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Mini-Workshop: Attraction to Solitary Waves and Related Aspects of Physics

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Abstract. The aim of the miniworkshop is the discussion of the solitary wave asymptotics for nonlinear Hamiltonian partial differential equations and relation to mathematical problems of Quantum Physics. While the existence and orbital stability of solitary waves is fairly well understood, the asymptotic stability of solitary waves is still not understood well. The global attraction of arbitrary solutions of finite energy to the set of solitary waves is not proved but in a few model cases. On the other side, there is now accumulating a great number of recent results that seem to enable us to make a crucial progress in this direction: namely, to prove the solitary asymptotics for the $\mathbf{U}(1)$-invariant nonlinear Klein-Gordon equation and similar dispersive Hamiltonian systems. On the Quantum Physics side, the workshop contained the thorough discussion of the quantum scattering and renormalization analysis.

Mathematics Subject Classification (2000): 83xx, 35xx.

Introduction by the Organisers

The workshop *Attraction to Solitary Waves and Related Aspects of Physics*, organised by Vladimir Buslaev (St. Petersburg University), Andrew Comech (Texas A&M), Alexander Komech (Universität Wien), and Boris Vainberg (UNC – Charlotte) was held February 10–16, 2008. This meeting was attended with 15 participants with broad geographic representation from Europe and America. This workshop was a blend of researchers with backgrounds in Partial Differential Equations, Harmonic Analysis, and Quantum Field Theory.

The aim of the miniworkshop has been the discussion of current state of the long-time asymptotics for nonlinear Hamiltonian partial differential equations and relation to mathematical problems of Quantum Physics.

The central themes were the orbital and asymptotic stability of solitary waves, quantum scattering, renormalization, and global attraction to solitary waves.

Bohr's transitions as global attraction to solitary waves. According to Bohr's postulates [Boh13], an unperturbed electron runs forever along certain *stationary orbit*, which we denote $|E\rangle$ and call *quantum stationary state*. Once in such a state, the electron has a fixed value of energy E, not losing the energy via emitting radiation. The electron can jump from one quantum stationary state to another,

$$|E_-\rangle \longmapsto |E_+\rangle, \tag{1}$$

emitting or absorbing a quantum of light with the energy equal to the difference of the energies E_+ and E_-. Bohr's second postulate states that the electrons can jump from one quantum stationary state (Bohr's *stationary orbit*) to another.

Bohr's *stationary orbits* were interpreted by Schrödinger as *quasistationary solitary wave solutions* of the form

$$\psi(x,t) = \phi(x)e^{-i\omega t}, \qquad \textit{with} \quad \omega \in \mathbb{R}, \quad \lim_{|x|\to\infty} \phi(x) = 0. \tag{2}$$

We will call such solutions *solitary waves*. Other appropriate names are *nonlinear eigenfunctions* and *quantum stationary states* (the solution (2) is not exactly stationary, but certain observable quantities, such as the charge and current densities, are time-independent indeed). As a consequence, the electron in such a state does not emit the energy and "circles" forever around the nucleus in an atom.

Bohr's *quantum jumps* can be interpreted dynamically as long-time asymptotics

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty, \tag{3}$$

for any trajectory $\Psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ generally depend on the trajectory. Then the quantum stationary states should be viewed as the points of the *global attractor* $\mathscr{A}$.

The attraction (3) takes the form of the long-time asymptotics

$$\psi(x,t) \sim \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{4}$$

that hold for each finite energy solution.

Now let us describe the existing results on solitary waves in the context of dispersive Hamiltonian systems.

Nonlinear wave equations. Well-posedness in the energy space. The nonlinear wave equations take their origin in Quantum Field Theory from the articles by Schiff [Sch51a, Sch51b], who considered the nonlinear Klein–Gordon equation in his research on the classical nonlinear meson theory of nuclear forces. The mathematical analysis of this equation is started by Jörgens [Jör61] and Segal [Seg63a, Seg63b], who studied its global well-posedness in the energy space.

Since then, this equation (alongside with the nonlinear Schrödinger equation) has been the main playground for developing tools to handle more general nonlinear Hamiltonian systems.

Local attraction to zero. The asymptotics of type (4) were discovered first with $\psi_\pm = 0$ in the scattering theory. Segal [Seg66] and then Morawetz and Strauss [Str68, MS72] studied the (nonlinear) scattering for solutions of nonlinear Klein–Gordon equation in $\mathbb{R}^3$. We may interpret these results as *local* (referring to small initial data) attraction to zero:

$$\psi(x,t) \sim \psi_\pm = 0, \qquad t \to \pm\infty. \tag{5}$$

The asymptotics (5) hold on an arbitrary compact set and represent the well-known local energy decay. These results were further extended in [GS79, Kla82, GV85, Hör91].

Solitary waves. Apparently, there could be no *global* attraction to zero (*global* referring to arbitrary initial data) if there are solitary wave solutions of the form $\phi_\omega(x)e^{-i\omega t}$. The existence of solitary wave solutions

$$\psi_\omega(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}^n),$$

with $H^1(\mathbb{R}^n)$ being the Sobolev space, to the nonlinear Klein–Gordon equation (and nonlinear Schrödinger equation) in $\mathbb{R}^n$, in a rather generic situation, was established in [Str77] (a more general result was obtained in [BL83a, BL83b]). Typically, such solutions exist for ω from an interval or a collection of intervals of the real line. We denote the set of all solitary waves by $\mathcal{S}_0$.

While all localized stationary solutions to the nonlinear wave equations in spatial dimensions $n \geq 3$ turn out to be unstable (the result known as "Derrick's theorem" [Der64]), *quasistationary* solitary waves can be orbitally stable. Stability of solitary waves takes its origin from [VK73] and has been extensively studied by Strauss and his school in [GSS87, Sha83, Sha85, SS85].

Local attraction to solitary waves. The asymptotic stability of solitary waves (convergence to a solitary wave for the initial data sufficiently close to it) has been studied by Soffer and Weinstein [SW90, SW92] in the context of nonlinear $\mathbf{U}(1)$-invariant Schrödinger equation with a potential. This theory has been further developed by Buslaev and Perelman [BP93, BP95] and then by others in [PW97, SW99, Cuc01a, Cuc01b, Cuc03, BS03] and other papers. Up to date, there are many open problems. While generically we expect that any orbitaly stable solitary wave is also asymptotically stable, the proof of this statement is only available for just a few cases and under very strong assumptions.

The existing results on orbital and asymptotic stability suggest that the set of orbitally stable solitary waves typically forms a *local attractor*, that is to say, attracts any finite energy solutions that were initially close to it. Moreover, a natural hypothesis is that the set of all solitary waves forms a *global attractor* of all finite energy solutions.

Global attraction to solitary waves. The *global attraction* of type (4) with $\psi_\pm \neq 0$ and $\omega_\pm = 0$ was established in certain models in [Kom91, Kom95, KV96, KSK97, Kom99, KS00] for a number of nonlinear wave problems. There the attractor is the set of all *static* stationary states. Let us mention that this set could be infinite and contain continuous components.

In [Kom03] and [KK07a], the attraction to the set of solitary waves (see Fig. 1) is proved for the Klein–Gordon field coupled to a nonlinear oscillator. In [KK07b], this result has been generalized for the Klein–Gordon field coupled to several oscillators. The paper [KK08] gives the extension to the higher-dimensional setting for a model with the nonlinear self-interaction of the mean field type.

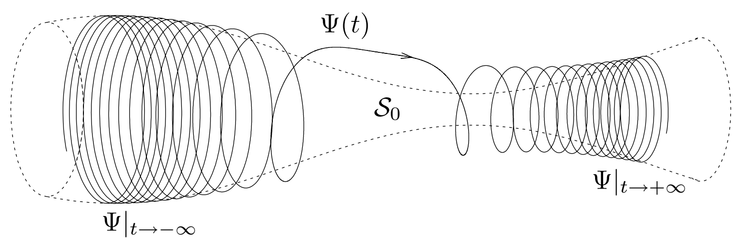

FIGURE 1. For $t \to \pm\infty$, a finite energy solution $\Psi(t)$ approaches (in local energy seminorms) the global attractor $\mathscr{A}$ which coincides with the set of all solitary waves $\mathcal{S}_0$.

Let us mention one more recent advance, [Tao07], in the field of nontrivial (nonzero) global attractors for Hamiltonian PDEs. In that paper, the global attraction for the nonlinear Schrödinger equation in dimensions $n \geq 5$ was considered. The dispersive (outgoing) wave was explicitly specified using the rapid decay of local energy in higher dimensions. The global attractor was proved to be compact, but it was not identified with the set of solitary waves [Tao07, Remark 1.18].

Relation to Quantum Physics. The Quantum Mechanics is formulated in terms of partial differential equations: coupled nonlinear Maxwell-Schrodinger, Maxwell-Dirac, Maxwell-Yang-Mills equations, etc. The Quantum Field Theory is formulated in terms of the correponding second quantized equations. The main goal of our workshop was to achieve the critical concentration of experts in Quantum Theory on one side and in PDEs on another side, to have a thorough discussion of recent advances in both areas and an exchange which could stimulate further progress.

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Mini-Workshop: Attraction to Solitary Waves and Related Aspects of Physics

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Abstracts

Global attraction to solitary waves

Andrew Comech

Convergence to a global attractor is well-known for dissipative systems, like Navier–Stokes equations (see [BV92, Hen81, Tem97]). For such systems, the global attractor is formed by the *static stationary states*, and the attraction

$$\psi(t) \longrightarrow \{\textit{stationary states}\}$$

only holds for $t \to +\infty$. We would like to know whether dispersive Hamiltonian systems could, in the same spirit, possess finite dimensional global attractors, and whether such attractors are formed by the solitary waves, so that

$$\psi(x,t) \underset{t\to\pm\infty}{\longrightarrow} \mathcal{S}_0 := \{\phi_\omega(x)e^{-i\omega t}\}, \tag{1}$$

or, even better,

$$\psi(x,t) \sim \phi_\pm(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty. \tag{2}$$

Although there is no dissipation per se, we expect that the attraction is caused by certain friction mechanism via the dispersion (local energy decay). Because of the difficulties posed by the system of interacting Maxwell and Dirac (or Schrödinger) fields, we will work with simpler models which share certain key properties of the coupled Maxwell–Dirac or Maxwell–Schrödinger systems. Let us try to single out these key features:

(1) *The system is* $\mathbf{U}(1)$*-invariant.*
This invariance leads to the existence of solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$.
(2) *The linear part of the system has a dispersive character.*
This property provides certain dissipative features in a Hamiltonian system, due to local energy decay via the dispersion mechanism.
(3) *The system is nonlinear.*
The nonlinearity is required to make sure that the superposition of solitary waves is not a part of the attractor. Bohr type transitions to pure eigenstates of the energy operator are impossible in a linear system because of the superposition principle.

We suggest that these are the very features responsible for the global attraction, such as (4), to "quantum stationary states".

We consider the Klein–Gordon equation with the nonlinearity concentrated at the origin:

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{3}$$

Above, $m > 0$ and F is a function describing an oscillator at the point $x = 0$. We assume that the oscillator force F admits a real-valued $\mathbf{U}(1)$-invariant potential,

$$F(\psi) = -\nabla_{\Re\psi,\Im\psi}V(|\psi|^2) = -2\psi V'(|\psi|^2), \qquad \psi \in \mathbb{C}, \quad V \in C^2(\mathbb{R}). \tag{4}$$

Equation (3) possesses the key properties we mentioned above: $\mathbf{U}(1)$-invariance, dispersive character, and the nonlinearity.

Remark 1. *If we identify a complex number $\psi = u + iv \in \mathbb{C}$ with the two-dimensional vector $(u, v) \in \mathbb{R}^2$, then, physically, equation* (3) *describes small crosswise oscillations of the infinite string in three-dimensional space (x, u, v) stretched along the x-axis. The string is subject to the action of an "elastic force" $-m^2\psi(x, t)$ and coupled to a nonlinear oscillator of force $F(\psi)$ attached at the point $x = 0$.*

Let us introduce the phase space $\mathscr{E}$ of finite energy states for equation (3).

Definition 2. $\mathscr{E} = H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ *is the Hilbert space of the states (ψ, π), with the norm*

$$\|(\psi, \pi)\|^2_{\mathscr{E}} := \|\pi\|^2_{L^2} + \|\nabla\psi\|^2_{L^2} + m^2\|\psi\|^2_{L^2}.$$

$\mathscr{E}_{loc}$ *is the space with the norm* $\|(\psi, \pi)\|_{\mathscr{E}_{loc}} = \sum_{R=1}^{\infty} 2^{-R}\|(\psi, \pi)\|_{\mathscr{E},R}$, *where*

$$\|(\psi, \pi)\|^2_{\mathscr{E},R} := \|\pi\|^2_{L^2_R} + \|\nabla\psi\|^2_{L^2_R} + m^2\|\psi\|^2_{L^2_R}, \qquad R > 0.$$

Above, $\|\cdot\|_{L^2_R}$ *the norm in* $L^2(-R, R)$.

Equation (3) can formally be written as a Hamiltonian system

$$\dot{\Psi}(t) = \mathcal{J}\, D\mathcal{H}(\Psi), \qquad \mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \Psi = (\psi, \pi), \tag{5}$$

with the Hamilton functional

$$\mathcal{H}(\psi, \pi) = \frac{1}{2}\int\limits_{\mathbb{R}} \left(|\pi|^2 + |\psi'|^2 + m^2|\psi|^2\right) dx + V(|\psi(0)|^2)$$

and with the phase space $\mathscr{E}$ from Definition 2.

Theorem 3 (Global well-posedness)**.** *Assume that the nonlinearity $F(\psi)$ is given by* (4) *with* $\inf\limits_{z\in\mathbb{C}} V(|z|^2) > -\infty$. *Then:*

- (*i*) *For every $(\psi_0, \pi_0) \in \mathscr{E}$, equation* (3) *with the initial data $(\psi, \dot{\psi})|_{t=0} = (\psi_0, \pi_0)$ has a unique global solution $\psi(t)$ such that $(\psi, \dot{\psi}) \in C(\mathbb{R}, \mathscr{E})$.*
- (*ii*) *The map $W(t):\ (\psi_0, \pi_0) \mapsto (\psi(t), \dot{\psi}(t))$ is continuous in $\mathscr{E}$ for each $t \in \mathbb{R}$.*
- (*iii*) *The energy is conserved:* $\mathcal{H}(\psi(t), \dot{\psi}(t)) = \mathrm{const}$, $t \in \mathbb{R}$.
- (*iv*) *There is the* a priori *bound* $\|(\psi(t), \dot{\psi}(t))\|_{\mathscr{E}} \le C(\psi_0, \pi_0)$, $t \in \mathbb{R}$.
- (*v*) *For* $0 \le \epsilon < 1/4$, $\psi \in C^{(\epsilon)}(\mathbb{R} \times \mathbb{R})$.

Definition 4. *The solitary waves of equation* (3) *are solutions of the form*

$$\psi(x, t) = \phi_\omega(x)e^{-i\omega t}, \qquad \text{where} \quad \omega \in \mathbb{R}, \quad \phi_\omega \in H^1(\mathbb{R}). \tag{6}$$

The solitary manifold is the set $\mathcal{S} = \{(\phi_\omega, -i\omega\phi_\omega)\} \subset \mathscr{E}$.

Let us note that there could be no (nonzero) solitary waves for $|\omega| \ge m$.

Theorem 5 (Main result)**.** *Assume that $F(\psi)$ is given by* (4) *where V is a polynomial of degree at least two (so that F is of polynomial type and strictly nonlinear). Then for any $(\psi_0, \pi_0) \in \mathscr{E}$ the solution $\psi(t)$ to equation* (3) *with the initial data $(\psi, \dot{\psi})|_{t=0} = (\psi_0, \pi_0)$ converges to $\mathcal{S}$:*

$$\lim_{t \to \pm\infty} \operatorname{dist}_{\mathscr{E}_{loc}}((\psi(t), \dot{\psi}(t)), \mathcal{S}) = 0. \tag{7}$$

Above, $\operatorname{dist}_{\mathscr{E}_{loc}}(\Psi, \mathcal{S}) := \inf_{\Phi \in \mathcal{S}} \|\Psi - \Phi\|_{\mathscr{E}_{loc}}$, with $\|\cdot\|_{\mathscr{E}_{loc}}$ from Definition 2.

Let us mention several important points.

(*i*) We prove the attraction of any finite energy solution to the solitary manifold $\mathcal{S}$:
$$(\psi(t), \dot{\psi}(t)) \xrightarrow{\mathscr{E}_{loc}} \mathcal{S}, \qquad t \to \pm\infty,$$
where the convergence holds in local energy seminorms. In this sense, $\mathcal{S}$ is a *weak* (convergence is local in space) *global* (convergence holds for arbitrary initial data) attractor.

(*ii*) $\mathcal{S}$ can be at most a *weak* attractor because we need to keep forgetting about the outgoing dispersive waves, so that the dispersion plays the role of friction. A *strong* attractor would have to consist of the direct sum of $\mathcal{S}$ and the space of outgoing waves.

(*iii*) We interpret the local energy decay caused by dispersion as a certain friction effect in order to clarify the cause of the convergence to the attractor in a Hamiltonian model. This "friction" does not contradict the time reversibility: if the system develops backwards in time, one observes the same local energy decay which leads to the convergence to the attractor as $t \to -\infty$ (unlike for the dissipative equations, for which the convergence to the attractor only holds for $t \to +\infty$).

(*iv*) Although we proved the attraction (1) to $\mathcal{S}$, we have not proved the attraction to a particular solitary wave, falling short of proving (2). Hypothetically, if $\mathcal{S}/\mathbf{U}(1)$ contains continuous components, a solution can be drifting along $\mathcal{S}$, keeping asymptotically close to it, but never approaching a particular solitary wave. (This could be viewed as the adiabatic modulation of solitary wave parameters.) Apparently, if $\mathcal{S}/\mathbf{U}(1)$ is discrete, a solution converges to a particular solitary wave.

(*v*) The requirement that the nonlinearity is polynomial allows us to apply the Titchmarsh convolution theorem. This step is vital in our approach. We do not know whether the polynomiality requirement could be dropped.

(*vi*) In the linear case, there can be no attraction to $\mathcal{S}$ since the global attractor contains the linear span of points of the solitary manifold: $\mathcal{A} \supset \langle \mathcal{S} \rangle \supsetneq \mathcal{S}$.

(*vii*) For the real initial data, we obtain a real-valued solution $\psi(t)$. Therefore, the convergence (7) of $\Psi(t) = (\psi(t), \dot{\psi}(t))$ to the set of pairs $(\phi_\omega, -i\omega\phi_\omega)$ with $\omega \in \mathbb{R}\backslash\{0\}$ implies that $\psi(t)$ locally converges to zero.

(*viii*) Our argument does not apply to the Schrödinger equation. The important feature of the Klein–Gordon equation is that the continuous spectrum corresponds to $|\omega| \geq m$, hence the spectral density of the solution is absolutely continuous for $|\omega| \geq m$, while the spectrum of the omega-limit trajectory is within the compact set $[-m, m]$. This is not so for the Schrödinger equation: since the continuous spectrum corresponds to $\omega \geq 0$, the resulting restriction on the spectrum of the omega-limit trajectory is $\omega \leq 0$. As a result, we do not know whether the spectrum is compact; the Titchmarsh convolution theorem does not apply, and the proof breaks down.

Sketch of the proof. First, we introduce a concept of the omega-limit trajectory $\beta(x, t)$ which plays a crucial role in the proof.

Definition 6. *The function* $\beta(x, t)$ *is called* omega-limit trajectory *if there is a global solution* $\psi \in C(\mathbb{R}, H^1)$ *and a sequence of times* $\{t_j \colon j \in \mathbb{N}\}$ *with* $\lim\limits_{j\to\infty} t_j = \infty$ *such that*

$$\psi(x, t + t_j) \underset{j\to\infty}{\longrightarrow} \beta(x, t),$$

where the convergence is in $C_b([-T, T] \times [-R, R])$ *for any* $T > 0$ *and* $R > 0$.

We are going to prove that all omega-limit trajectories are solitary waves; that is, that $\beta(x, t) = \phi_\omega(x)e^{-i\omega t}$, for some $\omega \in \mathbb{R}$ and $\phi_\omega \in H^1(\mathbb{R})$. It suffices to prove that the time spectrum of any omega-limit trajectory β consists of at most one frequency.

To complete this program, we study the time spectrum of solutions, that is, their complex Fourier–Laplace transform in time. First, we prove that the spectral density of a solution is absolutely continuous for $|\omega| > m$ hence the corresponding component of the solution disperses completely. It follows that the time-spectrum of omega-limit trajectory β is contained in a finite interval $[-m, m]$.

Second, we notice that β also satisfies the original nonlinear equation. Since the spectral support of β is compact and the nonlinearity is polynomial, we may apply the Titchmarsh convolution theorem. This theorem allows to conclude that the spectral support of the nonlinearity would be strictly larger than the spectral support of the linear terms in the equation (which would be a contradiction!) except in the case when the spectrum of the omega-limit trajectory consists of a single frequency $\omega_+ \in [-m, m]$.

Thus, any omega-limit trajectory is a solitary wave. We conclude that any finite energy solution converges to the set of solitary waves (but not necessarily to a particular wave).

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On asymptotic stability in energy space of ground states for Nonlinear Schrödinger equations

Scipio Cuccagna

We consider the nonlinear Schrödinger equation

$$iu_t + \Delta u + \beta(|u|^2)u = 0\,,\ (t,x) \in \mathbb{R} \times \mathbb{R}^3,$$

with the initial data $u(0,x) = u_0(x)$. We assume $\beta(0) = 0$, β sufficiently regular and with not too fast growth at ∞. We consider standing waves $e^{it\omega}\phi_\omega$ and the question of asymptotic stability of ground states, i.e. the case $\phi_\omega(x) > 0$, and of orbital stability of excited standing waves, i.e. when $\phi_\omega(x)$ changes signs. Our analysis is based on the linearization

$$H_\omega = \sigma_3 \left[-\Delta + \omega - \beta(\phi_\omega^2) - \beta'(\phi_\omega^2)\phi_\omega^2\right] + i\beta'(\phi_\omega^2)\phi_\omega^2\sigma_2$$

where σ_j are Pauli matrices. We introduce a more stringent notion of linear stability. The standard notion consists in the requirement that $\sigma(H_\omega) \subset \mathbb{R}$. We add two more requirements. We require that the generalized kernel $N_g(H_\omega)$ be non-degenerate and that the "signature" of all positive eigenvalues be positive:

(1) $N_g(H_\omega - \lambda) = \ker(H_\omega - \lambda)$;
(2) For any $\xi \in \ker(H_\omega - \lambda)$ with $\xi \neq 0$ we have $\langle \xi, \sigma_3\xi\rangle > 0$.

This definition of linear stability is essentially the classical one in the case of ground states, because in that case H_ω has no positive eigenvalues of negative signature. However we prove:

Excited states are never linearly stable in the above more stringent sense.

We then conjecture that the above stringent definition of linear stability is a necessary condition for orbital stability. We prove the conjecture, if we assume the Fermi Golden Rule (FGR), see later. Next, we conjecture that orbitally stable ground states are asymptotically stable, in the sense that for a finite energy solution $u(t,x)$ close to ground states we have for some $\omega_0 > 0$, $\theta(t) \in C^1(\mathbb{R}^+, \mathbb{R})$ and $h_0(x) \in H^1_x$ (we restrict attention to symmetric solutions $u(t,-x) = u(t,x)$),

$$\lim_{t\to+\infty} \|u(t,x) - e^{i\theta(t)}\phi_{\omega_0}(x) - e^{it\Delta}h_0\|_{H^1_x} = 0.$$

Also this conjecture is proved, under some generic conditions, assuming the FGR. The problem of asymptotic stability is classical, but the first results are of the early 90's by Soffer & Weinstein and by Buslaev & Perelman. It is standard to consider an ansatz

$$u(t,x) = e^{i\int_0^t \omega(s)ds + i\gamma(t)}(\phi_{\omega(t)}(x) + r(t,x))$$

with ω and γ determined by modulation. One then splits r in terms of the spectral decomposition of the linearization H_ω. The discrete modes which I will denote with $z_j(t)$ satisfy a system which is a perturbation of what looks like a conservative system in the z's. The coupling with the continuous modes is responsible for the asymptotic stability or for the orbital instability. Ideally after some changes of variables we have for the continuous modes

$$i\partial_t f \approx H_\omega f + z^{N+1} R_{N+1,0}(\omega)$$

and for the discrete modes (where ξ generates $\ker(H_\omega - \lambda)$ and is normalized)

$$i\dot{z}\xi - \lambda(\omega) z\xi = \overline{z}^N A_{0,N}(\omega) f.$$

(This as an illustration only, i.e. the interesting case involves more than one discrete modes.) Ideally, if the system is Hamiltonian (which is not any more because the changes of variables are not canonical) $R_{N+1,0}(\omega)$ and $A_{N,0}(\omega)$ are derivatives of the Hamiltonian. We introduce

$$z^{N+1} R_{H_\omega}((N+1)\lambda(\omega) + i0) R_{N+1,0}(\omega) + f$$

and, after substitution, we get

$$i\dot{z}\xi - \lambda(\omega) z\xi = |z|^{2N} z A_{0,N}(\omega) R_{H_\omega}((N+1)\lambda(\omega) + i0) R_{N+1,0}(\omega).$$

We apply $\overline{z}\langle \quad, \sigma_3\xi\rangle$ and get

$$\frac{1}{2}\frac{d}{dt}|z|^2 = \mp\pi \langle \delta(H_\omega - (N+1)\lambda) R_{N+1,0}, A_{0,N}^* \sigma_3 \xi\rangle |z|^{2N+2}$$

where we have the negative sign if λ has positive signature (linear stability), and the positive sign if λ has negative signature (linear instability). Since the FGR conjecture claims that

$$\langle \delta(H_\omega - (N+1)\lambda) R_{N+1,0}, A_{0,N}^* \sigma_3 \xi\rangle \geq 0,$$

when the latter is nonzero we get dissipation of z for linear stability and excitation of z for linear instability. The FGR should follow from the Hamiltonian structure of the NLS but is yet unproven. Basically, the FGR should be true because $R_{N+1,0}$ and $\sigma_3 A_{0,N}^* \sigma_3 \xi$ should be like derivatives of a Hamiltonian differing only by the order of differentiation, and so should be equal (up to some factorial factor). For the details, see [1].

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Scattering theory in nonrelativistic quantum theory

Jan Dereziński

My minicourse was devoted to scattering theory for a certain class of Hamiltonians motivated by nonrelativistic quantum field theory. The plan of my minicourse was as follows:

(1) Basic abstract scattering theory.
(2) Scattering of 2-body Schrödinger operators.
(3) Second quantization.
(4) Nonrelativistic QED.
(5) Scattering of Hamiltonians of QFT.
(6) Scattering of van Hove Hamiltonians.
(7) Spectrum of Pauli-Fierz Hamiltonians.
(8) Scattering of Pauli-Fierz Hamiltonians I.
(9) Representations of the CCR.
(10) Scattering of Pauli-Fierz Hamiltonians II.

Section 1 described definitions of Møller and scattering operators and their basic properties. The standard definitions usually do not work in the case of quantum field theory. Therefore some non-standard definitions were also discussed, such as Abelian Møller operators.

Section 2 was devoted to scattering theory for Schrödinger operators. This is the best known example of the application of the idea of scattering.

Section 3 and 4 were devoted to the formalism of second quantization (Fock spaces, creation/annihilation operators, etc.) and its applications.

In Section 5, I introduced two basic formalisms of scattering theory in quantum field theory with localized interactions. The first is based on the renormalization of Abelian wave operators. The second is the so-called LSZ formalism, which uses the so-called asymptotic fields [Fr, H, Sch].

In Section 6, I discussed scattering theory for a simple but instructive class of models, the so-called van Hove Hamiltonians [De]. They are Hamiltonians on a bosonic Fock space of the form

$$H = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \bar{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi.$$

In the case of van Hove Hamiltonians both approaches to scattering theory mentioned above are possible: one can renormalize the Abelian Møller operators or one can use the LSZ approach and construct asymptotic fields. The latter approach works also in the case of the infrared problem, when the Abelian Møller operators are zero and hence of no use, but asymptotic fields exist. The infrared problem is expressed by the fact that the representations of the asymptotic fields are non-Fock.

In Section 7 and 8, I discussed basic elements of scattering theory for the so-called Pauli-Fierz Hamiltonians, that is Hamiltonians of the form

$$\begin{aligned} H &= K \otimes 1 + 1 \otimes \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi \\ &\quad + \int v(\xi) \otimes a_\xi^* \mathrm{d}\xi + \mathrm{hc}, \end{aligned}$$

H is a self-adjoint operator on the tensor product of the Hilbert space of a small quantum system and a bosonic Fock space. An appropriately modified version of the LSZ formalism works in this case. If the single-boson kinetic energy is massive, that is if it has a mass gap, then under very general assumptions one can prove a kind of the asymptotic completeness [DG1], see also [DG2, FGS]. More precisely, one can show that the asymptotic representations of the CCR are Fock and all the Fock vacua coincide with linear combinations of bound states.

In the massless case the asymptotic completeness may break down, which is related to a possible appearance of the so-called non-Fock representations of the CCR [DG3]. This possibility was discussed in the last two sections of the mini-course.

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Stability of Solitary Waves For The Hartree Type Equations

Vladimir Georgiev

(joint work with Jimmy Mauro and George Venkov)

In this note we study the 3-D Hartree equation (with external potential $V(x) = -\frac{e^2}{|x|}$)

$$i\partial_t \psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + \left(e^2 \int_{\mathbb{R}^3} \frac{|\psi(t,y)|^2}{|x-y|} dy + V(x) \right) \psi(t,x). \tag{1}$$

We impose the following initial data

$$\psi(0,x) = \chi(x) + u_0(x), \tag{2}$$

where $\chi(x)$ is such that

$$\psi_s(t,x) = e^{-i\omega t}\chi(x)$$

is a solitary wave, i.e. $\chi(x)$ is a rapidly decaying smooth function, so that $\psi_s(t,x)$ is a solution of (1).

The existence of nontrivial radial solutions to this problem is constructed in [1] for the case, when $\|\psi_s(t)\|_{L^2} = \|\chi(t)\|_{L^2} = 1$. The function $u_0(x)$ is assumed to be compactly supported in the Sobolev space $H^s(\mathbb{R}^3)$, where $s < 3/2$ is sufficiently close to $3/2$ and its H^s norm is sufficiently small. We shall look for solution of the Cauchy problem (1) of type

$$\psi(t,x) = e^{-i\omega t}\chi(x) + u(t,x).$$

Consider the operator

$$f \in \mathcal{S}(\mathbb{R}^3) \Rightarrow q(f)(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$$

and

$$\begin{aligned} Q(f)(x) &= \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy - \frac{1}{|x|} \int_{\mathbb{R}^3} f(y) dy \\ &= q(f)(x) - \frac{1}{|x|} \int_{\mathbb{R}^3} f(y) dy \end{aligned}$$

and extend these operators in the Sobolev space $W^{1,1}(\mathbb{R}^3)$.

The Hartree equation (1) for a function $\psi(t,x) = u(t,x) + \psi_s(t,x) = u(t,x) + e^{-i\omega t}\chi(x)$ is

$$i\partial_t \psi + \frac{1}{2}\triangle\psi = bQ(|\psi|^2)\psi, \tag{3}$$

where $b = e^2 > 0$ will be considered as a sufficiently small constant.

Then u solves the equation

$$i\partial_t u + \frac{1}{2}\triangle u = bQ(|u+\psi_s|^2)(u+\psi_s) - bQ(|\psi_s|^2)\psi_s,$$

since $\psi_s(t,x) = e^{-i\omega t}\chi(x)$ is a solitary wave and $\chi(x) \in \mathcal{S}(\mathbb{R}^3)$.

We shall choose initial data for $u(t)$ as follows

$$u(0) = u_0 \in H^s(\mathbf{R^3}),\ 1 < s < 3/2,\ \ \|u_0\|_{H^s} \leq \delta,$$

$\delta > 0$ small enough, u_0 is compactly supported and the charge conservation law

$$\|u_0 + \chi\|_{L^2(\mathbb{R}^3)} = \|\chi\|_{L^2(\mathbb{R}^3)} \tag{4}$$

is satisfied.

Our main stability result is the following.

Theorem. *Let $\chi(x)$ be a smooth rapidly decaying solution such that $\psi_s(t,x) = e^{-i\omega t}\chi(x)$ is a solitary solution of* (1)*. There exist small positive numbers $\delta > 0$ and $b_0 > 0$, so that for any compactly supported initial data*

$$u_0 \in H^s(\mathbf{R^3}),\ 1 < s < 3/2, \quad \|u_0\|_{H^s} \leq \delta$$

satisfying (4)*, the Cauchy problem for* (3) *with $0 < b \leq b_0$ has a global solution*

$$\psi(t,x) = e^{-i\omega t}\chi(x) + u(t,x),$$

where

$$u(t,x) \in C(\mathbb{R}_t; H^s),$$

and $u(t,x)$ satisfies the dispersive estimate

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{L^p} = 0 \tag{5}$$

for some $p, 2 < p < 5$.

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Analytic Perturbation Theory and Renormalization Analysis of Matter Coupled to Quantized Radiation

Marcel Griesemer
(joint work with David Hasler)

When a neutral atom or molecule made from static nuclei and non-relativistic electrons is coupled to the (UV-cutoff) quantized radiation field, the least point of the energy spectrum becomes embedded in the continuous spectrum due to the absence of a photon mass, but it remains an eigenvalue [8, 10]. This *ground state energy* E depends on the parameters of the system, such as the fine-structure constant, the positions of static nuclei, or, in the center of mass frame of a translation invariant model, the total momentum. The regularity of E as a function of these parameters is of fundamental importance. For example, the accuracy of the Born-Oppenheimer approximation, a pillar of quantum chemistry, depends on the regularity of E and on the regularity of the ground state projection as functions of the nuclear coordinates. If E were an isolated eigenvalue, like it is in quantum mechanical description of molecules without radiation, then analyticity of E with respect to any of the aforementioned parameters would follow from regular perturbation theory. But in QED the energy E is not isolated and the analysis of its regularity is a difficult mathematical problem.

In this talk we report on a recently discovered solution to the above problem of regularity in a large class of models of matter and radiation where the Hamiltonian $H(s)$ depends analytically on complex parameters $s = (s_1, \dots, s_\nu) \in \mathbb{C}^\nu$ from a complex neighborhood of a compact set $K \subset \mathbb{R}^\nu$. Important properties of $H(s)$ are, that $H(\bar{s}) = H(s)^*$ and that, for $s \in K$, the lowest point, $E(s)$, of the spectrum of $H(s)$ is a non-degenerate eigenvalue. These and some further assumptions concerning the class of admissible Hamiltonians imply that the eigenvalue $E(s)$ and the projection operator associated with the eigenspace of $E(s)$ are *real-analytic* functions of s in a neighborhood of K [7]. In particular, they are of class C^∞ in this neighborhood. We apply this result to the Hamiltonian of a molecule with static nuclei and non-relativistic electrons that are coupled to the quantized radiation field in the dipole approximation. In suitably chosen atomic units, this Hamiltonian depends on the fine-structure constant α only though a factor of $\alpha^{3/2}$ in front of the dipole interaction operator. Hence the role of s may be played by $\alpha^{3/2}$, or, after a well-known unitary deformation argument [9], by the nuclear coordinates. It follows that the ground state energy, if it is a non-degenerate eigenvalue, depends analytically on $\alpha^{3/2}$ and the nuclear coordinates. The ground state projection is analytic in $\alpha^{3/2}$ and twice continuously differentiable with respect to the nuclear coordinates [7]. We remark that the dipole approximation seems necessary for the analyticity with respect to a power of α [4].

A further consequence of the new theorem concerns the accuracy of the adiabatic approximation to the time evolution U_τ generated by the Schrödinger equation

$$i\frac{d}{dt}\varphi_t = H(t/\tau)\varphi_t, \qquad t \in [0,\tau],$$

in the limit $\tau \to \infty$. If $H(s)$ satisfies the assumptions described above with $K = [0,1]$, then the ground state projection $P(s)$ is of class $C^\infty([0,1])$ and hence the adiabatic theorem without gap assumption implies that $\sup_{t\in[0,\tau]} \|(1-P(t))U_\tau(t)P(0)\| = o(1)$ as $\tau \to \infty$ [12, 2]. Previously, in all applications of the adiabatic theorem without gap assumption the differentiability of $P(s)$ was enforced or provided by the special form $H(s) = U(s)HU(s)^{-1}$ of $H(s)$ where $U(s)$ is a unitary and (strongly) differentiable operator [1, 2, 11].

The proof of the new theorem is based on the renormalization technique of Bach et al. [3, 5], in a new version taken from [6]. Like the authors of [6] we use a simplified renormalization map that consists of a Feshbach-Schur map and a scaling transformation only. In the corresponding spectral analysis the Hamiltonian is diagonalized, with respect to H_f, in a infinite sequence of renormalization steps. In each step the off-diagonal part becomes smaller, and the spectral parameter is adjusted to enforce convergence of the diagonal part. This method provides a fairly explicit construction of an eigenvector of $H(s)$, even for complex s, where $H(s)$ is not self-adjoint. We argue first, that the parameters of the renormalization analysis can be chosen independent of s and g in neighborhoods of $K \subset \mathbb{R}^n$ and $g = 0$, second, that all steps of the renormalization analysis preserve analyticity, and third, that all limits taken are uniform in s, which implies analyticity of the limiting functions.

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On asymptotic stability of solitary waves for Schödinger equation coupled to nonlinear oscillator

Elena Kopylova

(joint work with Alexander Komech and David Stuart)

We study the large time asymptotics for a model $U(1)$-invariant nonlinear Schrödinger equation

$$i\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)F(\psi(0,t)), \quad x \in \mathbb{R}, \tag{1}$$

Here $\psi(x,t)$ is a continuous complex-valued wave function and F is a continuous function. Our main focus is on the role that certain solitary waves (nonlinear bound states) play in the description of the solution for large times. In [1] the asymptotic stability of solitary waves was proved under a condition on the nonlinearity which ensures that the linearization about the solitary wave consists entirely of continuous spectrum, except for the two dimensional generalized null space which is always present due to the $U(1)$ symmetry of the equation. Now this result is extended to the case that the spectrum of the linearization includes an additional discrete component, which satisfies a non-degeneracy condition related to the Fermi Golden rule.

It will be convenient to rewrite (1) in real form: we identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector $(\psi_1, \psi_2) \in \mathbb{R}^2$ and rewrite (1) in the vectorial form

$$j\dot{\psi}(x,t) = -\psi''(x,t) - \delta(x)\mathbf{F}(\psi(0,t)), \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

where $\mathbf{F}(\psi) \in \mathbb{R}^2$ is the real vector version of $F(\psi) \in \mathbb{C}$.

We assume that the oscillator force $\mathbf{F}$ admits a real-valued potential

$$\mathbf{F}(\psi) = -\nabla U(\psi), \quad \psi \in \mathbb{R}^2, \quad U \in C^2(\mathbb{R}^2). \tag{3}$$

Then (2) is formally a Hamiltonian system with Hamiltonian

$$\mathcal{H}(\psi) = \frac{1}{2} \int |\psi'|^2 \, dx + U(\psi(0))$$

which is conserved for sufficiently regular finite energy solutions. We assume that the potential $U(\psi)$ satisfies the inequality

$$U(z) \geq A - B|z|^2 \quad \text{with some} \quad A \in \mathbb{R}, \quad B > 0. \tag{4}$$

We also assume that $U(\psi) = u(|\psi|^2)$ with $u \in C^2(\mathbb{R})$. Therefore, by (3),

$$F(\psi) = a(|\psi|^2)\psi, \quad \psi \in \mathbb{C}, \quad a \in C^1(\mathbb{R}), \tag{5}$$

where $a(|\psi|^2)$ is real.Then $F(e^{i\theta}\psi) = e^{i\theta}F(\psi), \quad \theta \in [0, 2\pi]$, and $e^{i\theta}\psi(x,t)$ is a solution to (1) if $\psi(x,t)$ is. Therefore, equation (1) is $U(1)$-invariant. Under these conditions the existence of global solutions to the Cauchy problem for (1) was proved in [2].

Equation (1) admits finite energy solutions of type $\psi_\omega(x)e^{i\omega t}$, called *solitary waves* or *nonlinear eigenfunctions*. The frequency ω and the amplitude $\psi_\omega(x)$ solve the following *nonlinear eigenvalue problem*:

$$-\omega\psi_\omega(x) = -\psi''_\omega(x) - \delta(x)F(\psi_\omega(0)), \quad x \in \mathbb{R}.$$

The set of all nonzero solitary waves consists of functions $C(\omega)e^{-\sqrt{\omega}|x|+i\theta}$, $C > 0$, $\omega > 0$, where

$$\sqrt{\omega} = a(C^2)/2 > 0,$$

and where $\theta \in [0, 2\pi]$ is arbitrary. The real form of the solitary wave is $e^{j\theta}\Phi_\omega$ where $\Phi_\omega = (\psi_\omega(x), 0)$.

Linearization at the solitary wave $e^{j\theta}\Phi_\omega$ leads to the operator

$$\mathbf{B} = -\frac{d^2}{dx^2} + \omega - \delta(x)[a(C^2) + 2a'(C^2)C^2 P_1] = \begin{pmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{pmatrix},$$

where P_1 is the projector in $\mathbb{R}^2$ acting as $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix}$,

$$\mathbf{D}_1 = -\frac{d^2}{dx^2} + \omega - \delta(x)[a + 2a'C^2], \quad \mathbf{D}_2 = -\frac{d^2}{dx^2} + \omega - \delta(x)a.$$

Let $\mathbf{C} = j^{-1}\mathbf{B}$. The continuous spectrum of $\mathbf{C}$ coincides with $(-i\infty, -i\omega] \cup [i\omega, i\infty)$. Previously, in [1] we considered the case when

$$a'(C^2) \in (-\infty, 0) \cup (0, a(C^2)/(\sqrt{2}C^2)).$$

In this case operator $\mathbf{C}$ has no discrete spectrum except zero Under this condition we proved asymptotic stability for initial data close to a solitary wave both in the energy norm and in the weighted Banach norm, L^p_β, defined by, $\|u\|_{L^p_\beta} = \|(1+|x|)^\beta u(x)\|_{L^p}$.

Now we consider the case when

$$a' \in \left(\frac{a}{\sqrt{2}C^2}, \frac{a\sqrt{2}(1+\sqrt{3})}{4C^2}\right). \tag{6}$$

In this case, there are, in addition, 2 simple eigenvalues $\pm i\mu$, which satisfy the property $2\mu > \omega$. If assumption (6) is true for a fixed value ω_0, it also true for values of ω in a small interval centered at ω_0. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $u^* := \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$ be eigenvectors of $\mathbf{C}$ associated to $i\mu$ and $-i\mu$ respectively, where the function $u_1(x)$ is real, and $u_2(x)$ is purely imaginary.

We consider the initial value ψ_0 to be of the form

$$\psi_0(x) = \Phi_{\omega_0}(x) + z_0 u(x,\omega_0) + \overline{z}_0 u^*(x,\omega_0) + f_0(x), \tag{7}$$

where f_0 belongs to the eigenspace associated to the continuous spectrum of $\mathbf{C}(\omega_0)$. We assume that z_0 and f_0 are sufficiently small. We also assume a non-degeneracy condition. Let $\langle\cdot,\cdot\rangle$ denote the Hermitian scalar product in L^2 of $\mathbb{C}^2$-valued function, and $(u,v) = u_1 v_1 + u_2 v_2$ for $u, v \in \mathbb{C}^2$. Let $E_2[f,f]$ be the quadratic terms coming from the Taylor expansion of the nonlinearity:

$$E_2[f,f] = \delta(x)[a'(C^2)(f,f)\Phi_\omega + 2a''(C^2)(\Phi_\omega, f)^2\Phi_\omega + 2a'(C^2)(\Phi_\omega, f)f],$$

where $f \in \mathbb{C}^2$. The non-degeneracy condition has the form

$$\langle E_2[u(\omega_0), u(\omega_0)], \tau_+(2i\mu_0)\rangle \neq 0, \tag{8}$$

where $\tau_+(2i\mu_0)$ is the eigenfunction associated to $2i\mu_0 = 2i\mu(\omega_0)$. This condition will be referred to as a nonlinear version of the Fermi Golden rule.

Our main theorem is following:

Theorem 7. *Let conditions* (3), (4), *and* (5) *hold,* $\beta > 2$ *and* $\psi(x,t) \in C(\mathbb{R}, H^1)$ *be the solution to the equation* (2) *with initial value* $\psi_0(x) = \psi(x,0) \in H^1 \cap L^1_\beta$ *of the form* (7) *which is close to a solitary wave* Φ_{ω_0}*:*

$$|z(0)| \le \varepsilon^{1/2}, \quad \|f_0\|_{L^1_\beta} \le c\varepsilon^{3/2}. \tag{9}$$

Assume further that the spectral condition (6) *and the non-degeneracy condition* (8) *hold for the solitary wave with* $C = C_0$*. Then for* ε *sufficiently small the solution admits the following scattering asymptotics in* $C_b(\mathbb{R}) \cap L^2(\mathbb{R})$*:*

$$\begin{aligned}\psi(x,t) = e^{j\varphi_\pm(t)}[\Phi_{\omega_\pm}(x) &+ z_\pm(t)u(x,\omega_\pm) \\ &+ \overline{z}_\pm(t)u^*(x,\omega_\pm)] + e^{j^{-1}Lt}\Psi_\pm + \mathcal{O}(t^{-\nu}), \quad t \to \pm\infty,\end{aligned} \tag{10}$$

with some $\nu > 0$*, where* $L = -\frac{\partial^2}{\partial x^2}$*,* $\Psi_\pm \in C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ *are the corresponding asymptotic scattering states,* $\varphi_\pm(t) = \omega_\pm t + p_\pm \log(1 + k_\pm t) + \varkappa_\pm$*,* $\omega_\pm$*,* $p_\pm$*,* $k_\pm$*,* $\varkappa_\pm$ *are constants.*

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Multipole Radiation in Classical Particle Systems

Markus Kunze

1. Models of Individual Particles. The motion of N charged classical particles in $\mathbb{R}^3$ under the influence of their self-generated electromagnetic fields can be described by the Abraham-Lorentz system

$$\frac{d}{dt}(m_{b\alpha}\gamma_\alpha v_\alpha(t)) = e_\alpha\Big(E_\varphi(q_\alpha(t),t) + c^{-1}v_\alpha(t)\wedge B_\varphi(q_\alpha(t),t)\Big),\ \ 1\le\alpha\le N,$$
$$c^{-1}\partial_t B = -\nabla\wedge E,\quad c^{-1}\partial_t E = \nabla\wedge B - c^{-1}j,\quad \nabla\cdot E=\rho,\quad \nabla\cdot B = 0,$$
$$\rho(x,t) = \sum_{\alpha=1}^N e_\alpha\varphi(x-q_\alpha(t)),\quad j(x,t) = \sum_{\alpha=1}^N e_\alpha\varphi(x-q_\alpha(t))v_\alpha(t),$$

where $v_\alpha(t) = \dot q_\alpha(t)$, $\gamma_\alpha = (1-(v_\alpha/c)^2)^{-1/2}$, and $m_{b\alpha}$ and e_α denote the bare mass and charge of the α'th particle, respectively. In order to avoid infinities of the self-energy of the particles, they are smeared out by a smooth form factor $\varphi = \varphi(x)$ of support radius R_φ, so that $E_\varphi(x,t) = \int\varphi(x-x')E(x',t)\,dx'$ and $B_\varphi(x,t) = \int\varphi(x-x')B(x',t)\,dx'$. Due to presence of the rigid charge distribution introduced by φ, the model is no longer covariant. However, as compared to the covariant models proposed in [9, 1], it is much easier accessible analytically. We require that initially the particles are far apart ($\sim\varepsilon^{-1}R_\varphi$) and move slowly ($\sim\varepsilon c$). Our aim is to derive an effective ODE whose solutions, as $\varepsilon\to 0$, very well approximate the full dynamics of the original PDE over long times. In order to do so, the Maxwell equations are rewritten as wave equations, e.g. $\Box E = -(\partial_t j + \nabla\rho)$ taking $c=1$. If the retarded solution is written out explicitly and put into the Lorentz force, then the terms showing up on the right-hand side are roughly of the form $\sum_{\beta=1}^N\int_0^t ds\,\ldots\rho(\ldots+q_\alpha(t)-q_\beta(s))$. The contribution of $\beta=\alpha$ accounts for the self-force of particle α, whereas the parts stemming from indices $\beta\neq\alpha$ are due to the interaction forces. In the following we consider the self-force only. Passing to the system rescaled by ε as mentioned before, the Taylor expansion of

$$q_\alpha(t) - q_\alpha(s) = -v_\alpha(t)(t-s) + \frac12\ddot q_\alpha(t)(t-s)^2 - \frac16\dddot q_\alpha(t)(t-s)^3 + \ldots$$

can be rigorously justified by estimating the errors [8]. Including only $-v_\alpha(t)(t-s)$ and integrating out the retarded integral $\int_0^t ds\,\ldots$ leads to an order zero effective ODE where the effective motion is governed by a Coulomb potential. To the next order (called the Darwin order), an effective velocity-dependent mass M_α shows up, since now additionally the term $\frac12\ddot q_\alpha(t)(t-s)^2$ has to be taken into account.

The radiation order is reached as soon as also $-\frac{1}{6}\dddot{q}_\alpha(t)(t-s)^3$ is included into the expansion. After the rescaling, the effective equation can be symbolically represented in the form $M_\alpha\ddot{q}_\alpha = -\nabla V(q) + \varepsilon\dddot{q}_\alpha$. It turns out that this equation admits very many undesirable solutions that run off to infinity very fast, the so-called run-away solutions [12]. Moreover, since it is a third-order equation and as it has to be compared to the solutions of the original second-order system, it is not clear how the initial data $\ddot{q}_\alpha(0)$ are to be specified. These problems have been resolved in [8]. It has been shown by using geometric singular perturbation theory that there is a center-like manifold $\mathcal{M}_\varepsilon = \{(q, v, h_\varepsilon(q, v)) : q, v \in \mathbb{R}^{3N}\}$, $\dim \mathcal{M}_\varepsilon = 6N$, in the phase space $\mathbb{R}^{9N}$ with the following properties: (i) $\mathcal{M}_\varepsilon$ is invariant under the flow of the effective equation, (ii) solutions not on $\mathcal{M}_\varepsilon$ tend to infinity very fast, (iii) if $\ddot{q}(0) = h_\varepsilon(q(0), \dot{q}(0))$ is chosen as the initial acceleration, then the effective solution approximates the full solution to good accuracy in ε over long times ($\sim \varepsilon^{-1}$), and (iv) there is an energy-like quantity (Schott energy) that decreases along the solutions on $\mathcal{M}_\varepsilon$, making manifest the radiative character of the effective equation. See [11] for much more background on this problem that basically goes back to Dirac and others.

2. Kinetic Models. If the number of charged particles is large, it is appropriate to pass to a kinetic description of the matter. For instance, one can consider the relativistic Vlasov-Maxwell system

$$\begin{aligned} &\partial_t f^\pm + v\cdot\nabla_x f^\pm \pm (E + c^{-1}v\wedge B)\cdot\nabla_p f^\pm = 0,\\ &c^{-1}\partial_t B = -\nabla\wedge E, \quad c^{-1}\partial_t E = \nabla\wedge B - c^{-1}j, \quad \nabla\cdot E = \rho, \quad \nabla\cdot B = 0,\\ &\rho = \int (f^+ - f^-)\,dp, \quad j = \int v(f^+ - f^-)\,dp, \quad v = (1 + c^{-2}|p|^2)^{-1/2}p, \end{aligned}$$

for two particle species $\pm$ of opposite unity charge and equal unity mass, the species being described by the phase space densities $f^\pm = f^\pm(t, x, p)$. We intend to derive an effective equation approximating the full system in the limit of slow motion $c \to \infty$. We start with a formal expansion of all quantities in powers of c^{-1}, $f^\pm = f_0^\pm + c^{-1}f_1^\pm + c^{-2}f_2^\pm + c^{-3}f_3^\pm + \ldots$, similarly for E, B, ρ, and j, and also $v = p - (c^{-2}/2)p^2p + \ldots$. Comparing the powers of c^{-1} yields a hierarchy of equations for the coefficient functions $f_j^\pm$ etc. At the order zero, the Vlasov-Poisson system of plasma physics

$$\partial_t f_0^\pm + p\cdot\nabla_x f_0^\pm \pm E_0\cdot\nabla_p f_0^\pm = 0, \quad E_0(t, x) = -\int |z|^{-2}\bar{z}\,\rho_0(t, x+z)\,dz,$$

is found, where $\bar{z} = |z|^{-1}z$ and $\rho_0 = \int(f_0^+ - f_0^-)\,dp$. This limit has been made rigorous in [10]. Note that due to the absence of a large data existence result for the Vlasov-Maxwell system, already some effort has to be put into the issue of proving the existence of solutions on some time interval $[0, T]$ that does not shrink as $c \to \infty$. Next it can be observed that one can take $f_{2l+1}^\pm = 0, \quad E_{2l+1} = 0, \quad B_{2l} = 0$ consistently in the hierarchy of equations for the coefficient functions. Then in [4] it has been shown that by passing to the 'Darwin approximation' $f^{\pm,\mathrm{D}} = f_0^\pm + c^{-2}f_2^\pm$, $E^\mathrm{D} = E_0 + c^{-2}E_2$, $B^\mathrm{D} = c^{-1}B_1$, the order of accuracy can be improved to $O(c^{-3})$. It comes as a certain surprise that this approach does not work any more at the next order, where radiation effects are known to occur. This is due to the fact that the resulting effective system would still be Hamiltonian, not accounting for the energy loss that is present in the system. To get a clue of how the radiation approximation $f^{\pm,\mathrm{R}}$, E^{R}, B^{R} has to be defined, the energy flux

$$\frac{d}{dt}\,\mathcal{E}_r(t) = \frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t,x)\,d\sigma(x) \sim -\frac{2}{3c^3}\,|\ddot{D}(t)|^2$$

over the surface of balls $B_r(0)\subset\mathbb{R}^3$ has rigorously been verified in [6] as $r,c\to\infty$; here $D(t)=\int x\rho_0(t,x)\,dx$ denotes the dipole moment of the Newtonian limit. As suggested in [7], the Vlasov equation for the Newtonian distribution is thus modified by incorporating a small correction into the force term as

$$\partial_t\tilde{f}_0^{\pm} + p\cdot\nabla_x\tilde{f}_0^{\pm} \pm \big(\tilde{E}_0 + \frac{2}{3c^3}\dddot{D}\big)\cdot\nabla_p\tilde{f}_0^{\pm} = 0. \tag{1}$$

The additional term is the generalization of the radiation reaction force used in particle models; see Section 1. We also note that for this system an energy-like quantity $\mathcal{E}_{\mathrm{S}}$ (Schott energy) is decreasing such that $\frac{d}{dt}\,\mathcal{E}_{\mathrm{S}}(t) = -\frac{2}{3c^3}|\ddot{D}(t)|^2$. However, analogously to the case of individual particles, the above Vlasov equation coupled to a Poisson equation for the potential does not yield a well-defined PDE system due to the presence of the third-order derivative in time. Therefore once again one has to pass to some kind of center manifold of the system, with the manifold now being infinite-dimensional. Although so far the existence of this manifold has not been verified in full detail, it is possible to use it formally to guess a certain well-defined approximation $D^{[3]}(t)$ to $\dddot{D}(t)$ such that if this replacement of $\dddot{D}$ by $D^{[3]}$ is made in (1), then a well-defined and globally solvable PDE system is obtained. Moreover, $f^{\pm,\mathrm{R}} = \tilde{f}_0^{\pm} + c^{-2}f_2^{\pm}$, $E^{\mathrm{R}} = \tilde{E}_0 + c^{-2}E_2 + (2/3)c^{-3}D^{[3]}$, $B^{\mathrm{R}} = c^{-1}B_1 + c^{-3}B_3$, can be shown to be an effective approximation of the full solutions to order $\mathcal{O}(c^{-4})$. See [5] for a more detailed review.

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The *g*-factor of the electron

Herbert Spohn

This note is based on material from the book [1]: H. Spohn, Dynamics of Charged Particles and Their Radiation Field, Cambridge University Press (2004), where the reader will find a more complete discussion and many references.

A charged particle with internal rotation (spin) is governed by the Bargmann, Michel, Telegdi (BMT) equation which, in nonrelativistic notation, reads

$$\begin{aligned}\dot{\omega} = \frac{e}{mc}\omega \wedge \big[&(\tfrac{1}{2}g - 1 + \gamma^{-1})B - (\tfrac{1}{2}g - 1)\gamma(1+\gamma)^{-1}c^{-2}(v\cdot B)v \\ &-(\tfrac{1}{2}g - \gamma(1+\gamma)^{-1})c^{-1}v \wedge E\big]\,. \end{aligned} \tag{1}$$

Here ω is the angular velocity of the charge and $\gamma = (1 - v^2/c^2)^{-1/2}$. The velocity v of the charge and the external electromagnetic fields E, B have to be evaluated along the actual space orbit of the charge. Clearly the material parameters for the BMT equation are the ratio e/m, i.e. charge over rest mass, and the dimensionless gyromagnetic ratio g. In essence, the form of Eq. (1) is imposed by relativistic invariance. (1) immediately suggests an experiment to determine g. One sets $E = 0$ and takes B to be a weak uniform magnetic field. Then the orbit of the charge is a circle perpendicular to B and is transversed with frequency $\omega_\mathrm{c} = e|B|/mc$. For small velocities it follows from (1) that ω precesses with frequency $\omega_\mathrm{s} = e|B|g/2mc$. Hence

$$g = \frac{2\omega_\mathrm{s}}{\omega_\mathrm{c}}\,. \tag{2}$$

The g-factor of the electron has been measured with great accuracy. A value dating from 1984 is

$$g/2 = 1.001\ 159\ 652\ 193\ (4)\,,$$

where the last digit is the experimental error bar. Even more astounding is the computation of the g-factor within the framework of relativistic quantum electrodynamics (QED), which yields

$$g/2 = 1.001\ 159\ 652\ 459\ (135)\,,$$

where the theoretical error in the round brackets results largely from the precision with which the fine-structure constant α is available. I refer to the informative review article by Brown and Gabrielse [2]. There has been an enormous effort to define QED as a relativistic quantum field theory satisfying the Wightman axioms. By consensus, this program remains uncompleted. The fact that from a mathematically undefined theory one can compute a number which is empirically so accurate may be regarded as either a mystery or a triumph depending on the point of view.

Our note has zero intention to compete with relativistic QED. Rather we plan to raise a different issue. The BMT equation is a very cleverly guessed macroscopic equation for the motion of a spinning charge. Our goal is to derive such an equation from an underlying more microscopic theory. Thus we would like to start from a mathematically well-defined theory of a charge coupled to the Maxwell field, even at the expense of loosing in empirical accuracy. Within such a framework we would like to prove that, in a suitable limit, the charge is governed by some version of the BMT equation, thereby also obtaining the g-factor within the theory under consideration.

Currently there are only three serious investigations.

1) *The Dirac equation.* It yields $g = 2$. The BMT equation results from a space-adiabatic limit, so to decouple the positron subspace, combined with a semiclassical limit, so to obtain a classical equation of motion. I refer to the lecture notes by Teufel [3], Chapter 4, for a detailed discussion, see also [4] .

2) *The Abraham model for a classical spinning charge.* In [5] we discuss the special case where the charge is standing still but rotates. Without coupling to the Maxwell field one has $g = 1$ (this is why $g = 2$ is called anomalous in the physics literature). The coupling renormalizes $g = 1$ to g_{eff} with an explicitly stated formula depending on the particular mass and charge distribution. g_{eff} can take a wide range of values including $g_{\mathrm{eff}} = 2$.

3) *Nonrelativistic quantum electrodynamics.* In this model the number of charges is conserved. For the g-factor it suffices to consider a single quantum particle with spin and interacting with the photons. For fixed total momentum one has to decouple the lowest energy states (dressed electron) from the multitude of possible excitations. This is again a space-adiabatic problem. A central point for such an investigation is to study of the spectral properties of the following Hamiltonian:

$$H(p) = \frac{1}{2m}(p - P_{\mathrm{f}} - eA_{\varphi})^2 - \frac{e}{2m}\sigma \cdot B_{\varphi} + H_{\mathrm{f}} \,. \tag{3}$$

$p \in \mathbb{R}^3$ is a parameter with the meaning of the total momentum and m is the bare mass of the electron. We will use units such that $\hbar = 1$ and $c = 1$. $H(p)$ is defined on the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathfrak{F}$. $\mathbb{C}^2$ refers to the spin of the electron with σ the 3-vector of Pauli spin matrices. $\mathfrak{F}$ refers to the photons and is the bosonic Fock space over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, $\mathbb{C}^2$ coming from the helicity of the photons. P_{f} is the total momentum and H_{f} is the field energy,

$$P_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk k a(k,\lambda)^* a(k,\lambda) \,, \tag{4}$$

$$H_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk\omega(k)a(k,\lambda)^* a(k,\lambda)\,, \quad \omega(k) = |k|\,, \tag{5}$$

where in the standard notation, $a(k,\lambda)$, $k \in \mathbb{R}^3$, $\lambda = 1,2$, is a two-component Bose field over $\mathbb{R}^3$. A_φ is the quantized vector potential and B_φ the corresponding magnetic field,

$$A_\varphi = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \frac{1}{\sqrt{2\omega}} \widehat{\varphi}(k) e_\lambda(k) \big(a(k,\lambda) + a(k,\lambda)^*\big)\,, \tag{6}$$

$$B_\varphi = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk \frac{1}{\sqrt{2\omega}} \widehat{\varphi}(k) e_\lambda(k) \wedge ik \big(a(k,\lambda) - a(k,\lambda)^*\big)\,. \tag{7}$$

Here $e_1(k)$, $e_2(k)$, $k/|k|$ form a left-handled dreibein and $\widehat{\varphi}$ is the Fourier transform of a rotation invariant charge distribution φ with rapid decay and $\int \varphi = 1$. Under our assumptions, $H(p)$ is bounded from below and self-adjoint.

To employ the space-adiabatic theory in its standard version one needs a spectral gap between the ground state of $H(p)$ and the rest of the spectrum. For $\omega(k) = |k|$ the excitations are gapless. For technical reasons we therefore introduce a small photon mass by setting

$$\omega(k) = \big(k^2 + (m_{\mathrm{ph}})^2\big)^{1/2}\,. \tag{8}$$

It is an interesting mathematical problem to investigate the physical case $m_{\mathrm{ph}} = 0$. If $m_{\mathrm{ph}} > 0$, then it is known that, for small e and small $|p|$, $H(p)$ has an exactly two-fold degenerate eigenvalue. However for large $|p|$, $H(p)$ looses its ground state which results in technical difficulties when applying the space-adiabatic theory.

To label the degeneracy we introduce the total angular momentum

$$J = \tfrac{1}{2}\sigma + J_{\mathrm{f}} + S_{\mathrm{f}}\,, \tag{9}$$

$$J_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk a(k,\lambda)^* (k \wedge i\nabla_k) a(k,\lambda)\,, \tag{10}$$

$$S_{\mathrm{f}} = i \int_{\mathbb{R}^3} dk \big(a(k,2)^* a(k,1) - a(k,1)^* a(k,2)\big)\,. \tag{11}$$

Then $[H(p), p \cdot J] = 0$ and, with $E(p) = \inf \sigma(H(p))$, for e, $|p|$ sufficiently small there exists a unique vector $\psi_+(p)$ such that

$$H(p)\psi_+(p) = E(p)\psi_+(p)\,, \quad p \cdot J\psi_+(p) = \tfrac{1}{2}|p|\psi_+(p)\,. \tag{12}$$

The g-factor resulting from space-adiabatic theory can be expressed through $\psi_+(p)$.

For $p = 0$ one defines

$$\frac{m}{m_{\mathrm{eff}}} = 1 - \tfrac{2}{3}\langle \psi_+, (P_{\mathrm{f}} + eA_\varphi) \cdot \frac{1}{H(0) - E(0)} (P_{\mathrm{f}} + eA_\varphi)\psi_+ \rangle\,, \tag{13}$$

where $\psi_+ = \psi_+(0)$ and the scalar product $\langle \cdot, \cdot \rangle$ refers to $\mathcal{H} = \mathbb{C}^2 \otimes \mathfrak{F}$. The g-factor is then obtained as

$$(14)\quad \tfrac{1}{2} g = \frac{m_{\text{eff}}}{m} \big(\langle \psi_+, \sigma_3 \psi_+ \rangle - i \langle \psi_+, (P_{\text{f}} + eA_\varphi) \frac{1}{m\big(H(0) - E(0)\big)} \wedge (P_{\text{f}} + eA_\varphi) \psi_+ \rangle \big) .$$

The extra factor $1/m$ on the right hand side looks dimensionally strange. In fact, for $\omega(k) = |k|$, it can be absorbed into φ whose scale is then set by the Compton length $\hbar/mc = 1/m$ in our units.

While well defined, (13) and (14) provide no easily accessible information of the actual value of g. One only notes that $g \to 2$ as $e \to 0$, as to be expected since the coupling to the photons is turned off. This suggests to compute g in an expansion in e. To second order one obtains, setting already $m_{\text{ph}} = 0$,

$$(15)\qquad g = 2\Big(1 + \tfrac{2}{3} e^2 \int dk |\widehat{\varphi}(k)|^2 \big(k^2 \big(1 + \tfrac{1}{2}|k|\big)^3\big)^{-1}\Big) + \mathcal{O}(e^4) .$$

Remarkably in this form one can remove the ultraviolet cutoff by letting the charge distribution $\varphi(x) \to \delta(x)$, hence $\widehat{\varphi}(k) \to (2\pi)^{-3/2}$. The limit is denoted by g_∞. Introducing the fine structure constant $\alpha = e^2/4\pi$ one finally obtains

$$(16)\qquad g_\infty = 2\Big(1 + \frac{8}{3}\big(\frac{\alpha}{2\pi}\big)\Big) + \mathcal{O}(\alpha^2)$$

which is to be compared with $2\big(1 + (\alpha/2\pi)\big) + \mathcal{O}(\alpha^2)$ from relativistic QED. Note that in the point charge limit $m_{\text{eff}}/m \to \infty$ while the second factor in (14) tends to 0. The finite answer in (16) comes only from fine cancellations.

Obvious questions result:

Does (14) have a finite limit when $m_{\text{ph}} \to 0$ and $\widehat{\varphi}(k) \to (2\pi)^{-3/2}$?

If the point charge limit of (14) exists, how does it depend on α?

What is the $\mathcal{O}(\alpha^2)$ in (16)?

Such information has to extracted from the Hamiltonian $H(0)$ of (3).

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Cherns-Simons Dynamics of Vortices

David Stuart
(joint work with Sophia Demoulini)

The nonlinear Schroedinger equation

$$i\partial_t \Phi = -\Delta\Phi - \frac{\lambda}{2}(1-|\Phi|^2)\Phi \tag{1}$$

on the plane $\mathbb{R}^2$ admits soliton type solutions called vortices, which are of the form $\Phi = f_N(r)e^{iN\theta}$ in polar coordinates, with f_N a function increasing from 0 to 1 as $r = |x|$ goes from 0 to $+\infty$. The number N is an integer called the degree. This system is invariant under the group of Galilean relativity transformations $(t, x) \mapsto (t, x - vt)$. We consider a gauge theoretic generalization of this equation which is also Galilean invariant and supports stable vortex solutions. The complex field Φ is coupled to an electromagnetic potential $A_0\, dt + A_1\, dx^1 + A_2\, dx^2$ through the covariant derivative $D = d - iA$, and they evolve according to the system:

$$\begin{aligned} &E + dB = *\langle i\Phi, D\Phi\rangle \\ &i(\partial_t - iA_0)\Phi = -\Delta_A\Phi - \frac{\lambda}{2}(1-|\Phi|^2)\Phi \\ &B = \frac{1}{2}(1-|\Phi|^2). \end{aligned} \tag{2}$$

Here $E_j = \frac{\partial A_j}{\partial t} - \frac{\partial A_0}{\partial x^j}$ is the electric field, and $B = *dA$ is the magnetic field on a two dimensional spatial domain Σ. The system is invariant under the infinite dimensional group of gauge transformations $\Phi \mapsto \Phi e^{i\chi}$, $A \mapsto A + d\chi$. The evolution of A is first order in time and is derived from a Chern-Simons term in a Lagrangian, which was introduced by Manton in 1997 ([1]). The system is also Hamiltonian, with the functional $\mathcal{V}_\lambda(A, \Phi) = \int_\Sigma v_\lambda(A, \Phi)$, where

$$v_\lambda(A, \Phi) = \frac{1}{2}\Big(B^2 + g^{jk}\langle D_j\Phi, D_k\Phi\rangle + \frac{\lambda}{4}(|\Phi|^2 - 1)^2\Big) \tag{3}$$

acting as Hamiltonian.

A global existence and regularity theorem for the system was proved in [2] for the case that Σ is a compact Riemann surface.

Compared to the nonlinear Schroedinger equation the system (2) has the advantage that the space of vortices is much larger, due to the presence of self-dual or Bogomolny structure, and some explicit solutions are available. To be precise, for $\lambda = 1$ the system admits static soliton solutions called Abelian Higgs, or Ginzburg-Landau, vortices which form a $2N$- dimensional manifold which can be identified with the symmetric N-fold product of Σ, where $N \in \mathbb{Z}$ is the degree; this is called the moduli space of self-dual vortices $\mathcal{M}_N$. This space inherits a Kaehler structure from the ambient infinite dimensional phase space in which (2) is a Hamiltonian flow; the associated symplectic form is called Ω.

For the case that Σ is a constant negative curvature space it is possible to write down the vortex solutions explicitly, by a reduction to the Liouville equation due to Witten.

An eventual aim of the study of the system (2) would be to show (e.g. for $\Sigma = \mathbb{R}^2$) that the large time behaviour of solutions can be described in terms of a scattering theory involving a finite dimensional Hamiltonian evolution on the moduli space, and a linear dispersive wave asymptotic to a solution of the free Schroedinger equation in some suitable sense.

As a starting point, we give theorems from [3] which describe the role of these static solutions in classes of solutions for the time dependent equations on long time scales, proving that the system may be approximated by a Hamiltonian system on a finite dimensional space, the moduli space of self-dual vortices:

> *For $\epsilon = |\lambda - 1|$ sufficiently small, the system* (2) *can be approximated, for times of order $\frac{1}{\epsilon}$, by the Hamiltonian flow on the phase space $\mathcal{M}_N = \mathrm{Sym}^N(\Sigma)$ associated to the Hamiltonian function $\mathcal{V}_\lambda|_{\mathcal{M}_N}$ via the symplectic form Ω.*

We also discuss the prospects for proving asymptotic stability of a single vortex, and also the very long time stability of bound states of vortices. In this regard, an interesting case is when the finite dimensional Hamiltonian motion on the moduli space appearing in the theorem just mentioned is in fact integrable. We would like to know whether the solutions of the original infinite dimensional system (2) remain close to the finite dimensional orbits indefinitely. As a first step in understanding this we consider a type of estimate called the Nekhoroshev estimate in classical mechanics, which gives stability on exponentially long time intervals. We discuss how these estimates appears to generalize to the problem of strongly constrained motion, in which a possibly infinite dimensional Hamiltonian system is constrained to a finite dimensional (symplectic) submanifold as some parameter μ becomes very large. This is close to the scenario in the theorem above in the description of vortex dynamics in (2) for λ close to one, with $\mu = |\lambda - 1|^{-1}$.

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Quasi-static Limits in Nonrelativistic Quantum Electrodynamics

Lucattilio Tenuta

The talk is based on the paper [7]. In the talk I consider a system of N nonrelativistic quantum particles of spin 1/2 interacting with the quantized Maxwell field (mass zero and spin one) in the limit when the particles have a velocity v small with respect to the speed of light c, imposing to the interaction an ultraviolet cutoff, but no infrared cutoff.

I first discuss the analogous classical model, where it's easier to present the different ways which are available to implement the concept of a slowly moving particle. I elaborate on three possibilities:

(1) $c \to \infty$ with the velocity v of the particles fixed, the case for which rigorous results for the quantum case have already been discussed in the literature [6] using methods of the weak coupling theory [2][3];
(2) $v \to 0$ with c fixed. This case can be rephrased as the limit of heavy particles, $m_j \to \varepsilon^{-2} m_j$, observed over a long time, $t \to \varepsilon^{-1} t$, $\varepsilon \to 0^+$, with kinetic energy $E_{\text{kin}} = \mathcal{O}(1)$.
(3) The third possibility is to consider special initial conditions for the classical equations of motion, as in the work by Kunze and Spohn [5]. They consider the solution to the classical equations of motion describing one particle moving with constant velocity v together with the electromagnetic field it generates (called *charge soliton*). They describe then the limit interaction between slowly moving charge solitons.

In the quantum case there is no obvious analogue of the classical charge soliton, because the Pauli-Fierz Hamiltonian without an infrared cutoff has no ground state when the total momentum is different from zero [1][4]. Therefore I focus on the second approach, the limit $\varepsilon \to 0^+$.

I construct subspaces which are invariant for the dynamics up to terms of order $\varepsilon\sqrt{\log(\varepsilon^{-1})}$ and describe effective dynamics, *for the particles only*, inside them. At the lowest order the particles interact through Coulomb potentials. At the second one, ε^2, the mass gets a correction of electromagnetic origin and a velocity dependent interaction, the Darwin term, appears.

Moreover, I calculate the radiated piece of the wave function, i. e., the piece which leaks out of the almost invariant subspaces and calculate the corresponding radiated energy.

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