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# Automorphic Forms, Geometry and Arithmetic

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Abstract. This meeting provided an overview of recent developments in the theory of automorphic forms and automorphic representations. In addition, new results in related areas including geometry, arithmetic geometry, moduli spaces, Galois theory and number theory, often involving the application to automorphic forms, were discussed.

*Mathematics Subject Classification (2000):* 11Fxx, 14Cxx Secondary: 14Gxx, 14Fxx, 11Gxx.

## Introduction by the Organisers

The theory of automorphic forms has its roots in the early ninteenth century in classical work of Euler, Gauss, Jacobi, Eisenstein, and others. The subject experienced a vast expansion and reformulation following the work of Selberg, Harish-Chandra, and Langlands, in the 1970's, and remains the focus of intense current activity. The goal of this meeting was two-fold, first to provide an overview of the most recent developments in the theory of automorphic forms and automorphic representations, and, second, to provide a glimpse of the many closely related topics involving geometry and arithmetic where automorphic forms play an important role. Thus, one subset of the lectures (Soudry, Waldspurger, Gan, Muic, Moeglin, and Henniart) focused on automorphic forms and automorphic representations, while a second subset ranged quite widely and included geometry (Burger), arithmetic geometry (Pink, Howard, Nekovar, Yang), moduli spaces (Rapoport, van der Geer, Görtz, Ngô), Galois theory (Savin) and L-functions (Harder, Shahidi).

Among the many fundamental insights of Langlands are the following:

(a) Automorphic representations of a given reductive group $G$ over a number field should occur in packets (L-packets or Arthur packets), parametrized by representations of the Weil-Deligne group into the Langlands dual group

$^LG$. A local version of this should describe the (irreducible, admissible) representations of the group $G(F)$ for any local field $F$.

(b) It is necessary to consider the automorphic representations of all reductive groups together and, in particular, their relations, the most important of which are predicted by the principle of functoriality.

These insights lie very deep and their complete realization is still a very distant dream. Nonetheless, they have provided a guide for much of the subsequent research in this area and a number of the most important techniques that have been brought to bear were discussed at the meeting. These included the Arthur-Selberg trace formula, fundamental lemma, local and global descent, converse theorems, local theta correspondence, and Eisenstein series.

The connections of automorphic forms with geometry and arithmetic are many and important. One such set of connections occurs in the theory of Shimura varieties. Here important topics include interpretation as moduli spaces and period domains, the arithmetic of Heegner points and their higher dimensional generalizations, including their arithmetic intersections and heights, and the structure of Shimura varieties in characteristic $p > 0$. Automorphic forms have a deep connection with the geometry of locally symmetric spaces, where, for example, the boundary behavior of cohomology classes and Eisenstein series can be applied to the study of special values of L-functions. Again, all of these aspects were discussed during the program.

The meeting revealed, once again, that the theory of automorphic forms continues to be a vibrant subject in which many exciting developments can be expected in the future.

**Special event**

On Friday afternoon, the Oberwolfach Prize was awarded to Ngô Bao-Chao for his work on the fundamental lemma. The award presentation, by Professor Reinhold Remmert, was followed by a Laudatio given by Michael Rapoport explaining the significance of Ngô's work and describing a basic case of the fundamental lemma. Rapoport's Laudatio is included at the end of this report. Ngô then gave a lecture in which he explained some of the fundamental ideas of his proof, for example, the use of the Hitchin fibration. In the evening, there was a festive dinner.

**Workshop: Automorphic Forms, Geometry and Arithmetic**

**Table of Contents**

David Soudry
*Local Descent from* GL($n$) *to Classical Groups* . . . . . . . . . . 247

Jean-Loup Waldspurger
*A propos du lemme fondamental pondéré tordu* . . . . . . . . . . 250

Marc Burger
*Bounded cohomology and applications: a panorama* . . . . . . . . . . 252

Michael Rapoport
*Period domains over finite fields and Deligne-Lusztig varieties* . . . . . . . . 255

Günter Harder
*Arithmetic properties of the constant term of Eisenstein classes* . . . . . . . 260

Richard Pink
*Applications of the Harder-Narasimhan filtration* . . . . . . . . . . 261

Gerard van der Geer
*Siegel Modular Forms of Genus* 2 . . . . . . . . . . 263

Freydoon Shahidi
*On Harder–Mahnkopf Periods* . . . . . . . . . . 266

Wee Teck Gan
*The local Langlands correspondence for* $GSp(4)$ . . . . . . . . . . 267

Ben Howard
*Intersection theory on Shimura surfaces* . . . . . . . . . . 268

Gordan Savin
*Functoriality and the Inverse Galois problem* . . . . . . . . . . 271

Jan Nekovář
*Growth of Selmer groups of Hilbert modular forms over ring class fields* 273

Goran Muić
*Degenerate Eisenstein Series on symplectic groups* . . . . . . . . . . 274

Tonghai Yang
*Arithmetic Intersection and a conjecture of Colmez* . . . . . . . . . . 279

Colette Mœglin
*Formes automorphes de carré intégrable non cuspidales* . . . . . . . . . . 282

Ulrich Görtz (joint with Chia-Fu Yu)
*The supersingular locus in Siegel modular varieties, and Deligne-Lusztig varieties* . . . . . . 283

Guy Henniart
*On Kim's exterior square functoriality for $GL(4)$* . . . . . . 286

Bao Châu Ngô
*Hitchin fibration and fundamental lemma* . . . . . . 287

Michael Rapoport
*Laudatio for Bao Châu Ngô* . . . . . . 287

# Abstracts

## Local Descent from GL($n$) to Classical Groups

David Soudry

The descent method of Ginzburg, Rallis and Soudry enabled us to construct, for an irreducible, self-dual, automorphic, cuspidal representation $\tau$ of $\mathrm{GL}_m(\rangle A)$ , an irreducible, automorphic, cuspidal and globally generic representation $\sigma$ on the Adele points of the appropriate symplectic or orthogonal (split or quasi-split) group $G$, such that $\sigma$ lifts to $\tau$ at almost all places. In case $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and hence $m = 2n$ is even, Jiang and the author showed that $\sigma$ lifts to $\tau$ at all places. This was done by local descent, which is the local counterpart of the global descent method, with almost complete analogy. It allows us to construct, for an irreducible, self-dual, supercuspidal, generic representation $\tau$ of $\mathrm{GL}_m(F)$, where $F$ is a $p$-adic field, an irreducible, supercuspidal, generic representation $\sigma$ of $G(F)$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, or equivalently, $L(\sigma \times \tau, s)$ has a pole at $s = 0$.

### 1. Local gamma factors

Let $\sigma$, $\tau$ be irreducible, generic representations of $G = G(F)$, $\mathrm{GL}_m(F)$, respectively. The local gamma factor $\gamma(\sigma \times \tau, s, \psi)$ ($\psi$ is a nontrivial character of $F$) is obtained via a local functional equation, which arises from the theory of global integrals (given by Ginzburg, Rallis and Soudry) of Rankin-Selberg type, or Shimura type, and represent the standard $L$-functions for $G \times \mathrm{GL}_m$. We restrict ourselves to $rank(G) < m$. The local functional equation has the form

$$\frac{\gamma(\sigma \times \tau, s, \psi)}{c(\tau, s, \psi)} \mathcal{L}(W_\sigma, D^\psi(f_{\tau,s})) = \mathcal{L}(W_\sigma, D^\psi(M(f_{\tau,s}))).$$

Here, $W_\sigma$ is in the Whittaker model of $\sigma$ (with respect to a given character), $f_{\tau,s}$ is a holomorphic section in $\rho_{\tau,s} = Ind_P^H \tau |\det\cdot|^{s-\frac{1}{2}}$, where $H$ is "essentially" a split classical group and $P \subset H$ is a Siegel type parabolic subgroup, with Levi part isomorphic to $\mathrm{GL}_m$, according to the following table, where we also specify $c(\tau, s, \psi)$.

| | $G$ | $H$ | $c(\tau, s, \psi)$ | $\rho$ |
|---|---|---|---|---|
| 1. | $\mathrm{SO}_{2n+1}(F)$ | $\mathrm{SO}_{2m}(F)$ | $\gamma(\tau, \Lambda^2, 2s-1, \psi)$ | $\Lambda^2$ |
| 2. | $\widetilde{Sp}_{2n}(F)$ | $\mathrm{Sp}_{2m}(F)$ | $\gamma(\tau, \Lambda^2, 2s-1, \psi)\gamma(\tau, s-\frac{1}{2}, \psi)$ | $\Lambda^2 \oplus st$ |
| 3. | $\mathrm{SO}_{2n}(F)$ | $SO_{2m+1}(F)$ | $\gamma(\tau, sym^2, 2s-1, \psi)$ | $sym^2$ |
| 4. | $\mathrm{Sp}_{2n}(F)$ | $\widetilde{Sp}_{2m}(F)$ | $\gamma(\tau, sym^2, 2s-1, \psi)$ | $sym^2$ |

Thus, we allow $G$ and $H$ to be metaplectic groups as well. In case (4) we have to consider $\rho_{\tau,s} = Ind_P^H \gamma_\psi \tau |\det\cdot|^{s-\frac{1}{2}}$ instead ($\gamma_\psi$ is the Weil factor); $M$ is the intertwining operator corresponding to the long Weyl element;

$$\mathcal{L}(W_\sigma, D^\psi(f_{\tau,s})) = \int\limits_{N\backslash G} W_\sigma(g) D^\psi(f_{\tau,s})(g)dg,$$

where $N$ is the (standard) maximal unipotent subgroup of $G$; $D^\psi(f_{\tau,s})$ is given as an integral along the unipotent radical of the standard parabolic subgroup, which preserves a maximal flag in a totally isotropic subspace of dimension $\ell$ , and factors through the Jacquet module of $\rho_{\tau,s}$, which furnishes a Gelfand-Graev (resp. Fourier-Jacobi) model of $\rho_{\tau,s}$, stabilized by $G$, when $H$ is orthogonal (resp. symplectic or metaplectic). In fact $D^\psi$ defines an isomorphism with this Jacquet module when $\tau$ is supercuspidal. Denote this Jacquet module by $\sigma_{\psi,\ell}(\rho_{\tau,s})$. The gamma factor thus defined is the same as the Shahidi gamma factor, at least up to a multiple by an exponential function, and hence it has the same set of poles and zeroes.

## 2. Descent

What is nice about this definition of the local factor is

**Theorem 1.** *Let $\sigma$, $\tau$ be irreducible, supercuspidal representations of $G$, $\mathrm{GL}_m(F)$ respectively. Then $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, if and only if $L(\tau, \rho, s)$ has a pole at $s = 0$ and $\sigma$ pairs with the Jacquet module above, where we replace $\rho_{\tau,1}$ with the image $\pi_\tau$ by the intertwining operator $M$ at $s = 1$.*

Since $L(\tau, \rho, s)$ has a pole at $s = 0$, $\pi_\tau$ is the Langlands quotient of $\rho_{\tau,1}$. We call $\sigma_{\psi,\ell}(\tau) = \sigma_{\psi,\ell}(\pi_\tau)$ the descent of $\tau$ to $G$. Consider the cases of functoriality.

| | $\mathrm{GL}_m(F)$ | *pole at* $s=0$ | $H$ | $G$ | *descent* |
|---|---|---|---|---|---|
| 1. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, \Lambda^2, s)$ | $\mathrm{SO}_{4n}(F)$ | $\mathrm{SO}_{2n+1}(F)$ | $\sigma_{\psi,n-1}(\tau)$ |
| 2. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, \Lambda^2, s)$ | $\mathrm{Sp}_{4n}(F)$ | $\widetilde{Sp}_{2n}(F)$ | $\sigma_{\psi,n-1}(\tau)$ |
| 3. | $\mathrm{GL}_{2n}(F)$ | $L(\tau, sym^2, s)$ | $\mathrm{SO}_{4n+1}(F)$ | $\mathrm{SO}_{2n,\alpha}(F)$ | $\sigma_{\psi,n,\alpha}(\tau),\ \omega_\tau = \chi_\alpha$ |
| 4. | $\mathrm{GL}_{2n+1}(F)$ | $L(\tau, sym^2, s)$ | $\widetilde{Sp}_{4n+2}(F)$ | $\mathrm{Sp}_{2n}(F)$ | $\sigma_{\psi,n}(\tau),\ \omega_\tau = 1$ |

Here, $\omega_\tau$ is the central quadratic character of $\tau$. If it corresponds to $\alpha \in F^*$, we denote it also by $\chi_\alpha$, and then we denote by $\mathrm{SO}_{2n,\alpha}(F)$ the corresponding quasi-split (or split when $\alpha$ is a square) orthogonal group in $2n$ variables.

**Theorem 2.** *In all these cases the descent of $\tau$ is a nontrivial, supercuspidal, multiplicity free representation of $G$, all of whose irreducible summands $\sigma$ are $\psi$-generic and are such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$. Moreover, any irreducible, supercuspidal and $\psi$-generic representation $\sigma$ of $G$, such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$ is isomorphic to a summand of the descent of $\tau$.*

Consider a self-dual, supercuspidal $\tau$ as above and an irreducible summand $\sigma$ of its descent to $G$. By globalizing $\sigma$ to an irreducible, automorphic, cuspidal, generic representation, and lifting it to $\mathrm{GL}_m(\rangle A)$ (by the theorem of Cogdell, Kim, Piatetski-Shapiro, Shahidi) we get

**Theorem 3.** *Let $\tau$ be an irreducible, self-dual, supercuspidal representation of* $\mathrm{GL}_m(F)$. *Assume that $L(\tau,\rho,s)$ has a pole at $s=0$, where $\rho=\Lambda^2,\ sym^2$. Then we can globalize $\tau$ to an irreducible, self-dual, automorphic, cuspidal representation $T$, such that $L(T,\rho,s)$ has a pole at $s=1$.*

In case (2) of the last table, Ginzburg, Rallis and the author proved that the descent is irreducible. Using this and the local theta correspondence, Jiang and the author proved that the descent is irreducible in case (1), as well. This means that $\sigma$ in the last theorem is unique. For a long time we tried to address the irreducibility question of the descent in cases (3), (4), without success. Here is our new idea. Let us add two more cases to the last table

| | $\mathrm{GL}_m(F)$ | *pole at* $s=0$ | $H$ | $G$ | *descent* |
|---|---|---|---|---|---|
| 5. | $\mathrm{GL}_{2n}(F)$ | $L(\tau,sym^2,s)$ | $\widetilde{Sp}_{4n}(F)$ | $\mathrm{Sp}_{2n}(F)$ | $\sigma'_{\psi,n-1}(\tau)$ |
| 6. | $\mathrm{GL}_{2n+1}(F)$ | $L(\tau,sym^2,s)$ | $\mathrm{SO}_{4n+3}(F)$ | $\mathrm{SO}_{2n+2}(F)$ | $\sigma'_{\psi,n}(\tau),\ \omega_\tau=1$ |

Here we denote the descent by $\sigma'_{\psi,\ell}$ in order to distinguish it from the one in cases (4), (5).

**Theorem 4.** *(Jiang and Soudry) Let $\tau$ be an irreducible, supercuspidal representation of* $\mathrm{GL}_m(F)$, *such that $L(\tau,sym^2,s)$ has a pole at $s=0$. In case $m$ is odd, assume that $\omega_\tau=1$. Then the descent in cases (5), (6) above is a nonzero irreducible, supercuspidal, $\psi$-generic representation $\sigma$ of $G$, such that $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$; these properties determine $\sigma$ uniquely.*

Consider case (5) and denote $\sigma=\sigma'_{\psi,n-1}(\tau)$. The local lift of $\sigma$ to $\mathrm{GL}_{2n+1}(F)$ must be $\tau\times\omega_\tau$. We conclude that $L(\sigma\times\omega_\tau,s)$ has a pole at $s=0$. We then prove that $\sigma$ is the local $\psi$-theta lift from an irreducible, supercuspidal, $\psi$-generic representation $\pi$ of $\mathrm{O}_{2n,\alpha}(F)$, where $\omega_\tau=\chi_\alpha$. We know that the restriction of $\pi$ to $\mathrm{SO}_{2n,\alpha}(F)$ is either irreducible or a direct sum of two irreducible representations of the form $\pi_1\oplus\pi_1^\varepsilon$, where $\varepsilon\in\mathrm{O}_{2n,\alpha}(F)$, with $\det(\varepsilon)=-1$. Thus,

**Theorem 5.** *(Jiang and Soudry) Let $\tau$ be an irreducible, supercuspidal representation of* $\mathrm{GL}_{2n}(F)$, *such that $L(\tau,sym^2,s)$ has a pole at $s=0$. Let $\omega_\tau=\chi_\alpha$. Then there is an irreducible, supercuspidal, $\psi$-generic representation $\sigma$ of* $\mathrm{SO}_{2n,\alpha}(F)$, *such that $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$, and it is unique up to outer conjugation by $\varepsilon$.*

Similarly, in case (6) we get

**Theorem 6.** *(Jiang and Soudry) Let $\tau$ be an irreducible, supercuspidal, self-dual representation of* $\mathrm{GL}_{2n+1}(F)$, *with $\omega_\tau=1$. Then there is a unique irreducible, supercuspidal, $\psi$-generic representation $\sigma$ of* $\mathrm{Sp}_{2n}(F)$, *such that $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$.*

With more work, quite similar to the work of Jiang and the author on the local converse theorem for $\mathrm{SO}_{2n+1}(F)$, this should give the local converse theorem for $\mathrm{SO}_{2n,\alpha}(F),\ \mathrm{Sp}_{2n}(F)$.

**Theorem 7.** *Let $G$ be one of the groups $\mathrm{SO}_{2n,\alpha}(F)$, $\mathrm{Sp}_{2n}(F)$. Let $\sigma_1$, $\sigma_2$ be two irreducible $\psi$-generic representations of $G$, such that*

$$\gamma(\sigma_1 \times \tau, s, \psi) = \gamma(\sigma_2 \times \tau, s, \psi),$$

*for all irreducible generic representations $\tau$ of $\mathrm{GL}_r(F)$, and $r < 2n$, or $r < 2n+1$, respectively. Then, in the first case $\sigma_1$ is isomorphic to either $\sigma_2$ or $\sigma_2^{\varepsilon}$, and in the second case, $\sigma_1 \cong \sigma_2$.*

Finally, the local converse theorem should enable us to prove rigidity (strong multiplicity one, up to isomorphism) of irreducible, automorphic, cuspidal and $\psi$-generic representations of $\mathrm{Sp}_{2n}(\rangle A)$, while for $\mathrm{SO}_{2n,\alpha}(\rangle A)$, two nearly equivalent such representations may be non-isomorphic in a finite number of places, where one representation is an outer conjugate of the other.

## A propos du lemme fondamental pondéré tordu

Jean-Loup Waldspurger

Soient $F$ un corps local non archimédien de caractéristique nulle, $G$ un groupe réductif défini sur $F$ et $(H, s, \hat{\xi})$ un triplet endoscopique de $G$. Cela signifie que $H$ est un groupe réductif connexe défini et quasi-déployé sur $F$, $s$ est un élément semi-simple du groupe dual $\hat{G}$ et $\hat{\xi} : {}^LH \to {}^LG$ est un $L$-plongement tel que $\hat{\xi}(\hat{H})$ soit la composante neutre du centralisateur de $s$ dans $\hat{G}$. Pour $f \in C_c^\infty(G(F))$ et $\gamma \in H(F)$ suffisamment régulier, Langlands et Shelstad ont défini l'intégrale orbitale endoscopique $J^{H,G}(\gamma, f)$. Pour $f^H \in C_c^\infty(H(F))$, on définit $J^{H,st}(\gamma, f^H)$: c'est le cas particulier où $G = H$.

**Conjecture de transfert**. *Pour toute $f \in C_c^\infty(G(F))$, il existe $f^H \in C_c^\infty(H(F))$ telle que $J^{H,st}(\gamma, f^H) = J^{H,G}(\gamma, f)$ pour tout $\gamma \in H(F)$ suffisamment régulier (on dit que $f^H$ est un transfert de $f$).*

Supposons $G$ et $H$ non ramifiés. On fixe des sous-groupes compacts hyperspéciaux $K \subset G(F)$ et $K^H \subset H(F)$. On définit l'algèbre de Hecke $\mathcal{H} \subset C_c^\infty(G(F))$ formée des fonctions biinvariantes par $K$ et l'algèbre similaire $\mathcal{H}^H$. De $\hat{\xi}$ se déduit un homomorphisme $b : \mathcal{H} \to \mathcal{H}^H$. Notons $\mathbf{1}_K$ et $\mathbf{1}_{K^H}$ les fonctions caractéristiques de $K$ et $K^H$. On a $b(\mathbf{1}_K) = \mathbf{1}_{K^H}$.

**Conjecture (lemme fondamental)**. *Pour toute $f \in \mathcal{H}$, $b(f)$ est un transfert de $f$.*

Une conjecture similaire concerne les algèbres de Lie. On note ci-dessous "transfert" l'assertion: "la conjecture de transfert est vraie pour toutes données $F$, $G$, $(H, s, \hat{\xi})$". On note "$LF$" l'assertion: "le lemme fondamental est vrai pour toutes les données $F$, $G$, $(H, s, \hat{\xi})$ telles que $G$ et $H$ sont non ramifiés, sous certaines hypothèses restrictives concernant la caractéristique résiduelle $p$ de $F$". Les autres assertions s'interprètent de façon similaire.

La situation au moment présent:

$LF$ pour les algèbres de Lie
et pour $car(F) > 0$ (Ngô Bao Chau)
$\Downarrow$
$LF$ pour les algèbres de Lie $\Rightarrow$ transfert
$\Downarrow$
$LF$ pour $f = \mathbf{1}_K$
$\Downarrow$
$LF$

La dernière implication est due à Clozel et Hales, l'avant-dernière à Langlands, Shelstad et Hales.

Soit $M$ un groupe de Lévi de $G$, c'est-à-dire que $M$ est une composante de Lévi définie sur $F$ d'un sous-groupe parabolique de $G$ défini sur $F$. Soit $(M', s, \hat{\xi})$ un triplet endoscopique de $M$. Pour $f \in C_c^\infty(G(F))$ et $\gamma' \in M'(F)$ suffisamment régulier, Arthur a défini l'intégrale orbitale pondérée endoscopique $J_{M'}^G(\gamma', f)$. Supposons $G$ et $M'$ non ramifiés. Arthur a posé une conjecture appelée lemme fondamental pondéré, qui calcule la fonction $J_{M'}^G(., \mathbf{1}_K)$ en termes de fonctions analogues pour les groupes endoscopiques elliptiques de $G$.

La situation au moment présent:

$LFP$ pour les algèbres de Lie
et pour $car(F) > 0$ (inconnu)
$\Downarrow$
$LFP$ pour les algèbres de Lie
$\Downarrow$
$LFP$

Pour obtenir les meilleurs résultats de la théorie de l'endoscopie, on doit utiliser la formule des traces tordue. Il intervient un automorphisme $\theta$ de $G$, défini sur $F$, et les intégrales orbitales sont remplacées par des intégrales orbitales tordues par cet automorphisme. D'où une conjecture de transfert tordue, un lemme fondamental tordu, un lemme fondamental pondéré tordu. La théorie fait aussi apparaître un lemme fondamental non standard et un lemme fondamental pondéré non standard: ce sont des assertions similaires aux précédentes mais qui concernent des couples de groupes $(G', G'')$ tels que l'un n'est pas un groupe endoscopique de l'autre. Par exemple $G' = Sp(2n)$ et $G'' = SO(2n+1)$.

La situation au moment présent:

$$\begin{array}{ccc}
LF \text{ pour les algèbres de Lie} & & \\
+ LF \text{ non standard} & & \\
\text{pour } car(F) > 0 \text{ (Ngô Bao Chau)} & & \\
\Downarrow & & \\
LF \text{ pour les algèbres de Lie} & & \\
+ LF \text{ non standard} & \Rightarrow & \text{transfert tordu} \\
\Downarrow & & \\
LFT \text{ pour } f = \mathbf{1}_K & & \\
\Downarrow \text{ (inconnu)} & & \\
LFT & &
\end{array}$$

(la dernière implication est connue dans certains cas particuliers, d'après Clozel, Labesse et Morel);

$$\begin{array}{c}
LFP \text{ pour les algèbres de Lie} \\
+ LFP \text{ non standard} \\
\text{pour } car(F) > 0 \text{ (inconnu)} \\
\Downarrow \\
LFP \text{ pour les algèbres de Lie} \\
+ LFP \text{ non standard} \\
\Downarrow \\
LFPT
\end{array}$$

## Bounded cohomology and applications: a panorama

Marc Burger

Bounded cohomology for groups and spaces is related to usual cohomology and in fact enriches it by providing stronger invariants. The aim of this talk is to illustrate certain aspects of this philosophy. General references for bounded cohomology are [12, 21, 22, 3].

*1. Definition, low degrees.* The continuous cohomology $\mathrm{H}^\bullet_{\mathrm{c}}(G, E)$ of a topological group $G$ acting continuously by linear isometries on a Banach space $E$ is defined using the resolution of $E$ by the complex of continuous $E$-valued cochains on $G$. Using the subcomplex of continuous cochains which are bounded in the supremum norm leads to the bounded continuous cohomology; it is equipped with a quotient seminorm and comes with a comparison map $\mathrm{H}^\bullet_{\mathrm{cb}}(G, E) \to \mathrm{H}^\bullet_{\mathrm{c}}(G, E)$. In degree zero both cohomology groups equal $E^G$. In degree one the comparison map is always injective; while $\mathrm{H}^1_{\mathrm{c}}(G, E)$ describes affine isometric $G$-actions on $E$ with given linear part, $\mathrm{H}^1_{\mathrm{cb}}(G, E)$ classifies those with bounded orbits. Starting with degree two this theory really exhibits new phenomena. The kernel $\mathrm{EH}^2_{\mathrm{cb}}(G, E)$ of the comparison map admits a description in terms of quasiactions. In the case of trivial coefficients, where we denote the corresponding objects by $\mathrm{H}^\bullet_{\mathrm{cb}}(G)$ and $\mathrm{H}^\bullet_{\mathrm{c}}(G)$, this kernel $\mathrm{EH}^2_{\mathrm{cb}}(G)$ is the quotient of the space

$$\mathrm{QH}(G) := \bigl\{f : G \to \mathbb{R} : f \text{ is continuous and } \sup_{x,y} \bigl|f(xy) - f(x) - f(y)\bigr| < \infty\bigr\}$$

of continuous quasimorphisms by the subspace $\mathrm{C^b}(G) \oplus \mathrm{Hom_c}(G, \mathbb{R})$, where $\mathrm{C^b}(G)$ is the space of continuous bounded functions on $G$.

*2. Examples.* This interpretation, together with the exploitation of certain hyperbolicity phenomena, leads to nonvanishing results on $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$; for instance, $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is infinite dimensional when $\Gamma$ is:

(a) a lattice in a real rank one Lie group [14]
(b) Gromov hyperbolic and nonelementary [13],
(c) a subgroup of the mapping class group $\mathcal{M}_g$ for $g \geq 2$ which is not virtually Abelian [2].

Many of these examples are fundamental groups of finite aspherical complexes; this illustrates the fact that there are no simple minded finiteness conditions on $\Gamma$ ensuring that $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is finite dimensional; indeed for the free group $\mathbb{F}_r$ on $r \geq 2$ generators, which is inherently a one-dimensional object, $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_r)$ and $\mathrm{H}^3_{\mathrm{b}}(\mathbb{F}_r)$ are infinite dimensional. This seems to be the price to pay for the advantage that bounded cohomology is the receptacle of rather refined invariants as the next section illustrates. Let's however mention that if $\Gamma$ is amenable $\mathrm{H}^n_{\mathrm{b}}(\Gamma) = 0$ for $n \geq 1$.

*3. Two important examples.*

*(1) Bounded Euler class:* The Euler class classifies the universal covering of the group $\mathrm{Homeo}^+(S^1)$ of orientation preserving homeomorphisms of the circle $S^1$; it admits a natural bounded representative $e^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}\bigl(\mathrm{Homeo}^+(S^1), \mathbb{Z}\bigr)$. For a minimal action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$, its bounded Euler class $\rho^*(e^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z})$ is then a complete invariant of conjugacy [16]. Let $e^{\mathrm{b}}_{\mathbb{R}}$ be the bounded class obtained by changing coefficients from $\mathbb{Z}$ to $\mathbb{R}$. Recently the author showed [4] that if $\Gamma < G$ is a lattice in a locally compact (second countable) group $G$, then for a minimal action which is not conjugate into the group of rotations, $\rho^*(e^{\mathrm{b}}_{\mathbb{R}})$ is in the image of the restriction map $\mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma)$ if and only if $\rho$ finitely covers an action which extends continuously to $G$.

*(2) Bounded Kähler class:* The integral of the Kähler form on triangles with geodesic sides in a hermitian symmetric space (of noncompact type) gives the bounded Kähler class $\kappa^{\mathrm{b}}_{\mathcal{X}} \in \mathrm{H}^2_{\mathrm{bc}}(G)$, where $G = (\mathrm{Aut}(\mathcal{X}))^\circ$. The bounded Kähler class of a representation $\rho : \Gamma \to G$ is then $\rho^*(\kappa^{\mathrm{b}}_{\mathcal{X}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma)$. When $\mathcal{X}$ is irreducible and not of tube type, it is a complete conjugacy invariant for representations with Zariski dense image [5, 10]. This invariant has served to define new types of embeddings between Hermitian symmetric spaces [8] and enters as well in the higher Teichmüller theory developed in [9, 7].

*4. The comparison map.* It encodes subtle information of algebraic and geometric nature.

*(1) Commutator length* [1]*:* The stable length $\ell_{st}$ on the commutator subgroup $\Gamma'$ of $\Gamma$ is $\ell_{st}(\gamma) := \lim_{n\to\infty} \|\gamma^n\|/n$, where $\|\cdot\|$ denotes the commutator length. Then $\ell_{st}$ vanishes identically if and only if $\mathrm{EH}^2_{\mathrm{b}}(\Gamma) = 0$, that is the comparison map in degree two is injective.

*(2) Hyperbolicity* [19]*:* A finitely generated group is (nonelementary) Gromov hyperbolic if and only if the comparison map $\mathrm{H}^n_{\mathrm{b}}(\Gamma, E) \to \mathrm{H}^n(\Gamma, E)$ is surjective for every Banach $\Gamma$-module $E$ and $n = 2$ (or equivalently $n \geq 2$).

*(3) Measure equivalence* [24, 20]*:* It preserves the property that $\mathrm{H}^2_{\mathrm{b}}\big(\Gamma, \ell^2(\Gamma)\big) \neq 0$. The latter holds for all (nonelementary) Gromov hyperbolic groups.

*(4) Higher rank* [11, 23]*:* The comparison map is injective with image the $G$-invariant classes if $\Gamma < G$ is an irreducible lattice in a connected semisimple Lie group $G$ with finite center and rank $\mathrm{rk}_G \geq 2$. This is also implied by the recent result [23] that $\mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is an isomorphism for $n < 2\,\mathrm{rk}_G$. Together with 2(3), 3(1) and 4(1) we obtain that:

- any homomorphism $\Gamma \to \mathcal{M}_g$ has finite image [18];
- any $\Gamma$-action by homeomorphisms of $S^1$ has a finite orbit [17];
- the stable length on commutators vanishes.

*(5) Geometry of central extensions* [15]*:* If a class $\alpha$ is in the image of the comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z})$ then the associated central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_\alpha \longrightarrow \Gamma \longrightarrow e$$

is quasiisometric to $\Gamma \times \mathbb{Z}$; here $\Gamma$ is a finitely generated group.

*(6) Differential forms* [6]*:* For a symmetric space of noncompact type $\mathcal{X}$ and a discrete subgroup $\Gamma < \mathrm{Iso}(\mathcal{X})$. there is a factorization

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma) & \longrightarrow & \mathrm{H}^\bullet(\Gamma) = \mathrm{H}^\bullet_{\mathrm{dR}}(\Gamma\backslash\mathcal{X}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}_{(\infty)}(\Gamma\backslash\mathcal{X}) & \end{array}$$

of the comparison map by a geometrically defined map to the cohomology $\mathrm{H}_{(\infty)}(\Gamma\backslash\mathcal{X})$ of the complex of smooth $\Gamma$-invariant differential forms on $\mathcal{X}$ which are bounded and $d$-bounded. In case $\Gamma$ is a lattice, $\mathrm{H}_{(\infty)}(\Gamma\backslash\mathcal{X})$ can be replaced by $\mathrm{L}^2$-cohomology.

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## Period domains over finite fields and Deligne-Lusztig varieties

Michael Rapoport

Let $G_0$ be a reductive group over $\mathbb{F}_q$. There are two classes of algebraic varieties over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$ attached to $G_0$. Let us recall their definition. We set $G = G_0 \times_{\mathbb{F}_q} \mathbb{F}$.

The *first class* is the class of Deligne-Lusztig varieties. Let $X$ be the algebraic variety of all Borel subgroups of $G$. Let $w \in W$. The DL-*variety associated to* $(G_0, w)$ is

$$X(w) = \{x \in X \mid inv(x, Fx) = w\} \ .$$

Here $F : X \to X$ denotes the Frobenius map over $\mathbb{F}_q$ and $inv(x, y) \in W$ denotes the $G$-orbit of $(x, y) \in X \times X$. Then $X(w)$ is a smooth quasi-projective variety of dimension $l(w)$, equipped with an action of $G_0(\mathbb{F}_q)$.

The *second class* is the class of period domains. Let $\mathcal{N}$ be a conjugacy class of cocharacters $\nu : \mathbb{G}_m \to G$. Any $\nu \in \mathcal{N}$ defines a parabolic subgroup $P_\nu$ of $G$, and any two such are conjugate. Let $X(\mathcal{N}) = G/P_\nu$ be the variety of parabolic subgroups obtained in this way. The *period domain associated to* $(G_0, \mathcal{N})$ is the open subset $X(\mathcal{N})^{ss}$ of semi-stable points. Here $\nu \in \mathcal{N}$ is *semi-stable* if

$$\deg(\mathcal{F}_\nu \mid U \otimes_{\mathbb{F}_q} \mathbb{F}) \leq 0 \ , \ \forall\, U \subset \mathrm{Lie}(G_0)\ \mathbb{F}_q\text{-subspace} \ .$$

Here $\mathcal{F}_\nu$ denotes the $\mathbb{Z}$-filtration on $\mathrm{Lie}(G)$ induced by the cocharacter $\nu$. The degree of a $\mathbb{Z}$-filtration $\mathcal{F}$ on a $\mathbb{F}$-vector space $V$ is defined as

$$\deg(\mathcal{F}) = \sum_i i \dim(gr^i_{\mathcal{F}}(V)).$$

The finite group $G_0(\mathbb{F}_q)$ acts on $X(\mathcal{N})^{ss}$.

**Examples 1.** a) Drinfeld space: $G_0 = \mathrm{GL}_n$.

• DL: $w = s_1 s_2 \dots s_{n-1} = (12 \dots n)$ special Coxeter element.
Then

$$X(w) \cong \Omega^n_{\mathbb{F}_q} = \mathbb{P}^{n-1} \setminus \bigcup\nolimits_{H/\mathbb{F}_q} H \ .$$

Here $H$ ranges through the $\mathbb{F}_q$-rational hyperplanes.

• PD: $\mathcal{N} = (x, y^{(n-1)}) \in (\mathbb{Z}^n)_+$ a dominant coweight with $x > y$. Then $X(\mathcal{N})^{ss} = \Omega^n_{\mathbb{F}_q}$. Similarly for $\mathcal{N} = (x^{(n-1)}, y)$.

b) $G_0 = \mathrm{GL}_3$: generic data.

• DL: $w = w_0$ (longest element).
Then

$$X(w) = \{\mathcal{V}_1 \subset \mathcal{V}_2 \subset V = \mathbb{F}^3 \mid F\mathcal{V}_1 + \mathcal{V}_2 = V, F\mathcal{V}_2 + \mathcal{V}_1 = V\} \ .$$

• PD: $\mathcal{N} = (x_1 > x_2 > x_3)$.
Then

$$X(\mathcal{N})^{ss} = \left\{ \ \mathcal{V}_1 \subset \mathcal{V}_2 \subset V \ \middle| \ \begin{array}{l} x_1 - x_2 > x_2 - x_3 : \mathcal{V}_1 + F\mathcal{V}_1 + F^2\mathcal{V}_1 = V \\ x_1 - x_2 < x_2 - x_3 : \mathcal{V}_2 \cap F\mathcal{V}_2 \cap F^2\mathcal{V}_2 = 0 \\ x_1 - x_2 = x_2 - x_3 : \mathcal{V}_1 \cap F\mathcal{V}_1 = 0, \mathcal{V}_2 + F\mathcal{V}_2 = V \end{array} \right\}$$

The motivation for looking at these varieties is as follows.

• DL: Deligne and Lusztig [DL] construct a $G_0(\mathbb{F}_q)$-equivariant Galois covering $\tilde{X}(w) \to X(w)$ with Galois group $T_w(\mathbb{F}_q)$. Here $T_w$ denotes the maximal torus of $G_0$ associated to $w$. Hence any character $\theta : T_w(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ defines an $\ell$-adic lisse sheaf $\mathcal{F}_\theta$ on $X(w)$ to which the action of $G_0(\mathbb{F}_q)$ is lifted. The $\ell$-adic cohomology groups (with compact supports) $H^*_c(X(w), \mathcal{F}_\theta)$ carry interesting representations of $G_0(\mathbb{F}_q)$.

• PD: The semi-stability concept is due to Faltings [F], in the context of Fontaine's $p$-adic Hodge theory. Period domains over finite fields are the little cousins of their $p$-adic counterparts ([DOR, R]), and are interesting testing grounds for questions on $p$-adic period domains.

In the talk I sketched some recent developments on the structure of these two classes of varieties and their interrelation. For this purpose we may and will assume in the sequel that $G_0$ is simple adjoint.

## 1. Deligne-Lusztig-varieties

The first result concerns the connectedness of these varieties.

**Proposition 2.** (Lusztig [L], Bonnafé/Rouquier [BR]). *$X(w)$ is connected iff $w$ is elliptic.*

Of course, at the opposite extreme, if $w = 1$, then $X(w) = X(\mathbb{F}_q)$ is a finite set of points. ■

We next address the question of affineness of these varieties. So far, no example of a non-affine DL-variety is known.

**Theorem 3.** (Deligne/Lusztig [DL], Haastert [H]).

a) *If $q \geq h - 1$, then $X(w)$ is affine.* [Here $h$ denotes the Coxeter number.]
b) *In general, $X(w)$ is a quasi-affine variety.*

Here the first statement for $q \geq h$ is proved in [DL], by verifying a combinatorial criterion. The strengthening to $q \geq h - 1$ is in [H']. The second statement is in [H]. ■

**Theorem 4.** (Orlik/Rapoport [OR], He [He], Bonnafé/Rouquier [BR]). *If $w$ is of minimal length in its $F$-conjugacy class, then $X(w)$ is affine.*

The theorem is proved in [OR] for split classical groups by checking case by case the Deligne-Lusztig combinatorial criterion mentioned above on a set of representatives of the *cyclic shift classes* of elements of minimal length in their conjugacy class (affineness is a property that only depends on the equivalence class of cyclic shifts). This method is extended in [He] to the general case. In [BR'] , affineness is proved for an a priori larger class of elements in $W$ (however, it seems that all elements in this larger class are $F$-cyclic shifts of elements of minimal length in their $F$-conjugacy class). Both methods ultimately rely on case-by-case considerations, and even on computer calculations.
Any Coxeter element satisfies the hypothesis made on $w$; in this case, the result is due to Lusztig [L']. ■

**Example 5.** Let $G_0$ be split of type $G_2$. Then $X(w)$ is always affine, except possibly if $q = 2$ and $w = s_1s_2s_1$ or $s_2s_1s_2$ [H']. I expect these last two DL-varieties to be non-affine.■

We next come to the cohomology of these varieties.

**Theorem 6.** *Let $\theta : T_w(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ with associated lisse sheaf $\mathcal{F}_\theta$ on $X(w)$. Then*

$$H^i_c(X(w), \mathcal{F}_\theta) = 0 \quad \textit{for} \quad 0 \leq i < l(w) \ .$$

*Furthermore, if $\theta$ is non-singular, then $H^i_c(X(w), \mathcal{F}_\theta) = 0$ for $i \neq l(w)$.*

Here the last statement follows from the first by Poincaré duality, since for non-singular $\theta$ the natural map

$$H_c^*(X(w), \mathcal{F}_\theta) \to H^*(X(w), \mathcal{F}_\theta)$$

is an isomorphism [DL]. If $q \geq h-1$, then $X(w)$ is affine and the statement is due to Deligne and Lusztig [DL] and follows from a general fact on the co-homological dimension of affine varieties. The last statement is proved in [H], using the quasi-affineness of $X(w)$. For $\theta =$ trivial, the first statement is due to Digne/Michel/Rouquier [DMR]. The general statement is in [OR]. ■

It should be pointed out that the cohomology of Deligne-Lusztig varieties is still very mysterious. Also, nothing seems to be known about the fundamental groups of DL-varieties.

## 2. Period domains

The cohomology of period domains has been completely determined by Orlik [O]. It turns out that all irreducible representations occurring in the cohomology are constituents of $\mathrm{Ind}_{B_0(\mathbb{F}_q)}^{G_0(\mathbb{F}_q)}\mathbf{1}$. As an upshot of his formula, Orlik obtains the following statement.

**Theorem 7.** (Orlik [O]). *Let $r_0 = \mathrm{rk}_{\mathbb{F}_q}(G_0)$. If $\mathcal{N}$ is non-trivial, then*

$$H_c^i(X(\mathcal{N})^{ss}, \mathbb{Q}_\ell) = \begin{cases} 0 & 0 \leq i < r_0 \\ \mathrm{St}_{G_0(\mathbb{F}_q)} & i = r_0 \ . \end{cases}$$

*Here $\mathrm{St}_{G_0(\mathbb{F}_q)}$ denotes the Steinberg representation.*■

We now use the following elementary observation:

*Let $G$ be a simple adjoint group over an algebraically closed field $k$. Then for any non-trivial parabolic subgroup $P$*

$$\mathrm{rk}(G) < \dim(G/P) \ ,$$

*unless $G = \mathrm{PGL}_n$ and $P$ is of type either $(1, n-1)$ or $(n-1, 1)$ (in which case equality holds).*

Applying this inequality and comparing with the vanishing theorem for affine varieties we obtain the following consequence, which answers the question of affineness for this class of varieties.

**Corollary 8.** (Orlik/Rapoport [OR]). *Assume for simplicity $G_0$ absolutely simple and $\mathcal{N}$ non-trivial. Then $X(\mathcal{N})^{ss}$ is never affine, unless $G_0 = \mathrm{PGL}_n$ and $\mathcal{N} = (x, y^{(n-1)})$ or $(x^{(n-1)}, y)$, in which case $X(\mathcal{N})^{ss} \cong \Omega^n_{\mathbb{F}_q}$ is affine.*■

The next result addresses the question of simple connectedness for these varieties.

**Theorem 9.** (Orlik [O']). *Assume for simplicity $G_0$ absolutely simple, and $\mathcal{N}$ non-trivial. Then $X(\mathcal{N})^{ss}$ is simply connected unless $G_0 = \mathrm{PGL}_n$ and $\mathcal{N} = (x_1, \ldots, x_n) \in (\mathbb{Z}^n)_+$ with either $x_2 < \frac{1}{n}\sum x_i$ or $x_{n-1} > \frac{1}{n}\sum x_i$. In these last two cases, $\pi_1(X(\mathcal{N})^{ss}) = \pi_1(\Omega^n_{\mathbb{F}_q})$.*

In these last two cases, $X(\mathcal{N})^{ss}$ is a fibering over $\Omega^n_{\mathbb{F}_q}$ with fibers in partial flag varieties which are simply connected.■

## 3. DL versus PD

The next result shows that the Drinfeld space is essentially the only intersection of these two classes of varieties attached to $G_0$.

**Proposition 10.** (Orlik/Rapoport [OR]). *Assume for simplicity $G_0$ absolutely simple. A DL-variety $X_{G_0}(w)$ is never homeomorphic to a PD of the form $X_{G_0}(\mathcal{N})^{ss}$, unless $G_0 = \mathrm{PGL}_n$, $w$ is a Coxeter element and $\mathcal{N} = (x, y^{(n-1)})$ or $(x^{(n-1)}, y)$, in which case both are homeomorphic to $\Omega^n_{\mathbb{F}_q}$.*■

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**Arithmetic properties of the constant term of Eisenstein classes**

Günter Harder

In my talk I explained certain principles which establish some connection between arithmetic properties of Eisenstein cohomology classes and arithmetic properties of special values of $L$-functions.

We consider the group $\mathrm{Sl}_3$ and a local coefficient system $\mathcal{M}_\lambda$ with highest weight $\lambda = n_\alpha\gamma_\alpha + n_\beta\gamma_\beta$. This is a specific example but I believe that the same ideas work in many more cases.

We are interested in the Eisenstein cohomology of $H^\bullet(\mathrm{Sl}_3(\mathbb{Z})\backslash X, \widetilde{\mathcal{M}}_\lambda)$, especially in that part which is obtained from the cuspidal cohomology in degree 2 of the two maximal strata $\partial_P(S), \partial_Q(S)$.

The cuspidal cohomology of these boundary strata is a direct sum of the cuspidal cohomology

$$H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash \mathrm{Gl}_2(\mathbb{R})/O(2), H^1(\mathfrak{u}_P, \mathcal{M}_\lambda)) \oplus H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash \mathrm{Gl}_2(\mathbb{R})/O(2), H^1(\mathfrak{u}_Q, \mathcal{M}_\lambda)).$$

These cohomology groups decompose into eigenspaces under the Hecke algebra, the eigenspaces are labelled by modular eigenforms $f$ of weight $k = n_\alpha + n_\beta + 3$. (Different choices of $\lambda$ lead to the same weight!) We denote these eigenspaces by $H_P(f), H_Q(f)$, we can find two canonical isomorphisms

$$T^{\text{ arith}} : H_P(f) \xrightarrow{\sim} H_Q(f), \quad T^{\text{ analytic}} : H_P(f)\otimes\mathbb{C} \xrightarrow{\sim} H_Q(f)\otimes\mathbb{C},$$

the ratio of these two isomorphisms is a (almost well defined) period $\Omega_f$.

The Eisenstein classes attached to such an eigenspace lie as a subspace

$$\mathrm{Eis}(f) \subset H_P(f) \oplus H_Q(f)$$

and the slope with respect to the arithmetic identification is given by the ratio of two critical values of the $L$-function attached to $f$. For $\omega \in H_P(f)$ we get from the expression for the constant term

$$\mathrm{Eis}(f)(\omega) = \omega + \frac{1}{\Omega(f)}\frac{L(f, n_\alpha - n_\beta + 1)}{L(f, n_\alpha - n_\beta + 2)} T^{\text{ arith}}(\omega).$$

From this we get rationality results for these ratios of special values, and we think that we also can get some insight into the arithmetic properties of these ratios, for instance, we get $p$-adic interpolation.

References:

For more details I refer to my home-page
`www.math.uni-bonn.de/people/harder/Manuscripts/Eisenstein`
and there to the two articles
Eiscoh-rank-one-3.pdf and Luminy-int.pdf
The subject is also closely related to the questions which are discussed in my Oberwolfach-report *OWR_2006_25.pdf*.

## Applications of the Harder-Narasimhan filtration

Richard Pink

The Harder-Narasimhan filtration was first constructed for vector bundles on a smooth projective curve by in the article [4]. Since then, analogues have been defined and used in many other contexts, such as:

• torsion free coherent sheaves on a polarized projective variety,

• hermitian vector bundles on $\overline{\operatorname{Spec}\mathcal{O}_K}$ for a number field $K$,

• vector bundles on a rigid analytic punctured disc with a Frobenius action,

• finite dimensional vector spaces over a field $k$ endowed with one or more filtrations defined over $k$ or over some extension(s) of $k$,

• filtered $(\varphi, N)$-modules, ...

It is also an indispensible tool in establishing the following new result:

**Theorem 1**. Let $K$ be a field which is finitely generated of transcendence degree $\leq 1$ over a finite field $\mathbb{F}_q$ with $q$ elements. Let $M$ be a semisimple $A$-motive of characteristic $\theta$ over $K$. Then there exist only finitely many isomorphism classes of $A$-motives $M'$ of characteristic $\theta$ over $K$ for which there exists a separable isogeny $M' \to M$.

**Remarks.** (a) The same result is expected when $K$ has arbitrary transcendence degree.

(b) The theorem is in precise analogy to known results for abelian varieties and for Drinfeld modules and will have strong consequences for the $\mathfrak{p}$-adic and adelic Galois representations associated to $M$.

(c) The concept of $A$-motives was invented by Anderson [1] in the case $A = \mathbb{F}_q[t]$ under the name of $t$-motives (see also Goss [3]). For simplicity we stick to this case here. The upcoming manuscript [5] will treat the general case.

**Definitions.** (a) A *$t$-motive over $K$ of characteristic $\theta \in K$* is a finitely generated projective $K[t]$-module $M$ together with a $K[t]$-linear map $\tau^{\text{lin}} : M \otimes_{K,\sigma} K \to M$, where $\sigma : x \mapsto x^q$ denotes the Frobenius map, such that $t - \theta$ is nilpotent on $\operatorname{coker}(\tau^{\text{lin}})$ .

(b) A *homomorphism* $f : M \to N$ is a $K[t]$-linear map that commutes with $\tau^{\text{lin}}$.

(c) An injective homomorphism with torsion cokernel is called an *isogeny*.

(d) An isogeny $f$ that induces an isomorphism $\operatorname{coker}(\tau_M^{\text{lin}}) \xrightarrow{\sim} \operatorname{coker}(\tau_N^{\text{lin}})$ is called *separable*.

(e) $M$ is called *simple* if it is non-zero and every non-zero injective homomorphism $M' \hookrightarrow M$ is an isogeny.

(f) $M$ is called *semisimple* if it is isogenous to a direct sum of simple $t$-motives.

The proof of Theorem 1 follows roughly the same strategy as in Faltings's corresponding result for abelian varieties. The idea is to define a height for $t$-motives, to show that $\text{height}(M') \leq c(M)$ for every separable isogeny $M' \hookrightarrow M$, and to show that for any bound $c$ there are only finitely many isomorphism classes of $t$-motives $M'$ with $\text{height}(M') \leq c$. But the definition of a height requires an extra structure analogous to the polarization of an abelian variety.

To describe this extra structure, let $\mathcal{G}$ be the locally free coherent sheaf on $\mathbb{A}^1_K := \mathbb{A}^1 \times K$ with $H^0(\mathbb{A}^1_K, \mathcal{G}^\vee) = M$. Then $\tau^{\text{lin}}$ translates into a homomorphism $\kappa: \mathcal{G} \hookrightarrow (\text{id}\times\sigma)^*\mathcal{G}$. The role of a polarization is played by an extension of $\mathcal{G}$ to a locally free coherent sheaf $\overline{\mathcal{G}}$ on $\mathbb{P}^1_K$. Then $\kappa$ extends to a homomorphism $\kappa: \overline{\mathcal{G}} \hookrightarrow (\text{id}\times\sigma)^*\overline{\mathcal{G}}(d)$ for some positive integer $d$ that can be viewed as the degree of the polarization.

Next, isogenies $M' \hookrightarrow M$ correspond to $\kappa$-equivariant inclusions of equal rank $\mathcal{G} \hookrightarrow \mathcal{G}'$. The chosen extension $\overline{\mathcal{G}}$ then yields a unique extension $\overline{\mathcal{G}'}$ of $\mathcal{G}'$ such that $\overline{\mathcal{G}} \hookrightarrow \overline{\mathcal{G}'}$ is an isomorphism at $\infty$.

A crucial step is to prove that if $M$ is simple of rank $r$, then the difference between the largest and the smallest slopes in the Harder-Narasimhan filtration of $\overline{\mathcal{G}'}$ is $\leq (r-1)d$. As a consequence, one can find an integer $n$ such that all slopes in the Harder-Narasimhan filtration of the twist $\overline{\mathcal{G}'}(n)$ lie within a bounded range. If $M$ is only semisimple, a decomposition into simple almost direct summands can be used to obtain a decomposition $\overline{\mathcal{G}'}_\infty = \bigoplus \overline{\mathcal{G}'_i}_\infty$, and then independent twists in all summands yield the same result.

With this result, one can already deduce Theorem 1 when $K$ is finite. When $K$ has transcendence degree 1, let $X$ be the irreducible smooth projective curve with function field $K$. To solve the remaining arithmetic problem one then extends $\mathcal{G}$ and $\mathcal{G}'$ further to locally free coherent sheaves on the surface $\mathbb{P}^1 \times X$. There is a canonical minimal extension (analogous to the maximal extension described by Gardeyn [2]), which must, however, be modified further to avoid a certain technical problem.

Let $\mathcal{F}$ be the resulting sheaf on $\mathbb{P}^1 \times X$ obtained by extending $\mathcal{G}'$. Then the role of the height of $\mathcal{G}'$ is played by the numerical invariants of $\mathcal{F}$ in the direction of $X$. We can show that these remain bounded whenever $\mathcal{F}$ is obtained from a separable isogeny. Theorem 1 then becomes a consequence of the following result.

Let $C$ and $X$ be irreducible smooth projective curves over the finite field $\mathbb{F}_q$, endowed with ample line bundles $\mathcal{O}(1)$. Abbreviate $\mathcal{F}(m,n) := \mathcal{F} \otimes \text{pr}_1^* \mathcal{O}(m) \otimes \text{pr}_2^* \mathcal{O}(n)$. Let $\eta_C$ and $\eta_X$ denote the generic points of $C$ and $X$. Fix constants $d$, $r$, $d_X$, $\mu_X$, $d_C$, and $\mu_C$.

**Theorem 2**. Up to isomorphism, there exist at most finitely many locally free coherent sheaves $\mathcal{F}$ on $C\times X$ together with an injective homomorphism $\kappa: \mathcal{F} \hookrightarrow (\text{id}\times\sigma)^*\mathcal{F}(d,0)$, such that

(a) $\mathcal{F}$ has constant rank $r$,

(b) the restriction $\mathcal{F}_c := \mathcal{F}|_{c\times X}$ has degree $d_X$ for all $c \in C$,

(c) all slopes in the Harder-Narasimhan filtration of $\mathcal{F}_c$ are $\geq \mu_X$ for all $c \in C$,

(d) the restriction $\mathcal{F}_{\eta_X} := \mathcal{F}|_{C\times\eta_X}$ has degree $d_C$,

(e) all slopes in the Harder-Narasimhan filtration of $\mathcal{F}_{\eta_X}$ are $\leq \mu_C$, and

(f) every $\kappa$-invariant coherent subsheaf of $\mathcal{F}$ of rank $r$ coincides with $\mathcal{F}$ along $\eta_C \times X$.

The proof of Theorem 2 involves an intricate study of the locally free coherent sheaves sheaves $\mathcal{G}_n := \text{pr}_{1*}\mathcal{F}(0,n)$ on $C$ for all integers $n$. An embedding

$\mathcal{O}(n) \hookrightarrow \mathcal{O}(n+1)$ induces an embedding $\mathcal{G}_n \hookrightarrow \mathcal{G}_{n+1}$, and from $\kappa$ one can construct embeddings $\mathcal{G}_{qn} \hookrightarrow \mathcal{G}_{n+a}(d)^{\oplus q}$ where $a$ depends only on $C$. Using these embeddings one can compare the Harder-Narasimhan filtrations of different $\mathcal{G}_n$, and a complicated induction argument proves that the slopes of any single $\mathcal{G}_n$ are in fact bounded above and below. This leaves only finitely many possibilities for the isomorphism class of $\mathcal{G}_n$, which allows one to conclude the same for $\mathcal{F}$ and $\kappa$.

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## Siegel Modular Forms of Genus 2

Gerard van der Geer

### 1. Eisenstein Cohomology

This is a report on joint work with Jonas Bergström and Carel Faber that has as its goal to obtain information about Siegel modular forms by calculating cohomology of local systems on moduli spaces using curves over finite fields.

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension $g$. Over the complex numbers it can be written as $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathcal{H}_g$ with $\mathcal{H}_g$ the Siegel upper half space of degree $g$. An irreducible representation of highest weight $\lambda$ of $\mathrm{GSp}(2g,\mathbb{Q})$ defines a local system $\mathbb{V}_\lambda$ on $\mathcal{A}_g$. The cohomology of such local systems is closely linked to Siegel modular forms. Therefore one considers the (motivic) Euler characteristic

$$e(\mathcal{A}_g,\mathbb{V}_\lambda) = \sum_i (-1)^i [H^i(\mathcal{A}_g,\mathbb{V}_\lambda)],$$

where this is taken in an appropriate category of mixed Hodge modules or Galois representations. We have an analogue $e_c(\mathcal{A}_g,\mathbb{V}_\lambda)$ for compactly supported cohomology. Cusp forms are related to the inner cohomology and the difference between the cohomology and inner cohomology is in some way encoded in the Eisenstein cohomology $e_{\mathrm{Eis}}(\mathcal{A}_g,\mathbb{V}_\lambda) := e(\mathcal{A}_g,\mathbb{V}_\lambda) - e_c(\mathcal{A}_g,\mathbb{V}_\lambda)$. The study of the Eisenstein cohomology was initiated by Harder [6] and continued by his students Schwermer and Pink, see [8, 9]. The Eisenstein cohomology can be broken up in parts coming from the boundary strata of the Satake compactification: if $j : \mathcal{A}_g \to \tilde{\mathcal{A}}_g$ is the embedding in a Faltings-Chai compactification and $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$ is the map to the

Satake compactification then $e_{\text{Eis}}(\mathcal{A}_g, \mathbb{V}_\lambda) = \sum_{i=0}^{g-1} e_c(q^{-1}(\mathcal{A}_i), Rj_*\mathbb{V}_\lambda - Rj_!\mathbb{V}_\lambda)$, where $j_!$ is the extension by zero.

**Theorem 1.** The contribution of the stratum $\mathcal{A}_{g-1}$ to the Eisenstein cohomology of $\mathbb{V}_\lambda$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is

$$\sum_{k=1}^{g}(-1)^k e_c(\mathcal{A}_{g-1}, \mathbb{V}_{\lambda_1+1,\ldots,\lambda_{k-1}+1,\lambda_{k+1},\ldots,\lambda_g})(1-\mathbb{L}^{\lambda_k+g+1-k}),$$

where $\mathbb{L} = h^2(\mathbb{P}^1)$ is the Lefschetz motive of weight 2.

The proof uses Faltings' BGG complex and is given in [4]. For $g = 2$ we can give a closed formula for the full Eisenstein cohomology of a local system, see [3, 4, 1]

We now restrict to $g = 2$. Our goal is to calculate the traces of the Hecke operators $T(p)$ on spaces of (vector-valued) Siegel modular cusp forms. If $\lambda = (l, m)$ is regular then $H^i(\mathcal{A}_2, \mathbb{V}_\lambda)$ vanishes if $i \neq 3$ and $H^3_!(\mathcal{A}_2, \mathbb{V}_\lambda)$ has a pure Hodge structure with Hodge filtration

$$0 \subset F^{l+m+3} \subset F^{l+2} \subset F^{m+1} \subset F^0 = H^3_!(\mathcal{A}_2, \mathbb{V}_\lambda)$$

and we know that $F^{l+m+3} \cong S_{l-m,m+3}(\mathrm{Sp}(4,\mathbb{Z}))$, where $S_{j,k}(\mathrm{Sp}(4,\mathbb{Z}))$ is the space of Siegel modular forms of weight $(j,k)$, i.e., with automorphy factor

$$\mathrm{Sym}^j(c\tau + d)\det(c\tau + d)^k$$

for a matrix $(a, b; c, d) \in \mathrm{Sp}(4,\mathbb{Z})$. We use the description of $\mathcal{A}_2$ as the moduli of stable curves of genus 2 of 'compact type.'

We fix a finite field $\mathbb{F}_q$ and try to calculate the trace of Frobenius on the compactly supported étale cohomology $e_{et,c}(\mathcal{A}_2, \mathbb{V}_{l,m})$ by listing all isomorphism classes of curves of genus 2 defined over $\mathbb{F}_q$, calculating for each such curve the characteristic polynomial of Frobenius on $H^1_{\text{et}}(C)$ and $\#\mathrm{Aut}_{\mathbb{F}_q}(C)$. Then from such a list we can calculate the trace of Frobenius for $q$ on the cohomology of $\mathbb{V}_{l,m}$ for all local systems $V_{l,m}$. We did this for example for level 1 for primes $p \leq 37$. But in view of $F^{l+m+3} \cong S_{l-m,m+3}(\mathrm{Sp}(4,\mathbb{Z}))$ we need to identify the 'strict endoscopic part', i.e. the part in the étale cohomology that has zero intersection with $F^{l+m+3}$. For this we formulated in [3] a conjecture.

**Conjecture 2.** For $\lambda = (l, m)$ regular we have

$$e_{\text{End}}(\mathcal{A}_2, \mathbb{V}_\lambda) = -s_{l+m+4} S[\mathrm{SL}(2,\mathbb{Z}), m+2]\, \mathbb{L}^{m+1},$$

where $s_n = \dim S_n(\mathrm{SL}(2,\mathbb{Z}))$ and $S[\mathrm{SL}(2,\mathbb{Z}), k]$ is the motive of cusp forms of weight $k$ on $\mathrm{SL}(2,\mathbb{Z})$.

We hope that the experts will be able to prove this. Now for example, for level 1 we subtract the Eisenstein part and the conjectured strict endoscopic part and what remains should give the trace of the Hecke operator $T(p)$ on $S_{l-m,m+3}((\mathrm{Sp}(4,\mathbb{Z}))$ for all $p \leq 37$. Does it work? There is a good test case:

**Conjecture 3.** (Harder) If $f = \sum_{n\geq 1} a(n)q^n \in S_{l+m+4}(\mathrm{SL}(2,\mathbb{Z}))$ is a normalized eigenform of weight $l+m+4$ and $\ell$ a 'large' prime in $\mathbb{Q}_f$ dividing a critical value $\Lambda(f, l{+}3)$ of the $L$-series of $f$ then there exists a Siegel modular form $F \in S_{l-m,m+3}$ with eigenvalues $\lambda(p) \in \mathbb{Q}_F$ such that

$$\lambda(p) \equiv p^{m+1} + a(p) + p^{l+2} \mod \ell'$$

for some prime $\ell'$ dividing $\ell$ in the compositum of the fields of eigenvalues $\mathbb{Q}_f$ and $\mathbb{Q}_F$.

For example, for the eigenform $f_{22} = \sum_{n\geq 1} a(n)q^n \in S_{22}$ we have that $41 \mid \Lambda(f_{22}, 14)$, hence there should exist a Siegel modular eigenform $F \in S_{4,10}$ with eigenvalues $\lambda(p)$ satisfying $\lambda(p) \equiv p^8 + a(p) + p^{13} \bmod 41$ and in cooperation with Harder we were able to confirm this congruence for all $p \leq 37$. We were able to check many other cases and this was great fun, see [7, 5].

Recently, we did similar things for level 2, incorporating also the action of $\mathrm{Sp}(4,\mathbb{Z}/2) \cong S_6$, see [1]. Let $\Gamma_2[2] = \ker(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/2))$. We calculated the Eisenstein cohomology and made conjectures about the strict endoscopic part. We also formulated a precise conjecture about Yoshida type liftings. We refer to [1] for the details.

**Conjecture 4.** Let $f \in S_{l+m+4}(\Gamma_0(2))^{\mathrm{new}}$ and $g \in S_{l-m+2}(\Gamma_0(2))^{\mathrm{new}}$ be newforms. Then there exists a Siegel modular form $F \in S_{l-m,m+3}(\Gamma_2[2])$ that is an eigenform for the Hecke algebra with spinor $L$-function $L(F,s) = L(f,s)L(g,s-m-1)$. It appears with multiplicity 5 if $f$ and $g$ have the same eigenvalue $\pm$ under $w_2$ and multiplicity 1 otherwise. Similarly, given newforms $f \in S_{l+m+4}(\Gamma_0(4))^{\mathrm{new}}$, $g \in S_{l-m+2}(\Gamma_0(4))^{\mathrm{new}}$ there exists a Siegel modular form $F \in S_{l-m,m+3}(\Gamma_2[2])$ with spinor $L$-function $L(F,s) = L(f,s)L(g,s-m-1)$. It appears with multiplicity 5 in $S_{l-m,m+3}(\Gamma_2[2])$.

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## On Harder–Mahnkopf Periods

Freydoon Shahidi

This semi-expository talk given during "Harder's day" aimed at defining certain periods for principal $L$–functions of cusp forms on $GL_n(\mathbb{A}_k)$ introduced by Harder [4] when $n = 2$, but k is any number field, and generalized by Mahnkopf [6] for arbitrary n, but only k = $\mathbb{Q}$.

These periods are obtained by comparing two rational structures, one on the cuspidal cohomology, through its embedding into Betti cohomology (cf. [3]), and the other on the $(\mathfrak{g}, K)$–cohomology [2] of its generic Whittaker model, generalizing ideas of Eichler and Shimura. More precisely, in [4] Harder introduced an adelic way of defining an action of Aut($\mathbb{C}$) on the Fourier coefficients of a regular algebraic representation of $GL_2(\mathbb{A}_k)$, which can be extended to $GL_n(\mathbb{A}_k)$ by means of their Whittaker models and using cyclotomic characters (cf. [4, 6]).

Our interest here is a joint project with A. Raghuram in which one exploits Mahnkopf's periods [6] to prove certain special cases of twisting conjectures of Blasius [1] and Panchishkin [7]. More precisely, one expects to prove the twisting conjecture for m-th symmetric power $L$–functions of holomorphic cusp forms on $GL_2(\mathbb{A}_k)$ upon realizing them as principal $L$–functions on $GL_{m+1}(\mathbb{A}_k)$ through Langlands functoriality conjecture (cf. [8, 10]). The twists are by means of algebraic Hecke characters. We note that each twist introduces a new $L$–function.

The results won't, of course, be as precise as those predicted by Deligne and Zagier for these $L$–functions, but are still new and interesting, and must be near in hand in the case of fourth symmetric power $L$–functions for $GL_2(\mathbb{A}_k)$, using the automorphy of $Sym^4(\pi)$ as an automorphic form on $GL_5(\mathbb{A}_k)$ established recently by H. H. Kim [5].

We concluded the talk by stating a result (cf. [9]) which shows that up to a Gauss sum defined by the finite part of the twisting character, the ratio of the period for the twisted representation on $GL_n(\mathbb{A}_k)$ to the original one belongs to the field of definition of the form and the character (cf. [3, 12]), as long as the character is any algebraic Hecke character for any number field. Moreover, the ratio of the period to the Gauss sum changes equivariently under the action of Aut ($\mathbb{C}$).

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## The local Langlands correspondence for $GSp(4)$

Wee Teck Gan

### 1. Introduction

In this talk, I discuss joint work with Shuichiro Takeda on the local Langlands correspondence for $G = GSp(4) = GSpin(5)$ over a p-adic field $F$, which gives a classification of the set $\Pi(G)$ of equivalence classes of irreducible smooth (complex) representations of $G(F)$ in terms of Galois theoretic data. Let $W_F$ be the Weil group of $F$ and let $WD_F = W_F \times SL_2(\mathbb{C})$ be the Weil-Deligne group. Let $\Phi(G)$ denote the set of equivalence classes of L-parameters for $G$, i.e.

$$\phi : WD_F \longrightarrow G^\vee = GSp_4(\mathbb{C}),$$

where $G^\vee$ is the Langlands dual group of $G$. The theorem is:

**Theorem 1.** There is a surjective map

$$L : \Pi(G) \longrightarrow \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

satisfying a number of expected conditions:

(i) (Fibers) For $\phi \in \Phi(G)$, the fiber $\Pi_\phi$ is in natural bijection with the set of irreducible characters of the component group $A_\phi = Z_{G^\vee}(Im(\phi))$.

(ii) (Discrete Series) $\pi \in \Pi(G)$ is discrete series representation iff $\phi_\pi$ does not factor through any proper parabolic subgroup.

(iii) (Central Character) The central character $\omega_\pi$ of $\pi$ corresponds to the similitude character $sim(\phi_\pi)$ of $\phi_\pi$.

(iv) (Twisting) The map is compatible with twisting by 1-dimensional characters.

(v) (Local Factors) For generic or non-supercuspidal representations, the map preserves $L$-factors and $\epsilon$-factors of pairs of representations in $\Pi(G) \times \Pi(GL_r)$.

(vi) (Plancherel Measures) For non-generic supercuspidal representations $\pi \in \Pi(G)$ and supercuspidal representations $\sigma$ of $GL_r(F)$, the map gives the expected formula for a certain Plancherel measure associated to $\pi \boldsymbol{\Theta} \sigma$.

(vii) (Genericity) A packet $\Pi_\phi$ contains a generic representation iff the adjoint L-function $L(s, Ad \circ \phi)$ is holomorphic at $s = 1$.

(viii) (Characterization) The map $L$ is uniquely characterized by (ii), (iii), (v) and (vi), with $r \leq 2$ in (v) and (vi).

The proof exploits the local theta correspondence for the groups

$$GSp(4) \times GSO(2,2), \quad GSp(4) \times GSO(4), \quad GSp(4) \times GSO(3,3),$$

and the local Langlands correspondence for $GL(2)$ and $GL(4)$. There is also an analogous theorem for the inner form $GSp(1,1) = GSpin(4,1)$ of $GSp(4)$. Finally, for supercuspidal $\pi \in \Pi(G)$, the formal degree of $\pi$ can be shown to be equal to an explicit multiple of the adjoint $\gamma$-factor $|\gamma(0, Ad \circ \phi_\pi, \psi)|$, thus confirming a conjecture of Hiraga-Ichino-Ikeda.

## Intersection theory on Shimura surfaces

Ben Howard

Let $B_0$ be an indefinite rational quaternion algebra, $\mathcal{O}_{B_0} \subset B_0$ a maximal order, $F$ a real quadratic field, and set $B = B_0 \otimes_{\mathbb{Q}} F$. Assume that $\mathcal{O}_F = \mathbb{Z}[\sqrt{\Delta}]$ for some integer $\Delta > 1$ and that all primes dividing the discriminant of $B_0$ are split in $F$. This second hypothesis implies that $\mathcal{O}_B = \mathcal{O}_{B_0} \otimes_{\mathbb{Z}} \mathcal{O}_F$ is a maximal order in $B$. Let $\mathcal{M}_0$ be the moduli stack of triples $(A_0, i_0, \lambda_0)$ in which $A_0$ is an abelian surface over a scheme, $i_0 : \mathcal{O}_{B_0} \to \mathrm{End}(A_0)$ is a ring homomorphism, and $\lambda_0$ is a polarization of $A_0$ which is compatible with the action of $\mathcal{O}_{B_0}$ in a sense which we will not make precise. Let $\mathcal{M}$ be the moduli stack of triples $(A, i, \lambda)$ in which $A$ is an abelian fourfold over a scheme, $i : \mathcal{O}_B \to \mathrm{End}(A)$ is a ring homomorphism, and $\lambda$ is a polarization of $A$ suitably compatible with the action of $\mathcal{O}_B$. Thus $\mathcal{M}$ is the integral model of a classical Shimura surface (i.e. a twisted analogue of a Hilbert modular surface) and $\mathcal{M}_0$ is the integral model of a classical Shimura curve (i.e. a twisted analogue of a modular curve). There is a closed immersion $\mathcal{M}_0 \to \mathcal{M}$ which is given on moduli, ignoring the polarizations, by $A_0 \mapsto A_0 \otimes \mathcal{O}_F$.

On the other hand, for any totally positive $\alpha \in \mathcal{O}_F$ one may also consider the moduli stack $\mathcal{Y}(\alpha)$ of quadruples $(A, i, \lambda, t_\alpha)$ in which $(A, i, \lambda)$ is as above, and $t_\alpha \in \mathrm{End}_{\mathcal{O}_B}(A)$ satisfies $t_\alpha^2 = -i(\alpha)$. Thus $\mathcal{Y}(\alpha)$ may be thought of as the locus of points in $\mathcal{M}$ for which the action of $\mathcal{O}_F$ can be extended to an action of the CM order $\mathcal{O}_F[\sqrt{-\alpha}]$. The generic fiber $\mathcal{Y}(\alpha) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finite étale stack over $\mathrm{Spec}(\mathbb{Q})$, and so may be viewed as a codimension two cycle on $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$, but the integral model $\mathcal{Y}(\alpha)$ is generally less pleasant: at a prime $p$ dividing the discriminant of $B_0$ the stack $\mathcal{Y}(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ typically has vertical components of codimension one in $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Using the Cerednik-Drinfeld uniformization of $\mathcal{M}$ at such a prime, one can modify the vertical components of $\mathcal{Y}(\alpha)$ in order to produce a codimension two cycle on $\mathcal{M}$. Using constructions of Kudla-Millson [2] one may then augment this cycle with a Green current for the complex fiber, resulting in a class

$$\widehat{\mathcal{Y}}(\alpha, v) \in \widehat{\mathrm{Ch}}^2(\mathcal{M})$$

in the codimension two Gillet-Soulé arithmetic Chow group [4] of $\mathcal{M}$. Here $v \in F \otimes_{\mathbb{Q}} \mathbb{R}$ is an auxiliary totally positive parameter.

For any $(A_0, i_0, \lambda_0) \in \mathcal{M}_0$ the $\mathbb{Z}$-module of trace zero elements of $\mathrm{End}_{\mathcal{O}_{B_0}}(A_0)$ is equipped with a natural positive definite quadratic form $Q_0$. Let $[-,-]$ be the bilinear form determined by $[x, x] = 2Q_0(x)$, and for any symmetric matrix $T$ with rational coefficients let $\mathcal{Z}(T)$ be the moduli stack of quintuples $(A_0, i_0, \lambda_0, s_1, s_2)$ in which $(A_0, i_0, \lambda_0) \in \mathcal{M}_0$ and $s_1$ and $s_2$ are trace zero elements of $\mathrm{End}_{\mathcal{O}_{B_0}}(A_0)$ which satisfy

$$T = \frac{1}{2} \begin{pmatrix} [s_1, s_1] & [s_1, s_2] \\ [s_2, s_1] & [s_2, s_2] \end{pmatrix}.$$

For sufficiently nice $T$ the stack $\mathcal{Z}(T)$ is zero-dimensional, and so defines a codimension two cycle on the arithmetic surface $\mathcal{M}_0$. For general $T$ Kudla-Rapoport-Yang [3] have defined an arithmetic cycle class

$$\widehat{\mathcal{Z}}(T, \mathbf{v}) \in \widehat{\mathrm{Ch}}^2(\mathcal{M}_0)$$

where $\mathbf{v}$, a symmetric positive definite $2 \times 2$ matrix with real entries, is an auxiliary parameter. For nice $T$ this arithmetic zero cycle is simply the cycle $\mathcal{Z}(T)$ equipped with the trivial Green current. Kudla-Rapoport-Yang have proved that the classes $\widehat{\mathcal{Z}}(T, \mathbf{v})$ are closely related to the Fourier coefficients of the derivative at $s = 0$ (the center point of the functional equation) of a genus two Siegel Eisenstein series $\mathcal{E}(\tau, s)$. More precisely, there is an isomorphism, the *arithmetic degree*,

$$\widehat{\deg} : \widehat{\mathrm{Ch}}^2(\mathcal{M}_0) \xrightarrow{\sim} \mathbb{R}$$

and Kudla-Rapoport-Yang [3] prove that $\mathcal{E}'(\tau, 0)$ has Fourier expansion

$$\mathcal{E}'(\tau, 0) = \sum_T \widehat{\deg}\ \widehat{\mathcal{Z}}(T, \mathbf{v}) \cdot q^T$$

where $\mathbf{v}$ is the imaginary part of $\tau$ and $q^T = e^{2\pi i \mathrm{Trace}(T\tau)}$.

The main result relates the classes $\widehat{\mathcal{Y}}(\alpha, v)$ to the cycle classes $\widehat{\mathcal{Z}}(T, \mathbf{v})$ of Kudla-Rapoport-Yang, and hence to the Fourier coefficients of automorphic forms. To state the result, fix an isomorphism $F \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}$ and a totally positive $v = (v_1, v_2) \in F \otimes_{\mathbb{Q}} \mathbb{R}$. Set

$$\mathbf{v} = \begin{pmatrix} \Delta(v_1 + v_2) & \delta(v_1 - v_2) \\ \delta(v_1 - v_2) & (v_1 + v_2) \end{pmatrix}.$$

For each $\alpha \in \mathcal{O}_F$ write $\alpha = x + y\sqrt{\Delta}$ with $x, y \in \mathbb{Z}$ and define

$$\Sigma(\alpha) = \left\{ \begin{pmatrix} z & \frac{y}{2} \\ \frac{y}{2} & x - \Delta z \end{pmatrix} \;\middle|\; z \in \mathbb{Z} \right\}.$$

**Theorem A.** *Suppose that $\alpha \in \mathcal{O}_F$ is totally positive and that at least one of the following holds: either*

(a) *$F(\sqrt{-\alpha})/\mathbb{Q}$ is not biquadratic, or*
(b) *for every prime $p$ nonsplit in $F$ (and writing $\mathfrak{p}$ for the prime above $p$)*

$$\mathrm{ord}_{\mathfrak{p}}(\alpha) \leq \begin{cases} 0 & \text{if } p\mathcal{O}_F = \mathfrak{p}^2 \\ 1 & \text{if } p\mathcal{O}_F = \mathfrak{p}. \end{cases}$$

*Then the arithmetic pullback $\widehat{\mathrm{Ch}}^2(\mathcal{M}) \to \widehat{\mathrm{Ch}}^2(\mathcal{M}_0)$ determined by the closed immersion $\mathcal{M}_0 \to \mathcal{M}$ satisfies*

$$\widehat{\mathcal{Y}}(\alpha, v) \mapsto \sum_{T \in \Sigma(\alpha)} \widehat{\mathcal{Z}}(T, \mathbf{v}).$$

To briefly indicate why the hypotheses (a) and (b) appear, hypothesis (a) ensures that $\mathcal{Y}(\alpha)(\mathbb{C}) \cap \mathcal{M}_0(\mathbb{C}) = \emptyset$, while hypothesis (b) ensures that $\mathcal{Y}(\alpha)$ has no vertical components except possibly at primes dividing the discriminant of $B_0$. Theorem A should be true without assuming either hypothesis. The proof of the theorem under hypothesis (a) may be found in [1]; the (more difficult) proof under hypothesis (b) will appear in a forthcoming work.

One can construct a twisted embedding $\mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathfrak{h}_2$ of the product of two upper half planes into the Siegel space of genus two in such a way that Siegel modular forms pull back to Hilbert modular forms for the real quadratic field $F$, and such that the Fourier expansion of the pull back of $\mathcal{E}'(\tau, 0)$ has the form

$$\mathcal{E}'(\tau_1, \tau_2, 0) = \sum_{\alpha \in \mathcal{O}_F} \left( \sum_{T \in \Sigma(\alpha)} \widehat{\deg}\ \widehat{\mathcal{Z}}(T, \mathbf{v}) \right) \cdot q^{\alpha}$$

where $v = (v_1, v_2)$ is the imaginary part of $(\tau_1, \tau_2)$ and $\mathbf{v}$ is related to $(v_1, v_2)$ as in Theorem A. Combining this with Theorem A allows us to relate the classes $\widehat{\mathcal{Y}}(\alpha, v)$ to the Fourier coefficients of a Hilbert modular form as follows.

**Corollary B.** The pullback of $\mathcal{E}'(\tau, 0)$ via the twisted embedding $\mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathfrak{h}_2$ has a Fourier expansion of the form

$$\mathcal{E}'(\tau_1, \tau_2, 0) = \sum_{\alpha \in \mathcal{O}_F} c(\alpha, v) \cdot q^{\alpha}$$

where $v = (v_1, v_2)$ is the imaginary part of $(\tau_1, \tau_2)$. If $\alpha$ satisfies the hypotheses of Theorem A then the Fourier coefficient $c(\alpha, v)$ is equal to the image of $\widehat{\mathcal{Y}}(\alpha, v)$ under the arithmetic pullback

$$\widehat{\mathrm{Ch}}^2(\mathcal{M}) \to \widehat{\mathrm{Ch}}^2(\mathcal{M}_0) \xrightarrow{\sim} \mathbb{R}.$$

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## Functoriality and the Inverse Galois problem

Gordan Savin

This is a report on a joint work with C. Khare and M. Larsen.

Let $\ell$ be a prime. In our previous work [KLS] Langlands functoriality principle was used to show that for infinitely many positive integers $k$ the finite simple group $\mathrm{C}_n(\ell^k) = \mathrm{PSp}_{2n}(\mathbb{F}_{\ell^k})$ is a Galois group over $\mathbb{Q}$ unramified outside $\{\infty, \ell, q\}$ where $q$ is a prime that depends on $k$. The construction is based on the following three steps. First, starting with a cuspidal automorphic representation on the split group $\mathrm{SO}_{2n+1}$ constructed using the Poincaré series, we use the global lift of Cogdell, Kim, Piatetski-Shapiro and Shahidi [CKPS] to obtain a self-dual cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_{2n}$ such that the following three conditions hold:

- $\Pi_\infty$ is cohomological.
- $\Pi_q$ is a supercuspidal representation of depth 0.
- $\Pi_v$ is unramified for all primes $v \neq \ell, q$.

Second, following Kottwitz, Clozel and Harris-Taylor [Ty], one can attach to $\Pi$ an $\ell$-adic representation $r_\Pi : G_\mathbb{Q} \to \mathrm{GL}_{2n}(\bar{\mathbb{Q}}_\ell)$ of the Galois group $G_\mathbb{Q}$ of $\mathbb{Q}$ such that for all primes $v \neq \ell$ the restriction of $r_\Pi$ to the decomposition group $G_{\mathbb{Q}_v}$ is the Langlands parameter of $\Pi_v$. The last step consists of reducing $r_\Pi$ modulo $\ell$. The key ingredient here is that the parameter of $\Pi_q$ can be picked so that $r_\Pi(G_{\mathbb{Q}_q})$ is a meta cyclic group deeply embedded in $r_\Pi(G_\mathbb{Q})$ [KW]. That is, for some large positive integer $d$, $r_\Pi(G_{\mathbb{Q}_q})$ is contained in every normal subgroup of $r_\Pi(G_\mathbb{Q})$ of index less than or equal to $d$. This property is crucial to ensure that the reduction modulo $\ell$ is a simple group of type $\mathrm{PSp}_{2n}(\mathbb{F}_{\ell^k})$.

The purpose of this work is to extend these results and to construct finite simple groups of type $\mathrm{B}_n$ and $\mathrm{G}_2$ as Galois groups over $\mathbb{Q}$. If $\ell$ is an odd prime then we can show that for infinitely many $k$ the finite simple group $\mathrm{B}_n(\ell^k) = \mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ or the classical group $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$ is a Galois group over $\mathbb{Q}$, unramified outside $\{\infty, 2, \ell, q\}$. In general we do not know which of the two is our Galois group, however, if $\ell \equiv 3, 5 \pmod 8$ then it is the finite simple group $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$. Finally and subject to a condition that will be explained in a moment, for infinitely many positive integers $k$ the finite simple group $\mathrm{G}_2(\ell^k)$ appears as a Galois group over $\mathbb{Q}$, unramified outside $\{\infty, \ell, q\}$.

The construction of Galois groups is based on the functorial lift from $\mathrm{Sp}_{2n}$ to $\mathrm{GL}_{2n+1}$ [CKPS] plus the lift from $\mathrm{G}_2$ to $\mathrm{Sp}_6$ using the theta correspondence arising from the minimal representation of the exceptional group $\mathrm{E}_7$ [Sa]. The main technical difficulty (when compared to the work [KLS]) is that $\mathrm{GL}_{2n+1}(\mathbb{Q}_p)$ has self-dual supercuspidal representations only if $p = 2$. Thus, while we can still construct a self-dual cuspidal automorphic representation $\Pi$ of $\mathrm{GL}_{2n+1}$ which should give rise to our desired Galois groups, the local component $\Pi_q$ cannot be supercuspidal. In particular, the existence of the corresponding $\ell$-adic representation $r_\Pi$ has not been established yet since no local component of $\Pi$ is square integrable. This restriction, in the moment, prevents us from completing the construction of Galois groups of type $\mathrm{G}_2$ over $\mathbb{Q}$. For groups of type $\mathrm{B}_n$ we can remedy the situation by requiring that the local component $\Pi_2$ be supercuspidal (which we pick to be of depth one). Existence of a global $\Pi$ with such local component $\Pi_2$ is again obtained using the global lift from $\mathrm{Sp}_{2n}$ plus recently announced (at this conference!) backward lift from $\mathrm{GL}_{2n+1}$ to $\mathrm{Sp}_{2n}$ by Jiang and Soudry.

The local component $\Pi_2$ not only assures us of the existence of the $\ell$-adic representation $r_\Pi$ but it also gives us a certain control over the Galois group obtained by reducing $r_\Pi$ modulo $\ell$. More precisely, $\Pi_2$ can be picked so that the image of the local Langlands parameter is a finite group $\Gamma(2)$ in $\mathrm{GL}_{2n+1}(\mathbb{C})$ with the following properties:

- $\Gamma(2)/[\Gamma(2), \Gamma(2)] \cong \mathbb{Z}/(2n+1)\mathbb{Z}$.
- $[\Gamma(2), \Gamma(2)] \cong (\mathbb{Z}/2\mathbb{Z})^{2n}$.

If $\ell \equiv 3, 5 \pmod 8$ then the first property of $\Gamma(2)$ implies that the Galois group is $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ and not $\mathrm{SO}_{2n+1}(\mathbb{F}_{\ell^k})$. If $n = 3$ then the second property of $\Gamma(2)$ implies that $\Pi_2$ is not a lift from $\mathrm{G}_2(\mathbb{Q}_2)$ and the Galois group is not $\mathrm{G}_2(\ell^k)$.

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## Growth of Selmer groups of Hilbert modular forms over ring class fields

Jan Nekovář

Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbb{Q}$, a prime number $p$ and embeddings $i_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $F$ be a totally real number field and $g$ a cuspidal Hilbert modular eigenform over $F$ of parallel weight $k$ and trivial character. Let $K$ be a totally imaginary quadratic extension of $F$ and $\chi : \mathbf{A}_K^*/K^*\mathbf{A}_F^* \longrightarrow \mathbf{C}^*$ a (continuous) character of finite order. Fix a number field $L \subset \overline{\mathbf{Q}}$ such that $i_\infty(L)$ contains all Hecke eigenvalues of $g$ and all values of $\chi$; denote by $\mathfrak{p}$ the prime of $L$ above $p$ induced by $i_p$. Let $V(g) = V_\mathfrak{p}(g)$ be the two-dimensional representation of $G_F = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$ with coefficients in $L_\mathfrak{p}$ attached to $g$. The Tate twist $V = V(g)(k/2)$ is self-dual in the sense that there exists a skew-symmetric isomorphism $V \stackrel{\sim}{\longrightarrow} V^*(1)$. Identify $\chi$ with the corresponding Galois character $\chi : G_K = \mathrm{Gal}(\overline{\mathbf{Q}}/K) \longrightarrow L_\mathfrak{p}^*$ via the reciprocity map $\mathrm{rec}_K : \mathbf{A}_K^*/K^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$; put $K_\chi = \overline{\mathbf{Q}}^{\mathrm{Ker}(\chi)}$. There are canonical isomorphisms

$$H^1_f(K, V \otimes \chi^{\pm 1}) \stackrel{\sim}{\longrightarrow} (H^1_f(K_\chi, V) \otimes \chi^{\pm 1})^{\mathrm{Gal}(K_\chi/K)} = H^1_f(K_\chi, V)^{(\chi^{\mp 1})}.$$

The conjectures of Bloch and Kato predict that

$$r_{\mathrm{an}}(K, g, \chi) \stackrel{?}{=} h^1_f(K, V \otimes \chi),$$

where $h^1_f(K, V \otimes \chi) := \dim_{L_\mathfrak{p}} H^1_f(K, V \otimes \chi)$ and $r_{\mathrm{an}}(K, g, \chi) := \mathrm{ord}_{s=k/2} L(g \otimes K, \chi, s)$.

**Theorem 1.** Assume that $g$ is potentially $p$-ordinary, i.e., that the base change of $g$ to a suitable finite solvable totally real extension of $F$ is $p$-ordinary (equivalently, that there exists a character of finite order $\varphi : \mathbf{A}_F^*/F^* \longrightarrow \overline{\mathbf{Q}}^*$ such that the newform associated to $g \otimes \varphi$ is $p$-ordinary). If $g$ has complex multiplication by a totally imaginary quadratic extension $K'$ of $F$, assume, in addition, that $p \neq 2$ and that $K' \not\subset K_\chi$. Then: if $2 \nmid r_{\mathrm{an}}(K, g, \chi)$, then $2 \nmid h^1_f(K, V \otimes \chi)$.

**Corollary 1.** Let $K[\infty] \subset K^{\mathrm{ab}}$ be the union of all ring class fields of $K$ (the Galois group $\mathrm{Gal}(K[\infty]/K)$ is the quotient of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ by $\mathrm{rec}_K(\mathbf{A}_F^*)$). Let $K_0/K$ be a finite subextension of $K[\infty]/K$. Assume that $g$ is potentially $p$-ordinary; if $g$ has complex multiplication by a totally imaginary quadratic extension $K'$ of $F$, assume, in addition, that $p \neq 2$ and that $K' \not\subset K_0$. Then

$$h_f^1(K_0, V) := \dim_{L_\mathfrak{p}} H_f^1(K_0, V) \geq |X^-(g, K_0)|,$$

where

$$X^{\pm}(g, K_0) = \{\chi : \mathrm{Gal}(K_0/K) \longrightarrow \mathbb{C}^* \mid \varepsilon(K, g, \chi) = \pm 1\}$$

(above, $\varepsilon(K, g, \chi) = \pm 1$ denotes the sign in the functional equation of $L(g \otimes K, \chi, s)$).

The proof [N 3] combines a deformation result [N 2] with an Euler system argument [N 1] and a non-triviality result for CM points [A-N] (a generalisation of the work of Cornut and Vatsal [C-V]).

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## Degenerate Eisenstein Series on symplectic groups

Goran Muić

In this talk we describe the generalization of usual notion of Siegel Eisenstein series to give a simple and natural construction of some classes of square–integrable automorphic representations. The construction of automorphic representations obtained in this paper is an automorphic version of the local construction of strongly negative unramified representations [4] [1] or of discrete series obtained by Tadić in early 90's. This is taken from our paper [5].

Let $G = \mathrm{Sp}_{2n}$ be a split symplectic group of rank $n$ over a number field $k$. Let $\mathbb{A}$ be the ring of adèles of $k$. Let $W_k$ be the Weil group of $k$. Let $\hat{G}(\mathbb{C}) = SO(2n+1, \mathbb{C})$

1. An unramified representation is strongly negative (resp. negative) if its exponents relative to a Borel lie in (resp. in the closure of) the obtuse Weyl chamber.

be the complex dual group of $G$. We write $B_n = T_n U_n$ for the Borel subgroup of $G$, $W$ for its Weyl group and $\Delta$ for the set of simple roots with respect to $B_n$.

## 1. Construction of Degenerate Eisenstein series

Let $P = MN$ be a maximal $k$–parabolic subgroup of $\mathrm{Sp}_{2n}$. Assume $M \simeq GL(m) \times \mathrm{Sp}_{2n'}$. Let

$$V \subset A(\mathrm{Sp}_{2n'}(k) \setminus \mathrm{Sp}_{2n'}(\mathbb{A}))$$

be an irreducible subspace of the space of automorphic forms. Let us call the corresponding representation $\Pi$. Assume that $V$ is concentrated on $B_{n'}$. (That is, there is a constant term along $B_{n'}$ that does not vanish.) Let $V_0$ be the space of constant terms along $B_{n'}$ of $V$. The map $V \to V_0$ defined by

$$\varphi \mapsto (g' \mapsto \int_{U_{n'}(k) \backslash U_{n'}(\mathbb{A})} \varphi(u'g')du')$$

is an intertwining operator. In particular, since $V$ is irreducible and concentrated on $B_{n'}$, the map is an isomorphism. For $t \in T_{n'}(\mathbb{A})$, we let

$$V_0^t = l(t)V_0,$$

where

$$l(t)F(g') = F(t^{-1}g').$$

The representation of $\prod_{v<\infty} \mathrm{Sp}_{2n'}(k_v) \times (\mathfrak{g}'_\infty, K'_\infty)$ on $V_0^t$ is irreducible and isomorphic to $V_0$ (and to $V$). The main point of that construction is that we can find $0 \neq F \in \sum_{t \in T_{n'}(\mathbb{A})} V_0^t$ and a character $\lambda' : T_{n'}(\mathbb{A}) \to \mathbb{C}^\times$, necessarily trivial on $T_{n'}(k)$, such that

$$F(t'g') = \delta_{B_{n'}}^{1/2}(t')\lambda'(t')F(g'),\ t' \in T_{n'}(\mathbb{A}),\ g' \in \mathrm{Sp}_{2n'}(\mathbb{A}).$$

Hence, we have the following:

$$F(t'u'g') = \delta_{B_{n'}}^{1/2}(t')\lambda'(t')F(g'),\ t' \in T_{n'}(\mathbb{A}),\ u' \in U_{n'}(\mathbb{A}),\ g' \in \mathrm{Sp}_{2n'}(\mathbb{A}). \tag{1}$$

The same identity holds for all functions in the $\prod_{v<\infty} \mathrm{Sp}_{2n'}(k_v) \times (\mathfrak{g}'_\infty, K'_\infty)$-subrepresentation $V' \subset \sum_{t \in T_{n'}(\mathbb{A})} V_0^t$ generated by $F$. Clearly, $V'$ is direct sum of irreducible representations all isomorphic to $V$. Therefore, we may assume that $V'$ is itself irreducible. Then (1) implies the embedding

$$\Pi \hookrightarrow \mathrm{Ind}_{B_{n'}(\mathbb{A})}^{\mathrm{Sp}_{2n'}(\mathbb{A})}(\lambda'). \tag{2}$$

Let $\mu : k^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$ be a (unitary) grössencharacter. The representation $\mu \mathbf{1}_{GL(m,\mathbb{A})}$ is an automorphic representation of $GL(m, \mathbb{A})$ on the one dimensional space $W \subset A(GL(m,k) \setminus GL(m,\mathbb{A}))$. The computation of the constant term $W_0$ of $W$ along Borel subgroup $B_m^{GL}$ gives an embedding:

$$\mu \mathbf{1}_{GL(m,\mathbb{A})} \hookrightarrow \mathrm{Ind}_{B_m^{GL}(\mathbb{A})}^{GL_m(\mathbb{A})} \left( |\ |^{s-(m-1)/2}\mu \otimes \cdots \otimes |\ |^{s+(m-1)/2}\mu \right).$$

Fixing above data, we can realize the induction in stages:

$$\mathrm{Ind}_{M(\mathbb{A})N(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} |\det|^s \mu \mathbf{1}_{GL(m,\mathbb{A})} \otimes \Pi) \hookrightarrow$$
$$\mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} \left( |\ |^{s-(m-1)/2} \mu \otimes \cdots \otimes |\ |^{s+(m-1)/2} \mu \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n'} \right)$$

which enables us to fix nice realization for $\mathrm{Ind}_{M(\mathbb{A})N(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(|\det|^s \mu \mathbf{1}_{GL(m,\mathbb{A})} \otimes \Pi)$ with analytic sections $f_s$. Then we define a *degenerate Eisenstein series* as follows:

$$E(f_s, g) = \sum_{\gamma \in P(k) \backslash \mathrm{Sp}_{2n}(k)} f_s(\gamma g). \tag{3}$$

This series converges for $Re(s)$ sufficiently large and continues to a meromorphic function in $s$. Obviously it as an automorphic form in $A(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$. Finally, its analytic behaviour is controlled by its constant term along Borel $B_n$. More precisely: The Eisenstein series given by (3) is concentrated on the Borel subgroup, and its constant term along $B_n$ is given by

$$E_0(f_s, g) = \int_{U_n(k) \backslash U_n(\mathbb{A})} E(f_s, ug) du = \sum_{w \in W,\ w(\Delta \backslash \{\alpha\}) > 0} M(\lambda(\underline{s}), w) f_s(g). \tag{4}$$

Here $\alpha$ is the unique simple root in $N$ and we write

$$\lambda(s) = |\ |^{s-(m-1)/2} \mu \otimes \cdots \otimes |\ |^{s+(m-1)/2} \mu \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n'}.$$

We remind the reader that $M(\lambda(\underline{s}), w)$ is the standard intertwining operator $\mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} (\lambda(\underline{s})) \to \mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} (w(\lambda(\underline{s})))$.

The expression (4) is studied using local methods of [4] by normalizing intertwining operators as in [3] or [6].

## 2. Construction of Automorphic representations

We say that an Arthur parameter $\varphi : W_k \times SL(2, \mathbb{C}) \to \hat{G}(\mathbb{C})$ is *spherical unipotent* if it is trivial on $W_k$. Thus, it is of the form $\varphi : SL(2, \mathbb{C}) \to \hat{G}(\mathbb{C})$. We can find a unique increasing sequence of positive integers $(m_1, \ldots, m_k)$,

$$\sum_{i=1}^{k} 2m_i + 1 = 2n + 1,$$

such that

$$\varphi = \oplus_{i=1}^{k} V_{2m_i+1}. \tag{5}$$

We remark that $k$ must be odd.

Let $\lambda(\varphi) : T_{min}(k) \setminus T_{min}(\mathbb{A}) \to \mathbb{C}^\times$ be defined by:

$$(|\ |^{-m_k} \otimes |\ |^{-m_k+1} \otimes \cdots \otimes |\ |^{m_{k-1}}) \otimes \cdots \otimes (|\ |^{-m_3} \otimes |\ |^{-m_3+1} \otimes \cdots \otimes |\ |^{m_2}) \otimes$$
$$\otimes (|\ |^{-m_1} \otimes |\ |^{-m_1+1} \otimes \cdots \otimes |\ |^{-1}).$$

We remark that for $k = 1$, the trivial representation $\mathbf{1}_{\mathrm{Sp}_{2m_1}(\mathbb{A})}$ has a unique automorphic realization

$$j(V_{2m_1+1}) : \mathbf{1}_{\mathrm{Sp}_{2m_1}(\mathbb{A})} \to A_2(\mathrm{Sp}_{2m_1}(k) \setminus \mathrm{Sp}_{2m_1}(\mathbb{A})).$$

The usual embedding

$$\mathbf{1}_{\mathrm{Sp}_{2m_1}(\mathbb{A})} \hookrightarrow \mathrm{Ind}_{B_{m_1}(\mathbb{A})}^{\mathrm{Sp}_{2m_1}(\mathbb{A})}(|\ |^{-m_1} \otimes \cdots \otimes |\ |^{-1})$$

is obtained computing the constant term along $B_{m_1}$ on the space of constant functions $Image(j(V_{2m_1+1}))$.

First, we describe the construction of the spherical component.

**Theorem 6.** Let $k > 0$ be an odd integer. Let $K$ be the usual maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{A})$. Under the above assumptions, the unique irreducible $K$–spherical subquotient $\sigma(\varphi)$ of the globally induced representation $\mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(\lambda(\varphi))$ is its subrepresentation, and there is a non–zero embedding

$$j(\varphi) : \sigma(\varphi) \to A_2(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$$

constructed recursively as follows. Let $k \geq 3$. Put $\varphi' = \oplus_{i=1}^{k-2} V_{2m_i+1}$ and $2n'+1 = \sum_{i=1}^{k-2} 2m_i + 1$. Consider the global induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})} \left(|\det|^s \mathbf{1}_{\mathrm{GL}(2m_{k-1}+1,\mathbb{A})} \otimes Image(j(\varphi'))\right), \tag{7}$$

where $P$ is a standard parabolic subgroup of $\mathrm{Sp}_{2n'+4m_{k-1}+2}$ with Levi factor $\mathrm{GL}(2m_{k-1}+1) \times \mathrm{Sp}_{2n'}$. (At $s = 0$ this representation is unitary and therefore semisimple (of infinite length).) Then the map obtained from a degenerate Eisenstein series

$$f_s \mapsto E(f_s, \cdot)|_{s=0} \tag{8}$$

is an intertwining operator

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})} \left(\mathbf{1}_{\mathrm{GL}(2m_{k-1}+1,\mathbb{A})} \otimes Image(j(\varphi'))\right) \to$$
$$A(\mathrm{Sp}_{2n'+4m_{k-1}+2}(k) \setminus \mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})),$$

which is non–trivial on the unique irreducible $K$–spherical subrepresentation of (7) for $s = 0$; let us write $X$ for the image of the $K$–spherical subrepresentation. Taking the constant term of $X$ along Borel $B_{n'+2m_{k-1}+1}$ we obtain the following embedding:

$$(9) \qquad X \hookrightarrow \mathrm{Ind}_{B_{n'+2m_{k-1}+1}(\mathbb{A})}^{\mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})} \left( |\ |^{-m_{k-1}} \otimes |\ |^{-m_{k-1}+1} \otimes \cdots \otimes |\ |^{m_{k-1}} \otimes \lambda(\varphi') \right)$$

which we use to construct degenerate Eisenstein series $f_s \mapsto E(f_s, g) = \sum_{\gamma \in P(k) \backslash \mathrm{Sp}_{2n}(k)} f_s(\gamma g)$ attached to the global induced representation

$$(10) \qquad \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} \left( |\det|^s \mathbf{1}_{\mathrm{GL}(m_k - m_{k-1}, \mathbb{A})} \otimes X \right),$$

where $P$ is a standard parabolic subgroup of $\mathrm{Sp}_{2n}$ with Levi factor $\mathrm{GL}(m_k - m_{k-1}) \times \mathrm{Sp}_{2n'+4m_{k-1}+2}$. Then the map

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} \left( |\det|^{\frac{m_{k-1}+m_k+1}{2}} \mathbf{1}_{\mathrm{GL}(m_k - m_{k-1}, \mathbb{A})} \otimes X \right)^K \to A(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$$

given by

$$(11) \qquad f_{\frac{m_{k-1}+m_k+1}{2}} \mapsto \left( s - \frac{m_{k-1} + m_k + 1}{2} \right)^2 E(f_s, \cdot)|_{s = \frac{m_{k-1}+m_k+1}{2}}$$

is well–defined and non–trivial. Let $E$ be a $\prod_{v<\infty} \mathrm{Sp}_{2n}(k_v) \times (\mathfrak{g}_\infty, K_\infty)$ – subrepresentation of $A(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$ generated by the image of the space of $K$–invariants. Then $E$ is irreducible, contained in the space of square–integrable automorphic forms $A_2(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$, and it induces the required embedding $j(\varphi) : \sigma(\varphi) \simeq E \subset A_2(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$. Finally, the embedding $\sigma(\varphi) \hookrightarrow \mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(\lambda(\varphi))$ is obtained computing the constant term of $E$ along $B_n$.

Second, we can describe the generalization of this theorem for non–spherical representations. This requires the construction of a particular local normalized intertwining operator. (See [5], Theorem 6-21 for details.)

We end by the following remark which follows from Theorem 6. Let $k = \mathbb{Q}$ and $\infty$ be the unique Archimedean place of $\mathbb{Q}$. Then $\sigma(\varphi)_\infty$ is automorphic and in fact

$$\sigma(\varphi)_\infty \hookrightarrow L^2(\mathrm{Sp}_{2n}(\mathbb{Q}) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$$

with the image contained in the space of residual representations.

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**Arithmetic Intersection and a conjecture of Colmez**

Tonghai Yang

Let $F = \mathbb{Q}(\sqrt{D})$ with $D \equiv 1 \mod 4$ being prime. Let $\mathcal{M}$ be the moduli stack over $\mathbb{Z}$ of abelian varieties with real multiplications $(A, \iota, \lambda)$ with Deligne-Pappas $\partial_F^{-1}$-polarization. Then $\mathcal{M}(\mathbb{C}) = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^2$. Let $T_m$ be the Hirzebruch-Zagier divisor in $M = \mathcal{M}(C)$ [HZ], and let $\mathcal{T}_m$ be the Zariski closure of $T_m$ in $\mathcal{M}$. When $m = q$ is a prime split in $F$, it is basically isomorphic to the open modular curve $\mathcal{Y}_0(q)$.

Let $K = F(\sqrt{\Delta})$ be a quartic CM number field with real quadratic subfield $F$, and let $\mathcal{CM}(K)$ be the moduli stack over $\mathbb{Z}$ of CM abelian varieties $(A, \iota, \lambda)$ such that $(A, \iota|_{\mathcal{O}_F}, \lambda) \in \mathcal{M}$ and the Rosati involution associated to $\lambda$ gives the complex conjugation in $K$. Notice that $\mathcal{CM}(K)$ and $\mathcal{T}_m$ intersect properly when $K$ is non-biquadratic. In this case, let $\tilde{K}$ be the reflex field of $K$ with respect to a CM type of $K$ with real quadratic field $\tilde{F}$. We proved in [Ya1] the following theorem.

**Theorem 1.** *Assume that $K$ is non-biquadratic and that $\mathcal{O}_K = \mathcal{O}_F + \mathcal{O}_F \frac{w+\sqrt{\Delta}}{2}$ is a free $\mathcal{O}_F$-module such that $\tilde{D} = \Delta\Delta' \equiv 1 \mod 4$ is a prime. Then*

$$\mathcal{T}_m.\mathcal{CM}(K) = \frac{1}{2} b_m$$

*where*

$$b_m = \sum_p b_m(p)$$

*is defined as follows:*

$$b_m(p) = \sum_{\mathfrak{p}|p} \sum_{t = \frac{n+m\sqrt{\tilde{D}}}{2D} \in d_{\tilde{K}/\tilde{F}}^{-1}, |n| < m\sqrt{\tilde{D}}} B_t(\mathfrak{p}) \tag{1}$$

*where*

$$B_t(\mathfrak{p}) = \begin{cases} 0 & \text{if } \mathfrak{p} \text{ is split in } \tilde{K}, \\ (\mathrm{ord}_{\mathfrak{p}} t_n + 1)\rho(t d_{\tilde{K}/\tilde{F}} \mathfrak{p}^{-1}) \log|\mathfrak{p}| & \text{if } \mathfrak{p} \text{ is not split in } \tilde{K}, \end{cases} \tag{2}$$

*and*

$$\rho(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_{\tilde{K}} : N_{\tilde{K}/\tilde{F}}\mathfrak{A} = \mathfrak{a}\}.$$

When $K = F(\sqrt{d})$ be biquadratic, let $\tilde{K} = \mathbb{Q}(\sqrt{d}) \oplus \mathbb{Q}(\sqrt{Dd})$ with real quadratic 'subfield' $\tilde{F} = \mathbb{Q} \oplus \mathbb{Q}$. Then the same formula holds when $\mathcal{CM}(K)$ and $\mathcal{T}_m$ intersect properly. This happens, for example, when $m = q$ is a prime split in $F$ but inert in $k = \mathbb{Q}(\sqrt{d})$.
The basic idea is to prove a weaker version of the formula when $m = q$ is a prime split in $F$, and derive general cases from the weaker version using Faltings's height machine and results in [BY] and [BBK].

Now we briefly describe an application of Theorem 1 to a conjecture of Colmez, which is a beautiful generalization of the celebrated Chowla-Selberg formula [CS]. In proving the famous Mordell conjecture, Faltings introduces the so-called Faltings height $h_{\mathrm{Fal}}(A)$ of an Abelian variety $A$, measuring the complexity of $A$ as a point in a Siegel modular variety. When $A$ has complex multiplication, it only depends on the CM type of $A$ and has a simple description as follows. Assume that $A$ is defined over a number field $L$ with good reduction everywhere, and let $\omega_A \in \Lambda^g \Omega_A$ be a Neron differential of $A$ over $\mathcal{O}_L$, non-vanishing everywhere. Then the Faltings' height of $A$ is defined as (our normalization is slightly different from that of [Co])

$$(3) \quad h_{\mathrm{Fal}}(A) = -\frac{1}{2[L:\mathbb{Q}]} \sum_{\sigma: L \hookrightarrow \mathbb{C}} \log \left| (\frac{1}{2\pi i})^g \int_{\sigma(A)(\mathbb{C})} \sigma(\omega_A) \wedge \overline{\sigma(\omega_A)} \right| + \log \# \Lambda^g \Omega_A / \mathcal{O}_L \omega_A .$$

Here $g = \dim A$. Colmez gives a beautiful conjectural formula to compute the Faltings height of a CM abelian variety in terms of the log derivative of certain Artin L-series associated to the CM type [Co]. When $A$ is a CM elliptic curve, the height conjecture is a reformulation of the well-known Chowla-Selberg formula. Colmez proved his conjecture up to a multiple of $\log 2$ when the CM field (which acts on $A$) is abelian, refining Gross's [Gr] and Anderson's [An] work. A key point is that such CM abelian varieties are quotients of the Jacobians of the Fermat curves, so one has a model to work with. The following theorem confirms the first non-abelian case of the conjecture.

**Theorem 2.** *Let the notation and the assumptions be as in 1. Let $A$ be a CM abelian variety of CM type $(\mathcal{O}_K, \Phi)$. Then*

$$(4) \qquad h_{Fal}(A) = \frac{1}{2} \beta(K/F).$$

Next let $\mathcal{A}$ be the moduli stack of principally polarized abelian surfaces, so that $\mathcal{A}(\mathbb{C}) = \mathrm{Sp}_4(\mathbb{Z}) \backslash \mathbb{H}_2$. There is a natural map

$$(5) \qquad j : \mathcal{M} \to \mathcal{A}, (A, \iota, \lambda) \mapsto (A, \lambda(\frac{\epsilon}{\sqrt{D}}))$$

which is proper and generically 2 to 1. Here $\epsilon$ is a fundamental unit so that $\lambda(\frac{\epsilon}{\sqrt{D}})$ is a principal polarization of $A$. Similar to Hirzebruch-Zagier curves, there are also so-called Humbert surfaces $G_m$ in $\mathcal{A}(\mathbb{Q})$ (see for example [Ge]). Let $\mathcal{G}_m$ be the Zariski closure of $G_m$ in $\mathcal{A}$. Then one has

$$j_*\mathcal{M} = 2\mathcal{G}_D,$$

and

$$j^*\mathcal{G}_m = \sum_{\frac{Dm-x^2}{4}\in\mathbb{Z}_{>0},x>0} \mathcal{T}_{\frac{Dm-x^2}{4}}$$

if $Dm$ is not a square. Notice also that $j_*\mathcal{CM}(K)$ is the moduli stack of principally polarized abelian surfaces. Theorem 1 has the following consequence:

**Theorem 3.** *Let the notation and assumption be as in Theorem 1, and assume $Dm$ is not a square. Then*

$$\mathcal{G}_m.j_*\mathcal{CM}(K) = \frac{1}{2} \sum_{\frac{Dm-x^2}{4}\in\mathbb{Z}_{>0},x>0} b_{\frac{Dm-x^2}{4}}.$$

Since $\operatorname{div}\chi_{10} = 2\mathcal{G}_1$, and $\mathcal{H}_1$ characterizes exactly the abelian surfaces which are not Jacobians of genus two curves. So we have

**Theorem 4.** *Let $K$ be the number field as in Theorem 1. Let $C$ be a genus two curve over a number field $L$ such that its Jacobian $J(C)$ has CM by $\mathcal{O}_K$ and has good reduction everywhere. Let $l$ be a prime. If $C$ has bad reduction at a prime $\mathfrak{l}|l$ of $L$, then*

$$\sum_{0<n<\sqrt{D},odd} b_{\frac{D-n^2}{4}}(l) \neq 0 \tag{6}$$

*In particular, $l \leq \frac{D\tilde{D}}{64}$. Conversely, if (6) holds for a prime $l \neq 2$, then there is a genus two curve $C$ over a number field $L$ such that*

*(1) $J(C)$ has CM by $\mathcal{O}_K$ and has good reduction everywhere, and*

*(2) $C$ has bad reduction at a prime $\mathfrak{l}$ above $l$.*

The theorem can be used to solve Lauter's conjecture on CM values of Igusa invariants and also has application to the bad reduction of CM genus two curves. We refer to [Ya1] for more details.

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## Formes automorphes de carré intégrable non cuspidales

Colette Mœglin

On fixe un groupe $G$, classique, et on admet que pour ce groupe on dispose d'une description à la Arthur des formes automorphes de carré intégrable utilisant l'endoscopie ordinaire et l'endoscopie tordue. Le but est alors de donner des conditions nécessaires et suffisantes à partir des données fournies par cette description pour qu'une représentation de carré intégrable irréductible de $G$ ne soit pas cuspidale. Le corps de base est ici un corps de nombres, noté $k$.

Fixons $\pi$ une représentation de carré intégrable irréductible de $G$. On rappelle les 2 points de la description d'Arthur qui nous serviront explicitement ici. Cette description inclut d'abord un paramètre global associé à $\pi$ que l'on peut voir comme une collection de couples $(\rho, b)$ où $\rho$ est une représentation cuspidale, unitaire, irréductible d'un groupe linéaire $GL(d_\rho)$ convenable et $b$ est un entier. On appelle $Jord(\pi)$ cette collection de couples. En toute place $v$ de $k$, on considère $\psi_v$ le morphisme, défini à conjugaison près, de $W'_v \times SL(2,\mathbb{C})$ dans un groupe linéaire convenable: ici $W'_v$ est le groupe de Weil-Deligne associé à la complétion $k_v$ et par définition

$$\psi_v := \oplus_{(\rho,b)\in Jord(\pi)} Langlands(\rho_v) \otimes [b],$$

où $Langlands(\rho_v)$ est l'homomorphisme de $W'_v$ dans un groupe linéaire convenable obtenue via la correspondance de Langlands (Harris-Taylor et Henniart) et où $[b]$ est la représentation de dimension $b$ de $SL(2,\mathbb{C})$. Dans la description d'Arthur $\psi_v$ est, à conjugaison près, à valeurs dans le groupe dual de $G$. De plus à un tel $\psi_v$, on associe un ensemble fini, $\Pi(\psi_v)$, de représentations irréductibles de $G$ avec des propriétés de transfert; ce paquet est maintenant bien connu si $v$ est non archimédien et moins bien connu, si $v$ est archimédien. Le résultat annoncé par Arthur est qu'en toute place $v$, la composante locale $\pi_v$ de $\pi$ est dans $\Pi(\psi_v)$.

Pour simplifier, on suppose que le caractère infinitésimal de $\pi$ est régulier (on renvoie aux papiers écrits pour le cas général). Le but de l'exposé est d'expliquer la conjecture suivante; $\pi$ n'est pas cuspidale si et seulement si il existe $(\rho, b) \in Jord(\pi)$ avec $b \geq 2$ tel que

pour tout $(\rho', b') \in Jord(\psi)$ avec $b' = b - 1$, $L(\rho \times \rho', 1/2) \neq 0$;

pour toute place $v$ de $k$, la représentation $\psi_v$ de $W'_v \otimes SL(2,\mathbb{C})$ contient la représentation (non nécessairement irréductible) $Langlands(\rho_v) \otimes [b]$ avec multiplicité au moins 1, ce qui permet de définir la représentation $\psi_v^-$ qui se déduit de

$\psi_v$ en remplaçant une copie de $Langlands(\rho_v) \otimes [b]$ par $Langlands(\rho_v) \otimes [b-2]$. Et il doit exister une représentation irréductible $\sigma_v$ dans $\Pi(\psi_v^-)$ tel que $\pi_v$ est un quotient de l'induite $\rho_v \otimes \sigma_v$.

Dans l'état actuel, on montre que ces conditions sont nécessaires si $\pi$ vérifie quelques conditions restrictives à l'infini, par exemple $\pi_\infty$ est dans le paquet de Langlands à l'intérieur du paquet d'Arthur, ou $G$ est un groupe unitaire et $\pi_\infty$ a de la cohomologie. On montre aussi que ces conditions sont suffisantes par exemple si toutes les représentations $\pi_v$ sont dans le paquet de Langlands à l'intérieur du paquet d'Arthur; montrer la suffisance de telles conditions nécessite d'utiliser la formule de multiplicité d'Arthur dont la description dépasse le cadre de ce résumé.

Le point nouveau est que l'on sait démontrer l'holomorphie en tous les points de la demi-droite réelle positive des opérateurs d'entrelacement normalisés à la Arthur c'est-à-dire en utilisant en toute place la normalisation qui permet de compléter les fonctions $L$ partielles qui sont les contributions des places non ramifiées aux termes constants des séries d'Eisenstein. On montre comment cette normalisation se comprend bien en termes de la normalisation de Langlands-Shahidi quand on considère les représentations des groupes linéaires associés aux $\psi_v^-$.

Cet exposé reprend les prépublications disponibles sur
http://www.math.jussieu.fr/˜moeglin:
Formes automorphes de carré intégrable non cuspidales
Holomorphie des opérateurs d'entrelacement normalisés

## The supersingular locus in Siegel modular varieties, and Deligne-Lusztig varieties

Ulrich Görtz
(joint work with Chia-Fu Yu)

Let $p$ be a prime number, and $g \geq 1$ an integer. We consider the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, and the following variant, the Siegel modular variety $\mathcal{A}_{g,I}$ with Iwahori level structure at $p$, which is much less well understood. By definition, $\mathcal{A}_I$ is the space of isomorphism classes of tuples

$$(A_0 \to A_1 \to \cdots \to A_g, \lambda_0, \lambda_g, \eta),$$

where the $A_i$ are abelian varieties of dimension $g$, the maps $A_i \to A_{i+1}$ are isogenies of degree $p$, $\lambda_0$ and $\lambda_g$ are principal polarizations of $A_0$ and $A_g$, respectively, such that the pull-back of $\lambda_g$ is $p\lambda_0$, and $\eta$ is a level structure away from $p$. We consider these spaces over an algebraic closure $k$ of $\mathbb{F}_p$. Both have dimension $g(g+1)/2$.

Inside $\mathcal{A}_g$, we have the *supersingular locus*, i. e. the closed subset of those abelian varieties which are isogeneous to a product of supersingular elliptic curves. There are a number of results describing the geometry of the supersingular locus. For instance, it was proved by Li and Oort [3] that the dimension of the supersingular locus is $[g^2/4]$. It is known to be connected if $g \geq 2$, and there is a formula for the number of irreducible components, in terms of a certain class number, also proved in loc. cit.

On the other hand, in the Iwahori case, currently very little is known about the supersingular locus. Even its dimension is known only for $g \leq 3$ (but see below). Note that the situation here is definitely more complicated than in the case of $\mathcal{A}_g$; as an example, in the case $g = 2$, the supersingular locus coincides with the $p$-rank 0 locus, but it is not contained in the closure of the $p$-rank 1 locus. In addition, it is not equi-dimensional (see [6, Prop. 6.3]).

The supersingular locus (and especially its cohomology) are interesting objects from the point of view of automorphic representations and the Langlands program.

On the other hand we have the Kottwitz-Rapoport stratification (KR stratification)

$$\mathcal{A}_I = \coprod_{x \in \mathrm{Adm}} \mathcal{A}_{I,x}$$

by locally closed subsets, which should be thought of as a stratification by singularities. It corresponds to the stratification by Schubert cells of the associated local model. In terms of abelian varieties, we can express this as follows: the strata are the loci where the relative position of the chain of de Rham cohomology groups $H^1_{DR}(A_i)$ and the chain of Hodge filtrations inside each $H^1_{DR}(A_i)$ is constant. This relative position "is" an element of the extended affine Weyl group $\widetilde{W}$ of the group $GSp_{2g}$, and the so-called admissible set $\mathrm{Adm} \subset \widetilde{W}$ is the (finite) set of relative positions which actually occur. The KR stratification on the space $\mathcal{A}_g$ consists of only one stratum, and hence does not provide any interesting information.

In general neither of these stratifications is a refinement of the other one. Nevertheless, there are some relations between them. For instance, the ordinary Newton stratum (which is open and dense in $\mathcal{A}_I$) is precisely the union of the maximal KR strata. At the other extreme, the supersingular locus is not in general a union of KR strata. However, it is our impression that those KR strata which are entirely contained in the supersingular locus make up a significant part of it. We call these KR strata *supersingular*.

We can produce a list of supersingular KR strata which admit a very simple geometric description in terms of Deligne-Lusztig varieties. To give a more precise description, we identify the elements of $\widetilde{W}$, and in particular of Adm with alcoves in the standard apartment (in the Bruhat-Tits building of $GSp_{2g}$). Each alcove $x$ is determined by its vertices $x_0, \dots, x_g$. In particular we have the base alcove $\tau = (\tau_0, \dots, \tau_g)$, which corresponds to the unique 0-dimensional KR stratum. Fix $0 \leq i \leq [g/2]$. Let

$$W_{\{i,g-i\}} = \{x \in \mathrm{Adm};\ x_i = \tau_i, x_{g-i} = \tau_{g-i}\}.$$

It is not hard to show that for $x \in W_{\{i,g-i\}}$, the KR stratum $\mathcal{A}_x$ is contained in the supersingular locus. We conjecture that the set $\bigcup_i W_{\{i,g-i\}}$ is the set of supersingular KR strata.

We denote by $G_{\{i,g-i\}}$ the algebraic group over $\mathbb{F}_p$ whose Dynkin diagram is obtained by removing the vertices $i$ and $g-i$ from the extended Dynkin diagram of $GSp_{2g}$, which splits over $\mathbb{F}_{p^2}$, and where the Frobenius (of $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$) acts by $j \mapsto g-j$. The set $W_{\{i,g-i\}}$ can be identified with the Weyl group of $G_{\{i,g-i\}}$ in a natural way.

**Theorem 2.** Fix a point $(A_i)_i$ in the minimal KR stratum $\mathcal{A}_\tau$. Let $0 \le i \le [g/2]$, and let $x \in W_{\{i,g-i\}}$. Denote by $\pi$ the projection from $\mathcal{A}_{g,I}$ to the analogous moduli space of partial lattice chains $(B_i \to B_{g-i})$. There is an isomorphism

$$\pi^{-1}((A_j)_{j\in\{i,g-i\}}) \xrightarrow{\cong} \mathrm{Flag}(G_{\{i,g-i\}}).$$

**Theorem 3.** Let $0 \le i \le [g/2]$, and let $x \in W_{\{i,g-i\}}$. We identify $W_{\{i,g-i\}}$ with the Weyl group of $G_{\{i,g-i\}}$, and let $x^{-1}$ be the inverse element of $x$ in this Weyl group. The KR stratum $\mathcal{A}_x$ is a disjoint union

$$\mathcal{A}_x \xrightarrow{\cong} \coprod X(x^{-1}),$$

of copies of $X(x^{-1})$, which is by definition the Deligne-Lusztig variety for $G_{\{i,g-i\}}$ associated with $x^{-1}$. The union ranges over the finite set $\pi(\mathcal{A}_\tau)$, where $\pi$ is as in the previous theorem.

As a direct corollary, we obtain

**Corollary 2.** (a) If $p \ge 2g$, and $w \in W_{\{i,g-i\}}$ for some $0 \le i \le [g/2]$, then the KR stratum associated with $w$ is affine.

(b) There is an explicit formula for the number of connected components of KR strata as above (see [1], section 6).

(c) The dimension of the supersingular locus is greater than or equal to $g^2/2$ if $g$ is even, and $g(g-1)/2$ if $g$ is odd.

Although the union of the supersingular KR strata is not all of the supersingular locus, we still get a significant part. The following table backs this up for small $g$. The dimension of the whole moduli space $\mathcal{A}_I$ is $g(g+1)/2$. The dimension of the union of all superspecial KR strata is $g^2/2$ if $g$ is even, and $g(g-1)/2$ otherwise (and this is how part (3) of the corollary is obtained). The numbers of KR strata, and of KR strata of $p$-rank 0 can be obtained from Haines' paper [2], Prop. 8.2, together with the results of Ngô and Genestier [4]; we indicate it to show the combinatorial complexity of these questions. The dimension of the $p$-rank 0 locus was obtained by a computer program. We conjecture that it is given by $[g^2/2]$ in general. This formula has been checked for all $g \le 9$. If the conjecture holds true, it follows in particular that for $g$ even, the dimension of the supersingular locus is $g^2/2$. Note that for $g = 5$ we do not know the dimension of the supersingular locus; for $g = 6$ we know it only because it has to lie between the dimension of the union of all superspecial KR strata and the dimension of the $p$-rank 0 locus. As a word of warning one should say that neither of these loci is equi-dimensional in general.

| $g$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| number of KR strata | 3 | 13 | 79 | 633 | 6331 | 75973 |
| number of KR strata of $p$-rank 0 | 1 | 5 | 29 | 233 | 2329 | 27949 |
| dim. of union of superspecial KR strata | 0 | 2 | 3 | 8 | 10 | 18 |
| dim. of supersingular locus | 0 | 2 | 3 | 8 | ? | 18 |
| dim. of $p$-rank 0 locus | 0 | 2 | 4 | 8 | 12 | 18 |
| $\dim \mathcal{A}_I$ | 1 | 3 | 6 | 10 | 15 | 21 |

Furthermore, it can be shown that any irreducible component of maximal dimension of the union of all superspecial KR strata is actually an irreducible component of the $p$-rank 0 locus, and hence in particular an irreducible component of the supersingular locus.

For further details we refer to our recent preprint [1].

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## On Kim's exterior square functoriality for $GL(4)$

Guy Henniart

Let $k$ be a number field, and let $R$ be a unitary cuspidal automorphic representation of $GL(4)$ over the adèle ring of $k$. For each place $v$ of $k$, let $S_v$ be the 4-dimensional representation of the Weil-Deligne group of $k_v$ corresponding, via the local Langlands correspondence, to the component of $R$ at $v$. Form the 6-dimensional representation of this Weil-Deligne group, by composing $S_v$ with the exterior square representation of $GL(4,\mathbb{C})$; it corresponds to a generic smooth irreducible representation $P_v$ of $GL(4,k_v)$.

**Theorem.** *There is a unique automorphic representation $T$ of $GL(6)$ over the adèle ring of $k$, which has component $P_v$ at each place $v$. It is parabolically induced from unitary cuspidals.*

In a recent preprint, Asgari and Raghuram have determined when that representation T is cuspidal.

The theorem above completes work of Henry Kim (JAMS 16, 2003), who proved the existence of such a representation $T$ with component $P_v$ at all places with residue characteristic greater than 4, or places where $R_v$ is not supercuspidal. What we show is that at places above 2 or 3 where $R_v$ is supercuspidal, $T$ also has component $P_v$. At such places, the component $T_v$ is uniquely determined by local $L$ and epsilon factors obtained via the Langlands-Shahidi method. The idea of proof is to insert the local situation into a global one as above, where $R$ has an associated system of $l$-adic representations, and it is known a priori, by the results of Harris and Taylor, that $T_v$ has to be equal to $P_v$.

## Hitchin fibration and fundamental lemma

Bao Châu Ngô

We give an overview of the proof of Langlands-Shelstad's fundamental lemma for Lie algebras based on the geometry of the Hitchin fibration. The basic ingredient is the description of the supports of simple perverse sheaves which occur in the decomposition of the $l$-adic cohomology of the Hitchin fibration. Based on the knowledge of these supports, we can realize general stable (or kappa) orbital integrals as a limit of product of very simple stable (or kappa) orbital integrals. This allows us to reduce the fundamental lemma for general groups essentially to the case of $SL(2)$ which is known thanks to Labesse-Langlands work.

### References

[1] Ngô, Bao Châu. Le lemme fondamental pour les algebres de Lie, preprint available at http://arxiv.org/abs/0801.0446

## Laudatio for Bao Châu Ngô

Michael Rapoport

Dear Ngô Bao Châu,
Dear Professor Remmert,
Ladies and Gentlemen,

It is a great pleasure to give the laudatory speech for Ngô Bao Châu who is the recipient of the 2007 Oberwolfach prize. This prize is awarded approximately every three years to a young European mathematician below the age of 36 by the Oberwolfach Foundation in cooperation with the Mathematical Research Institute Oberwolfach and its Scientific Committee. The field of mathematics within which the recipient of this year's prize was selected is algebra and number theory. Ngô Bao Châu was chosen for his work on the Fundamental Lemma conjecture of Langlands and Shelstad. With his proof of this long standing conjecture, Ngô has established himself as a leader in a central area of mathematics at the crossroads between algebraic geometry and automorphic forms.

I have structured my talk as follows. First, I will give a short curriculum vitae of Ngô in the form of a table. Then I will place the result of Ngô in its historical context. Finally, I will state a special case of his result and give some comments on his proof.

1. **Short curriculum vitae of Ngô Bao Châu**
   - 1972 born in Hanoi, Vietnam
   - 1990 moves to France
   - 1992-1995 student at the ENS, rue d'Ulm
   - 1993-1997 doctoral studies at U. de Paris Sud, with G. Laumon
   - 1997 dissertation 'Le lemme fondamental de Jacquet et Ye'
   - 1998-2004 chargé de recherches au CNRS, at Univ. de Paris Nord
   - 2004 Habilitation
   - 2004– Professor U. de Paris-Sud
   - 2006– IAS, Princeton
   - *distinctions*: Clay Research Award 2004, Speaker at ICM 2006.
2. **Background**

The conjecture of Langlands and Shelstad lies in the field of automorphic forms. In the beginning of the $20^{\text{th}}$ century this theory was the theory of modular forms, i.e., of holomorphic functions on the upper half plane transforming in a prescribed way under the action of discrete groups of conformal motions. It was only in the 1950's, under the influence of I. Gelfand and Harish-Chandra, that the theory of automorphic forms on arbitrary semi-simple Lie groups, or semi-simple algebraic groups, was developed. In the 1960's the theory was dramatically refocused through the introduction by R. Langlands of his functoriality principle. This principle is a conjecture that stipulates correspondences between automorphic forms on semi-simple groups which are related by a homomorphism between their Langlands dual groups. This principle is surely among the most ingenious ideas of the last century and constitutes the deepest statement about automorphic forms known to us today (as a conjecture!). Langlands himself also showed how his functoriality principle bears upon one of the central problems of arithmetic algebraic geometry, that of calculating the zeta function of Shimura varieties and of determining the $\ell$-adic Galois representations defined by their cohomology.

At the same time, Langlands emphasized the importance of the Selberg trace formula as a tool for a proof of the functoriality principle in many cases, for instance for establishing correspondences between automorphic forms on classical groups. He also pointed to the relevance of the Selberg trace formula for the zeta function problem.

One of the first tests of these radically new ideas is contained in the paper by J.-P. Labesse and Langlands on $SL_2$. At a certain point in their paper they prove an innocuous-looking statement that later turned out to be an instance of a general phenomenon. This result allowed them to construct a transport of certain functions between groups, dual to the desired transport of automorphic forms.

Langlands soon recognized the importance of this statement in the general context of the functoriality principle, and named this conjecture 'fundamental lemma'; a more appropriate name would have been the *fundamental matching conjecture*. In joint work with D. Shelstad, he formulated a precise conjecture in the general case. Already the formulation of this conjecture is very complicated, and, in fact, the conjecture comes in several variants (e.g., endoscopic version, or base change version, etc.), depending on which homomorphism on the Langlands dual group one uses to transport automorphic forms.

In the ensuing 25 years the matching conjecture has turned out to be absolutely essential in achieving progress on the functoriality principle. Furthermore, R. Kottwitz showed that the matching conjecture is also crucial in the zeta function problem. In spite of its importance and its proof in special cases, the fundamental lemma resisted intense efforts and its proof seemed out of reach. Indeed, quite a number of papers were written during this period which were conditional on the fundamental matching conjecture.

Ngô has now finally proved this conjecture and has thereby removed this major stumbling block to further progress. More precisely, he proved the endoscopic fundamental lemma for unitary groups in joint work with G. Laumon. Very recently, he posted a 188 page manuscript with a solution in the general endoscopic case.

What is the fundamental lemma about? As indicated above, it arises in the comparisons of trace formulas. The trace formula is an identity, where on one side, the 'geometric side', there appear sums of orbital integrals. The fundamental lemma is an identity between orbital integrals of simple functions, like characteristic functions of open compact subgroups.

The field of automorphic forms, and in particular the fundamental lemma, has the reputation of being impenetrable, with results only appreciable by an insider. In the rest of my talk I want to show that this is not necessarily so. I will state a special case of the FL theorem of Laumon/Ngô which is highly non-trivial, yet can be understood by many. And my hope is that the beauty of the statement, if not of its proof, can be appreciated by all.

## 3. The theorem

As mentioned above, the result of Laumon/Ngô concerns orbital integrals for unitary groups. As a warm-up, let us first consider orbital integrals for $\mathrm{GL}(n)$:

$$\mathrm{O}^G_\gamma(1_K) = \int_{G_\gamma\backslash G} 1_K(g^{-1}\gamma g)\frac{\mathrm{d}g}{\mathrm{d}g_\gamma}\ ,$$

where we used the following notation.

- $F$ non archimedean local field, $O_F$ the ring of integers of $F$
- $G = \mathrm{GL}(n, F)$, $K = \mathrm{GL}(n, O_F)$ maximal compact open subgroup.
- $1_K =$ the characteristic function of $K$
- $\gamma \in G$ regular semi-simple, hence its centralizer $G_\gamma$ is a maximal torus in $G$
- $\mathrm{d}g$ and $\mathrm{d}g_\gamma$ Haar measures on $G$ and $G_\gamma$.

This orbital integral has a combinatorial description as the cardinality of a set of lattices, as follwos.

$$\mathrm{O}_\gamma^G = |X_\gamma/\Lambda_\gamma|.$$

Here:

- $X_\gamma = \{O_F\text{-lattices } M \subset F^n \mid \gamma(M) = M\}$,
- $\gamma$ is regular semi-simple, i.e., the $F$-subalgebra $F[\gamma]$ of $\mathrm{M}_n(F)$ generated by $\gamma$ is commutative semi-simple of dimension $n$, hence $F[\gamma] = \prod_{i\in I} E_i$, where $(E_i)_{i\in I}$ is a finite family of finite separable extensions of $F$,
- after choosing uniformizers $\pi_i = \pi_{E_i}$ in the $E_i$ we have $F[\gamma]^\times \cong \Lambda_\gamma \times K_\gamma$, where $\Lambda_\gamma = \mathbb{Z}^I$ and $K_\gamma = \prod_{i\in I} O_{E_i}^\times$ is a maximal compact open subgroup of $G_\gamma = F[\gamma]^\times$,
- $\Lambda_\gamma \subset G_\gamma$ acts freely on $X_\gamma$,
- we normalized the Haar measures by $\mathrm{vol}(K, \mathrm{d}g) = \mathrm{vol}(K_\gamma, \mathrm{d}g_\gamma) = 1$.

Thus we see that this simple orbital integral unwinds as a cardinality, namely the number of lattices fixed under translation by $\gamma$, taken up to the obvious homotheties commuting with the action of $\gamma$.

Next, we want to describe the orbital integrals for unitary groups. We will use the following general notation to describe the relevant unitary groups.

- $F$ is a local field of equal characteristic $p$
- $F'$ is an unramified quadratic field extension of $F$, with Galois group $\mathrm{Gal}(F'/F) = \{1, \tau\}$.
- $E$ is a totally ramified separable extension of $F$.
- $\Phi_{(\alpha)}$ is a non degenerate hermitian form on the $F'$-vector space $E' = E \otimes_F F'$
$$\Phi_{(\alpha)}(x, y) = \mathrm{tr}_{E'/F'}(\alpha x^\tau y),$$
$(\alpha \in E^\times)$.
- The discriminant of $\Phi_{(\alpha)}$ only depends on the valuation of $\alpha$. Fix $\alpha^+$, resp. $\alpha^-$ such that $\Phi_{E'}^+ = \Phi_{(\alpha^+)}$ has *even* parity of the order of the discriminant, and $\Phi_{E'}^- = \Phi_{(\alpha^-)}$ has *odd* parity of the order of the discriminant.

Now we can exhibit a typical *endoscopic subgroup* of a unitary group. Fix totally ramified separable finite extensions $E_1$ and $E_2$ of $F$ of degrees $n_1$ and $n_2$. Let $E_1'$ and $E_2'$ denote the unramified quadratic field extensions $E_1F'$ and $E_2F'$ of $E_1$ and $E_2$.

Let $E' = E_1' \oplus E_2'$ (a $F'$-vector space of dimension $n_1 + n_2$). Endow $E'$ with the non degenerate hermitian forms

$$\Phi^+ = \Phi_{E_1'}^+ \oplus \Phi_{E_2'}^+$$

and

$$\Phi^- = \Phi_{E_1'}^- \oplus \Phi_{E_2'}^-.$$

These two forms are equivalent. Therefore we can find $g \in GL_{F'}(E')$ such that

$$\Phi^-(x, y) = \Phi^+(gx, gy) \qquad (\forall x, y \in E').$$

Let us now fix $\gamma_1 \in E_1'^{\times}$ and $\gamma_2 \in E_2'^{\times}$ such that $\gamma_1\gamma_1^{\sigma} = \gamma_2\gamma_2^{\sigma} = 1$. We assume that $E_i' = F'[\gamma_i]$, i.e. the minimal polynomial $P_i(T) \in F'[T]$ of $\gamma_i$ has degree $n_i$. We assume moreover that the polynomials $P_1(T)$ and $P_2(T)$ are separable and prime to each other.

The diagonal element $(\gamma_1, \gamma_2) \in GL_{F'}(E')$ may be simultaneously viewed as

- an elliptic regular semi-simple element $\gamma^+$ in the unitary group

$$G \overset{\text{dfn}}{=\!=} U(E', \Phi^+) = gU(E', \Phi^-)g^{-1} \subset GL_{F'}(E'),$$

- as an elliptic regular semi-simple element $\gamma^-$ in the unitary group

$$U(E', \Phi^-) \subset GL_{F'}(E')$$

- and as an elliptic $(G, H)$-regular semi-simple element $\delta$ in the endoscopic group

$$H = U(E_1', \Phi_1^+) \times U(E_2', \Phi_2^+) \subset GL_{F'}(E').$$

The elements $\gamma^+$ and $g\gamma^-g^{-1}$ of $G$ are conjugate in $GL_{F'}(E')$ but are not conjugate in $G$. The conjugacy class of $\delta$ in $H$ is equal to its stable conjugacy class. To see this, note that an element of $U(E_i', \Phi_i^+) \subset GL_{F'}(E_i')$ is stably conjugate to $\gamma_i$ if and only if it has the same minimal polynomial as $\gamma_i$.

Define subgroups

$$K = \mathrm{Fix}_G(O_{E_1'} \oplus O_{E_2'}), \quad K^H = \mathrm{Fix}_H(O_{E_1'} \oplus O_{E_2'}) \ .$$

These are hyperspecial maximal open compact subgroups of $G$ and $H$ respectively.

Now we define *stable* and *unstable* orbital integrals. Let

- The *$\kappa$-orbital integral*,

$$O_\gamma^\kappa(1_K) = |\{L' \subset E' \mid L'^{\perp^+} = L' \text{ and } (\gamma_1,\gamma_2)L' = L'\}| \\ -|\{L' \subset E' \mid L'^{\perp^-} = L' \text{ and } (\gamma_1,\gamma_2)L' = L'\}|$$

( $L'$'s are $O_{F'}$-lattices, $(\cdot)^{\perp^\pm}$ denotes the duality for such lattices with respect to the hermitian form $\Phi^\pm$).

- The *stable orbital integral*,

$$SO_\delta^H(1_{K^H}) = |\{M_1' \subset E_1' \mid M_1'^{\perp_1^+} = M_1' \text{ and } \gamma_1 M_1' = M_1'\}| \\ \times|\{M_2' \subset E_2' \mid M_2'^{\perp_2^+} = M_2' \text{ and } \gamma_2 M_2' = M_2'\}|.$$

($M_i'$'s are $O_{F'}$-lattices and $(\cdot)^{\perp_i^+}$ denotes the duality for such lattices with respect to the hermitian form $\Phi_i^+$).

Before we can state the main theorem, we need to define an additional numerical invariant of the situation. Let

$$r = r(\gamma_1, \gamma_2) = \mathrm{val}\big(\mathrm{Res}(P_1, P_2)\big),$$

where

$$\mathrm{Res}(P_1, P_2) = \prod_{k_1=0}^{n_1-1}\prod_{k_2=0}^{n_2-1}(\gamma_1^{(k_1)} - \gamma_2^{(k_2)}) \in O_{F'}$$

is the resultant of the minimal polynomials $P_1(T), P_2(T) \in F'[T]$ of $\gamma_1, \gamma_2$. Here $\gamma_i = \gamma_i^{(0)}, \ldots, \gamma_i^{(n_i-1)}$ are the roots of $P_i(T)$ in some algebraic closure of $F'$ containing $E'_1$ and $E'_2$.

A special case of the theorem of Laumon and Ngô (which confirms the matching conjecture of Langlands-Shelstad in this particular case) is now the following statement.

**Theorem.** *Under the above hypotheses, assume that the characteristic $p$ of $F$ is bigger than $n$. Then*

$$O_\gamma^\kappa(1_K) = (-q)^r SO_\delta^H(1_{K^H}),$$

*where $q$ is the number of elements in the residue field $k$.*

As is obvious, the theorem is a purely combinatorial statement. However, the combinatorics are quite difficult. In earlier attempts, methods of combinatorial geometry based on Bruhat-Tits buildings were used; and these methods are successful in low-dimensional cases. In the proof of Laumon/Ngô, the whole arsenal of modern algebraic geometry is brought to bear on the problem. The starting point is the observation that $G/K = (LG/L^+G)(k)$ is the set of $k$-points of the *affine Grassmannian* of $G$, an ind-algebraic variety of infinite dimension. I cannot go here into this proof.

In the end, I stress that I have not given the history of the problem. Any such history would have to mention at least the following names, which are ordered here alphabetically : Chaudouard, Clozel, Goresky, Haines, Hales, Kazhdan, Kottwitz, Labesse, Langlands, MacPherson, Rogawski, Saito, Schröder, Shelstad, Shintani, Waldspurger, Weissauer, Whitehouse, ...

And now I ask you all to join me in congratulating Ngô Bao Châu for his brilliant achievement.

*Reporters: Gerald Gotsbacher, Harald Grobner*

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