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Representation Theory of Finite Dimensional Algebras

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Abstract. Methods and results from the representation theory of finite dimensional algebras have led to many interactions with other areas of mathematics. Such areas include the theory of Lie algebras and quantum groups, commutative algebra, algebraic geometry and topology, and in particular the new theory of cluster algebras. The aim of this workshop was to further develop such interactions and to stimulate progress in the representation theory of algebras.

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Introduction by the Organisers

Representation theory of finite dimensional algebras has always been inspired by interactions with other subjects, and Oberwolfach meetings traditionally serve as a forum for such exchange of ideas. The main source of interactions are the many problems in representation theory and in other parts of mathematics which can be formulated in terms of representations of finite dimensional associative algebras. The study of non-semisimple representations took off in the late 20th century with key advances, such as the link to Lie algebras and quantum groups via quivers and Hall algebras, and the use of tilting theory and derived categories to pass from known algebras to new classes of algebras.

In modern work, instead of studying an algebra through its category of representations, or derived category, one may study similar but more general categories. Thus the classification of some classes of hereditary abelian categories or Calabi-Yau triangulated categories fits into this setup. Another recent development, which had just started at the time of the last Oberwolfach meeting in February 2005, and is still being played out, is the interaction with cluster algebras.

At the workshop, there were 46 participants. Among them, there were experts from neighbouring subjects like commutative algebra, algebraic topology, and combinatorics. Compared to previous meetings, the number of participants was reduced, which made it difficult to include representatives of many other fields with close links to representation theory of finite dimensional algebras. What follows is a quick survey of the main themes of the 23 lectures given at the meeting.

Cluster combinatorics and Calabi-Yau categories arising from representations of algebras. Cluster algebras were invented by Fomin and Zelevinsky in 2000 with motivations coming from the study of canonical bases in quantum groups and total positivity in algebraic groups. The combinatorics of these algebras were soon recognized to be closely related to those of tilting theory for hereditary algebras. A collective effort over the last few years has led to a good understanding of these relations for certain classes of cluster algebras. This was made possible by the use of 2-Calabi-Yau categories constructed from representations of algebras. The introductory talks by Reiten and Iyama were devoted to these developments as well as to the impact of recent important work by Derksen-Weyman-Zelevinsky. In an informal evening presentation, Keller put Derksen-Weyman-Zelevinsky's work into a beautiful homological framework. The talk by Geiss presented cutting-edge results towards the construction of 'dual PBW-bases' in large classes of cluster algebras. The proofs are based on subtle techniques from the study of quasi-hereditary algebras, as demonstrated in Schröer's talk. Marsh analyzed fine points of the correspondence between cluster variables and rigid indecomposables and disproved a recent conjecture by Fomin-Zelevinsky. A powerful representation-theoretic model for 'higher cluster combinatorics' was presented in the talk by Bin Zhu.

Categorification via representations. The method of categorification has been developed and studied successfully in representation theory by Chuang and Rouquier. They constructed sl_2-categorifications for blocks of symmetric groups and used them to establish Broué's abelian defect group conjecture for the symmetric groups. A similar philosophy led to the categorification of cluster algebras via certain 2-Calabi-Yau categories, where the multiplication in the cluster algebra is modeled by direct sums. A more recent and very promising approach due to Leclerc was presented by Keller. In this case the multiplication is modeled by the tensor product in certain categories of representations of quantum affine algebras. Categorifications also play an important role in low dimensional topology, thanks to important work of Khovanov. This connection was the motivation for Stroppel's talk on convolution algebras arising from Springer fibres.

Representation dimension of algebras and complexity of triangulated categories. The representation dimension of an algebra is a homological invariant which Auslander introduced in 1971 and which remained mysterious for many years thereafter. Some of the modern techniques in representation theory provide now a better understanding. An introductory talk by Ringel discussed the basic ideas and some interesting new phenomena for hereditary algebras. Dimensions of triangulated categories were introduced by Rouquier to obtain lower bounds for representation dimensions and Iyengar presented some new techniques to compute them. The talk of Buchweitz provided a more general perspective for the computation of these dimensions by reviewing the work of Beligiannis and Christensen on projective classes and ghosts in triangulated categories. A description of triangulated structures on additive categories in terms of Hochschild cohomology was presented by Pirashvili.

Hereditary categories of geometric origin. Hereditary categories are in some sense the building blocks for many interesting structures in modern representation theory. Typical examples are categories of coherent sheaves which come equipped with some additional geometric structure. Using this extra structure, Lenzing presented a new description of the stable category of vector bundles on a weighted projective line. The talk of Burban discussed an intriguing connection between vector bundles on elleptic curves and solutions of Yang-Baxter equations. A complete classification of abelian 1-Calabi-Yau categories up to derived equivalence was presented by van Roosmalen.

Representations of quivers. Quivers and their representations have always played a central role in the representation theory of finite dimensional algebras. They provide the link to Lie theory, either through the theorems of Gabriel and Kac, relating possible dimension vectors of indecomposable representations to positive roots, or more directly via Ringel's construction of quantum groups using Ringel-Hall algebras. Progress since the last meeting includes Hausel's announcement of a positive solution of Kac's conjecture that the constant term of the polynomial counting the number of absolutely indecomposable representations over a finite field is the corresponding root multiplicity. Hausel was invited to the meeting, but sadly in the end it was not possible for him to attend. Hausel's result involves hyper-Kähler geometry, and in his talk Reineke also used geometry, namely the cohomology of moduli spaces of quiver representations, to prove a formula similar to one conjectured by Kontsevich and Soibelman concerning Donaldson-Thomas type invariants. Chapoton and Hille both gave intriguing talks involving tilting modules for quivers, exceptional sequences and braid group actions. Hubery discussed the connections between Hall algebras and cluster algebras and the existence of Hall polynomials for non-simply laced affine diagrams, using species rather than quivers.

Further aspects of algebras and their representations. Representation theory of finite dimensional algebras has developed immensely since its origin, and it has now, as demonstrated above, profound connections to many other fields. However, the 'internal' theory of representation theory is still pushed forward: The talk of Skowroński presented results on algebras with generalized standard almost cyclic coherent Auslander-Reiten components. Representation theory of

Lie algebras and algebraic groups is intimately related to finite dimensional algebras which are cellular or quasi-hereditary. These are algebras given by a specific filtration of ideals. König presented work on how to generalize such a filtration further in order to deal with possibly infinite dimensional building blocks. Benson, Carlson and others developed a theory of support varieties for finitely generated modules over a finite group, and they obtained deep structural information about modular representations of finite groups in terms of the group cohomology ring. These results found their analogous twin results for Lie algebras and Steenrod algebras arising in topology. Similar support varieties have since then been defined for instance for complete intersections, quantum groups and arbitrary finite dimensional algebras. A common denominator for these situations is the presence of a ring of cohomological operations, and in the latter case this is provided by the Hochschild cohomology ring. The talk of Avramov gave an overview over recent results and questions on the Hochschild cohomology ring of an algebra arising in this context. Nakano presented results on the cohomology and support varieties for quantum groups in a quest to find relationships between representations for quantum groups and geometric constructions in complex Lie theory.

The format of the workshop has been a combination of introductory survey lectures and more specialized talks on recent progress. In addition there was plenty of time for informal discussions. Thus the workshop provided an ideal atmosphere for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

Workshop: Representation Theory of Finite Dimensional Algebras

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Abstracts

Cluster tilting in 2-Calabi-Yau categories I

IDUN REITEN

This is the first in a series of two lectures by Osamu Iyama and myself. We give an introduction to the subject of triangulated 2-Calabi-Yau categories with cluster tilting objects, and discuss some of the more recent developments. In this lecture we start with definitions and basic examples. Then we concentrate on a class of examples associated with elements in Coxeter groups of graphs, based on [15][3][4], with some related material in [12].

The work on triangulated 2-Calabi-Yau (2-CY for short) categories was, via cluster categories, inspired by the theory of cluster algebras by Fomin-Zelevinsky, starting with [9]. There is a lot of interesting work by many authors on the interplay between cluster algebras and 2-CY categories, but we will not discuss these aspects here.

1. 2-CY CATEGORIES AND 2-CY TILTED ALGEBRAS

2-CY categories. Let $\mathcal{C}$ be a Hom-finite triangulated category over an algebraically closed field K. Then $\mathcal{C}$ is 2-CY if we have a functorial isomorphism $D\operatorname{Ext}^1_{\mathcal{C}}(A,B) \simeq \operatorname{Ext}^1_{\mathcal{C}}(B,A)$ for A, B in $\mathcal{C}$, where $D = \operatorname{Hom}_K(-,K)$. We have the following important examples.

(i) Let Q be a finite connected quiver without oriented cycles, let KQ be the path algebra of Q over K, and denote by τ the AR-translation for KQ. Then the orbit category $\mathcal{C} = \mathcal{C}_Q = D^b(KQ)/\tau^{-1}[1]$ is called the cluster category [6], and was shown to be triangulated in [17]. This construction was inspired by [9], via the connections with quiver representations from [21]. An equivalent category in the case A_n was investigated in [7]. Loosely speaking there are two crucial differences between the category $\operatorname{mod} KQ$ of finite dimensional KQ-modules and the cluster category $\mathcal{C}_Q$. The category $\mathcal{C}_Q$ has a finite number of additional indecomposable objects $P_1[1], \dots, P_n[1]$, where $P_1, \dots, P_n$ are the indecomposable projective KQ-modules, and there are more maps between the indecomposable KQ-modules when viewed as objects in $\mathcal{C}_Q$.

(ii) When Λ is the preprojective algebra of a Dynkin diagram, the stable module category $\underline{\operatorname{mod}\Lambda}$ is known to be Hom-finite triangulated 2-CY. This category, or rather the associated abelian category $\operatorname{mod}\Lambda$, has been extensively studied by Geiss-Leclerc-Schröer.

(iii) Let R be an odd-dimensional isolated hypersurface singularity with residue field K, and denote by $\operatorname{CM}(R)$ the category of maximal Cohen-Macaulay R-modules. Then $\underline{\operatorname{CM}(R)}$ is a triangulated category [13] which is Hom-finite 2-CY by work of Auslander [1] and Eisenbud [8] (See [2]).

Cluster tilting objects. An object T in $\mathcal{C}$ is *cluster tilting* if $\mathrm{Ext}^1_{\mathcal{C}}(T,T)=0$, and $\mathrm{Ext}^1_{\mathcal{C}}(T,X)=0$ for X in $\mathcal{C}$ implies that X is in the additive category $\operatorname{add} T$ generated by T [19]. Such objects were investigated in the context of cluster categories in [6], under the name of Ext-configuration, which was shown to be equivalent to T being maximal rigid, that is, $\mathrm{Ext}^1_{\mathcal{C}}(T,T)=0$ and T is maximal with this property. In cluster categories these objects are exactly those induced by tilting KQ'-modules for some algebra KQ' derived equivalent to KQ. The concept of cluster tilting also appeared in the work of Iyama, under the name *maximal* 1*-orthogonal*, in a completely different setting.

Tilting KQ-modules were natural candidates for modelling clusters in cluster algebras. One drawback was that almost complete tilting modules have at most two complements, but not necessarily exactly two [22] [23] [14]. However, when considering cluster tilting objects in cluster categories, one obtains exactly two. These complements are connected via special exchange triangles [6]. That $\mathrm{Ext}^1_{\mathcal{C}}(A,B)=0$ if and only if $\mathrm{Ext}^1_{\mathcal{C}}(B,A)=0$ was an essential ingredient for these results, and so many arguments carry over to the general 2-CY case (see also [11]). Still some further work was needed for the generalisation in [16].

2-CY tilted algebras. The 2-CY tilted algebras are by definition the algebras $\Gamma=\mathrm{End}_{\mathcal{C}}(T)$ where T is a cluster tilting object in a Hom-finite triangulated 2-CY category $\mathcal{C}$. When $\mathcal{C}$ is a cluster category we have the cluster tilted algebras [5].

Some interesting properties of 2-CY tilted algebras are the following, with the above notation.

(i) $\mathcal{C}/\operatorname{add}\tau T\simeq \operatorname{mod}\Gamma$ [5] [19]
(ii) $\operatorname{id}_\Gamma\Gamma\leq 1$ and $\operatorname{id}\Gamma_\Gamma\leq 1$ [19]
(iii) If $\operatorname{Sub}\Gamma$ denotes the full subcategory of $\operatorname{mod}\Gamma$ whose objects are the submodules of projective modules, then $\underline{\operatorname{Sub}\Gamma}$ is 3-CY, that is $D\,\mathrm{Ext}^2_\Gamma(A,B)\simeq \mathrm{Ext}^1_\Gamma(B,A)$ for any A,B in $\underline{\operatorname{Sub}\Gamma}$ [10] [19].

2. Examples associated with Coxeter groups

Let Q be a finite connected non Dynkin quiver with no loops and vertices $1,\dots,n$, let Λ be the completion of the preprojective algebra of Q over K, and W the associated Coxeter group with generators $s_1,\dots,s_n$. For $i=1,\dots,n$, let I_i be the ideal $\Lambda(1-e_i)\Lambda$ where e_i is the trivial path at vertex i. If for $w\in W$, the expression $w=s_{i_1}\cdots s_{i_t}$ is reduced, let $I_w=I_{i_1}\cdots I_{i_t}$. Then I_w is independent of the choice of reduced expression. Further $\Lambda_w=\Lambda/I_w$ is a finite dimensional K-algebra with $id_{\Lambda_w}\Lambda_w\leq 1$ and $\mathcal{C}=\underline{\operatorname{Sub}\Lambda_w}$ is Hom-finite triangulated 2-CY [15] [3]. Some of these categories are described in a different way in [12].

All these 2-CY categories have some nice cluster tilting objects. Namely, for each reduced expression $w=s_{i_1}\cdots s_{i_t}$ of w, the object $T=\Lambda/I_{i_1}\oplus\Lambda/I_{i_1}I_{i_2}\oplus\cdots\oplus\Lambda/I_{i_1}I_{i_2}\cdots I_{i_t}$ is cluster tilting in $\underline{\operatorname{Sub}\Lambda_w}$. There is a combinatorial rule depending on the sequence of integers $i_1,\dots,i_t$ for describing the quiver of $\mathrm{End}_{\mathcal{C}}(T)$ [3]. In addition we have shown in [4] that these $\mathrm{End}_{\mathcal{C}}(T)$ are given by quivers with potentials, where we actually give an explicit description of the potentials. There is some related work by Keller [18].

3. Special cases

It is interesting to note that the two cases of 2-CY categories which have been most extensively investigated fit into the general setup in section 2.

(a) Let Q be a finite connected quiver with no oriented cycles and with vertices labeled $1, \ldots, n$ such that if there is an arrow $i \to j$, then $i > j$. If Q is not Dynkin, let $w = (s_1 \cdots s_n)^2$. (The Dynkin case is treated separately). Then for the corresponding cluster tilting object T in $\mathcal{C} = \underline{\operatorname{Sub} \Lambda_w}$, the quiver of $\operatorname{End}_{\mathcal{C}}(T)$ is Q, which has no oriented cycles. Hence $\mathcal{C}$ is equivalent to the cluster category $\mathcal{C}_Q$ [20]. (See [3], and [12] for an independent related approach).

(b) Let Λ' be the preprojective algebra of a Dynkin quiver Q' and Q an extended Dynkin quiver containing Q'. Let W be the Coxeter group of Q and W' the subgroup generated by the s_i for $i \in Q'_0$. Let w_0 be the longest element in W'. Then $\underline{\operatorname{mod} \Lambda'}$ is equivalent to $\underline{\operatorname{Sub} \Lambda_{w_0}}$ [3].

References

[1] M. Auslander, *Functors and morphisms determined by objects*, Representation of Algebras, Proc. Conf., Temple Univ. Philadelphia, PA, 1979, in: Lecture Notes in Pure and Appl. Math., vol. 37, Dekker, New York, 1978, pp. 1-244.

[2] I. Burban, O. Iyama, B. Keller, I. Reiten, *Cluster tilting for one-dimensional hypersurface singularities*, Adv. Math.

[3] A. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, arXiv:math/0701551.

[4] A. Buan, O. Iyama, I. Reiten, D. Smith, *Mutation of cluster tilting objects and potentials*, in preparation.

[5] A. Buan, R. Marsh, I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 1, 323-332.

[6] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572-618.

[7] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358 (2)(2006), 1347-1364.

[8] D. Eisenbud, *Homological Algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. 260 (1980), no.1, 35-64.

[9] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. 15, no. 2 (2002), 497-529.

[10] C. Geiß, B. Keller, *Quadrangulated categories*, Oberwolfach talk, February 2005.

[11] C. Geiss, B. Leclerc, J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math. 165 (2006), no. 3, 589-632.

[12] C. Geiss, B. Leclerc, J. Schröer, *Rigid modules over preprojective algebras II: The Kac-Moody case*, preprint arXiv:math.RT/0703039.

[13] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988. x+208 pp.

[14] D. Happel, L. Unger *Almost complete tilting modules*, Proc. Amer. Math. Soc. 107 (3)(1989), 603-610.

[15] O. Iyama, I. Reiten, *Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras*, to appear in Amer. J. Math., arXiv:math/0605136.

[16] O. Iyama, Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, to appear in Invent. Math., arXiv:math/0607736.

[17] B. Keller, *On triangulated categories*, Documenta Math. 10 (2005), 551-581.

[18] B. Keller, *CY-completions and their duals*, in preparation.

[19] B. Keller, I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. 211 (1)(2007), 123-151.

[20] B. Keller, I. Reiten, *Acyclic Calabi-Yau categories*, preprint arXiv: math.RT/0610594, to appear in Compositio Math.

[21] R. Marsh, M. Reineke, A. Zelevinsky, *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. 355 (2003), no. 10, 4171-4186.

[22] C. Riedtmann, A. Schofield, *On open orbits and their complements*. J. Algebra 130 (2)(1990), 388-411.

[23] L. Unger, *Schur modules over wild, finite-dimensional path algebras with three simple modules*, J. Pure Appl. Algebra 64 (2)(1990), 205-222.

Cluster tilting in 2-Calabi-Yau categories II

OSAMU IYAMA

This is the second part in a series of two lectures with Idun Reiten. We shall show that cluster tilting mutation is compatible with quiver mutation and QP mutation. Throughout let K be an algebraically closed field, and let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category over K with the suspension functor Σ. Let T be a basic cluster tilting object in $\mathcal{C}$ with an indecomposable decomposition $T = T_1 \oplus \cdots \oplus T_n$, and let $1 \leq k \leq n$. The following result [BMRRT, IY] is fundamental.

Theorem 1 (cluster tilting mutation)

(a) *There exists a unique indecomposable object* $T_k^* \in \mathcal{C}$ *such that* $T_k^* \not\simeq T_k$ *and* $\mu_k(T) := (T/T_k) \oplus T_k^*$ *is a basic cluster tilting object in* $\mathcal{C}$.

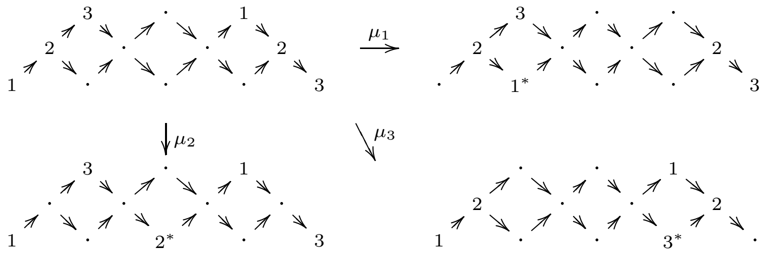
(b) *There exist triangles (called* exchange sequences*)*

$$T_k^* \xrightarrow{g} U_k \xrightarrow{f} T_k \to \Sigma T_k^* \quad \textit{and} \quad T_k \xrightarrow{g'} U_k' \xrightarrow{f'} T_k^* \to \Sigma T_k$$

such that f *and* f' *are right* $\operatorname{add}(T/T_k)$*-approximations and* g *and* g' *are left* $\operatorname{add}(T/T_k)$*-approximations.*

Clearly we have $\mu_k \circ \mu_k(T) \simeq T$.

Example 2 Let $\mathcal{C}$ be a cluster category of type A_3.



Following [FZ], we introduce mutation of quivers.

Definition 3 (*quiver mutation*) Let Q be a quiver[1] without loops. Assume that $k \in Q_0$ is not contained in 2-cycles. Define a quiver $\widetilde{\mu}_k(Q)$ by applying the following (i)-(iii) to Q.

(i) For each pair (a, b) of arrows in Q with $e(a) = k = s(b)$, add a new arrow $[ab] : s(a) \to e(b)$.

(ii) Replace each arrow $a \in Q_1$ with $e(a) = k$ by a new arrow $a^* : k \to s(a)$.

(iii) Replace each arrow $b \in Q_1$ with $s(b) = k$ by a new arrow $b^* : e(b) \to k$.

Define a quiver $\mu_k(Q)$ by applying the following (iv) to $\widetilde{\mu}_k(Q)$.

(iv) Remove a maximal disjoint collection of 2-cycles.

Then $\mu_k(Q)$ has no loops, k is not contained in 2-cycles in $\mu_k(Q)$, and $\mu_k \circ \mu_k(Q) \simeq Q$ holds.

Example 4 For the following quiver Q of type A_3, we calculate $\mu_1(Q)$, $\mu_2(Q)$ and $\mu_2 \circ \mu_2(Q)$. (For simplicity we denote a^{**} and b^{**} by a and b respectively.)

$$\begin{tikzcd}
Q = \Bigg(& 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "\mu_2"] & 3 & \Bigg) \arrow[rr, "\mu_1"] & & \Bigg(& 1 & 2 \arrow[l, "a^*"'] \arrow[r, "b"] & 3 & \Bigg) & & & & & & \\
\Bigg(& 1 \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[l, "a^*"'] & 3 \arrow[l, "b^*"'] & \Bigg) \arrow[rr, "\widetilde{\mu}_2"] & & \Bigg(& 1 \arrow[r, "a"] \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend right, "{[b^*a^*]}"'] & \Bigg) \arrow[rr, "\text{(iv)}"] & & \Bigg(& 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 & \Bigg)
\end{tikzcd}$$

From now on, we assume that $\mathcal{C}$ has a *cluster structure* [BIRSc]. This means that the quiver Q_T of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ of any cluster tilting object T in Q has no loops and 2-cycles. In this case we have the following.

Observation 5 Combining the exchange sequences in Theorem 1, we have a complex[2]

$$T_k \xrightarrow{g'} U'_k \xrightarrow{f'g} U_k \xrightarrow{f} T_k$$

such that the following sequences are exact for the Jacobson radical $J_{\mathcal{C}}$ of $\mathcal{C}$.

$$(T, U'_k) \xrightarrow{f'g} (T, U_k) \xrightarrow{f} J_{\mathcal{C}}(T, T_k) \to 0,$$
$$(U_k, T) \xrightarrow{f'g} (U'_k, T) \xrightarrow{g'} J_{\mathcal{C}}(T_k, T) \to 0.$$

Thus the quiver and relations of $\mathrm{End}_{\mathcal{C}}(T)$ can be controlled by exchange sequences.

Using Observation 5, we have the following result [BMR, BIRSc] which asserts that cluster tilting mutation is compatible with quiver mutation.

Theorem 6 $Q_{\mu_k(T)} \simeq \mu_k(Q_T)$.

Using Theorem 6, we can show the following result [BIRSm].

Corollary 7 *Cluster tilted algebras are determined by their quivers.*

Following [DWZ], we introduce quivers with potentials.

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[1] We use the convention $a : s(a) \to e(a)$ for each $a \in Q_1$.

[2] Such a complex is called a *2-almost split sequence* in [I] and an *AR 4-angle* in [IY].

Definition 8 Let Q be a quiver. We denote by A_i the K-vector space with the basis consisting of paths of length i, and by $A_{i,\mathrm{cyc}}$ the subspace of A_i spanned by all cycles. We denote by $\widehat{KQ} := \prod_{i\geq 0} A_i$ the complete path algebra. Its Jacobson radical is given by $J_{\widehat{KQ}} = \prod_{i\geq 1} A_i$.

A *quiver with a potential* (or *QP*) is a pair (Q, W) consisting of a quiver Q without loops and an element $W \in \prod_{i\geq 1} A_{i,\mathrm{cyc}}$ (called a *potential*). It is called *reduced* if $W \in \prod_{i\geq 3} A_{i,\mathrm{cyc}}$. Define $\partial_a W \in \widehat{KQ}$ by

$$\partial_a(a_1 \cdots a_\ell) := \sum_{a_i = a} a_{i+1} \cdots a_\ell a_1 \cdots a_{i-1}$$

and extend linearly and continuously. The *Jacobian algebra* is defined by

$$\mathcal{P}(Q, W) := \widehat{KQ}/\overline{\langle \partial_a W \mid a \in Q_1 \rangle}$$

where $\overline{I}$ is the closure of I with respect to the $(J_{\widehat{KQ}})$-adic topology on $\widehat{KQ}$.

Two potentials W and W' are called *cyclically equivalent* if $W - W' \in \overline{[KQ, KQ]}$. Two QP's (Q, W) and (Q', W') are called *right-equivalent* if $Q_0 = Q'_0$ and there exists a continuous K-algebra isomorphism $\phi : \widehat{KQ} \to \widehat{KQ'}$ such that $\phi|_{Q_0} = \mathrm{id}$ and $\phi(W)$ and W' are cyclically equivalent. In this case ϕ induces an isomorphism $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$.

It was shown in [DWZ] that for any QP (Q, W), there exists a reduced QP (Q', W') such that $\mathcal{P}(Q, W) \simeq \mathcal{P}(Q', W')$, and such (Q', W') is uniquely determined up to right-equivalence. We call (Q', W') a *reduced part* of (Q, W).

Example 9 Let (Q, W) be the QP below. Its reduced part is given by the QP (Q', W') below.

$$(Q, W) = \left(\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right, "c"'] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend right, "d"'] \end{tikzcd}, cd + abd \right) \qquad (Q', W') = \left(\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \end{tikzcd}, 0 \right)$$

Definition 10 (*QP mutation*) Let (Q, W) be a QP. Assume that $k \in Q_0$ is not contained in 2-cycles. Replacing W by a cyclically equivalent potential, we assume that no cycles in W start at k. Define a QP $\widetilde{\mu}_k(Q, P) := (\widetilde{\mu}_k(Q), [W] + \Delta)$ as follows:

- $\widetilde{\mu}_k(Q)$ is given in Definition 3.
- $[W]$ is obtained by substituting $[ab]$ for each factor ab in W with $e(a) = k = s(b)$.
- $\Delta := \sum_{a,b \in Q_1,\ e(a)=k=s(b)} a^*[ab]b^*$.

Define a QP $\mu_k(Q, P)$ as a reduced part of $\widetilde{\mu}_k(Q, P)$.

Then k is not contained in 2-cycles in $\mu_k(Q, W)$, and it was shown in [DWZ] that $\mu_k \circ \mu_k(Q, W)$ is right-equivalent to (Q, W).

Example 11 For a QP (Q,W) below, we calculate $\mu_2(Q,W)$ and $\mu_2 \circ \mu_2(Q,W)$. (The reduced part of $\widetilde{\mu}_2 \circ \mu_2(Q,W)$ was calculated in Example 9.)

$$(Q,W) = \left(\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \end{tikzcd}, 0 \right) \xrightarrow{\mu_2} \left(\begin{tikzcd} 1 \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[l, "a^*"'] & 3 \arrow[l, "b^*"'] \end{tikzcd}, a^*[ab]b^* \right)$$

$$\xrightarrow{\widetilde{\mu}_2} \left(\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right, "{[ab]}"'] & 2 \arrow[r, "b"] & 3 \arrow[ll, bend right, "{[b^*a^*]}"'] \end{tikzcd}, [ab][b^*a^*] + b[b^*a^*]a \right) \xrightarrow{\text{reduced}} \left(\begin{tikzcd} 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \end{tikzcd}, 0 \right)$$

Using Observation 5, we have the following result [BIRSm] which asserts that cluster tilting mutation is compatible with QP mutation.

Theorem 12 *If* $\operatorname{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q,W)$*, then* $\operatorname{End}_{\mathcal{C}}(\mu_k(T)) \simeq \mathcal{P}(\mu_k(Q,W))$*.*

Immediately we have the following conclusion.

Corollary 13 *If* $\operatorname{End}_{\mathcal{C}}(T)$ *is a Jacobian algebra of a QP, then so is* $\operatorname{End}_{\mathcal{C}}(T')$ *for any cluster tilting object* $T' \in \mathcal{C}$ *reachable from* T *by successive mutation.*

We have the following applications [BIRSm] of Corollary 13 (see also [K]).

Example 14 (a) Cluster tilted algebras are Jacobian algebras of QP's.

(b) Let Λ be a preprojective algebra and W the corresponding Coxeter group. For any $w \in W$, we have a 2-CY triangulated category $\mathcal{C} := \underline{\mathrm{Sub}}\Lambda_w$ [BIRSc]. For any cluster tilting object $T \in \mathcal{C}$ reachable from a cluster tilting object given by a reduced expression of w by successive mutation, $\operatorname{End}_{\mathcal{C}}(T)$ is a Jacobian algebra of a QP.

We end this report by the following *nearly Morita equivalence* for Jacobian algebras [BMR2, BIRSm], where f.l. is the category of modules with finite length.

Theorem 15 *For a QP* (Q,W)*, we have an equivalence*

$$\text{f.l.}\, \mathcal{P}(Q,W)/\operatorname{add} S_k \simeq \text{f.l.}\, \mathcal{P}(\mu_k(Q,W))/\operatorname{add} S'_k,$$

where S_k *and* S'_k *are simple modules associated with the vertex* k*.*

REFERENCES

[BIRSc] A. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, arXiv:math/0701557.

[BIRSm] A. Buan, O. Iyama, I. Reiten, D. Smith, *Mutation of cluster tilting object and quiver with potentials*, in preparation.

[BMR] A. Buan, R. Marsh, I. Reiten, *Cluster mutation via quiver representations*, Comment. Math. Helv. 83 (2008), no. 1, 143–177.

[BMR2] A. Buan, R. Marsh, I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 1, 323–332.

[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572–618

[DWZ] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, arXiv:0704.0649.

[FZ] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[I] O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 (2007), no. 1, 22–50.

[IY] O. Iyama, Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, to appear in Invent. Math., arXiv:math/0607736.

[K] B. Keller, in preparation.

An introduction to the representation dimension of artin algebras

Claus Michael Ringel

Let Λ be an artin algebra (this means that Λ is a module-finite k-algebra, where k is an artinian commutative ring). The modules to be considered will be left Λ-modules of finite length. Given a module M we denote by $\operatorname{add} M$ the class of modules which are direct summands of direct sums of copies of M.

The representation dimension of artin algebras was introduced by M.Auslander in his famous Queen Mary Notes, but remained a hidden treasure for a long time. Only very recently some basic questions concerning the representation dimension have been solved by Iyama and Rouquier, and now there is a steadily increasing interest in this dimension (in particular, see papers by Oppermann, and also Krause-Kussin, Avramov-Iyengar, and Bergh). This introduction will recall the basic setting and outline a general scheme in order to understand some of the artin algebras with representation dimension at most 3. But we should stress that the main focus at present lies on the artin algebras with representation dimension greater than 3.

1. Some basic results.

A module M is called a *generator* if any projective module belongs to $\operatorname{add} M$; it is called a *cogenerator* if any injective module belongs to $\operatorname{add} M$. It was Auslander who stressed the importance of the global dimension d of the endomorphism rings $\operatorname{End}(M)$, where M is both a generator and a cogenerator. Note that d is either 0 (this happens precisely when Λ is semisimple) or greater or equal to 2 (of course, it may be infinite). The representation dimension of an artin algebra Λ which is not semisimple is the smallest possible such value d; whereas the representation dimension of a semisimple artin algebra is defined to be 1.

The main tool for calculating the representation dimension is the following criterion due to Auslander (implicit in the Queen Mary Notes). Given modules M, X, denote by $\Omega_M(X)$ the kernel of a minimal right $(\operatorname{add} M)$-approximation $M' \to X$. By definition, the *M-dimension* $M\text{-}\dim X$ is the minimal value i such that $\Omega^i_M(X)$ belongs to $\operatorname{add} M$.

(A) **Theorem** (Auslander). *Let M be a Λ-module which is both a generator and a cogenerator and let $d \geq 2$. The global dimension of* $\operatorname{End}(M)$ *is less or equal to d if and only if $M\text{-}\dim X \leq d-2$ for all Λ-modules X.*

An immediate consequence is:

(B) **Theorem** (Auslander). *An artin algebra Λ is of finite representation type if and only if* $\operatorname{rep.dim.} \Lambda \leq 2$. This result was the starting observation and indicates that the representation dimension may be considered as a measure for the distance of being representation-finite.

There is the following characterization of the endomorphisms rings of modules which are both generators and cogenerators; its proof provides an important bicentralizer situation:

(C) **Theorem** (Morita-Tachikawa). *If M is a Λ-module which is a generator and cogenerator, then* $\operatorname{End}(M)$ *is an artin algebra with dominant dimension at least* 2 *and any artin algebra with dominant dimension at least* 2 *arises in this way.*

(D) **Theorem** (Iyama). *The representation dimension is always finite.* This asserts, in particular, that any artin algebra Λ can be written in the form $\Lambda = e\Lambda' e$, where Λ' is an artin algebra with finite global dimension; thus many homological questions concerning Λ-modules can be handled by dealing with modules for an algebra with finite global dimension.

(E) **Theorem** (Igusa-Todorov). *If* $\operatorname{rep.dim.} \Lambda \le 3$, *then Λ has finite finitistic dimension.*

Until 2001, for all artin algebras Λ where the representation dimension was calculated, it turned out that $\operatorname{rep.dim.} \Lambda \le 3$. Thus, there was a strong feeling that all artin algebras could have this property. If this would have been true, the finitistic dimension conjecture and therefore a lot of other homological conjectures would have been proven by (E).

(F) **Example** (Rouquier). *Let V be a finite-dimensional k-space, where k is a field, and $\Lambda(V)$ the corresponding exterior algebra. Then* $\operatorname{rep.dim.} \Lambda(V) = 1 + \dim V$.

2. Endomorphism rings of generator-cogenerators in case Λ is hereditary.

In case Λ is hereditary, one can determine the set of all possible values of the global dimension of endomorphism rings of Λ-modules which are generator-cogenerators. Let τ_Λ denote the Auslander-Reiten translation for the category $\operatorname{mod} \Lambda$.

Theorem (Dlab-Ringel). *Let Λ be a hereditary artin algebra and let $d \ge 3$ be in $\mathbb{N} \cup \{\infty\}$. There exists a Λ-module M which is both a generator and a cogenerator such that the global dimension of* $\operatorname{End}(M)$ *is equal to d if and only if there is a τ_Λ-orbit of cardinality at least d.*

3. Torsionless-finite artin algebras.

We call an artin algebra Λ *torsionless-finite* provided there are only finitely many isomorphism classes of indecomposable modules which are torsionless (i.e. submodules of projective modules).

Theorem. *If Λ is torsionless-finite, then its representation dimension is at most* 3.

The proof follows again arguments by Auslander presented in the Queen Mary Notes. According to Auslander-Bridger a torsionless-finite artin algebra has also only finitely many isomorphism classes of indecomposable modules which are factor modules of injective modules. Let L be an additive generator for the subcategory of all torsionless modules, and F an additive generator for the subcategory of all factor modules of injective modules. Given any Λ-module X, let X' be the F-trace in X, thus the inclusion map $X' \to X$ is a right $(\operatorname{add} F)$-approximation of X. Let $p : X'' \to X$ be a right $(\operatorname{add} L)$-approximation of X. Then there is an exact sequence of the form $0 \to p^{-1}(X') \to X'' \oplus X' \to X \to 0$ which shows that $\Omega_{L\oplus F}(X)$ is a direct summand of $p^{-1}(X')$. Since $p^{-1}(X')$ is a submodule of X'', it follows that $\Omega_{L\oplus F}(X)$ is in $\operatorname{add} L$.

Many classes of artin algebras are known to be torsionless-finite: the hereditary algebras (Auslander), the algebras with $J^n = 0$ such that Λ/J^{n-1} is representation-finite, where J is the radical of Λ (Auslander), in particular: the algebras with $J^2 = 0$, but also the minimal representation-infinite algebras, then the artin algebras stably equivalent to hereditary algebras (Auslander-Reiten), the right glued algebras and the left glued algebras (Coelho, Platzeck; an artin algebra is right glued provided almost all indecomposable modules have projective dimension 1), as well as the special biserial algebras (Schröer). Also, if Λ is a local algebra of quaternion type, then $\Lambda/\operatorname{soc}\Lambda$ is torsionless-finite, so that again its representation dimension is equal to 3 (Holm).

But it should be stressed that there are many classes of artin algebras with representation dimension 3 which are not necessarily torsionless-finite: for example the tilted algebras (Assem-Platzeck-Trepode), the trivial extensions of hereditary algebras (Coelho-Platzeck) as well as the canonical algebras (Oppermann).

Basic references

- M. Auslander: The representation dimension of artin algebras. Queen Mary College Mathematics Notes (1971)
- M. Auslander, M. Bridger: Stable module theory. Mem. Amer. Math. Soc. 94 (1969).
- K. Igusa, G. Todorov: On the finitistic global dimension conjecture for artin algebras. In: Representations of algebras and related topics. Fields Inst. Commun. 45 (2005), 201-204.
- O. Iyama: Finiteness of representation dimension. Proc. Amer. Math. Soc. 131 (2003), 1011-1014.
- R. Rouquier: Representation dimension of exterior algebras. Invent. Math. 168 (2006), 357-367.
- V. Dlab, C.M. Ringel: The global dimension of the endomorphism ring of a generator-cogenerator for a hereditary artin algebra. Preprint (2008).

Some aspects of Hochschild (co)homology

LUCHEZAR AVRAMOV

In the talk I surveyed certain results and given questions connected to the Hochschild homology $\mathsf{HH}_*(A/K,B)$ and cohomology $\mathsf{HH}^*(A/K,B)$ of an associative algebra A over a commutative noetherian ring K with coefficients in an A-bimodule B. The following topics were discussed:

(1) Links of vanishing of $\mathsf{HH}^n(A/K,B)$ with smoothness,
(2) Finite generation of the K-algebra $\mathsf{HH}^*(A/K,A)$, and the K-algebra $\mathsf{HH}_*(A/K,A)$ (the latter when A is commutative),
(3) Generalizations and related theories, such as Schinkler (co)homology, André-Quillen (co)homology, cyclic homology.

In general, the case when A is commutative and finitely generated as a K-algebra are better understood, and offer guidelines about possible results in the case of finite dimensional algebras over a field K.

Poisson automorphisms, quiver moduli, and wall-crossing for Donaldson-Thomas invariants

MARKUS REINEKE

1. NOTATION

Let Q be a finite quiver without oriented cycles, with set of vertices $I = \{1,\ldots,n\}$ and arrows denoted by $\alpha : i \to j$. We order the vertices in such a way that $i > j$ provided there exists an arrow $i \to j$ in Q. Let $\Lambda = \mathbf{Z}I$ be the free abelian group on I, and let $\Lambda^+ = \mathbf{N}I$ be the set of dimension vectors. On Λ, we have the Euler form of Q given by $\langle d,e\rangle = \sum_{i\in I} d_ie_i - \sum_{\alpha:i\to j} d_ie_j$, and its antisymmetrization $\{d,e\} = \langle d,e\rangle - \langle e,d\rangle$.

Choose a linear map $\Theta : \Lambda \to \mathbf{Z}$ (a stability), and define the slope of $d \in \Lambda^+ \setminus 0$ by $\mu(d) = \Theta(d)/\dim d$, where $\dim d = \sum_{i\in I} d_i$. The set of all $d \in \Lambda^+ \setminus 0$ of slope μ, together with 0, forms a sublattice Λ^+_μ of Λ^+, for all $\mu \in \mathbf{Q}$.

We consider the category $\mathrm{Rep}_{\mathbf{C}}Q$ of complex representations of Q. The Auslander-Reiten translation on $\mathrm{Rep}_{\mathbf{C}}Q$ induces a linear map τ on Λ. The slope of a representation X is defined as the slope of its dimension vector $\underline{\dim}X$. Using the stability Θ, we can define notions of (semi-)stable representations in $\mathrm{Rep}_{\mathbf{C}}(Q)$ as follows: a representation X is called semistable if $\mu(U) \leq \mu(X)$ for all non-zero subrepresentations U of X, and it is called stable if $\mu(U) < \mu(X)$ for all non-zero proper subrepresentations.

By [1], there exists a smooth manifold $M^{st}_d(Q)$ parametrizing isomorphism classes of stable representations of dimension vector $d \in \Lambda^+$. We are interested in the Euler characteristic $\chi(M^{st}_d(Q))$ in singular cohomology.

2. STATEMENT OF THE RESULT

Given a quiver as above, we can define a Possion algebra structure on the formal power series ring $B = \mathbf{Q}[[x_1, \dots, x_n]]$ by $\{x^d, x^e\} = \{d, e\}x^{d+e}$ on monomials $x^d = \prod_{i \in I} x_i^{d_i} \in B$ for $d \in \Lambda^+$. We consider Poisson automorphisms T_i of B for $i \in I$ given by $T_i(x^d) = x^d(1 + x_i)^{\{i,d\}}$.

Theorem 2.1. *There exists a factorization*

$$T_1 \circ \ldots \circ T_n = \prod_{\mu \in \mathbf{Q} \ \mathit{decreasing}} (T_\mu : x^d \mapsto x^d \cdot F_\mu^{-(\mathrm{id}+\tau)d}(x)),$$

where formal series $F_\mu^d(x) \in B$ for $\mu \in \mathbf{Q}$ and $d \in \Lambda$ are defined by the functional equations

$$F_\mu^d(x) = \prod_{e \in \Lambda_\mu^+ \setminus 0} (1 - x^e F_\mu^e(x))^{\langle e,d \rangle \chi(M_e^{st}(Q))}.$$

3. RELATION TO [3]

On the formal power series ring $\mathbf{Q}[[x, y]]$ with Possion bracket $\{x, y\} = xy$, define Poisson automorphisms $T_{a,b}$ for $(a, b) \neq (0, 0)$ by

$$T_{a,b}(x) = x(1 - (-1)^{ab}x^a y^b)^b, \quad T_{a,b}(y) = y(1 - (-1)^{ab}x^a y^b)^{-a}.$$

The following is conjectured in [3]:

Conjecture 3.1. *There exists a factorization*

$$T_{0,1}^k T_{1,0}^k = \prod_{a/b \ \mathit{decreasing}} T_{a,b}^{kd(a,b,k)}$$

for $d(a, b, k) \in \mathbf{Z}$.

This is interpreted in [3] as a formula describing the behaviour of Donaldson-Thomas type invariants of a polarized noncommutative Calabi-Yau threefold (a 3-Calabi-Yau category endowed with a certain stability structure) under a change of stability condition (more precisely, a wall-crossing in a space of stability structures); the exponents $d(a, b, k)$ are viewed as universal local Donaldson-Thomas type invariants.

Specializing Theorem 2.1 to the k-arrow Kronecker quiver $K_k : 1 \overset{(k)}{\Leftarrow} 2$ with stability $\Theta(d, e) = e$, one can derive:

Corollary 3.2. *There exists a factorization*

$$T_{0,1}^k T_{1,0}^k = \prod_{\substack{a/b \ \mathit{decreasing} \\ a,b \ \mathit{coprime}}} \begin{pmatrix} x & \mapsto & xF_{a,b}(x^a y^b)^b \\ y & \mapsto & yF_{a,b}(x^a y^b)^{-a} \end{pmatrix}^k,$$

where $F_{a,b}(t) = F(t) \in \mathbf{Z}[[t]]$ is given by the functional equation

$$F(t) = \prod_{i \geq 1} (1 - (tF(t)^N)^i)^{-i\chi_i}$$

for $N = kab - a^2 - b^2$ *and* $\chi_i = \chi(M^{st}_{(ia,ib)}(K_k))$.

Conjecture 3.1 would follow from this provided all series $F(t)$ defined by such a functional equation admit a product factorization $F(t) = \prod_{i\geq 1}(1 - ((-1)^N t)^i)^{id_i}$ for integer d_i.

4. INGREDIENTS OF THE PROOF

Let $K = \mathbf{Q}(q)$ be the field of rational functions in q, and let $R = \mathbf{Q}[q]_{(q-1)}$ be the subring of functions without pole at $q = 1$. We consider the skew formal power series algebra $A = K_q[[x_1, \ldots, x_n]]$ with multiplication $x^d x^e = q^{-\langle e,d\rangle} x^{d+e}$. The natural R-lattice A_R in A (topologically) spanned by the x^d quantizes the Poisson algebra R, since $A_R(q-1) \simeq R$.

Let $H_k((Q))$ be the (completed) Hall algebra of Q for a finite field k, with topological basis $[M]$ indexed by the isomorphism classes of k-representations of Q, and multiplication

$$[M][N] = \sum_{[X]} |\{U \subset X \,:\, U \simeq M,\ X/U \simeq N\}| \cdot [X].$$

It admits a $\mathbf{Q}$-algebra morphism $\int : H((Q)) \to A_k$ to a $\mathbf{Q}$-algebra A_k defined in the same way as A, but with q replaced by $|k|$; this map is given by

$$\int [X] = \frac{1}{|\mathrm{Aut}_Q(X)|} x^{\underline{\dim} X}.$$

. Define series P and P_μ for $\mu \in \mathbf{Q}$ in $H((Q))$ by

$$P = \sum_{[X]} [X], \quad P_\mu = \sum_{\substack{X\, semistable \\ \mu(X)=\mu}} [X].$$

By [4], we have an identity (the Harder-Narasimhan recursion)

$$P = \prod_{\mu \text{ decreasing}} P_\mu, \text{ and thus } \int P = \prod_{\mu \text{ decreasing}} \int P_\mu$$

in A_k. It is also shown in [4] that $\int P$, $\int P_\mu$ admit generic versions E, E_μ in the K-algebra A.

Using results of [2], one can prove that conjugation by E, E_μ induces Poisson automorphisms T, T_μ of B, where T equals $T_1 \circ \ldots \circ T_n$, and T_μ is given by a power series involving Euler characteristics $\chi(M_d^P(Q))$ of so-called smooth models $M_d^P(Q)$. These are manifolds parametrizing pairs consisting of a semistable representation X of dimension vector d, together with a map $P \to X$ from a projective representation whose image avoids all subrepresentations U of X with $\mu(U) = \mu(X)$. The analysis of a stratification of $M_d^P(Q)$ in [2] allows to characterize this generating function of Euler characteristics as the solution to the functional equation of Theorem 2.1.

References

[1] A. King, *Moduli of representations of finite-dimensional algebras.* Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 515–530.

[2] J. Engel, M. Reineke, *Smooth models of quiver moduli.* Preprint 2007.

[3] M. Kontsevich, Y. Soibelman, *Stability structures, Donaldson-Thomas invariants and cluster transformations.* In preparation

[4] M. Reineke, *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli.* Invent. Math. 152 (2003), 349–368.

Quasi-hereditary algebras arising from preprojective algebras

Jan Schröer
(joint work with Christof Geiß, Bernard Leclerc)

Our aim is the construction of a new class of quasi-hereditary algebras: We start with the preprojective algebra Λ associated to a quiver Q, then we pass to a Frobenius subcategory $\mathcal{C}_M$ of $\mathrm{mod}(\Lambda)$. Inside $\mathcal{C}_M$ one can find a certain maximal rigid module T_M. Then $B := \mathrm{End}_\Lambda(T_M)$ is the quasi-hereditary algebra we want to study. The algebra B has many unusual and interesting properties. As an application, one can use B to "categorify" a certain cluster algebra associated to B. In this way, we obtain a new categorification of all acyclic cluster algebras using only classical tilting theory.

(Cluster algebras were introduced by Fomin and Zelevinsky [3]. They are combinatorially defined commutative algebras. As an introduction to this beautiful and rapidly developing area, we recommend to look at the survey article [4] and also at Sergey Fomin's *Cluster Algebra Portal.*)

Let Q be a finite quiver without oriented cycles, and let

$$\Lambda = \Lambda_Q = K\overline{Q}/(c)$$

be the associated *preprojective algebra.* We assume that Q is connected and has vertices $\{1, \ldots, n\}$ with n at least two. Here K is an algebraically closed field, $K\overline{Q}$ is the path algebra of the *double quiver* $\overline{Q}$ of Q which is obtained from Q by adding to each arrow $a\colon i \to j$ in Q an arrow $a^*\colon j \to i$ pointing in the opposite direction, and (c) is the ideal generated by the element

$$c = \sum_{a \in Q_1} (a^*a - aa^*)$$

where Q_1 is the set of arrows of Q. Preprojective algebras first appeared in work of Gelfand and Ponomarev. These algebras occur in many different contexts, for example there are close links with the theory of canonical bases for quantum group.

Clearly, the path algebra KQ is a subalgebra of Λ. By

$$\pi_Q\colon \mathrm{mod}(\Lambda) \to \mathrm{mod}(KQ)$$

we denote the corresponding restriction functor.

Let $\tau = \tau_Q$ be the Auslander-Reiten translation of KQ, and let $I_1, \ldots, I_n$ be the indecomposable injective KQ-modules. A KQ-module M is called *preinjective* if M is isomorphic to a direct sum of modules of the form $\tau^j(I_i)$ where $j \geq 0$ and $1 \leq i \leq n$.

A KQ-module $M = M_1 \oplus \cdots \oplus M_r$ with M_i indecomposable and $M_i \ncong M_j$ for all $i \neq j$ is called a *terminal KQ-module* if the following hold:

(i) M is preinjective;

(ii) If X is an indecomposable KQ-module with $\mathrm{Hom}_{KQ}(M, X) \neq 0$, then $X \in \mathrm{add}(M)$;

(iii) $I_i \in \mathrm{add}(M)$ for all indecomposable injective KQ-modules I_i.

In other words, the indecomposable direct summands of M are the vertices of a subgraph of the preinjective component of the Auslander-Reiten quiver of KQ which is closed under successor. We define

$$t_i := t_i(M) := \max\left\{j \geq 0 \mid \tau^j(I_i) \in \mathrm{add}(M) \setminus \{0\}\right\}.$$

Let M be a terminal KQ-module, and let

$$\mathcal{C}_M := \pi_Q^{-1}(\mathrm{add}(M))$$

be the subcategory of all Λ-modules X with $\pi_Q(X) \in \mathrm{add}(M)$. Notice that if Q is a Dynkin quiver and M is the sum of all indecomposable representations of Q then $\mathcal{C}_M = \mathrm{mod}(\Lambda)$. This case is studied intensively in [5].

Theorem 1 ([6]). *Let $M = M_1 \oplus \cdots \oplus M_r$ be a terminal KQ-module. Then the following hold:*

(i) *$\mathcal{C}_M$ is a Frobenius category with n indecomposable $\mathcal{C}_M$-projective-injectives;*

(ii) *The stable category $\underline{\mathcal{C}}_M$ is a 2-Calabi-Yau category;*

(iii) *If $t_i(M) = 1$ for all i, then $\underline{\mathcal{C}}_M$ is triangle equivalent to the cluster category (introduced in [1]) associated to Q.*

A Λ-module T is *rigid* if $\mathrm{Ext}^1_\Lambda(T, T) = 0$. Recall that for all $X, Y \in \mathrm{mod}(\Lambda)$ we have $\dim \mathrm{Ext}^1_\Lambda(X, Y) = \dim \mathrm{Ext}^1_\Lambda(Y, X)$. Assume that T is a rigid Λ-module in $\mathcal{C}_M$. Then T is called *$\mathcal{C}_M$-maximal rigid* if $\mathrm{Ext}^1_\Lambda(T \oplus X, X) = 0$ with $X \in \mathcal{C}_M$ implies $X \in \mathrm{add}(T)$.

Let A be a finite-dimensional algebra. By $P_1, \ldots, P_r$ and $S_1, \ldots, S_r$ we denote the indecomposable projective and simple A-modules, respectively, where $S_i = \mathrm{top}(P_i)$.

For a class $\mathcal{U}$ of A-modules let $\mathcal{F}(\mathcal{U})$ be the class of all A-modules X which have a filtration

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_t = 0$$

of submodules such that all factors X_{j-1}/X_j belong to $\mathcal{U}$ for all $1 \leq j \leq t$. Such a filtration is called a *$\mathcal{U}$-filtration* of X. We call these modules the *$\mathcal{U}$-filtered modules.*

Let Δ_i be the largest factor module of P_i in $\mathcal{F}(S_1, \ldots, S_i)$, and set

$$\Delta = \{\Delta_1, \ldots, \Delta_r\}.$$

The modules Δ_i are called *standard modules.* By $\mathcal{F}(\Delta)$ we denote the category of Δ-filtered A-modules.

The algebra A is called *quasi-hereditary* if $\mathrm{End}_A(\Delta_i) \cong K$ for all i, and if ${}_AA$ belongs to $\mathcal{F}(\Delta)$. Quasi-hereditary algebras first occured in Cline, Parshall and Scott's [2] study of highest weight categories.

To any terminal KQ-module $M = M_1 \oplus \cdots \oplus M_r$, we can construct a $\mathcal{C}_M$-maximal rigid module T_M such that for $B := \mathrm{End}_\Lambda(T_M)$ the following hold:

Theorem 2 ([6]). (i) *B is a quasi-hereditary algebra;*

(ii) *The restriction of the contravariant functor* $\mathrm{Hom}_\Lambda(-, T_M)\colon \mathrm{mod}(\Lambda) \to \mathrm{mod}(B)$ *induces an anti-equivalence* $F\colon \mathcal{C}_M \to \mathcal{F}(\Delta)$ *where* $\mathcal{F}(\Delta)$ *is the category of* Δ*-filtered* B*-modules and*

$$\Delta := \{F(M_i) \mid 1 \leq i \leq r\}$$

is the set of standard modules. (We interpret M_i *as a* Λ*-module using the obvious embedding functor.);*

(iii) *For a short exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $\mathcal{C}_M$ *the following are equivalent:*
  (a) *The short exact sequence* $0 \to \pi_Q(X) \to \pi_Q(Y) \to \pi_Q(Z) \to 0$ *splits;*
  (b) *The sequence* $0 \to F(Z) \to F(Y) \to F(X) \to 0$ *is exact.*

The quasi-hereditary algebras $B = \mathrm{End}_\Lambda(T_M)$ have many interesting properties. For example, the modules in $\mathcal{F}(\Delta)$ all have projective dimension at most one. Furthermore, the indecomposable projective B-modules have a unique(!) Δ-filtration. We can describe the characteristic tilting module in great detail, which is also quite rare.

The category $\mathcal{F}(\Delta)$ of Δ-filtered B-modules "categorifies" the cluster algebra $\mathcal{A}(\mathcal{C}_M)$ defined by the quiver of B. This includes all acyclic cluster algebras. There is a bijection between the set of clusters of $\mathcal{A}(\mathcal{C}_M)$ and the set of isomorphism classes of classical tilting B-modules in $\mathcal{F}(\Delta)$.

References

[1] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

[2] E. Cline, B. Parshall, L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[3] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[4] S. Fomin, A. Zelevinsky, *Cluster algebras: notes for the CDM-03 conference*, Current developments in mathematics, 2003, 1–34, Int. Press, Somerville, MA, 2003.

[5] C. Geiß, B. Leclerc, J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), no. 3, 589–632.

[6] C. Geiß, B. Leclerc, J. Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*, Preprint (2008), 114pp.

PBW and semicanonical bases for cluster algebras

Christof Geiss
(joint work with Bernard Leclerc and Jan Schröer)

Introduction

This is the continuation of the talk given by Jan Schröer. Our aim is to show that we have a particular good understanding of cluster algebras associated to the "initial seed" T_M coming from a preinjective $\mathbb{C}Q$-module M.

1. Review of Lusztig's geometric construction of $U(\mathfrak{n})$

Let Q be a quiver without oriented cycles and vertex set $Q_0 = \{1, 2, \dots, n\}$. Associated to the underlying graph $|Q|$ we have a Kac-Moody Lie algebra $\mathfrak{g}$ with *symmetric* generalized Cartan matrix $C_{|Q|}$. We have the usual triangular decomposition $\mathfrak{h} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$, and the decomposition

$$\mathfrak{n} = \oplus_{\alpha \in \Phi^+} \mathfrak{n}_\alpha$$

into root spaces, where Φ^+ denotes the positive roots associated to the Weyl group $W \subset \mathrm{GL}(\mathfrak{h}^*)$.

For each dimension vector $\beta \in \mathbb{N}_0^n$ let Λ_β be the (affine) variety of nilpotent representations of Λ with dimension vector β. On Λ_β acts the algebraic group

$$\mathrm{GL}_\beta = \prod_{i=1}^n \mathrm{GL}_{\beta(i)}(\mathbb{C})$$

by conjugation. Thus the orbits are in bijection with the isoclasses of nilpotent representations with dimension vector β.

We consider $\widetilde{\mathcal{M}}(\beta)$ the space of $\mathbb{C}$-valued constructable GL_β-invariant functions on Λ_β. The direct sum

$$\widetilde{\mathcal{M}} = \oplus_{\beta \in \mathbb{N}_0^n} \widetilde{\mathcal{M}}(\beta)$$

becomes a (graded) associative algebra with multiplication for $f' \in \widetilde{\mathcal{M}}(\beta')$ and $f'' \in \widetilde{\mathcal{M}}(\beta'')$ given by

$$(f' * f'')(X) := \int_{U \in \mathrm{Gr}^{\mathrm{rep}}(X, \beta')} f'(U) f''(X/U) \text{ for all } X \in \Lambda_{\beta' + \beta''},$$

where $\mathrm{Gr}^{\mathrm{rep}}(X, \beta')$ denotes the (projective) variety of all sub-representations of X with dimension vector β'. The integral is defined with the topological Euler characteristic as measure. This is quite similar to Ringel's Hall-algebra construction, however we work over the complex numbers rather than with finite fields.

In $\widetilde{\mathcal{M}}$ we consider the subalgebra $\mathcal{M}$ which is generated by the functions $\mathbb{1}_i \in \widetilde{\mathcal{M}}\alpha_i$ for $1 \leq i \leq n$ where α_i denotes the corresponding simple root, so $\Lambda_{\alpha_i} = \{\mathrm{pt}\}$.

Theorem (Lusztig)**.** *The assignation* $e_i \mapsto 1\!\!1_i$ *induces an isomorphism*

$$U(\mathfrak{n}) \to \mathcal{M},$$

where the e_i *denote the Chevalley generators of* $U(\mathfrak{n})$*.*

See [5]. Henceforth we will identify these two algebras. In order to prove this result the following is needed. Let $\Lambda_\beta = C_1 \cup \cdots \cup C_p$ be the decomposition into irreducible components. Then we have:

Proposition 1.1. *There exist dense open subsets* $U_i \subset C_i$ *and a basis* $\mathcal{S}(\beta) = (s_i)_{1,\dots,p}$ *of* $\mathcal{M}(\beta)$ *such that*

$$s_i\,|_{U_j} = \begin{cases} 1\!\!1_{U_i} & \text{if } i = j \\ 0 & \text{else} \end{cases}.$$

The union of the bases $\mathcal{S}(\beta)$ for all possible β is the *semicanonical basis* of $U(\mathfrak{n})$.

2. Cluster Character

Proposition 2.1. *Consider the usual comultiplication* $\Delta\colon U(\mathfrak{n}) \to U(\mathfrak{n}) \otimes U(\mathfrak{n})$*. In terms of constructable functions we have* $f(x' \oplus x") = \Delta(f)(x', x")$

See [2] for a proof.

Corollary 2.2. *For* $f \in \mathfrak{n}_\alpha$ *we have* $\mathrm{supp}(f) \subset \Lambda_\alpha^{ind}$

This follows from the classical fact that for $f \in U(\mathfrak{n})$ we have $f \in \mathfrak{n}$ if and only if $f = 1 \otimes f + f \otimes f$ and the above proposition.

Now consider the graded dual $U(\mathfrak{n})^*_{\mathrm{gr}} = \oplus_{\alpha \in \mathbb{N}_0^n} \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n})_\alpha, \mathbb{C})$. This is is a commutative Hopf algebra. Note that the dual $\mathcal{S}^*$ of the semicanoical basis $\mathcal{S}$ is a basis of $U(\mathfrak{n})^*_{\mathrm{gr}}$, the *dual semicanoical basis.*

Via our identification $U(\mathfrak{n})_\alpha = \mathcal{M}(\alpha)$ we obtain an evaluation map

$$\delta_?\colon \Lambda\text{-mod}_0 \to U(\mathfrak{n})^*_{\mathrm{gr}}, \quad x \mapsto \delta_x$$

i.e. if $\underline{\dim} X = \alpha$ then $\delta_X(f) = f(x)$ for all $f \in \mathcal{M}(\alpha)$. It is easy to see that in case $\mathrm{Ext}^1_\Lambda(X, X) = 0$ we have $\delta_X \in \mathcal{S}^*$. By the following result δ is a cluster character in the sense of Y. Palu.

Theorem 1. *We have*

(a) *For* $X, Y \in \Lambda\text{-mod}_0$ *we have* $\delta_{X \oplus Y} = \delta_X \delta_Y$*.*
(b) *If* $\dim_{\mathbb{C}} \mathrm{Ext}^1_\Lambda(X, Y) = 1$ *and*

$$0 \to Y \to E' \to X \to 0 \text{ and } 0 \to X \to E" \to Y \to 0$$

are the corresponding non-split short exact sequences, then

$$\delta_X \delta_Y = \delta_{E'} + \delta_{E"}.$$

Part (a) of the Theorem follows from 2.1 above, while part (b) follows from an adaption [3] of the corresponding result for the Caldero-Keller map [1]. In fact, we have a more general result (without the restriction to $\dim_{\mathbb{C}} \mathrm{Ext}^1_\Lambda(X, Y) \leq 1$), however we do not need this here.

3. A CLUSTER ALGEBRA ASSOCIATED TO $\mathcal{C}_M$

We consider $\mathcal{R}(\mathcal{C}_M, T_M)$ the *subalgebra of* $U(\mathfrak{n})^*_{gr}$ *generated by the* δ_R *with* R *(indecomposable), rigid and reachable from* T_M *via mutation.* So it is roughly speaking the cluster algebra which via δ is categorified by $(\mathcal{C}_M, T_M)$. We know:

$$\mathbb{C}[\delta_{M_1}, \ldots, \delta_{M_r}] \subset \mathcal{R}(\mathcal{C}_M, T_M) \subset \operatorname{span}_{\mathbb{C}}\langle \delta_X \mid X \in \mathcal{C}_M\rangle$$

The first inclusion holds since all M_i appear on the mutation path from T_M to $T_M^\vee$, see Schröer's talk. The second inclusion is trivial however, we should note that the last space is in fact a ring. This follows since $\mathcal{C}_M$ is additive and by Theorem 1 (a). Note, that we can consider $\mathbb{C}Q$-modules as Λ-modules with all arrows a^* acting trivially.

Theorem 2. *We have the following:*

(a) *The monomials* $(\delta_{M'})_{M' \in \mathrm{Add}(M)}$ *form a basis of* $\operatorname{span}_{\mathbb{C}}\langle \delta_X \mid X \in \mathcal{C}_M\rangle$ *in fact, they belong to the dual of a PBW-basis. In particular,* $\mathcal{R}(\mathcal{C}_M, T_M)$ *is a polynomial ring.*

(b) *The elements* $(\delta_{(M', g_{M'})} \subset \mathcal{S}^*$ *span also* $\mathcal{R}(\mathcal{C}_M, T_M)$*. Thus we have completed the cluster monomials to a basis of* $\mathcal{R}(\mathcal{C}_M, T_M)$*.*

Proof. (Sketch) Since the summands M_i of M are indecomposable preinjective $\mathbb{C}Q$-modules we may assume $\operatorname{Ext}^*_Q(M_i, M_j) = 0$ if $i \geq j$. It is easy to see that

$$\mathfrak{n}(M) := \oplus_{i=1}^r \mathfrak{n}_{\underline{\dim} M_i} \subset \mathfrak{n}$$

is a Lie algebra. We can choose a basis $p_1, p_2, \ldots$ of $\mathfrak{n}$ consisting of root vectors such that p_i spans $\mathfrak{n}_{\underline{\dim} M_i}$ for $1 \leq i \leq r$. Build from this the (scaled) PBW-basis of $U(\mathfrak{n})$ consisting of the elements

$$p_{\mathbf{m}} = p_1^{(m_1)} * \cdots * p_l^{(m_l)}$$

with $\mathbf{m} \in \mathbb{N}_0^{(\mathbb{N})}$ and $p_i^{(m)} := \frac{1}{m!} p_i^m$.

Our claim follows now essentially from the following observations:

(a) If $m_l \neq 0$ for some $l > r$ then $p_{\mathbf{m}}(X) = 0$ for all $X \in \mathcal{C}_M$.

(b) Let $M' = \oplus_{i=1}^r M_i^{m_i} \in \mathrm{Add}(M)$, then

$$p_{\mathbf{m}} \mid_{\mathrm{Rep}(Q, \underline{\dim} M')} = \mathbb{1}_{\mathcal{O}(M')}.$$

For (i) one reduces easyly to the claim $p_l(X) = 0$ for $l > r$ and $X \in \mathcal{C}_M$ since our category is closed under factor modules. Now, the claim follows by a dimension argument.

For (ii) we note that the affine space $\mathrm{Rep}(Q, \underline{\dim} M')$ of representations of Q can be viewed as an irreducible component of $\Lambda_{\underline{\dim} M'}$. Our claim follows now from the order of the M_i. ☐

REFERENCES

[1] Ph. Caldero, B. Keller: *From triangulated categories to cluster algebras*, `arXiv:math.RT/0506018v2`, to appear Invent. Math.

[2] Ch. Geiss, B. Leclerc, J. Schröer: *Semicanonical bases and preprojective algebras*, Ann. Sci. École Norm. Sup. (4) **38** (2005), 193-253.

[3] Ch. Geiss, B. Leclerc, J. Schröer: *Semicanonical baseses and preprojective algebras II, a multiplication formula*, Compositio Math. **143** (2007), 131–1334.

[4] Ch. Geiss, B. Leclerc, J. Schröer: *Cluster algebra structures and semicanonical bases for unipotent groups*, `arXiv:math/0703039v2`.

[5] G. Lusztig: *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129-139.

Denominators of cluster variables

Robert Marsh

(joint work with Aslak Bakke Buan, Idun Reiten)

1. The Laurent Phenomenon

Cluster algebras were introduced by Fomin-Zelevinsky [10], and have links with many topics, from Poisson geometry to Teichmüller theory. They are of particular interest because of links to the canonical basis (introduced by Kashiwara and Lusztig) of a quantized enveloping algebra and totally positive matrices. Here we consider connections to the representation theory of finite dimensional algebras.

Let $\mathbf{x}$ be a free generating set for the field $\mathbb{F}$ of rational functions in n indeterminates over $\mathbb{Q}$. Let Q be a quiver with vertices indexed by $\mathbf{x}$. Such a pair $(\mathbf{x}, Q)$ is known as a *seed*. (We consider here cluster algebras without coefficients and restrict to the skew-symmetric case.) For each $z \in \mathbf{x}$, $(\mathbf{x}, Q)$ can be *mutated* to a new seed $\mu_z(\mathbf{x}, Q)$. Let $\mathcal{S}(\mathbf{x}, Q)$ be the set of seeds obtained by arbitrary iterated mutation of $(\mathbf{x}, Q)$. The *cluster algebra* $\mathcal{A}(\mathbf{x}, Q)$ is the subring of $\mathbb{F}$ generated by the union of all free generating sets in the seeds in $\mathcal{S}(\mathbf{x}, Q)$. Such free generating sets are known as *clusters* and their elements are known as *cluster variables*.

By the definition each cluster variable is a rational function of the cluster variables in any fixed cluster. In fact, more is true. The *Laurent Phenomenon* of Fomin and Zelevinsky [10, 3.1] (which was proved in wider generality) is as follows:

Theorem 1.1. *Any cluster variable of* $\mathcal{A}(\mathbf{x}, Q)$ *can be written as a Laurent polynomial in the elements of a fixed cluster* $\mathbf{y}$.

2. The cluster category

Motivated by connections between cluster algebras and the representation theory of finite dimensional algebras developed in [13], the *cluster category* $\mathcal{C}_Q$ associated to the cluster algebra above was introduced independently in [8] (for type A_n) and [2]. The construction in [8] was combinatorial, given in terms of diagonals of a regular polygon with $n + 3$ vertices, and the construction in [2] was given in terms of the derived category of the path algebra kQ.

An object X of $\mathcal{C}_Q$ is said to be *exceptional* if $\operatorname{Ext}^1_{\mathcal{C}_Q}(X, X) = 0$. We now collect together some important results linking the properties of $\mathcal{A}(\mathbf{x}, Q)$ and $\mathcal{C}_Q$:

Theorem 2.1. *Suppose that Q is an acyclic quiver, i.e. it has no oriented cycles.*
(a) There is a bijection $M \mapsto u_M$ between the exceptional indecomposable objects of $\mathcal{C}_Q$ and the cluster variables of $\mathcal{A}(\mathbf{x}, Q)$.
(b) The bijection in (a) induces a bijection between the seeds of $\mathcal{A}(\mathbf{x}, Q)$ and the maximal rigid *(cluster-tilting) objects of $\mathcal{C}_Q$, maximal direct sums of nonisomorphic indecomposable objects with no self-extensions.*
(c) The quiver in a seed is the same as the quiver of the endomorphism algebra of the corresponding cluster-tilting object.
(d) Let M be an exceptional indecomposable object of $\mathcal{C}_Q$. Then:

$$u_M = \frac{f(\mathbf{x})}{\prod_{x \in \mathbf{x}} x^{d_x}},$$

where $\mathbf{d} = (d_x)_{x \in \mathbf{x}}$ is the dimension vector of M and f is a polynomial not divisible by any $x \in \mathbf{x}$.

Proof. Suppose first that Q is an alternating orientation of a Dynkin quiver. Fomin-Zelevinsky proved in [11] that there is a bijection between the cluster variables of $\mathcal{A}(\mathbf{x}, Q)$ and the almost positive roots of the corresponding root system (i.e. the positive roots together with the negative simple roots), such that the cluster variable u_α corresponding to a root α can be written in the form:

$$u_\alpha = \frac{f(\mathbf{x})}{\prod_{x \in \mathbf{x}} x^{d_x}},$$

where the d_x are the coefficients of α written in terms of the simple roots and f is not divisible by any $x \in \mathbf{x}$. This can be regarded as a root system-theoretic version of (a) and (d) (a version of (b) is also provided).

In [8] (a) and (b) were shown for type A and in [2] (using results from [13]), (a) and (b) were shown for simply-laced Dynkin quivers. In [16], (a) and (b) were shown for the non-simply-laced case (see also [15]). Part (d) then follows from [11] if Q is an alternating orientation of a simply-laced Dynkin quiver. Part (d) was shown for arbitrary orientations in type A in [8] and for arbitrary orientations of a simply-laced Dynkin quiver in [9, 14].

In the general case, (a) and (b) were shown in [7] (then also in [1] using results from [6]). That denominators of cluster variables are given by dimension vectors was shown in [6]. Part (d) was shown in [7]. Part (c) follows from results in [3] using (a) and (b) and the fact that in the bijection in (a) exchange of complements of an almost complete cluster-tilting object in $\mathcal{C}_Q$ corresponds to cluster mutation. □

A natural question is how the denominator of a cluster variable can be interpreted in the case where the initial seed does not contain an acyclic quiver. Let $T = \oplus_{y \in \mathbf{y}} T_y$ be the the image under τ^{-1} of the cluster-tilting object corresponding to a seed $(\mathbf{y}, R)$ of $\mathcal{A}(\mathbf{x}, Q)$. We say that u_M has a *T-denominator* if $u_M \in \mathbf{y}$ or if

$$u_M = \frac{f(\mathbf{y})}{\prod_{y \in \mathbf{y}} y^{d_y}},$$

where $d_y = dim\,\mathrm{Hom}_{\mathcal{C}_Q}(T_y, M)$ for $y \in \mathbf{y}$ and $f(\mathbf{y})$ is not divisible by any $y \in \mathbf{y}$. It follows from [5] that there is a bijection between cluster variables not in $\mathbf{y}$ and the indecomposable modules over the cluster-tilted algebra $\Gamma_T := \mathrm{End}_{\mathcal{C}_Q}(T)^{\mathrm{opp}}$. The vector $\mathbf{d} = (d_y)_{y\in\mathbf{y}}$ can be interpreted as the dimension vector of the Γ_T-module corresponding to u_M.

Theorem 2.2. [9, 14] *Suppose that $\mathcal{A}(\mathbf{x}, Q)$ has simply-laced Dynkin type. Then, for all indecomposable exceptional objects M of $\mathcal{C}_Q$, u_M has a T-denominator.*

3. MAIN RESULTS

We can now state the main results of [4], using the above notation.

Theorem 3.1. [4] *(a) If no summand of T is regular then every cluster variable of $\mathcal{A}(\mathbf{x}, Q)$ has a T-denominator.*
(b) If all cluster variables have T-denominators then $\mathrm{End}_{\mathcal{C}_Q}(T_y) \cong k$ for all $y \in \mathbf{y}$.

Theorem 3.2. [4] *Suppose that kQ is tame. The following are equivalent:*
(a) Every cluster variable of $\mathcal{A}(\mathbf{x}, Q)$ has a T-denominator.
(b) No regular summand T_y of T with quasilength $r-1$ lies in a tube of rank r.
(c) We have $\mathrm{End}_{\mathcal{C}_Q}(T_y) \cong k$ for all $y \in \mathbf{y}$.

Corollary 3.3. [4] *Every cluster variable of $\mathcal{A}(\mathbf{x}, Q)$ has a T-denominator for every cluster-tilting object if and only if Q is Dynkin or has exactly two vertices.*

An interesting open question is how to interpret the exponents in the denominator of a cluster variable representation-theoretically in the general case.

REFERENCES

[1] A. B. Buan, P. Caldero, B. Keller, R. J. Marsh, I. Reiten and G. Todorov, *Appendix to Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. **135**, No. 10 (2007), 3049–3060.

[2] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204** (2) (2006), 572–618.

[3] A. B. Buan, R. J. Marsh and I. Reiten, *Cluster mutation via quiver representations*, Comment. Math. Helv. **83** Issue 1 (2008), 143-177.

[4] A. B. Buan, R. J. Marsh and I. Reiten, *Denominators of cluster variables*, preprint arXiv:math.RT/0710.4335 (2007).

[5] A. B. Buan, R. Marsh and I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc., **359**, no. 1 (2007), 323–332.

[6] A. B. Buan, R. J. Marsh, I. Reiten and G. Todorov, *Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. **135**, No. 10 (2007), 3049–3060.

[7] P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*, Ann. Sci. Ecole Norm. Sup, 4eme serie, **39**, (2006), 983–1009.

[8] P. Caldero, F. Chapoton and R. Schiffler *Quivers with relations arising from clusters (A_n case)*, Transactions of the American Mathematical Society **358** (2006), 1347–1364.

[9] P. Caldero, F. Chapoton and R. Schiffler *Quivers with relations and cluster-tilted algebras*, Algebras and Representation Theory, **9**, No. 4 , (2006), 359–376.

[10] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[11] S. Fomin and A. Zelevinsky, *Cluster Algebras II: Finite type classification*, Invent. Math. **154**(1) (2003), 63–121.

[12] S. Fomin and A. Zelevinsky. *The Laurent phenomenon.* Adv. in Appl. Math. **28** (2002), no. 2, 119–144.

[13] R. Marsh, M. Reineke and A. Zelevinsky, *Generalized associahedra via quiver representations*, Trans. Amer. Math. Soc. **355** (2003), no. 1, 4171–4186.

[14] I. Reiten and G. Todorov, unpublished.

[15] Dong Yang, *Non-simply-laced Clusters of Finite Type via Frobenius Morphism*, preprint arxiv:math.RT/0608114 (2006).

[16] B. Zhu, *BGP-reflection functors and cluster combinatorics*, J. Pure and Applied Alg. **209** (2007), 497–506.

Vector bundles on cubic curves and Yang-Baxter equations

Igor Burban

(joint work with Bernd Kreußler)

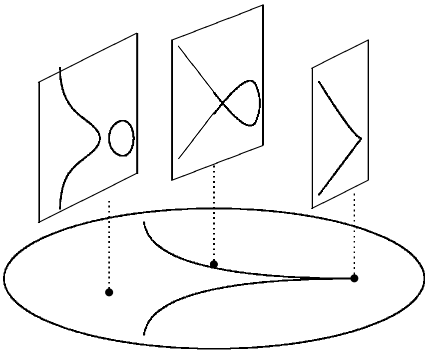
My talk is based on a joint article with Bernd Kreußler [3]. It is devoted to applications of methods of homological algebra in the theory of the classical Yang-Baxter equation

$$\big[r^{12}(x), r^{23}(y)\big] + \big[r^{12}(x), r^{13}(x+y)\big] + \big[r^{13}(x+y), r^{23}(y)\big] = 0,$$

where $r(z)$ is the germ of a meromorphic function of one complex variable z in a neighborhood of 0 taking values in $U\big(\mathfrak{sl}_n(\mathbb{C})\big) \otimes U\big(\mathfrak{sl}_n(\mathbb{C})\big)$.

Due to a classification of Belavin and Drinfeld, there are three types of non-degenerated solutions of this equation: elliptic, trigonometric and rational [1]. From that time an open question is to study degenerations of elliptic solutions into trigonometric and then further into rational ones. In order to attack this problem, we use a construction of solutions of the Yang-Baxter equation which was introduced by Polishchuk [4, 5]. The quintessence of his method is the following.

Let E be an irreducible projective curve of arithmetic genus one over $\mathbb{C}$, i.e. a plane projective curve given by the equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$. It is singular if any only if $\Delta := g_2^3 - 27g_3^2 = 0$.



Unless $g_2 = g_3 = 0$, the singularity is a node, whereas for $g_2 = g_3 = 0$ it is a cusp.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two non-isomorphic stable vector bundle on E of the same rang n and degree d, and $\mathbb{C}_{y_1}$ and $\mathbb{C}_{y_2}$ be two different sky-scraper sheaves. Then the tensor describing the triple Massey product

$$m_3 : \mathsf{Hom}(\mathcal{E}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}(\mathbb{C}_{y_1}, \mathcal{E}_2[1]) \otimes \mathsf{Hom}(\mathcal{E}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}(\mathcal{E}_1, \mathbb{C}_{y_2})$$

in the derived category of coherent sheaves $D^b\big(\mathsf{Coh}(E)\big)$ gives rise to a solution of the classical Yang-Baxter equation for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, see [4, Theorem 4]. In the case of a smooth elliptic curve E it was shown by Polishchuk that in such a way one gets all elliptic solutions. In the case of a nodal respectively cuspidal Weierstraß cubic curve one gets certain trigonometric or rational solutions respectively.

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be a basis of $\mathfrak{g}$. Using an explicit description of vector bundles on Weierstraß cubic curves (see for example [2]), we can carry out an explicit computation of triple Massey products, which leads to the following solutions of the classical Yang-Baxter equation.

- For a smooth elliptic curve, we get the elliptic solution obtained and used by Baxter, Belavin and Sklyanin:

$$r_{\mathrm{ell}}(y) = \frac{\mathrm{cn}(y)}{\mathrm{sn}(y)} h \otimes h + \frac{1 + \mathrm{dn}(y)}{\mathrm{sn}(y)}\big(e \otimes f + f \otimes e\big) + \frac{1 - \mathrm{dn}(y)}{\mathrm{sn}(y)}\big(e \otimes e + f \otimes f\big).$$

- For a nodal cubic curve, we obtain the trigonometric solution of Cherednik:

$$r_{\mathrm{trg}}(y) = \frac{1}{2}\cot(y) h \otimes h + \frac{1}{\sin(y)}\big(e \otimes f + f \otimes e\big) + \sin(y) e \otimes e.$$

- Finally, in the case of the cuspidal cubic curve, we get the rational solution of Stolin:

$$r_{\mathrm{rat}}(y) = \frac{1}{y}\left(\frac{1}{2} h \otimes h + e \otimes f + f \otimes e\right) + y\big(e \otimes h + h \otimes e\big) - y^3 e \otimes e.$$

REFERENCES

[1] A. Belavin, V. Drinfeld, *Solutions of the classical Yang-Baxter equation for simple Lie algebras*, Funct. Anal. Appl. **16**, 159-180 (1983).

[2] L. Bodnarchuk, I. Burban, Yu. Drozd, G.-M. Greuel, *Vector bundles and torsion free sheaves on degenerations of elliptic curves*, Global aspects of complex geometry, 83–128, Springer, Berlin, 2006.

[3] I. Burban, B. Kreussler, *Vector bundles on degenerations of elliptic curves and Yang-Baxter equations*, arXiv:0708.1685.

[4] A. Polishchuk, *Classical Yang-Baxter equation and the A_∞–constraint,* Adv. Math. **168** (2002), no. **1**, 56–95.

[5] A. Polishchuk, *Massey products on cycles of projective lines and trigonometric solutions of the Yang-Baxter equations*, arXiv:math/0612761.

The stable category of vector bundles on a weighted projective line

Helmut Lenzing

(joint work with José Antonio de la Peña, Dirk Kussin, Hagen Meltzer)

We work over an algebraically closed field k. It is known that hereditary categories catch a significant part of the representation theory of finite dimensional algebras. By a result of Happel [5] there are — up to derived equivalence — only two types of (connected, Hom-finite) hereditary, abelian categories with a tilting object: the categories $\mathrm{mod}(H)$ of finite dimensional modules over the path algebra of a finite quiver and the categories $\mathrm{coh}(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X}$. We extend the range of representation theoretic phenomena controlled by hereditary categories by introducing the stable category of vector bundles on a weighted projective line.

For a weighted projective line $\mathbb{X}$ of weight type $(p_1,\dots,p_t)$ let $\mathrm{coh}(\mathbb{X})$ denote its category of coherent sheaves and $\mathrm{vect}(\mathbb{X})$ its category of vector bundles (i.e. locally free coherent sheaves). Recall from [2] that $\mathrm{coh}(\mathbb{X})$ is a Hom-finite abelian category which is hereditary, noetherian and satisfies Serre duality $\mathrm{D}\,\mathrm{Ext}^1(X,Y) = \mathrm{Hom}(Y,\tau X)$ with a self-equivalence τ serving as the Auslander-Reiten translation for $\mathrm{coh}(\mathbb{X})$. Moreover, $\mathrm{coh}(\mathbb{X})$ has a tilting object whose endomorphism ring is a canonical algebra in the sense of Ringel [14].

Depending on the Euler characteristic $\chi_{\mathbb{X}} = 2 - \sum_{i=1}^{t}(1-1/p_i)$ of $\mathbb{X}$, we define the *distinguished class of line bundles* $\mathcal{L}$ on $\mathbb{X}$ as follows: If $\chi_{\mathbb{X}} \neq 0$, then $\mathcal{L}$ is the closure under isomorphism of the τ-orbit $\tau^{\mathbb{Z}}\mathcal{O}$ of the structure sheaf, otherwise $\mathcal{L}$ is the system of all line bundles. Note that $\mathcal{L}$ is always closed under Auslander-Reiten translation. A sequence $0 \to A \to B \to C \to 0$ of vector bundles on $\mathbb{X}$ is called a *distinguished exact sequence* if for each distinguished line bundle the sequence

$$0 \to \mathrm{Hom}(L,A) \to \mathrm{Hom}(L,B) \to \mathrm{Hom}(L,C) \to 0$$

is exact. Each distinguished exact sequence in $\mathrm{vect}(\mathbb{X})$ is an exact sequence in $\mathrm{coh}(\mathbb{X})$, the converse does not hold.

Theorem 1. *(i) The distinguished exact sequences define an exact structure $\mathcal{E}$ (in the sense of Quillen) on the category* $\mathrm{vect}(\mathbb{X})$.

(ii) With respect to this exact structure $\mathrm{vect}(\mathbb{X})$ *is a* Frobenius category, *that is, $\mathcal{E}$-injectives agree with $\mathcal{E}$-projectives and there are sufficiently many $\mathcal{E}$-injectives ($\mathcal{E}$-projectives).*

(iii) The distinguished line bundles are exactly the indecomposable $\mathcal{E}$-injectives (or $\mathcal{E}$-projectives) of $\mathrm{vect}(\mathbb{X})$.

By definition the *stable category of vector bundles* $\underline{\mathrm{vect}}(\mathbb{X})$ on $\mathbb{X}$ is the factor category of $\mathrm{vect}(\mathbb{X})$ by the two-sided ideal of all morphisms factoring through a finite direct sum of distinguished line bundles. As a consequence of [4] we thus obtain:

Corollary 2. *The stable category* $\underline{\mathrm{vect}}(\mathbb{X})$ *of vector bundles on* $\mathbb{X}$ *is a triangulated category. Its triangles are induced from the distinguished exact sequences from* $\mathrm{vect}(\mathbb{X})$.

Each Auslander-Reiten sequence in $\mathrm{vect}(\mathbb{X})$ whose end terms do not belong to $\mathcal{L}$ is a distinguished exact sequence. This yields:

Corollary 3. *The triangulated category* $\underline{\mathrm{vect}}(\mathbb{X})$ *has Auslander-Reiten triangles. Moreover, the Auslander-Reiten translation in* $\underline{\mathrm{vect}}(\mathbb{X})$ *is induced from the Auslander-Reiten translation of* $\mathrm{vect}(\mathbb{X})$.

The proof of Theorem 1 is not obvious. It relies on an analysis of the graded surface singularities attached to a weighted projective line as summarized in the next proposition, which combines results of [2], [3] and [9], where more explicit information is given.

Proposition 4. *(i) For* $\chi_{\mathbb{X}} > 0$, *the orbit algebra* $R = \bigoplus_{n\geq 0} \mathrm{Hom}(\mathcal{O}, \tau^{-n}\mathcal{O})$ *is a commutative affine algebra of the form* $k[x_1, x_2, x_3]/(f)$, *in particular graded complete intersection.*

(ii) For $\chi_{\mathbb{X}} = 0$ *the category* $\mathcal{L}$ *is determined completely by the* $\mathrm{L}(p)$*-graded coordinate algebra* S *of* $\mathbb{X}$ *(see* [2]*) which is commutative affine and graded complete intersection.*

(iii) For $\chi_{\mathbb{X}} < 0$ *the orbit algebra* $R = \bigoplus_{n\geq 0} \mathrm{Hom}(\mathcal{O}, \tau^{n}\mathcal{O})$ *is a commutative affine algebra which is graded Gorenstein.* $\square$

By way of example the weight type $(2,3,5)$, where $\chi_{\mathbb{X}} > 0$, yields the simple singularity $R = k[x_1, x_2, x_3]/(x_1^2 + x_2^3 + x_3^5)$ with degrees $\bar{x}_1 = 15$, $\bar{x}_2 = 10$, $\bar{x}_3 = 6$. For $(2,3,7)$, where $\chi_{\mathbb{X}} < 0$, we obtain the exceptional unimodal singularity $R = k[x_1, x_2, x_3]/(x_1^2 + x_2^3 + x_3^7)$ with degrees $\bar{x}_1 = 21$, $\bar{x}_2 = 14$, $\bar{x}_3 = 6$. For $(2,3,6)$, where $\chi_{\mathbb{X}} = 0$, we obtain the elliptic singularity $S = k[x_1, x_2, x_3]/(x_1^2 + x_2^3 + x_3^6)$ where the degrees $\vec{x}_1$, $\vec{x_2}$, $\vec{x_3}$ generate the rank one abelian group $\mathrm{L}(2,3,6) \cong \mathbb{Z} \times \mathbb{Z}_6$ with relations $2\vec{x}_1 = 3\vec{x}_2 = 6\vec{x}_3$.

If $\chi_{\mathbb{X}} > 0$, the weight sequence (p_1, p_2, p_3) describes a Dynkin diagram Δ, by $k[\vec{\Delta}]$ we denote the corresponding path algebra for some quiver $\vec{\Delta}$ with underlying graph Δ. For a finite dimensional k-algebra A and a right A-module M we define the one-point extension $A[M]$ as the matrix algebra $\begin{bmatrix} A & 0 \\ M & k \end{bmatrix}$. For a canonical algebra Λ all its one-point extensions with an indecomposable projective module P are derived-equivalent [11]. For P the indecomposable projective corresponding to the sink vertex of Λ the algebra $\hat{\Lambda} = \Lambda[P]$ is called the *extended canonical algebra* attached to $\mathbb{X}$. We obtain the following interesting trichotomy.

Theorem 5. *There are equivalences of triangulated categories*

$$\underline{\mathrm{vect}}(\mathbb{X}) = \begin{cases} \mathrm{D}^b(\mathrm{mod}(k[\vec{\Delta}])) & \textit{if } \chi_{\mathbb{X}} > 0, \\ \mathrm{D}^b(\mathrm{coh}(\mathbb{X})) & \textit{if } \chi_{\mathbb{X}} = 0, \\ \mathrm{D}^b(\mathrm{mod}(\hat{\Lambda})) & \textit{if } \chi_{\mathbb{X}} < 0. \end{cases}$$

Invoking [10] we obtain:

Corollary 6. *Assume* $\chi_{\mathbb{X}} < 0$. *Then each component of the Auslander-Reiten quiver of* $\mathrm{D}^b(\mathrm{mod}(\hat{\Lambda}))$ *has type* $\mathbb{Z}\mathbb{A}_\infty$.

This provides the first instance of a finite dimensional algebra having an Auslander-Reiten quiver of this shape.

We briefly discuss the ingredients for the proof of Theorem 2. The encountered trichotomy is related to Orlov's theorem [13] dealing with the *triangulated category of graded singularities* $\mathrm{D}^{\mathbb{Z}}_{\mathrm{Sg}}(R)$ of a graded singularity R, defined as the quotient category of the derived category $\mathrm{D}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}R)$ of finitely generated graded modules modulo the subcategory $\mathrm{D}^b(\mathrm{proj}^{\mathbb{Z}}\text{-}R)$ of perfect complexes. The category $\mathrm{D}^{\mathbb{Z}}_{\mathrm{Sg}}(R)$ has an interesting alternative interpretation, due to Buchweitz [1] as the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ of maximal graded Cohen-Macaulay R-modules (where one factors out all projective R-modules). An analogous result holds for the $\mathrm{L}(p)$-graded case. By means of [2], [3] and [9] we obtain that $\underline{\mathrm{vect}}(\mathbb{X})$ is equivalent to $\mathrm{D}^{\mathbb{Z}}_{\mathrm{Sg}}(\mathrm{mod}(R))$ for $\chi_{\mathbb{X}} \neq 0$ and to $\mathrm{D}^{\mathrm{L}(p)}_{\mathrm{Sg}}(\mathrm{mod}(S))$ for $\chi_{\mathbb{X}} = 0$.

For $\chi_{\mathbb{X}} > 0$ the assertion of the theorem now follows from work of Kashiura, Saito and Takahashi [6]. For $\chi_{\mathbb{X}} = 0$ the result is due to Ueda [15]. For an alternative treatment in case $\chi_{\mathbb{X}} \geq 0$ see also [12, 8]. For case $\chi_{\mathbb{X}} < 0$ the result is due to work of de la Peña and the author [11, 12]. We point to [7] for a related investigation.

REFERENCES

[1] R. O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings.* Unpublished manuscript (1987) 155 pp.

[2] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras.* In *Singularities, Representations of Algebras, and Vector Bundles*, Lecture Notes in Math. **1273** (1987), Springer-Verlag, 265–297.

[3] W. Geigle and H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra **144** (1991), 273–343.

[4] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras.* London Mathematical Lecture Notes Series **119**, Cambridge University Press, Cambridge 1988.

[5] D. Happel, *A characterization of hereditary categories with tilting object*, *Invent. Math.* **144** (2001), 381–398.

[6] H. Kajiura, K. Saito and A. Takahashi, *Matrix factorizations and representations of quivers II: Type ADE case*, Adv. Math. **211** (2007), 327–362.

[7] H. Kajiura, K. Saito and A. Takahashi, *Triangulated categories of matrix factorizations for regular systems of weights with* $\epsilon = 1$, arXiv:0708.0210v1 [math.AG] (2007).

[8] D. Kussin, H. Meltzer, H. Lenzing and J.A. de la Peña, Stable categories of vector bundles, I. In preparation.

[9] H. Lenzing, *Wild canonical algebras and rings of automorphic forms*, *In Finite Dimensional Algebras and Related Topics*, NATO ASI Series C. Math. Phys. Sci. **424**, Kluwer Academic Publishers, Dordrecht, 1994, 191–212.

[10] H. Lenzing and J.A. de la Peña, *Wild canonical algebras*, Math. Z. **224** (1997), 403–425.

[11] H. Lenzing and J.A. de la Peña, *Extended canonical algebras and Fuchsian singularities*, arXiv math.RT/0611532 (2006).

[12] H. Lenzing and J.A. de la Peña, *Spectral analysis of finite dimensional algebras and singularities*, In *Trends in Representation Theory of Algebras and Related Topics*, ed. A. Skowroński, EMS Publishing House, Zürich, to appear.

[13] D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, arXiv:math.AG/0503632v2 (2005).

[14] C.M. Ringel, *Tame Algebras and Integral Quadratic Forms.* Lecture Notes in Math. **1099**, Springer, Berlin-Heidelberg-New York, 1984.

[15] K. Ueda, *Homological mirror symmetry and simple elliptic singularities*, arXiv:math.AG/0604361 (2006).

Cohomology for Quantum Groups: A Bridge between Algebra and Geometry

Daniel K. Nakano

(joint work with C. Bendel, Z. Lin, B. Parshall, C. Pillen)

Cohomology theories were developed throughout the 20th century by topologists to construct algebraic invariants for the investigation of manifolds and topological spaces. During this time, cohomology was also defined for algebraic structures like groups and Lie algebras to determine ways in which their representations can be glued together.

The purpose of this talk will be to demonstrate how cohomology theories for algebraic structures can be used to reintroduce the underlying geometry. For finite groups, these ideas started with the work of D. Quillen and J. Carlson. My talk with focus on the situation for the (small) quantum group $u_q(\mathfrak{g})$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and q is a primitive lth root of unity. For $l > h$, Ginzburg and Kumar proved that the cohomology ring identifies with the coordinate algebra of the nilpotent cone $\mathcal{N}$.

In this talk, I will present results which extend this result in two directions. The first direction encompasses the computation of the cohomology for quantum groups when $l \leq h$. This computation entails many beautiful results which include powerful vanishing results on line bundle cohomology and normality of nilpotent orbit closures. Moreover, our results show that the cohomology ring is finitely generated. This allows us to define support varieties and compute the support varieties for quantum Weyl modules in the case when (l, p)=1 where p is any bad prime for the underlying root system.

The second direction will include a discussion on how to realize rings of regular functions on nilpotent orbits and their closures via the cohomology of the (small) quantum group. These results answer an old question of Friedlander and Parshall posed in the mid 1980's. Our results have direct applications in relating classical multiplicity formulas due to McGovern and Graham to the Kazhdan-Lusztig theory for the (small) quantum groups which was established first in the 1990's by Kazhdan-Lusztig, and Kashiwara-Tanisaki, and in more recent work by Arkhipov, Bezrukavnikov and Ginzburg.

This talk represents joint work with C. Bendel, B. Parshall, C. Pillen (first part), and Z. Lin (second part).

References

[ABG] S. Arkhipov, R. Bezrukavnikov, V. Ginzburg, Quantum groups, the loop Grassmannian and the Springer resolution, *J. Amer. Math. Soc.*, 17, (2004), 595–678.

[CLNP] J. F. Carlson, Z. Lin, D. K. Nakano, B. J. Parshall, The restricted nullcone, *Cont. Math.*, 325, (2003). 51–75.

[BNPP] C.P. Bendel, D.K. Nakano, B.J. Parshall, C. Pillen, Quantum group cohomology via the geometry of the nullcone, preprint (2007).

[GK] V. Ginzburg, S. Kumar, Cohomology of quantum groups at roots of unity, *Duke Math. Journal*, 69 (1993), 179–198.

[KLT] S. Kumar, N. Lauritzen, J. Thomsen, Frobenius splitting of cotangent bundles of flag varieties, *Invent. Math.*, 136, (1999), 603-621.

[LN] Z. Lin, D.K. Nakano, Realizing rings of regular functions via the cohomology of quantum groups, in preparation.

Polynomial Invariants for Tilted Algebras and Cluster Mutations

Lutz Hille

We consider three related problems, all three lead to the notion of a polynomial invariant. We always work over an algebraically closed field for simplicity.

1) Assume A is a finite dimensional hereditary algebra with upper triangular Cartan matrix $C_A = (y_{i,j})$. Assume B is a tilted algebra for A and we consider its upper triangular Cartan matrix $C_B = (z_{i,j})$. A polynomial invariant is an element $F \in R = k[x_{i,j} \mid 1 \leq i < j \leq n]$ (the polynomial ring in variables corresponding to the non–trivial entries in an upper triangular Cartan matrix) satisfying $F(y_{i,j}) = F(z_{i,j})$ for each pair (A, B) of tilted algebras as above.

2) We consider again the Polynomial ring R as above and an action of a group Γ generated by $n-1$ elements. This group action corresponds to the exceptional mutations on the level of the Grothendieck group. We are also interested in an extension $\overline{\Gamma}$ of this group by allowing certain sign changes in R (this extended action corresponds to the shift in the derived category). The action of Γ corresponds to the braid group action on exceptional sequences (see [7], [3], and [6]).

Theorem 1. *An element F is a polynomial invariant (according to 1)), precisely if it is a Γ–invariant polynomial in R (with respect to the action defined in 2)).*

3) Assume we consider a quiver with n vertices and $q_{i,j}$ arrows (we assume Q has no loops and no 2–cycles). Then we can define the cluster mutations μ_i for $i = 1, \ldots, n$. Depending on the various orientations h of a quiver Q (we forget the number of arrows $q_{i,j}$ and consider only the underlying orientation) we can look for polynomials F^h (for different orientations h we may have different polynomials) so that $F^h(q_{i,j}) = F^{\mu_l(h)}(\mu_l(q_{i,j}))$. In a similar way as above we ask for polynomials invariant under cluster mutations.

The first problem concerns the existence of those polynomials. We first present the case with 3 vertices, it leads to the well–known Markov equation $x^2+y^2+z^2-xyz$ (in the three variables x, y, z). Moreover, we show that each other invariant F is a polynomial in this equation. Finally, we apply this result to the cluster mutations (see [1]). For $n = 4$ we can construct an invariant of degree 4 and one of degree 2. We generalize the construction of these two invariants: we can construct them in a purely combinatorial way for all n.

Theorem 2. *For arbitrary n we obtain an invariant*

$$F_1 := \sum_{r=2}^{n} \sum_{i_1<i_2<\ldots<i_r} (-1)^r x_{i_1,i_2} x_{i_2,i_3} \ldots x_{i_{r-1},i_r} x_{i_1,i_r}.$$

For n even the Pfaffian of the skew–symmetric matrix $C - C^t$

$$\sum_{I=\{\{i_1,i_2\},\{i_3,i_4\},\ldots,\{i_{n-1},i_n\}\}} (-1)^{|I|} x_{i_1,i_2} x_{i_3,i_4} \ldots x_{i_{n-1},i_n}$$

is Γ–invariant, where the sum runs through all sets I of $n/2$ sets of disjoint two–element sets. The number $|I|$ counts the number of crossings of two pairs of sets: a pair $i < j$ and $k < l$ crosses if $i < k < j < l$ or $k < i < l < j$, otherwise it does not cross.

The first invariant is, up to sign and a constant, the Euler characteristic of the Hochschild cohomology. This was proven by Happel in [4].

The main result of this talk concerns the generalization to arbitrary n. We can explicitely construct further invariants D_i for $i = 1, \ldots, \lfloor n/2 \rfloor$ using the Coxeter transformation. We can define new polynomials F_i (as a certain linear combination of the D_j for $j = 1, \ldots, i$) of degree n and minimal degree $2i$. In particular, for n even, the polynomial $F_{\lfloor n/2 \rfloor}$ is homogeneous of degree n. For the construction of the D_i we also refer to [2] and for further properties concerning the Coxeter transformation to [5].

Theorem 3. *For a given n there exist algebraically independent polynomials F_i for $i = 1, \ldots, \lfloor n/2 \rfloor$ of degree n and minimal degree $2i$ with $F_i \in R^{\Gamma}$. If n is even then $\sqrt{F_{n/2}}$ is already Γ–invariant.*

We conjecture that these invariants form a generating set of all invariants.

For $n = 4$ we obtain the following two invariants for the Γ–action

$$\sum_{1\le i<j\le 4} x_{i,j}^2 - \sum_{1\le i<j<k\le 4} x_{i,j} x_{j,k} x_{i,k} + x_{1,2} x_{2,3} x_{3,4} x_{1,4},$$

$$x_{1,2} x_{3,4} + x_{2,3} x_{1,4} - x_{1,3} x_{2,4}.$$

Finally, we use the result above to obtain polynomial invariants for cluster mutations. For $n = 4$ we can show, that polynomial invariants in the sense of 3) above can not exist. The reason is the existence of non–admissible orientations. So we can ask question 3) only for admissible orientations of a quiver Q. We explain this notion and the construction of invariants as well as some applications.

References

[1] A. Beineke, T. Brüstle, L. Hille, *Cluster-Cyclic Quivers with three Vertices and the Markov Equation*, with an appendix by Otto Kerner, arXiv:math/0612213, to appear in Algebras and Representation Theory

[2] A. Bondal, *A symplectic groupoid of triangular bilinear forms and the braid group.* (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 4, 19–74; translation in Izv. Math. 68 (2004), no. 4, 659–708

[3] W. Crawley–Boevey, *Exceptional Sequences of Representations of Quivers.* In Proceedings of ICRA VI, Carleton-Ottawa, 1992. Math. LNS Vol. 14.

[4] D. Happel, *The trace of the Coxeter matrix and Hochschild cohomology.* Linear Algebra Appl. 258 (1997), 169–177.

[5] H. Lenzing, *Coxeter transformations associated with finite-dimensional algebras.* Computational methods for representations of groups and algebras (Essen, 1997), 287–308, Progr. Math., 173, Birkhäuser, Basel, 1999.

[6] C. M. Ringel, *The braid group action on the set of exceptional sequences of a hereditary Artin algebra.* Abelian group theory and related topics (Oberwolfach, 1993), 339–352, Contemp. Math., 171, Amer. Math. Soc., Providence, RI, 1994.

[7] A. Rudakov, *Exceptional collections, mutations and helices.* Helices and vector bundles, 1–6, London Math. Soc. Lecture Note Ser., 148, Cambridge Univ. Press, Cambridge, 1990.

Convolution algebras, coherent sheaves and "embedded TQFT"

Catharina Stroppel
(joint work with Ben Webster)

The aim of this talk will be to construct finite dimensional convolution algebras using cohomology rings arising from Springer fibres. We then give a purely diagrammatical description of these algebras in terms of what we call an "embedded" 2-dimensional TQFT. Finally we connect these with algebras of extensions of certain coherent sheaves on resolutions of Springer fibres. (For details and precise references we refer to [5]).

In this talk we restrict to the very special case of 2-block Springer fibres. Let $N : \mathbb{C}^n \to \mathbb{C}^n$ be a fixed nilpotent endomorphism in Jordan Normal with two blocks of size $n-k$ and k. For simplicity we assume $n-k \geq k$. The Springer fibre $Y = Y(N)$ associated with N is the variety of all full flags $\mathcal{F}$ in $\mathbb{C}^n$ fixed under N (i.e. for any space F_i of the full flag $\mathcal{F}$, we have the property $NF_i \subset F_{i-1}$ is satisfied). The irreducible components of Y were described by Spaltenstein and Vargas who put them in natural bijection with the standard tableaux of shape $(n-k,k)$, and described the components as closures of explicitly given locally closed subspaces.

A standard tableau S of shape $(n-k,k)$ is by definition a Young diagram of shape $(n-k,k)$, filled with the numbers $\{1,2,\dots n\}$ decreasing from the left to the right and from top to bottom. We can associate a crossingless matching $\mathbf{m}(S)$ of n points with k cups and $n-k$ orphaned points, such that the bottom row of the tableau contains all the numbers $S_\vee$ which are at the left end of a cup, and the top row of the diagram contains all the numbers $S_\wedge$ which are at the right endpoint of a cup, or are orphaned. This defines a bijection between standard tableaux of shape (n-k,k) and crossingless matchings of n points with k cups and $n-2k$ orphaned points. In the special case of two blocks the result of Spaltenstein and Vargas has the following handy description due to Fung ([2]):

Proposition 1 (Fung). *A complete flag* $\{0\} = F_0 \subset \cdots \subset F_n = V$ *lies in* Y_S *if and only if the following holds: If there is a cup in* $\mathbf{m}(S)$ *connecting* i *and* $j > i$, *then* $N^{\frac{j-i+1}{2}}(F_j) = F_{i-1}$, *and if* i *is orphaned then* $F_i = N^{-b_i}(\operatorname{im} N^{t_i})$ *where* b_i *and* t_i *are uniquely defined nonnegative integers associated with* i *by some easy combinatorial rule. Moreover, any component is an iterated* $\mathbb{P}^1$*-bundle.*

Our first result is the following

Theorem 2. *Let* Y_A, Y_B *be irreducible components of* Y *and assume* $Y_A \cap Y_B$ *is non-empty. Then* $Y_A \cap Y_B$ *is an iterated* $\mathbb{P}^1$*-bundle (in particular smooth). The cohomology ring* $H^\bullet(Y_A \cap Y_B)$ *is isomorphic to* $\mathbb{C}[x]/(x^2)^{\otimes r}$, *where* r *denotes the number of closed circles in the diagram* $A\overline{B}$ *obtained by putting* $\mathbf{m}(B)$ *upside down on top of* $\mathbf{m}(A)$. *The pull back map from the cohomology of the full flag variety surjects onto* $H^\bullet(Y_A \cap Y_B)$ *such that the x's are the images of the first Chern classes of the tautological line bundles associated with the leftmost points of the circles.*

1. The convolution algebra, TQFT and embedded TQFT

Consider the space $\bigoplus H^*(Y_A \cap Y_B)$, where the sum is taken over all pairs (A, B) of standard tableaux of shape $(n-k, k)$. It has a natural *convolution product structure*: the product of two classes $\alpha \in H^\bullet(Y_A \cap Y_B)$ and $\beta \in H^\bullet(Y_B \cap Y_C)$, is given by first taking their pullbacks to $H^\bullet(Y_A \cap Y_B \cap Y_C)$, then taking their cup product and afterwards pushing forward to obtain the product $\alpha * \beta \in H^\bullet(Y_A \cap Y_C)$. Let $d(A,B) = n - c(A,B)$, where c denotes the number of circles in $A\overline{B}$ and let $H^*(Y_A \cap Y_B)\langle d(A,B)\rangle$ be the vector space $H^*(Y_A \cap Y_B)$ with its cohomological grading shifted up by $d(A,B)$.

Theorem 3. *The convolution product turns* $H^* := \bigoplus H^*(Y_A \cap Y_B)\langle d(A,B)\rangle$ *into a positively graded algebra.*

For simplicity we restrict now to the case $n = 2k$. Recall that a 2-dimensional TQFT is a monoidal functor F_R from the category $2 - Cob$ to vector spaces. It associates to a union of circles (=oriented compact 1-manifolds) a vector space R, to a pair of pants joining two circles to one circle, the multiplication $m : R \otimes R \to R$ and to the reverse cobordism, splitting one circle into two circles a comultiplication map $\Delta : R \to R \otimes R$. Functoriality means exactly that R becomes a commutative Frobenius algebra. We are interested in the easiest case where $R = \mathbb{C}[x]/(x^2)$ and $\Delta : R \to R \otimes R$, $1 \mapsto X \otimes 1 + 1 \otimes X$, $X \mapsto X \otimes X$. The space $H^*(Y_A \cap Y_B)$ can then be realized as F_R applied to the diagram $A\overline{B}$ (which is a union of circles by our assumption $n = 2k$). Under this identification the minimal cobordism from the union of $A\overline{B}$ and $B\overline{C}$ to $A\overline{C}$ gives rise to a multiplication map

$$H^*(Y_A \cap Y_B) \otimes H^*(Y_B \cap Y_C) \to H^*(Y_A \cap Y_C)$$

which can be used to turn H^* into a positively graded associative algebra with primitive idempotents naturally labelled by the irreducible components of the Springer fibre Y. This algebra is exactly Khovanov's arc algebra H_n [3] which he introduced to categorify the Jones polynomial and to obtain bigraded link and tangle invariants. The connections with Lie theory is given by the following

Theorem 4. *([6]) Let $\mathcal{O}_0^{k,k}(\mathfrak{gl_n})$ be the principal block of the parabolic BGG category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n$ with respect to the partition $n = k + k$. Then H_n is isomorphic to the endomorphism algebra of the sum of all indecomposable projective-injective modules in $\mathcal{O}_0^{k,k}(\mathfrak{gl_n})$. The endomorphism algebra of a minimal projective generator is the quasi-hereditary cover of H_n in the sense of Rouquier.*

The algebras H_n and $H^\bullet$ are related as follows:

Theorem 5. *(1) When considered with coefficients in $\mathbb{Z}/2\mathbb{Z}$, the algebras H^* and H_n are isomorphic.*

(2) When considered with coefficients in $\mathbb{C}$ (in fact, for any field of characteristic $\neq 2$), for all $n \geq 3$, there is no algebra isomorphism respecting the direct sum decomposition given by the $H^*(Y_A \cap Y_B)$'s.

However, our convolution algebra has a description using a refined version of TQFT, which should keep track of the nestedness of the circles. In particular, there will be two types of pair of pants cobordisms, namely one which connects one circle with two disjoint, *not nested* circles in the usual embedding for trousers and a second "unusual" one which connects one circle with two disjoint, but *nested* circles, with one of the trouser legs pushed down the middle of the other or rather arranged such that the orientation of the nested circle is swapped. The "unusual" maps involve twist by minus signs according to the nestedness of the circles.

Theorem 6. *The algebra H^* can be described via an "embedded TQFT".*

There is an analog of the quasi-hereditary cover of H_n for H^* constructed from stable manifolds with respect to a $\mathbb{C}^*$-action (generalising irreducible components).

2. Coherent sheaves on varieties associated with Slodowy slices

Finally we study how all this is related to the sheaf-theoretic model of Khovanov homology given by Cautis and Kamnitzer [1]. Their model associates a certain coherent sheaf $i_*\Omega(A)^{1/2}$ on a certain compact smooth variety $S_{k,k}$ related with Slodowy slices to each crossingless matching A of $2k$ points. The variety naturally contains the Springer fiber Y, and the sheaf $i_*\Omega(A)^{1/2}$ is supported on the component associated with A. As our notation suggest, these sheaves arise from square roots of canonical bundles. As a vector space, the Ext-algebra of these sheaves can be identified with our algebra $H^\bullet$ (and thus also with Khovanov's algebra):

Theorem 7. *There is an isomorphism of graded vector spaces*

$$\mathrm{Ext}^{\bullet}_{\mathrm{Coh}(S_{k,k})}(i_*\Omega(A)^{1/2}, j_*\Omega(B)^{1/2}) \cong H^{\bullet}(A\cap B)\langle d(a,b)\rangle.$$

Conjecture 8. *There is an isomorphism of algebras*

$$\bigoplus_{A,B}\mathrm{Ext}^{\bullet}_{\mathrm{Coh}(S_{k,k})}(i_*\Omega(A)^{1/2}, j_*\Omega(B)^{1/2}) \cong H^{\bullet}.$$

This would be the first step to clarify the precise relationship between the geometric models [1] (using coherent sheaves) and [4] (using symplectic geometry) and the algebro-representation theoretic versions of Khovanov homology ([3], [7]).

References

[1] S. Cautis, J. Kamnitzer, *Knot homology via derived categories of coherent sheaves I, sl(2) case.* arXiv:math/0710.3216 (2007).

[2] F. Fung, *On the topology of components of some Springer fibers and their relation to Kazhdan-Lusztig theory*, Adv. Math., 178, 2003, no. 2, 244–276.

[3] M. Khovanov, *A categorification of the Jones polynomial.* Duke Math. J. 101 (2000), no. 3, 359–426.

[4] P. Seidel, Paul; I. Smith, *A link invariant from the symplectic geometry of nilpotent slices.* Duke Math. J. 134 (2006), no. 3, 453–514.

[5] C. Stroppel, B. Webster, *2-block Springer fibers: convolution algebras, coherent sheaves, and embedded TQFT*, arXiv:math/0802.1943, (2008).

[6] C. Stroppel, *Parabolic category O, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology*, arXiv:math/0608234, (2006), to appear in Compositio Math.

[7] C. Stroppel, *Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors*, Duke Math. J., 126, 2005, no.3, 547–596.

Atiyah Classes, Ghosts and Levels of Perfection

Ragnar-Olaf Buchweitz

1. If C is a class of objects in a triangulated category $\mathcal{T}$, (with shift, suspension, or translation functor Σ) and M is any object from $\mathcal{T}$, one may ask whether and, if so, how efficiently M can be built inside $\mathcal{T}$ from objects in C through the standard operations of forming direct sums, taking direct summands, shifting objects, or completing a morphism to an exact triangle.

The objects in $\mathrm{add}_{\Sigma}(C)$, the essential closure of C under formation of (finite) direct sums, direct summands, or (de-)suspensions, form the basic "building blocks", and the *level* of M with respect to C records the minimal "cost" of building an object M out of C, where taking direct sums or summands as well as (de-)suspending are for free, but attaching an object from $\mathrm{add}_{\Sigma}(C)$ by completing a morphism to a triangle raises the cost by one unit.

With the zero objects the only ones at level zero, the objects at level at most one are precisely those in $\mathrm{add}_{\Sigma}(C)$. The level of an object M, noted $\mathsf{level}_C M$, is then *finite* if, and only if, M belongs to the *thick subcategory* spanned by C in $\mathcal{T}$.

This approach to measuring the complexity of an object M relative to C is formalized and used in [1], and we refer to it for precise definitions and an account of the history of the notion. We mention here just that it is as well a crucial ingredient in Rouquier's [8] treatment of the *dimension* of a triangulated category.

Although simple enough as a concept, the fundamental question is how to determine the level of a given object, or, less ambitiously, how to decide whether that level is finite, that is, whether the object is contained in the thick subcategory generated by C?

2. If C generates a projective class, in the sense of [2] or [5], then the answer to the above question is easier, as the level with respect to C can be read off from an *Adams resolution* (perhaps more appropriately called an *ABC resolution*, as in [7]) in form of the *ghost index* of that object.

Recall that C *generates a projective class*, with $\mathcal{P} = \operatorname{add}_\Sigma(C)$ its category of *relative projectives*, and the ideal $\mathcal{I}$, given by $\mathcal{I}(X,Y) = \{f : X \to Y \mid \forall p : P \to X; P \in \mathcal{P}; fp = 0\}$ for $X, Y \in \mathcal{T}$, its *ghost ideal*, if these data satisfy

($*$) For each $X \in \mathcal{T}$, there exists an exact triangle

$$\Omega X \xrightarrow{q} P \xrightarrow{p} X \xrightarrow{a} \Sigma\Omega X$$

with $P \in \mathcal{P}$ and $a \in \mathcal{I}$, where ΩX denotes a representative from the isomorphism class of objects that complete p to an exact triangle.

We call the morphism $a \in \operatorname{Ext}^1_{\mathcal{T}}(X, \Omega X)$ *an Atiyah class* for X relative to C. Note the following:

(a) While ΩX is generally not given functorially, the functor to abelian groups, or *right $\mathcal{P}$–module*, such a choice defines through

$$(-, \Omega X) = \operatorname{Hom}_{\mathcal{T}}(-, \Omega X)|_{\mathcal{P}} : \mathcal{P}^{\mathrm{op}} \to \mathfrak{Ab}$$

is indeed uniquely determined by p. It equals $\operatorname{Ker}(-,p) : (-,P) \to (-,X)$, due to the assumption that a is a ghost morphism. Replacing X by ΩX in ($*$), it follows that each $(-,X)$ is already a *coherent* $\mathcal{P}$–module, an object in $\mathbf{Coh}(\mathcal{P})$. Induction yields further that each such module admits a *projective resolution* by finite projectives in $\mathbf{Coh}(\mathcal{P})$.

(b) Clearly, the Atiyah class *vanishes*, $a = 0$, if, and only if, p *splits*, if, and only if, $X \in \mathcal{P}$, if, and only if, $\mathsf{level}_C X \leq 1$.

3. If C generates a projective class $(\mathcal{P}, \mathcal{I})$, one can successively construct a diagram

$$\begin{tikzcd}
X \arrow[rr, "a_1", "+1"'] & & \Omega X \arrow[rr, "a_2", "+1"'] \arrow[dl, "q_0"] & & \Omega^2 X \arrow[rr, "a_3", "+1"'] \arrow[dl, "q_1"] & & \cdots \\
& P_0 \arrow[ul, "p_0"] & & P_1 \arrow[ll, "q_0p_1"'] \arrow[ul, "p_1"] & & P_2 \arrow[ll, "q_1p_2"'] \arrow[ul, "p_2"] & & \cdots \arrow[ll]
\end{tikzcd}$$

with the triangles containing the indicated morphisms of degree $+1$ being *exact*, the triangles pointing upwards being *commutative*, and $P_i \in \mathcal{P}, a_i \in \mathcal{I}$, for each $i \geq 0$. This diagram constitutes an *Adams* (or *ABC*) *resolution* of X in $\mathcal{T}$.

Mapping this diagram into $\mathbf{Coh}(\mathcal{P})$ yields a projective resolution of $(-,X)$ by finite projectives,

$$\begin{tikzcd} 0 & (-,X) \arrow[l] & (-,P_0) \arrow[l, "(-{,}p_0)"'] & (-,P_1) \arrow[l, "(-{,}p_1)"'] & \cdots \arrow[l] \end{tikzcd}$$

and, conversely, any such projective resolution can be lifted to an Adams resolution of X in $\mathcal{T}$. Note, however, that passing to $\mathbf{Coh}(\mathcal{P})$, the Atiyah classes a_i get lost: $(-,a_i) = 0$, as each a_i is a ghost!

4. The key invariant now is the *ghost index* of X with respect to C, (or to the projective class generated by it.) Setting $a_0 = \mathrm{id}_X$, it is defined as

$$\mathrm{gin}_C X = \min\{i \geq 0 \mid a_i \cdots a_1 a_0 = 0\} \quad \in \mathbb{N} \cup \{\infty\}$$

That this notion is independent of the choice of the Adams resolution, or of the projective resolution of $(-,X)$, is the content of a main result from [5]:

5. **Theorem.** *Given an Adams resolution of X as above, denote*

$$\mathrm{at}^i_C(X) = a_i a_{i-1} \cdots a_0 \in \mathrm{Ext}^i_{\mathcal{T}}(X, \Omega^i X)$$

the i^{th} Atiyah class *of X with respect to C. The following are then equivalent for each $i \geq 0$.*

(1) $\mathrm{at}^i_C(X) = 0$
(2) $\mathrm{gin}_C(X) \leq i$
(3) $\mathsf{level}_C X (= \mathsf{level}_{\mathcal{P}} X) \leq i$

If these equivalent conditions are satisfied for some $i > 0$, then already

$$\mathsf{level}_{\{P_0,\ldots,P_{i-1}\}} X \leq i$$

for any sequence of objects $P_0, ..., P_{i-1}$ occurring in the corresponding initial segment of some Adams resolution of X.

Note that

(a) $\mathsf{level}_{\mathcal{P}} X \leq \mathrm{projdim}_{\mathbf{Coh}(\mathcal{P})}(-,X)$, but the inequality is usually strict.
(b) With (3), as well the other two assertions are independent of the choice of an Adams resolution of X.

6. To apply these results, we need suitable projective classes. The simplest class of examples arises from pairs of adjoint functors. If $F^* : \mathcal{S} \to \mathcal{T}, F_* : \mathcal{T} \to \mathcal{S}$ is a pair of exact adjoint functors, with F^* left adjoint to F_*, then $C = F^*(\mathcal{S})$ generates a projective class in $\mathcal{T}$, its ghost ideal consisting of all morphisms f such that $F_*(f) = 0$. Indeed, for any $X \in \mathcal{T}$ the co-unit of the adjunction $p : F^*F_*X \to X$ will complete to an exact triangle as required. For a typical application, let A be a noetherian ring, $X \in D(A)$ a complex in the (full) derived category of (right) A–modules, such that its (total) homology $H(X)$ is a finitely generated (graded) A–module. The complex X is then *perfect* if $\mathsf{level}_A X < \infty$, and one has the following test for perfection.

7. **Theorem.** *With A and X as just described, let $\rho : B \to A$ be a ring homomorphism and $\rho^* = ? \otimes^{\mathbb{L}}_B A : D(B) \to D(A)$ the left adjoint to the restriction of scalars ρ_*. Let $\operatorname{at}^i_{A/B}(X)$ denote the relative Atiyah classes associated to the class $\rho^*(D(B))$. If* $\operatorname{gldim} B < \infty$*, then*

$$\mathsf{level}_{\rho^*(D(B))}(X) < \infty \iff X \textit{ is perfect} \iff \operatorname{at}^i_{B/A}(X) = 0 \textit{ for } i \gg 0.$$

8. If we pass to DG algebras, and replace A by a DG model $\mathcal{A}$ over B, noting that $D(\mathcal{A}) \simeq D(A)$ as triangulated categories, then we can realise Ω as an endofunctor on $D(\mathcal{A})$ to get an exact triangle of exact functors

$$\Omega \xrightarrow{q} \rho^*\rho_* \xrightarrow{p} \operatorname{id}_{D(\mathcal{A})} \xrightarrow{\operatorname{at}_{\mathcal{A}/B}} \Sigma\Omega$$

The group $\operatorname{HH}^n_{A/B} = \operatorname{Hom}(\Omega^n, \operatorname{id}_{D(\mathcal{A})})/(q\Omega^{n-1})^* \operatorname{Hom}(\rho^*\rho_*\Omega^{n-1}, \operatorname{id}_{D(\mathcal{A})})$ serves as the n^{th} *Hochschild cohomology* of the adjoint pair (ρ^*, ρ_*) and composition with the corresponding powers of the functorial relative Atiyah classes defines a canonical homomorphism of graded commutative rings to the graded centre of $D(A)$, the *relative Hochschild-Chern character*,

$$\operatorname{ch}^\bullet_{A/B} : \operatorname{HH}^\bullet_{A/B} \to \mathcal{Z}^\bullet(D(A))$$

Following this ring homomorphism with evaluation in some object $X \in D(A) \simeq D(\mathcal{A})$ shows that the components of the Hochschild-Chern character yield lower bounds for levels (of perfection, if $\operatorname{gldim} B < \infty$ and $H(X)$ is finitely generated over the noetherian ring A),

$$\mathsf{level}_{\rho^*(D(B))}(X) \geq \min\{n \geq 0 \mid \operatorname{ch}^n_{A/B}(X) = 0\}$$

As in the classical case, the theory of Atiyah classes here can be made explicit through differential forms and connections in a way entirely analogous to our joint work with Flenner [3],[4], in view of the treatment of Hochschild cohomology in [6].

References

[1] L. L. Avramov, R. -O. Buchweitz, S. B. Iyengar and C. Miller, *Homology of perfect complexes*, preprint, http://www.citebase.org/abstract?id=oai:arXiv.org:math/0609008, 40pp., 2006

[2] A. Beligiannis, *Relative homological algebra and purity in triangulated categories.* J. Algebra 227 (2000), no. 1, 268–361.

[3] R.-O. Buchweitz and H. Flenner, *Global Hochschild (co-)homology of singular spaces.* Adv. Math. 217 (2008), no. 1, 205–242.

[4] R.-O. Buchweitz and H. Flenner, *The global decomposition theorem for Hochschild (co-)homology of singular spaces via the Atiyah-Chern character.* Adv. Math. 217 (2008), no. 1, 243–281.

[5] J. .D. Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta.* Adv. Math. 136 (1998), no. 2, 284–339.

[6] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity.* J. Amer. Math. Soc. 8 (1995), no. 2, 251–289.

[7] R. Meyer, *Homological algebra in bivariant K-theory and other triangulated categories. II*, preprint http://www.citebase.org/abstract?id=oai:arXiv.org:0801.1344, 32pp., 2008

[8] R. Rouquier, *Derived equivalences and finite dimensional algebras.* International Congress of Mathematicians. Vol. II, 191–221, Eur. Math. Soc., Zürich, 2006.

Dimensions of triangulated categories via Koszul objects

Srikanth B. Iyengar

My aim in this talk was to describe how some simple, and not so simple, techniques from commutative algebra can be used to obtain lower bounds for dimensions of triangulated categories.

Consider a triangulated category T. Given an object G in T, the thick subcategory, $\operatorname{thick}_{\mathsf{T}}(G)$, it generates admits a natural filtration

$$\{0\} = \operatorname{thick}^0_{\mathsf{T}}(G) \subseteq \operatorname{thick}^1_{\mathsf{T}}(G) \subseteq \cdots \subseteq \bigcup_{n\geqslant 0} \operatorname{thick}^n_{\mathsf{T}}(G) = \operatorname{thick}_{\mathsf{T}}(G)$$

where $\operatorname{thick}^1_{\mathsf{T}}(G)$ consists of retracts of finite direct sums of suspensions of G, and $\operatorname{thick}^n_{\mathsf{T}}(G)$ consists of retracts of n-fold extensions of $\operatorname{thick}^1_{\mathsf{T}}(G)$.

The *dimension* of T is then the number

$$\dim \mathsf{T} = \inf\{n \mid \text{there exists a } G \in \mathsf{T} \text{ with } \operatorname{thick}^{n+1}_{\mathsf{T}}(G) = \mathsf{T}\}.$$

This number was introduced by Bondal and Van Den Bergh [6]. Rouquier [8, 9] used this invariant to calculate the representation dimension of exterior algebras; this was discussed by Ringel in his lecture at this meeting. Buchweitz in his lecture discussed methods to obtain *upper bounds* on $\dim \mathsf{T}$, at least when $\operatorname{thick}^1_{\mathsf{T}}(G)$ is a projective class, in the sense of Christensen [7].

Most *lower bounds* on $\dim \mathsf{T}$ obtained thus far have concerned the case where T is the derived category, or the stable derived category, of some ring. Moreover, the arguments typically involve some commutative ring lurking in the background. One way to formalize this situation is to consider a triangulated category with an action of a commutative noetherian ring, as follows.

Let $R = \bigoplus_{i\geqslant 0} R^i$ be a graded-commutative ring where the ring R^0 artinian; the ring R need not be noetherian. As usual, set

$$\operatorname{Proj} R = \{\mathfrak{p} \mid \ \mathfrak{p} \text{ a homogenous prime ideal in } R \text{ with } \mathfrak{p} \not\supseteq R^+\}$$

For any graded R-module M, we set $\operatorname{Supp}^+_R M = \{\mathfrak{p} \in \operatorname{Proj} R \mid M_{\mathfrak{p}} \neq 0\}$, and let

$$\dim \operatorname{Supp}^+_R M = \sup \left\{ d \;\middle|\; \begin{array}{c} \text{there exists a chain of prime ideals} \\ \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d \text{ in } \operatorname{Supp}^+_R M \end{array} \right\}$$

We say M is *eventually noetherian* if the R-module $M^{\geqslant n}$ is noetherian for some integer n. In this case $\operatorname{Supp}^+_R M$ is a closed subset of $\operatorname{Proj} R$, in the Zariski topology.

The category said to be T is *R-linear* if there are homomorphisms of rings

$$R \to \operatorname{End}^*_{\mathsf{T}}(X) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(X, \Sigma^n X)$$

for each X in T, such that the R-module structures on $\operatorname{Hom}^*_{\mathsf{T}}(X, Y)$ induced via $\operatorname{End}^*_{\mathsf{T}}(X)$ and $\operatorname{End}^*_{\mathsf{T}}(Y)$ coincide, up to the usual sign-rule.

Henning Krause and I recently proved:

Theorem 1. *Let* T *be an* R*-linear triangulated category. If* $\operatorname{thick}_{\mathsf{T}}(G) = \mathsf{T}$ *and the* R*-module* $\operatorname{End}^*_{\mathsf{T}}(G)$ *is eventually noetherian, then one has an inequality:*

$$\dim \mathsf{T} \geq \dim \operatorname{Supp}^+_R \operatorname{End}^*_{\mathsf{T}}(G)\,.$$

In my talk I explained how this specializes to a recent result of Bergh and Oppermann, who proved it under rather more restrictive hypotheses on T. As pointed out by them, it yields lower bounds on the dimension of the stable derived category, and hence on the representation dimension, of certain classes of Artin algebras. The four of us are preparing a joint article [5] containing these results.

I gave a fairly complete proof of Theorem 1 in the lecture. A crucial idea is the systematic use of properties of Koszul objects in T, which are analogues of Koszul complexes in commutative algebra. This builds on the work in [2], where they were used to realize objects with prescribed cohomological varieties. Another important tool is a variation, due to Bergh [4], of the 'Ghost Lemma', which has appeared in the work of many authors; see [5] for references.

The rest of the talk was focussed on the derived category of a (commutative, noetherian) local ring R, with maximal ideal $\mathfrak{m}$. When R is a *complete intersection* which is complete in the $\mathfrak{m}$-adic topology, Theorem 1 implies that dimension of the stable derived category, $\underline{\mathbf{D}}(R)$, of R, is at least $\operatorname{codim} R - 1$.

Such a bound holds for all complete intersection local rings. This follows from the next result, proved in joint work with Avramov [3]. Here $\operatorname{cf-rank} R$ is the *conormal free rank* of R; when R has a conormal module, for example, when it is finitely generated over a field, then it is the rank of the largest free summand of the conormal module of R; see [1, Appendix A] for details.

Theorem 2. *An inequality* $\dim \underline{\mathbf{D}}(R) \geq \operatorname{cf-rank} R - 1$ *holds for each local ring* R.

The proof of this result uses [1, (5.1)], which is a vast generalization of the New Intersection Theorem for rings containing fields, and [1, (7.4)], which is a Differential Graded Algebra analogue of the Bernstein-Gelfand-Gelfand correspondence.

REFERENCES

1. L. L. Avramov, R.-O. Buchweitz, S. B. Iyengar, and C. Miller: Homology of perfect complexes. preprint 2007; `arxiv:math/0609008`.
2. L. L. Avramov and S. B. Iyengar: Modules with prescribed cohomological support. Ill. Jour. Math. **51** (2007), 1–20.
3. L. L. Avramov and S. B. Iyengar: The dimension of the stable derived category of a local ring; in preparation.
4. P. Bergh: Representation dimension and finitely generated cohomology. preprint 2007.
5. P. Bergh, S. B. Iyengar, H. Krause, and S. Oppermann: Dimension of triangulated categories via Koszul objects; in preparation.
6. A. Bondal and M. Van den Bergh: Generators and representability of functors in commutative and noncommutative geometry. Moscow Math. J. **3** (2003), 1-36.
7. D. J. Christensen: Ideals in triangulated categories: phantoms, ghosts and skeleta. Adv. Math. **136** (1998), 284–339.
8. R. Rouquier: Representation dimension of exterior algebras. Invent. Math. **165** (2006), 357–367.

9. R. Rouquier: Dimensions of triangulated categories. preprint 2005; `arXiv:math/0310134`.

Hochschild Cohomology and models of triangulated categories

Teimuraz Pirashvili

Our work was inspired by the work of Fernando Muro and his coauthors [4], [5] where they find interesting examples of triangulated categories without "models". Though they did not gave a rigorous definition of what it means a triangulated category to have a model. We give the definition of a Gabriel-Zisman model of a triangulated category and we give a cohomological characterization of a triangulated categories having such a model.

The canonical class of a triangulated category. Let $\mathbb{T}$ be a triangulated category. We do not assume the octahedron axiom to hold in $\mathbb{T}$. We consider the category $\mathsf{Triangles}(\mathbb{T})$ of distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{u_f} C_f \xrightarrow{v_f} A[1] .$$

while morphisms are commutative diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "u_f"] \arrow[d, "b"] & C_f \arrow[r, "v_f"] \arrow[d, "c"] & A[1] \arrow[d, "a[1]"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "u_{f'}"] & C_{f'} \arrow[r, "v_{f'}"] & A'[1]. \end{tikzcd}$$

We consider the ideal Θ of $\mathsf{Triangles}(\mathbb{T})$ consisting with maps (a, b, c) such that $a = 0 = b$. One easily sees that $\Theta^2 = 0$ and the quotient category $\mathsf{Triangles}(\mathbb{T})/\Theta$ is equivalent to the category $\mathbb{T}^{[1]}$. It follows that Θ can be considered as a bifunctor $\Theta : (\mathbb{T}^{[1]})^{\mathrm{op}} \times \mathbb{T}^{[1]} \to \mathsf{Ab}$ (in fact as a τ-bifunctor) and the extension

$$0 \to \Theta \to \mathsf{Triangles}(\mathbb{T}) \xrightarrow{\pi} \mathbb{T}^{[1]} \to 0$$

defines an element $\vartheta \in \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Theta)$ and therefore the triangulated category structure on the category $\mathbb{T}$ is completely determined by a bifunctor Θ and the corresponding class ϑ. Here $\mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Theta)$ is a variant of Hochschild cohomology of additive categories equipped with auto-equivalences constructed in [2].

Toda bifunctor and natural transformations β and θ. Let $\mathbb{T}$ be a triangulated category. Let $\mathbb{T}^{[1]}$ be the category of arrows of $\mathbb{T}$. For morphisms $f : A \to B$ and $f' : A' \to B'$ we consider the homomorphism of abelian groups $\phi_{f,f'} : \mathsf{Hom}_{\mathbb{A}}(A[1], A') \oplus \mathsf{Hom}_{\mathbb{A}}(B[1], B') \to \mathsf{Hom}_{\mathbb{A}}(A[1], B')$ given by $\phi_{f,f'}(g, h) = f'_*(g) - (f[1])^*(h) = f' \circ g - h \circ (f[1])$. Here $g : A[1] \to A'$ and $h : B[1] \to B'$ are morphisms of $\mathbb{T}$. The *Toda bifunctor* Δ is a bifunctor $\Delta : (\mathbb{T}^{[1]})^{\mathrm{op}} \times \mathbb{T}^{[1]} \to \mathsf{Ab}$ given by $\Delta(f, f') := \mathsf{Coker}(\phi_{f,f'})$, where $f : A \to B$ and $f' : A' \to B'$ are morphisms in $\mathbb{A}$. According to Baues [1] there is a natural homomorphism:

$$\beta : \mathsf{HML}^3_\Sigma(\mathbb{A}, \mathsf{Hom}^{10}) \to \mathsf{HML}^2_\Sigma(\mathbb{A}^{[1]}, \Delta),$$

where $\mathsf{Hom}^{10} : \mathbb{A}^{op} \times \mathbb{A} \to \mathsf{Ab}$ is a bifunctor given by $\mathsf{Hom}^{10}(X, Y) = \mathsf{Hom}(\Sigma X, Y)$. We now define the transformation

$$\theta : \varDelta_{\mathbb{T}} \to \varTheta_{\mathbb{T}}$$

as follows. Let $f : A \to B$ and $f' : A' \to B'$ be morphisms in $\mathbb{T}$. For any morphism $x : A[1] \to B'$ we have the following morphism of distinguished triangles:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "0"] & B \arrow[r, "u_f"] \arrow[d, "0"] & C_f \arrow[r, "v_f"] \arrow[d, "c_x"] & A[1] \arrow[d, "0"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "u_{f'}"] & C_{f'} \arrow[r, "v_{f'}"] & A'[1],
\end{tikzcd}$$

where $c_x = u_{f'} x v_f$. One easily sees that the assignment $x \mapsto (0, 0, c_x)$ yields the homomorphism $\theta(f, f') : \varDelta(f, f') \to \varTheta(f, f')$, hence a natural transformation $\theta : \varDelta \to \varTheta$.

Lemma 1. *The maps $\theta(f, f')$ is an isomorphism if f or f' is split.*

Gabriel-Zisman category. Let $F : \mathbf{G} \to \mathbf{H}$ be a morphism of small groupoids. Then F is called *connective* provided F is full and for any object $u \in \mathbf{H}$ there exist an object $x \in \mathbf{G}$ and an isomorphism $f : F(x) \to u$.

Let $F : (\mathbf{G}, x_0) \to (\mathbf{H}, y_0)$ be a morphism of pointed groupoids. We define the *homotopy fiber* of F to be the groupoid $\Gamma(F, y_0)$ (or simply $\Gamma(F)$). The objects of $\Gamma(F)$ are pairs (x, g), where x is an object of $\mathbf{G}$ and $g : y_0 \to F(x)$ is a morphism of $\mathbf{H}$; a morphism from (x, g) to (x', g') is a morphism $f : x \to x'$ in $\mathbf{G}$ such that $g' = F(f)g$.

A *track category* is a category enriched in the category of small groupoids. In other words a track category $\mathbb{C}$ consists of a class of objects $Ob(\mathbb{C})$, a collection of small groupoids $\mathbb{C}(A, B)$ for $A, B \in Ob(\mathbb{C})$ called *hom-groupoids* of $\mathbb{C}$, identities $1_A \in \mathbb{C}(A, A)_0$ and composition functors $\mathbb{C}(B, C) \times \mathbb{C}(A, B) \to \mathbb{C}(A, C)$ satisfying the usual equations of associativity and identity morphisms. Objects of the groupoid $\mathbb{C}(B, C)$ are called morphisms of $\mathbb{C}$, while morphisms of the groupoid $\mathbb{C}(B, C)$ are called tracks. Two morphisms f and g are *homotopic* provided there exists a track $\alpha : f \Rightarrow g$. In this case we write $f \sim g$. We let $\mathbf{Ho}(\mathbb{C})$ be the category whose objects are $Ob(\mathbb{C})$ while morphisms are homotopy classes of morphisms of $\mathbb{C}$. For a map f we let $[f]$ be denote the homotopy class of f.

We will say that a track category has finite coproducts if for any objects A and B there exists an object $A \vee B$ and morphisms $A \to A \vee B$ and $B \to A \vee B$ such that the induced functor

$$\mathbb{C}(A \vee B, X) \to \mathbb{C}(A, X) \times \mathbb{C}(B, X)$$

is an equivalence of categories.

We will say that a track category has a zero object if it posses an object 0 such that the groupoids $\mathbb{C}(0, X)$ and $\mathbb{C}(X, 0)$ are equivalent to the category with one object and one arrow.

A *Gabriel-Zisman category* is a track category $\mathbb{C}$ with zero, such that for any arrow $f : A \to B$ there is an object C_f, an arrow $q_f : B \to C$ and a track $\alpha_f : 0 \to fq$ such that the induced functor from the category $\mathbb{C}(C, X)$ to the homotopy fiber of the functor $\mathbb{C}(f, X) : \mathbb{C}(A, X) \to \mathbb{C}(B, X))$:

$$\mathbb{C}(C, X) \to \Gamma(\mathbb{C}(f, X))$$

is connective. Then C_f is defined uniquely up to an isomorphism in the quotient category $\mathbf{Ho}(\mathbb{C})$. In particular if one defines ΣX to be C_f for $f : X \to 0$, then $\Sigma : \mathbf{Ho}(\mathbb{C}) \to \mathbf{Ho}(\mathbb{C})$ is a functor.

A *stable Gabriel-Zisman category* is a Gabriel-Zisman category $\mathbb{C}$ with finite coproduct, such that $\Sigma : \mathbf{Ho}(\mathbb{C}) \to \mathbf{Ho}(\mathbb{C})$ is an equivalence of categories.

Proposition 2. *Let $\mathbb{C}$ be a stable Gabriel-Zisman category, then the category $\mathbf{Ho}(\mathbb{C})$ has a triangulated category structure such that the distinguished triangles are*

$$A \to B \to C_f \to \Sigma X.$$

Definition 3. *We will say that a triangulated category $\mathbb{T}$ has a* Gabriel-Zisman model*, if there exist a Gabriel-Zisman category $\mathbb{C}$ and a triangulate equivalence $\mathbf{Ho}(\mathbb{C}) \to \mathbb{T}$.*

The main result. We finally are in the position to announce our main result.

Theorem 4. *A small triangulated category $\mathbb{T}$ has a Gabriel-Zisman model iff the class $\vartheta \in \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Theta)$ lies in the image of the homomorphism*

$$\vartheta_* \circ \beta : \mathsf{HML}^3_\Sigma(\mathbb{T}, \mathsf{Hom}^{10}) \to \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Theta)$$

where $\theta_* : \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Delta) \to \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Theta)$ is the homomorphism induced by the natural transformation $\theta : \Delta \to \Theta$.

One can use this result to prove that the examples of constructed in [4], [5] does not have a Gabriel-Zisman models. In fact for such examples the class ϑ does not lies even in the image of the homomorphism $\theta_* : \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Delta) \to \mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \Theta)$ because the image of ϑ in the group $\mathsf{HML}^2_\Sigma(\mathbb{T}^{[1]}, \mathsf{Coker}(\theta))$ is nonzero.

REFERENCES

[1] H.-J. Baues. On the cohomology of categories, universal Toda brackets and homotopy pairs. K-Theory 11 (1997), no. 3, 259–285.

[2] H.-J. Baues. F.Muro. The homotopy category of pseudofunctors and translation cohomology. J. Pure Appl. Algebra 211 (2007), no. 3, 821–850.

[3] P. Gabriel and M. Zisman. Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35 Springer-Verlag New York, Inc., New York 1967 x+168 pp.

[4] F. Muro. A triangulated category without models. **arXiv: math.KT/0703311**.

[5] F. Muro, S. Schwede and N. Strickland. Triangulated categories without models. Invent. Math. 170 (2007), no. 2, 231–241.

Classification of abelian 1-Calabi-Yau categories

Adam-Christiaan van Roosmalen

This talk is based on a recent classification of abelian 1-Calabi-Yau categories ([6]). This classification works up to derived equivalence, but since it is easy to give all derived equivalent abelian categories, we will end with a short discussion of these abelian categories.

1. Definitions and main result

We start with some definitions. Throughout, let k be an algebraically closed field and let $\mathcal{A}$ be a k-linear abelian category.

- We say $\mathcal{A}$ is Ext-finite if $\dim_k \mathsf{Ext}^i(X,Y) < \infty$, for all $X, Y \in \mathsf{Ob}\,\mathcal{A}$, and for all $i \geq 0$.
- An Ext-finite abelian category $\mathcal{A}$ is n-Calabi-Yau if for all $X, Y \in \mathsf{Ob}\,\mathsf{D}^b\,\mathcal{A}$ there are isomorphisms
$$\mathsf{Hom}_{\mathrm{D}^b\,\mathcal{A}}(X,Y) \cong \mathsf{Hom}_{\mathrm{D}^b\,\mathcal{A}}(Y, X[n])^*$$
natural in X and Y, where $(-)^*$ is the vector space dual.

Due to results in [5], the category $\mathcal{A}$ is 1-Calabi-Yau if and only if $\mathsf{D}^b(A)$ has Auslander-Reiten triangles and the translation τ is naturally isomorphic to the identity functor. One may show that in this case, all components of the Auslander-Reiten quiver are standard homogeneous tubes, as in Figure 1.

As a first property, we wish to state following well-known result.

Proposition 1.1. *An abelian n-Calabi-Yau category $\mathcal{A}$ has global dimension n.*

In particular, all abelian 1-Calabi-Yau categories are hereditary. We now come to the main result.

Theorem 1.2. *Let $\mathcal{A}$ be an abelian 1-Calabi-Yau category, then $\mathcal{A}$ is derived equivalent to either*

(i) the category $\mathsf{mod}^{\mathsf{fd}}\, k[[t]]$ of finite dimensional representations of $k[[t]]$, or
(ii) the category $\mathsf{coh}\, X$ of coherent sheaves over an elliptic curve X.

2. Derived equivalences

In order to describe the different categories derived equivalent to $\mathsf{coh}\, X$, we will give a short review of the stability theory of elliptic curves, and torsion theories.

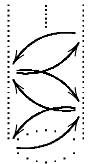


Figure 1. A homogeneous tube.

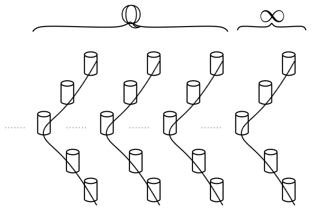


FIGURE 2. Auslander-Reiten quiver of $\mathsf{coh}\, X$

Stability theory on elliptic curves. Let X be an elliptic curve. We denote by $\mathcal{O}$ the structure sheaf of X and by $k(P)$ the skyscraper sheaf associated to a point $P \in X$. The rank, degree, and slope of a coherent sheaf $\mathcal{E}$ are defined as

$$\begin{aligned} \deg \mathcal{E} &= \chi(\mathcal{O}, \mathcal{E}), \\ \operatorname{rk} \mathcal{E} &= \chi(\mathcal{E}, k(P)), \\ \mu(\mathcal{E}) &= \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}, \end{aligned}$$

respectively, where $\chi(\mathcal{E}, \mathcal{F}) = \dim \mathsf{Hom}(\mathcal{E}, \mathcal{F}) - \dim \mathsf{Hom}(\mathcal{E}, \mathcal{F})$ is the Euler form. One may show that $\mu(\mathcal{E}) \in \mathbb{Q} \cup \{\infty\}$.

A coherent sheaf $\mathcal{F}$ is called stable or semi-stable if for every nontrivial subobject $\mathcal{E} \subset \mathcal{F}$, we have $\mu(\mathcal{E}) < \mu(\mathcal{F})$ or $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$, respectively.

It is well-known that, for an elliptic curve, all indecomposables are semi-stable. The stable sheafs are exactly those sheaves $\mathcal{E}$ with $\operatorname{Hom}(\mathcal{E}, \mathcal{E}) \cong k$.

Every Auslander-Reiten component is a homogeneous tube (Figure 1), and the peripheral objects of these tubes correspond to the stable objects of $\mathsf{coh}\, X$.

The full subcategory $\mathcal{A}_\theta$ generated by the semi-stable objects of a given slope θ is an abelian subcategory of $\mathsf{coh}\, X$. Furthermore, for any two slopes, the corresponding subcategories are equivalent. The category $\mathcal{A}_\theta$ is a direct sum of homogeneous tubes; there are no nonzero maps between two element lying in different tubes of $\mathcal{A}_\theta$.

For any two indecomposable (and hence semi-stable) objects $\mathcal{E}, \mathcal{F} \in \mathsf{coh}\, X$ with $\mu(\mathcal{E}) < \mu(\mathcal{F})$, we have $\mathsf{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$, $\mathsf{Ext}(\mathcal{F}, \mathcal{E}) \neq 0$, and $\mathsf{Hom}(\mathcal{F}, \mathcal{E}) = \mathsf{Ext}(\mathcal{E}, \mathcal{F}) = 0$.

The category $\mathcal{A}_\infty$ may be described as follows: for every point $P \in X$ there is a skyscraper sheaf $k(P)$ lying in a homogeneous tube in $\mathcal{A}_\infty$, and every tube in $\mathcal{A}_\infty$ is obtained in this way. The tubes of $\mathcal{A}_\infty$ are thus parametrized by the points of X. Since for every $\theta \in \mathbb{Q} \cup \{\infty\}$, the categories $\mathcal{A}_\theta$ and $\mathcal{A}_\infty$ are equivalent, we may sketch the Auslander-Reiten quiver as in Figure 2.

Torsion theories. We will recall some definitions and results about torsion theories from [2]. Let $\mathcal{A}$ be any hereditary abelian category. A *torsion theory* $(\mathcal{F}, \mathcal{T})$ on $\mathcal{A}$ is a pair of full additive subcategories of $\mathcal{A}$, such that $\mathsf{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and that for every $X \in \mathsf{Ob}\, \mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $F \in \mathcal{F}$ and $T \in \mathcal{T}$. We will say the torsion theory $(\mathcal{F}, \mathcal{T})$ is *split* if $\mathsf{Ext}(\mathcal{F}, \mathcal{T}) = 0$. In case of a split torsion theory we obtain, by *tilting*, a hereditary category $\mathcal{H}$ derived equivalent to $\mathcal{A}$ with an induced split torsion theory $(\mathcal{T}, \mathcal{F}[1])$. The category $\mathcal{H}$ will only be hereditary if and only if $(\mathcal{F}, \mathcal{T})$ is a split torsion theory.

Abelian 1-Calabi-Yau categories. The Auslander-Reiten quiver of $\mathsf{mod}^{\mathsf{fd}}\, k[[t]]$ consists of just one homogeneous tube. It is easily verified that all categories derived equivalent to $\mathsf{mod}^{\mathsf{fd}}\, k[[t]]$ are, in fact, equivalent to $\mathsf{mod}^{\mathsf{fd}}\, k[[t]]$.

Hence, we need only to discuss all possible torsion theories when $\mathcal{A}$ is equivalent to $\mathsf{coh}\, X$. Note that, since every category $\mathcal{H}$ derived equivalent to $\mathcal{A}$ will be 1-Calabi-Yau and hence hereditary, all torsion theories on $\mathcal{A}$ will be split.

Let $(\mathcal{F}, \mathcal{T})$ be a torsion theory on $\mathcal{A}$, and let $\mathcal{E}$ be an indecomposable of $\mathcal{T}$. Then every indecomposable $\mathcal{F}$ with $\mu(\mathcal{E}) < \mu(\mathcal{F})$ has to be in $\mathcal{T}$ since $\mathsf{Hom}(\mathcal{E}, \mathcal{F}) \neq 0$.

We may now give a characterization of all possible torsion theories.

Theorem 2.1. [1] *Let X be an elliptic curve. Every category $\mathcal{H}$ derived equivalent to $\mathcal{A} = \mathsf{coh}\, X$ may be obtained by tilting with respect to a torsion theory. All torsion theories on $\mathsf{coh}\, X$ are split and may be described as follows. Let $\theta \in \mathbb{R} \cup \{\infty\}$. Denote by $\mathcal{A}_{>\theta}$ and $\mathcal{A}_{\geq\theta}$ the subcategory of $\mathcal{A}$ generated by all indecomposables $\mathcal{E}$ with $\mu(\mathcal{E}) > \theta$ and $\mu(\mathcal{E}) \geq \theta$, respectively. All full subcategories $\mathcal{T}$ of $\mathcal{A}$ generated by tubes, and with $\mathcal{A}_{\geq\theta} \subseteq \mathcal{T} \subseteq \mathcal{A}_{>\theta} \subseteq \mathcal{A}$ give rise to a torsion theory $(\mathcal{F}, \mathcal{T})$, with $\mathsf{ind}\, \mathcal{F} = \mathsf{ind}\, \mathcal{A} \setminus \mathsf{ind}\, \mathcal{T}$, and all torsion theories are obtained in this way.*

We give some examples of torsion theories. Let X be an elliptic curve, and $\mathcal{A} = \mathsf{coh}\, X$. In here $\mathcal{H}$ always stands for the category obtained from $\mathcal{A}$ by tilting with respect to the described torsion theory.

(i) If $\theta \in \mathbb{Q} \cup \{\infty\}$ and $\mathcal{T} = \mathcal{A}_{>\theta}$, then the tilted category $\mathcal{H}$ is equivalent to $\mathsf{coh}\, X$. Indeed, it follows from the proof of Theorem 1.2 that $\mathcal{H} \cong \mathsf{coh}\, E$ for an elliptic curve E. Results from [3] then yield $E \cong X$.

(ii) If $\mathcal{T} = \mathcal{A}_{\geq\theta}$, then $\mathcal{H}$ is dual to $\mathcal{A}$. This follows from Grothendieck duality.

(iii) If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathcal{T} = \mathcal{A}_{>\theta} = \mathcal{A}_{\geq\theta}$ then $\mathcal{H}$ is equivalent to the category of holomorphic bundles on a noncommutative two-torus ([4]).

REFERENCES

[1] Igor Burban and Bernd Kreußler, *Derived categories of irreducible projective curves of arithmetic genus one*, Compos. Math. **142** (2006), no. 5, 1231–1262.

[2] Dieter Happel, Idun Reiten, and Sverre O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575, viii+ 88.

[3] Lutz Hille and Michel Van den Bergh, *Fourier-Mukai transforms*, Contribution to the Handbook of Tilting Theory, Cambridge University Press (2007), LMS Lecture Notes Series 332.

[4] Alexander Polishchuk, *Classification of holomorphic vector bundles on noncommutative two-tori*, Doc. Math. **9** (2004), 163–181.

[5] Idun Reiten and Michel Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[6] Adam-Christiaan van Roosmalen, *Abelian 1-Calabi-Yau categories*, Int. Math. Res. Not. IMRN (2008) Vol. 2008 : article ID rnn003, 20 pages, doi:10.1093/imrn/rnn003.

Exceptional sequences and posets of tilting modules

Fréderic Chapoton

Let Q be a finite quiver without oriented cycles. Let k be a ground field and $\operatorname{mod} kQ$ be the category of finite dimensional modules over the path algebra kQ of the quiver Q. Let $\operatorname{Tilt} Q$ be the set of isomorphism classes of tilting modules in the category $\operatorname{mod} kQ$.

Then there is a natural partial order on the set $\operatorname{Tilt} Q$ (due to Riedtmann-Schofield and Happel-Unger [RS91, HU05]). This partial order is defined as follows: $T \leq T'$ if and only if $T^{\perp} \supseteq T'^{\perp}$, where $T^{\perp}$ is the perpendicular subcategory $\{M \in \operatorname{mod} kQ \mid Ext^1(T, M) = 0\}$.

Let $\operatorname{mod} k \operatorname{Tilt} Q$ be the category of finite dimensional modules over the incidence algebra of the poset $\operatorname{Tilt} Q$. This can also be thought of as the category of representations of the Hasse diagram of $\operatorname{Tilt} Q$, seen as a quiver with all possible commuting relations.

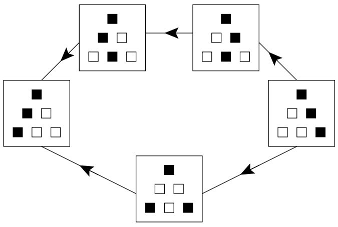


Figure 1. Hasse diagram of $\operatorname{Tilt} Q$ where $Q = 1 \to 2 \to 3$.

One has the following result, due to Ladkani [Lad07].

Theorem *Assume that the vertex i is a source in Q and that $Q' = \mu_i(Q)$ is the quiver obtained from Q by reversing all arrows incident to i. Then the bounded derived categories $D^b \operatorname{mod} k \operatorname{Tilt} Q$ and $D^b \operatorname{mod} k \operatorname{Tilt} Q'$ are triangle equivalent.*

This theorem is very similar to the classical statement due to Bernstein, Gelfand and Ponomarev [BGP73], which says that, in the same situation, the bounded derived categories $D^b \operatorname{mod} kQ$ and $D^b \operatorname{mod} kQ'$ are triangle equivalent, the equivalence being given by a reflection functor.

Let Q_0 be the set of vertices of Q. Let K be the field $\mathbb{Q}((u_i)_{i \in Q_0})$ of rational functions in the indeterminates $(u_i)_{i \in Q_0}$.

One can map the set $\operatorname{Tilt} Q$ to the field K as follows. Let T be a tilting module. It can be written uniquely (up to permutation of the summands) as a direct sum of indecomposable modules $T_1 \oplus \cdots \oplus T_n$. Then one defines

$$T \mapsto \psi(T) = \frac{1}{\prod_{j=1}^{n} \sum_{i \in Q_0} \dim(T_j)_i u_i}. \tag{0.0.1}$$

Note that the value of $\psi(T)$ when all $u_i = 1$ is the volume of T introduced in [Hil06].

One can then extend this map to a linear map ψ from $K_0(\operatorname{mod} k \operatorname{Tilt} Q)$ to K by sending the class $[T]$ of the simple module corresponding to a tilting module T to the fraction $\psi(T)$. One can then define the value of ψ on any module in $\operatorname{mod} k \operatorname{Tilt} Q$, in particular for intervals.

One expects the following property to hold.

CONJECTURE *The map ψ is injective. Equivalently, the images $\psi(T)$ of all tilting modules are linearly independent.*

This is known for the equioriented quiver of type $\mathbb{A}_n$. The same proof (using iterated residues) should also work for any quiver of type $\mathbb{A}$.

Assume now that Q is a Dynkin quiver.

Exceptional sequences up to permutations (exceptional sets) should appear in the following way.

CONJECTURE *There is a bijection between the set $\mathtt{E}_Q$ of exceptional sets E in $\operatorname{mod} kQ$ and the set $\mathtt{I}_Q$ of intervals I in $\operatorname{Tilt} Q$ such that $\psi(I)$ is the inverse of a polynomial. The exceptional set E can be recovered from the interval I by factorization of the denominator of the fraction $\psi(I)$. Tilting modules correspond to singleton intervals.*

For the equioriented quiver of type $\mathbb{A}_n$, one has proved that there is an injective map from $\mathtt{E}_Q$ to $\mathtt{I}_Q$ with these properties.

REFERENCES

[BGP73] I. N. Bernšteĭn, I. M. Gel′fand, and V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[Hil06] Lutz Hille. On the volume of a tilting module. *Abh. Math. Sem. Univ. Hamburg*, 76:261–277, 2006.

[HU05] Dieter Happel and Luise Unger. On a partial order of tilting modules. *Algebr. Represent. Theory*, 8(2):147–156, 2005.

[Lad07] Sefi Ladkani. Universal derived equivalences of posets of cluster tilting objects, 2007. http://www.citebase.org/abstract?id=oai:arXiv.org:0710.2860.

[RS91] Christine Riedtmann and Aidan Schofield. On a simplicial complex associated with tilting modules. *Comment. Math. Helv.*, 66(1):70–78, 1991.

$d-$Cluster tiltings in $d-$cluster categories and their combinatorics

BIN ZHU

(joint work with Yu Zhou)

Cluster categories are introduced by Buan-Marsh-Reineke-Reiten-Todorov [BMRRT] for a categorification understanding of cluster algebras introduced by Fomin-Zelevinsky in [FZ], see also [CCS] for type A_n.

$d-$cluster categories $\mathcal{D}/\tau^{-1}[d]$ as a generalization of cluster categories, are introduced by Keller [Ke] for $d \in \mathbf{N}$. They are studied recently in [Th], [KR1], [Zh], [BaM1, BaM2], [IY], [KR2], [HoJ1, HoJ2], [J], [ABST], [T], [Wr]. $d-$cluster categories are triangulated categories with Calabi-Yau dimension $d+1$ [Ke]. When $d=1$, the cluster categories are recovered.

We study the cluster combinatorics of $d-$cluster tilting objects in $d-$cluster categories. By using mutations of maximal $d-$rigid objects in $d-$cluster categories which are defined similarly for $d-$cluster tilting objects, we prove the equivalences between $d-$cluster tilting objects, maximal $d-$rigid objects and complete $d-$rigid objects. Using the chain of $d+1$ triangles of $d-$cluster tilting objects in [IY], we prove that any almost complete $d-$cluster tilting object has exactly $d+1$ complements, compute the extension groups between these complements, and study the middle terms of these $d+1$ triangles. All results are the extensions of corresponding results on cluster tilting objects in cluster categories established in [BMRRT] to $d-$cluster categories. They are applied to the Fomin-Reading's generalized cluster complexes of finite root systems defined and studied in [FR] [Th] [BaM1-2], and to that of infinite root systems [Zh].

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References

[ABST] I. Assem, T. Brüstle, R. Schiffler and G. Todorov, m-cluster categories and m-replicated algebras. To appear in J. Pure and applied Alg., see also arXiv:math.RT/0608727, 2006.

[BaM 1] K.Baur and R.Marsh. A Geometric description of $m-$Cluster categories. Preprint, math.RT/0610512. To appear in Transactions of the AMS.

[BaM 2] K Baur and RJ Marsh, A Geometric description of $m-$Cluster categories. Arxiv preprint math.RT/0610512, IMRN(2007).

[BMRRT] A.Buan, R.Marsh, M.Reineke, I.Reiten and G.Todorov. Tilting theory and cluster combinatorics. Advances in Math. **204**, 572-618, 2006.

[CCS] P.Caldero, F.Chapoton and R.Schiffler. Quivers with relations arising from clusters (A_n case). Transactions of the AMS. **358**, 1347-1364, 2006.

[FR] S.Fomin and N.Reading. Generalized cluster complexes and Coxeter combinatorics. IMRN. **44**, 2709-2757, 2005.

[FZ] S.Fomin and A.Zelevinsky. Cluster Algebras I: Foundations. Jour. Amer. Math. Soc. **15**, no. 2, 497–529, 2002.

[HoJ1] T.Holm and P.Jørgensen. Cluster actegories and selfinjective algebras: Type A. Arxiv preprint math.RT/0610728, 2006.

[HoJ2] T.Holm and P.Jørgensen. Cluster actegories and selfinjective algebras: Type D. Arxiv preprint math.RT/0612451, 2006.

[IY] O.Iyama and Y. Yoshino Mutations in triangulated categories and rigid Cohen-Macaulay modules, Preprint. arXiv:math.RT/0607736.

[J] P. Jørgensen. Quotients of cluster categories, arXiv:0705.1117 [math.RT].

[Ke] B.Keller. Triangulated orbit categories. Documenta Math. **10**, 551-581, 2005.

[KR1] B.Keller and I.Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211, 123-151, 2007.

[KR2] B.Keller and I.Reiten. Acyclic Calabi-Yau categories, with an appendix by Van.Den Bergh. Arxiv preprint math.RT/0610594, 2006.

[T] G.Tabuada, On the structure of Calabi-Yau catgeories with a cluster tilting subcategories. Doc.Math. 12 (2007), 193-213.

[Th] H.Thomas. Defining an $m-$cluster category. Journal of Algebra, 318(2007), 37-46.

[Wr] Anette Wrålsen, Rigid objects in higher cluster categories, arXiv.0712.2970 [math.RT].

[Zh] B.Zhu. Generalized cluster complexes via quiver representations. Journal of Algebraic Combinatorics 27, 35-54, 2008.

Cluster algebras and quantum affine algebras, after B. Leclerc

Bernhard Keller

This talk, based on [14], is a report on recent work by B. Leclerc on a new type of categorification for cluster algebras.

Cluster algebras were invented by Fomin and Zelevinsky [8] at the beginning of this decade. Since then, a major effort has gone into their categorification (*cf.* for example [15] [1] [2] [3] [10]). Namely, in many cases, it was proved that for a given cluster algebra $\mathcal{A}$, there exists a triangulated (or Frobenius) category $\mathcal{C}$, such that

- the cluster variables x of $\mathcal{A}$ correspond to certain indecomposables T_x of $\mathcal{C}$,
- two cluster variables x and y belong to the same cluster if and only if there are no non split extensions between the corresponding objects T_x and T_y,
- the cluster monomial $m = xy \cdots z$ corresponds to the the object $M = T_x \oplus T_y \oplus \cdots T_z$ of $\mathcal{C}$,
- the exchange relations $xx^* = m + m'$ of $\mathcal{A}$ correspond to triangles

$$T_x \to M \to T_{x^*} \to \Sigma T_x \text{ and } T_{x^*} \to M' \to T_x \to \Sigma T_{x^*}$$

of $\mathcal{C}$.

It was shown that in certain cases, the objects T_x are precisely the indecomposable *rigid* objects of $\mathcal{C}$, *i.e.* those without selfextensions. For example, when $\mathcal{A}$ has only a finite number of cluster variables, then all indecomposable objects of $\mathcal{C}$ are rigid and the cluster variables are in bijection with the indecomposables of $\mathcal{C}$. In this case, it was also shown that the cluster algebra $\mathcal{A}$ can be realized as a sort of dual Hall algebra of the triangulated category $\mathcal{C}$ and that its commutativity reflects the fact that $\mathcal{C}$ is 2-Calabi-Yau, *i.e.* the space $\mathsf{Ext}^1_{\mathcal{C}}(L, M)$ is in natural duality with $\mathsf{Ext}^1_{\mathcal{C}}(M, L)$ for all objects L and M of $\mathcal{C}$.

This type of categorification is very useful: it has allowed to prove properties of cluster algebras which appear to be beyond the reach of the purely combinatorial methods, *cf.* for example [4]. However, it is perhaps not the most natural notion of categorification which we could expect for a cluster algebra.

In order to categorify an algebra $\mathcal{A}$ defined over the integers and endowed with a distinguished $\mathbb{Z}$-basis B, one would rather look for an abelian category $\mathcal{M}$ which is monoidal (*i.e.* endowed with a tensor product) and whose Grothendieck ring is isomorphic to $\mathcal{A}$ in such a way that the elements of B correspond to the classes of the simple objects of $\mathcal{M}$, *cf.* for example [12]. The definition of a 'canonical basis' for a general cluster algebra is still an open problem (*cf.* for example [18]) but in many cases, this basis is known, for example when there is only a finite number of clusters or when the algebra already admits a canonical basis in the sense of Kashiwara and Lusztig. One then expects [8] that the cluster monomials, and in particular the cluster variables, belong to this canonical basis.

The natural notion of 'tensor-indecomposability' is primality: an object of $\mathcal{M}$ is *prime*, if it does not admit a non trivial tensor factorization. In order to categorify a cluster algebra $\mathcal{A}$, one would therefore look for an abelian monoidal category $\mathcal{M}$ whose Grothendieck ring is $\mathcal{A}$ and such that

- the cluster variables x of $\mathcal{A}$ are the classes of certain prime simple objects S_x of $\mathcal{M}$,
- two cluster variables x and y belong to the same cluster if and only if $S_x \otimes S_y$ is simple,
- the cluster monomial $m = xy \cdots z$ in $\mathcal{A}$ is the class of the simple object $M = S_x \otimes S_y \otimes \ldots \otimes S_z$ of $\mathcal{M}$,
- the exchange relations $xx^* = m + m'$ come from exact sequences

$$0 \to M \to S_x \otimes S_{x*} \to M' \to 0.$$

This last condition lacks in symmetry. But if we remember that the cluster algebra is commutative, and thus the tensor product induces a commutative multiplication in the Grothendieck group, we can save symmetry by also requiring the existence of an exact sequence

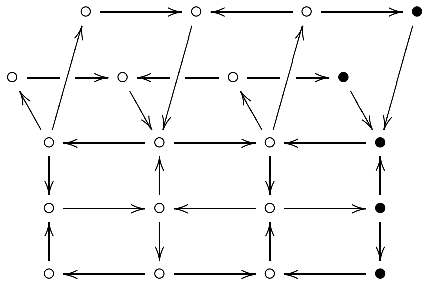
$$0 \to M' \to S_{x^*} \otimes S_x \to M \to 0.$$

The natural notion which replaces rigidity in a monoidal category appears to be 'reality': an object of $\mathcal{M}$ is *real* if its tensor square is simple (*cf.* [13]). The objects S_x should exactly be the real prime simple objects of $\mathcal{M}$. When the cluster algebra $\mathcal{A}$ has only finitely many cluster variables, all the prime simple objects of $\mathcal{M}$ should be real and the cluster variables of $\mathcal{A}$ should be in bijection with the prime simples.

Using classical results on representations of quantum affine algebras [5] [6] [9] [16] [17] B. Leclerc has shown [14] that the cluster algebras of types A_n, $n \in \mathbb{N}$, and D_4 (with suitable coefficients) do admit monoidal categorifications given by tensor abelian subcategories of categories of finite-dimensional representations of quantum affine algebras. He conjectures that this holds in many more cases. More precisely, the main conjecture of [14] is the following.

Conjecture (Leclerc)**.** *Let Δ be a Dynkin diagram and $l \geq 1$ an integer. Let $\mathfrak{g}$ be the complex simple Lie algebra of type Δ, q a non zero complex number which is not a root of unity and $U_q(\widehat{\mathfrak{g}})$ the corresponding quantum affine algebra. Then the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ admits a monoidal abelian subcategory $\mathcal{M}_{\Delta,l}$ which is a monoidal categorification of the cluster algebra associated with a quiver $Q_{\Delta,l}$.*

In [14], Leclerc explicitly describes the subcategory $\mathcal{M}_{\Delta,l}$ and the quiver $Q_{\Delta,l}$. For example, if $\Delta = D_5$ and $l = 3$, then the quiver $Q_{\Delta,l}$ is as follows



The vertices marked by $\bullet$ correspond to 'frozen variables' of the initial cluster. For $\Delta = A_1$ and $l = 3$, the quiver $Q_{\Delta,l}$ is

$$\circ \longleftarrow \circ \longrightarrow \circ \longleftarrow \bullet$$

In this last case, the subcategory $\mathcal{M}_{\Delta,l}$ is the full subcategory on the finite-dimensional $U_q(\widehat{sl_2})$-modules all of whose simple subfactors have Drinfeld polynomials with roots in q^4, q^2, q^0, q^{-2}. The isomorphism between the cluster algebra $\mathcal{A}(Q_{A_1,3})$ and the Grothendieck group $K_0(\mathcal{M}_{A_1,3}) \otimes_{\mathbb{Z}} \mathbb{Q}$ sends the variables x_1, x_2, x_3, x_4 of the initial cluster to the classes of the Kirillov-Reshetikhin modules W_{1,q^0}, $W_{2,q^{-2}}$, $W_{3,q^{-2}}$ and $W_{4,q^{-4}}$. The complete list of the prime simples (up to isomorphism) is

$$\begin{tikzcd}
 & & & & W_{4,q^{-4}} \arrow[dr] & & & & \\
 & {} \arrow[dr, "\alpha"] & & W_{3,q^{-4}} \arrow[ur] \arrow[dr] & & W_{3,q^{-2}} \arrow[dr] & & {} & \\
{} \arrow[dr, "\beta"] & & W_{2,q^{-4}} \arrow[ur] \arrow[dr] & & W_{2,q^{-2}} \arrow[ur] \arrow[dr] & & W_{2,q^{0}} \arrow[dr] \arrow[ur, "\beta"] & & {} \\
 & W_{1,q^{-4}} \arrow[ur] & & W_{1,q^{-2}} \arrow[ur] & & W_{1,q^{0}} \arrow[ur] & & W_{1,q^{2}} \arrow[ur, "\alpha"] &
\end{tikzcd}$$

The arrows do not indicate morphisms but serve to identify the vertices other than $W_{4,q^{-4}}$ with those of the Auslander-Reiten quiver of the cluster category of type A_3 (the arrows on the left and on the right of the diagram are identified as indicated by their labels). Every simple module in $\mathcal{M}_{\Delta,l}$ is a tensor product of modules in this list. A given tensor product of modules in the list other than $W_{4,q^{-4}}$ is simple iff the corresponding direct sum of indecomposables of the cluster category is rigid.

Thus, at least in certain examples, one obtains two rather different categorifications of a given cluster algebra. Table 1 sums up the correspondences. The category $\mathcal{C}$ is much 'smaller' than $\mathcal{M}$ and $\mathcal{M}$ is much less well understood than $\mathcal{C}$. It does not seem to be known whether it has enough projectives, for example. The table suggests that $\mathcal{M}$ should be an 'exponential' of $\mathcal{C}$ or $\mathcal{C}$ a 'linearisation' of $\mathcal{M}$ . . .

cluster algebra $\mathcal{A}$	additive categorification $\mathcal{C}$	monoidal categorification $\mathcal{M}$
$+$	?	$\oplus$
$\times$	$\oplus$	$\otimes$
cluster monomial	rigid object	real simple object
cluster variable	rigid indecomposable	real prime simple

Table 1. Correspondences between categorifications

Finally, let us point out [11] [7] for a very different link between cluster algebras and quantum affine algebras, which does not seem to be related to categorification.

References

[1] Aslak Bakke Buan, Robert J. Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Advances in Mathematics **204 (2)** (2006), 572–618.

[2] Aslak Bakke Buan, Robert J. Marsh, Idun Reiten, and Gordana Todorov, *Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3049–3060 (electronic), With an appendix coauthored in addition by P. Caldero and B. Keller.

[3] Philippe Caldero and Bernhard Keller, *From triangulated categories to cluster algebras*, arXiv:math.RT/0506018, to appear in Inv. Math.

[4] Philippe Caldero and Markus Reineke, *On the quiver Grassmannian in the acyclic case*, arXiv:math/0611074.

[5] Vyjayanthi Chari and Andrew Pressley, *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), no. 2, 261–283.

[6] ______, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[7] Philippe Di Francesco and Rinat Kedem, *Q-systems as cluster algebras II: Cartan matrix of finite type and the polynomial property*, arXiv:0803.0362.

[8] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic).

[9] Edward Frenkel and Evgeny Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, Comm. Math. Phys. **216** (2001), no. 1, 23–57.

[10] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), no. 3, 589–632.

[11] Rinat Kedem, *Q-systems as cluster algebras*, arXiv:0712.2695.

[12] Mikhail Khovanov, Volodymyr Mazorchuk, and Catharina Stroppel, *A brief review of abelian categorification*, arXiv:math.RT/0702746.

[13] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(\mathfrak{n})$*, Transform. Groups **8** (2003), no. 1, 95–104.

[14] Bernard Leclerc, *Algèbres affines quantiques et algèbres amassées*, in French, Notes from a talk at the Algebra Seminar at the Institut Henri Poincaré, January 14, 2008, available at `www.institut.math.jussieu.fr/projets/tg/Seminaires/2007-08/LeclercExpoIHP08.pdf`.

[15] Robert Marsh, Markus Reineke, and Andrei Zelevinsky, *Generalized associahedra via quiver representations*, arXiv:math.RT/0205152.

[16] Hiraku Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238 (electronic).

[17] ______, *Quiver varieties and t-analogs of q-characters of quantum affine algebras*, Ann. of Math. (2) **160** (2004), no. 3, 1057–1097.

[18] Paul Sherman and Andrei Zelevinsky, *Positivity and canonical bases in rank 2 cluster algebras of finite and affine types*, Mosc. Math. J. **4** (2004), no. 4, 947–974, 982.

Cluster Multiplication, Hall Polynomials and species

ANDREW HUBERY

In [3] Caldero and Chapoton described a map from isomorphism classes of $\mathbb{C}$-representations of an acyclic quiver Q to the function field on n variables, where n is the number of vertices Q_0 of Q. This map involves the Euler characteristic of quiver Grassmannians, where, for a quiver representation M and a dimension vector $\underline{d}$, the quiver Grassmannian

$$\mathrm{Gr}\tbinom{M}{\underline{d}} = \{U \leq M \mid \underline{\dim}U = \underline{d}\} \subset \prod_{i \in Q_0} \mathrm{Gr}\tbinom{\dim M_i}{d_i}$$

is the closed subvariety of the product of Grassmannians consisting of those collections of subspaces yielding a subrepresentation of M.

The Caldero-Chapoton map is now given as

$$M \mapsto u_M := \sum_{\underline{d}} \chi\big(\mathrm{Gr}\tbinom{M}{\underline{d}}\big) \underline{x}^{m(\underline{\dim}M,\underline{d})-\underline{\dim}M} \in \mathbb{Q}(\{x_i \mid i \in Q_0\}),$$

where $m(\underline{\dim}M, \underline{d})$ is defined as follows. Recall that the Euler form of the category of quiver representations

$$\langle M, N\rangle := \dim \mathrm{Hom}(M, N) - \dim \mathrm{Ext}^1(M, N)$$

is a bilinear form depending only on the dimension vectors of M and N. We can represent this by the matrix $I - R$. Then

$$m(\underline{\dim}M, \underline{d}) = R\underline{d} + R^t(\underline{\dim}M - \underline{d}).$$

This map induces a bijection between the isomorphism classes of exceptional reresentations ($\mathrm{End}(M) = \mathbb{C}$ and $\mathrm{Ext}^1(M, M) = 0$) and the cluster variables (other than the x_i) in the cluster algebra $\mathcal{A} \subset \mathbb{Q}(\underline{x})$ (Fomin and Zelevinsky [7]).

This was shown in [3] when Q is Dynkin by comparing u_M with $u_{\tau M}$, where τ is the Auslander-Reiten translate of Q. This corresponds to a special form of cluster mutation in the cluster algebra. In this way one shows that each u_M for exceptional M is a cluster variable. One knows from Gabriel's Theorem [9] that there are precisely $|\Delta^+|$ such exceptional representations up to isomorphism, where Δ is the root system of Q, and similarly from [8] that there are $n + |\Delta^+|$ cluster variables, including the x_i, so we are done.

In general there are infinitely many exceptional objects, so one would like to compute $u_M u_N$ and compare this to cluster mutation. This was done by Caldero and Keller in the two papers [4, 5], working in the cluster category (Buan, Marsh, Reineke, Reiten and Todorov, [2]). In the first paper, they construct a general cluster multiplication formula for a Dynkin quiver, and in the second paper they prove a cluster multiplication formula just for the two complements of cluster-tilting object. One drawback is that the Caldero-Chapoton map can only be defined for the hereditary category, and not for the cluster category directly.

In the preprint [11] I showed how one could obtain a cluster multiplication formula in the Dynkin case using the theory of Ringel-Hall algebras. The Ringel-Hall algebra [14] has as basis the isomorphism classes of quiver representations over a finite field, and with structure constants given by the Hall numbers

$$F_{AB}^{M} = |\{U \leq M \mid U \cong B, M/U \cong A\}|.$$

In particular, we see that

$$|\mathrm{Gr}\tbinom{M}{\underline{d}}| = \sum_{A,B;\underline{\dim}B=\underline{d}} F_{AB}^{M}.$$

In the particular case of a Dynkin quiver, we know that these numbers are given by Hall polynomials, which are universal polynomials in the size of the base field [15]. Thus, for a finite field k, the number of k-rational points of the quiver Grassmannian (viewed as a $\mathbb{Z}$-scheme) is also given by a universal polynomial in $|k|$. Evaluating this polynomial at 1 therefore gives the Euler characteristic of the quiver Grassmannian over $\mathbb{C}$ [13].

In the cluster multiplication formula of Caldero and Keller there are two terms, given by taking extensions of M by N and of N by M in the cluster category. Using our Ringel-Hall algebra interpretation of the quiver Grassmannians, the first of these is a natural consequence of Green's formula for Ringel-Hall algebras. Green's formula is a beautiful result which proves that the Ringel-Hall algebra is a (twisted) bialgebra [10]. In fact, the Ringel-Hall algebra is isomorphic to the positive part of the quantised enveloping algebra of a Borcherd's Lie algebra [16], containing as a subalgebra the positive part of the quantised enveloping algebra of the Kac-Moody Lie algebra of type Q [10].

This Ringel-Hall algebra approach has the advantage that one remains in the original hereditary category of quiver representations, but we lose the symmetry in the two terms of the cluster multiplication formula. On the other hand, these two terms can be compared to the multiplication in Toën's derived Hall algebra [17]. More precisely, the first term is given by extensions $\mathrm{Ext}^1(M,N)$ in the category of representations, whereas for the second term we take the kernel and cokernel of a homomorphism $M \to \tau^{-1}N$. These correspond precisely to the multiplications of N by M and of M by $\tau^{-1}N[1]$ in the derived Hall algebra. We note that $\tau^{-1}N[1] \cong N$ in the cluster category.

Moreover, we obtain the result for all non-simply laced Dynkin diagrams, since Hall polynomials also exist for species of finite representation type [15].

The main difficulty in generalising this approach to arbitrary acyclic quivers is that we require the existence of universal polynomials for the Hall numbers. This is a very strong property, recently shown to hold for all affine quivers [12].

The main idea is to use the partition of Bongartz and Dudek [1], which generalises the Segre classes for $k[T]$-modules to all affine quivers. What is important is that this partition is defined combinatorially, so that we can talk about the same partition irrespective of the base field. The result is now that, given any three such classes α, β, μ in this partition, there exists a universal polynomial $F_{\alpha\beta}^{\mu}$ such that, over a field with q elements,

$$F_{\alpha\beta}^{\mu}(q) = \sum_{A\in\alpha, B\in\beta} F_{AB}^{M} \quad \text{for all } M \in \mu.$$

One cannot do any better, since it is easy to construct examples (even for $k[T]$) where no polynomial exists if we are allowed to fix two out of three representations.

One should remark that the first part of the proof again uses Green's formula, this time as a basis for an induction. In particular, given an extension $0 \to B \to M \to A \to 0$ with M decomposable with respect to some torsion pair, one can apply Green's formula to write F^M_{AB} as a sum involving Hall numbers for representations having strictly smaller dimension.

The final part of my talk involved generalising this result on Hall polynomials to all non-simply laced affine diagrams. The main case to study is that of type $\widetilde{\mathbb{A}}_{11}$, which plays the same role for tame species as the Kronecker algebra does for tame path algebras. Explicitly, we need to study the tame bimodule ${}_KK_k$, or equivalently the matrix algebra $\Lambda := \begin{pmatrix} K & K \\ 0 & k \end{pmatrix}$, for a field extension K/k of degree 4. (The Kronecker algebra corresponds to the k-bimodule k^2.) In this case, as for the Kronecker algebra, all regular components of the Auslander-Reiten quiver are homogeneous tubes [6]. We can parameterise the tubes using the orbit algebra

$$\mathcal{O}(P) := \bigoplus_{n\geq 0} \mathrm{Hom}(P, \tau^{-n}P), \quad f\cdot g := (\tau^{-|g|}f)g\colon P \to \tau^{-|f|-|g|}P,$$

where P is the projective cover of the simple module k. If we do this for the Kronecker algebra we obtain the polynomial ring $k[X,Y]$, and hence we can parameterise the tubes by the scheme $\mathbb{P}^1_k$. In our situation we obtain a Brauer-Severi curve, since if L/K is any algebraic extension, then $\Lambda \otimes_k L$ is isomorphic to the path algebra over L of a quiver of type $\widetilde{\mathbb{D}}_4$, hence has tubes parameterised by $\mathbb{P}^1_L$. In fact, using Hilbert polynomials it is easy to calculate that

$$\mathcal{O}(P) \otimes_k L = L\langle w,x,y,z\rangle/(w+x+y+z, w^2, x^2, y^2, z^2) =: R,$$

which is eight dimensional over its centre

$$Z(R) := L[(x+y)^2, (x+z)^2].$$

Thus $\mathcal{O}(P) = R^F$, where F is the induced action of the Frobenius automorphism of L/k. More precisely, for $\lambda \in L$ we have

$$F\colon \lambda w \mapsto \lambda^q x \mapsto \lambda^{q^2} y \mapsto \lambda^{q^3} z \mapsto \lambda^{q^4} w.$$

The difficulty is that, if M is a regular simple module of dimension vector $(n, 2n)$, then $\dim_k \mathrm{End}(M) = n$ for all but two modules. For these modules we have $\underline{\dim} M_i = (i, 2i)$ but $\dim_k \mathrm{End}(M_i) = 2i$. Therefore, if we wish to form an analogue of the Bongatrz-Dudek partition, then we must also be able to distinguish these two tubes.

References

[1] K. Bongartz and D. Dudek, *Decomposition classes for representations of tame quivers*, J. Algebra **240** (2001), 268–288.

[2] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204**, 572–618.

[3] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), 595–616.

[4] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*, to appear in Invent. Math.

[5] P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*, Ann. Sci. École Norm. Sup. **39** (2006), 983–1009.

[6] V. Dlab and C.M. Ringel, *Indecomposable representations of groups and algebras*, Mem. Amer. Math. Soc. **6** (1976).

[7] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.

[8] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121.

[9] P. Gabriel, *Unzerlegbare Darstellungen, I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. **6** (1972), 309.

[10] J.A. Green, *Hall algebras, hereditary algebras and quantum groups*, Invent. Math. **120** (1995), 361–377.

[11] A. Hubery, *Acyclic cluster algebras via Ringel-Hall algebras*, preprint `http://www.math.uni-paderborn.de/~hubery/Cluster.pdf` .

[12] A. Hubery, *Hall polynomials for affine quivers*, preprint `math.RT/0703178` .

[13] M. Reineke, *Counting rational points of quiver moduli*, Internat. Math. Research Notices (2006).

[14] C.M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–591.

[15] C.M. Ringel, *Hall polynomials for the representation-finite hereditary algebras*, Adv. Math. **84** (1990), 137–178.

[16] B. Sevenhant and M. Van den Bergh, *A relation between a conjecture of Kac and the structure of the Hall algebra*, J. Pure Appl. Algebra **160** (2001), 319–332.

[17] B. Toën, *Derived Hall algebras*, Duke Math. J. **135** (2006), 587–615.

Additive categories of generalized standard Auslander-Reiten components of algebras

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Let A be a finite dimensional algebra over a field K. Denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules, by Γ_A the Auslander-Reiten quiver of $\operatorname{mod} A$, by τ_A the translation $D\operatorname{Tr}$ in Γ_A, and identify the vertices of Γ_A with the corresponding indecomposable A-modules. For a module M in $\operatorname{mod} A$, denote by $[M]$ the image of M in the Grothendieck group $K_0(A) = K_0(\operatorname{mod} A)$. Thus $[M] = [N]$ if and only if the modules M and N have the same simple composition factors including the multiplicities. For modules M and N in $\operatorname{mod} A$, we abbreviate $[M, N] = \dim_K \operatorname{Hom}_A(M, N)$.

Let $\mathscr{C}$ be a family of (connected) components of Γ_A. Following [6] $\mathscr{C}$ is said to be *generalized standard* if $\operatorname{rad}^\infty(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$, where $\operatorname{rad}^\infty(\operatorname{mod} A)$ is the infinite Jacobson radical of $\operatorname{mod} A$. It is known that if $\mathscr{C}$ is generalized standard then $\mathscr{C}$ is *almost periodic*, that is all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic [6]. During the talk, homological and geometric properties of modules from the additive categories $\operatorname{add}(\mathscr{C})$ of generalized standard families $\mathscr{C}$ of components in Γ_A were discussed.

For modules M and N in $\operatorname{mod} A$ with $[M] = [N]$, the following partial orders are of special interest:

- $M \leq_{ext} N$: $\iff$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural s.
- $M \leq_R N$: $\iff$ there exists in $\operatorname{mod} A$ an exact sequence of the form $0 \to N \to M \oplus Z \to Z \to 0$ (equivalently $0 \to Z' \to Z' \oplus M \to N \to 0$).
- $M \leq N$: $\iff$ $[M, X] \leq [N, X]$ (equivalently $[X, M] \leq [X, N]$) for all modules X in $\operatorname{mod} A$.

Then for modules M and N in $\operatorname{mod} A$, the following implications hold:

$$M \leq_{ext} N \quad \Longrightarrow \quad M \leq_R N \quad \Longrightarrow \quad M \leq N.$$

Unfortunately, the reverse implications are not true in general. We also mention that, for K algebraically closed, $\leq_R$ coincides with the degeneration order $\leq_{deg}$, where $M \leq_{deg} N$ means that N belongs to the Zariski closure of the orbit of M under the action of the general linear group $GL_d(K)$ on the variety of A-modules of dimension $d = \dim_K M = \dim_K N$ (see [5], [9]). For modules M and N from $\operatorname{add}(\mathscr{C})$ with $[M] = [N]$, we have also the partial order

- $M \leq_{\mathscr{C}} N$: $\iff$ $[M, X] \leq [N, X]$ (equivalently $[X, M] \leq [X, N]$) for all modules X in $\operatorname{add}(\mathscr{C})$.

The following homological properties of modules from the additive categories of Auslander-Reiten components have been established.

Theorem 1 (Skowroński–Zwara [8]). *Let $\mathscr{C}$ be a generalized standard family of components of Γ_A.*

(i) $\operatorname{add}(\mathscr{C})$ *is closed under extensions.*
(ii) If $M \in \operatorname{mod} A$, $N \in \operatorname{add}(\mathscr{C})$, with $M \leq_R N$, then $M \in \operatorname{add}(\mathscr{C})$.
(iii) If $M, N \in \operatorname{add}(\mathscr{C})$, $V \in \operatorname{mod} A$, with $M \leq V \leq N$, then $V \in \operatorname{add}(\mathscr{C})$.
(iv) If $M, N \in \operatorname{add}(\mathscr{C})$, the following are equivalent:
  (a) $M \leq_R N$.
  (b) There exists an exact sequence $0 \to N \to M \oplus Z \to Z \to 0$ in $\operatorname{add}(\mathscr{C})$.
  (c) There exists an exact sequence $0 \to Z' \to Z' \oplus M \to N \to 0$ in $\operatorname{add}(\mathscr{C})$.
  (d) $M \leq_{\mathscr{C}} N$.
(v) Assume $\operatorname{Ext}^1_A(X, X) = 0$ for all indecomposable modules X in $\mathscr{C}$. Then, for $M, N \in \operatorname{add}(\mathscr{C})$, $M \leq_{ext} N$ if and only if $M \leq_{\mathscr{C}} N$.

We note that (5) applies to all generalized standard families of components without oriented cycles.

A family $\mathscr{C}$ of components of Γ_A is said to be *almost cyclic* if all but finitely many modules of $\mathscr{C}$ lie on oriented cycles. Moreover, $\mathscr{C}$ is said to be *coherent* if every projective module P in $\mathscr{C}$ is the starting module of an infinite sectional path and every injective module I in $\mathscr{C}$ is the ending module of an infinite sectional path. An important class of almost cyclic coherent components is formed by the *quasi-tubes*, for which the projective modules coincide with the injective modules and all modules lie on oriented cycles. Clearly, all stable tubes are quasi-tubes. We refer to [1], [2] for the structure of almost cyclic coherent components. Moreover, a sequence $X \to Y \to Z$ of nonzero morphisms between indecomposable modules in $\operatorname{mod} A$ is called a *short external path* with respect to a family $\mathscr{C}$ of components in Γ_A if X and Z lie in $\mathscr{C}$ but Y is not in $\mathscr{C}$.

Theorem 2 (Skowroński–Zwara [7]). *Let $\mathscr{C}$ be a generalized standard family of quasi-tubes in Γ_A, and M, N modules in $\operatorname{add}(\mathscr{C})$. Then $M \leq_{ext} N$ if and only if $M \leq_{\mathscr{C}} N$.*

We note that all quasi-tubes in Γ_A have infinitely many indecomposalbe modules X with $\operatorname{Ext}^1_A(X, X) \neq 0$.

Theorem 3 (Malicki–Skowroński [2], [3]). *Let $\mathscr{C}$ be a generalized standard family of almost cyclic coherent components in Γ_A without external short paths, and M a module in $\operatorname{add}(\mathscr{C})$. Then $\operatorname{Ext}^i_A(M, M) = 0$ for all $i \geq 2$.*

Theorem 4 (Malicki–Skowroński [3]). *Let K be algebraically closed and $\mathscr{C}$ be a generalized standard family of almost cyclic coherent components of Γ_A, and M a module in $\operatorname{add}(\mathscr{C})$. Then $\dim_K \operatorname{Ext}^1_A(M, M) \leq \dim_K \operatorname{End}_A(M)$.*

We do not know if the above inequality holds an arbitrary field K. In the proof of the above theorem some algebraic geometry arguments are essentially applied.

Let A be a basic finite dimensional algebra over an algebraically closed field K, $A = KQ/I$ its bound quiver presentation, $Q = Q_A$ the quiver of A, with the set of vertices Q_0 and the set of arrows Q_1. Assume A is triangular (Q has no oriented cycles). Then $K_0(A) = \mathbb{Z}^{Q_0}$ and $[M] = \mathbf{dim}\, M$ (the dimension vector of M). The *Tits quadratic form* $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ of A is defined by

$$q_A(\mathbf{x}) = \sum_{i \in Q_0} x^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i,j \in Q_0} r_{ij} x_i x_j$$

where $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0}$ and r_{ij} is the number of relations from i to j in a minimal admissible set of relations generating the ideal I. We denote by $\chi_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ the *Euler quadratic form* of A such that

$$\chi_A(\mathbf{dim}\, M) = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}^i_A(M, M)$$

for any module M in $\operatorname{mod} A$. It is known that q_A and χ_A coincide if $\operatorname{gl.dim} A \leq 2$ but in general there are different. For $\mathbf{d} \in \mathbb{Z}^{Q_0}$, denote by $\operatorname{mod}_A(\mathbf{d})$ the variety of A-modules of dimension vector $\mathbf{d}$. Then the algebraic group $G(\mathbf{d}) = \prod_{i \in Q_0} GL_{d_i}(K)$ acts on $\operatorname{mod}_A(\mathbf{d})$ in such a way that the $GL(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ correspond to the isomorphism classes of A-modules of dimension vector $\mathbf{d}$. For a module M in $\operatorname{mod}_A(\mathbf{d})$, denote by $\dim_M \operatorname{mod}_A(\mathbf{d})$ the local dimension of $\operatorname{mod}_A(\mathbf{d})$ at M (maximal dimension of the irreducible components containing M).

Theorem 5 (Malicki–Skowroński [3]). *Let A be a basic algebra over an algebraically closed field K, $\mathscr{C}$ a generalized standard family of almost cyclic coherent components of Γ_A without external short paths, M a module in* $\operatorname{add}(\mathscr{C})$*, and* $\mathbf{d} = \mathbf{dim}\, M$*. Then*

(i) M *is a nonsingular point of* $\operatorname{mod}_A(\mathbf{d})$.
(ii) $q_A(\mathbf{d}) \geq \chi_A(\mathbf{d}) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M) \geq 0$.
(iii) $\dim_M \operatorname{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - \chi_A(\mathbf{d})$.

We note that $\dim G(\mathbf{d}) - \chi_A(\mathbf{d})$ (respectively, $\chi_A(\mathbf{d})$) can be arbitrary large [4].

References

[1] P. Malicki and A. Skowroński, *Almost cyclic coherent components of an Auslander-Reiten quiver*, J. Algebra **229** (2000), 695–749.

[2] P. Malicki and A. Skowroński, *Algebras with separating almost cyclic coherent Auslander-Reiten components*, J. Algebra **291** (2005), 208–237.

[3] P. Malicki and A. Skowroński, *On the additive categories of generalized standard almost cyclic coherent Auslander-Reiten components*, J. Algebra **316** (2007), 133–146.

[4] J. A. de la Peña and A. Skowroński, *The Tits and Euler forms of a tame algebra*, Math. Ann. **315** (1999), 37–59.

[5] C. Riedtmann, *Degenerations for representations of quivers with relations*, Ann. Sci. École Norm. Sup. **4** (1986), 275–301.

[6] A. Skowroński, *Generalized standard Auslander-Reiten components*, J. Math. Soc. Japan **46** (1994), 517–543.

[7] A. Skowroński and G. Zwara, *On degenerations of modules with nondirecting indecomposable summands*, Canad. J. Math. **48** (1996), 1091–1120.

[8] A. Skowroński and G. Zwara, *Degeneration-like orders on the additive categories of generalized standard Auslander-Reiten components*, Archiv Math. (Basel) **74** (2000), 11–21.

[9] G. Zwara, *Degenerations of finite dimensional modules are given by extensions*, Compositio Math. **121** (2000), 205–218.

Cells and matrices

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(joint work with Changchang Xi)

Graham and Lehrer [2] defined cellular algebras in order to capture common structural and combinatorial features of symmetric groups, Hecke algebras and various diagram algebras such as Brauer algebras. Cellular structures provide parameter sets of simple modules up to isomorphism, and there is also a homological theory of cellular algebras, see for example [6]. We are proposing a definition of affine cellular algebras that works for infinite dimensional algebras, such as the affine Temperley-Lieb algebras and, in particular, the extended affine Hecke algebras of type $\widetilde{A}$.

Definition. *Let A be a k-algebra with a k-involution i on A. A two–sided ideal J in A is called an* affine cell ideal *if and only if $i(J) = J$ and there exist a free k-module V of finite rank, a commutative affine k-algebra B with identity and with a k-involution σ such that $\Delta := V \otimes_k B$ is an A-B-bimodule (on which the* *right B-module structure is induced by B_B) and an A-A- bimodule isomorphism $\alpha : J \longrightarrow \Delta \otimes_B \Delta'$, where $\Delta' = B \otimes_k V$ is a B-A-bimodule with the left B-structure induced by ${}_B B$ and with the right A-structure via i, that is, $(b \otimes v)a := \tau(i(a)(v \otimes b))$ for $a \in A, b \in B$ and $v \in V$), such that the following diagram is commutative:*

$$\begin{tikzcd}
J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_B \Delta' \arrow[d, "v_1 \otimes b_1 \otimes_B b_2 \otimes v_2 \mapsto v_2 \otimes \sigma(b_2) \otimes_B \sigma(b_1) \otimes v_1"] \\
J \arrow[r, "\alpha"] & \Delta \otimes_B \Delta'
\end{tikzcd}$$

The algebra A (with the involution i) is called affine cellular *if and only if there is a k-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J'_l$ gives a chain of two sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j / J_{j-1}$ is an affine cell ideal of A/J_{j-1} (with respect to the involution induced by i on the quotient).*

We call this chain a cell chain *for the affine cellular algebra A. The module Δ will be called a* cell lattice *for the affine cell ideal J.*

This definition insists on the finiteness of the cell chain. The main new ingredient is the algebras B associated with the cells. These algebras are not a priori related to the algebra A or its centre. In practice it is necessary to choose different B for different cells. The lack of a relation between A and B poses the main problem when developing a theory of affine cellular algebras. Nevertheless, it is possible to describe a parameter set of simple modules, as an affine variety.

The first step is to view a cell ideal J as an algebra without unit and to identify it with a generalized matrix ring (see also [1, 5]) with entries in B (note that in general this matrix ring is not an algebra over B). The multiplication in such a generalized matrix ring is controlled by a 'sandwich matrix' ψ_{st} (which corresponds to a bilinear form on the cell lattice). Once the cells J_j/J_{j-1}, their algebras B_j and their sandwich matrices $(\psi_{st}^{(j)})$ have been determined, the classification of simples is as follows:

Theorem. *Let A be an affine cellular algebra with a cell chain*

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

such that J_j/J_{j-1} has sandwich matrix $(\psi_{st}^{(j)})$.

Then there is a bijection between the set of isomorphism classes of simple A-modules and the set

$$\{(j, \mathsf{m}) \mid 1 \leqslant j \leqslant n, \text{ there is a maximal ideal } \mathsf{m} \text{ of } B_j \\ \text{such that there is some } \psi_{st}^{(j)} \notin \mathsf{m}\}.$$

In particular, the parameter set of simple A-modules is a disjoint union of affine varieties Var_j, and each variety Var_j is contained in an affine space $Spec(B_j)$.

Extended affine Hecke algebras of type $\widetilde{A}$ are affine cellular; this uses results of Lusztig [8, 9, 10] and of Nanhua Xi [11]. Working over a field and assuming that the quantum parameter q is not a root of the Poincaré polynomial (as in Nanhua Xi's [12] extension of the Deligne-Langlands classification) it follows that cells are idempotent and that *the extended affine Hecke algebra has finite global dimension.*

References

[1] W.P.Brown, Generalized matrix algebras. *Canad. J. Math.* **7** (1955), 188-190.

[2] J.J.Graham and G.I.Lehrer, Cellular algebras. *Invent. Math.* **123** (1996), 1-34.

[3] J.J.Graham and G.I.Lehrer, The representation theory of affine Temperley-Lieb algebras. *L'Enseign. Math.* **44** (1998), 173-218.

[4] S.Koenig and C.C.Xi, On the structure of cellular algebras. In: I.Reiten, S.Smalø and Ø.Solberg (Eds.): Algebras and Modules II. Canadian Mathematical Society Conference Proc., vol. 24 (1998), 365–386.

[5] S.Koenig and C.C.Xi, A characteristic-free approach to Brauer algebras. *Trans. Amer. Math. Soc.* **353** (2001), 1489-1505.

[6] S.Koenig and C.C.Xi, When is a cellular algebra quasi-hereditary ? *Math. Ann.* **315** (1999), 281-293.

[7] S.Koenig and C.C.Xi, Affine cellular algebras. Preprint, 2008.

[8] G.Lusztig, Cells in affine Weyl groups, in "Algebraic groups and related topics", Advanced Studies in Pure Math., vol. 6, Kinokunia and North Holland, 1985, 255-287.

[9] G.Lusztig, Cells in affine Weyl groups. II. *J. Algebra* **109** (1987), 536-548.

[10] G.Lusztig, Cells in affine Weyl groups. III. *J.Fac.Sci.Univ.Tokyo Sect.IA, Math.* **34** (1987),223-243.

[11] N.H.Xi, The based ring of two sided cells of affine Weyl groups. *Memoirs of Amer. Math. Soc.* **152** (2002), No. 749.

[12] N.H.Xi, Representations of affine Hecke algebras and based rings of affine Weyl groups. *J. Amer. Math. Soc.* **20**(1) (2007), 211-217.

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