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Noncommutative Geometry

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Abstract. Many of the various aspects of Noncommutative Geometry were represented in the talks. This includes in particular index theory on foliated spaces, a number of results on spectral triples, noncommutative manifolds and deformation theory, Hopf cyclic theory and K-theoretic and L^2- invariants attached to certain classes of groups. The connections to physics were covered in three talks on quantum field theory on noncommutative spaces such as the Moyal plane, one of the talks giving an attractive survey of the results and techniques in that domain.

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Introduction by the Organisers

Noncommutative geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. Within mathematics, it is a highly interdisciplinary subject drawing ideas and methods from many areas of mathematics and physics. Natural questions involving noncommuting variables arise in abundance in many parts of mathematics and quantum mathematical physics. On the basis of ideas and methods from algebraic topology and Riemannian geometry, as well as from the theory of operator algebras and from homological algebra, an extensive machinery has been developed which permits the formulation and investigation of the geometric properties of noncommutative structures. This includes K-theory, cyclic homology and the theory of spectral triples. Areas of intense research in recent years are related to topics such as index theory, quantum groups and Hopf algebras, the Novikov- and Baum-Connes conjectures as well as to the study of specific questions in other fields such as number theory, modular forms, topological dynamical systems, renormalization theory, theoretical high-energy physics and string theory. Many results elucidate important properties of fascinating specific classes of examples that arise in many applications. The talks covered substantial new results and insights in several of the different areas in Noncommutative Geometry. The workshop was attended by 53 participants including 6 young researchers supported by the European Union.

Workshop: Noncommutative Geometry

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Abstracts

Semifinite noncommutative geometry and invariants from KMS states

Alan Carey

This talk summarised several projects that involve some joint work with Matilde Marcolli, Ryszard Nest, Sergey Neshveyev, John Phillips and Adam Rennie. At the moment the principal motivation comes from a proposal by Marcolli and collaborators that the C*-algebras of certain directed graphs associated to Mumford curves may contain information about the curves. Our calculations to date support this view.

The examples we have studied share some common features, namely that the algebras in question do not admit faithful traces in general but do admit faithful states or weights that are KMS for an action of the circle group. For such an algebra A with faithful KMS state τ we construct spectral flow invariants associated to certain unitaries acting on the GNS Hilbert space H_τ. In brief our construction works as follows.

We let Δ^{it} be the one parameter unitary group that implements the modular automorphism group for the circle action and we let $D = \log \Delta$. Strictly speaking we have to work with matrix algebras over A, however we will suppress this additional complexity in the discussion. In all of the examples we have been able to associate to a pair (D, u) where u is a unitary on H_τ, an auxiliary semifinite von Neumann algebra $\mathcal{M}$ such that

(1) D is affiliated to $\mathcal{M}$ and and defines an $L^{(1,\infty)}$-summable spectral triple,
(2) uDu^* is also affiliated to $\mathcal{M}$ and differs from D by a bounded self adjoint element of $\mathcal{M}$.

With these hypotheses, the von Neumann spectral flow introduced by J. Phillips and denoted $sf_{\mathcal{M}}(D, uDu^*)$, between D and uDu^* along the straight line path joining them is well defined. In our examples, when u is a unitary from A or from a matrix algebra over A, satisfying the additional conditions that both $u\Delta u^*\Delta^{-1}$ and $u^*\Delta u\Delta^{-1}$ lie in the fixed point algebra for the circle action, we are able to compute this spectral flow. This idea proved successful for graph algebras that admit faithful traces (see [7, 8] and led to a proof [9] that the C^*-algebras of directed graphs may be regarded as one dimensional noncommutative geometries in the sense of a slight generalisation of the axioms of Alain Connes [4].

In our work the formula we use for these spectral flow computations is derived from a spectral flow formula for unbounded Breuer-Fredholm operators affiliated to a semifinite von Neumann algebra proved in [1]. We modify this formula so that the spectral flow is expressed as a residue along the lines of the local index formula in noncommutative geometry [2, 3, 5, 6]. The numerical invariants that we obtain in this fashion exhibit, for all of the graph algebra cases we have considered, a similar pattern.

With Nest and Neshveyev, Rennie and I have been looking for an explanation of this pattern and we have recently found a way to understand it. If F denotes the fixed point algebra for the automorphic circle action on A then following [10] we may introduce the mapping cone algebra $M(F, A)$. What we find is that in some cases we may explain the pattern of the spectral flow invariants using the K-theory of $M(F, A)$. For the tracial case considered in [7] we are preparing a paper which explains how this works. In the case of faithful KMS states we have a less complete picture. In some cases the map that assigns to a unitary u the spectral flow $sf_{\mathcal{M}}(D, uDu^*)$ can be seen to factor through the $\mathbb{T}$-equivariant K-theory of the mapping cone.

For this idea to work for the examples proposed by Matilde Marcolli a deeper understanding of our residue formula for spectral flow is needed. Specifically the difficulty is that with respect to the trace on $\mathcal{M}$, which we construct using the KMS state τ on A, the truncated von Neumann eta invariants of the endpoints D and uDu^* which contribute to the spectral flow do not cancel. Note that the formal argument which would lead one to think that these eta invariants should be equal as they are unitarily equivalent fails because u is not in $\mathcal{M}$ and so when we try to use cyclicity of the trace we find that it fails and a twist is introduced from the modular group. The problem we are now investigating is the study Cuntz-Krieger systems that admit a type of eta cocycle associated with the KMS state τ and the auxliary trace on $\mathcal{M}$.

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Homological algebra for dense subalgebras

Ralf Meyer

The C^*-algebra of continuous functions on the circle $C(S^1)$ contains the subalgebra of smooth functions $C^\infty(S^1)$ and the subalgebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials. Many non-commutative C^*-algebras contain similar subalgebras of smooth and

polynomial elements.
As an example, we consider the rotation algebra $C(\mathbb{T}_\vartheta)$ with parameter $\vartheta \in [0,1]$; this is the universal C^*-algebra u, v generated by two unitaries that satisfy the relation $uv = \exp(2\pi i\vartheta)vu$. Its elements have Fourier or Laurent series of the form $\sum_{k,n\in\mathbb{Z}} a_{kn}u^k v^n$ with $a_{kn} \in \mathbb{C}$, which converge in a sense that we do not make precise here. The Laurent series $\sum_{k,n\in\mathbb{Z}} a_{kn}u^k v^n$ for which (a_{kn}) is rapidly decreasing form a dense subalgebra $C^\infty(\mathbb{T}_\vartheta)$ that plays the role of smooth functions; if $\vartheta = 0$, we do indeed get the algebra of smooth functions in the C^*-algebra of continuous functions on the usual 2-torus. Elements of $C(\mathbb{T}_\vartheta)$ with finite Laurent series form another dense subalgebra that plays the role of polynomial functions on the non-commutative torus $\mathbb{T}_\vartheta$.
Let G be a discrete group G. The reduced group C^*-algebra $C^*_{\mathrm{r}}(G)$ contains the group ring $\mathbb{C}[G]$ as a dense subalgebra, which plays the role of the polynomial functions in this situation. More generally, if G is totally disconnected, then we should replace $\mathbb{C}[G]$ by the Hecke algebra of the group. In general, there is no good analogue of the algebra of smooth functions. For reductive p-adic groups, we can use the Harish-Chandra–Schwartz algebra of G. Similar Schwartz algebras exist for Abelian groups and, more generally, for groups of polynomial growth. In some cases, a construction of Jolissaint provides a good candidate for an algebra of smooth functions (but this example is not so nice because the embedding of the group ring in the Jolissaint algebra of a free group is not isoradial).
In the examples above, the algebras of polynomial, smooth, and continuous functions are related in various ways. The purpose of this lecture is to identify some relations between them that can be expected in reasonable generality. We cannot expect completely general statements because already group C^*-algebra for general discrete groups are too wild for this.
First we study the relationship between smooth and continuous functions. If X is a compact manifold, then the algebra $C^\infty(X)$ of smooth functions on X is closed under holomorphic functional calculus in $C(X)$. This implies that $C(X)$ and $C^\infty(X)$ have the same K-theory. The notion of being closed under functional calculus is an intrinsically commutative one because it only involves commutative subalgebras of the algebras in question. We get a more satisfactory theory if we refine this notion to take into account the non-commutativity of the algebras in question. This is done most easily in the setting of bornological algebras, that is, algebras with a family of bounded subsets (see [3]). In this lecture, all bornological vector spaces and bornological algebras are tacitly assumed to be complete.

Definition 1. Let A be a bornological algebra and let $S \subseteq A$ be bounded. The *spectral radius* of S is defined by

$$\varrho_A(S) = \inf\Big\{ \mathrm{r} \in \mathbb{R}_{>0} \;\Big|\; \bigcup_{\mathrm{n}=1}^{\infty} (\mathrm{r}^{-1}\,\mathrm{S})^{\mathrm{n}} \text{ is bounded} \Big\}.$$

This is ∞ if $\bigcup_{n=1}^\infty (r^{-1}\,S)^n$ is unbounded for all $r \in \mathbb{R}_{>0}$.

Definition 2. Let A and B be complete bornological algebras and let $f\colon A \to B$ be an injective bounded homomorphism with dense range. Suppose that $\varrho_B(S) < \infty$ for all bounded subsets $S \subseteq B$. We call f *isoradial* if $\varrho_B\bigl(f(S)\bigr) = \varrho_A(S)$ for all bounded subsets $S \subseteq A$.

It can be expected that a "smooth" subalgebra is isoradial in the enveloping C^*-algebra. This happens in the following cases:

- for the subalgebra of smooth functions $C^\infty(X)$ in $C(X)$;
- for the subalgebra $C^\infty(\mathbb{T}_\vartheta)$ in $C(\mathbb{T}_\vartheta)$;
- for the Schwartz algebra of a polynomial growth group;
- for the Schwartz algebra of a reductive p-adic group.

The Jolissaint algebras of groups with rapid decay also satisfy this condition. Here we equip the algebras that occur with certain natural bornologies. For applications, it is important to use precompact instead of von Neumann bounded subsets in Banach spaces (see [3]).

Theorem 3. *Let $f\colon A \to B$ be an isoradial bounded algebra homomorphism. Then f induces an isomorphism in* K*-theory.*

Suppose, in addition, that B satisfies a certain approximation property explained in [3]*. Then the embedding f is invertible in bivariant local cyclic homology and hence induces an isomorphism on local cyclic homology and cohomology.*

These two statements are proved in [4, 3], respectively. For the first one, much weaker assumptions suffice. The second statement is due to Michael Puschnigg and really uses the isoradiality condition.

Another advantage of the isoradiality notion is that it behaves well with respect to tensor products and extensions (see [3]). This is useful, for instance, to check that the obvious candidate for a smooth subalgebra in the Toeplitz C^*-algebra is isoradial.

Thus local cyclic homology addresses a shortcoming of periodic cyclic homology. Since the latter yields pathological results for C^*-algebras, we are forced to pass to smooth subalgebras when studying it. But the result we get depends on the choice of the smooth subalgebra, and it is hard to be sure that we have got the right result. In contrast, local cyclic homology does not depend on the choice of smooth subalgebra. Therefore, a smooth subalgebra is a good choice if its local and periodic cyclic homology agree. Unfortunately, local cyclic homology is hard to compute by hand, and few examples have been treated so far.

Now we study the relationship between polynomial and smooth functions. In many cases, it happens that these two algebras have the same periodic cyclic homology and the same *topological* K-theory. Here it is important to use topological instead of algebraic K-theory because algebraic K-theory, say, for Laurent polynomials in several variables, is quite different from the K-theory of a torus. Topological K-theory for bornological algebras, including algebras with the fine bornology, is defined and studied in [4].

In all the examples mentioned at the beginning, the periodic cyclic homology agrees for the polynomial and smooth algebras. For K-theory, it is comparatively easy to prove this for functions on the circle and on non-commutative tori and for group algebras of groups of polynomial growth. The case of Hecke and Schwartz algebras of reductive p-adic groups has not been looked at so far, but I would expect things to work out in this case as well.

Unfortunately, we do not yet understand satisfactorily why K-theory or periodic cyclic homology agree so often for algebras of polynomial and smooth functions. In the cases where this is known, the proof works by computing the theories for both algebras and checking that the results agree.

In representation theory or algebraic geometry, we are interested in finer invariants than periodic cyclic homology or K-theory, namely, derived categories and derived functors. In all the examples mentioned at the beginning, it turns out that the embedding from the polynomial into the smooth algebra induces a fully faithful functor between the associated derived categories, provided the derived categories are defined correctly. It is important to incorporate some functional analysis into their definitions, even for algebras such as $\mathbb{C}[t, t^{-1}]$ that carry a fine bornology.

The embedding $\mathbb{C}[t, t^{-1}] \to C^\infty(S^1)$ serves as a trivial example to see the the problem. If V is a module over $A = \mathbb{C}[t, t^{-1}]$, then the following (Koszul) resolution provides a free resolution of V:

$$V \leftarrow A \otimes V \leftarrow A \otimes V \leftarrow 0 \leftarrow \cdots ,$$

where the map $A \otimes V \to V$ is the multiplication map $a \otimes v \mapsto a \cdot v$ that describes the module structure and the map $A \otimes V \to A \otimes V$ maps $a \otimes v \mapsto at \otimes v - a \otimes tv$. Similarly, if V is a bornological module over $B = C^\infty(S^1)$, then the following is a free resolution of V:

$$V \leftarrow B \hat{\otimes} V \leftarrow B \hat{\otimes} V \leftarrow 0 \leftarrow \cdots ,$$

where the maps are defined by the same formulas. Here we use the completed projective bornological tensor product. In contrast, if we use the purely algebraic tensor product, then already the chain complex

$$B \leftarrow B \otimes B \leftarrow B \otimes B \leftarrow 0 \leftarrow \cdots , \tag{4}$$

fails to be exact. The issue is that the proof of exactness of the two resolutions above involves division; this usually produces Laurent series with infinitely many summands, which cannot be accommodated in the algebraic tensor product. The homology groups of the chain complex (4) are $\mathrm{Tor}_1^A(B, B)$ and the kernel of the canonical map $\mathrm{Tor}_0^A(B, B) = B \otimes_A B \to B$. If homological algebra for B-modules would reduce to homological algebra for A-modules, we would expect $\mathrm{Tor}_*^A(B, B) \cong \mathrm{Tor}_*^B(B, B)$, which would predict (4) to be exact. Since it is not exact, the canonical functor on derived categories $\mathsf{Der}(B) \to \mathsf{Der}(A)$ cannot be fully faithful.

To avoid this problem, we should replace the vector space tensor product $\otimes$ by the completed projective bornological tensor product. This can be incorporated as follows. For a unital bornological algebra A, let $\mathsf{Mod}(A)$ be the category of (complete) bornological A-modules with bounded A-module homomorphisms as morphisms. It is possible to extend this to certain non-unital algebras (see [1]), but we do not discuss this here to avoid complications. The category $\mathsf{Mod}(A)$ becomes an exact category if the admissible extensions are the extensions with a bounded linear section, that is, extensions that split if we ignore the A-module structures. In this exact category, free modules of the form $A \hat{\otimes} V$ are projective, and bar resolutions with completed tensor products provide projective resolutions. Given any exact category such as $\mathsf{Mod}(A)$, there is a standard recipe to form a derived category. First consider the category of chain complexes (say, unbounded, but it does not matter much). A chain map is called a quasi-isomorphism if its mapping cone is admissibly exact, that is, obtained by splicing admissible exact sequences. The localisation at the quasi-homomorphisms is the derived category $\mathsf{Der}(A)$. Since $\mathsf{Mod}(A)$ has enough projective and injective objects, we can compute morphisms in $\mathsf{Der}(A)$ and derived functors using projective or injective resolutions.

Theorem 5. *Let $f\colon A \to B$ be a bounded algebra homomorphism between two bornological algebras. Then the following are equivalent:*

(1) *the (forgetful) functor $f^*\colon \mathsf{Der}(B) \to \mathsf{Der}(A)$ is fully faithful;*
(2) *let $P_A \to A$ be a projective A-bimodule resolution of A; then*

$$B \hat{\otimes}_A P_A \hat{\otimes}_A B \to B$$

is again an admissible resolution.

If this is the case, we call f isocohomological*.*

Various other equivalent characterisations of isocohomological algebra homomorphisms can be found in [1]. The main point of the proof is that an A-bimodule resolution of A allows us to compute all kinds of derived functors. The chain complex $B \hat{\otimes}_A P_A \hat{\otimes}_A B$ is automatically a chain complex of projective B-modules. If it is a resolution as well, then we get a projective bimodule resolution for B from one for A.
The embedding of the polynomial algebra in the smooth algebra is isocohomological for all the examples considered at the beginning by the results in [1, 2]. To see why this is useful, let us consider one of the applications in [2]. If f is isocohomological and V, W are two B-modules, then we get

$$\mathrm{Ext}^*_A(V, W) \cong \mathrm{Ext}^*_B(V, W). \tag{6}$$

Now let A and B be the Hecke and the Schwartz algebra of a reductive p-adic group. If V is a discrete series representation, then V is projective as a B-module, but not necessarily as an A-module. Nevertheless, (6) yields $\mathrm{Ext}^n_A(V, W) = 0$ for $n > 0$ if V is discrete series and W is tempered, even if V is not projective. This vanishing result also has other interesting applications. For instance, together with previous work of Schneider and Stuhler, it yields a quantisation of formal dimensions of square-integrable representations (see [2]).
Such vanishing results were observed in all examples where the relevant Ext-groups were computed, and this led representation theorists to conjecture that there should be a comparison of derived categories. But to be able to prove such a

statement, it is necessary to incorporate functional analysis into the derived categories because otherwise the result fails already in the trivial case of functions on the circle, corresponding to the group of integers.

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Duality and analogues of the Lefschetz formula in operator algebras

Heath Emerson

K-theoretic duality of one kind or another plays a significant role in operator algebras. There are three good examples of the the kind of duality I have in mind.

(1) The Baum-Connes duality (see *e.g.* [1]) whose statement I extend slightly to assert that $KK^G(C_0(\underline{EG}) \otimes A, B) \cong KK(A, B \rtimes G)$ where A is a trivial G-C^*-algebra.

(2) Noncommutative Poincaré duality for C^*-algebras, asserting in various special cases A, B that there is a natural family of isomorphisms
$$KK(A \otimes P, Q) \cong KK(P, B \otimes Q),$$
one for each P, Q.

(3) Kasparov duality (see [4]) which says that
$$\mathrm{RKK}^G(X; A, B) \cong KK^G(\mathcal{P} \otimes A, B)$$
for a given G-space X and some G-C^*-algebra $\mathcal{P}$.

It turns out that there is a connection between duality and what I loosely term Lefschetz formulae. I will only discuss the examples 2) and 3) here because 1) is more or less equivalent to 3) modulo the Baum-Connes conjecture, and the Baum-Connes conjecture understood in terms of the Dirac dual Dirac method is not itself a duality but a question about whether or not the γ-element is the identity.

In reference to example 3) and based on ideas from [3] and in collaboration with R. Meyer, I have extracted an invariant from a Kasparov duality (in the above sense) for a G-space X which we call the Lefschetz invariant of that G-space. The Lefschetz invariant is really a map

$$\mathrm{Lef}\colon \mathrm{RKK}^G(X; C_0(X), \mathbb{C}) \to KK^G(C_0(X), \mathbb{C}). \tag{1}$$

This map should be thought of as producing from a morphism $X \to X$ in KK^G a class in the equivariant K-homology of X. (For these purposes, the RKK^G-group is in fact the 'correct' home for morphisms $X \to X$, when X is noncompact). The idea is that Lef in some sense takes the intersection of the graph of the morphism and the diagonal in $X \times X$; this produces a linear combination of 'geometric cycles' for the (equivariant) K-homology of X. This is a pretty accurate account of what happens when X is a smooth manifold. Another situation, where differential topology is absent, is when X is a G-simplicial complex. The method still works, since the Lefschetz invariant is determined by a duality, and there is a duality available in this setting based on work of Kasparov and Skandalis, we can describe the Lefschetz map in combinatorial terms. One assumes one has a G-equivariant morphism $X \to X$ which maps simplices of X to unions of simplices. Then one can extract from each simplex the number of times that simplex occurs in the chain which is its image under the map. These multiplicities make up a G-equivariant (zero-dimensional, in this case) K-homology class in a fairly obvious way. This is the Lefschetz invariant of the skeletal map in question. Thus, in particular, we have assigned Lefschetz data to morphisms in a unified way for both simplicial complexes and for smooth manifolds.

The point of the construction is that Lef only depends on X as a G-proper-homotopy class of spaces, and not on further structure. But one cannot *describe* Lef at the level of cycles without utilizing further structure (the structure implicit in a dual for X), for example a differential structure, or a triangulation. The corresponding description of Lef then is in terms of the extra structure. So one can relate, for instance, combinatorial and differential-geometric invariants, if, say, X admits both a smooth Riemannian structure, and a simplicial structure. The simplest such application asserts that if X is a proper G-manifold then the class of the de Rham operator on X is essentially 0-dimensional (a sum of point K-homology classes), a nontrivial assertion previously proven by Witten's technique of perturbing the de Rham complex by a generic vector field, but rather trivial given our set-up. The classical Lefschetz fixed-point formula is also a special case. It is fortunate that, dictated by the requirements of equivariant KK-theory, we are forced to talk not about Lefschetz numbers, but Lefschetz classes. For as mentioned above, our set-up applies in principle not just to maps $X \to X$ but to general KK-elements. Conventional Lefschetz fixed-point theory picks off the zero-dimensional part of our Lefschetz classes. When the morphism in question is in a sense a higher-dimensional object than a map from X to X, there is correspondingly higher-dimensional fixed-set information which is lost by passing to numbers. This information is retained in our set-up. Thus, we have 'higher Lefschetz formulae' associated to smooth and G-equivariant correspondences from X to X; they are too complicated to state here. It would be nice if there was a notion of combinatorial correspondence: this could give higher dimensional versions of the Lück-Rosenberg theorem.

We are also working on generalizing the framework to where the group G is allowed to be a groupoid. For instance, one should be able to easily prove that the class of the longitudinal de Rham operator on a compact, foliated manifold, is equal to the zero-dimensional class of an appropriately chosen transversal (of course 0-dimensional in this case means intersection with each leaf should be 0-dimensional.) Connes has stated a theorem quite close to this.

The rest of my work so far in this area is concerned with the example 2) above. Thus, here one also has an appropriate notion of duality and correspondingly an abstract Lefschetz formula. Suppose one has two C^*-algebras A and B, assumed nuclear, separable and satisfying the Kunneth theorem. Suppose they are Poincaré dual in the sense of Connes. So there are classes $\Delta \in KK(A \otimes B, \mathbb{C})$ and $\widehat{\Delta} \in KK(\mathbb{C}, A \otimes B)$ are the fundamental classes of the duality, satisfying the 'zigzag equations' of adjoint functors. One of course wants classes with interesting descriptions (as with duals for G-spaces above.) The abstract Lefschetz theorem in this case is easy to state: if $\alpha \colon A \to A$ is a morphism in KK, then the *graded trace of* $\alpha_* \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}} \colon \mathrm{K}_*(A) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{K}_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is equal to the index pairing* $< \widehat{\Delta} \otimes_A \alpha, \Delta >$. The point is that the intersection product of $\widehat{\Delta} \otimes_A \alpha$ and Δ admits a description related to the geometry of Δ and $\widehat{\Delta}$. A very simple but very illustrative example is when $A = B$ is the C^*-algebra of a finite group and α comes from a group automorphism. The tracial side turns out to be the number of fixed points of the induced permutation of the irreducible dual of the group. The geometric side is the number of twisted conjugacy classes. (In particular, these two invariants are equal.) With S. Echterhoff and Hyun-Jeong Kim I have worked out in some detail (based on material from [2]) the case of $A = C_0(X) \rtimes G$ where G is a countable group acting properly (but *not* necessarily freely) on a manifold X. The morphisms we study are covariant pairs for (X, G). The result is a rather beautiful formula for the trace involving fixed orbits of ϕ, twisted conjugacy classes, and orientation data. The computation seems to strongly recommend looking at other examples.

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The Higher Harmonic Signature for Foliations

Moulay-Tahar Benameur

(joint work with James Heitsch)

1. The main theorem

One goal of index theory is to develop and relate analytic and topological invariants. A paradigm for this sort of result is the Atiyah-Singer index theorem for families of elliptic operators defined along the fibers of a compact fiber bundle N. The index of such a family is the difference of families of spaces (i.e. a K-theory class) over the base space of the bundle. The families index theorem relates the Chern character of this index bundle to the characteristic classes of the tangent bundle along the fibers of N and the symbol bundle of the family of operators. A major application of this index theorem was Lusztig's proof of the Novikov conjecture for free abelian groups, [9].

In the foliation case, it is an important problem to develop a theory which incorporates index bundles, when they exist, into the general index theory. In this work, we define the higher harmonic signature, $\sigma(F)$, of an 2ℓ-dimensional oriented foliation F of a compact Riemannian manifold M, and prove that, for Riemannian foliations, it is a leafwise homotopy invariant. As $\sigma(F)$ is equal to the Chern character of the index bundle of the leafwise signature operator, this work is a contribution to the goal of integrating the index bundle into general non-commutative index theory.

A leafwise metric determines Laplace operators Δ and Hodge $*$-operators on the differential forms on the leaves and their covers. The Hodge operator allows a splitting $\Delta_\ell = \Delta_\ell^+ + \Delta_\ell^-$. To each leaf L of F, we associate the difference of the (in general, infinite dimensional) spaces $\mathrm{Ker}(\Delta_\ell^+)$ and $\mathrm{Ker}(\Delta_\ell^-)$ on its holonomy (or monodromy) cover. Assume that the Schwartz kernels of the projections onto these two spaces (thought of as sections of a certain bundle over the groupoid) vary *smoothly transversely* in a sense that is made precise in [3]. In our previous work [2], we defined a Chern-Connes character ch_a for such "bundles", which takes values in the Haefliger cohomology of F. The higher harmonic signature of F is then defined as

$$\sigma(F) = \mathrm{ch}_a(\mathrm{Ker}(\Delta_\ell^+)) - \mathrm{ch}_a(\mathrm{Ker}(\Delta_\ell^-)).$$

The main theorem that I explained in my talk in this conference is the following.

Theorem 1. *Suppose that M is a compact Riemannian manifold, with oriented Riemannian foliation F of dimension 2ℓ. Then $\sigma(F)$ is a leafwise homotopy invariant.*

It is an open question whether the projection to the leafwise harmonic forms always has transversely smooth Schwartz kernel. This assumption is satisfied for all foliations with compact leaves and Hausdorff graph [6, 7], and in particular for fibrations. We have assumed that the foliation is Riemannian to ensure this condition, see [4]. Simple examples of fibrations of Grassmannians over Grassmannians show that the higher components of the higher harmonic signature are non-trivial in general. See also [1].

We point out that this theorem with a more general twisted version can be deduced from an extension of the analytic method developped by Connes in [5], but the method we develop here is completely geometric and new, and is in the spirit of the classical proofs using connections on bundles. We believe that besides the result itself, this approach can be applied to a wider class of topological problems.

Let us explain briefly the main steps of the proof. We fix a smooth leafwise homotopy equivalence $f : (M, F) \to (M', F')$ with homotopy inverse g.

(1) We introduce the notion of smooth bundle with connection over the space of leaves and prove that the Chern-Connes character of such bundles is well defined and can be recovered using any such connection. The connections we use extend the Bismut approch and correspond grossomodo to the notion of connection over the space of leaves.
(2) We prove that the pull-back under the leafwise homotopy equivalence f of a smooth bundle with connection is a smooth bundle with connection. It is worth pointing out that the pull-back of a leafwise L^2-form is not L^2 in general and hence that one needs to adapt the Hilsum-Skandalis (HS) method to overcome this difficulty and to define a new notion of pull-back map. It is this HS pull-back map that is used to define the pull-back bundle and the pull-back connection. As an important step, we quote here that this HS pull-back map is proved to be uniformly bounded between Sobolev spaces and not only L^2.
(3) We show by using the properties of leafwise maps that the Chern-Connes character of the pull-back bundle coincides in Haefliger cohomology with the pull-back of the Chern-Connes character of the given bundle. As a corollary, we deduce an interpretation of the pull-back of the higher signature $\sigma(F')$ under the equivalence relation f, using a smooth bundle $\pi^f_\pm$ over (M, F).
(4) To complete the proof, one needs to relate $\mathrm{ch}_a(\pi^f_\pm)$ with $\sigma(F)$. This is achieved using a smooth isomorphism between the two smooth bundles. While this isomorphism is easy to construct, a non trivial result is the smoothness of its inverse.

The consequence of our method is also the leafwise homotopy invariance of the higher Betti numbers.

2. Discussions and open questions

As previously mentioned, the geometric method that we have adopted applies as well to the higher harmonic signature with coefficients in a leafwise flat bundle. We are currently working on an extension of this work to the twist by leafwise almost flat bundles as studied by Hilsum and Skandalis in [8]. We now explain the relation of such theorem with the leafwise Novikov conjecture.
In [2], we proved that given a closed holonomy invariant current C of dimension $2k$ on the foliation, the pairing of the twisted, by a vector bundle E, higher harmonic signature with C, coincides with the characteristic number

$$\int_F \mathbb{L}(TF)\,\mathrm{ch}(E) \quad \in H^*_c(M/F),$$

whenever the Novikov-Shubin invariant is larger than k. As for a large class of discrete groups and according to Gromov's observation, almost flat bundles along the leaves are suspected to generate, for a large class of foliations, the cohomology of the classifying space of the groupoid. The precise conjectured statement is given in [4]. In [3], we conjectured that

- the Novikov-Shubin invariants of Riemannian foliations are always positive;
- the above equality between the Chern-Connes character of the index bundle of a leafwise elliptic operator and the expected characteristic classes is always true, under the weaker assumption that the index bundle of this operator exists.

We observe that by using the Hilsum-Skandalis theorem [8], and modulo the generalization of a famous extension theorem of Connes [5] to all closed invariant currents, one can deduce an analytic proof of our main theorem. Many experts think that this generalization is straightforward.

Another approach that would replace Connes' extension method was indicated to us by Joachim Cuntz and is a consequence of the following probably true statement for Riemannian foliations. Denote by HL_* Puschnigg's local cyclic homology. Then there should exit a morphism

$$\Phi_L : HL_*(C^*(M,F)) \longrightarrow H_c^*(M/F),$$

such that $\Phi_L \circ \mathrm{Ch}_L = \mathrm{ch}_a$ on $K_*(C_c^\infty(G))$, where

$$\mathrm{Ch}_L : K_*(C^*(M,F)) \to HL_*(C^*(M,F))$$

is Puschnigg's Chern-Connes character. This approach is somehow close to Connes' extension approach.

I would like to thank the organizers for giving me the opportunity to present and discuss this work. I am grateful to P. Baum, A. Carey, J. Cuntz, A. Gorokhovsky, V. Mathai, R. Meyer, H. Moscovici, P. Piazza, M. Puschnigg and T. Schick, for several useful discussions during the workshop.

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Algebraic vs. topological K-theory of locally convex algebras and Karoubi's conjecture

Guillermo Cortiñas
(joint work with Andreas Thom)

This talk was about our joint paper [3] on the comparison between algebraic and topological K-theory. This is a classical subject with numerous applications, which has been considered by several authors, for different classes of topological algebras, and using a wide variety of tools (see J. M. Rosenberg's excellent survey [9]).
Here we are concerned with the comparison between algebraic and topological K-theory of not necessarily unital locally convex $\mathbb{C}$-algebras. By a locally convex algebra we understand a complete locally convex vector space L together with an associative multiplication map $L\hat{\otimes}L \to L$; here $\hat{\otimes}$ is the projective tensor product of A. Grothendieck. We establish a six-term exact sequence relating algebraic and topological K-theory with algebraic cyclic homology. We show that if $\mathcal{J}$ is a Fréchet operator ideal and L a locally convex algebra, then there is an exact sequence

$$\begin{array}{ccccc} K_1^{\mathrm{top}}(L\hat{\otimes}\mathcal{J}) & \longrightarrow & HC_{2n-1}(L\hat{\otimes}\mathcal{J}) & \longrightarrow & K_{2n}(L\hat{\otimes}\mathcal{J}) \\ \uparrow & & & & \downarrow \\ K_{2n-1}(L\hat{\otimes}\mathcal{J}) & \longleftarrow & HC_{2n-2}(L\hat{\otimes}\mathcal{J}) & \longleftarrow & K_0^{\mathrm{top}}(L\hat{\otimes}\mathcal{J}). \end{array} \tag{1}$$

Here K_* is algebraic K-theory. There are several possible definitions for topological K-theory of general locally convex algebras; however we show that they all coincide for algebras of the form $L\hat{\otimes}\mathcal{J}$ as above. Thus for example in the sequence above we can define K^{top} in terms of Cuntz' bivariant K-theory for locally convex algebras, see [4],

$$K_*^{\mathrm{top}}(L\hat{\otimes}\mathcal{J}) = kk_*^{\mathrm{top}}(\mathbb{C}, L\hat{\otimes}\mathcal{J}).$$

The algebraic cyclic homology groups are taken over $\mathbb{Q}$; this means that the (algebraic) tensor products appearing in the complex we use for defining HC (there are several quasi-isomorphic such complexes) must be taken over $\mathbb{Q}$. For example, we have $HC_*(A) = H_*(C^\lambda(A), b)$, where C^λ is Connes' complex, see [8],

$$C_n^\lambda(A) = (A^{\otimes_{\mathbb{Q}}^{n+1}})_{\mathbb{Z}/\langle n+1\rangle}.$$

The meaning of the sequence (1) is clear; it says that, for locally convex algebras stabilized by a Fréchet operator ideal, algebraic cyclic homology measures the obstruction for the comparison map $K_* \to K_*^{\mathrm{top}}$ to be an isomorphism. As an immediate application of this, and of the fact that, by definition, cyclic homology vanishes in negative degrees, we get

$$K_n(L\hat{\otimes}\mathcal{J}) = K_n^{\mathrm{top}}(L\hat{\otimes}\mathcal{J}) \qquad (n \le 0). \tag{2}$$

The particular case of (2) when $\mathcal{J} = \mathcal{L}_p$, and $p > 1$ (or, more generally, when $\mathcal{J}$ is harmonic) was proved in [5, Thm. 6.2.1].
It is also clear from (1) that the vanishing of cyclic homology in *all* degrees is

equivalent to the isomorphism between algebraic and topological K-theory. For example, we show that if $\mathcal{J} = \mathcal{L}_\infty$ is the ideal of all compact operators and L is a unital Banach algebra then $HC_*(L\hat{\otimes}\mathcal{J}) = 0$, whence

$$K_*(L\hat{\otimes}\mathcal{J}) = K_*^{\mathrm{top}}(L\hat{\otimes}\mathcal{J}). \tag{3}$$

This establishes Karoubi's conjecture (as stated in [7]). In fact we show (3) holds more generally when L is a Fréchet algebra whose topology is generated by a countable family of sub-multiplicative seminorms and admits an approximate right or left unit which is totally bounded with respect to that family. We point out that Mariusz Wodzicki is credited with the solution of Karoubi's conjecture, both the original one and the generalization just mentioned, as well as with other results proved in this paper. He has lectured on these results in several places, including Heidelberg and Paris, giving full details of his proofs. However, although some of these results have been announced in [10], his proofs have not been published in print except in some particular cases, see [6]. Our proofs use some of the published results of Wodzicki, as well as other results which are independent of his work. For example, most of our proofs rely heavily on the diffeotopy invariance theorem from [5] –which we generalize– and the excision theorem for infinitesimal K-theory from [2], none of which were available at the time when Wodzicki's pioneering work [10] appeared.
Another result announced in [10, Thm. 5] is the existence of a 6-term exact sequence
(4)

$$\begin{array}{ccccc}
K_{-1}(A\otimes\mathcal{J}) & \longrightarrow & HC_{2n-1}(A\otimes\mathcal{B}:A\otimes\mathcal{J}) & \longrightarrow & K_{2n}(A\otimes\mathcal{B}:A\otimes\mathcal{J}) \\
\uparrow & & & & \downarrow \\
K_{2n-1}(A\otimes\mathcal{B}:A\otimes\mathcal{J}) & \longleftarrow & HC_{2n-2}(A\otimes\mathcal{B}:A\otimes\mathcal{J}) & \longleftarrow & K_0(A\otimes\mathcal{J}).
\end{array}$$

Here A is an H-unital $\mathbb{C}$-algebra, $\otimes$ is the algebraic tensor product over the complex numbers $\mathbb{C}$, $\mathcal{B}$ is the algebra of bounded operators in an infinite dimensional, separable Hilbert space, and $\mathcal{J}$ is what we call a sub-harmonic operator ideal. We prove (4) for all algebras A and for all sub-harmonic operator ideals $\mathcal{J}$. Thus we generalize Wodzicki's sequence from H-unital algebras to all algebras. Furthermore, we also prove a variant of (4), which is still valid under the same hypothesis, and which involves absolute, rather than relative K-theory and cyclic homology. We show that there is an exact sequence

$$\begin{array}{ccccc}
K_{-1}(A\otimes\mathcal{J}) & \longrightarrow & HC_{2n-1}(A\otimes\mathcal{J}) & \longrightarrow & K_{2n}(A\otimes\mathcal{J}) \\
\uparrow & & & & \downarrow \\
K_{2n-1}(A\otimes\mathcal{J}) & \longleftarrow & HC_{2n-2}(A\otimes\mathcal{J}) & \longleftarrow & K_0(A\otimes\mathcal{J}).
\end{array} \tag{5}$$

Examples of sub-harmonic ideals include all Fréchet ideals as well as some ideals, such as the Schatten ideals $\mathcal{L}_p$ with $0 < p < 1$, which are not even locally convex. In the particular case when $A = \mathbb{C}$ both (4) and (5) simplify. Indeed we show that, as stated without proof in [10, Prop. on p. 491], we have

$$K_{-1}(\mathcal{J}) = 0, \quad K_0(\mathcal{J}) = \mathbb{Z} \tag{6}$$

for any proper operator ideal $\mathcal{J}$. Thus (5) becomes

$$0 \to HC_{2n-1}(\mathcal{J}) \to K_{2n}(\mathcal{J}) \to \mathbb{Z} \stackrel{\alpha_n}{\to} HC_{2n-2}(\mathcal{J}) \to K_{2n-1}(\mathcal{J}) \to 0. \tag{7}$$

We show moreover that if $\mathcal{I} \subset \mathcal{L}_p$ $(p \geq 1)$ then α_n is injective for $n \geq (p+1)/2$. As an application of this we obtain a new description of the multiplicative character of a p-summable Fredholm module defined by A. Connes and M. Karoubi in [1]. One can also combine (6) with (4) to obtain a sequence similar to (7), but involving relative, instead of absolute K-theory and cyclic homology:

$$0 \to HC_{2n-1}(\mathcal{J}) \to K_{2n}(\mathcal{B} : \mathcal{J}) \to \mathbb{Z} \to HC_{2n-2}(\mathcal{B} : \mathcal{J}) \to K_{2n-1}(\mathcal{J}) \to 0. \tag{8}$$

This is the sequence announced in [10, Thm. 6]; Dale Husemöller took the work to write down Wodzicki's proof in [6].
As indicated above we obtain a generalization of the diffeotopy invariance theorem proved by J. Cuntz and the second author in [5]. The latter implies that if E is a functor from the category $\mathfrak{LocAlg}$ of locally convex algebras to abelian groups which is split exact and stable with respect to 2×2-matrices, and $\mathcal{J}$ is a harmonic ideal, then

$$L \mapsto E(L \hat{\otimes} \mathcal{J}) \tag{9}$$

is diffeotopy invariant, i.e. sends the two evaluation maps $A \hat{\otimes} C^{\infty}(\Delta^1) \to A$ to the same morphism. Recall that a harmonic ideal is a Banach operator ideal $\mathcal{J} \subset \mathcal{B}$ with continuous inclusion, which contains a compact operator whose sequence of singular values is the harmonic sequence $(\frac{1}{n})_n$, and which is multiplicative. We prove that, under an extra hypothesis on E, the functor (9) is diffeotopy invariant for any Fréchet ideal $\mathcal{J}$. The extra hypothesis essentially says that E sends certain homomorphisms whose kernel and cokernel are both square-zero algebras into isomorphisms. For example the functors K_n for $n \leq 0$ as well as the infinitesimal and (polynomial) homotopy K-theory groups K_n^{inf} and KH_n for $n \in \mathbb{Z}$, all satisfy these hypothesis. Hence if A is any $\mathbb{C}$-algebra, and $\mathcal{J}$ any Fréchet ideal, then the functors

$$KH_*(A \otimes (? \hat{\otimes} \mathcal{J})) \text{ and } K_*^{\mathrm{inf}}(A \otimes (? \hat{\otimes} \mathcal{J})), \tag{10}$$

are diffeotopy invariant. Using this we show, for example, that

$$A \otimes (L \hat{\otimes} \mathcal{J}) \text{ is } K_0\text{-regular}. \tag{11}$$

In particular

$$K_n(A \otimes (L \hat{\otimes} \mathcal{J})) = KH_n(A \otimes (L \hat{\otimes} \mathcal{J})) \quad (n \leq 0).$$

Further, we prove that topological and homotopy K-theory agree on stable locally convex algebras. We have

$$KH_*(L \hat{\otimes} \mathcal{J}) = K_*^{\mathrm{top}}(L \hat{\otimes} \mathcal{J}) \tag{12}$$

for every Fréchet ideal $\mathcal{J}$. In particular both (11) and (12) hold when $\mathcal{J} = \mathcal{L}_p$, $p \geq 1$.

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Secondary invariants of Dirac operators on foliated spaces

Paolo Piazza
(joint work with Moulay Benameur, Hitoshi Moriyoshi, Thomas Schick)

Index theory for Dirac operators is not only a fascinating subject but it is also a very useful tool in establishing purely geometric results. Still, there are situations where one needs invariants associated to Dirac operators that are more sophisticated than the index. Rho-invariants, introduced by Atiyah, Patodi and Singer in [1] are such secondary invariants; they are defined in terms of a Dirac-type operator D on a compact odd dimensional manifold M and the choice of two unitary representations $\alpha, \beta : \pi_1(M) \to U(\ell)$. More precisely: $\rho_{\alpha-\beta}(D) := \eta(D_\alpha) - \eta(D_\beta)$, with η indicating the classic eta invariant and D_α, D_β indicating the Dirac operators twisted by the flat bundles defined by α and β. The rho-invariant for the signature operator of an orientable manifold can be used in order to distinguish non-diffeomorphic lens spaces that are homotopy equivalent; the rho invariant for the spin-Dirac operator on a spin manifold is useful in distinguishing metrics of positive scalar curvature that are not path-connected or even non-bordant (results by Botvinnik-Gilkey).
Now, index theory for Dirac operators has been developed in geometric situations of increasing complexity:

- Galois coverings with structure group Γ
- measured foliations
- general foliations

In the first two cases one can define a Von Neumann index and corresponding formulae have been proved respectively by Atiyah and Connes. In the third case one pairs the index class of a longitudinal Dirac operator with suitable cyclic cocycles, a very interesting particular example being given by the so-called Godbillon-Vey cyclic cocycle; an index formula has been established in this case by Connes with alternative proofs given later by Moriyoshi-Natsume and Gorokhovsky-Lott.
The basic themes of my talk can be summarized as follows:

- is it possible to define secondary invariants of type rho in these more complex situations ?
- are these invariants interesting ?
- can one give purely geometric applications of these invariants ?

The rho-invariant on Galois coverings has been introduced by Cheeger-Gromov as the difference of the Von Neumann eta invariants upstairs, i.e. for the lifted Dirac operator, and the usual eta invariants downstairs. I explained how the fundamental dichotomy for L^2-invariants on Γ-coverings, namely whether Γ has torsion or whether Γ is torsion-free, presents itself also in the case of the Cheeger-Gromov rho-invariant. Indeed, if Γ is torsion-free and, in addition, satisfies the Baum-Connes hypothesis for the maximal group C^*-algebra, then one can prove that the Cheeger-Gromov rho-invariant for the signature operator is a homotopy-invariant, whereas it is equal to zero for the spin-Dirac operator associated to a metric of positive scalar curvature. These results are originally due to Keswani, see [3], and have been reproved by Piazza and Schick in [5], where an analogous result was also proved for Lott's delocalized eta invariant (the latter result involves the weaker hypthesis of Baum-Connes valid for the *reduced* C^*-algebra).
The message that we should retain here is that when Γ is torsion-free (and under a Baum-Connes assumption), *the Cheeger-Gromov rho-invariant behaves exactly as an index.* This is a rather interesting phenomenon, given that the rho-invariant is defined in terms of the heat-kernel for all times.
When Γ has torsion, the situation is completely different and the rho-invariant becomes a true secondary invariant. *Geometric applications* of the rho-invariant in this case are due to Chang-Weinberger [2] (who proved the following: given a manifold M of dimension $4k+3$ with fundamental group with torsion then there is an *infinite number* of manifolds that are homotopy-equivalent to M but not diffeomorphic to it) and to Piazza and Schick [6] (who proved that in the same dimensions a spin manifolds with fundamental group with torsion with one metric of positive scalar curvature will admit an *infinite number* of metrics of positive scalar curvature that are paiwise non-bordant).
In the rest of my talk I then concentrated on foliated spaces admitting an invariant transverse measure ν. In this case, given a longitudinal Dirac operator D one can introduce a Von Neumann eta invariant upstairs, defined in terms of the monodromy groupoid, and a von Neumann eta invariant downstairs, defined in terms of the groupoid associated to the equivalence relation fixed by the foliation. The first definition is due to Peric [4] whereas the second is due to Ramachandran [7]. We define the foliated rho-invariant, $\rho_\nu(D)$, as the difference of these two eta

invariants. For such an invariant one can ask the usual fundamental questions: is this an interesting invariant? Can one give geometric applications of it?
I reported on a recent result in collaboration with Moulay Benameur:

Theorem 1. *Let X be a foliated space and assume that X is a foliated flat bundle: $X = \tilde{M} \times_\Gamma T$, with $\Gamma \to \tilde{M} \to M$ the universal covering of a compact manifold and T a Γ-topological space endowed with a Γ-invariant measure ν. If Γ is torsion free and the maximal Baum-Connes map with coefficients in $C(T)$ is an isomorphism, then $\rho_\nu(D^{\text{sign}})$ is a foliated homotopy invariant.*

In other words, we see once again that the rho-invariant for the signature operator behaves like an index in a torsion-free situation (and under a Baum-Connes assumption).
The attempt to answer to the second question, i.e. to give geometric applications of this invariant when Γ is not torsion free, is work in progress with Benameur.
Finally, in the very last part of my talk I reported briefly about the problem of defining secondary invariants in the general case, for example for the Godbillon-Vey cyclic cocyle. It should be said that it is already unclear how to define a Godbillon-Vey eta invariant. Now, eta invariants are boundary correction terms to index theorems on manifolds with boundary and one strategy is to proceed as Atiyah Patodi and Singer did, namely to prove a Godbillon-Vey index theorem on manifolds with boundary and find the corresponding eta invariant directly out of the formula. I reported on joint work with Hitoshi Moriyoshi in this direction: what we proved is the existence of the Godbillon-Vey index on a foliated flat bundle with boundary under the assumption that the londitudinal Dirac operator induced on the boundary-foliation is invertible. Because of the non-locality of the parametrix this theorem is not at all obvious and in fact employs heavily an extension of the the b-calculus of Melrose to the foliated context.

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Noncommutative Field Theory

Razvan Gurau and Vincent Rivasseau

1. Introduction

Noncommutative (NC) quantum field theory (QFT) may be important for physics beyond the standard model and for understanding the quantum Hall effect [1]. It also occurs naturally as an effective regime of string theory [2, 3].
The simplest NC field theory is the ϕ^4_4 model on the Moyal space. Its perturbative renormalizability at all orders has been proved by Grosse, Wulkenhaar and followers [4]. Grosse and Wulkenhaar solved the difficult problem of ultraviolet/infrared mixing by introducing a new harmonic potential term inspired by the Langmann-Szabo (LS) duality [5] between positions and momenta.
An amazing discovery was made in [6]: the non commutative ϕ^4_4 model does not exhibit any Landau ghost at one loop. It is not asymptotically free either. For any renormalized Grosse-Wulkenhaar harmonic potential parameter $\Omega_{ren} > 0$, the running Ω tends to the special LS dual point $\Omega_{bare} = 1$ in the ultraviolet. As a result the RG flow of the coupling constant is simply bounded[1]. This result was extended up to three loops in [7].
We compute the flow at the special LS dual point $\Omega = 1$, and check that the beta function vanishes at all orders using a kind of Ward identity inspired by those of the Thirring or Luttinger models [8]. Note however that in contrast with these models, the model we treat has quadratic (mass) divergences.
We denote $\Gamma^4(0,0,0,0)$ the amputated one particle irreducible four point function and $\Sigma(0,0)$ the amputated one particle irreducible two point function with external indices set to zero. The wave function renormalization is $\partial_L\Sigma = \partial_R\Sigma = \Sigma(1,0) - \Sigma(0,0)$ [7]. Our main result is [9]:

Theorem 1. *The equation*

$$\Gamma^4(0,0,0,0) = \lambda(1 - \partial_L\Sigma(0,0))^2 \tag{2}$$

holds up to irrelevant terms to **all** *orders of perturbation, either as a bare equation with fixed ultraviolet cutoff, or as an equation for the renormalized theory. In the latter case λ should still be understood as the bare constant, but reexpressed as a series in powers of λ_{ren}.*

The proof of this result resides on a Ward identy (see fig.1) which writes:

$$(a-b) < [\bar\phi\phi]_{ab}\phi_{\nu a}\bar\phi_{b\nu} >_c = < \phi_{\nu b}\bar\phi_{b\nu} >_c - < \bar\phi_{a\nu}\phi_{\nu a} >_c \tag{3}$$

(repeated indices are not summed).
The proof of this identity consists of making the change of variables:

$$\phi^U = \phi U; \bar\phi^U = U^\dagger\bar\phi \ .$$

and following some steps parallel to those one takes in commutative quantum field thaory when deriving the usual Ward identities.

[1] The Landau ghost can be recovered in the limit $\Omega_{ren} \to 0$.

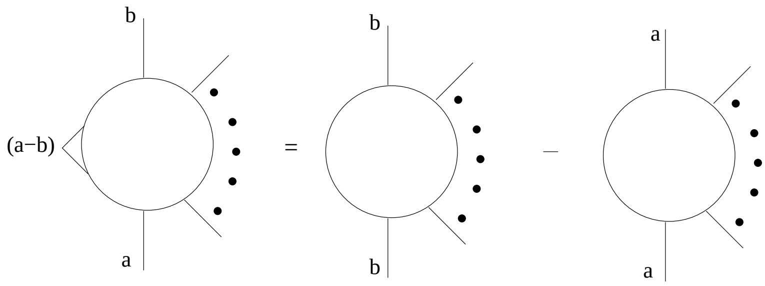

Figure 1. The Ward identity

2. Constructive Aspects

Constructive field theory build functions whose Taylor expansion is perturbative field theory [10, 11]. Formal power series being asymptotic to infinitely many smooth functions, perturbative field theory in a deep sense is no theory at all. The main advantage of perturbative field theory is that connected functions are simply the sum of the connected Feynman graphs. But it diverges because there are too many such graphs.

In fact connectedness only requires the (classical) notion of a spanning tree. To summarize constructive theory, let's say that it is all about working as much as possible with the trees only. This is the constructive golden rule:

"Thou shall not know all the loops, or thou shall diverge!"

The constructive program of A. Wightman, J. Glimm, A. Jaffe and followers partly failed! It was partly a physical failure because no natural four dimensional field theory could be identified and fully built within this program. This is because only non-Abelian gauge theories are UV asymptotically free. The constructive program went on, but mostly as a set of rigorous techniques applied to many different areas of mathematical physics [12].

It was also partly a mathematical failure because the main tool of constructive theory, the cluster amd Mayer expansion of Glimm, Jaffe and Spencer uses a discretization through a lattice (e.g. of cubes) which violates the natural rotation invariance of the theory. Non-canonical intermediate tools are not very appealing to mathematicians.

Some constructive revival could come from NCQFT, because of the absence of the Landau ghost explained in the previous section. But it does not seem that NCQFT with its non-local vertices can be treated through cluster expansion. The difficulty can be traced to the inabilty of the usual constructive methods to treat matrix models uniformly as the size of the matrix increases. A matrix field has N^2 components and at a given vertex four different indices meet. The scaling of the vertex is only $1/N$, but this is because each propagator identifies *two* matrix field indices with two others, rather than one. Therefore matrix models apparently clash with the constructive golden rule. The knowledge of the full loop structure of the graph, not only of a tree, seems necessary to recover the correct power counting, for instance a single global N^2 factor for vacuum graphs.

A new idea to solve this problem has been provided recently [13]. Matrix ϕ^4 models can be decomposed with respect to an intermediate matrix field. Integrating over the initial field leads to a perfect gas of so called *loop vertices* for this intermediate field. Performing the tree expansion directly on these loop vertices, all indices loops then appear as the correct number of traces of products of interpolated resolvents, which can be bounded because of the anti-Hermitian character of the intermediate field insertions.

This idea in turn provides a way to repack the ordinary ϕ^4 model into a convergent series without using any cluster and Mayer expansion, hence any "non-canonical" tools [14].

The main idea is to use a canonical symmetric tree formula [12], not on the ordinary lines of the graph but on the dual lines which correspond to the intermediate field propagators. In this way Bosonic constructions of ϕ^4 fields can be brought almost to the same level of simplicity than the Fermionic ones.

In conclusion solving NCQFT problems can also provide the right way to better understand ordinary QFT.

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Fredholm modules over higher rank lattices

Michael Puschnigg

We addressed th following question: Let Γ be a discrete group and let $\alpha \in KK_*(C_r^*(\Gamma), \mathbb{C})$ be a K-homology class of the reduced group C^*-algebra $C_r^*(\Gamma)$. Under which conditions can α be represented by a bounded Fredholm module which is finitely summable over the group ring $\mathbb{C}\Gamma$?

Such "regular" K-cycles are much more accessible than general ones, there is a single character formula for the index due to Connes, etc.

It is known (by results of Connes) that there are lots of finitely summable K-cycles (bounded ones) over hyperbolic groups. It turns out that the situation for higher rank lattices is quite different:

Theorem 1. *Let Γ be a lattice in a product of simple Lie groups of real rank ≥ 2. Then no nonzero class of $KK_*(C_r^*(\Gamma), \mathbb{C})$ can be represented by a finitely summable Fredholm module.*

More generally

Theorem 2. *Every finitely summable Fredholm module over the group ring of a higher rank lattice as in theorem 1 is homotopic to a Fredholm module whose underlying Hilbert space is finite dimensional.*

Both results are more or less straightforward consequences of the work of Bader, Furman, Gelander and Monod on the rigidity of affine actions of higher rank lattices on L^p-spaces. Their work leads to

Theorem 3. *Let Γ be a higher rank lattice and let $\mathcal{E} = (H, \rho, F)$ be a p-summable Fredholm module over $\mathbb{C}\Gamma$ for $1 < p < \infty$. Then $H^1(\Gamma, l^p(H)) = 0$.*

One observes that theorem 3 $\Rightarrow$ theorem 2 $\Rightarrow$ theorem 1.

Non-commutative algebraic geometry and the representation theory of p-adic groups

Paul Baum

(joint work with Anne-Marie Aubert, Roger Plymen)

Let G be a reductive p-adic group. Examples are $GL(n, F), SL(n, F)$ etc. where F can be any finite extension of the p-adic numbers. The smooth dual of G is the set of equivalence classes of smooth irreducible representations of G. Equivalently, the smooth dual of G is the set of isomorphism classes of non-degenerate (left) $\mathcal{H}G$ modules where $\mathcal{H}G$ is the Hecke algebra of G. Thus $\mathcal{H}G$ is the convolution algebra of all locally-constant complex-valued compactly-supported functions defined on G. This a dense, but not holomorphically closed, subalgebra of the reduced C^*-algebra of G. The main topological invariant of $\mathcal{H}G$ is its (purely algebraic) periodic cyclic homology.

Associated to any irreducible $\mathcal{H}G$ module is its null-space. This determines a bijection between the smooth dual of G and the set of primitive ideals of $\mathcal{H}G$.

On the set of primitive ideals of $\mathcal{H}G$ impose the Jacobson topology, and consider the connected components. J. Bernstein assigns to each connected component a complex torus and a finite group acting on the torus by automorphisms. He then defines a map (known as the infinitesimal character or the central character) from the connected component to the quotient of the complex torus by the action of the finite group. This quotient affine variety will be referred to as the ordinary quotient.

A conjecture, due to A. M. Aubert, P. F. Baum, and R. J. Plymen asserts that there is a certain resemblance between the infinitesimal character and the projection of the extended quotient onto the ordinary quotient. More precisely, the conjecture asserts that the ideal in $\mathcal{H}G$ corresponding to the connected component is equivalent via the equivalence relation introduced in [1] to the coordinate algebra of the ordinary quotient. In particular, the isomorphism of periodic cyclic homology so obtained makes the infinitesimal character and the projection of the extended quotient onto the ordinary quotient topologically indistinguishable. Also, the conjecture asserts that the elementary steps connecting the ideal to the coordinate algebra of the extended quotient can be chosen so that when the resulting bijection (between the connected component and the extended quotient) is used to give the connected component the structure of a complex affine variety, then the infinitesimal character becomes a morphism of algebraic varieties. Finally, the conjecture states that the infinitesimal character algebraically deforms to the projection of the extended quotient onto the ordinary quotient, and that this deformation is given by a finite set of cocharacters of the complex torus.

Due to results of M. Solleveld, the above conjecture is closely related to Baum-Connes for G, and hence the above conjecture can be viewed as an attempt to build a bridge between BC and the representation theory of p-adic groups.

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On the Index Theory of $\mathbf{SL(3,\mathbb{C})}$

Robert Yuncken

This work concerns an application of noncommutative harmonic analysis to noncommutative geometry. The particular motivation here is the Baum-Connes Conjecture with coefficients for semisimple Lie groups. In this particular instance, we will work entirely with the groups $\mathrm{SL}(n,\mathbb{C})$, and particularly with $\mathrm{SL}(3,\mathbb{C})$ which is the simplest group for which the conjecture is unknown. However, the methods used here seem quite general and should have broader application.

Let us begin with the motivation. Let G be a semisimple Lie group. We will not give a description of the Baum-Connes conjecture here—for good introductions to the conjecture and its many applications, we refer the reader to [2]. What is important here is a method of proof for the conjecture which was initiated by G. Kasparov in the 1980's, and which has been successful in proving the conjecture for all the rank-one semisimple Lie groups. Kasparov's approach can be broken into two broad steps as follows:

(1) Construct a 'nice' model for a canonical idempotent $\gamma \in KK^{\mathsf{G}}(\mathbb{C},\mathbb{C})$. (Again, see [2] for a description and properties of the γ-element.)
(2) Show, by means of a homotopy argument, that $\gamma = 1$ in $KK^{\mathsf{G}}(\mathbb{C},\mathbb{C})$.

If these two steps can be carried out, then the conjecture follows.
This method was successfully carried out for all the rank-one semisimple Lie groups: for the Lorentz groups $\mathrm{SO}_0(n,1)$ in [5], for $\mathrm{SU}(n,1)$ in [4], and for the groups $\mathrm{Sp}(n,1)$ in [6], [3][1]. The question is whether this method can be carried out for the higher-rank groups. To this end, we describe a solution to step one of the above recipe in the case of $\mathrm{SL}(3,\mathbb{C})$. It should be noted from the outset that step two will be impossible in KK-theory, since Kazhdan's property T is an obstruction to having $\gamma = 1$. One would have to look for some weaker kind of homotopy in order to complete a proof of the conjecture.
The following two theorems, when juxtaposed, strongly suggest an approach to constructing γ for any semisimple Lie group G. Let B denote the minimal parabolic ('Borel') subgroup of G, and let $\mathcal{X}$ be the homogeneous space G/B.

Theorem 1 (Kasparov). *Suppose $\theta \in KK^{\mathsf{G}}(C(\mathcal{X}),\mathbb{C})$ is such that $\theta \mapsto 1 \in KK^{\mathsf{K}}(\mathbb{C},\mathbb{C})$ under the forgetful map. Then $\theta \mapsto \gamma \in KK^{\mathsf{G}}(\mathbb{C},\mathbb{C})$.*

Theorem 2 (Bernstein-Gelfand-Gelfand [1]). *There is a differential complex consisting of direct sums of homogeneous line bundles over $\mathcal{X}$ and G-equivariant differential operators between them which resolves the trivial representation 1 of G.*

Indeed, the proof of the conjecture in each of the rank-one cases begins with the construction of an equivariant KK-element based upon some variant of the Bernstein-Gelfand-Gelfand (BGG) complex of Theorem 2. The difficulty in each specific case lies in the analysis: how to package the unbounded differential operator provided by the BGG complex into the form of a bounded equivariant Fredholm module.
Looking to the case of higher rank groups, the homogeneous space $\mathcal{X}$ admits a number of natural fibrations, and the BGG complex involves differential operators tangential to these various fibrations. Roughly speaking, in order to carry out the analysis one needs a theory for directionally elliptic operators which can simultaneously take account of each fibration direction in some compatible way. There are strong reasons which suggest that the pseudodifferential calculus approach of [5], [4] will not work in the higher rank case (see, for instance, [7, Chapter 5]). Instead, we appeal to noncommutative harmonic analysis.
For simplicity, let us now restrict to the case $\mathsf{G} = \mathrm{SL}(3,\mathbb{C})$. Let $\mathsf{K} = \mathrm{SU}(3)$ be the

[1]The first proof, due to Lafforge, uses slightly different methods. The unpublished alternative proof later announced by Julg is more sympathetic to the approach being discussed here.

maximal compact subgroup of G, and consider the two subgroups

$$\mathsf{K}_1 = \begin{pmatrix} \begin{matrix} \mathrm{SU}(2) \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

and

$$\mathsf{K}_2 = \begin{pmatrix} 1 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \mathrm{SU}(2) \end{pmatrix}.$$

By using the representation theory of the lattice of subgroups $\mathsf{K} \geq \mathsf{K}_1, \mathsf{K}_2 \geq 1$, we are able to define a lattice of operator ideals $\mathcal{A} \geq \mathcal{K}_1, \mathcal{K}_2 \geq \mathcal{K}$ on the L^2-section spaces of the BGG-line bundles. In this construction, $\mathcal{K}_1$ and $\mathcal{K}_2$ play the role of operators 'directionally compact' along the fibres of the two natural fibrations of $\mathcal{X}$, while $\mathcal{K}$ is the ideal of genuinely compact operators.
We now apply these constructions to index theory. We prove that the operators of multiplication by a continuous function $f \in C(\mathcal{X})$, and of pull-back by a group element $g \in \mathsf{G}$, all belong to $\mathcal{A}$. Next, we construct a bounded ('normalized') version of the directional BGG-operators, and prove that their commutators with multiplication and pullback operators are 'directionally compact', meaning that they belong to $\mathcal{K}_1$ or $\mathcal{K}_2$, according to their directionality. Finally, using a variation of the Kasparov product, all of the above data can be used to produce an equivariant K-homology element which represents the γ-element, as desired.

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Chern character for equivariant K-theory

Thomas Schick
(joint work with Paul Baum)

Equivariant homology theories play a more and more important role in modern geometric topology. In particular, they feature in various isomorphism conjectures, to compute e.g. the K- or L-theory of groups.
We are interested in the case of proper (and cocompact) actions of discrete or of totally disconnected group G on a space X. In the later case, we require that all orbits are discrete (then the action is called "smooth").
We want to calculate (via an explicit and simple Chern character) the equivariant topological K-theory of X, by definition the K-theory of the crossed product C^*-algebra $C^*(G, C_0(X))$. A general, but less explicit version of the Chern character is given by Voigt in [6, 7, 5], for discrete groups by Lück and Oliver in [4]. We first show that this group is equal to the set of equivariant homotopy classes of maps from X to $Fred(H_G)$, the space of Fredholm operators on $H_G = L^2(G) \otimes l^2(N)$.
Replacing $Fred(H_G)$ by an equivariantly homotopy equivalent Banach Lie group model $Freed(H_G)$ as in [3], we can use a canonically defined explicit and G-invariant Alexander-Spanier cochain representing the Chern character.
Using this explicit cochain, we define for each $f\colon X \to Freed(H_G)$ (i.e. each element of $K^*_G(X)$) an invariant Alexander-Spanier cochain $\mathrm{ch}(f)$ (with complex coefficients) on the space

$$\hat{X} := \{(g, x) \in G \times X \mid gx = x, < g > \subset G \text{ precompact}\}.$$

Such an invariant cochain can also be considered as an Alexander-Spanier cochain on $\hat{X}/G$. The map $[f] \mapsto [\mathrm{ch}(f)]$ is by definition the Chern character, an isomorphism after complexification.
Note: this is worked out in detail if G is discrete; a couple of details need to be checked for G totally disconnected but not necessarily discrete (like $Sl_n(Q_p)$). The special case of compact totally disconnected G has been treated by Paul Baum and Peter Schneider [1], using the description of equivariant K-theory of X in terms of equivariant vector bundles.
Given a continuous equivariant map $f\colon X \to Freed(H_G)$ and $g \in G$, the cocycle $ch(f)$ is on the component $X^g := \{(x, g) \in \hat{X} \mid gx = x\}$ given as follows: the image of $f|_{X^g}\colon X^g \to Freed(H_G)$ is contained in the set of g-invariant operators. On this subset, the Chern character cochain c canonically splits as a sum

$$c|_{Freed(H_G)^g} = \sum_{\lambda \in \hat{<g>}} c^\lambda$$

labeled by the characters of the cyclic group $< g >$. The invariant Alexander-Spanier cochain is then on X^g given as

$$\sum_{\lambda \in \hat{<g>}} \lambda(g) f^* c^\lambda.$$

For a geometric Chern character in the dual theory, i.e in equivariant K-homology, a suitable geometric model should be used. In the non-equivariant case, such a model is given in [2]. Equivariant extensions, in particular to totally disconnected groups, are work in progress.

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Four variations on spectral triples

Alain Connes

Let us recall the basic paradigm of noncommutative geometry:

Definition 1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ represented in Hilbert space $\mathcal{H}$ and a self-adjoint operator D with compact resolvent such that the commutators $[D, a]$ are bounded for all $a \in \mathcal{A}$.

An even spectral triple is endowed with a $\mathbb{Z}/2$-grading

$$\gamma = \gamma^* \in \mathcal{L}(\mathcal{H}), \quad \gamma^2 = I$$

which commutes with the action of $\mathcal{A}$, and fulfills

$$D\gamma = -\gamma D. \tag{2}$$

Before doing any variation on this theme I explained how the classification of finite noncommutative geometries of KO-dimension 6 modulo 8 leads naturally to the standard Model after crossing with a 4 dimensional manifold and applying the spectral action principle. This is joint work with Ali Chamseddine ([1]) based on our previous collaboration with Matilde Marcolli ([4]). The Standard Model is based on the gauge invariance principle with gauge group $U(1) \times SU(2) \times SU(3)$ and suitable representations for fermions and bosons. We propose a purely gravitational explanation: space-time has a fine structure given as a product of a four dimensional continuum by a finite noncommutative geometry F. The raison d'etre for F is to correct the K-theoretic dimension from four to ten (modulo eight). Our road to F is through the following steps

(1) We classify the irreducible triplets $(\mathcal{A}, \mathcal{H}, J)$.

(2) We study the $\mathbb{Z}/2$-gradings γ on $\mathcal{H}$.
(3) We classify the subalgebras $\mathcal{A}_F \subset \mathcal{A}$ which allow for an operator D that does not commute with the center of $\mathcal{A}$ but fulfills the "order one" condition:

$$[[D,a],b^0] = 0 \qquad \forall\, a,b \in \mathcal{A}_F\,. \tag{3}$$

The classification of the irreducible finite noncommutative geometries of K-theoretic. dimension six shows that the dimension (per generation) is a square of an integer k. Under an additional hypothesis of quaternion linearity, the geometry which reproduces the Standard Model is singled out (and one gets $k = 4$) with the correct quantum numbers for all fields. The spectral action applied to the product $M \times F$ delivers the full Standard Model, with neutrino mixing, coupled to gravity, and makes predictions (the number of generations is still an input).
The four variations are

(1) Spectral triples with boundary
(2) Type II and spectral triples
(3) Type III and spectral triples
(4) The role of the $g_{\mu\nu}$

The first variation is joint work with Chamseddine ([3]). We showed that using natural local boundary conditions for the Dirac operator on manifolds with boundary, the spectral action gives the key combination of curvatures

$$-\int_M R\sqrt{g}d^4x - 2\int_{\partial M} K\sqrt{h}d^3y$$

needed for gravity. The abstract notion that emerges comes from the failure of (2) for the self-adjoint extension. Thus the first variation is based on the following

Definition 4. A spectral triple is *∂-even* if $\mathcal{H}$ is endowed with a $\mathbb{Z}/2$- grading γ such that $[\gamma,a] = 0$ for $a \in \mathcal{A}$ and $\mathrm{Dom}\,D \cap \gamma\mathrm{Dom}\,D$ is dense in $\mathcal{H}$ with

$$(D\gamma + \gamma D)\,\xi = 0, \quad \forall \xi \in \mathrm{Dom}\,D \cap \gamma\mathrm{Dom}\,D. \tag{5}$$

The second variation is joint work with M. Marcolli and part of our forthcoming book ([5]). The main point is that type II spectral triples allow one to construct noncommutative spaces X_z which are pure of dimension z, *i.e.* whose operator $D = D_z$ fulfills:

$$\mathrm{Tr}(e^{-\lambda D^2}) = \pi^{z/2}\lambda^{-z/2}, \quad \forall \lambda \in \mathbb{R}_+^*\,, \tag{6}$$

corresponding to the basic rule of dimensional regularization in physics:

$$\int e^{-\lambda\, k^2}\, d^z k = \pi^{z/2}\lambda^{-z/2}, \quad \forall \lambda \in \mathbb{R}_+^*\,. \tag{7}$$

Then, at least for one loop fermionic graphs the DimReg procedure in the presence of γ_5 is geometrically encoded by the product of the space-time by the above spaces X_z *i.e.* by replacing D by D'' with

$$\bar{D} = D \otimes 1\,, \quad \hat{D} = \gamma \otimes D_z\,, \quad D'' = \bar{D} + \hat{D}\,. \tag{8}$$

The aim of the third variation is to reconcile spectral triples with type III situations. This is joint work with Henri Moscovici ([6]). We explain a simple twisting of the notion of spectral triple which allows to incorporate type III examples, such as those arising from the transverse geometry of codimension one foliations. We show that the classical cyclic cohomology valued Chern character of finitely summable spectral triples extends to the twisted case and lands in ordinary (untwisted) cyclic cohomology. The index pairing with ordinary (untwisted) K-theory continues to make sense and the index formula is given by the pairing of the Chern characters. This opens the road to extending the local index formula to the type III case. The basic definition is:

Definition 9. Let σ be an automorphism of $\mathcal{A}$, then an odd σ-spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an action of $\mathcal{A}$ in the Hilbert space $\mathcal{H}$, while D is a self-adjoint operator with compact resolvent and such that

$$D\,a - \sigma(a)\,D \text{ is bounded } \forall\, a \in \mathcal{A}\,. \tag{10}$$

The content of the last variation is the following Theorem:

Theorem 11. *Let $g = (g^{\mu\nu}) \in M_n(\mathcal{L}(\mathcal{H}))$ be an invertible positive matrix of operators in Hilbert space $\mathcal{H}$.*
Let D_μ be invertible operators in $\mathcal{H}$ and

$$D^2 = \sum_{\mu,\nu} D_\mu g^{\mu\nu} D_\nu^*\,. \tag{12}$$

Let $dz^\mu = D_\mu^{-1}$. Then $ds^2 = D^{-2}$ is given by

$$\langle \xi, ds^2\,\xi\rangle = \inf_{\sum_\mu \xi^\mu = \xi} \sum_{\mu,\nu} \langle \mathrm{dz}^\mu \xi^\mu, \mathrm{g}_{\mu\nu} \mathrm{dz}^\nu \xi^\nu\rangle, \qquad \forall \xi \in \mathcal{H} \tag{13}$$

where $g^{-1} = (g_{\mu\nu})$ is the inverse of $g = (g^{\mu\nu})$ in $M_n(\mathcal{L}(\mathcal{H}))$.

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Noncommutative tori and the Riemann–Hilbert correspondence

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(joint work with Snigdhayan Mahanta)

We study the interplay between noncommutative tori and noncommutative elliptic curves. The first have their origin in noncommmtuative differential geometry à la Connes, and the second are more affiliated to noncommutative algebraic geometry. We link the two via a category of equivariant differential modules on $\mathbb{C}^*$ which turns out to be equivalent to $\text{Rep}(\mathbb{Z}^2)$ [3].
We adopt the philosophy that a noncommutative space can be described by the category of its representations (restricted to the setting one works with: continuous, smooth, algebraic etc.). For instance, in noncommutative geometry one thinks of an algebra as describing a noncommutative space. It suffices to consider the category of representations $\text{Mod}(A)$ of this algebra A, since the algebra itself can be reconstructed as the automorphisms of the identity functor. Another instance of this philoshophy can be found as a starting point of noncommutative algebraic geometry. Classically, it is possible to reconstruct a scheme from the category of its coherent sheaves (this result is known as the Gabriel-Rosenberg theorem).
The noncommutative torus $\mathbb{T}_\theta$ was extensively studied ever since its invention [1, 6, 7]; it is described by the (smooth) algebra $\mathcal{A}_\theta$ generated by two unitaries U_1 and U_2 that satisfy the relation

$$U_1 U_2 = e^{2\pi i\theta} U_2 U_1,$$

where θ is an irrational real number. It turns out that the algebra $\mathcal{A}_\theta$ gives a noncommutative description of the ill-defined quotient space $S^1/\theta\mathbb{Z}$; in fact, $\mathcal{A}_\theta \simeq C^\infty(S^1) \rtimes_\theta \mathbb{Z}$ so that in the case that θ is rational, $\mathcal{A}_\theta$ is in fact Morita equivalent to $C^\infty(S^1/\theta\mathbb{Z})$. We consider the category introduced recently by Polishchuk and Schwarz which turned out to be well-suited for a connection to algebraic geometry [5, 4]. The objects in this category $\text{Vect}(\mathbb{T}_\theta^\tau)$ are finite projective modules M over the noncommutative torus, equipped with a connection ∇ associated to the basic derivation $\delta_\tau = \tau\delta_1 + \delta_2$ on $\mathbb{T}_\theta$. The category $\text{Vect}(\mathbb{T}_\theta^\tau)$ is an abelian category.
Noncommutative elliptic curves $\mathcal{B}_q$ were introduced by Soibelman and Vologodsky [8] as a description of the ill-defined quotient $\mathbb{C}^*/q^{\mathbb{Z}}$ with $q = e^{2\pi i\theta}$ and θ still irrational. The category $\mathcal{B}_q$ consists of $q^{\mathbb{Z}}$-equivariant coherent sheaves on $\mathbb{C}^*$ and turns out to be abelian as well.
Let us introduce the category $\mathcal{B}_q^\tau$ of $q^{\mathbb{Z}}$-equivariant coherent sheaves on $\mathbb{C}^*$ equipped with a $q^{\mathbb{Z}}$-equivariant connection ∇ that is associated to $\delta = \tau z d/dz$. Equivalently, it can be described as finite presentable modules over $\mathcal{O}(\mathbb{C}^*)$ with a $q^{\mathbb{Z}}$-action and a connection. Clearly, forgetting the connection defines a faithful and exact functor to $\mathcal{B}_q$ (it turns out that $\mathcal{B}_q^\tau$ is abelian as well). For the construction of a functor to $\text{Vect}(\mathbb{T}_\theta^\tau)$ we introduce the following map:

$$\psi : \mathcal{O}(\mathbb{C}^*) \to \mathcal{A}_\theta, \qquad \sum_{n\in\mathbb{Z}} f_n z^n \mapsto \sum_{n\in\mathbb{Z}} f_n U_1^n.$$

The map is essentially restricting a holomorphic function on $\mathbb{C}^*$ to the unit circle. In fact, it is injective since, if a holomorphic function vanishes on the unit circle, it must vanish on the whole of $\mathbb{C}^*$.
With this homomorphism, $\mathcal{A}_\theta$ can be given the structure of a module over $\mathcal{O}(\mathbb{C}^*)$ and the induced functor ψ^* defines a faithful and exact functor from $\mathcal{B}_q^\tau$ to $\mathrm{Vect}(\mathbb{T}_\theta^\tau)$. It is given explicitly by the association $(M, \sigma, \nabla) \mapsto (\tilde{M}, \tilde{\nabla})$ where

$$\tilde{M} = M \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{A}_\theta, \qquad \tilde{\nabla} = \nabla \otimes 1 + 1 \otimes \delta_\tau.$$

Next, we study the properties of the category $\mathcal{B}_q^\tau$. It is not only an abelian category as mentioned above, but even a Tannakian category. Although it is possible to show this abstractly, it is more illustrative to directly establish an equivalence with the category of finite dimensional representations of a group. After all, that is the characteristic property of a Tannakian category. Via a $q^{\mathbb{Z}}$-equivariant version of the Riemann–Hilbert correpondence on $\mathbb{C}^*$ we derive that

$$\mathcal{B}_q^\tau \simeq \mathrm{Rep}(\mathbb{Z}^2).$$

The group $\mathbb{Z}^2$ arises essentially in the following way: one copy of $\mathbb{Z}$ is the fundamental group of $\mathbb{C}^*$. This is the usual Riemann–Hilbert correspondence between (germs of) solutions to the differential equation $\nabla U = 0$ and a representation of the fundamental group $\pi^1(\mathbb{C}^*, z_0) = \mathbb{Z}$, which is established via the monodromy representation (see for instance [2]). The second copy of $\mathbb{Z}$ comes from the action of $q^{\mathbb{Z}}$ on the module M, or, in other words on each fiber of the corresponding sheaf on $\mathbb{C}^*$. The compatibility between the connection and the $\mathbb{Z}$-action implies that one gets the direct product $\mathbb{Z} \times \mathbb{Z}$.

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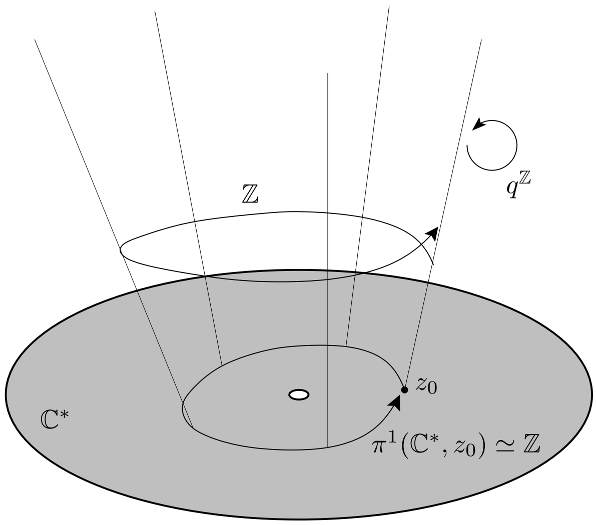

FIGURE 1. The $q^{\mathbb{Z}}$-equivariant Riemann–Hilbert correspondence on $\mathbb{C}^*$. Vertical lines denote germs of solutions of $\nabla U = 0$; there are two commuting actions of $\mathbb{Z}$ by the monodromy representation of $\pi^1(\mathbb{C}^*, z_0)$ and the fiberwise action of $q^{\mathbb{Z}}$.

The Dirac operator on compact quantum groups

SERGEY NESHVEYEV
(joint work with Lars Tuset)

Let G be a compact simple simply connected Lie group, $\mathfrak{g}$ its complexified Lie algebra. Fix a maximal torus in G and choose a system of simple roots. For each integral dominant weight λ we fix an irreducible unitary representation $\pi_\lambda\colon G \to B(V_\lambda)$ with highest weight λ. Then the group von Neumann algebra $W^*(G)$ of G is the C^*-product of the algebras $B(V_\lambda)$. The algebra $\mathcal{U}(G)$ of unbounded operators affiliated with $W^*(G)$ is the algebraic product $\prod_\lambda B(V_\lambda)$. Denote by $\hat{\Delta}$ the comultiplication $W^*(G) \to W^*(G)\bar{\otimes}W^*(G)$.
Consider the invariant form $(\cdot,\cdot)$ on $\mathfrak{g}$ normalized such that for the dual form on $\mathfrak{g}^*$ we have $(\alpha,\alpha) = 2$ for short roots α. Let $\{x_k\}_k$ be a basis in $\mathfrak{g}$ such that $(x_k, x_l) = -\delta_{kl}$. Put

$$t = -\sum_k x_k \otimes x_k \in \mathfrak{g} \otimes \mathfrak{g} \subset \mathcal{U}(G \times G).$$

Denote by $\mathrm{Cl}(\mathfrak{g})$ the complex Clifford algebra of $\mathfrak{g}$ and by $\gamma\colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ the canonical embedding, so $\mathrm{Cl}(\mathfrak{g})$ is generated by $\gamma(x)$, $x \in \mathfrak{g}$, and $\gamma(x)^2 = (x,x)1$. Fix a spin module, that is, an irreducible representation $s\colon \mathrm{Cl}(\mathfrak{g}) \to B(\mathbb{S})$. Then the

Dirac operator on G can be written as

$$D = (\partial \otimes s)(\mathcal{D}),$$

where ∂ is the representation of $U\mathfrak{g}$ by left-invariant differential operators,

$$\mathcal{D} = \sum_k \left(x_k \otimes \gamma(x_k) + \frac{1}{2} \otimes \gamma(x_k)\widetilde{\mathrm{ad}}(x_k) \right) \in U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$$

and

$$\widetilde{\mathrm{ad}}(x) = \frac{1}{4} \sum_k \gamma(x_k)\gamma([x, x_k])$$

is a lifting of the adjoint action of $\mathfrak{g}$ onto itself to a homomorphism $\mathfrak{g} \to \mathfrak{spin}(\mathfrak{g})$.
For $q \in (0,1)$ consider now the q-deformation G_q of G. Denote by $\mathcal{U}(G_q)$ the algebra of unbounded operators affiliated with the von Neumann algebra $W^*(G_q)$. Recall that it can be considered as a completion of the quantized universal enveloping algebra $U_q\mathfrak{g}$. Denote by $\hat{\Delta}_q$ the comultiplication on $W^*(G_q)$ and by $\mathcal{R} \in \mathcal{U}(G_q \times G_q)$ the universal R-matrix.
The irreducible representations of G_q are again classified by integral dominant weights, so we have a canonical identification of the centers of $W^*(G)$ and $W^*(G_q)$. Choose a $*$-isomorphism $\varphi\colon W^*(G_q) \to W^*(G)$ extending this identification. Fix $\hbar \in i\mathbb{R}$ such that $q = e^{\pi i \hbar}$. It follows from the work of Kazhdan and Lusztig [4] that there exists a unitary element $\mathcal{F} \in W^*(G)\bar{\otimes}W^*(G)$ such that

1. $(\varphi \otimes \varphi)\hat{\Delta}_q = \mathcal{F}\hat{\Delta}\varphi(\cdot)\mathcal{F}^{-1}$;
2. $(\hat{\varepsilon} \otimes \iota)(\mathcal{F}) = (\iota \otimes \hat{\varepsilon})(\mathcal{F}) = 1$, where $\hat{\varepsilon}$ is the trivial representation of G;
3. $(\varphi \otimes \varphi)(\mathcal{R}) = \mathcal{F}_{21} q^t \mathcal{F}^{-1}$;
4. the unitary $\Phi = (\iota \otimes \hat{\Delta})(\mathcal{F}^{-1})(1 \otimes \mathcal{F}^{-1})(\mathcal{F} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{F})$ coincides with Drinfeld's KZ-associator $\Phi(\hbar t_{12}, \hbar t_{23})$ [2, 3].

Define the universal quantum Dirac operator $\mathcal{D}_q \in \mathcal{U}(G_q) \otimes \mathrm{Cl}(\mathfrak{g})$ by

$$\mathcal{D}_q = (\varphi^{-1} \otimes \iota)((\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F})\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F}^*)).$$

Then we define the quantum Dirac operator D_q by

$$D_q = (\partial_q \otimes s)(\mathcal{D}_q),$$

where ∂_q is the left regular representation of $W^*(G_q)$ on $L^2(G_q)$. Therefore D_q is an unbounded self-adjoint operator on $L^2(G_q) \otimes \mathbb{S}$. In the particular case $G = SU(2)$ this is exactly the operator defined in [1].
Our main result is now formulated as follows.

Theorem 1. [5] *The triple $(\mathbb{C}[G_q], L^2(G_q) \otimes \mathbb{S}, D_q)$ is a G_q-equivariant spectral triple of the same parity as the dimension of G.*

The key point is boundedness of commutators. It turns out it is equivalent to the following property: the commutator

$$[(\pi \otimes \iota \otimes \gamma)(t_{23}), (\pi \otimes \iota \otimes \widetilde{\mathrm{ad}})(\Phi)]$$

is a bounded element of $B(V_\pi) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ for any finite dimensional representation $\pi \colon G \to B(V_\pi)$. Using that Φ is defined via monodromy of the Knizhnik-Zamolodchikov equations, this property is proved by expressing the commutators above in terms of solutions of appropriate differential equations.

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A diagram calculus for KK and the noncommutative Riemann-Roch theorem

Jonathan Rosenberg

(joint work with Jacek Brodzki, Mathai Varghese, and Richard Szabo)

This talk is motivated by the desire to:

- develop some of the formalism for dealing with noncommutative space-times in mathematical physics;
- establish a general formula for D-brane charges; and
- find a version of Grothendieck-Riemann-Roch suited to the noncommutative world.

However, before getting to these topics, I would like to explain a convenient shorthand or "diagram calculus," found in our papers [1] and [2], for keeping track of the [generalized] Kasparov product in KK theory, or similar product structures in other bivariant theories as described, for example, in the books [4] and [5].

The idea is to represent an element of a bivariant group such as $KK(A \otimes B, C \otimes D)$ by a diagram with input nodes labeled by the tensor factors A and B in the first variable, and with output nodes labeled by the tensor factors C and D in the second variable. Two such diagrams can be multiplied (concatenated) if some subset of the output nodes of the first diagram matches a corresponding subset of input nodes of the second diagram. Then the associativity property of the Kasparov product amounts to the assertion that one can do concatenation in any order. (When one considers also KK^1 and not just KK^0, then sometimes some sign changes are also involved, but these follow the usual rule for graded commutativity.) The diagram calculus is especially convenient when dealing with multiple products, such as those that occur in the KK proof of the Atiyah-Singer Index Theorem.

The next section of the talk concerns Poincaré duality in KK theory. This was also discussed in Heath Emerson's talk. Two (separable) C^*-algebras A and B

are said to be a (strong) *PD pair* (of mod-2 dimension d) if there are elements $\Delta \in KK_d(A \otimes B, \mathbb{C})$, $\Delta^\vee \in KK_{-d}(\mathbb{C}, A \otimes B)$ with the properties

$$\begin{aligned} \Delta^\vee \otimes_B \Delta &= 1_A \in KK_0(A, A) , \\ \Delta^\vee \otimes_A \Delta &= (-1)^d \, 1_B \in KK_0(B, B) . \end{aligned}$$

The algebra A is called simply a *PD-algebra* if (A, A°) is a PD pair, A° the opposite algebra to A. Poincaré duality in this sense is not especially rare—if A is KK-equivalent to a commutative C^*-algebra, as when A is an inductive limit of type I algebras, then A is always part of a PD pair provided $K_\bullet(A)$ is finitely generated as an abelian group, and when this is the case, A is a PD algebra if and only if either $\operatorname{rank} K_0(A) = \operatorname{rank} K_1(A)$ (in this case we can take $d = 1$) or $\operatorname{Tors} K_0(A) \cong \operatorname{Tors} K_1(A)$ (in this case we can take $d = 0$).

Poincaré duality makes it possible to define Gysin maps. Following Connes-Skandalis [3], if $f\colon A \to B$ is a $*$-homomorphism of C^*-algebras, a *K-orientation* for f is a functorial way of defining a KK-element $f! \in KK(B, A)$. If A and B are PD algebras, such a K-orientation can be defined as a Gysin map,

$$f! = (-1)^{d_A} \Delta_A^\vee \otimes_{A^\circ} [f^\circ] \otimes_{B^\circ} \Delta_B,$$

where $[f^\circ]$ is the KK class of the $*$-homomorphism $f^\circ\colon A^\circ \to B^\circ$.

Now we can explain the Todd class and the noncommutative Riemann-Roch theorem. If HL is Puschnigg's local cyclic homology [5], there is a functorial Chern character $KK \to HL$. So if A is a PD algebra for KK with fundamental class Δ and dual fundamental class $\Delta^\vee$, $\operatorname{Ch}(\Delta)$ and $\operatorname{Ch}(\Delta^\vee)$ define a PD structure for A in HL. (We are ignoring a technical issue concerning tensor products pointed out to us by Ralf Meyer, but usually one can get around this.) However, often one wants to use a *different* pair of fundamental classes in HL, say Ξ and $\Xi^\vee$. For example, if $A = C(M)$, M a spin^c manifold, the Chern character sends the K-theory fundamental class to the Poincaré dual of the Todd class, and not to the usual choice of a fundamental class in rational homology. Then we define the [*noncommutative*] *Todd class* of A, which depends on these choices, to be

$$\operatorname{Todd}\big(A\big) := \Xi^\vee \otimes_{A^\circ} \operatorname{Ch}\big(\Delta\big)$$

in the ring $HL^0(A, A)$.

Theorem 1 (Noncommutative Riemann-Roch)**.** *Suppose A and B are strong PD algebras with given KK and HL fundamental classes. Then one has the Grothendieck-Riemann-Roch formula,*

$$\operatorname{Ch}(f!) = (-1)^{d_B} \ \operatorname{Todd}(B) \otimes_B f^{HL}! \otimes_A \operatorname{Todd}(A)^{-1}.$$

We conclude with a noncommutative version of the "formula for D-brane charges" of [6]. A fundamental class Δ of a strong PD algebra A is said to be *symmetric* if $\sigma(\Delta)^\circ = \Delta \in KK^d(A \otimes A^\circ, \mathbb{C})$, where

$$\sigma : A \otimes A^\circ \longrightarrow A^\circ \otimes A$$

is the involution $x \otimes y^\circ \mapsto y^\circ \otimes x$ and σ also denotes the induced map on K-homology.

Theorem 2. *Suppose that A satisfies the UCT for local cyclic homology, and that $HL_\bullet(A)$ is a finite dimensional vector space. If A has symmetric (even-dimensional) fundamental classes in both K-theory and in cyclic theory, then the modified Chern character*

$$\mathrm{Ch} \otimes_A \sqrt{\mathrm{Todd}(A)} : K_\bullet(A) \to HL_\bullet(A)$$

is an isometry with respect to the inner products

$$\langle \alpha, \beta \rangle = (\alpha \times \beta^\circ) \otimes_{A \otimes A^\circ} \Delta$$

and

$$(x, y) = (x \times y^\circ) \otimes_{A \otimes A^\circ} \Xi.$$

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Group cocycles and the ring of affiliated operators

Andreas Thom

(joint work with Jesse Peterson)

1. Introduction

The computations of ℓ^2-homology have been algebraized through the seminal work of W. Lück, which is summarized and explained in detail in his nice compendium [8]. This extend abstract is a report about results obtained in [10].

Our first theorem gives an identification of dimensions of cohomology groups, where the coefficients vary among the canonical choices $LG, \ell^2 G$ and $\mathcal{U}G$.

Theorem 1. *Let G be a countable discrete group.*

$$\beta_k^{(2)}(G) = \dim_{LG} H^k(G, \mathcal{U}G) = \dim_{LG} H^k(G, \ell^2 G) = \dim_{LG} H^k(G, LG).$$

Moreover, if $\beta_k^{(2)}(G) = 0$ for some k, then $H^k(G, \mathcal{U}G) = 0$.

In [3] it is shown that for a finitely generated non-amenable discrete group, the first ℓ^2-Betti number vanishes if and only if the first cohomology group with values in the left regular representation vanishes, (see also Corollary 3.2 in [9]). As a corollary we may drop the assumption that the group is finitely generated.

Corollary 1. *Let G be a non-amenable countable discrete group, then $\beta_1^{(2)}(G) = 0$ if and only if $H^1(G, \ell^2 G) = 0$.*

The following theorem is an affirmative answer to Conjecture 2 in [5].

Theorem 2. *Let G be an countable and discrete group which is amenable. Every 1-cocycle with values in $\ell^2 G$ is either bounded or proper.*

There is a partial converse to the preceding result.

Theorem 3. *Let G be a group with $\beta_1^{(2)}(G) \neq 0$ and assume that there exists an infinite amenable sub-group. There exists a 1-cocycle with values in $\ell^2 G$ on G which is neither bounded nor proper.*

2. Free subgroups

2.1. **Restriction maps and free subgroups.** Throughout this section, we are assuming that G is a torsionfree discrete countable group and most of the time also that it satisfies the following condition:

($\star$) Every non-trivial element of $\mathbb{Z}G$ acts without kernel on $\ell^2 G$.

Condition ($\star$) is known to hold for all right orderable groups and all residually torsionfree elementary amenable groups. No counterexample is known.
Let G be a discrete group, we use the notation $\dot{G}$ to denote the set $G \setminus \{e\}$. The main result here is the following theorem.

Theorem 4. *Let G be a torsionfree discrete countable group. There exists a family of subgroups $\{G_i \mid i \in I\}$, such that*

(i) *We can write G as the disjoint union:*

$$G = \{e\} \cup \bigcup_{i \in I} \dot{G}_i.$$

(ii) *The groups G_i are mal-normal in G, for $i \in I$.*
(iii) *If G satisfies condition ($\star$), then G_i is free from G_j, for $i \neq j$.*
(iv) *$\beta_1^{(2)}(G_i) = 0$, for all $i \in I$.*

Remark 1. It follows from Theorem 7.1 in [10], that the set I is infinite if the first ℓ^2-Betti number of G does not vanish.

Corollary 2. *Let G be a discrete countable group satisfying condition ($\star$). Assume that the first ℓ^2-Betti number does not vanish. Let F be a finite subset of G. There exists $g \in G$, such that g is free from each element in F. In particular, G contains a copy of F_2.*

Remark 2. Corollary 2 confirms the feeling that a sufficiently non-amenable group contains a free subgroup. Note, that various weaker conditions like *non-amenability* itself or *uniform non-amenability* have been proved to be insufficient to ensure the existence of free subgroups, at least in the presence of torsion.

Using results from [2] we obtain the following result.

Corollary 3. *Let G be a torsionfree discrete countable group satisfying condition $(\star)$. If the first ℓ^2-Betti number does not vanish, then the reduced group C^*-algebra $C^*_{red}(G)$ is simple and carries a unique trace.*

The following result is a generalization of the main result of J. Wilson in [12] for torsionfree groups which satisfy $(\star)$. For this, note that a group G with n generators and m relations satisfies $\beta_1^{(2)}(G) \geq n - m - 1$.

Corollary 4 (Freiheitssatz)**.** *Let G be a torsionfree discrete countable group which satisfies $(\star)$. Assume that $a_1, \ldots, a_n \in G$ generate G and $\lceil \beta_1^{(2)}(G) \rceil \geq k$. There exist $k+1$ elements $a_{i_0}, \ldots, a_{i_k}$ among the generators such that the natural map*

$$\pi \colon F_{k+1} \to \langle a_{i_0}, \ldots, a_{i_k} \rangle \subset G$$

is an isomorphism.

Corollary 5. *Let G be a finitely generated torsionfree discrete countable group which satisfies $(\star)$. Then*

$$e_S(G) \geq 2\lceil \beta_1^{(2)}(G) \rceil + 1,$$

for any generating set S. Here, $e_S(G)$ denotes the exponential growth rate w.r.t. the generating set S.

In particular, a torsionfree group satisfying condition $(\star)$ has uniform exponential growth if its first ℓ^2-Betti number is positive.

3. Notions of normality

We now want to review some notions of normality of subgroups which are more or less standard, and introduce some notation. A subgroup $H \subset G$ is called:

(i) normal iff $gHg^{-1} = H$, for all $g \in G$,
(ii) s-normal iff $gHg^{-1} \cap H$ is infinite for all $g \in G$, and
(iii) q-normal iff $gHg^{-1} \cap H$ is infinite for elements $g \in G$, which generate G.

We say that a subgroup inclusion $H \subset G$ satisfies one of the normality properties from above *weakly*, iff there exists an ordinal number α, and an ascending α-chain of subgroups, such that $H_0 = H$, $H_\alpha = G$, and $\mathcal{U}p_{\beta<\gamma} H_\beta \subset H_\gamma$ has the required normality property.

Example 1. The inclusions

$$GL_n(\mathbb{Z}) \subset GL_n(\mathbb{Q}), \qquad \mathbb{Z} = \langle x \rangle \subset \langle x, y \mid yx^p y^{-1} = x^q \rangle = BS_{p,q}$$

are inclusions of s-normal subgroups. The inclusion

$$F_2 = \langle a, b^2 \rangle \subset \langle a, b \rangle = F_2$$

is q-normal but not s-normal.

3.1. **ℓ^2-invariants and normal subgroups.** The two main results in this subsection are Theorem 5 and Theorem 6. We derive several corollaries about the structure of groups G with $\beta_1^{(2)}(G) \neq 0$.

Theorem 5. *Let G be a countable discrete group and suppose H is an infinite wq-normal subgroup. We have $\beta_1^{(2)}(H) \geq \beta_1^{(2)}(G)$.*

Corollary 6. *Let $H \subset K \subset G$ be a chain of subgroups and assume that $H \subset G$ is wq-normal and $[K : H] < \infty$. Then*

$$[K : H] \cdot \beta_1^{(2)}(G) \leq \beta_1^{(2)}(H).$$

Corollary 7. *Let G be a torsionfree discrete countable group and let $H \subset G$ be an infinite subgroup. If $\beta_1^{(2)}(H) < \beta_1^{(2)}(G)$, then there exists a proper malnormal subgroup $K \subset G$, such that $H \subset K$.*

Corollary 8. *Let G be a countable discrete group and let $H \subset G$ be an infinite wq-normal subgroup. Let $K \subset G$ be a subgroup with $H \subset K$ and assume that $\beta_1^{(2)}(G) > n$. Then, K is not generated by n or less elements.*

The second main result in this section is the following.

Theorem 6. *Let G be a countable discrete group and suppose H is an infinite index, infinite ws-normal subgroup. If $\beta_1^{(2)}(H) < \infty$, then $\beta_1^{(2)}(G) = 0$.*

Corollary 9. *Let G be a countable discrete group with $\beta_1^{(2)}(G) > 0$. Suppose that $H \subset G$ is an infinite, finitely generated ws-normal subgroup. Then H has to be of finite index.*

Note that the result applies in case H contains an infinite normal subgroup. Hence, this result is a generalization of the classical results by A. Karass and D. Solitar in [7], H. Griffiths in [6], and B. Baumslag in [1]. A weaker statement with additional hypothesis was proved as Theorem 1(2) in [4].

Corollary 10 (Gaboriau)**.** *Let G be a group with an infinite normal subgroup of infinite index, which is either finitely generated or amenable. Then $\beta_1^{(2)}(G) = 0$.*

Remark 3. A generalization of Gaboriau's result to higher ℓ^2-Betti numbers was obtained by R. Sauer and the second author in [11]. There it was shown that for a normal subgroup $N \subset G$ with all $\beta_p^{(2)}(N) = 0$, for $p < q$, and $\beta_q^{(2)}(N)$ finite, it follows that $\beta_p^{(2)}(G) = 0$, for $p \leq q$. The proof uses a Hochschild-Serre spectral sequence for discrete measured groupoids. For more results in this direction, see [11].

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Hopf algebras of infinite primitive Lie pseudogroups and their cyclic cohomology I

Henri Moscovici
(joint work with Bahram Rangipour)

The joint work with A. Connes [3] on the transverse index class of the hypoelliptic signature operator revealed the key role played by a certain Hopf algebra $\mathcal{H}_n$ in the transverse geometry of codimension n foliations. Furthermore, the inner structure of the computation proper, performed by applying the local index formula [2], turned out to be controlled by a specific type of cyclic cohomology proper to Hopf algebras. For the case of $\mathcal{H}_n$, we proved that its Hopf-cyclic cohomology is canonically isomorphic to the Gelfand-Fuks cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^n$, fact that allowed us to express the outcome of the index calculation in terms of classical geometric characteristic classes of foliations. This talk reports on joint work with Bahram Rangipour, which refines the above developments and extends their scope to cover all types of transverse geometric structures. These are the classical geometries resulting from Elie Cartan's classification [1] of infinite primitive Lie pseudogroups: general, volume preserving, symplectic, and contact. It should be noted that the proof of Cartan's classification was completed only in mid-1960's, cf. Guillemin, Quillen and Sternberg [4], after being recast in the setting of linearly compact Lie algebras, cf. Singer and Sternberg [7]. Along with the Lie algebra $\mathfrak{L}(\mathcal{G})$ of germs of $\mathcal{G}$-vector fields, *we associate to an infinite primitive Lie pseudogroup $\mathcal{G}$ a Hopf algebra $\mathcal{H}(\mathcal{G})$, and prove that its periodic Hopf-cyclic cohomology is canonically isomorphic to the Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{L}(\mathcal{G})$.* In the particular case of $\mathcal{H}_n$

we also exhibit an explicit basis for its periodic cyclic cohomology, obtained from a Hopf-algebraic version of the truncated Weil complex for $\mathfrak{gl}(n,\mathbb{R})$.

Our construction of the Hopf algebra $\mathcal{H}(\mathcal{G})$, modeled after and extending that of $\mathcal{H}_n$ to all infinite primitive Lie pseudogroups, can be viewed as an infinite-dimensional analogue of the 'bicrossed product' for finite groups introduced by G. I. Kac [5]. It relies on splitting the group $\mathbf{G} = \mathcal{G} \cap \mathrm{Diff}(\mathbb{R}^{\mathbf{n}})$ of globally defined diffeomorphisms of type $\mathcal{G}$ (with $n = 2m$ in the symplectic case and $n = 2m+1$ in the contact case), as a set-theoretical product $\mathbf{G} = G \cdot N$, where G is a Lie group containing the linear isotropy group, and N is the (prounipotent) subgroup consisting of those $\psi \in \mathbf{G}$ that preserve the origin to first order; G is actually a subgroup of the group of affine motions in $\mathbb{R}^n$ in all cases but one, namely that of the contact diffeomorphisms, when the corresponding pseudogroup is not 'flat', *i.e.* does not contain the translations. Thus, any $\phi \in \mathbf{G}$ factors uniquely in the form $\phi = \varphi \cdot \psi$, with $\varphi \in G$ and $\psi \in N$. Reversing the order and then applying the canonical decomposition,

$$\psi \cdot \varphi \,=\, (\psi \triangleright \varphi) \cdot (\psi \triangleleft \varphi),$$

one obtains a left action $\psi \mapsto \tilde{\psi}(\varphi) := \psi \triangleright \varphi$ of N on G, along with a right action $\triangleleft$ of G on N, both fixing the unit $e \in \mathbf{G}$.

We define the Hopf algebra $\mathcal{H}(\mathcal{G})$ via its action on the crossed product algebra $\mathcal{A}(\mathcal{G}) = C^\infty(G) \rtimes N$. Fix a basis $\{X_i\}_{1\le i\le m}$ of the Lie algebra $\mathfrak{g}$ of G, and thus a framing of G by left-invariant vector fields $\{\tilde{X}_i\}_{1\le i\le m}$. Let $U_\psi(g) = g \circ \tilde{\psi}^{-1}$, $g \in C^\infty(G)$, $\psi \in N$, and view $\mathcal{A}(\mathcal{G})$ as spanned by monomials $f\,U_\psi$, $f \in C^\infty(G)$. Extend $\tilde{X}_i$ to linear operators on $\mathcal{A}(\mathcal{G})$, $\tilde{X}_i(f\,U_\psi) = \tilde{X}_i(f)\,U_\psi$, and define $\gamma_i^j(\psi) \in C^\infty(G)$, with $\psi \in N$, by $U_\psi^{-1}\,\tilde{X}_i\,U_\psi \,=\, \sum_{j=1}^m \gamma_i^j(\psi)\,\tilde{X}_j$. Next, define the linear operators Γ_i^j on $\mathcal{A}(\mathcal{G})$, by $\Gamma_i^j(f\,U_\psi^{-1}) = (\gamma(\psi))^{-1})_i^j\,f\,U_\psi^{-1}$. As algebra, $\mathcal{H}(\mathcal{G})$ is by definition *the subalgebra of linear operators on* $\mathcal{A}(\mathcal{G})$ *generated by the above operators* $\tilde{X}_k$'s and Γ_i^j's, $i,j,k = 1,\dots,m$. In particular, it contains the iterated commutators $\Gamma^j_{i,k_1\dots k_r} := [\tilde{X}_{k_r},\dots,[\tilde{X}_{k_1},\Gamma_i^j]\dots]$, which are multiplication operators by functions on G, $\gamma^j_{i,k_1\dots k_r}(\psi) := \tilde{X}_{k_r}\dots\tilde{X}_{k_1}(\gamma_i^j(\psi))$, $\psi \in N$. On the other hand, for any $a,b \in \mathcal{A}(\mathcal{G})$, one has

$$\tilde{X}_k(ab) \,=\, \tilde{X}_k(a)\,b + \sum_k \Gamma_k^j(a)\,\tilde{X}_j(b), \qquad \Gamma_i^j(ab) \,=\, \sum_k \Gamma_i^k(a)\,\Gamma_k^j(b).$$

By multiplicativity, every $h \in \mathcal{H}(\mathcal{G})$ satisfies a *Leibniz rule* of the form

$$h(ab) \,=\, \sum h_{(1)}(a)\,h_{(2)}(b) \qquad \text{(Sweedler notation)}.$$

We note that if c^i_{jk} are the structure constants of $\mathfrak{g}$, $[X_j,X_k] = \sum_i c^i_{jk}X_i$, the following analogue of the *Bianchi identity* holds

$$\Gamma^k_{j,i} - \Gamma^k_{i,j} \,=\, \sum_{p,r} c^k_{pr}\Gamma^p_j\Gamma^r_i \,-\, \sum_s c^s_{ji}\Gamma^k_s;$$

iterated commutators with $\tilde{X}_k$'s generate higher order identities.

Theorem A. *Let $\mathfrak{G}$ be the Lie algebra generated by the operators $\tilde{X}_k$ and $\Gamma^j_{i,k_1\dots k_r}$.*

(1) *The algebra $\mathcal{H}(\mathcal{G})$ is the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{G})$ by the ideal generated by the above "Bianchi identities" of all orders.*
(2) *The Leibniz rule determines uniquely a multiplicative coproduct* $\Delta : \mathcal{H}(\mathcal{G}) \to \mathcal{H}(\mathcal{G}) \otimes \mathcal{H}(\mathcal{G})$.
(3) *$\mathcal{H}(\mathcal{G})$ is a Hopf algebra and $\mathcal{A}(\mathcal{G})$ an $\mathcal{H}(\mathcal{G})$-module algebra.*

We next describe the *bicrossed product* realization of $\mathcal{H}(\mathcal{G})$. Let $\mathcal{F}(N)$ = *algebra of functions on* N generated by the functions $\eta^j_{i,k_1\dots k_r}(\psi) := \gamma^j_{i,k_1\dots k_r}(\psi)(e)$, $\psi \in N$. This definition is independent of the choice of basis for $\mathfrak{g}$. Moreover, $\mathcal{F}(N)$ *is a Hopf algebra* with coproduct $\Delta F(\psi_1, \psi_2) := F(\psi_1 \circ \psi_2)$, and with antipode $SF(\psi) := F(\psi^{-1})$, $\psi \in N$.
One defines a *right $\mathcal{F}(N)$-comodule coalgebra* structure $\blacktriangledown : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{F}(N)$ as follows. Let $\{X_I\}$ be the PBW basis of $\mathcal{U}(\mathfrak{g})$ induced by the basis of $\mathfrak{g}$. Then $U_\psi^{-1}\, \tilde{X}_I\, U_\psi = \sum_J \beta^J_I(\psi)\, \tilde{X}_J$, with $\beta^J_I(\psi)$ in the algebra of functions on G generated by $\gamma^j_{i,K}(\psi)$, with I, J, K multi-indices. Evaluating at $e \in G$, one defines

$$\blacktriangledown(\tilde{X}_I) = \textstyle\sum_J \tilde{X}_J \otimes \beta^J_I(\cdot)(e),$$

which extends by linearization to a map $\blacktriangledown : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{F}(N)$. Again, the definition is independent of the choice of basis.
The right action $\triangleleft$ of G on N induces an action of G on $\mathcal{F}(N)$ and hence a left action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{F}(N)$; this turns $\mathcal{F}(N)$ into a *left $\mathcal{U}(\mathfrak{g})$-module algebra.*

Theorem B. *With the above operations,*

(1) *$\mathcal{U}(\mathfrak{g})$ and $\mathcal{F}(N)$ form a matched pair of Hopf algebras;*
(2) *the Hopf algebra $\mathcal{H}(\mathcal{G})^{\mathrm{cop}}$ is canonically isomorphic to the bicrossed product* $\mathcal{F}(N) \blacktriangleright\!\!\triangleleft\, \mathcal{U}(\mathfrak{g})$.

We conclude by stating the relationship between Hopf-cyclic cohomology and Lie algebra cohomology, associated to the Cartan-Lie pseudogroups.

Theorem C. *Let $\mathcal{G}$ be a primitive Lie pseudogroup of infinite type.*

(1) *There is a canonical Hopf algebra $\mathcal{H}(\mathcal{G})$ with modular character $\delta \in \mathcal{H}(\mathcal{G})^*$ extending* $\mathrm{tr} \circ \mathrm{ad} : \mathfrak{g} \to \mathbb{C}$, *such that $(\delta, 1)$ is a modular pair in involution.*
(2) *The $\mathbb{Z}/2\mathbb{Z}$-graded periodic Hopf-cyclic cohomology groups $HP^*(\mathcal{H}(\mathcal{G}); \mathbb{C}_\delta)$ of the Hopf algebra $\mathcal{H}(\mathcal{G})$ are canonically isomorphic to the Gelfand-Fuks cohomology groups $H^*_{\mathrm{GF}}(\mathfrak{L}(\mathcal{G}); \mathbb{C})$ of the Lie algebra $\mathfrak{L}(\mathcal{G})$.*
(3) *More generally, for any Lie subalgebra $\mathfrak{h}$ of the linear isotropy Lie algebra $\mathfrak{g}_o \subset \mathfrak{g}$, one has a canonical isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded groups*

$$HP^*(\mathcal{H}(\mathcal{G}), \mathcal{U}(\mathfrak{h}); \mathbb{C}_\delta) \cong H^*_{\mathrm{GF}}(\mathfrak{L}(\mathcal{G}), \mathfrak{h}; \mathbb{C}).$$

An outline of the methods of proof, which build on those initiated in [6], is given in B. Rangipour's talk.

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Hopf algebras of infinite primitive Lie pseudogroups and their cyclic cohomology II

Bahram Rangipour

(joint work with Henri Moscovici)

In his talk Henri Moscovici outlined a procedure by which to every infinite primitive Cartan-Lie pseudogroup $\mathcal{G}$ one can associate a canonical noncommutative noncocommutative Hopf algebra $\mathcal{H}(\mathcal{G})$, equipped with a modular character $\delta \in \mathcal{H}(\mathcal{G})^*$. The prototypical example is that of the Hopf algebra $\mathcal{H}_n$, first introduced in his joint work with Alain Connes [2], that corresponds to the pseudogroup of all local diffeomorphisms of $\mathbb{R}^n$.

The main goal of the present talk is to explain the proof of the fundamental fact that the Hopf cyclic cohomology of the Hopf algebra $\mathcal{H}(\mathcal{G})$ with coefficients in the modular pair in involution $(\delta, 1)$, as well as its relative version, is canonically isomorphic to the Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{L}(\mathcal{G}) := Lie(\mathcal{G})$, resp. its relative version.

We recall that $\mathcal{H}(\mathcal{G})$ is a bicrossed product Hopf algebra $\mathcal{U} \blacktriangleright\!\!\triangleleft \mathcal{F}$, where $\mathcal{U} = U(\mathfrak{g})$ the enveloping Hopf algebra of a Lie algebra $\mathfrak{g} = Lie(G)$ and $\mathcal{F} = \mathcal{F}(N)$ is a Hopf algebra of polynomial functions on a prounipotent group N. The groups G and N are respectively the analogues of the affine and unipotent groups in the definition of $\mathcal{H}_n$. However, since we are also interested in relative Hopf cyclic cohomology, it does not suffice to restrict ourselves to bicrossed product Hopf algebras. The more general setup and the method of computation are as follows.

Let D be a left $\mathcal{H}$ comodule coalgebra and let C be a left module coalgebra. One forms the cocrossed product coalgebra $C \blacktriangleright\!\!< D$ in the usual way. In addition, if C is $\mathcal{H}$ module coalgebra and via its action and coaction becomes YD module, then $\mathcal{H}$ acts diagonally on $C \blacktriangleright\!\!< D$ and makes it a $\mathcal{H}$-module coalgebra. So $HC_{\mathcal{H}}(C \blacktriangleright\!\!< D, M)$, for any SAYD M, is well defined (for the definition, see [5, 6]). The main idea for its computation is to use the natural twisting map

$$\mathrm{top} : \mathrm{C} \otimes \mathrm{D} \to \mathrm{D} \otimes \mathrm{C}, \quad \mathrm{top}(\mathrm{c} \otimes \mathrm{d}) = \mathrm{c}_{<-1>}\mathrm{d} \otimes \mathrm{c}_{<0>},$$

to identify the cyclic complex of $C \blacktriangleright\!\!< D$ with the diagonal of a natural bicocyclic module made of $\mathcal{H}$, C, D and M. The next step is to apply the cyclic Eilenberg-Zilber theorem [7] to homologically exchange diagonal cyclic module for the total mixed complex.

This idea has a long history, at least going back to the Künneth formula. However in each new case, e.g, for groups, Lie algebras, algebras, coalgebras, Hopf algebras, module coalgebras, etc. is difficult to derive the corresponding homological machinery, and one has to start anew from scratch. It is quite remarkable that this method of computation somehow fits very well with the setting of Hopf algebras. To wit, this machinery never brings up a bicocylic module, except in the case of bicrossed product Hopf algebras.

Now if $\mathcal{K}$ is a Hopf subalgebra of $\mathcal{H}$, then $\mathcal{H}$ acts on the coalgebra $\mathcal{H} \otimes_{\mathcal{K}} \mathbb{C}$ and makes it a module coalgebra. In addition if $\mathcal{H}$ acts on $\mathcal{K}$ as a module coalgebra, then by considering the natural coaction of $\mathcal{H}$ on C defined by coadjoint coaction one completes the needed ingredients for the machinary to compute the Hopf cyclic cohomology of the cocrossed product $C \blacktriangleright\!\!< \mathcal{K}$. If one is as lucky, as we are in the case of $\mathcal{H}(\mathcal{G})$, then $\mathcal{H} \simeq C \blacktriangleright\!\!< \mathcal{K}$ and hence there is a spectral sequence available to compute the Hopf cyclic cohomology of $\mathcal{H}$. To make a long story short, one deals with the following quadricomplex derived from our bicocyclic module:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & \\
\mathbb{C} \otimes \mathcal{U}^{\otimes 2} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \mathcal{F} \otimes \mathcal{U}^{\otimes 2} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \mathcal{F}^{\otimes 2} \otimes \mathcal{U}^{\otimes 2} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \cdots \\
b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & \\
\mathbb{C} \otimes \mathcal{U} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \mathcal{F} \otimes \mathcal{U} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \mathcal{F}^{\otimes 2} \otimes \mathcal{U} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \cdots \\
b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & b_{\mathcal{U}} \uparrow\downarrow B_{\mathcal{U}} & & \\
\mathbb{C} \otimes \mathbb{C} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \mathcal{F} \otimes \mathbb{C} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \mathcal{F}^{\otimes 2} \otimes \mathbb{C} & \underset{B_{\mathcal{F}}}{\overset{b_{\mathcal{F}}}{\rightleftarrows}} & \cdots
\end{array}$$

One then uses a series of homological arguments, including the following: (1) the Hopf cyclic cohomology of $U(\mathfrak{g})$ coincides with the Lie algebra cohomology of $\mathfrak{g}$ [2]; (2) the Connes boundary operator B is zero on the Hochschild cohomology if the Hopf algebra under consideration is commmutative [8]. This simplifies the above quadricomplex to the bicomplex

$$(C^p(N, \wedge^q \mathfrak{g}^*), \partial, b),$$

where $C^\bullet(N, \wedge^\bullet \mathfrak{g}^*)$ stands for the complex of algebraic cochains. One uses the van Est isomorphism to identify the group cohomology of N with the Gelfand-Fuks cohomology of $\mathfrak{n} := Lie\, N$. This essentially proves the first part of following theorem, while the rest is proved by similar but more elegant arguments.

Theorem 1. *Let $\mathcal{G}$ be a primitive Lie pseudogroup of infinite type, and let $\mathbf{G} = G \cdot N$ be the canonical decomposition of the corresponding group $\mathbf{G}$ of globally defined transformations.*

(1) *There is a canonical isomorphism*

$$H^*_{\mathrm{GF}}(\mathfrak{L}(\mathcal{G}); \mathbb{C}) \cong HP^*(\mathcal{H}(\mathcal{G}); \mathbb{C}_\delta)$$

that extends the canonical isomorphisms

$$H_*(\mathfrak{g}; \mathbb{C}_\delta) \cong HP^*(\mathcal{U}(\mathfrak{g}); \mathbb{C}_\delta), \quad H^*(\mathfrak{n}; \mathbb{C}) \cong HP^*(\mathcal{F}(N); \mathbb{C}),$$

where δ is the modular character of $\mathfrak{g}$.

(2) *For any Lie subalgebra $\mathfrak{h}$ of the isotropy Lie algebra $\mathfrak{g}_0$, there is a canonical isomorphism in relative cohomology*

$$HP^*(\mathcal{H}(\mathcal{G}), \mathcal{U}(\mathfrak{h}); \mathbb{C}_\delta) \cong H^*_{\mathrm{GF}}(\mathfrak{L}(\mathcal{G}), \mathfrak{h}; \mathbb{C}).$$

One knows that the Gelfand-Fuks cohomology of the Lie algebra formal vector fields is finite-dimensional and explicitly computed in most cases [3]. The first expectation out of Theorem 1 is to transfer the Lie algebra cohomology classes to get their counterparts in Hopf cyclic cohomology. Connes and Moscovici have shown in [2] that, in principle, this can be done in the case of $\mathcal{H}_n$, but they also showed how difficult the task is. To overcome the main difficulty, one needs to avoid reliance on the van Est isomorphism. To this end, we use the method outlined above, but this time we apply the machinery to the isotropy Hopf subalgebra $\mathcal{K} := U(gl_n)$ instead of $\mathcal{U}$. We then show that the truncated Weil complex is quasi isomorphism with the Hopf cyclic complex, which provides an easier way to transfer the classes.

So far we have talked only about the periodic Hopf cyclic cohomology. The main obstacle in computing nonperiodic cyclic cohomology is not knowing the Hochschild cohomology of the Hopf algebra under question. In the case of $\mathcal{H}_1$ one uses the result of Gončarova [4], $HH^i(\mathcal{F}) = \mathbb{C} < \xi_i > \oplus \mathbb{C} < \xi'_i >$, to compute the cyclic cohomology as follows:

$$HC^0(\mathcal{H}_1, \mathbb{C}_\delta) = 0,\ HC^1(\mathcal{H}_1, \mathbb{C}_\delta) = \mathbb{C} < 1 \otimes \xi_1 > \oplus \mathbb{C} < 1 \otimes \xi'_1 > .$$

For $p \geq 2, HC^p(\mathcal{H}_1, \mathbb{C}_\delta)$ is generated by:

$$S^q(T),\ 1 \otimes \xi_p,\ 1 \otimes \xi'_p,\ X \otimes \xi_{p-1} - \frac{1}{|\xi_{p-1}|} Y \otimes X\xi_{p-1},$$

$$X \otimes \xi'_{p-1} - \frac{1}{|\xi_{p-1}|} Y \otimes X\xi'_{p-1};$$

where $T := X \wedge Y \otimes 1$ if $p = 2q + 2$, and $T := 1 \otimes \xi_1$, if $p = 2q + 1$.

The above method of computation represents a significant enhancement of our techniques in [9], where we could only treat those Hopf algebras that are cocrossed product as coalgebras, but whose underlying algebra structure is just that of a 'straight' tensor product.

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Noncommutative geometry and quantum group symmetries

Francesco D'Andrea
(joint work with L. Dąbrowski, G. Landi and E. Wagner)

A key notion in Connes' noncommutative geometry [2] is the one of unital spectral triple, which is the data of an involutive complex unital algebra $\mathcal{A}$ of bounded operators on a (separable) Hilbert space H and an unbounded self-adjoint operator D on H with compact resolvent and bounded commutators with the algebra. This notion, together with some further conditions, captures the essence of a spin structure over a spin manifold and generalizes it to the noncommutative framework.
The recent constructions of spectral triples for the manifold of the quantum $SU(2)$ group in [1, 3, 7, 8] and for some of its quantum homogeneous spaces (the equatorial [9] and standard [10] Podleś spheres) have provided a number of examples showing that spaces with quantum group symmetries can be successfully studied using the tools of noncommutative geometry.
We completed this analysis in [11] by constructing regular spectral triples for all Podleś quantum spheres S^2_{qt}, which are two-parameter ($q \in (0,1)$ and $t \in [0,1]$) noncommutative deformations of the round sphere S^2 [12]. The representation of the algebra $\mathcal{A}(S^2_{qt})$ by left multiplication on two components vectors in $\mathcal{A}(S^2_{qt}) \otimes \mathbb{C}^2$ can be explicitly described in a basis of harmonic spinors which are the 'q-analogue' of Penrose chiral spinors. With this representation, and with a (generalized) Dirac operator D with the same spectrum of the classical Dirac operator of the round sphere S^2, we constructed a spectral triple which is 'equivariant' under the natural action of the Drinfeld-Jimbo deformation $U_q(su(2))$. More generally, for any fixed $N \in \frac{1}{2}\mathbb{Z}$, by using the representations in [13] we constructed [5] an equivariant spectral triple whose Dirac operator D_N has always undeformed spectrum,

$\mathrm{Sp}(D_N) = \{1, 2, 3, \ldots\}$. We checked the non-triviality of the Fredholm modules canonically associated to these spectral triples by pairing their Chern-Connes character with a deformation of the Bott projection: the result of the pairing, which is $2N$, proves in particular that these spectral triples are all inequivalent.
Among the conditions for a 'real' spectral triple, a particular role is covered by the regularity condition, which is the starting point for the development of an abstract pseudo-differential calculus. If a spectral triple (with finite metric dimension) satisfies this condition, one can define its 'dimension spectrum' $\Sigma \subset \mathbb{C}$ as the set of singularities of zeta-type functions associated to the triple. Connes-Moscovici local index formula [4] yields a 'local representative' of the periodic cyclic cohomology class of a regular spectral triple, provided the dimension spectrum of the triple is known. Unfortunately there is no a canonical way to compute the dimension spectrum in the case of algebras coming from quantum groups.
In [3] the dimension spectrum for the spectral triple of [1] on $SU_q(2)$ was worked out by constructing a 'symbol map' from order zero pseudodifferential operators on $SU_q(2)$ to a noncommutative version of the cosphere bundle. The same strategy has been used in [8] for the spectral triple on $SU_q(2)$ given in [7].
On Podleś spheres the idea which allowed the computation of the dimension spectrum was to use the embedding of the algebra $\mathcal{A}(S^2_{qt})$ into the algebra $\mathcal{A}(SU_q(2))$ of the quantum $SU(2)$ group. Using this embedding we obtained a (quite surprising) simplification of the spinorial representation: it turns out that modulo an ideal of 'smoothing operators' the spinorial representation can be approximated by a very simple representation coming from the infinite-dimensional irreducible representations of $\mathcal{A}(SU_q(2))$. This approximation allowed to express zeta-type functions of operators in $\mathcal{A}(S^2_{qt})$ in terms of the Riemann zeta-function, and to prove that the dimension spectrum is $\Sigma = \{1, 2\}$ as expected from an analysis of the 'commutative' limit S^2.
A similar work has been performed on a higher dimensional example, the 4-dimensional orthogonal quantum sphere [6].

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Hopf-Hochschild (co)homology of module algebras

Atabey Kaygun

Our goal in this talk is twofold. First, we would like to draw attention to a class of algebras admitting compatible actions of bialgebras, called "module algebras" by formulating two interesting moduli problems and solve them. Second, we would like to define a version of Hochschild homology and cohomology suitable for module algebras.

We demonstrate that the moduli space of all Ore extensions on a fixed k-algebra A is the space of all $\mathcal{O}$-module algebra structures on A over a fixed Hopf algebra $\mathcal{O}$, which we call the Hopf-Ore algebra. We also determine the moduli space of all $\mathcal{H}_1$-module algebra structures on the algebra of meromorphic functions on the complex plane where $\mathcal{H}_1$ is the Connes-Moscovici Hopf algebra of codimension-1 foliations. The computation of this last moduli space is the first step in determining the moduli space of $\mathcal{H}_1$-module structures on the space of analytic functions, thus the moduli space of all codimension-1 analytic foliations, on arbitrary Riemannian manifolds and then other (pseudo-) manifolds.

Our second goal is motivated by the work of Tsygan et. al. [7, 15, 16, 17] which indicate that the Hochschild cochain complex of an algebra (i.e. a noncommutative space) is the replacement of the bundle of multi-vector fields over that noncommutative space. We show that for a module algebra, i.e. a noncommutative space which carries a Hopf symmetry, the underlying Hopf symmetry can be lifted to the level of Hochschild cochain complex. Thus we arrive at the right noncommutative analogue of the bundle of equivariant multi-vector fields. Then we define the Hopf-Hochschild cohomology as the cohomology of the Hopf-invariant cochains. We show that in low dimensions, these Hochschild cohomology groups are

$$HH^0_{\rm Hopf}(A,V) \cong {\rm Sym}_A(V^B) \quad\text{and}\quad HH^1_{\rm Hopf}(A,V) \cong {\rm Der}_B(A,V)/{\rm Inn}(A,V^B)$$

the symmetric A-sub-bimodule of B-invariant elements in V, and the B-equivariant arbitrary V-valued derivations modulo the subspace of B-equivariant V-valued inner derivations on A.

In the context of cyclic (co)homology and K–Theory, one of the most commonly used tools dealing with module algebras has been crossed product algebras. There is a large body of work dealing with algebras admitting actions of discrete groups and compact Lie groups, e.g. [9, 11, 2, 5, 12, 8, 4], which utilize this tool to its fullest extent. Also, there have been successful attempts in defining equivariant

cyclic (co)homology and K–Theory for module algebras [3, 1, 14] again by using crossed product algebras. We show that Hopf-Hochschild cohomology can also be defined as the Ext-groups on the category of modules of the equivariant analogue of the enveloping algebra of a crossed product algebra $A^e \rtimes B$.
However, defining Hochschild homology of module algebras is not as straight forward as defining Hochschild cohomology. The difficulty lies in the simple fact that for a B-module algebra A, unlike the Hochschild cochain complex, the Hochschild chain complex associated with this algebra need not be a differential graded B–module. The obstruction which prevents this complex from being B–linear is trivial whenever the bialgebra B is cocommutative, as in the case of group rings and universal enveloping algebras. Yet the same obstruction is far from being trivial if the underlying bialgebra is non-cocommutative. We investigate how much of the Hochschild homology is retained after dividing this obstruction out. To this end, we construct a new differential graded B–module $\mathrm{QCH}_*(A, B, V)$ for a B–module algebra A and a B–equivariant A–bimodule V. We define $HH_*^{\mathrm{Hopf}}(A, B, V)$ the Hopf–Hochschild homology of A with coefficients in V as the homology of the complex $k \otimes_B \mathrm{QCH}_*(A, B, V)$.
We would like to point out that the same strategy worked remarkably well in the case of cyclic cohomology of module coalgebras. In [13] we show that if we start with the cocyclic bicomplex of a module coalgebra twisted by a stable anti-Yetter–Drinfeld module, dividing the analogous obstruction results in the Hopf cyclic complex of [10] which was an extension of the Hopf cyclic cohomology of Connes and Moscovici [6]. At the end of the talk we show that the Hopf-Hochschild homology groups are isomorphic to certain higher Tor-groups over the category of $A^e \rtimes B$-modules as expected. We also compute these groups in low dimensions.

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Chern-Connes character and twisted equivariant cohomology

Ping Xu
(joint work with Jean-Louis Tu)

Twisted K-theory has attracted a lot of attention recently due to its close connection with mathematical physics. See [9] for a recent survey. In [11], we developed the twisted K-theory for a differentiable stack $\mathfrak{X}$, where the twisted class is given by a class in $H^3(\mathfrak{X},\mathbb{Z})$ when the stack is proper. Our theory contains two important special cases: (1) if the stack is an orbifold $\mathfrak{X}$, this defines the twisted K-theory of an orbifold by a general class $\alpha \in H^3(\mathfrak{X},\mathbb{Z})$ [1]; (2) if the stack is the one corresponding to the transformation groupoid $M\times G \rightrightarrows M$ for a proper G action on M, we obtain the twisted equivariant K-theory $K^*_{G,\alpha}(M)$ where $\alpha \in H^3_G(M,\mathbb{Z})$ (see also [2, 7, 8]). General properties, including the Mayer-Vietoris exact sequence, Bott periodicity, and the product structure $K^i_\alpha \otimes K^j_\beta \to K^{i+j}_{\alpha+\beta}$ are derived.
Our approach is essentially to follow the classical one using C^*-algebras due to Rosenberg [10]. The main idea here is to transform the geometric question to a question of C^*-algebras, for which many powerful K-theoretic techniques such as KK-theory have been developed.
To obtain a C^*-algebra from a class $H^3(\mathfrak{X},\mathbb{Z})$, we applied the general theory of S^1-gerbes over differentiable stacks developed in [4, 5]. Roughly speaking, differential stacks can be considered as Lie groupoids up to Morita equivalence. Under this picture, for a proper differentiable stack $\mathfrak{X}$, there is a bijection between elements in $H^3(\mathfrak{X},\mathbb{Z})$ and Morita equivalence classes of Lie groupoid S^1-central extensions $\tilde{\Gamma} \to \Gamma \rightrightarrows M$. Now for a proper differentiable stack $\mathfrak{X}$ and any $\alpha \in H^3(\mathfrak{X},\mathbb{Z})$, the twisted K-groups $K^*_\alpha(\mathfrak{X})$ are defined to be the K-groups of the C^*-algebra $C^*_r(\Gamma,\alpha)$. In [11], we proved that $C^*_r(\Gamma,\alpha)$ contains a natural smooth subalgebra $C^\infty_c(\Gamma,L)$ which is stable under the holomorphic functional calculus. Here $\tilde{\Gamma} \to \Gamma \rightrightarrows M$ is a groupoid S^1-central extension representing $\alpha \in H^3(\mathfrak{X},\mathbb{Z})$, $L = \tilde{\Gamma} \times_{S^1} \mathbb{C}$ is the associated complex line bundle, and $C^\infty_c(\Gamma,L)$ is the space of compactly supported sections of $L \to \Gamma$. Thus the Chern-Connes character is a

map $K^*_\alpha(\mathfrak{X}) \otimes \mathbb{C} \xrightarrow{\text{ch}} HP_*(C_c^\infty(\Gamma, L))$.
As an important situation, we study this Chern-Connes character map for the quotient stack $[M/G]$, where G is a compact Lie group. In particular, for any $\alpha \in H^3_G(M, \mathbb{Z})$, using the geometry of equivariant gerbes, we introduce *global twisted equivariant cohomology* $H^*_{G,global}(M, \alpha)$, which is a twisted version of the global equivariant cohomology in the sense of Baum-Brylinski-MacPherson [3] and Block-Getzler [6]. We also construct an explicit chain map from the periodic cyclic chain complex of the algebra $C_c^\infty(\Gamma, L)$ to this global twisted equivariant cohomology cochain complex and conjecture that this is a quasi-isomorphism. An important consequence of this conjecture is the following relation

$$K^*_{G,\alpha}(M) \otimes_{R(G)} R^\infty(G) \cong H^*_{G,global}(M, \alpha),$$

where $R(G)$ is the representation ring of G and $R^\infty(G)$ is the ring of smooth conjugation invariant functions on the group G.

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Universal Deformation Formulas for Non-Abelian Lie Group Actions and Symplectic Symmetric Spaces

Pierre Bieliavsky

I am currently interested in universal deformation formulas ('UDF') for non-abelian Lie group actions on topological algebras, in the same spirit as Rieffel's work in the framework of deformation quantization for actions of $\mathbb{R}^d$. Within this context, I am particularly concerned with the interplay between geometrical, representation theoretical and harmonic analytical aspects.
Roughly, a UDF for a given group $\mathbb{S}$ within a topological category of associative algebras $\mathcal{A}$ is a procedure which, for every data of an algebra $\mathbb{A} \in \mathcal{A}$ which the group $\mathbb{S}$ acts on by automorphisms, produces a field of algebras $\{\mathbb{A}_\theta\}$ in the initial

category $\mathcal{A}$ and deforming $\mathbb{A} =: \mathbb{A}_o$ (here θ is a parameter running in some topological space Θ and o is some base point in Θ).
The basic example [10], due to Rieffel, applies to the abelian group $\mathbb{S} = \mathbb{R}^d$ and relies on Weyl's quantization: starting with a Fréchet algebra $\mathbb{A}$ together with a (sufficiently regular) action $\alpha : \mathbb{R}^d \times \mathbb{A} \to \mathbb{A}$ and a skewsymmetric matrix $\theta \in \mathfrak{so}(d)$, the following formula:

$$a \star_\theta b \;:=\; \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i\,x\cdot y}\,\alpha_x(a)\,\alpha_{\theta(y)}(b)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

defines an associative product, $\star_\theta$, on the space $\mathbb{A}^\infty$ of smooth vectors of α in $\mathbb{A}$. The above integral sign denotes an oscillatory integral defined through a standard integration by parts technique involving the Laplace operator $\Delta_{\mathbb{R}^d}$ on $\mathbb{R}^d$. Observe that the integral expression for Dirac's distribution suggests that at $\theta = 0$ the deformed product coincides with the original one from $\mathbb{A}$. When $C^\star$, the family $\{(\mathbb{A}^\infty, \star_\theta)\}$ can be completed into a continuous field $\mathbb{A}_\theta$ of $C^\star$-algebras over the space $\Theta = \mathfrak{so}(d)$ deforming $\mathbb{A} = \mathbb{A}_0$.
On the formal algebraic side, such a UDF corresponds to the data of a Drinfeld twisting element based on a universal enveloping algebra, or more generally on a Hopf algebra. Explicit expressions for such twists are very rare in the literature. However, one finds the examples due to Zagier [12] and Giaquinto-Zhang [9] for the case of the affine group $G = ax + b$. Zagier's example comes from analytical number theory (Rankin-Cohen brackets on modular forms). It has been used and developed by Connes and Moscovici in their work on codimension one foliations [8]. Giaquinto-Zhang's example may be interpreted as the restriction of Moyal's product on $\mathbb{R}^2$ to an open orbit of the affine group. Both examples are formal with respect to a finite dimensional parameter space.
The groups I consider are solvable Lie groups (see [1], and [5, 3] for developments), more precisely, Iwasawa components $\mathbb{S} = AN$ of real semi-simple Lie groups G of hermitian type (e.g. the affine group $ax+b$ appears here as the Iwasawa component of $SL(2,\mathbb{R})$). A geometrical study of various symplectic symmetric spaces attached to such a semi-simple Lie group G led me to consider specific two- and three-point functions on its Iwasawa component $\mathbb{S}$. It turns out that the latter functions constitute the phase and amplitude of oscillatory kernels on $\mathbb{S} \times \mathbb{S}$ defining, in an analogous manner as in (1), non-formal UDF's for the actions of $\mathbb{S}$ valid in the category of Fréchet algebras[2].
The phase and amplitude functions admit elementary geometrical interpretations. For instance, in the rank one situation[1], I find UDF's such as:

$$a \star b = \int_{\mathbb{S}\times\mathbb{S}} K_\theta\,\alpha(a)\otimes\alpha(b)\,,$$

where K_θ is an oscillating kernel of the form:

$$K_\theta \;=\; \mathbf{A}\,\mathbf{b}_\theta\,e^{\frac{i}{\theta}S}\,.$$

[1] Higher ranks can be obtained by applying a split extension technique which is essentially standard.

The phase S of the oscillating kernel equals the symplectic area of a geodesic triangle in the group manifold $\mathbb{S}$ with respect to a symmetric space geometry obtained by partially contracting the curvature of the Hermitian symmetric space G/K [2]. In contrast, the amplitude $\mathbf{A}$ is only partially fixed by the geometry: it can be modified arbitrarily by a factor[3] which is the coboundary (for a certain natural differential on the multiple-point function complex) of a pseudo-differential operator symbol $\mathbf{b}_\theta$ living on the boundary of G/K. In particular, the deformation parameter space Θ is infinite dimensional. Surprisingly, the differential operator playing the role of the flat Laplacian in the integration by parts procedure allowing to define the oscillatory integral is here a power (depending on the dimension) of the *uncontracted* Laplacian on the Hermitian symmetric space G/K [4]. The latter fact reveals a concrete geometrical interplay between the contracted and uncontracted geometries considered here.
The contracted symmetric space whose geometry underlies the oscillating kernel is a symplectic solvable analogue of conformally flat Riemannian (or pseudo-Riemannian) symmetric spaces (i.e. with vanishing Weyl tensor)[5]. Its transvection group being a solvable Lie group, one may want, in order to attend the semi-simple context of the Hermitian space G/K, to modify the above construction in such a way that the automorphism group of the resulting kernel would contain G rather than its solvable contraction. In other words, one may want to 'decontract' the situation. In a collaboration with Detournay and Spindel, we realized this program for the case of $G = SL(2,\mathbb{R})$ i.e. in the case of the hyperbolic plane $\Delta = SL(2,\mathbb{R})/SO(2)$ [6]. The key is to relate the associative kernels on Δ to the space of solutions for the evolution of a certain second order hyperbolic differential operator. Typically, the resulting $SL(2,\mathbb{R})$-equivariant quantization kernels have the form:

$$K_\theta^\Delta \;=\; \int_0^\infty t^2 J_{\frac{1}{\theta}}(t)\, e^{it\, S_\Delta}\, \mathrm{d}t\,,$$

where J_μ denotes the Bessel function of first kind and where S_Δ denotes a specific three-point function on the hyperbolic plane closely related with a hyperbolic triangle area. An interesting challenge consists in implementing an action of an arithmetic Fuchian group $\Gamma \subset SL(2,\mathbb{R})$. I plan to investigate this question in a near future.
Actually, the above method works not only in the negatively curved case. Any surface admitting a local structure of a coadjoint orbit of $SL(2,\mathbb{R})$ can be treated in the same way. At the formal level, every $\mathfrak{sl}(2,\mathbb{R})$-invariant star product can be

[2]Note that the Iwasawa decomposition of G yields a canonical $\mathbb{S}$-equivariant symplectomorphism between G/K and $\mathbb{S}$.

[3]Strictly speaking, the modifying factor being complex valued, the phase S could be modified as well. But this modification is cohomologically inessential.

[4]More generally a power of a Damek-Ricci metric Laplacian on $\mathbb{S}$ could be chosen as well.

[5]These symplectic spaces are said to be of *Ricci type* in the literature.

[6]Unpublished; see Detournay's PhD thesis (arXiv:hep-th/0611031 pp. 64-80). The higher dimensional case is currently under investigation by one of my students. The rank one case ($G = SU(1,n)$) is very likely to follow the exact same scheme as $SL(2,\mathbb{R})$.

produced by the above method, and, in particular, Zagier's formal modular form product may be derived from the flat nilpotent conical orbit case. The case of the one-sheeted hyperboloids has not been investigated, however it would not be surprising if it would produce an equivalent version of Unterberger's Weyl calculus on non-holomorphic automorphic forms [11].

I'll finish with mentioning applications of the above non-abelian UDF's as well as possible developments. Analogously to noncommutative spherical manifolds obtained from quantum torus actions [7], a first natural application consists in producing new examples of noncommutative manifolds (spectral triples) such as deformed locally anti de Sitter black holes (see [4] and [3] for developments) or noncommutative hermitian symmetric spaces (in progress, jointly with Gayral and Iochum). In a locally compact quantum group context, it is very likely that the non-formal twists obtained (based on the full Borel in the Hermitian case) will yield multiplicative unitaries in the sense of Baaj-Skandalis or Woronowicz. I would like to study this question as well as the possibility of defining in this non-formal context a notion of twisted Dirac operator in a similar spirit as Kostant's cubic Dirac operator in the formal case.

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The harmonic oscillator, its noncommutative dimension and the vacuum of noncommutative gauge theory

Raimar Wulkenhaar
(joint work with Harald Grosse)

Renormalisable scalar quantum field theories on noncommutative Moyal space are characterised by the appearance of a harmonic oscillator potential in the action functional [1]. To make contact with noncommutative geometry, and also to extend the model to gauge theory, one must understand the Moyal-oscillator space as a spectral triple. This amounts to construct a Dirac operator such that its square is the Schrödinger operator of the harmonic oscillator. All attempts to find such a spectral triple in agreement with the original set of axioms [2] failed so far. Recently, it was realised that many interesting noncommutative geometries, such as the standard model [3], have different KO and spectral dimensions.
With this flexibility in mind, it it now not difficult to understand the harmonic oscillator as a spectral triple. We describe here the four-dimensional case. For details we refer to [4]. As the algebra we take the Schwartz class functions $\mathcal{A}_4 = \mathcal{S}(\mathbb{R}^4)$ either with the commutative product or with the Moyal product

$$(1) \qquad (f\star g)(x) = \int d^4y\,\frac{d^4k}{(2\pi)^4}\,f(x+\tfrac{1}{2}\Theta\cdot k)\,g(x+y)\,\mathrm{e}^{i\langle k,y\rangle}\;, \qquad f,g\in\mathcal{A}_4\;.$$

The Dirac operator $\mathcal{D}_4$ is constructed in the *eight-dimensional Clifford algebra* with generators $\Gamma_1,\dots,\Gamma_8$. In the Moyal case we take

$$(2) \qquad \mathcal{D}_4 = i\Gamma^\mu\partial_\mu + \Omega\Gamma^{\mu+4}\tilde{x}_\mu\;,$$

where $\tilde{x}_\mu := 2(\Theta^{-1}_{\mu\nu}x^\nu$. As usual greek indices run from 1 to 4 and Einstein's sum convention is used. In the commutative case, we replace $\Omega\Gamma^{\mu+4}\tilde{x}_\mu$ by $\omega\Gamma^{\mu+4}x_\mu$. Accordingly, the Hilbert space is $\mathcal{H}_4 = L^2(\mathbb{R}^4)\otimes\mathbb{C}^{16}$. Equivalently, one can regard $\mathbb{C}^{16}\simeq \mathrm{Cliff}(\mathbb{C}^4)$ and realise the first set $\Gamma^1,\dots,\Gamma^4$ as usual Clifford multiplication with standard $4D$ gamma matrices, whereas the action of $\Gamma^{\mu+4}$ is constructed from γ^μ and a graded sign. In the Moyal case, the algebra acts on $\mathcal{H}_4$ by componentwise left Moyal multiplication $L_\star : \mathcal{A}_4\times\mathcal{H}_4\to\mathcal{H}_4$.
The only possibility for the grading operator is $\chi_4 = \Gamma_9 := \Gamma_1\cdots\Gamma_8$. Then, choosing the Clifford generators such that $\Gamma_1,\dots,\Gamma_4$ are real and $\Gamma_5,\dots,\Gamma_8$ purely imaginary, the only possible real structure is $J_4\psi := \Gamma_9\bar{\psi}$ for $\psi\in\mathcal{H}_4$. This means that the geometry has KO-dimension 0Mod8. At first sight, this seems to be related with the dimension of the Clifford algebra and the phase space dimension. However, the KO-dimension is always zero for any harmonic oscillator dimension. On the other hand, the metric dimension of the triple $(\mathcal{A}_4,\mathcal{D}_4,\mathcal{H}_4)$ is four—if defined in non-compact sense. First, if Θ is given by θ times the standard symplectic form, one finds $\mathcal{D}_4^2 = H_4\otimes 1_{16} + \tilde{\Omega}\otimes\Sigma_4$, where $H_4 = -\partial^\mu\partial_\mu + \tilde{\Omega}^2x^\mu x_\mu$, $\tilde{\Omega} := \frac{2\Omega}{\theta}$, and $\Sigma\in M_{16}(\mathbb{C})$ is the traceless spin matrix. Using the Mehler kernel

$$(3) \qquad e^{-tH_4}(x,y) = \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^2 e^{-\frac{\omega}{4}\coth(\omega t)\|x-y\|^2-\frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}\;,$$

for $x, y \in \mathbb{R}^4$, and the kernel representation of the Moyal product (1), one can compute the Dixmier trace of $L_\star(f)(\mathcal{D}_4^2+1)^{-\frac{\delta}{2}}$ as a residue:

$$\operatorname{Tr}_\omega\big(L_\star(f)(\mathcal{D}_4^2+1)^{-\frac{\delta}{2}}\big) = \lim_{s\to 1}\frac{(s-1)}{\Gamma(\frac{\delta s}{2})}\int_0^\infty dt\, t^{\frac{\delta s}{2}-3}e^{-t}\,\frac{(\tilde{\Omega}t)^2}{\pi^2(1+\Omega^2)^2\tanh^2(\tilde{\Omega}t)}$$
$$\times\int d^4x\, f(x)\, e^{-\frac{\tilde{\Omega}\tanh(\tilde{\Omega}t)}{1+\Omega^2}\|x\|^2}\,. \tag{4}$$

For Schwartz-class functions f, the dimension spectrum is given by the even integers ≤ 4, so that we regard the maximal value $d=4$ as the spectral dimension. The corresponding trace theorem allows one to express the scalar product for smooth spinors by the Dixmier trace and the standard hermitean structure. However, if we adjoin a unit to the algebra and take $f=1$, then the compact dimension would be 8. The non-compact version is the correct one, because the non-compact L^2-Hilbert space is essential in the quantisation of the spectrum of H_4.

Now we can derive the spectral action [2]. To make the model more interesting, we tensor $(\mathcal{A}_4,\mathcal{D}_4,\mathcal{H}_4,\chi_4)$ with the Connes-Lott two-point spectral triple $(\mathbb{C}\oplus\mathbb{C},\mathbb{C}^2,M\sigma_1)$. Then, using $[\mathcal{D}_4,L_\star(f)] = i(\Gamma^\mu+\Omega\Gamma^{\mu+4})(\partial_\mu f)$ due to $[x^\nu,f]_\star = i\Theta^{\nu\rho}\partial_\rho f$, the fluctuated Dirac operators $\mathcal{D}_A = \mathcal{D}+\sum_i a_i[\mathcal{D},b_i]$ are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4+(\Gamma^\mu+\Omega\Gamma^{\mu+4})L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_4+(\Gamma^\mu+\Omega\Gamma^{\mu+4})L_\star(B_\mu)\end{pmatrix}, \tag{5}$$

for real Yang-Mills fields $A_\mu, B_\mu \in \mathcal{A}_4$ and a complex Higgs field $\phi\in\mathcal{A}_4$.

According to the spectral action principle [2], the most general form of the bosonic action is

$$S(\mathcal{D}_A) = \operatorname{Tr}\big(\chi(\mathcal{D}_A^2)\big) = \int_0^\infty dt\ \operatorname{Tr}(e^{-t\mathcal{D}_A^2})\hat{\chi}(t)\,, \tag{6}$$

where $\hat{\chi}$ is the (inverse) Laplace transform of the weight function χ. We write $\mathcal{D}_A^2 =: H_4 1_{32}+\tilde{\Omega}\Sigma_4 1_2 - V$ and iterate the Duhamel expansion

$$e^{-t_0(\mathrm{H}_0-V)} = e^{-t_0\mathrm{H}_0} + \int_0^{t_0} dt_1\,\big(e^{-(t_0-t_1)(\mathrm{H}_0-V)}Ve^{-t_1\mathrm{H}_0}\big) \tag{7}$$

to obtain an asymptotic expansion $e^{-t\mathcal{D}_A^2} = \sum_{z=-4}^\infty a_z(\mathcal{D}_A^2)t^z$. After long calculation, we obtain the following spectral action

$$\begin{aligned} S &= \frac{\theta^4\chi_{-4}}{\Omega^4}+\frac{2\theta^2\chi_{-2}}{3\Omega^2}+\frac{52\chi_0}{45}\\ &+\frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{\big(\tfrac{(1-\Omega^2)^2}{2}-\tfrac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\big)\,(F^A_{\mu\nu}\star F^{\mu\nu}_A + F^B_{\mu\nu}\star F^{\mu\nu}_B)\\ &+\Big(\phi\star\bar{\phi}+\tfrac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A\star\tilde{X}_{A\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2+\Big(\bar{\phi}\star\phi+\tfrac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B\star\tilde{X}_{B\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2\\ &-2\Big(\tfrac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0\star\tilde{X}_{0\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2+2D^\mu\phi\star\overline{D_\mu\phi}\Big\}(z)+\mathcal{O}(\chi_1)\,, \end{aligned} \tag{8}$$

where $\chi_z = \int_0^\infty dt\, t^z\,\hat{\chi}(t)$. In (8), $\tilde{X}^\mu_A(x) := \frac{\tilde{x}^\mu}{2}+A^\mu(x)$ is a covariant coordinate with gauge transformation $X^\mu_A \mapsto u_A\star X^\mu_A\star\overline{u_A}$. Similarly for $\tilde{X}^\mu_B$. By $D_\mu\phi =$

$\partial_\mu\phi - iA\star\phi + i\phi\star B$ we denote the covariant coordinate of the Higgs field. The gauge transformation of the latter is $\phi\mapsto u_A\star\phi\star\overline{u_B}$.
Some conclusions and comments:

- The square of covariant derivatives combines with the Higgs field to a non-trivial potential. This was not noticed in [5, 6] where gauge theory induced by scalar fields was derived. We observe here a much deeper unification of the continuous geometry described by Yang-Mills fields and discrete geometry described by the Higgs field than previously in almost-commutative geometry. The distinction into discrete and continuous part is no longer possible in general noncommutative geometries. Therefore, the Higgs potential cannot be restricted to the Higgs field, it must include the gauge field, too.
- The coefficient in front of the Yang-Mills action is positive for all $\Omega\in[0,1[$. In the bosonic model of [5, 6] there was only the analogue of the negative part, which leads to problems with the field equations.
- Most importantly, (8) is translation-invariant if we forget how $\tilde{X}$ is constructed: The transformation $\phi(x)\mapsto\phi(x+a)$ and $X_{G\mu}(x)\mapsto X_{G\mu}(x+a)$, for $G\in A,B,0$, leaves the action invariant. Thus, a frequent objection against the renormalisable ϕ^4_4-models disappears for the Yang-Mills-Higgs spectral action.

We have found several solutions of the classical field equations resulting from the spectral action. For pure Yang-Mills theory (i.e. $\phi=0$), there is an interesting radial solution in terms of modified Bessel and Struve functions:

$$(\tilde{X}^\mu\star\tilde{X}_\mu)(x)=\frac{\eta^2(1+\Omega^2)}{4\Omega^2}\Big(1+\frac{\pi}{2}\big(I_1(\gamma\|x\|^2)-\mathbf{L}_{-1}(\gamma\|x\|^2)\big)\Big)\,, \tag{9}$$

where $\frac{\chi_{-1}}{\chi_0}=\eta^2$ and $\frac{4\sqrt{2}g\Omega^2}{\theta(1+\Omega^2)}$ for the coupling constant given by $\frac{1}{4g^2}:=\frac{(1-\Omega^2)^2}{2}-\frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}$. For small distances $\|x\|$ we have $\tilde{X}^\mu\star\tilde{X}_\mu\sim\|x\|^2$, which provides the oscillator potential needed for renormalisation. At large distances, $\tilde{X}^\mu\star\tilde{X}_\mu$ approaches its asymptotic value where the Higgs type potential vanishes. The influence of this behaviour on renormalisation remains to be studied.

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