admits a stable torus bundle series* $(\mathcal{S}^i\,\mathrm{lg})_{i=0,\dots,t}$.
(2) *The nilmanifolds of the type* $(\mathcal{S}^{t-1}\,\mathrm{lg}, J, \Gamma \cap \exp(\mathcal{S}^{t-1}\,\mathrm{lg}))$ *constitute a good fibre class. (Examples are Tori or Kodaira surfaces.)*

Then any deformation in the large M' of a nilmanifold with left-invariant complex structure of type $M = (\Gamma\backslash G, J)$ carries a left-invariant complex structure.

In complex dimension 3 there are only 16 cases to check [Sal01]:

Theorem 3 (Theorem C)**.** *Let $M = (\Gamma\backslash G, J)$ be an iterated, principal holomorphic torus bundle which has complex dimension at most 3.*

If not $\dim_{\mathbb{R}} \mathcal{Z}(G) = \dim_{\mathbb{R}}[G, G] = 3$ *then every deformation in the large of M is gain an iterated principal holomorphic torus bundle.*

In higher dimension there are several conditions on the structure of the Lie group under which the same conclusion as in Theorem C holds.

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Azumaya algebras and Artin stacks

Stefan Schröer
(joint work with Jochen Heinloth)

Our main result ist:

Theorem. *Let X be a noetherian scheme. Then the inclusion of the bigger Brauer group $\widetilde{\mathrm{Br}}(X) \subset H^2(X, \mathbb{G}_m)$ into étale cohomology is an equality.*

Here $\widetilde{\mathrm{Br}}(X)$ denotes Taylor's *bigger Brauer group* [4], which is defined in terms of quasicoherent associative $\mathcal{O}_X$-algebras that are étale locally of the form $\mathcal{E} \otimes \mathcal{F}$ for some quasicoherent sheaves $\mathcal{E}$ and $\mathcal{F}$, with multiplication law given by

$$e \otimes f \cdot e' \otimes f' = e \otimes f' \Phi(f, e')$$

for some surjective pairing $\Phi : \mathcal{F} \otimes \mathcal{E} \to \mathcal{O}_X$.

Our result generalizes a theorem of Raeburn and Taylor [3]. Our proof relies on the theory of algebraic stacks as developed in the book of Laumon and Moret-Bailly [2], and is closely related to de Jong's proof [1] that the Brauer group $\mathrm{Br}(X)$ equals the torsion subgroup of $H^2(X, \mathcal{O}_X^\times)$ if X carries an ample invertible sheaf. The main idea it to show that a $\mathbb{G}_m$-gerbe lies in the bigger Brauer group if and only if the associated algebraic stack carries a coherent sheaf of weight $w = 1$ that locally contains invertible direct summands. Then we use some general direct limit and descend arguments to see that such sheaves always exists.

The result does not only hold for noetherian schemes X, but also for noetherian algebraic stacks whose diagonal is quasiaffine.

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On the Quantisation of Completely Integrable Hamiltonian Systems

Duco van Straten
(joint work with Mauricio Garay)

Classical mechanics is described by a hamiltonian function that induces a flow in a phase space. The mathematical model is that of a symplectic manifold M, where the symplectic form ω defines an identification ϕ between the cotangent bundle Ω_M and the tangent bundle Θ_M; a function H on M defines a flow by integrating the hamiltonian vector field $\phi(dH)$, [1].

We consider the case $M = \mathbb{C}^{2n}$ with canonical coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ such that $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. The dynamics is described by the Hamilton equations

$$\dot{p_i} = -\partial H/\partial q_i, \;\; \dot{q_i} = \partial H/\partial p_i$$

where the hamiltonian H is a function of the $2n$ coordinates (p, q). The time derivative of an arbitrary function is then given by $\dot{F} = \{H, F\}$, where

$$\{F, G\} = \sum_{i=1}^n \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}$$

is the *Poisson-bracket* of F and G. F is called a *conserved quantity* if $\dot{F} = 0$, or, what is the same F *Poisson commutes* with H, $\{F, H\} = 0$.

In general we call $I_1, I_2, \ldots, I_n \in R := \mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$ which are functionally independent and with $\{I_i, I_j\} = 0$ for all i, j a *(polynomial classical) integrable system*. Although they are rare and hard to construct, several examples are known, like the tops of Euler, Lagrange, Kovalevskaya; special cases of the Henon-Heiles system, the Calogero-Moser systems, to mention a few. In many cases the fibres of the map $I := (I_1, \ldots, I_n) : \mathbb{C}^{2n} \longrightarrow \mathbb{C}^n$ are affine pieces of abelian varieties, see [6] for an overview. In algebraic geometry one encounteres the integrable Hitchin system, the systems of Beauville-Mukai, which correspond to the global situation of a Lagrangian fibrations on a hyperkähler manifold.

In their 1925 paper [2], Born and Jordan realised that quantum mechanics is a *non-commutative deformation of classical mechanics*: the ring $R = \mathbb{C}[p, q]$ is replaced by the non-commutative Heisenberg algebra $Q := \mathbb{C} < \hbar, p, q >$ with the relation

$$pq - qp = \hbar, \hbar := \frac{h}{2\pi i}, \quad (h \approx 6.10^{-34} Js)$$

$\hbar$ should be considered as a central element, and classical mechanics is recovered by putting $\hbar = 0$. Indeed, one can consider R as a quotient of Q: $Q/\hbar Q = R$. It was observed by Dirac, that the Poisson-bracket is recovered from the commutator via

$$\{f, g\} := \frac{1}{\hbar}[F, G] \mod \hbar Q$$

Question: *Given a integrable system $I_1, \ldots, I_n \in R$, do there exist $J_1, \ldots, J_n \in Q$ such that $[J_i, J_j] = 0$ and $J_i = I_i \mod \hbar$?*

If we can find such commuting $J_1, \ldots, J_n$, we will say the system is *quantum completely integrable*. We have no general answer to this question, but for many integrable systems explicit quantisations are known. The quantisation of the Hitchin system plays a central role in the *geometric Langlands program* [3].

It is natural to work order by order in $\hbar$ and put $Q_k := Q/\hbar^k Q$ and replace Q by the completion $\hat{Q} = \lim_{\leftarrow k} Q_k$. We consider the polynomial ring $A = \mathbb{C}[I_1, \ldots, I_n] \overset{\iota_1}{\hookrightarrow} Q_1 = R$ which we try to lift ι_1 order by order to $A \overset{\iota_2}{\hookrightarrow} Q_2$, $\ldots$, $A \overset{\iota_k}{\hookrightarrow} Q_k$. The Poisson-commutativity of the I_i is equivalent to the liftability of ι_1 to ι_2.

Let $\Theta_A := Der(A, A) = \bigoplus_{i=1} A \frac{\partial}{\partial I_i}$ and put $C^p := R \otimes_A \wedge^p \Theta_A$. We have n commuting derivations $f \mapsto \{I_i, f\}$ of R, which combine to define a differential

$$\delta : C^p \longrightarrow C^{p+1}, \;\; fw \mapsto \sum_{i=1}^{n} \{f, I_i\} \frac{\partial}{\partial I_i} \wedge w$$

Proposition [5]**:** Consider $\iota_k : A \longrightarrow Q_k$ and a lifting to $\iota_{k+1} : A \longrightarrow Q_{k+1}$. Then there exists a well-defined obstruction element

$$\Xi = \Xi(\iota_k) \in H^2(C^\bullet, \delta).$$

with the following property: ι_k can be lifted to $\iota_{k+2} : A \longrightarrow Q_{k+2}$ by changing the lift ι_{k+1} if and only if $\Xi(\iota_k) = 0$.

We put $X = Spec(R) = \mathbb{C}^{2n}$, $S = Spec(A) = \mathbb{C}^n$ and let $I : X \longrightarrow S$ the corresponding map. There is a discriminant set $\Sigma \subset S$, such that the pull-back $I' : X' \longrightarrow S' := S \backslash \Sigma$ is smooth and for $s \in S'$ the fibre X_s is a smooth Lagrangian subvariety of X. The complex $(C^\bullet, \delta)$ can be sheafied to a sheaf complex $\mathcal{C}^\bullet$ on X.

Proposition [5]**:** There is a natural map of complexes

$$\rho : (\Omega^{\bullet}_{X/S}, d) \longrightarrow (\mathcal{C}^{\bullet}, \delta)$$

which is an isomorphism on X'.

As a consequence, the obstruction class Ξ induces for $s \in S'$ an element

$$\Xi_s \in H^2(\Omega_{X_s}) = H^2(X_s, \mathbb{C})$$

If one makes reasonable assumptions on the structure of the singularities, one can show coherence of the cohomology, using the classical Kiehl-Verdier approach:

Theorem [4]**:** If $I : X \longrightarrow S$ is *pyramidal*, then $H^i(\mathcal{C}^{\bullet}, \delta)$ are $\mathcal{O}_S$-coherent.

Corollary: If $H^2(\mathcal{C}^{\bullet}, \delta)$ is torsion free, then the obstruction Ξ is zero if and only if $\Xi_s = 0$ for generic $s \in S'$.

In fact, the modules H^i are in fact free modules in all examples we calculated.

The classical Darboux-Givental'-Weinstein theorem says that in the C^{∞} context, a neighbourhood of a Lagrange submanifold L is symplectomorphic to a neighbourhood in the cotangent bundle T^*L. The same is true in our situation for $L = X_s \subset X$, because L is a Stein space. As a consequence of the rigidity of of the Poisson structure, it seems one can construct a formal quantisation on a formal generic fibre. This *quantum Darboux theorem* would imply the vanisishing of Ξ_s for s generic. One would obtain the following corollary: If $I : X \longrightarrow S$ is pyramidal and $H^2(C^{\bullet}, \delta)$ is torsion free, then there I lifts to a formal quantum integrable system: we find $J_i \in \hat{Q}$, $[J_1, J_j] = 0$ and $J_i = I_i \mod \hbar$.

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