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Mini-Workshop: Complex Approximation and Universality

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Abstract. The mini-workshop was devoted to the study of universality in connection with approximation theory in complex analysis and related notions within the framework of Banach spaces and various function spaces. Topics discussed included invariance of universality, Baire category and universality, the size and structure of the set of universal functions, universality for composition with respect to a sequence of mappings, universal Taylor series and Faber series, overconvergent series, lacunary series, holomorphic continuation of series, universal series with restricted coefficients, universality for solutions of the Laplace equation, the heat equation and Burgers' equation, hypercyclicity, and superhypercyclicity. Topics in approximation included polynomial approximation, approximation and maximum principles, approximating inner functions by Blaschke products, and approximation of the Riemann zeta function. A list of open problems is included in the report.

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Introduction by the Organisers

The notion of universality covers a wide range of phenomena in complex analysis. Generally speaking, a universal object is one which, when subjected to some limiting process, approximates every object in some universe. For example, universality occurs when the translates of an entire function can approximate any other entire function, or when the partial sums of a formal power series or a formal trigonometric series approximate all functions in some natural class. For a long time, existing approximation theorems were used in constructions of universal functions and universal series. In recent years, however, constructions have required the development of new approximation theorems, thereby also enriching the area of complex approximation.

Universal functions. There is no single definition of a universal function. What they have in common is the following. One considers a suitable sequence $\mathcal{T} = (T_n)$ of operators acting on a space X, for example, of holomorphic functions with values in another space Y of holomorphic functions. Then a function $f \in X$ is called universal with respect to $\mathcal{T}$ if the sequence $(T_n f)$ is dense in Y . One of the earliest examples of a universal function is due to Birkhoff (1929) who showed that there exists an entire function f whose translates $f(z+n), n \geq 1$, can approximate any other entire function, uniformly on compact sets. In that case we have $(T_n f)(z) = f(z+n)$, and $X = Y$ is the space of entire functions with the usual compact-open topology.

Seidel and Walsh showed that an analogue of Birkhoff's universality theorem holds for functions holomorphic in the unit disc, if we replace translates by "non-euclidian translates", that is $T_n f = f \circ \phi_n$ is the composition of f with an automorphism ϕ_n of the unit disc D. At the heart of the study of holomorphic functions in the disc D is the class $H^\infty(D)$ of bounded holomorphic functions on the disc. Chee showed the existence of universal functions for the class $H^\infty(B)$ of bounded holomorphic functions on the unit ball of C^N. Richard Aron's talk was concerned with the size and the structure of the set of such universal functions. In the study of the space $H^\infty(B)$ a fundamental role is played by inner functions. These are also of importance in engineering control theory. Recently, Gauthier and Xiao have shown the existence of universal inner functions in the unit ball of C^N. Geir Arne Hjelle and Raymond Mortini gave talks concerned with approximating inner functions in the unit disc D by simpler inner functions, namely Blaschke products.

Extending the study of functions in the unit disc, which are universal with respect to composition with automorphisms of the disc, Mortini talked about the universality of functions f holomorphic on a domain Ω with respect to a sequence $(f \circ \phi_n)$ of compositions, where (ϕ_n) are self-maps of Ω (not necessarily automorphisms).

Universal series. In 1918 Jentzsch gave an example of a power series Σ for which a subsequence of the partial sums of Σ converges outside of its disc D of convergence. Such a power series is said to be overconvergent. Luh, Chui and Parnes showed the existence of such an overconvergent series Σ which is universal in the sense that, for each compact set K in the complement of $\overline{D}$, and for each f holomorphic on K, there are partial sums of Σ which converge uniformly to f. Nestoridis showed that one can even allow K to meet the boundary of D.

The main focus of the mini-workshop was in fact on universal series of one sort or another. The talks by Wolfgang Luh and Tatevik Gharibyan dealt with the strong relation between universality and lacunarity for power series and also the relation between various forms of summability and holomorphic continuation. Lacunary power series are ones for which many of the coefficients are zero. Another restriction which one can impose on the coefficients is that the sequence of coefficients lie in some sequence space. Vassili Nestoridis, in his first lecture, showed that there are universal series, whose sequence of coefficients are in every ℓ^p-space, for each

$p > 1$. In his second talk, he spoke on the recent work of Mouze concerning universality of the geometric series. Vagia Vlachou gave a talk on universal Faber series. Jürgen Müller, in his talk "From polynomial approximation to universal Taylor series and back again" gave specific examples of the interplay between theorems on approximation by polynomials of a given class and the existence of universal Taylor series whose coefficients satisfy a given restriction.

Potential theory. The extension of some universality results to harmonic functions is due to Armitage (2002, 2003, 2005). Innocent Tamptse's talk was concerned with universal series of harmonic functions. Universality for harmonic functions is based on approximation theory just as universality for holomorphic functions. One reason that harmonic universality was developed much more recently than holomorphic universality is that harmonic approximation theory attained its full development only recently, largely due to Stephen Gardiner. In his talk, Gardiner discussed the relation between approximation theorems and maximum principles in potential theory. The classical approximation theorem of Runge was extended not only to harmonic functions but also to solutions of elliptic partial differential equations (Lax-Malgrange). This leads to universality results for solutions of such equations. Paul Gauthier spoke on universality for solutions of the heat equation (which of course is not elliptic) and also for solutions to Burgers' equation, which is one of the simplest non-linear parabolic equations. Burgers' equation has applications in aerodynamics.

General theory of universality. Several of the talks were not so much concerned with one particular type of universality as with phenomena related to universality in general. For example, the first talk of Nestoridis as well as the talks by Tamptse and Gauthier made use of the recently developed "abstract theory of universality".

A bounded operator T defined on some separable Banach space X is called *hypercyclic* if there exists some vector $x \in X$ such that the orbit of x under T, namely $\{T^n x;\ n \geq 0\}$ is dense in X. A theorem of Kitai, Gethner and Shapiro asserts that an operator satisfying a certain criterion called the *Hypercyclicity Criterion* is always hypercyclic. The Hypercyclicity Criterion is a very powerful tool to prove that an operator is hypercyclic. And even, until recently, every hypercyclic operator was hypercyclic... because it satisfies the assumptions of the Hypercyclity Criterion. Thus, a natural question was to know whether every hypercyclic operator satisfies the assumptions of the Hypercyclicity Criterion. Recently, De La Rosa and Read proved that there exist a Banach space X and a hypercyclic operator T on X that does not satisfy the hypercyclicity criterion, but this space cannot be identified with some "classical" Banach space. Fréderic Bayart, in his talk proved that in fact such an operator exists on the separable Hilbert space.

A bounded linear operator T defined on a separable Banach space X is said to be supercyclic if there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. It is called weakly supercyclic if the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is weakly dense in X.

Fernando León-Saavedra, in his talk proposed a method to prove non-supercyclicity and non-weak supercyclicity which is less computational than previous methods and for which the proofs turn out to be simpler.

George Costakis, spoke on the question as to whether there exist maps between spaces of holomorphic functions which preserve certain notions of universality.

Karl-G. Grosse-Erdmann named his talk "Construction versus Baire category in universality". He could also have named it "Bare hands versus Baire category in universality". The existence of universal functions is usually proved by one of two methods: by an explicit construction or by the use of the Baire category theorem. In his talk, Grosse-Erdmann argued that the two methods are largely equivalent and ended his talk by recalling a pronouncement of T. W. Körner

The Baire Category Theorem is a profound triviality.

The Riemann zeta function. Although universality is a generic phenomenon, the only explicit function which is known to have universality properties is the Riemann zeta function and its close cousins (Voronin, 1975). Markus Nieß extended recent results showing that it is possible to approximate the Riemann zeta function by functions which fail to satisfy the conclusion of the Riemann hypothesis. That is, they do have zeros which are not on the critical axis.

An additional lecture was given by Sophie Grivaux from Lille, who was at Oberwolfach as a participant in the RIP program. She talked about the relation between hypercyclicity and the invariant subset problem.

Problem session. A problem session was held in which problems were presented by Grosse-Erdmann, Aron, Nestoridis, Vlachou, Gauthier, Luh, Gardiner, and Mortini. Gardiner presented some problems in the name of David H. Armitage, who unfortunately could not attend.

Participants found the mini-workshop extremely stimulating. Mathematical and social bonds were reinforced which will surely prolong existing collaborations and develop new ones.

The organizers were Paul M. Gauthier (Montréal), Karl-Goswin Grosse-Erdmann (Mons), and Raymond Mortini (Metz). The participants greatly appreciated the hospitality and the stimulating atmosphere of the Forschungsinstitut Oberwolfach.

Mini-Workshop: Complex Approximation and Universality

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Abstracts

Universal series in $\bigcap_{p>1} \ell^p$

V. Nestoridis
(joint work with S. Koumandos, G.-S. Smyrlis and V. Stefanopoulos)

The present is mainly based on [3], which in turn makes use of the recently developed "Abstract theory of universal series" ([2]).

Let X, ρ denote a topological vector space (on the field $\mathbb{K} = R$ or $\mathbb{K} = C$) endowed with a translation invariant metric ρ. Let $x_0, x_1, \dots$ be a fixed sequence of vectors in X.

Definition. A scalar sequence $a = (a_j)_{j=0}^{\infty}$ in $\mathbb{K}^{N_0}$ belongs in the class U, if the sequence $\sum_{j=0}^{n} a_j x_j$, $n = 0, 1, 2, \dots$ in dense in X.

Proposition. *$U \neq \emptyset$, if and only if, for every $p \in N$ the linear span $\langle x_p, x_{p+1}, x_{p+2}, \dots \rangle$ is dense in X.*

If this holds, then U is G_δ and dense in $\mathbb{K}^{N_0}$ endowed with its cartesian topology and contains a dense vector subspace except 0.

Let A, d be a vector subspace of $\mathbb{K}^{N_0}$ endowed with a translation invariant metric d. We assume that the following postulates hold.

i) (A, d) is a complete vector space.

ii) The projections $A \ni a = (a_j)_{j=0}^{\infty} \to a_m \in \mathbb{K}$ are continuous for all $m \in N$.

iii) The set $c_{00} = \{(a_j)_{j=0}^{\infty} \in \mathbb{K}^N : \exists\ j_0$ so that $a_j = 0$ for all $j \geq j_0\}$ is contained in A.

iv) $\overline{c_{00}} = A$.

A stronger postulate than iv) is the following

iv)′: For every $a = (a_j)_{j=0}^{\infty}$ in A we have $\sum_{j=0}^{n} a_j e_j \xrightarrow{d} a$, as $n \to +\infty$, where $e_j = (\delta_{i_\gamma})_{i=0}^{\infty}$ is the usual base.

Definition. A sequence $a = (a_j)_{j=0}^{\infty}$ in A belongs to the class U_A, if, for every $x \in X$, there exists a sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{N}$ so that $\sum_{j=0}^{\lambda_n} a_j x_j \xrightarrow{\rho} x$ and $\sum_{j=0}^{\lambda_n} a_j e_j \xrightarrow{d} a$ as $n \to +\infty$.

Obviously $U_A \subset U \cap A$. If iv)′ holds then $U_A = U \cap A$. An example where $U_A \neq U \cap A$ can be found in [2] part C.

Theorem. *$U_A \neq \emptyset$, if and only if, for every $x \in X$ and $\varepsilon > 0$, there exist $M \in N$ and $\beta_0, \beta_1, \dots, \beta_M \in \mathbb{K}$ so that $\rho\left(x, \sum_{j=0}^{M} \beta_j x_j\right) < \varepsilon$ and $d\left(\sum_{j=0}^{M} \beta_j e_j, 0\right) < \varepsilon$.*

If this holds then U_A is G_δ and dense in A and contains a dense vector subspace of A except 0.

In [2] one can find variants of the previous theorem where the space X is replaced by a denumerable family of spaces.

In the sequel we assume that the sequence x_j, $j = 0, 1, 2, \dots$ satisfies the following.

Condition D. For every finite set $I \subset \mathbb{N}$, there exist distinct indices $j_n(i)$, $n \in N$, $i \in I$ such that $x_{j_n(i)} \to x_i$, as $n \to +\infty$.

Clearly it is equivalent that condition D holds for all singletons I.

Proposition. *If $U \neq \emptyset$ and condition D holds then $U_{\bigcap_{p>1} \ell^p} \neq \emptyset$.*

Remark. The space $\bigcap_{p>1} \ell^p$ is endowed with a metric compatible with the norms $\| \ \|_{1+\frac{1}{k}}$, $k = 1, 2, \dots$ and it is easily seen that it satisfies the postulates for A.

Theorem. *Suppose that (E, ρ) is a vector space endowed with a translation invariant metric ρ and $x_0, x_1, x_2, \dots$ is a fixed sequence of vectors in E. We assume that condition D is satisfied. We set X to be the linear span $\langle x_0, x_1, \dots \rangle$. Then $U_{\bigcap_{p>1} \ell^p} \neq \emptyset$.*

Remark. The approximation results guaranteed by the previous theorem remain valid if the space X, ρ is replaced by it closure or completion. In the applications we need to use an approximation result in order to identify the completion of X. In most of our applications φ is a continuous function (in some set) and x_j are the translations $\varphi(\cdot - b_j)$, where b_j, $j = 0, 1, \dots$ is a sequence without isolated points. Condition D is then guaranteed by the continuity of the translation $b \to \varphi(\cdot - b)$.

Applications yield approximation results by translates of the Riemann zeta function [6], or by translates of a fundamental solution of suitable elliptic operators with constant coefficients in $\mathbb{R}^n$ [4], or by translates of approximate identities in $\mathbb{R}^d$, as for example by normal distributions [5]. Another application yields universal trigonometric series in $\mathbb{R}^d$ (non-periodic case) with frequencies with finite accumulation points. Improvements of these results are obtained by using universal Dirichlet series in the sense of Bayart ([1]) in one or several variables, where the only accumulation point of the frequencies is ∞.

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From polynomial approximation to universal Taylor series and back again

Jürgen Müller

For a compact set $E \subset \mathbb{C}$ let $(A(E), \|\cdot\|_E)$ denote the Banach space of all $f : E \to \mathbb{C}$ continuous on E and holomorphic in E^0 with the uniform norm. Moreover, let $P(E)$ be the closure of the polynomials in $A(E)$ and let $H(E)$ be the set of functions on E extending holomorphically to some neighborhood of E. For E^c connected, Runge's theorem says that $H(E) \subset P(E)$ and Mergelian's theorem shows that, more precisely, $A(E) = P(E)$.

For a domain $\Omega \subset \mathbb{C}$, which is always supposed to contain the unit disk $\mathbb{D}$ but not its closure, and for a function $f \in H(\Omega)$ we consider the Taylor sections

$$(S_n f)(z) = \sum_{\nu=0}^{n} \frac{f^{(\nu)}(0)}{\nu!} z^\nu \qquad (n \in \mathbb{N},\, z \in \mathbb{C})$$

and ask for results on

$$\omega(f, E) := \{g \in A(E) : \exists\, (n_m) : S_{n_m} f \to g \text{ in } A(E)\}.$$

For $K \subset \mathbb{D}^c$ compact with K^c connected it may happen that $\omega(f, K)$ is maximal. We set

$$U_K(\Omega) := \{f \in H(\Omega) : \omega(f, K) = A(K)\}.$$

Applying the universality criterion (see [GE]), we give a short proof of (cf. [Ne])

Proposition 1. Let Ω be simply connected. Then $U_K(\Omega)$ is residual in $H(\Omega)$ for all compact sets $K \subset \Omega^c$ with K^c connected.

Proof. We consider $S_n : H(\Omega) \to A(K)$ with the compact-open topology on $H(\Omega)$. Then the S_n are continuous. From Mergelian's theorem it follows that, for a suitable sequence of polynomials (h_j), the sequence $V_j := \{\psi \in A(K) : \|\psi - h_j\| < 1/j\}$ forms a countable base of the topology of $A(K)$. According to the universality criterion we have to guarantee that for all $j \in \mathbb{N}$, all compact sets $L \subset \Omega$ with L^c connected and all $g \in H(\Omega)$ there are arbitrary large $n \in \mathbb{N}$ with $S_n(\{\varphi \in H(\Omega) : \|\varphi - g\|_L < 1/j\}) \cap V_j \neq \emptyset$.

Let $E := L \cup K$. Since $f : E \to \mathbb{C}$ with $f_{|L} := g$ and $f_{|K} := h_j$ is in $H(E)$, and since E has connected complement, Runge's theorem offers a polynomial p with $\|p - g\|_L < 1/j$ and $p \in V_j$. Noting that $S_n(p) = p$ for all $n \geq \deg(p)$ we are done. □

We remark that the proof equally works for arbitrary sequences $T_n : H(\Omega) \to A(K)$ of continuous projections to the set of polynomials of degree $\leq n$, as e. g. Faber sections or sequences of suitable interpolating polynomials.

By using variants of Runge's theorem it is possible to impose further conditions on universal Taylor series. We consider lacunary series: For $\Lambda \subset \mathbb{N}_0$ let

$$H_\Lambda(\Omega) := \{f \in H(\Omega) : f^{(\nu)}(0) = 0\,(\nu \notin \Lambda)\},\ U_{K,\Lambda}(\Omega) := H_\Lambda(\Omega) \cap U_K(\Omega)$$

and let $P_\Lambda(E)$ be the closed linear span of the monomials $z \mapsto z^\nu$ ($\nu \in \Lambda$) in $A(E)$. If $0 \in E^0$, then $f \in P_\Lambda(E)$ implies $f^{(\nu)}(0) = 0$ for all $\nu \notin \Lambda$. Conversely, we have the following Runge type result (see [LMM]):

Suppose that E is compact with E^c connected, $0 \in E^0$ and such that the component of E containing 0 is starlike with respect to 0. If Λ has upper density $\bar{d}(\Lambda) = 1$, then every $f \in H(E)$ with $f^{(\nu)}(0) = 0$ for all $\nu \notin \Lambda$ is in $P_\Lambda(E)$.

Since no extra conditions are imposed on components of E not containing 0, the same proof as for Proposition 1 gives (cf. [Sch])

Proposition 2. Let Ω be starlike with respect to 0 and suppose that $\bar{d}(\Lambda) = 1$. Then $U_{K,\Lambda}(\Omega)$ is residual in $H_\Lambda(\Omega)$ for all compact sets $K \subset \Omega^c$ with K^c connected.

Remarks 3. By topological arguments and a further application of Mergelian's theorem (only on the "K-side") it can be shown (see [Ne]) that for Ω simply connected there is a sequence (K_j) in Ω^c with K_j^c connected and

$$U(\Omega) := \bigcap\{U_K(\Omega) : K \subset \Omega^c,\ K^c \text{ connected}\} = \bigcap_{j\in\mathbb{N}} U_{K_j}(\Omega)$$

Therefore, $U(\Omega)$ is still residual in $H(\Omega)$. The same arguments lead to the residuality of $U_\Lambda(\Omega) := \bigcap\{U_{K,\Lambda}(\Omega) : K \subset \Omega^c,\ K^c \text{ connected}\}$ in $H_\Lambda(\Omega)$ for Ω starlike and $\bar{d}(\Lambda) = 1$.

From a result in [MM], it follows that the condition $\bar{d}(\Lambda) = 1$ turns out to be sharp. More precisely, given $d < 1$, there is a compact sector $S_d \subset \mathbb{D}^c$ such that for all K with $K^0 \supset S_d$ and all $f \in H_\Lambda(\mathbb{D})$ with $\bar{d}(\Lambda) \leq d$ the condition $0 \in \omega(f, K)$ implies $f \equiv 0$. Therefore, in particular, for Λ with $\bar{d}(\Lambda) < 1$ always $U_\Lambda(\mathbb{D}) = \emptyset$.

That polynomial approximation has impact on the existence of universal Taylor series is well known. On the other hand, universality properties of (S_n) lead to certain overconvergence and thus to extra approximation of f in $\Omega \setminus \mathbb{D}$. In [MY], the following result on reduced growth of sequences of polynomials is found:

Lemma 4. Let $B \subset \mathbb{C}$ be closed and non-thin at ∞. If (p_m) is a sequence of polynomials with

$$\limsup_{m\to\infty} |p_m(z)|^{1/d_m} \leq 1 \qquad (z \in B),$$

where $\deg(p_m) \leq d_m$, then for all compact $E \subset \mathbb{C}$

$$\limsup_{m\to\infty} \|P_m\|_E^{1/d_m} \leq 1.$$

If $f \in H(\mathbb{D})$ is so that for some B as in the Lemma

$$\limsup_{m\to\infty} |S_{n_m}(z)|^{1/n_m} \leq 1 \qquad (z \in B),$$

an application of the two-constants-theorem (as in the proof of the classical Ostrowski overconvergence theorem, see e. g. [Hi], Theorem 16.7.2) shows that f

has a maximal domain of existence Ω_f, that Ω_f is simply connected, and that $S_{n_m}f \to f$ in $H(\Omega_f)$ (thus $f \in \omega(f, L)$ for all $L \subset \Omega_f$ compact). In particular, this is satisfied if Ω is simply connected and $f \in U(\Omega)$ (the complement of a simply connected domain is non-thin at ∞). In this case, also $\Omega = \Omega_f$, that is, all $f \in U(\Omega)$ have Ω as natural boundary (cf. [MVY]).

From results in [Ge] it follows that some overconvergence of (S_n) already occurs under weaker conditions: If $f \in H(\Omega)$ for $\Omega \neq \mathbb{D}$ and if $\omega(f, E) \neq \emptyset$ for some compact set $E \subset \mathbb{C}$ with $\text{cap}(E) > 1$, then there is a domain $\Omega_E \not\supseteq \mathbb{D}$ with $S_{n_m}f \to f$ in $H(\Omega_E)$ for some (n_m).

On the other hand, Taylor sections of functions in $H(\mathbb{C} \setminus \{1\})$ cannot exhibit overconvergence (or, equivalently, cannot have Hadamard-Ostrowski gaps). This follows from the classical Wigert theorem in connection with [Po], Theorem V. However, in [Me], it is shown that $U_K(\mathbb{C} \setminus \{1\})$ is residual in $H(\mathbb{C} \setminus \{1\})$, for all $K \subset \mathbb{D}^c$ finite.

In view of the above results a reasonable guess is that K finite might be replaced by $\text{cap}(K) = 0$.

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Densities connected with overconvergence

Wolfgang Luh
(joint work with Tatevik Gharibyan)

Suppose that $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ is a power series with radius of convergence 1 and denote by $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ its partial sums. The sequence $\{s_n(z)\}$ converges compactly in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and diverges in every point $z \in \overline{\mathbb{D}}^c$. However, exactly a century ago Porter [15] constructed such a power series with the property that a certain subsequence $\{s_{n_k}(z)\}$ of its partial sums converges compactly in a domain which contains points of $\overline{\mathbb{D}}^c$. This phenomenon is called overconvergence.

Definition 1. A power series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \qquad \text{with} \qquad \overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} = 1$$

is called overconvergent along the sequence $\{n_k\} \subset \mathbb{N}$ if there exists a domain $G \not\subset \mathbb{D}$ such that the subsequence $\{s_{n_k}(z)\}$ of its partial sums converges compactly on G.

Other examples of overconvergent power series were given by Jentzsch [5], and in the early twentieth century Ostrowski [10] - [14] made a thorough investigation of overconvergence phenomena. One of his main results consists in the correlation between overconvergence and the existence of certain gaps in the sequence of coefficients; nowadays they are designated as Hadamard-Ostrowski-gaps (H.O.-gaps for short) and defined in the following way.

Definition 2. Let be given a power series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \qquad \text{with} \qquad \overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} = 1. \tag{1.1}$$

(1) We say that (1.1) has a sequence $\{p_k, q_k\}$ of H.O.-gaps if the following holds:
   a) p_k, q_k are natural numbers with $p_1 < p_2 < \ldots$,
   b) $\underline{\lim}_{k\to\infty} \frac{q_k}{p_k} > 1$,
   c) for $I := \bigcup_{k=1}^{\infty} \{p_k + 1, p_k + 2, \ldots, q_k - 1\}$ we have $\overline{\lim}_{\substack{\nu\to\infty \\ \nu\in I}} |a_\nu|^{1/\nu} < 1$.

(2) In the case that $\frac{q_k}{p_k} \to \infty$ and $\lim\limits_{\substack{\nu\to\infty \\ \nu\in I}} |a_\nu|^{1/\nu} = 0$ we say that the power series has Ostrowski-gaps.
(3) By $\mathscr{O}(f)$ we denote the set of all H.O.-gaps which the series (1.1) has and we set $\mathscr{O}(f) = \emptyset$ if there are no H.O.-gaps.

The most essential results of Ostrowski's theory on overconvergence are the following.

Theorem O₁. *Suppose that the power series* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *with radius of convergence 1 possesses H.O.-gaps* $\{p_k, q_k\}$. *Let the function* f *be holomorphic at* $z_0, |z_0| = 1$. *Then there exists a neighborhood* $U(z_0)$ *of* z_0 *such that* $\{s_{n_k}(z)\}$ *where*

$$s_{n_k}(z) = \sum_{\nu=0}^{n_k} a_\nu z^\nu \quad \text{with } p_k \le n_k < q_k$$

converges compactly on $U(z_0)$.

If the power series has Ostrowski-gaps $\{p_k, q_k\}$ *then the domain* G *of holomorphy of* f *is simply connected and* $\{s_{n_k}(z)\}$ *converges compactly on* G.

Theorem O₂. *Suppose that* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *has radius of convergence 1 and that it has an overconvergent subsequence* $\{s_{n_k}(z)\}$ *of its partial sums. Then there exist constants* ϑ, θ *with* $0 < \vartheta < 1 < \theta$ *such that* $\{[\vartheta n_k], [\theta n_k]\}$ *is a sequence of H.O.-gaps.*

Sharp estimates for the constant ϑ and θ were given by Müller [7] and Gehlen [2].

In recent years the investigation of overconvergence has had a revival since its connection with universal properties of Taylor series has been detected. For details we refer to the excellent survey of Große-Erdmann [4], where also a synopsis of the relevant literature is given.

In this note we consider overconvergent power series and investigate questions related to densities. Among others we deal with the following problems:

- What are the "weakest" conditions on $\{n_k\}$ to generate overconvergence.
- Which information – in terms of density-properties – is available for the intervals of gaps $\{p_k, q_k\}$ and the intervals of "non-gaps".

2. Sequences $\{\mathbf{n_k}\}$ generating overconvergence

In this talk we discuss characteristics of sequences $\{n_k\}$ with the property that $\{s_{n_k}(z)\}$ is an overconvergent sequence of partial sums of a power series. Our first result is the following.

Theorem 1. *Let be prescribed a sequence* $\{n_k\} \subset \mathbb{N}$ *with*

$$\underline{\lim}_{k\to\infty} \frac{n_{k+1}}{n_k} > 1.$$

Then there exists a power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *with radius of convergence 1 such that the sequence* $\{s_{n_k}(z)\}$ *with*

$$s_{n_k}(z) = \sum_{\nu=0}^{n_k} a_\nu z^\nu$$

is compactly convergent in a domain $G \supset \mathbb{D}, G \neq \mathbb{D}$.

Remark. The construction of this overconvergent power series follows along lines similar to the classical examples of Porter and Jentzsch. However, here the sequence $\{n_k\}$ can be prescribed with $\underline{\lim}_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$ but arbitrarily close to 1.

By Ostrowski's Theorem O_2 this density property on $\{n_k\}$ is sharp in the sense that it cannot be replaced by the condition $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = 1$, in order that overconvergence of a power series along the sequence $\{n_k\}$ be possible.

For an $R > 0$ we now consider the disk $\mathbb{D}_R := \{z : |z| < R\}$. Let γ be an arc of $\partial\mathbb{D}_R$ and suppose that Γ is the complementary arc of $\partial\mathbb{D}_R$. Then there exists a uniquely defined bounded harmonic function on $\mathbb{D}_R$ with boundary values 0 on γ and 1 on Γ (both without their common end points). This function will be denoted by $\omega(z, \Gamma, R)$ and is called the harmonic measure of Γ with respect to a point $z \in \mathbb{D}_R$.

Theorem 2. *Suppose that* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *is a power series with radius of convergence 1. Let* $\{s_{n_k}(z)\}$ *be a sequence of partial sums which converges uniformly on a closed subarc* γ *of* $\partial\mathbb{D}_R$*, where* $R > 1$*. Then we have*

$$\overline{\lim_{k\to\infty}} \frac{n_{k+1}}{n_k} \geq \frac{1}{\max\limits_{|z|=1} \omega(z, \Gamma, R)} =: M(\gamma).$$

Obviously $M(\gamma)$ is a "geometric" constant, depending only on γ, it increases, when the length of γ increases.

As an application of the preceding theorem one easily obtains

Theorem 3. *Let* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *be a power series with radius of convergence 1 and suppose that the sequence* $\{s_{n_k}(z)\}$ *of partial sums is compactly convergent in a domain containing for an* $\alpha \in (0, \pi)$ *the sector*

$$S_\alpha := \{z \in \mathbb{C} : -\alpha \leq \arg z \leq +\alpha\}.$$

Then we have

$$\overline{\lim_{k\to\infty}} \frac{n_{k+1}}{n_k} \geq \frac{\pi}{\pi-\alpha}.$$

3. The structure of H.O.-gaps

Suppose that $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ has radius of convergence 1 and that $\mathscr{O}(f) \neq \emptyset$, so that there is a sequence $\{p_k, q_k\}$ of H.O.-gaps. In addition let us assume that f has an analytic extension to a domain $G \supset \mathbb{D}, G \neq \mathbb{D}$, which implies that the power series under consideration is overconvergent. Then, by combining Theorem O_1 and Theorem O_2 we find constants $0 < \vartheta < 1 < \theta$ so that $\{[\vartheta p_k], [\theta q_k]\}$ is also a sequence of H.O.-gaps. It may happen that $[\vartheta p_{k+1}] < [\theta q_k]$ so that the index-range $\{p_k + 1, \ldots, q_k - 1\}$ of "small-coefficients" may be extended to $\{[\vartheta p_k] + 1, \ldots, [\theta q_{k+1}] - 1\}$. Of course, for infinitely many k we must have $[\vartheta q_k] \leq [\theta p_{k+1}]$, otherwise the power series would not have radius of convergence 1.

These observations motivate our introducing the following notion.

Definition 3. Suppose that $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ has radius of convergence 1.

(1) If $\mathscr{O}(f) \neq \emptyset$ then

$$\Omega(f) := \sup_{\{p_k,q_k\}\in\mathscr{O}(f)} \left\{ \overline{\lim_{k\to\infty}} \frac{q_k}{p_k} \right\}$$

is called the order of $\mathscr{O}(f)$.

(2) If $\mathscr{O}(f) = \emptyset$ then we define $\Omega(f) := 1$.

It is clear that $1 \leq \Omega(f) \leq \infty$ and that $\Omega(f) = 1$ if and only if the power series under consideration does not have H.O.-gaps.

Example. The power series $f(z) = \sum_{k=0}^{\infty} z^{2^k}$ has H.O.-gaps $p_k = 2^k$ and $q_k = 2^{k+1}$; we have

$$\lim_{k\to\infty} \frac{q_k}{p_k} = 2 = \Omega(f)$$

and for $g(z) = \sum_{k=1}^{\infty} z^{k!}$ we get $p'_k = k!$ and $q'_k = (k+1)!$ and therefore

$$\lim_{k\to\infty} \frac{q'_k}{p'_k} = \infty = \Omega(g).$$

For the geometric series $h(z) = \sum_{\nu=0}^{\infty} z^\nu$ we obtain $\Omega(h) = 1$.

Theorem 4. *Suppose that the power series* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *has radius of convergence 1 and that* $1 < \Omega(f) < \infty$. *If* f *is analytically extendable, then any sequence* $\{p_k, q_k\} \in \mathscr{O}(f)$ *satisfies*

$$\overline{\lim_{k\to\infty}} \frac{q_k}{p_k} < \Omega(f).$$

Example. The function f in the above example shows that

$$\overline{\lim_{k\to\infty}} \frac{q_k}{p_k} = \lim_{k\to\infty} \frac{q_k}{p_k} = \Omega(f)$$

is possible under the assumption that f has no analytic extension. It is also clear that the finiteness of the order is a necessary condition in Theorem 4.

4. Density properties of H.O.-gaps and non-gaps

We first prove the following result which is an easy consequence of Theorem 5.

Theorem 5. *Suppose that* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *has radius of convergence 1 and that* $\{p_k, q_k\} \in \mathscr{O}(f)$. *Define*

$$I := \bigcup_{k=1}^{\infty} \{p_k + 1, p_k + 2, \ldots, q_k - 1\}.$$

If f *is analytically extendable, then*

$$\sum_{\substack{\nu=1 \\ \nu \notin I}}^{\infty} \frac{1}{\nu} = \infty.$$

Remark. If under the same conditions as in Theorem 6

$$\sum_{\substack{\nu=1 \\ \nu \notin I}}^{\infty} \frac{1}{\nu} < \infty,$$

then the function f cannot have an analytic extension.

In order to get information about density properties of H.O.-gaps we introduce the following notions.

Suppose that S is a subset of the natural numbers and let $N_S(n)$ be the number of elements of S in the interval $[1, n]$. Then the upper and lower density of S are defined by

$$\overline{d}(S) := \overline{\lim_{n\to\infty}} \frac{N_S(n)}{n}, \qquad \underline{d}(S) := \underline{\lim_{n\to\infty}} \frac{N_S(n)}{n}.$$

Theorem 6 *Suppose that* $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ *has radius of convergence 1 and* $\{p_k, q_k\} \in \mathscr{O}(f)$. *Define*

$$I := \bigcup_{k=1}^{\infty} \{p_k + 1, p_k + 1, \ldots, q_k - 1\}.$$

If f *is analytically extendable then we obtain*

(1) $\underline{d}(I) < 1$ *in the case that* $1 < \Omega(f) \leq \infty$;
(2) $\overline{d}(I) < 1$ *in the case that* $1 < \Omega(f) < \infty$.

Remark. The second statement of this Theorem does not hold for any power series with $\Omega(f) = \infty$. In this case we find a sequence $\{p_q, q_k\} \in \mathscr{O}(f)$ with $\frac{q_k}{p_k} \to \infty$ for $k \to \infty$ and it follows from (4.1)

$$\begin{aligned} \frac{N(q_k)}{q_k} &= \frac{1}{q_k}\Big\{ \sum_{\nu=1}^{k} (q_\nu - p_\nu) - k \Big\} \geq \\ &\geq \frac{q_k - p_k - k}{q_k} = 1 - \frac{p_k + k}{q_k} \to 1\,, \end{aligned}$$

so that $\overline{d}(I) = 1$.

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Lacunary summability and analytic continuation of power series

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(joint work with Wolfgang Luh and Jürgen Müller)

1. Introduction

1.1. **Preliminaries.** Let $p = \{p_\nu\}$ be a fixed sequence of complex numbers and suppose that there exists a subsequence $M = \{m_n\}$ of the natural numbers such that

$$P_n := \sum_{\nu=0}^{m_n} p_\nu \neq 0 \qquad \text{for all } n.$$

The matrix $A = [\alpha_{n\nu}]$ with the entries

$$\alpha_{n\nu} = \frac{p_\nu}{P_n} \qquad \text{if} \quad 0 \leq \nu \leq m_n\,,$$
$$\alpha_{n\nu} = 0 \qquad \text{if} \quad \nu > m_n\,,$$

generates a summability method (R, p, M) of so-called weighted mean type. The (R, p, M) methods were introduced by Faulstich [9] and may be considered as refinements of the well known Riesz methods. The (R, p, M) transforms of a sequence $\{z_n\}$ are given by

$$\sigma_n := \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu z_\nu\,. \tag{1.1}$$

Such a method is called regular (in terms of summability theory) if for all convergent sequences $\{z_n\}$ also $\{\sigma_n\}$ converges and $\lim\limits_{n\to\infty} \sigma_n = \lim\limits_{n\to\infty} z_n$ holds. The Toeplitz-Silverman theorem (see for instance [22]) shows that a method (R, p, M) is regular if and only if the following conditions hold

$$\lim_{n\to\infty} P_n = \infty\,, \tag{1.2}$$

$$\sup_n \sum_{\nu=0}^{m_n} \left|\frac{p_\nu}{P_n}\right| < \infty\,. \tag{1.3}$$

A method (R, p, M) is called ps-regular ("regular for power series") if for all power series $f(z) = \sum\limits_{\nu=0}^{\infty} f_\nu z^\nu$ with radius of convergence r, $0 < r \leq \infty$ the sequence of its (R, p, M) transforms

$$\sigma_n^f(z) := \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu\, s_\nu^f(z)\,, \quad \text{where} \quad s_\nu^f(z) = \sum_{\mu=0}^{\nu} f_\mu z^\mu$$

converges compactly to $f(z)$ in the circle of convergence. (For the general concept and characteristic properties of the ps-regularity of a summability method we refer to [16].) The (R, p, M) method is ps-regular if and only if (1.2) and instead of (1.3)

$$\sup_n \sum_{\nu=0}^{m_n} \left|\frac{p_\nu}{P_n}\right| \rho^\nu < \infty \qquad \text{for all } \rho \in (0,1) \tag{1.3'}$$

are satisfied.

In the present talk we deal with (R, p, M) methods which are generated by "lacunary" sequences $p = \{p_\nu\}$. This means that there exists a subsequence $S \subset \mathbb{N}_0$ such that $p_\nu = 0$ for all $\nu \notin S$, where we will suppose that S satisfies certain density properties. In a reasonable manner we call such a method an S-lacunary (R, p, M) method. The corresponding matrix $A = [\alpha_{n\nu}]$ has the significant property that any column $[\alpha_{n\nu}]_{n=0}^{\infty}$ is the null-vector for each $\nu \notin S$ and therefore the transforms (1.1) are independent of the subsequence $\{z_\nu\}_{\nu\notin S}$ of $\{z_n\}$.

1.2. **Some notation.** We here consider subsequences S of $\mathbb{N}_0$, with upper density $\overline{d}(S) = 1$, where $\overline{d}(S)$ is defined by

$$\overline{d}(S) := \overline{\lim_{t\to\infty}} \frac{N_S(t)}{t},$$

and $N_S(t)$ denotes the number of elements of S in the interval $[0, t]$.

The symbol $\mathcal{M}$ is used for the family of all compact sets $K \subset \mathbb{C}$ which have connected complement K^c. For $K \in \mathcal{M}$ the notation $A(K)$ stands for the set of all functions which are continuous on K and holomorphic in its interior K^0.
We denote, as usual, by $\mathbb{D} := \{z : |z| < 1\}$ the unit disk in the complex plane.
If a sequence $\{f_n\}$ of functions converges uniformly to a limit function f on a set $B \subset \mathbb{C}$ we write

$$f_n(z) \underset{B}{\rightrightarrows} f(z).$$

If $\mathcal{O} \subset \mathbb{C}$ is an open set then $\{f_n\}$ is said to be compactly convergent to f if $f_n(z) \underset{B}{\rightrightarrows} f(z)$ for all compact sets $B \subset \mathcal{O}$; we abbreviate this by

$$f_n(z) \underset{\mathcal{O}}{\Rrightarrow} f(z).$$

For a fixed parameter $\alpha \in \mathbb{R}$ we consider the logarithmic α-spiral

$$L_\alpha := \{z : z = e^{(1+i\alpha)t}, t \in \mathbb{R}\} \cup \{0\}.$$

A set $M \subset \mathbb{C}$ with $0 \in M$ is called an α-starlike set (with respect to the origin) if

$$M \cdot (L_\alpha \cap \overline{\mathbb{D}}) = M.$$

If $\alpha = 0$ then M is starlike in the traditional sense.
If in particular $M = G$ is a domain which is α-starlike with respect to the origin

then G is automatically simply connected. It has been shown by Arakelian [1], [2] that α-starlike domains play an essential role in the theory of the analytic continuation of functions which are given by power series (see also [4]).

1.3. **Statement of the main result.** Our main result is the following Theorem 1, which is actually a result on universal summability and approximation properties by (R, p, M) transforms of the geometric sequence $\{z^\nu\}$. Admittedly it seems to be of a special character at first sight. But, it is well known that the behavior of the geometric sequence has an important effect on the summation and analytic continuation of general power series. For instance we refer to the theorem of Okada [21] and its improvements by Gawronski-Trautner [12] and Große-Erdmann [14]; see also Lemma 2.3 in Arakelian's article [1] and sections 2.2 and 3 of [4]. Relations with the Hadamard multiplication theorem have been investigated in [20].
In addition we shall show in section 2 of the talk that Theorem 1 has several interesting consequences, for instance: with respect to overconvergence; to the analytic continuation of function elements; as well as to universal properties of trigonometric series in the sense of Menšov.

Theorem 1. Suppose that there are given:

- a subsequence S of $\mathbb{N}_0$ with $\overline{d}(S) = 1$,
- a domain G which is α-starlike with respect to the origin and satisfies $\mathbb{D} \subset G, 1 \notin G$,
- a sequence $\{P_n\}$ in $\mathbb{C}$ with $0 \neq P_n \to \infty$.

Then there exists a sequence $\{p_\nu\}$ in $\mathbb{C}$ with $p_\nu = 0$ for all $\nu \notin S$ and a subsequence $\{m_n\}$ of $\mathbb{N}_0$ with

$$P_n = \sum_{\nu=0}^{m_n} p_\nu \qquad \text{for all } n \in \mathbb{N}_0 ,$$

such that the following properties hold.

(1) There exists a function P which is holomorphic exactly on G with the power series expansion

$$P(z) = \sum_{\nu=0}^{\infty} p_\nu z^\nu$$

(of radius of convergence $R = 1$) and the properties

$$s_{m_n}(z) := \sum_{\nu=0}^{m_n} p_\nu z^\nu \underset{G}{\Rrightarrow} P(z) ,$$

$$\tau_n(z) := \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu z^\nu \underset{G}{\Rrightarrow} 0 .$$

(2) For every $K \in \mathcal{M}$ with $K \subset G^c \backslash \{1\}$ and every function $f \in A(K)$ there exists a subsequence $\{n_s\}$ of $\mathbb{N}$ with

$$\tau_{n_s}(z) \underset{K}{\Longrightarrow} f(z) .$$

The proof of Theorem 1 depends on a lacunary version of Runge's approximation theorem and exhausting properties for G and $G^c\backslash\{1\}$ respectively.

2. Consequences

2.1. **Overconvergence.** Let G and $\{p_\nu\}$ be the same as in Theorem 1. Then it was shown that there exists a function P which is holomorphic exactly on G (i.e. ∂G is the natural boundary for P) with a power series representation

$$P(z) = \sum_{\nu=0}^{\infty} p_\nu z^\nu \quad \text{with } p_\nu = 0 \text{ for all } \nu \notin S, \quad \overline{\lim_{\nu\to\infty}} |p_\nu|^{1/\nu} = 1$$

and the property that the sequence $\{s_{m_n}(z)\}$ of its partial sums converges to $P(z)$ compactly on G. It follows that in addition this sequence diverges for all $z_0 \in G^c$. Indeed if $z_0 = 1$ then $s_{m_n}(1) = \sum_{\nu=0}^{m_n} p_\nu = P_n \to \infty$, and if $z_0 \in G^c\backslash\{1\}$ then (by the universal properties of $\{\tau_n(z)\}$) there exists a sequence $\{n_j\} \subset \mathbb{N}$ with

$$\tau_{n_j}(z_0) = \frac{1}{P_{n_j}} \sum_{\nu=0}^{m_{n_j}} p_\nu z^\nu = \frac{1}{P_{n_j}} s_{m_{n_j}}(z_0) \to 2 \qquad (s \to \infty)\,,$$

and we can choose j_0 with

$$|s_{m_{n_j}}(z_0)| > |P_{n_j}| \qquad \text{for all } j > j_0\,,$$

which proves the assertion.

2.2. **Lacunary summability of power series.** 1. Suppose that G and $\{p_\nu\}$ are again the same as in Theorem 1. Then we have in particular

$$\tau_n(z) = \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu z^\nu \underset{\mathbb{D}}{\Rrightarrow} 0\,.$$

Let be given any $\varrho \in (0,1)$; we choose $\varrho_0 = \frac{1+\varrho}{2}$ and obtain with a constant $C_1(\varrho)$ for $0 \le \nu \le m_n$

$$\Big|\frac{p_\nu}{P_n}\Big| = \Big|\frac{1}{2\pi i} \int\limits_{|z|=\varrho_0} \frac{\tau_n(z)}{z^{\nu+1}}\,dz\Big| \le \frac{1}{\varrho_0^\nu} \cdot \max_{|z|=\varrho_0} |\tau_n(z)| = \frac{C_1(\varrho)}{\varrho_0^\nu}\,,$$

which implies with some constant $C_2(\varrho)$

$$\sum_{\nu=0}^{m_n} \Big|\frac{p_\nu}{P_n}\Big| \varrho^\nu \le \sum_{\nu=0}^{m_n} \Big|\frac{p_\nu}{P_n}\Big| \varrho_0^\nu \Big(\frac{\varrho}{\varrho_0}\Big)^\nu \le C_1(\varrho) \sum_{\nu=0}^{m_n} \Big(\frac{\varrho}{\varrho_0}\Big)^\nu \le C_2(\varrho)$$

for all $n \in \mathbb{N}_0$. Since on the other hand $P_n \to \infty$ the (R,p,M) method under consideration satisfies (1.2) and (1.3') and is therefore ps-regular. This means (cf. section 1.1) that for all power series $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ with radius of convergence

$r, 0 < r \leq \infty$ the sequence of its (R, p, M) transforms

$$\sigma_n^f(z) := \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu s_\nu^f(z), \qquad \text{where } s_\nu^f(z) = \sum_{\mu=0}^{\nu} f_\mu z^\mu$$

converges to $f(z)$ compactly in the circle of convergence $\{z : |z| < r\}$.

However, the (R, p, M) method under consideration cannot be regular in the case that there exists a $z_0 \in G^c \backslash \{1\}$ with $|z_0| = 1$. Otherwise, by (1.3) there would be a constant M with

$$\frac{1}{|P_n|} \sum_{\nu=0}^{m_n} |p_\nu| \leq M \qquad \text{for all } n \in \mathbb{N}_0. \tag{2.1}$$

On the other hand there exists (by the universal properties according to Theorem 1) a sequence $\{n_s\}$ with

$$\frac{1}{P_{n_s}} \sum_{\nu=0}^{m_{n_s}} p_\nu z_0^\nu \to M + 1\,,$$

which contradicts (2.1).

2. We consider the geometric series $f_0(z) = \sum\limits_{\nu=0}^{\infty} z^\nu$ with partial sums $s_n(z) = \frac{1-z^{n+1}}{1-z}$ for $z \neq 1$. The application of the (R, p, M) method from Theorem 1 gives

$$\sigma_n^{f_0}(z) = \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu \frac{1 - z^{\nu+1}}{1 - z} = \frac{1}{1-z} - \frac{z}{1-z}\, \tau_n(z)$$

and hence $\sigma_n^{f_0}(z) \underset{G}{\Rrightarrow} f_0(z) = \frac{1}{1-z}$.

3. Let now be given any power series $f(z) = \sum\limits_{\nu=0}^{\infty} f_\nu z^\nu$ with radius of convergence $R, 0 < R \leq \infty$.
The α-Mittag-Leffler star $A_\alpha[f]$ of f is the union of all α-starlike domains to which f is analytically continuable. Then $A_\alpha[f]$ is also α-starlike and f is holomorphic on $A_\alpha[f]$.

If the domain G is α-starlike and if as in 2

$$\sigma_n^{f_0}(z) \underset{G}{\Rrightarrow} f_0(z) = \frac{1}{1-z}\,,$$

then, according to generalized versions of Okada's theorem (see [21], [14] or [20]), the sequence of (R, p, M) transforms

$$\sigma_n^f(z) := \frac{1}{P_n} \sum_{\nu=0}^{m_n} p_\nu s_\nu(z), \quad \text{where } s_\nu(z) = \sum_{\mu=0}^{\nu} f_\mu z^\mu$$

converges to $f(z)$ compactly in the domain

$$G * A_\alpha[f] := (G^c \cdot (A_\alpha[f])^c)^c\,,$$

which turns out to be also α-starlike.

From the point of view of the theory of analytic continuation the most effective methods of summation are those which sum a given power series compactly in its Mittag-Leffler star. A number of such methods is available in the literature of which the methods of Lindelöf, Mittag-Leffler, and Le Roy are best known.

If we choose in Theorem 1 in particular the α-starlike domain $G = \mathbb{C}\backslash(L_\alpha \cap \mathbb{D}^c)$, then we obtain by the considerations above the existence of a "universal" lacunary (R, p, M) method (which is ps-regular) and which is effective for the analytic continuation of all power series in their α-Mittag-Leffler-star. In comparison with the classical methods mentioned above, this method has an elementary structure. We have

Theorem 2. Suppose that there are given:

- a subsequence S of $\mathbb{N}_0$ with $\overline{d}(S) = 1$,
- a sequence $\{P_n\}$ in $\mathbb{C}$ with $0 \neq P_n \to \infty$.

Then there exists a sequence $\{p_\nu\}$ in $\mathbb{C}$ with $p_\nu = 0$ for all $\nu \neq S$ and a subsequence $\{m_n\}$ of $\mathbb{N}_0$ with $P_n = \sum\limits_{\nu=0}^{m_n} p_\nu$ satisfying the following property.

For all power series $f(z) = \sum\limits_{\nu=0}^{\infty} f_\nu z^\nu$ with radius of convergence r, where $0 < r \leq \infty$ the sequence of (R, p, M)-transforms

$$\sigma_n^f(z) \;=\; \frac{1}{P_n}\sum_{\nu=0}^{m_n} p_\nu s_\nu^f(z)\,, \quad \text{where } s_\nu^f(z) = \sum_{\mu=0}^{\nu} f_\mu z^\mu$$

converges to $f(z)$ compactly in the α-Mittag-Leffler star $A_\alpha[f]$.

This follows immediately from the considerations above and the identity

$$G * A_\alpha[f] = A_\alpha[f]\,.$$

2.3. **Universal lacunary summability of trigonometric series.** In 1945 Menšov [17], [18] proved the existence of a universal (real) trigonometric series

$$\sum_{\nu=0}^{\infty}\{a_\nu \cos \nu t + b_\nu \sin \nu t\}$$

with the property that for every (Lebesgue-) measurable real valued function φ on $[0, 2\pi]$ there exists a sequence $\{n_k\}$ of natural numbers such that the corresponding sequence of partial sums

$$s_{n_k}(t) \;=\; \sum_{\nu=0}^{n_k}\{a_\nu \cos \nu t + b_\nu \sin \nu t\}$$

converges to $\varphi(t)$ almost everywhere on $[0,2\pi]$. For a proof we refer to [5].

The trigonometric series $\sum_{\nu=0}^{\infty}\{\cos\nu t + i\sin\nu t\} = \sum_{\nu=0}^{\infty} e^{i\nu t}$ is obviously not universal in the sense of Menšov. However, by applying Theorem 1, we can show that its (R,p,M) transforms satisfy universal properties with respect to measurable functions. In addition the generating sequence $p=\{p_\nu\}$ may be chosen lacunary.

We shall prove (as a simple application of Theorem 1) the following result; by $\mu(A)$ we denote the (Lebesgue-) measure of a measurable set $A\subset\mathbb{R}$.

Theorem 3. Suppose that there are given

- a subsequence S of $\mathbb{N}_0$ with $\overline{d}(S)=1$,
- a sequence $\{P_n\}$ in $\mathbb{C}$ with $0\neq P_n\to\infty$.

Then there exists a sequence $\{p_\nu\}$ in $\mathbb{C}$ with $p_\nu=0$ for all $\nu\notin S$ and a subsequence $\{m_n\}$ of $\mathbb{N}_0$ with $P_n=\sum_{\nu=0}^{m_n}p_\nu$ satisfying the following property.

Let be given two real valued measurable functions φ and ψ on $[0,2\pi]$.

(1) There exists a subsequence $\{r_k\}$ of $\mathbb{N}_0$ such that

$$\left.\begin{aligned}\mathrm{Re}\Big\{\frac{1}{P_{r_k}}\sum_{\nu=0}^{m_{r_k}}p_\nu e^{i\nu t}\Big\} &\longrightarrow \varphi(t)\\ \mathrm{Im}\Big\{\frac{1}{P_{r_k}}\sum_{\nu=0}^{m_{r_k}}p_\nu e^{i\nu t}\Big\} &\longrightarrow \psi(t)\end{aligned}\right\}\quad\text{almost everywhere on } [0,2\pi].$$

(2) There exists a subsequence $\{t_k\}$ of $\mathbb{N}_0$ such that

$$\left.\begin{aligned}\mathrm{Re}\Big\{\frac{1}{P_{t_k}}\sum_{\nu=0}^{m_{t_k}}p_\nu \sum_{\mu=0}^{\nu}e^{i\mu t}\Big\} &\longrightarrow \varphi(t)\\ \mathrm{Im}\Big\{\frac{1}{P_{t_k}}\sum_{\nu=0}^{m_{t_k}}p_\nu \sum_{\mu=0}^{\nu}e^{i\mu t}\Big\} &\longrightarrow \psi(t)\end{aligned}\right\}\quad\text{almost everywhere on } [0,2\pi].$$

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Universal Faber series

Vagia Vlachou

Let $\Gamma \subset \mathbb{C}$ be a compact and connected set with connected complement, containing more than one point (for simplicity, we shall use the term Faber-expansion set for a set $\Gamma \subset \mathbb{C}$, having this property) and let ϕ_Γ be the Riemann-mapping which maps the unit disk $\mathbb{D}$ onto $\hat{\mathbb{C}} \setminus \Gamma$, with $\phi_\Gamma(0) = \infty$ and $\rho_\Gamma = \lim_{z \to 0} z\phi_\Gamma(z) > 0$. We consider, in addition, the conformal mapping $\psi_\Gamma : \{w \in \mathbb{C} : |w| > \rho_\Gamma\} \to \Gamma^c$, defined by $\psi_\Gamma(w) = \phi_\Gamma(\frac{\rho_\Gamma}{w})$, which is of the form $\psi_\Gamma(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \ldots$. Finally, we denote the inverse mapping of ψ_Γ with φ_Γ. For $n \in \mathbb{N}$, the n^{th} Faber polynomial $F_n(\Gamma, \cdot)$ is the part of the Laurent series of $(\varphi_\Gamma(z))^n$, containing the non-negative powers of z. Each $F_n(\Gamma, \cdot)$ is a polynomial of degree n, and the coefficient of z^n is one. Hence, every polynomial can be expressed in a unique way as a linear combination of the $F_n(\Gamma, \cdot)$'s.

Now, let Ω be a domain such that $\Gamma \subset \Omega$ and let f be a function holomorhic in Ω. Then f admits a unique representation as a Faber series

$$f(z) = \sum_{n=0}^{\infty} c_n(f,\Gamma) F_n(\Gamma, z)$$

which converges uniformly to f on those compact sets of Ω, which are contained in the interior of some level curve of φ_Γ inside $\overline{\Omega}$ (see [8],[7]).

We are interested in functions, which have universal Faber series. To be more specific, let $H(\Omega)$ be the space of holomorphic functions in Ω endowed with the topology of uniform convergence on compacta and let

$$S_N(f,\Gamma)(z) = \sum_{n=0}^{N} c_n(f,\Gamma) F_n(\Gamma, z), \quad N = 1, 2 \ldots.$$

Let, in addition,

$$\mathcal{M}_\Omega = \{K \subset \Omega^c : K \text{ compact set and } K^c \text{ connected set}\}$$

and

$$\mathcal{A}(K) = \{g \in H(K^o) : \ g \text{ is continuous in } K\}.$$

The following definition was introduced in [8]. The same definition was first given in [6], for more restricted domains Ω.

Definition 1. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, \Gamma)$ (i.e it has a universal Faber expansion with respect to Γ), if for every set $K \in \mathcal{M}_\Omega$ and for every function $g \in \mathcal{A}(K)$, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of natural numbers such that

$$\sup_{z\in K} |S_{\lambda_n}(f,\Gamma)(z) - g(z)| \to 0 , \quad \text{as } n \to \infty.$$

In [6], it was proved that if Ω is a Jordan domain with analytic boundary, then the class of functions with universal Faber expansion is a G_δ and dense subset of $H(\Omega)$ (in this case only the most natural Faber expansion was considered namely the one that is valid in the interior of Ω). Five years later, D. Mayenberger and J. Müller (see [8]) presented an impressive result that holds for all simply connected domains. Namely, they proved that if Ω is a simply connected domain[1] and $\Gamma \subset \Omega$ is a Faber-expansion set, then $U(\Omega, \Gamma) = U(\Omega, \zeta)$ for every point $\zeta \in \Omega$, where $U(\Omega, \zeta)$ is the class of universal Taylor series with center ζ, which was known to be a G_δ and dense subset of $H(\Omega)$ (see [13], see also [12] and [10]). Thus, in this case the class is independent of the choice of the compact set of expansion. The next step was made by F. Bayart and V. Nestoridis in [2], who introduced a topology on the set of $\Gamma's$ and proved that the class $U(\Omega, \Gamma)$ (for simply connected domains Ω), not only is independent of Γ, but coincides with the class $U_{Fab}(\Omega)$,

[1] In fact, their proof holds for all domains Ω, whose complement is non-thin at ∞, although this is not mentioned by the authors.

where the convergence is uniform in respect to Γ, when this varies on a compact set[2].

Recently (see [15]), non-simply connected domains Ω have been considered. In particular, if $K \subset \mathbb{C}$ is a compact and connected set having connected complement and if $\Omega = \mathbb{C} \setminus K$, then the class $U(\Omega, \Gamma)$ is a G_δ and dense subset of $H(\Omega)$, for every choice of Faber-expansion set $\Gamma \subset \Omega$.

We work on the same type of non-simply connected domains as above; we set

$$Y = \{\Gamma \subset \Omega : \Gamma \text{ is a Faber-expansion set}\}$$

and our main result is:

Theorem 2. The class $\bigcap_{\Gamma \in Y} U(\Omega, \Gamma)$ is residual in $H(\Omega)$.

Thus, we prove that for this choice of domain Ω, there exist functions with universal Faber series with respect to any choice of Faber-expansion set Γ.

Remark 1: If Ω is a non-simply connected domain, such that Ω^c has an unbounded component (or more generally Ω^c is non-thin at ∞), then all the classes of functions we mentioned are empty (see [11] and [8], see also [10]).
Remark 2: If Ω is a non-simply connected domain with complement thin at ∞, then known results from the theory of universal Taylor series yield that the class probably depends on the choice of Γ (though this has been proved only for a specific type of Ω [9] and not for the type of domains we work with). Moreover, it is impossible to achieve the result of F. Bayart and V. Nestoridis in this case, i.e. to find a function f, with universal Faber expansion with respect to any compact set, such that the approximation is uniform when Γ varies on a compact set. This happens because the class they worked with $U_{Fab}(\Omega)$ is contained in $U(\Omega)$, which in this case is empty (see [10]).
Remark 3: Similar results as ours have been proved for universal Taylor series in [4], [1] and [5]. In the articles [1] and [5], where the problem studied is closer to the present work, the authors had another approach. In particular, they considered a partition of the set of centers and then they tried to solve the problem by choosing appropriate poles. As a consequence, they were obliged to include a finite induction in their proof (see proposition 1 in [1] and lemma 2.2 in [5]), which was complicated. In our proof, on the other hand, we begin with the choice of appropriate poles and then form the partition of Y.
Open Problem: It is known that if Γ is a closed disk, then the Faber-expansion of a function holomorphic in a neighborhood of Γ, coincides with the Taylor expansion of the function around the center of the disk. Thus, $\bigcap_{\Gamma \in Y} U(\Omega, \Gamma) \subset$

[2]This result also holds for all domains whose complement is non-thin at ∞ even though it is not mentioned in the article.

$\bigcap_{\zeta\in\Omega} U(\Omega,\zeta)$. The class $\bigcap_{\zeta\in\Omega} U(\Omega,\zeta)$ was known to be residual in $H(\Omega)$, for this type of doubly connected domains Ω (see [1]). We do not know whether $\bigcap_{\Gamma\in Y} U(\Omega,\Gamma) \neq \bigcap_{\zeta\in\Omega} U(\Omega,\zeta)$, but we believe that it is true.

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Hypercyclic operators failing the Hypercyclicity Criterion

Fréderic Bayart
(joint work with É. Matheron)

A bounded operator T defined on some separable Banach space X is called *hypercyclic* if there exists some vector $x \in X$ such that the orbit of x under T, namely $\{T^n x;\ n \geq 0\}$ is dense in X. It was a long-standing open problem to know whether, for every hypercyclic operator T, its direct sum with itself $T \oplus T$ is hypercyclic.

This problem can be rewritten in an other way. We say that T satisfies the *Hypercyclicity Criterion* if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:

(1) $T^{n_k}(x) \to 0$ for any $x \in \mathcal{D}_1$;
(2) $S_{n_k}(y) \to 0$ for any $y \in \mathcal{D}_2$;
(3) $T^{n_k} S_{n_k}(y) \to y$ for any $y \in \mathcal{D}_2$.

A theorem of Kitai, Gethner and Shapiro asserts that an operator satisfying this criterion is always hypercyclic.

The Hypercyclicity Criterion is a very powerful tool to prove that an operator is hypercyclic. And even, until recently, every hypercyclic operator was hypercyclic... because it satisfies the assumptions of the Hypercyclity Criterion. Thus, a natural question was to know whether every hypercyclic operator satisfies the assumptions of the Hypercyclicity Criterion.

It turns out that the two questions are in fact equivalent (this is a result of Bès and Peris). A negative answer has been given recently by De La Rosa and Read to both questions. They prove that there exist a Banach space X and a hypercyclic operator T on X such that $T \oplus T$ is not hypercyclic.

Although the space constructed by De La Rosa and Read is not extremely complicated, it cannot be identified with some “classical” Banach space. We prove that in fact such an operator does exist on a large class of Banach spaces, including the separable Hilbert space. Our strategy to prove the theorem is the following. We first construct some operator on $\ell^2(\mathbb{N})$ which is hypercyclic *by definition*. This is easy by a purely algebraic construction, the difficulty being to prove that the operator is indeed hypercyclic. We then use a lemma on multiplicative linear forms to show that $T \oplus T$ is not hypercyclic.

We then give several variants of the construction. In particular, we emphasize the role of some arithmetical sequences in the construction, in order to obtain hypercyclic operators failing the Hypercyclicity Criterion with an orbit which is as *frequently* dense as possible. We also show that we may construct such an operator which is also invertible.

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A new method for non-supercyclicity

Fernando León-Saavedra

A bounded linear operator T defined on a separable Banach space X is said to be supercyclic if there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X. It is called weakly supercyclic if the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is weakly dense in X.

In general, to prove that an operator is supercyclic (or weakly supercyclic) is actually easier than proving non-supercyclicity or non weakly supercyclicity. The reason is that the methods have not been sufficiently developed. For instance, the only effective method to prove non-supercyclicity was discovered by Montes and Salas in [11]. It is known as the "Angle Criterion" :

> Let $T \in B(H)$ be a bounded Hilbert space operator and let us suppose that there exists $x_0 \in H$ such that for any $x \in H$
>
> $$\sup_n \frac{|\langle T^n x, x_0\rangle|}{\|T^n x\| \|x_0\|} < 1.$$
>
> Then T is not supercyclic.

This statement doesn't need proof. Roughly speaking, it says that if there exists a cone C such that $x_0 \in C$ and $T^n x \notin C$ for all n then T is not supercyclic. This method was used by Montes-Rodríguez and co-workers to prove non-supercyclicity (see [1], [9]). A similar method can be extended to prove non-weakly supercyclicity (see [14]).

The method that we propose is less computational and the proofs turn out to be more simple. Let us denote by $[T]$ the commutant of T. Suppose that we know that T has many non weakly supercyclic vectors. Our method relies on the size of the commutant of T. If the commutant of T is very big then T has less opportunity to be weakly supercyclic. Suppose that for any supercyclic vector x there exists $R \in [T]$ such that Rx is not weakly supercyclic, then, in such a case T should not be a weakly supercyclic operator.

Let us illustrate the method with some examples. Let $T = V$ be the Volterra operator acting on $L^p[0,1]$ defined by

$$\int_0^x f(s)\, ds,$$

and let us denote by $[V]_p$ its commutant. In most of the cases the commutant of an operator is exactly the weak closure of the algebra $\mathcal{A}_T$ generated by T and the identity (for instance this is the case for the Volterra operator [4]). The biggest commutant of an operator occurs when T is strictly cyclic. Let us recall that an operator $T \in \mathcal{B}(X)$ is said to be strictly cyclic if there exists $x \in X$ such that

$$\{Ax : A \in \mathcal{A}_T\} = X.$$

In this sense, the following related question emerges:

Question: *Is V strictly cyclic on $L^p[0,1]$?*

The answer is no; V is even not strictly cyclic on the Sobolev spaces $W^{k,p}[0,1]$ (see [8]). This result complements the results which appeared in [10]. Let us mention that the problem of characterizing the commutant of V was only solved by Dixmier for $p = 1, \infty$ (see [3]) and by D. Sarason for $p = 2$ (see [12]), and this problem remains open for $p \neq 1, 2, \infty$.

Let us continue with our example. Let us observe that if $g \in L^p[0,1]$ is real valued then g cannot be weakly supercyclic for V, because each element in the orbit $\{V^n g\}$ is a real valued function. Moreover, the same is true for operators which map real valued functions into real valued functions. Now, pick $f \in L^p[0,1]$, and let us denote by $\overline{f}$ its complex conjugate. The convolution operator $V_{\overline{f}}$ with kernel $\overline{f}$ defined by

$$V_{\overline{f}}g(x) = \int_0^x \overline{f}(x-t)g(t)\,dt$$

belongs to the commutant of V. On the other hand, since the convolution commutes $V_{\overline{f}}f(x) \in \mathbb{R}$ for all $x \in [0,1]$ as a consequence we obtain the following result (see [8]):

Theorem. *Let $T \in [V]_p$ and let us suppose that T preserves real valued functions. Then T is not weakly supercyclic on $L^p[0,1]$, $1 \leq p < \infty$.*

Curiously the same trick appears in other situations, for instance, for Cesàro type operators. Boundedness for Cesàro type operators was provided by Hardy, Littlewood and Pólya (see [6] Chapter IX). We refer to the paper [2] by Brown, Halmos and Shields, where the authors, among other results provide a less computational proof of the boundedness and determine their norms and various parts of the spectrum. The interest in the cyclic phenomena of Cesàro type operators was initiated in [7] and it was continued later in [5].

For instance, the infinite Cesàro operator is defined on $L^p[0,1]$ by

$$C_\infty f(x) = \frac{1}{x}\int_0^x f(s)\,ds,$$

for $p = 2$, C_∞ is similar to the identity plus the unweighted bilateral shift B, defined on $L^2(\mathbb{Z})$. The commutant of a bilateral shift was characterized by A. Shields (see [13]). Using the same method we can prove the following result (see [5])

Theorem. *Let B be the unweighted bilateral shift defined on $L^p(\mathbb{Z})$. If p is a polynomial with real coefficients then $p(B)$ is not weakly supercyclic for $1 \leq p \leq 2$.*

Let us mention that the above result is very sharp because the unweighted shift B is weakly supercyclic on $L^p(\mathbb{Z})$ for $p > 2$ (see [15]). As far we know to prove non-weakly supercyclicity of $p(B)$ using the Angle Criterion is quite complicated because it is very difficult to compute the norms $\|(p(B))^n x\|$. Finally, as a consequence we obtain that C_∞ is not weakly supercyclic on $L^2(0,\infty)$.

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Which maps preserve universal functions?

George Costakis

In the present paper we give some partial answers to the following question: Do there exist maps, between spaces of holomorphic functions, which preserve certain notions of universality?

Let us first introduce some standard notation and the relevant definitions. By $\mathbb{D}$ we denote the open unit disk with center 0 in the complex plane. For a holomorphic function f in $\mathbb{D}$, $f \in H(\mathbb{D})$ and $\zeta \in \mathbb{D}$ the symbol $S_n(f,\zeta)(z) = \sum_{k=0}^{n} a_k(\zeta)(z-\zeta)^k$, $n = 0, 1, 2, \ldots$ stands for the partial sums of the Taylor development of f with center ζ.

Definition 1. (Luh [8] and Chui and Parnes [2]) A holomorphic function $f \in H(\mathbb{D})$ is called a universal Taylor series in $\mathbb{D}$ with respect to $\zeta \in \mathbb{D}$ if the following holds. For every compact set $K \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ with connected complement, i.e. K^c connected, and every function $h : K \to \mathbb{C}$ continuous on K and holomorphic in the interior, K^o, of K there exists a subsequence $\{\lambda_n\}$ of positive integers such that

$$\sup_{z\in K} |S_{\lambda_n}(f,\zeta)(z) - h(z)| \to 0, \text{ as } n \to \infty.$$

The class of universal Taylor series (with respect to the center ζ) in the sense of Luh and Chui and Parnes will be denoted by $U_1(\mathbb{D},\zeta)$.

Definition 2. (Nestoridis [11]) If in the previous definition the compact set K is allowed to contain pieces of the boundary, that is $K \subset \mathbb{C} \setminus \mathbb{D}$ then the function f is said to belong to the class $U(\mathbb{D},\zeta)$.

In [10] it is proved that the classes $U_1(\mathbb{D},\zeta)$, $U(\mathbb{D},\zeta)$ do not coincide, therefore the obvious inclusion $U(\mathbb{D},\zeta) \subset U_1(\mathbb{D},\zeta)$ is strict. On the other hand it is well known that both classes are G_δ and dense subsets of $H(\mathbb{D})$, where $H(\mathbb{D})$ is endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$. It is a hard problem to find maps which preserve universal Taylor series. For example, although the derivative f' of a universal Taylor series f in the class $U_1(\mathbb{D},\zeta)$ still belongs to this class, see [3], it is still unknown if the same holds within the class $U(\mathbb{D},\zeta)$. I will next discuss what happens if we multiply a universal Taylor series by a polynomial.

Theorem 3. i) If $f \in U_1(\mathbb{D},0)$ and $a \in \overline{\mathbb{D}}$ then $(z-a)f(z) \in U_1(\mathbb{D},\zeta)$.
ii) If $f \in U(\mathbb{D},0)$ and $a \in \mathbb{D}$ then $(z-a)f(z) \in U(\mathbb{D},0)$.
iii) For any $a \in \mathbb{C}$ there exists a residual subset G of $H(\mathbb{D})$ such that $G \subset U(\mathbb{D},0)$ and for every $f \in G$ we have $(z-a)f(z) \in U(\mathbb{D},0)$.

Let me briefly discuss the main ingredients of the proof of the above theorem. The proofs of items i) and ii) are based on the use of H-O gaps (Hadamard-Ostrowski gaps), see for example [5], on the property $U_1(\mathbb{D},0) = U_1(\mathbb{D},\zeta)$ for every $\zeta \in \mathbb{D}$ and on a simple formula which relates the Taylor development of the function $(z-1)f(z)$ with center 0 with the Taylor development of the function f with center 0. On the other hand the proof of item iii) relies on a Baire category argument and on the fact that the map $T_a : H(\mathbb{D}) \to H(\mathbb{D})$, $T_a f(z) = (z-a)f(z)$ for $f \in H(\mathbb{D})$ is a homeomorphism. The following questions seem to be completely open.

Question 1. Take $a \in \mathbb{C}$ with $|a| \geq 1$ and $f \in U(\mathbb{D},0)$. Is it true that $(z-a)f(z) \in U(\mathbb{D},0)$?

A more general question than the previous one is the following.

Question 2. Take a non-constant polynomial p and $f \in U(\mathbb{D},0)$. Is it true that $pf \in U(\mathbb{D},0)$?

Question 3. Let $f \in U(\mathbb{D},0)$. Is it true that $f' \in U(\mathbb{D},0)$?

It is well known that universal Taylor series in the sense of Nestoridis cannot be bounded. On the other hand there are universal Taylor series in the sense of Luh and Chui and Parnes which belong to the disk algebra. Therefore the next question arises naturally.

Question 4. Does there exist an inner function f such that $f \in U_1(\mathbb{D},0)$?

I would like to stress that there are outer functions which belong to the class $U_1(\mathbb{D}, 0)$.

Let us turn our attention to universal functions with respect to derivatives. We recall the definition which is due to G. R. MacLane.

Definition 4. (MacLane [9]) An entire function $f \in H(\mathbb{C})$ is called universal with respect to derivatives if the sequence

$$\{f^{(n)} : n = 0, 1, 2, \ldots\}$$

of its derivatives is dense in $H(\mathbb{C})$.

The set of universal functions with respect to derivatives is denoted by $HC(D)$. In modern terminology this means that the differentiation operator D acting on the space of entire functions $H(\mathbb{C})$ is hypercyclic. For a general discussion and an account on results related to the notions of universality and hypercyclicity we refer to [6]

Theorem 5. i) If $f(z) \in HC(D)$ then $f(z) + e^{az} \in HC(D)$ for every $a \in \mathbb{C}$ with $|a| \leq 1$.

ii) There exists $g(z) \in HC(D)$ such that $g(z) + e^{az} \notin HC(D)$ for every $a \in \mathbb{C}$ with $|a| > 1$.

In order to prove item i) we consider two cases. If $|a| < 1$ the result is trivial since the n-th derivative of e^{az} is equal to $a^n e^{az}$ and the latter goes to 0 as n tends to infinity uniformly on compact subsets of $\mathbb{C}$. If $a = e^{2\pi i\theta}$ then again we consider two cases. If θ is rational we use Ansari's theorem, which tells us that $HC(D) = HC(D^m)$ for every $m = 2, 3, \ldots$, where

$$HC(D^m) = \{f \in H(\mathbb{C}) : \overline{\{f^{(mn)} : n = 1, 2, \ldots\}} = H(\mathbb{C})\}.$$

If θ is irrational we make use of Ansari's theorem together with the minimality of the irrational rotation. The proof of item ii) depends on a result due to Grosse-Erdmann and Shkarin (proved independently) concerning the optimal growth of universal functions with respect to derivatives. To be precise fix $g \in HC(D)$ such that $|g(z)| = O(e^r)$ for $|z| \leq r$ as $r \to \infty$, see [7], [12]. Then it is not difficult to show that $g(z) + e^{az} \notin HC(D)$ for every $a \in \mathbb{C}$ with $|a| > 1$.

Let us finally deal with the problem of whether the multiplication of a function $f \in HC(D)$ with a non-zero polynomial p still belongs to $HC(D)$. We do not have a complete answer. What we can prove is the following.

Theorem 6. There exists a residual subset G of $H(\mathbb{C})$ such that $G \subset HC(D)$ and for every non-zero polynomial p we have $pf \in HC(D)$ for every $f \in G$.

The proof uses in an essential way a result from [4] related to the existence of common universal-hypercyclic vectors.

Question 5. Take $f \in HC(D)$ and let p be a non-constant polynomial. Is it true that $pf \in HC(D)$?

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Universal functions on $H^\infty(B^n)$

Richard M. Aron
(joint work with Pamela Gorkin)

We report on a paper with Pamela Gorkin [1], whose principal result is a theorem about universal functions on $H^\infty(B^n)$, where B^n is the ℓ_2−ball in $\mathbb{C}^n$. P. S. Chee showed that there is a sequence (L_k) of automorphisms of B^n to which one can associate a universal function $f \in H^\infty(B^n)$, $\|f\| = 1$. That is, the set $\{f \circ L_k \mid k \in \mathbb{N}\}$ is dense in $H^\infty(B^n)$ when this space is endowed with the compact-open topology. Here, each L_k corresponds to a point $z_k \in B^n$.

Our interest is the size and structure of the set of such universal functions.
Theorem: There is a sequence $(z_k) \subset B^n$ for which one can find an infinite dimensional closed subspace $V \subset H^\infty(B^n)$ with the following property: Every $f \in V$, $\|f\| = 1$, is universal with respect to the sequence (L_k).

This work has been extended by F. Bayart, P. Gorkin, S. Grivaux, and R. Mortini to holomorphic self-maps of the disk [2] and to more general domains in $\mathbb{C}^n$ by F. Bayart and P. Gorkin [3] .

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Universal functions for composition operators

Raymond Mortini
(joint work with Karl-Goswin Grosse-Erdmann)

Let Ω be a planar domain and let $H(\Omega)$ be the Fréchet space of all holomorphic functions on Ω. Let X denote either $H(\Omega)$ or the unit ball $\mathcal{B} = \{f \in H(\Omega) : \sup_{z\in\Omega} |f(z)| \leq 1\}$ of $H^\infty(\Omega)$. A function $f \in X$ is said to be X-universal for a sequence, (ϕ_n), of selfmaps of Ω if $\{f \circ \phi_n : n \in \mathbb{N}\}$ is (locally uniformly) dense in X. Whereas the case of ϕ_n being an automorphism has been successfully dealt with by many authors (see e.g. [2, 7, 4, 3]), we will study here the general case. It will be shown that for every domain $\Omega \subseteq \mathbb{C}$ for which $H^\infty(\Omega)$ is dense in $H(\Omega)$ there exists a sequence (ϕ_n) such that the family (C_{ϕ_n}) of composition operators admits $H(\Omega)$-universal functions. Moreover, if Ω is finitely connected, but not simply connected, then such a sequence of selfmaps cannot be eventually injective. On the other hand, if Ω is a domain of infinite connectivity, then a sequence of eventually injective selfmaps of Ω admits $H(\Omega)$-universal functions if and only for every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$ there is some $n \geq N$ such that $\phi_n(K)$ is Ω-convex and $\phi_n(K) \cap K = \emptyset$. Simply connected domains in $\mathbb{C}$ are considered, too. The case of the unit disk appeared in [1].

The problem of characterizing $\mathcal{B}$-universality for selfmappings of a non-simply connected domain is still open. In the case of the unit disk, it was shown in [1] that a sequence of holomorphic self-maps (ϕ_n) in $\mathbb{D}$ with $\phi_n(0) \to 1$ admits a $\mathcal{B}$-universal function if and only if

$$\limsup_{n\to\infty} \frac{|\phi_n'(0)|}{1 - |\phi_n(0)|^2} = 1. \tag{0.1}$$

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Close universal approximants of the Riemann zeta-function

Markus Niess

The Riemann zeta-function $\zeta(z)$ has the following well-known properties, cf. [4]:

(1) It is a meromorphic with a single pole at $z = 1$ with residue 1.
(2) The symmetry relation $\zeta(z) = \overline{\zeta(\overline{z})}$ holds for $z \neq 1$.
(3) The functional equation $\zeta(z)\Gamma(z/2)\pi^{-z/2} = \zeta(1-z)\Gamma((1-z)/2)\pi^{-(1-z)/2}$ holds.
(V) It has a universality property due to Voronin (1975): For every compact set $K \subset \{z\colon \frac{1}{2} < \Re z < 1\}$ with connected complement, every function $f \in A(K) := \{f\colon f$ is continuous on K and holomorphic in the interior of $K\}$ zero-free on K and every $\varepsilon > 0$:

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]\colon \max_{z\in K} |\zeta(z+i\tau) - f(z)| < \varepsilon\right\} > 0,$$

where *meas* stands for the Lebesgue measure.

We use the following notations: Let $b = \{b_n\} \subset \mathbb{C}$ be a sequence without finite accumulation point and $a = \{a_n\}$ with $a_n \to 0$. A function φ is said to satisfy the property (B_b) or $(L_{a,b})$ respectively, if

(B_b) For every compact set K with connected complement, every function $f \in A(K)$, there exists a subsequence $\{n_k\} \subset \mathbb{N}$ with $\varphi(z + b_{n_k}) \to f(z)$ uniformly on K.
$(L_{a,b})$ For every compact set K with connected complement, every function $f \in A(K)$, there exists a subsequence $\{n_k\} \subset \mathbb{N}$ with $\varphi(a_{n_k} z + b_{n_k}) \to f(z)$ uniformly on K.

We say that b satisfies

(A) if $b = \{b_n\} \subset \{z\colon \Re z \notin [0,1], \Im z \neq 0\}$,
(B) if $\text{dist}(b_n; \{z\colon \Re z \in [0,1] \text{ or } \Im z = 0\}) \to \infty$, as $n \to \infty$.

Our aim is to construct functions satisfying several of these properties and being "close" to ζ. This led to the following results:

Theorem 1. If b has (A) and (B), then there exists a meromorphic function ζ_1 with (1)-(3),(V) and (B_b). The functions ζ_1 and ζ have the same zeros in the critical strip.

Theorem 2. Let $\Lambda > 0$ and b with (A). There exists a closed set S (of tangential approximation) such that the area of the complement of S is less than Λ. Moreover, for every continuous, positive function ε on S, there exists a meromorphic function ζ_2 with (1)-(3), $(L_{a,b})$ for some sequence a and $|\zeta(z) - \zeta_2(z)| < \varepsilon(z)$ on S.
The function ζ_2 has the same zeros as ζ plus additional ones that are located by construction, and ζ_2 does not satisfy the Riemann hypothesis.

If we consider the sets M of all meromorphic functions with (1)-(3) and $C(\mathbb{C}, \mathbb{C}_\infty)$ of all continuous functions from $\mathbb{C}$ to the extended complex plane, each one endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$, we obtain

Theorem 3. Let b with (A) and (B). We consider the operators $T_n : M \to C(\mathbb{C}, \mathbb{C}_\infty), f(z) \mapsto f(z + b_n)$. Then the set of universal functions

$$\mathcal{U} := \left\{ \varphi \in M : (H(\mathbb{C}) \cup \{f : f \equiv \infty\}) \subset \overline{\{T_n \varphi : n \in \mathbb{N}\}} \right\}$$

is a dense G_δ-set in M.

For details and proofs we refer the interested reader to [2]. Previous results in this context can be found in [3] and [1].

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Construction versus Baire category in universality

Karl-G. Grosse-Erdmann

The existence of universal functions is usually proved by one of two methods: by an explicit construction or by the use of the Baire category theorem. In the talk we argued that the two methods are largely equivalent. The talk was therefore divided into two parts.

A. From Baire category to construction. The Baire category theorem was used as early as 1937 in existence proofs of universal functions: S. Mazurkiewicz [9] showed that there is a universal Taylor series on $[0, 1]$. Since then this method has been used in a myriad of related situations.

However, the first existence proofs of universal functions were based on explicit constructions. For example, G. D. Birkhoff [1] showed that there is an entire function whose translates are dense in the space of all entire functions. Such constructions have the advantage that modifications of the proof may provide the existence of universal functions with additional properties. This has led some researchers to wonder if, in cases where the Baire category theorem was applied to obtain existence, a constructive proof may also be possible. In a recent paper, C. Kariofillis [4, Section 4] was the first to pose a corresponding problem explicitly in the literature. A related, and stronger, problem was posed to the speaker by V. Nestoridis [10].

The following observation leads to the solution of these problems: *The Baire category theorem itself has a constructive proof.*

As an illustration, we sketched the usual Baire category proof of Birkhoff's universality theorem. We then gave a (constructive) proof of the Baire category theorem in the special case of a complete metric linear space. When combining these two proofs one is led in a natural way to a constructive proof of Birkhoff's theorem, which, as it turns out, is very close to Birkhoff's original proof.

B. From construction to Baire category. On the other hand, there are constructive existence proofs of universalities that so far have not been turned into Baire category proofs. The latter, however, is of interest because it automatically gives a much better result: not only will the set of universal functions in that case be non-empty, it will be residual in the underlying space. This will then provide, for example, easy proofs for the existence of common universalities.

As mentioned above, constructions of universal functions are of particular interest in cases where one tries to impose additional properties on these functions. Put abstractly, this means that if a universality is described by a family $(T_j)_{j\in I}$ of continuous mappings $T_j : X \to Y$, where X is a complete metric space, then one searches for universal functions in a certain subset A of X.

A study of the relevant literature has shown that such restricted universalities commonly occur in one of three situations.

First, if A is *a closed subset* of X then A is a complete metric space under the induced metric, so that the Baire category theorem can be applied to the restrictions $T_j|_A : A \to Y$. As an illustration we noted that recent theorems of Luh, Martirosian and Müller [6], [7] and of Martirosian and Martirosyan [8] on universal entire functions with lacunary power series can be given Baire category proofs. These proofs even lead to stronger results with no additional effort.

Secondly, if A is not necessarily closed but *a G_δ-subset* of X then, in fact, A can be given an equivalent complete metric, so that the Baire category theorem can again be applied to the mappings $T_j|_A$. Moreover, in many situations, a very useful theorem of Herzog [3] can be employed.

Thirdly, if A is *completely metrizable* in its own right, with a metric that is usually strictly stronger than the original metric, then one applies the Baire category theorem to the mappings $T_j|_A$ with the new topology on A. As an illustration we gave a new proof of a recent result of Gharibyan, Luh and Nieß [2] on the existence of Birkhoff universal functions that are bounded on a prescribed sector.

C. Concluding remarks. We stress that we did not give and did not intend to give a logically precise definition of what is meant by a constructive proof. We adopted the naive, and in universality generally accepted, position that a constructive proof is one that is formulated within the framework of classical analysis.

We ended the talk by recalling a pronouncement of T. W. Körner [5] that reflects perfectly the speaker's opinion:

The Baire Category Theorem is a profound triviality.

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An improvement of the universality of the geometric series

Vassili Nestoridis

Recently A. Mouze obtained an improvement of a result of Bernal–Gonzalez, Calderon-Moreno and W. Luh concerning the universality of the geometric series. Mouze proved the following: Let $S = \sum_{n=0}^{\infty} c_n z^n$ be a formal power series with $c_n \neq 0$ for all $n \in \mathbb{N}$. Then there exists a matrix $A = [a_{n,v}]_{n,v \geq 0}$, satisfying some properties, such that the sequence of the A-transforms of S has universal properties in $\{z : |z| \geq 1\} - \{1\}$. The set of universal series with respect to this matrix A is G_δ and dense in natural spaces of series.

Similar results are obtained replacing power series by Dirichlet series.

Universal series from fundamental solution of the Laplace operator

Innocent Tamptsé

Let ϕ be the standard fundamental solution of the Laplace operator on $\mathbf{R}^N$, $(N \geq 2)$. We prove the existence of universal series of the form

$$\sum_{k=0}^{\infty} c_k \phi(x - a_k) \tag{0.1}$$

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha D^\alpha \phi(x - a) \tag{0.2}$$

in the space of functions that are harmonic in the neighborhood of a fixed compact set $K \subset \mathbf{R}^N$ with connected complement, or the space of functions that are harmonic on an open set $\Omega \subset \mathbf{R}^N$ that have an exhaustion by compact sets with connected complements. Here, $a, a_k, k \in \mathbf{N}$ are fixed points lying outside the domain of definition of the harmonic functions.

We also prove the existence of a series of the form (0.2) which is convergent in $\mathbf{R}^N \setminus B(0,r)$ and is universally overconvergent in $B(a,r) \setminus \{a\}$. Moreover, we give some conditions for series in the form

$$\sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha u_\alpha \tag{0.3}$$

to be universal in a metrizable topological linear space X.

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A universal solution for a quasi-linear parabolic equation occuring in aerodynamics

Paul M. Gauthier

(joint work with Nikolai Tarkhanov)

We present results on universal functions as well as universal (formal) series for solutions to a quasi-linear parabolic equation arising in aerodynamics, Burgers' equation. Burgers' equation is one of the simplest examples of a non-linear partial differential equation and also perhaps the simplest equation describing waves under the influence of diffusion. The crucial role in our investigation is due to the so-called Cole-Hopf transformation.

Asymptotic maximum principles for subharmonic functions

Stephen J. Gardiner

Let Ω be a domain in the complex plane $\mathbb{C}$, or Euclidean space $\mathbb{R}^n$ $(n \geq 2)$. By a *boundary path* in Ω we mean a continuous function $\gamma : [0, \infty) \to \Omega$ such that $\gamma(t)$ lies outside any given compact set $K \subset \Omega$ for all sufficiently large t. A complex (or real) function f on Ω is said to have *asymptotic value* ∞ if there is a boundary path γ such that $f(\gamma(t)) \to \infty$ as $t \to \infty$.

A classical result of Valiron says that a function which is holomorphic and unbounded on the unit disc $\mathbb{D}$, but bounded on a spiral that accumulates at all boundary points, must have asymptotic value ∞. Barth and Rippon [1] have recently obtained several generalizations of this result. For example, they show that a function which is holomorphic and unbounded on $\mathbb{D}$, but bounded on a set that accumulates nontangentially at almost all boundary points, must have asymptotic value ∞. This abstract presents asymptotic maximum principles for subharmonic functions which were inspired by, and generalize, their work. Full details may be found in [3].

From now on $\Omega \subset \mathbb{R}^n$. If $n = 2$, we assume that Ω has nonpolar complement. Thus we can define harmonic measure ν_0 for Ω and a fixed point $x_0 \in \Omega$. The boundary of Ω in compactified space $\mathbb{R}^n \cup \{\infty\}$ will be denoted by $\partial^\infty\Omega$. A prototype asymptotic maximum principle for subharmonic functions is as follows.

Theorem 1 *Let s be a subharmonic function on Ω. If s does not have asymptotic value ∞ and*

$$\limsup_{x\to z} s(x) \leq 0 \quad \text{for } \nu_0\text{-almost every } z \in \partial^\infty\Omega, \tag{1}$$

then $s \leq 0$.

This is only a slight extension of a result of Fuglede [2], and can be proved by known arguments based on the *fine topology*, namely the coarsest topology which makes all subharmonic functions continuous. However, the work of Barth and Rippon suggests that the boundary condition (1) might be weakened. For simplicity we will state our results for the case where Ω is the unit ball B, and add some concluding remarks about the form they take in general domains.

Let s be a subharmonic function on B. An *(asymptotic) tract of s for ∞* is a decreasing sequence (W_k) of sets where, for each k, the set W_k is a finely connected component of the finely open set $\{x \in B : s(x) > k\}$. We call the (Euclidean) closed set $E = \cap_k \overline{W}_k$ the *end* of the tract. Let σ denote surface area measure on ∂B.

Theorem 2 *Let s be a subharmonic function on B. Suppose that s has only finitely many tracts for ∞, with ends $E_1, ..., E_m$, say, and that*

$$\liminf_{r\to 1-} s(rz) \leq 0 \quad \text{for } \sigma\text{-almost every } z \in \partial B. \tag{2}$$

Then

$$\limsup_{x\to z} s(x) \leq 0 \quad \text{for all } z \in \partial B \backslash (\cup_j E_j)\,.$$

Corollary. *Let s be a subharmonic function on B. If s does not have asymptotic value ∞ and (2) holds, then $s \leq 0$.*

Approximation arguments can be used to construct a harmonic function s on B such that the inequality in (2) holds at *every* point of ∂B, and all tracts of s

for ∞ end at the point $(1,0,...,0)$, say, yet $\limsup_{x\to z} s(x) > 0$ for every $z \in \partial B$. Thus the finiteness hypothesis on the number of tracts in Theorem 2 is crucial.

Finally, analogues of Theorem 2 and the Corollary hold for general domains. These are expressed in terms of the behaviour of s/h, where h is an arbitrary positive harmonic function on Ω. The Martin boundary is used in place of the Euclidean boundary, and the representing measure for h is used in place of σ. Further, the role of radial convergence is taken by convergence with respect to a natural extension of the fine topology to the Martin compactification, namely the minimal fine topology.

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Uniform approximation by interpolating Blaschke products

Raymond Mortini

Let $\mathfrak{I}$ be the class of all inner functions that can be uniformly approximated on $\mathbb{D}$ by interpolating Blaschke products. It is well known (see [3]) that finite products of interpolating Blaschke products are contained in $\mathfrak{I}$. Moreover, any infinite Blaschke product whose zeros lie in a cone belongs to $\mathfrak{I}$ (see [4]). In the early eighties P.W. Jones and J. B. Garnett asked whether $\mathfrak{I}$ coincides with the class of all inner functions. In my talk I sketch the proof that any inner function u for which there exists a level set $\{|u| < \eta\}$ that can be controlled in a certain way by the zero *set* of u belongs to $\mathfrak{I}$. To be more precise, let $D_\rho(a,\varepsilon) = \{z \in \mathbb{D} : \rho(z,a) < \varepsilon\}$ be the pseudohyperbolic disk of center a and radius ε and let $Z_{\mathbb{D}}(u) = \{z \in \mathbb{D} : u(z) = 0\}$ be the zero set of u in $\mathbb{D}$. Then the condition

$$(M) \qquad \exists \varepsilon \in]0,1[\ \exists \eta \in]0,1[\text{ such that } \{|u| < \eta\} \subseteq \bigcup_{\lambda \in Z_{\mathbb{D}}(u)} D_\rho(\lambda,\varepsilon)$$

implies that $u \in \mathfrak{I}$. The proof is done with maximal ideal space techniques; the main feature will be to show that an inner function u satisfying M will have the property that the zeros of u belonging to the set of trivial points in the maximal ideal space $M(H^\infty)$ are in the closure of the zeros of u in $\mathbb{D}$. This in turn will imply that $(u-a)/(1-\overline{a}u)$ is a finite product of interpolating Blaschke products whenever $|a|$ is small, $a \neq 0$.

In particular, we will notice that $\mathfrak{I}$ contains the set of inner functions satisfying the weak embedding property; a set that appeared in recent work of Gorkin, Nikolski and the speaker on H^∞-quotient algebras.

Let us also note that, whenever in condition (M) above, the disks $D_\rho(\lambda,\varepsilon)$ are pairwise disjoint then u is an interpolating Blaschke product, as was shown a long time ago by Kerr-Lawson [2].

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Approximating inner functions

Geir Arne Hjelle
(joint work with Artur Nicolau)

Let H^∞ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$. A function in H^∞ is called inner if it has radial limit of modulus one at almost every point of the unit circle. A Blaschke product is an inner function of the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

where m is a non-negative integer and $\{z_n\}$ is a sequence of points in $\mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum_n (1 - |z_n|) < \infty$. A classical result of O. Frostman says that for any inner function f, there exists an exceptional set $E = E(f) \subset \mathbb{D}$ of logarithmic capacity zero such that the Möbius shift

$$\frac{f - \alpha}{1 - \bar{\alpha} f}$$

is a Blaschke product for any $\alpha \in \mathbb{D} \setminus E$. See [3] or [4, II.6.4]. It follows that any inner function can be uniformly approximated by a Blaschke product.

A Blaschke product B is called an interpolating Blaschke product if its zero set $\{z_n\}$ forms an interpolating sequence, that is, for any bounded sequence of complex numbers $\{w_n\}$, there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$, $n = 1, 2, \ldots$. A celebrated result by L. Carleson tells us that this holds precisely when the following two conditions are satisfied:

(1) $\inf_{n \neq m} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| > 0$,

(2) there exists a constant C such that $\sum_{z_n \in Q} (1 - |z_n|) < C\ell(Q)$ for any Carleson square Q of the form

$$Q = \left\{ re^{i\theta} \colon 0 < 1 - r < \ell(Q),\ |\theta - \theta_0| < \pi \ell(Q) \right\}$$

where $\theta_0 \in [0, 2\pi)$ and $0 < \ell(Q) < 1$.

See [1] or [4, VII.1.1]. Although the interpolating Blaschke products comprise a small subset of all Blaschke products, they play a central role in the theory of the algebra H^∞. See the last three chapters of [4].

In [9] D. Marshall proved that any function $f \in H^\infty$ can be uniformly approximated by finite linear combinations of Blaschke products. That is, for any $\varepsilon > 0$ there are constants $c_1, \dots, c_N$ and Blaschke products $B_1, \dots, B_N$ such that

$$\Big\| f - \sum_{i=1}^{N} c_i B_i \Big\|_\infty < \varepsilon.$$

This result was improved in [5] by showing that one can take each of $B_1, \dots, B_N$ to be an interpolating Blaschke product. However the following problem remains open.

(1) For any inner function B and $\varepsilon > 0$, is there an interpolating Blaschke product I such that $\|B - I\|_\infty < \varepsilon$?

This question was posed in [4, X.5.4], [6, pp. 268–269], [8] and [12, p. 202]. If one restricts attention to the modulus, the question has a positive answer.

Theorem 1. Let B be an inner function and $\varepsilon > 0$. Then there exists an interpolating Blaschke product I such that

$$\big| |B(z)| - |I(z)| \big| < \varepsilon$$

for all $z \in \mathbb{D}$.

The following is a quick sketch of the proof. Details may be found in [7]. The first step consists of constructing a system $\Gamma = \bigcup_i \Gamma_i$ of disjoint closed curves $\Gamma_i \subset \mathbb{D}$ such that arclength of Γ is a Carleson measure, and verifying that

(a) $|B(z)|$ is uniformly small on hyperbolic disks of fixed radius centered at points of Γ,
(b) in any hyperbolic disk of fixed radius centered at a point outside the union of the interiors of Γ_i, $\bigcup_i \operatorname{int} \Gamma_i$, there is a point z where $|B(z)|$ is not small.

Decompose $B = B_1 \cdot B_2$ where B_1 is the Blaschke product formed with the zeros of B which are in $\bigcup_i \operatorname{int} \Gamma_i$. Statement (b) gives that B_2 is a finite product of interpolating Blaschke products. D. Marshall and A. Stray proved in [10] that any finite product of interpolating Blaschke products may be approximated by a single interpolating Blaschke product. Therefore only the zeros of B_1 concern us. The construction of Γ is a variation due to Nicolau and D. Suarez [11] of the original Corona construction introduced by L. Carleson. See [2] or [4, VIII.5].

Next, for each $i = 1, 2, \dots$, let μ_i be the sum of harmonic measures in $\operatorname{int} \Gamma_i$ from the zeros of B_1 contained in $\operatorname{int} \Gamma_i$. Then the mass $\mu_i(\Gamma_i)$ is the total number of zeros of B_1 contained in $\operatorname{int} \Gamma_i$. The next step consists of splitting $\Gamma_i = \bigcup_k \Gamma_{i,k}$, into pieces $\Gamma_{i,k}$ with $\mu_i(\Gamma_{i,k}) = 1$, $k = 1, 2, \dots$ and choosing points $\xi_{i,k} \in \Gamma_{i,k}$ which match a certain moment of the measure μ_i on $\Gamma_{i,k}$. Let I_1 be the Blaschke product with zeros $\xi_{i,k}$, $i, k = 1, 2, \dots$. The last step of the proof is to use (b) above to show that I_1 is an interpolating Blaschke product and to use the location of $\{\xi_{i,k}\}$, as well as (a) above, to show that $|I_1(z) \cdot B_2(z)|$ approximates $|B(z)|$.

Besides the individual problem mentioned above, some questions concerning approximation by arguments of interpolating Blaschke products remain open. Let B be an inner function.

(2) Given $\varepsilon > 0$, is there an interpolating Blaschke product I such that

$$\| \operatorname{Arg} B - \operatorname{Arg} I \|_{\mathrm{BMO}(\partial \mathbb{D})} < \varepsilon?$$

(3) Is there an interpolating Blaschke product I such that $\operatorname{Arg} B - \operatorname{Arg} I = \widetilde{v}$ where $v \in L^\infty(\partial \mathbb{D})$?
(4) Is there an interpolating Blaschke product I such that $\operatorname{Arg} B - \operatorname{Arg} I = u + \tilde{v}$ where $u, v \in L^\infty(\partial \mathbb{D})$ and $\|u\|_\infty < \frac{\pi}{2}$?

A positive answer to Problem 2 would imply Theorem 1. Problem 4 was posed by N. K. Nikol′skiĭ in [6] and [12] in connection with Toeplitz operators and complete interpolating sequences in model spaces. Problem 3 and Problem 4 are also discussed in the nice monograph by K. Seip [13, p. 92].

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List of open problems

Problem session

1. Richard M. Aron: Compositions of universal or hypercyclic operators

Example. Consider the collection $\mathcal{O}(\mathbb{C}^n) = \{T : \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^n) \mid T$ is linear, continuous, and T commutes with partial derivatives $\}$. For all $T \in \mathcal{O}$ such that T is not a multiple of the identity, T is hypercyclic (Godefroy and Shapiro). Note that $\mathcal{O}$ is an algebra under composition, and we have precise information about when the composition of two operators in $\mathcal{O}$ is hypercyclic.

The general question is whether this occurs in other contexts? Specifically, consider the following situations:

1. Let $\lambda, b \in \mathbb{C}$. It is known that for $|\lambda| \geq 1$, the operator $T_{\lambda,b} : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ given by $T_{\lambda,b}(f)(z) = f'(\lambda z + b)$ is hypercyclic. Question: Given two such operators, $T_{\lambda,b}$ and $T_{\mu,c}$, is their composition hypercyclic?

2. There are by now a number of situations where one has a sequence (φ_n) of holomorphic self-maps of the unit disk (or, more generally, some domain in $\mathbb{C}$ or $\mathbb{C}^n$) for which one can find a universal function f relative to $\mathcal{H}^\infty$. Are there 'natural,' 'interesting' situations where one can take the composition of two such sequences of holomorphic self-maps, (φ_n) and (ϑ_n), obtaining $(\varphi_n \circ \vartheta_n)$ for which there is again a universal function?

2. Vagia Vlachou

Let $\overline{D(\zeta_1, r)}$, $r > 0$, be a closed disk of the complex plane and let $\Omega = \mathbb{C} \setminus \overline{D(\zeta_1, r)}$. If we fix any point $\zeta_0 \in \Omega$, than it is known that there exist functions, holomorphic in Ω, which are universal Taylor series centered at ζ_0.
Question: Is it possible for a universal Taylor series in this domain to be analytically continuable across a point $z_1 \in \partial\Omega$? We believe the answer is negative.

3. Wolfgang Luh: Three problems

Suppose that f is a holomorphic function in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ with the power series expansion

$$(*) \qquad f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu \qquad \text{where} \qquad \overline{\lim_{\nu\to\infty}} |f_\nu|^{1/\nu} = 1.$$

A point $z_0 \in \mathbb{C}$ is called a limit point of zeros of the partial sums $s_n(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$, if for each $\delta > 0$ there are infinitely many s_n having a zero in $U_\delta(z_0) = \{z : |z - z_0| < \delta\}$. The set of all those limit points is denoted by $Z(f, \{s_n\})$. The analogous definition is used for each subsequence $\{s_{n_k}\}$. It is clear that $Z(f,\{s_n\})$ and $Z(f,\{s_{n_k}\})$ are closed sets.

It has been shown by Jentzsch [3,4] that always $\partial\mathbb{D} = \{z : |z| = 1\}$ is a subset of $Z(f,\{s_n\})$.

The following results were proved by W. Gehlen and W. Luh [2]; for a refinement see also W. Gehlen [3].

Theorem 1. *Suppose that S is a prescribed closed set satisfying $\partial\mathbb{D} \subset S \subset \mathbb{D}^c$. Then there exists a power series $(*)$ such that $Z(f,\{s_n\}) \cap \mathbb{D}^c = S$.*

Theorem 2. *There exists a power series $(*)$ with the following property. Given any closed set $S \subset \mathbb{D}^c$ then there exists a subsequence $\{n_k\}$ of $\mathbb{N}_0$ (depending on S) such that $Z(f,\{s_{n_k}\}) = S$.*

Problems:

- Are there generic proofs for Theorem 1 or Theorem 2.
- How "many" of those functions f exist.
- Are there (connected with Theorem 2) other universalities simultaneously.

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4. Raymond Mortini: The Image of disk-algebra functions

Give a geometric/topological characterization of the images $f(\overline{\mathbb{D}})$ of functions in the disk-algebra $A(\mathbb{D})$ and the associated real algebra $A_{\mathbb{R}}(\mathbb{D}) = \{f \in A(\mathbb{D}) : f(z) = \overline{f(\overline{z})}\}$.

For example, as was observed by V. Müller, for $a > 0$, the compact set

$$\{|z-a| \leq a\} \cup \{|z+a| \leq a\}$$

is not the image of any $f \in A(\mathbb{D})$.

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