Mathematisches Forschungsinstitut Oberwolfach

Report No. 10/2008

Mini-Workshop: The Mathematics of Electro-Active Smart Materials

Organised by
Luis Dorfmann, Medford
Ray Ogden, Glasgow
Giuseppe Saccomandi, Lecce & Perugia

February 24th – March 1st, 2008

Abstract. Science and technology have produced amazing developments in the design of electronics and machinery using smart materials. Some everyday items are already incorporating smart materials as electro-rheological and magneto-rheological fluids. Such materials are capable of a rapid and significant change in their material properties on the application of a magnetic or electric field. The workshop brings together leading experts from various fields to address the mathematical and computational problems that pervade research, development, testing in the field of smart materials.

Mathematics Subject Classification (2000): 74F15.

Introduction by the Organisers

The mini-workshop *The Mathematics of Electro-Active Smart Materials*, was attended by 15 participants (7 from Europe, 4 from USA and 1 from Israel). This workshop contained a nice blend of researchers with various backgrounds: mathematics, physics, engineering science and chemistry. The interdisciplinarity of the invited participants was essential because smart materials, such as piezoceramics, magnetostrictives and electroactive polymers, are characterized by strong couplings between their mechanical properties and applied electric, magnetic or thermal fields. Therefore skills in several areas of science are needed to understand the complicated and highly nonlinear behavior of such materials.

Science and technology have produced amazing developments in the design of electronics and machinery using smart materials. Some everyday items are already incorporating smart materials (cars, the International Space Station, eyeglasses, coffee pots, speakers, for example) and the number of applications is growing steadily. For example, electro-rheological (ER) and magneto-rheological (MR)

materials are fluids that can experience a dramatic change in their viscosity. They can change from a thick fluid (similar to motor oil) to nearly solid substances within the span of a millisecond when exposed to a magnetic or electric field; the effect can be completely reversed just as quickly when the field is removed. MR fluids are being developed for use in car shock absorbers, damping of machine vibration, prosthetic limbs, exercise equipment, and surface polishing of machine parts, in particular. ER fluids have been developed mainly for use in clutches and valves, as well as in engine mounts where they are designed to reduce noise and vibration in vehicles. More recently, magneto- and electro-sensitive elastomers have been developed and commercialized. Such materials are capable of large elastic deformations and a rapid and significant change in their material properties on the application of a magnetic or electric field. The coupling in the material response is achieved and optimized by distributing within an elastomeric matrix micron size ferrous particles. Cross-linking may occur, for example, in the presence of an applied field, whereby particles form chain-like structures aligned along the applied field direction. The material is then characterized by two families of preferred direction and may result in highly controllable smart materials.

Mathematical and computational methods pervade research, development, testing, and evaluation problems encountered by researchers in the field of smart materials. Furthermore, increasing demands are being placed on research in the mathematical sciences because of their fundamental roles in the analysis and modelling issues that arise in such complex materials.

In this Mini-Workshop we have discussed recent mathematical and computational questions related to the modelling of smart materials used in structures, actuators and sensors from the point of view of material properties and control of mechanical properties.

The first talk by G. A. Maugin was on overview of existing theories capable of describing the strong coupling between mechanical properties and applied fields in electro- and magneto-active smart materials. This overview was extremely interesting also because it was punctuated with historical notes and remarks.

On the other hand M. Zrinyi gave an exhaustive review of the phenomenology of such materials and of experimental data. This was extremely useful for facilitating a clear identification of further key experiments that should be conducted in order to provide more comprehensive data for improving the modelling process.

The other talks have been devoted to

- Recently developed specific and realistic constitutive laws for describing the response of smart materials, with particular reference to the highly nonlinear deformation, polarization and magnetization.
- Analytical formulations and numerical solution methods for a range of specific boundary-value problems for both idealized geometries and geometries that are of practical interest in the design of smart sensors and actuators.

- Illustrative numerical solutions of the coupled field equations for boundary value problems with complex geometries with particular emphasis on electric and magnetic boundary conditions at the material interfaces.
- Mathematical and numerical theories that include incremental deformations superimposed on a finite deformation as a basis for studying the effect of electro-magnetic fields on the stability of finitely deformed elastic solids, both in general terms and with a view to analysis of stability in specific boundary-value problems.
- Mathematical theories that include time dependence in the formulation of the equations in order to examine vibration and wave propagation characteristics of magneto- and electro-sensitive materials. Numerical methods for the corresponding equations of motion.

Mini-Workshop: The Mathematics of Electro-Active Smart Materials

Table of Contents

Abstracts

On the role of the microstructure in electroactive polymer composites

Gal deBotton

Introduction. The class of electroactive polymers provide attractive advantages: they are soft, light-weight, undergo large deformations, possess fast response time and are resilient. However, wide-spread application has been hindered by their limitations: the need for large electric field, relatively small forces and energy density. It is now recognized that the limitations arise from poor ratio of dielectric to elastic moduli in typical polymers. Recent experimental findings [2] suggest that these difficulties can be resolved with the aid of composites made out of flexible matrices with inclusions of high dielectric materials.

The objective of this work is to address the first of the two main difficulties associated with the characterization of the behavior of electroactive polymer composites (EAPCs). The first is associated with the nonlinear coupling between the applied electric fields and the induced mechanical fields. This nonlinearity can result in amplification of the mechanical fields by local concentrations in the electric field. The second difficulty results from the main advantage of these actuators namely the large actuation strains. Here, however, we limit the investigation to the response of heterogeneous media undergoing *infinitesimal* deformations due to nonlinear electrostatic excitations. This will provide the required tool for analyzing the electromechanical coupling in EAPCs.

Theory. The general variational principle characterizing the behavior of *heterogeneous* hyperelastic dielectrics under combined mechanical and electrical loads was considered in [1]. In this section this, more general result, is specialized to the limit of small deformations elasticity [3, 4].

Consider a heterogeneous n-phase elastic dielectric material in an external electric field generated by thin electrodes with fixed potential, $\hat{\phi}$ (Fig. 1). The electrodes are attached on a portion S_v of the composite's boundary and move with the composite. In addition, external mechanical forces, $\mathbf{t}$ are acting on the boundaries of the sample. The composite occupies a volume region Ω, with boundary $\partial\Omega$. Each homogeneous phase occupies a volume $\Omega^{(r)}$, and the space surrounding the sample is $\Omega^{(0)} \equiv \mathbf{R}^3 \setminus \bigcup_{r=1}^{n} \Omega^{(r)}$. The charge on the surface of the electrodes induces an electric field in all space. Within each phase the electric field is continuous and can be express in terms of a scalar potential field ϕ such that

$$\mathbf{E}^{(r)} = -\nabla\phi \qquad (r = 1, 2, \ldots, n). \tag{1}$$

The electrostatic potential ϕ is continuous throughout. $\mathbf{p}^{(r)}$ are the electric polarizations generated in the phases in response to the electric field. The corresponding electric displacement fields are

$$\mathbf{D}^{(r)} = \epsilon_0 \mathbf{E}^{(r)} + \mathbf{p}^{(r)}, \tag{2}$$

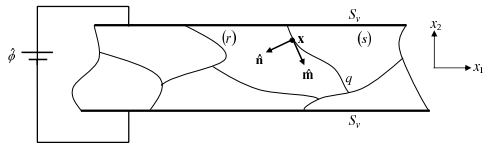

FIGURE 1. A heterogeneous elastic dielectric in an external field generated by thin electrodes with fixed potential

where ϵ_0 is the dielectric constant of the vacuum. In the absence of charge distribution in the phases, everywhere within the phases the electric displacement fields are governed by the equation

$$\nabla \cdot \mathbf{D}^{(r)} = 0, \tag{3}$$

subjected to the boundary condition $\phi = \hat{\phi}$ on S_v.

The deformation within the homogeneous phases is characterized in terms of the strain tensors

$$\varepsilon^{(r)} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right], \tag{4}$$

where the displacement vector $\mathbf{u}$ is continuous in Ω. In the absence of body forces the equilibrium equations are

$$\nabla \cdot \left(\mathbf{T}^{(r)} + \mathbf{T}_M^{(r)}\right) = 0 \quad \text{in} \quad \Omega^{(r)}, \tag{5}$$

subjected to the boundary condition $(\mathbf{T}^{(r)} + \mathbf{T}_M^{(r)} - \mathbf{T}_M^{(0)})\hat{\mathbf{n}} = \mathbf{t}$ on $\partial\Omega \cap \partial\Omega^{(r)}$. Here $\mathbf{T}$ is the mechanical Cauchy stress tensor, and the electromechanical coupling forces are characterized in terms of *Maxwell stress* [5]

$$\mathbf{T}_M = \mathbf{E} \otimes \mathbf{D} - \frac{\epsilon_0}{2}\mathbf{E} \cdot \mathbf{E}\mathbf{I}, \tag{6}$$

where $\mathbf{I}$ is the identity matrix.

The jump across the interface between phases r and s is $[\![\xi]\!] \equiv \xi^{(s)} - \xi^{(r)}$, with ξ being some variable defined in both phases. We denote by $\hat{\mathbf{n}}$ an unit normal of the interface pointing from phase s to r. $\hat{\mathbf{m}}$ is an arbitrary unit vector tangent to the interface. Consider a point $\mathbf{x}$ on an interface charged with surface charge density, q. For the electrostatic fields the jump conditions at this point are

$$[\![\mathbf{E}]\!] = ([\![\mathbf{E}]\!] \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}}, \qquad [\![\mathbf{D}]\!] \cdot \hat{\mathbf{n}} = -q, \tag{7}$$

and for the mechanical fields

$$[\![\nabla \mathbf{u}]\!]\,\hat{\mathbf{m}} = 0, \qquad [\![\mathbf{T} + \mathbf{T}_M]\!]\,\hat{\mathbf{n}} = 0. \tag{8}$$

Application. We consider a linear composite dielectric in which, within the homogeneous phases, the stress-strain relations are $\mathbf{T}^{(r)} = \mathbf{L}^{(r)} : \varepsilon^{(r)}$, and the relations between the electric displacement tensor and the electric field tensor are $\mathbf{D}^{(r)} = \epsilon_0 \mathbf{k}^{(r)} \mathbf{E}^{(r)}$. Here, $\mathbf{L}^{(r)}$ and $\mathbf{k}^{(r)}$ are the elastic and the *dielectric* tensors of the r-phase, respectively.

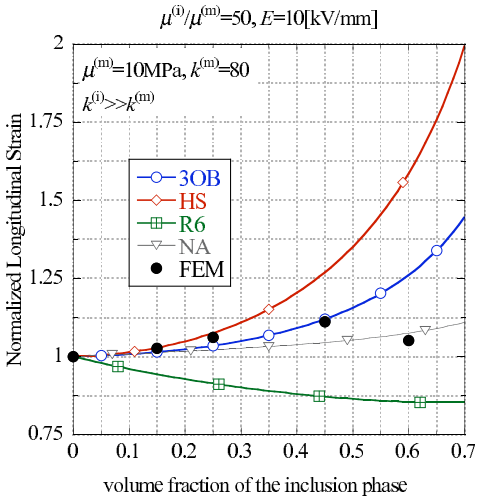

FIGURE 2. The actuation strains of a rank-6 sequentially laminated composite and a composite with hexagonal unit cell shown together with corresponding HS, third-order and naive estimates as functions of the inclusions' volume fraction.

Representative results for the actuation strains developing in heterogeneous dielectrics are depict in Fig. 2. Shown in the figure are exact predictions for the actuation strains of a rank-6 composite and finite element simulations of a composite with hexagonal unit cell. Also shown are the Hashin-Shtrikman and the third-order estimates for the same class of composites. We note that by adding stiff inclusions with high dielectric modulus to a more compliant matrix material the actuation strain can be improved.

Acknowledgment. This work was supported by the United States-Israel Binational Science Foundation (Grant No. 2004146).

REFERENCES

[1] deBotton, G., Tevet-Deree, L., and Socolsky, E. A., *Electroactive heterogeneous polymers: analysis and applications to laminated composites*, Mechanics of Advanced Materials and Structures, **14**, (2007), 13–22.

[2] Huang, C., Zhang, Q. M., deBotton, G., and Bhattacharya, K., *All-organic dielectric-percolative three-component composite materials with high electromechanical response*, Applied Physics Letters, **84**, (2004), 4391–4393.

[3] Tevet-Deree, L., *Electroactive polymer composites - analysis and simulation*, PhD thesis, Ben-Gurion University, (2008).

[4] Tian, L., Bhattacharya, K., Tevet-Deree, L., and deBotton, G., *Electroactive polymer composites - analysis and simulation*, In preparation.

[5] Toupin, R. A., *The elastic dielectric*, Arch. Rational. Mech. Anal., **5**, (1956), 849–915.

Three numerical experiments on models of nematic elastomers

Antonio DeSimone

Nematic elastomers consist of networks of cross–linked polymeric chains, each of which contains nematic rigid rod-like molecules (nematic mesogens). The interaction between nematic order and the underlying rubbery solid results in unusual elastic properties, similar to those exhibited by shape-memory alloys.

It is rather well accepted that the free-energy density of nematic elastomers has a multi-well character, due to the existence of spontaneous states of distortion depending on the orientation of the nematic director [6]. This energy can be used to model both the static and the dynamic response of the material. The analysis of the static response has been pursued in joint work with Sergio Conti and Georg Dolzmann [2, 3]. The dynamic response has been studied in collaboration with Antonio DiCarlo and Luciano Teresi [5].

Accepting to model the static response through the study of global energy minimizers, we are naturally led to the study of the convexity properties of the free-energy of the system. The following expression for the free-energy density has been proposed in [1]

$$W(\mathbf{F}) = \begin{cases} \lambda_1^2(\mathbf{F}) + \lambda_2^2(\mathbf{F}) + a\lambda_3^2(\mathbf{F}) - 3a^{1/3} & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{else} \end{cases} \tag{1}$$

where $\lambda_1(\mathbf{F}) \le \lambda_2(\mathbf{F}) \le \lambda_3(\mathbf{F})$ are the ordered principal stretches associated with the deformation gradient $\mathbf{F} = \nabla \mathbf{y}$, $\mathbf{y}$ being the deformation. The scalar $a < 1$ is a material parameter related to the distortion that the polymeric chains suffer with the establishment of nematic order. The quasiconvex envelope of (1) has been obtained in [5] and it is given by

$$W_{\mathrm{qc}}(\mathbf{F}) = \begin{cases} 0 & \text{if } \lambda_1 \ge a^{1/6} \quad \text{(phase L)} \\ W(\mathbf{F}) & \text{if } a^{1/2}\lambda_3^2\lambda_1 > 1 \quad \text{(phase S)} \\ \lambda_1^2 + 2a^{1/2}\lambda_1^{-1} - 3a^{1/3} & \text{else} \quad \text{(phase Sm)} \end{cases} \tag{2}$$

while it is infinite if $\det \mathbf{F} \neq 1$. This expression has been used to simulate numerically the stretching of clamped sheets of nematic elastomers in [2] and [2] using the Finite Element Method (FEM).

Moving to dynamics, a mechanism to describe viscous dissipation associated with rotations of the nematic director has been proposed in [4]. The resulting model has been used to examine the dependence on the loading rate of the stress-strain response in a purely mechanical experiment, and to model the dynamic response of nematic gels to applied electric fields.

References

[1] Bladon, P., Terentjev, E. M., Warner, M., 1993. Transitions and instabilities in liquid-crystal elastomers. Phys. Rev. E 47, R3838–R3840.

[2] S. Conti, A. DeSimone, and G. Dolzmann, Soft elastic response of stretched sheets of nematic elastomers: a numerical study. *J. Mech. Phys. Solids*, **50** (2002), 1431-1451.

[3] S. Conti, A. DeSimone, and G. Dolzmann, Semi-soft elasticity and director reorientation in stretched sheets of nematic elastomers. *Phys. Rev. E*, **66** (2002), 061710-1-8.
[4] A. DeSimone, A. DiCarlo, and L. Teresi, Critical voltages and blocking stresses in nematic gels. *Eur. Phys. J. E*, **24** (2007), 303-310.
[5] A. DeSimone and G. Dolzmann, Macroscopic Response of Nematic Elastomers via Relaxation of a Class of SO(3)-Invariant Energies. *Arch. Rat. Mech. Anal.*, **161** (2002), 181-204.
[6] M. Warner and E. Terentjev, Liquid Crystal Elastomers, Clarendon Press, Oxford 2003.

Nonlinear electroelasticity: finite deformations, incremental equations and stability

Luis Dorfmann
(joint work with Roger Bustamante, Ray W. Ogden)

In this talk we first provide an overview on a recently developed theory on the interactions between electric fields and mechanical deformations of electro-sensitive materials that are capable of large deformations [1]. Attention is focused on isotropic electroelastic materials and on time-independent electric fields. We summarize the basic equations for the mechanical and electric field variables and their interactions in Eulerian and Lagrangian forms. The general constitutive law for an isotropic electroelastic material is based on the existence of a free energy function expressed in terms of the deformation and one of the electric field vectors. Following Kovetz [2] we begin by considering the deformation gradient $\mathbf{F}$ and the electric field $\mathbf{E}$ as the independent variables, and we write $\psi = \psi(\mathbf{F}, \mathbf{E})$. For a material with no internal mechanical constraints, the Cauchy-like stress tensor $\boldsymbol{\sigma}$ and the polarization density $\mathbf{P}$, are given by

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \psi}{\partial \mathbf{F}}, \quad \mathbf{P} = -\rho \frac{\partial \psi}{\partial \mathbf{E}}. \tag{1}$$

The electric displacement $\mathbf{D}$, defined in terms of $\mathbf{E}$ and $\mathbf{P}$, is then obtained by the standard relation

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{2}$$

where the constant ε_0 is the electric permittivity. These are Eulerian vector fields defined on the current configuration and are regarded as a function of the deformed position vector $\mathbf{x}$. For time-independent phenomena and in the absence of magnetic fields, free currents and free electric charges, the vector fields $\mathbf{D}$ and $\mathbf{E}$ satisfy Maxwell's equations

$$\mathrm{div}\mathbf{D} = 0, \qquad \mathrm{curl}\mathbf{E} = \mathbf{0}. \tag{3}$$

We denote by $\mathbf{D}_l$ and $\mathbf{E}_l$ the Lagrangian counterparts of $\mathbf{D}$ and $\mathbf{E}$, respectively. These are given by

$$\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathbf{E}_l = \mathbf{F}^T\mathbf{E}, \tag{4}$$

and its follows that equations (3) can be written in the Lagrangian forms

$$\mathrm{Div}\mathbf{D}_l = 0, \quad \mathrm{Curl}\mathbf{E}_l = \mathbf{0}, \tag{5}$$

where Div and Curl are the div and curl operators in the reference configuration.

The equilibrium equation for electro-sensitive materials, in the absence of mechanical body forces, has the form

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}_e = \mathbf{0}, \tag{6}$$

where the electric body force (per unit volume) is given by $\mathbf{f}_e = (\operatorname{grad}\mathbf{E})^T\mathbf{D}$. The influence of the electric field on a deformable body may be incorporated either through a stress tensor or through a body force term. Alternative stress tensors and electric body forces can be defined that allow for many different ways to formulate the equilibrium equation, see [3] for a detailed discussion. Following Dorfmann and Ogden [1], it is convenient to introduce the *total stress tensor* $\boldsymbol{\tau}$, which is symmetric and includes the contribution of the electric body forces. The stress $\boldsymbol{\tau}$ is related to $\boldsymbol{\sigma}$ by

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mathbf{D}\otimes\mathbf{E} - \frac{1}{2}\varepsilon_0(\mathbf{E}\cdot\mathbf{E})\mathbf{I}, \tag{7}$$

where $\mathbf{I}$ is the identity tensor. Let the material be surrounded by vacuum where $\mathbf{P} = \mathbf{0}$. In the surrounding space we denote the electric displacement and electric field vectors by $\mathbf{D}^\star$ and $\mathbf{E}^\star$ respectively, which are related by $\mathbf{D}^\star = \varepsilon_0\mathbf{E}^\star$. In the surrounding space, the total stress tensor $\boldsymbol{\tau}$ reduces to

$$\boldsymbol{\tau}^\star = \varepsilon_0\left[\mathbf{E}^\star\otimes\mathbf{E}^\star - \frac{1}{2}(\mathbf{E}^\star\cdot\mathbf{E}^\star)\mathbf{I}\right], \tag{8}$$

which is known as the *Maxwell stress*. The electric displacement and electric field vectors as well as Maxwell's stress tensor satisfy the equations $\operatorname{div}\mathbf{D}^\star = 0$, $\operatorname{curl}\mathbf{E}^\star = \mathbf{0}$ and $\operatorname{div}\boldsymbol{\tau}^\star = \mathbf{0}$.

The total nominal stress tensor, here denoted $\mathbf{T}$, is defined by $\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}$. It follows that equilibrium, in the current and reference configurations, requires that

$$\operatorname{div}\boldsymbol{\tau} = \mathbf{0}, \qquad \operatorname{Div}\mathbf{T} = \mathbf{0}, \tag{9}$$

where $(9)_1$ replaces equation (6). Next, appropriate continuity conditions need to be prescribed on the boundary of the body for the electric displacement and electric field vectors and stress tensors. In the current configuration the continuity condition of the total stress tensor $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_a + \mathbf{t}_e, \tag{10}$$

where $\mathbf{n}$ is the unit outward normal vector, $\mathbf{t}_a$ the applied mechanical traction per unit deformed area and $\mathbf{t}_e$ includes the contribution of the electric field exterior to the body, i.e. $\mathbf{t}_e = \boldsymbol{\tau}^\star\mathbf{n}$. In the reference configuration the traction boundary has the form

$$\mathbf{T}^T\mathbf{N} = \mathbf{t}_A + \mathbf{t}_E, \tag{11}$$

where $\mathbf{N}$ is the unit outward normal, $\mathbf{t}_A$ and $\mathbf{t}_E$ are defined by $\mathbf{t}_A\mathrm{d}A = \mathbf{t}_a\mathrm{d}a$ and $\mathbf{t}_E\mathrm{d}A = \mathbf{t}_e\mathrm{d}a$, respectively. The area elements $\mathrm{d}a$ and $\mathrm{d}A$ are related by Nanson's formula via $\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A$.

Boundary conditions for the electric field $\mathbf{E}$ and displacement vector $\mathbf{D}$ take the Eulerian forms

$$(\mathbf{D}-\mathbf{D}^{\star})\cdot\mathbf{n}=0,\quad (\mathbf{E}-\mathbf{E}^{\star})\times\mathbf{n}=\mathbf{0}, \tag{12}$$

with $\mathbf{D}^{\star}$ and $\mathbf{E}^{\star}$ being evaluated on the boundary.

Following Dorfmann and Ogden [4], it is convenient to introduce a *total energy function*, denoted Ω (as distinct from the notation Ω^* used in [1, 3]). The total energy is defined as a function of $\mathbf{F}$ and $\mathbf{D}_l$ per unit volume in the undeformed configuration.

For unconstrained materials, the total nominal stress $\mathbf{T}$ and the Lagrangian electric field $\mathbf{E}_l$ are then given by

$$\mathbf{T}=\frac{\partial\Omega}{\partial\mathbf{F}},\quad \mathbf{E}_l=\frac{\partial\Omega}{\partial\mathbf{D}_l}, \tag{13}$$

and the Eulerian counterparts are

$$\boldsymbol{\tau}=J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}},\quad \mathbf{E}=\mathbf{F}^{-T}\frac{\partial\Omega}{\partial\mathbf{D}_l}. \tag{14}$$

Let increments be signified by superposed dots. From the current configuration consider an incremental deformation $\dot{\mathbf{F}}$ combined with an increment in the electric displacement $\dot{\mathbf{D}}$. We denote by $\dot{\mathbf{T}}$, $\dot{\mathbf{E}}_l$ increments in total nominal stress $\mathbf{T}$ and electric field $\mathbf{E}_l$, respectively. For an unconstrained material these are given by

$$\dot{\mathbf{T}}=\boldsymbol{\mathcal{A}}\dot{\mathbf{F}}+\boldsymbol{\Gamma}\dot{\mathbf{D}}_l,\quad \dot{\mathbf{E}}_l=\boldsymbol{\Gamma}\dot{\mathbf{F}}+\boldsymbol{\mathcal{K}}\dot{\mathbf{D}}_l, \tag{15}$$

where $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\mathcal{K}}$ are, respectively, fourth-, third- and second-order tensors, which we refer to as *electroelastic moduli tensors*. The incremental forms of the field equations (5) and the equilibrium equation $(9)_2$ are

$$\mathrm{Div}\,\dot{\mathbf{D}}_l=0,\quad \mathrm{Curl}\,\dot{\mathbf{E}}_l=\mathbf{0},\quad \mathrm{Div}\,\dot{\mathbf{T}}=\mathbf{0}. \tag{16}$$

For the derivation and component representation of the electroelastic moduli tensors we refer to [4].

Increments in the deformation and electric displacement need to be accounted for when evaluating the fields exterior to the material body. The associated increments $\dot{\mathbf{D}}^{\star}$ and $\dot{\mathbf{E}}^{\star}$ satisfy Maxwell's equations

$$\mathrm{div}\,\dot{\mathbf{D}}^{\star}=0,\quad \mathrm{curl}\,\dot{\mathbf{E}}^{\star}=\mathbf{0}. \tag{17}$$

The increment of $\boldsymbol{\tau}^{\star}$ is obtained by writing equation (8) in incremental form

$$\dot{\boldsymbol{\tau}}^{\star}=\varepsilon_0[\dot{\mathbf{E}}^{\star}\otimes\mathbf{E}^{\star}+\mathbf{E}^{\star}\otimes\dot{\mathbf{E}}^{\star}-(\mathbf{E}^{\star}\cdot\dot{\mathbf{E}}^{\star})\mathbf{I}], \tag{18}$$

which satisfies the equilibrium equation $\mathrm{div}\,\dot{\boldsymbol{\tau}}^{\star}=\mathbf{0}$.

Finally, on taking the increment of the traction boundary (11), we obtain

$$\dot{\mathbf{T}}^{T}\mathbf{N}=\dot{\mathbf{t}}_A+J\dot{\boldsymbol{\tau}}^{\star}\mathbf{F}^{-T}\mathbf{N}-J\boldsymbol{\tau}^{\star}\mathbf{F}^{-T}\dot{\mathbf{F}}^{T}\mathbf{F}^{-T}\mathbf{N}+\dot{J}\boldsymbol{\tau}^{\star}\mathbf{F}^{-T}\mathbf{N}, \tag{19}$$

where we used Nanson's formula to write $\mathbf{T}_E=J\boldsymbol{\tau}^{\star}\mathbf{F}^{-T}\mathbf{N}$. Similarly, incremental equations can be derived for the electric boundary conditions (12). We refer to [4] for a detailed discussion of this topic.

References

[1] A. Dorfmann and R. W. Ogden, *Nonlinear electroelasticity*, Acta Mechanica **174** (2005), 167–183.
[2] A. Kovetz, *Electromagnetic Theory*, Oxford University Press (2000).
[3] R. Bustamante, A. Dorfmann and R. W. Ogden, *Nonlinear electroelastostatics: a variational framework*, Zeitschrift für angewandte Mathematik und Physik, in press (2008).
[4] A. Dorfmann and R. W. Ogden, *Nonlinear electroelasticity: incremental equations and stability*, International Journal of Engineering Science, submitted (2008).

Electromagnetic effects in deformable solids: historical perspectives and current challenges

Gerard Maugin

A detailed and historical survey of the development of the various theories of electromagnetic effects in deformable solids is provided.

References

[1] G. A. Maugin. *Nonlinear electromechanical effects and applications*. Series in Theoretical and Applied Mechanics, 1. World Scientific Publishing Co., Philadelphia, 1985.
[2] G. A. Maugin. *Continuum mechanics of electromagnetic solids*. North-Holland, Amsterdam 1988.
[3] A. C. Eringen, G. A. Maugin. *Electrodynamics of continua*; Vol. I: Foundations and Solid Media; Vol. II: Fluids and Complex Media. Springer-Verlag, New York, 1989.
[4] G. A. Maugin, J. Pouget, B. Collet, R. Drouot. *Nonlinear electromechanical couplings*. John Wiley and Sons, New York 1993.

The Principle of virtual work for combined electrostatic and mechanical loading

Robert M. McMeeking

(joint work with Chad M. Landis, Salomon M.A. Jimenez)

The equations governing mechanics and electrostatics are formulated for a system in which the material deformations and electrostatic polarizations are arbitrary. A mechanical/electrostatic energy balance is formulated in terms of the electric enthalpy, in which the electric potential and the electric field are the independent variables. This energy statement is presented in the form of a principle of virtual work (PVW), in which external virtual work is equated to internal virtual work. The resulting expression involves an internal material virtual work in which (1) material polarization is work-conjugate to increments of electric field and (2) a combination of Cauchy stress, Maxwell stress and a product of polarization and electric field is work-conjugate to increments of strain. This principle of virtual work is valid for all material types, including those that are conservative and those that are dissipative. The internal virtual work expression can be used to develop the structure of conservative constitutive laws governing, e.g. electroactive elastomers and piezoelectrics, thereby determining the form of the

Maxwell or electrostatic stress. The Maxwell or electrostatic stress has a form fully constrained by the constitutive law and cannot be chosen independently of it. The structure of constitutive laws for dissipative materials, such as viscoelastic electroactive polymers and switchable ferroelectrics, is similarly determined, and it can be shown that the Maxwell or electrostatic stress for these materials is identical to that for a material having the same conservative response when the dissipative processes in the material are shut off. The form of the internal virtual work can be used further to develop the structure of dissipative constitutive laws controlled by rearrangement of material internal variables.

Consider a system consisting of dielectric materials, perfect conductors and free space, without any implication that the dielectricity is conservative or linear. In the current configuration, the system occupies the volume V. The perimeter of the system plus interfaces within it are designated S in the current configuration. The internal interfaces separate the dielectric materials, the conductors and free space from each other. In addition, sectors of dielectric with homogeneous or heterogeneous properties may be separated by surfaces included within S, as may sectors of free space.

Let the free charge per unit volume within V be $q(x_i, t)$ where x_i is the position of material points in the current configuration and t is time. Furthermore, let $\omega(x_i, t)$ be the free charge per unit area on the surfaces S and define $\phi(x_i, t)$ to be the electrical potential everywhere within the system such that it is continuous everywhere in space. Further variables to be considered in the PVW are the velocity $v_j(x_i, t)$ of material points, the surface traction $T_j(x_i, t)$ defined as the force per unit area acting on S and $b_j(x_i, t)$, which is the body force per unit volume acting at points in V. Note that the surface traction T_j and the body force b_j arise from sources other than electrical effects and therefore, do not represent the influence of charges interacting at a distance or electrical fields exerting forces on charges. The surface traction T_j and the body force b_j and any equivalent quantity defined in the current state will be designated mechanical.

Consider the physical laws governing the electromechanical fields in the material. In the quasi-static limit, Maxwell's laws state that the electric field must be curl-free and Gauss's law states that the divergence of the electric displacement must be equal to the volume density of free charge. Therefore, for the electric field, E_i, and the electric displacement, D_i,

$$\epsilon_{ijk}\frac{\partial E_j}{\partial x_i} = 0 \Rightarrow E_i = -\frac{\partial \phi}{\partial x_i}, \tag{1}$$

$$\frac{\partial D_i}{\partial x_i} = q \text{ in } V, \tag{2}$$

$$n_i \left\| D_i \right\| = \omega \text{ on } S. \tag{3}$$

Here, n_i are the Cartesian components of the unit normal to the surface S pointing from the " $-$ " side of the surface out toward the " $+$ " side, and ϵ_{ijk} are the components of the permutation symbol. The notation $\| \ \|$ represents the difference

or jump in the included quantity across the surface S such that

$$\|D_i\| = D_i^+ - D_i^-. \tag{4}$$

The electric displacement can be decomposed into two parts such that

$$D_i = \kappa_0 E_i + P_i, \tag{5}$$

where κ_0 is the dielectric permittivity of free space and P_i are the Cartesian components of the material polarization. Conservation of mass implies that for a given material volume

$$\frac{d}{dt}\int_V \rho dV = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial x_i} = 0, \tag{6}$$

where $\rho(x_i, t)$ is the mass density of the material. The principles of conservation of linear momentum is stated as

$$\int_V (b_i + b_i^E) dV + \int_S (T_i + T_i^E) dS = \frac{d}{dt}\int_V \rho v_i dV. \tag{7}$$

Here the components of the electrical body force b_i^E and surface traction T_i^E have been introduced. These forces arise directly from electric fields acting in the material and are in addition to the mechanical body force and surface traction. Furthermore, it is assumed that the electrical body force can be derived from the Maxwell stress tensor σ_{ij}^M such that

$$b_i^E = \frac{\partial \sigma_{ij}^M}{\partial x_j} \text{ in } V \tag{8}$$

where the electrical body force is the effect of charges interacting at a distance or equivalently, the force per unit volume arising from electric fields acting on charges. The traction relationship for Maxwell stress is then

$$T_i^E = n_j \left\| \sigma_{ij}^M \right\| \text{ on } S. \tag{9}$$

Then, in order to satisfy the principle of conservation of linear momentum for a small surface element, the Cauchy stress in the material, σ_{ij}, must balance the total surface traction such that along with Eq. (9)

$$T_i + T_i^E = -n_j \|\sigma_{ij}\| \Rightarrow T_i = -n_j \left\| \sigma_{ij} + \sigma_{ij}^M \right\|, \tag{10}$$

The sum of the Cauchy and Maxwell stresses will be termed the total true stress.

Next, application of Eqs. (7), (8) and (10) within the principles of conservation of linear and angular momentum and recognition that the resultant integrals must be valid for any arbitrary volume yields

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma_{ij}^M}{\partial x_j} + b_i = \rho \frac{dv_i}{dt} \text{ in } V, \tag{11}$$

and

$$\sigma_{ij} + \sigma_{ij}^M = \sigma_{ji} + \sigma_{ji}^M \text{ in } V. \tag{12}$$

Thus, for the balance of angular momentum to be satisfied, the total true stress must be symmetric, but there is no requirement that the Maxwell and Cauchy stress tensors are individually symmetric.

An asterisk indicates, respectively, a virtual differentiation with respect to time, and a virtual velocity. Introduction of the material virtual rate of change of the electric field then provides a desirable form of the PVW, namely

$$\int_V \left(b_i v_i^* - q\frac{d\phi}{dt} \right) dV + \int_S \left(T_i v_i^* - \omega \frac{d\phi}{dt} \right) dS = \int_V \left[\left(\sigma_{ij} + \sigma_{ij}^M - D_j E_i \right) \frac{\partial v_i^*}{\partial x_j} - D_i \frac{E_i^*}{dt} + \rho \frac{dv_i}{dt} v_i^* \right] dV. \tag{13}$$

Since no constitutive information has been assumed or applied in this derivation, the forms here derived, are valid for both conservative and dissipative materials.

Magnetostriction in a hard ferromagnetic thin-film beam-plate theory

Andrea Nobili

The theory of magnetoelastic interaction is concerned with the deformation of a continuum endowed with some magnetic property, generally expressed in terms of the magnetization vector per unit volume, in the presence of a magnetic field, either it be external, self generated or both. The treatment comprises the equations of continuum mechanics together with Maxwell equations, which take care of the magnetic part, coupled with suitable constitutive equations both in terms of the elastic and the magnetic response. In the simplest case, the magnetic constitutive equations define the so-called soft ferromagnetic materials, which are such that the magnetization is completely defined in terms of the local magnetic field. Accordingly, the magnetic field evaluated at a point, which depends in a potential-theory fashion from the whole magnetization distribution, acquires a direct kinship with the magnetization vector at that same point. Thereby, the generally non local dependence of the magnetic field on the magnetization is overridden by a special local dependence. Special cases are the so-called paramagnetic and diamagnetic materials, wherein the dependence is furthermore linear. In soft ferromagnets, the magnetic problem is almost decoupled from the mechanical problem, the only interaction being represented by the current configuration over which the Maxwell equations are to be solved. An interesting and distinctive feature of such an approach lies in that the magnetic aspect, both in terms of magnetic distribution and in terms of the action on the continuum, is completely expressible through a suitably chosen strain energy function, which depends only on local terms, be they mechanical or magnetic. The total energy of the body may then be expressed as the sum of the strain energy function and of the so-called self energy, which is the energy associated with the magnetic field over the entire domain. Yet the latter, in the case of soft ferromagnets, may be recast as a integral over the body

of the magnetization field (squared) and, as such, as an integral of a local property of the body. As a consequence, the complete phenomenon is reduced to the interplay of local quantities and the non local character is lost. Indeed, the same strain energy function, differentiated, yields the magnetic field as a function of the mechanical and magnetic quantities, locally evaluated. Furthermore, a local magnetic constitutive equation gives rise to multiple (and mutually inequivalent) force expressions, according to the mechanical aspect which is being emphasized (for instance, the couple distribution in the body). Such force expressions are fundamentally dependent on the local form of the strain energy function and it turns out that the deformation is ultimately driven by the strain energy expression. Hard ferromagnetic bodies, on the contrary, possess a constitutive response wherein the magnetization is at equilibrium with the local magnetic field, which, in turn, is given by an external field, a self field and a constitutive field. In other words, the magnetic constitutive response is given in terms of the latter contribution, which sums up with the potential theory self contribution (whence the non local effect) and the external field (which acts as an external load) to yield the total magnetic field at a point. In such approach, the strain energy function is responsible for the constitutive field alone and the energy of the system cannot be expressed in terms of purely local quantities. Such aspect is brought into light by the self energy, which is no longer the integral of a local energy density. Such an approach warrants that different magnetic force expressions yield different local actions which, however, are all equal inasmuch as observable quantities are concerned. Indeed, the discrepancy among the force expressions is shown to be accounted for by local terms only, which means that such expressions all imply a different choice of the strain energy function with respect to the magnetic contribution. The overall balance is thereby consistent. In this paper, a simple beam-plate thin-film cantilever structure is studied within the theory of hard ferromagnetic and elastic interaction. The key step lies in assuming a preferred direction for the magnetization distribution which, following Maxwell equations, allows to recast the usual force expressions in terms of the magnetic field rather than its derivative. This provides a substantial advantage in evaluating the otherwise singular integrals which give the magnetic action. Notwithstanding the simplifying hypothesis, all the relevant features of a hard ferromagnetic and elastic interaction problem are encountered and it is possible to show the nature of the magnetic force (which is non local and singular at the corner points in the domain) and to assess the deformation and the interaction between the layers through an asymptotic analysis. A numerical pseudo-spectral approach lends the full solution.

References

[1] A. Nobili and A. M. Tarantino, *Magnetostriction of a hard ferromagnetic and elastic thin-film structure*, Mathematics and Mechanics of Solids **13** (2008), 95–123.

Body forces and Maxwell stresses in nonlinear electroelasticity

Ray W. Ogden
(joint work with Roger Bustamante, Luis Dorfmann)

There are several different theories concerned with describing the interaction of electromagnetic fields with the deformation of solid continuous materials. In particular, there are different definitions for the 'Cauchy stress', the Maxwell stress and the electromagnetic body forces, some which are summarized in [1] for the electro-elastostatic specialization. These arise because of the fundamentally different theoretical frameworks on which the theories of continuum mechanics and electromagnetism are based, and from the point of view of the mathematical formulation of the governing equations there is considerable flexibility in the way in which these two theories can be combined.

In a recent paper [2], Dorfmann and Ogden have developed, in the quasi-static context, a formulation of the equations of nonlinear electroelasticity that is applicable to electro-sensitive materials that are capable of large deformations, in particular to the electroelastic response of elastomers. Therein, use is made of a 'total' (Cauchy) stress tensor and an associated 'total' energy density function. The latter is the key to providing a very simple, but general, mathematical structure of the governing equilibrium and constitutive equations, simpler than in previously available formulations. This formulation has the advantage that within the material it does not involve the notion of Maxwell stress, a quantity that is in any case not uniquely defined and not accessible to direct experimental measurement.

The purpose of this talk is an attempt to clarify these issues, based partly on the formulation in [2] and the discussion in a forthcoming paper [3]. We consider the influence of an applied electric field on an electro-sensitive material and the resultant 'self' field, which combine to give the total electric field $\mathbf{E}$. In free space the electric displacement vector is given by $\mathbf{D} = \epsilon_0\mathbf{E}$, where the constant ϵ_0 is the permittivity therein, but in an electro-sensitive material this is replaced by the connection $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, where $\mathbf{P}$ is the polarization density of the material. In free space the Maxwell stress tensor $\boldsymbol{\tau}_m$ is defined uniquely by

$$\boldsymbol{\tau}_m = \mathbf{D} \otimes \mathbf{E} - \tfrac{1}{2}\epsilon_0(\mathbf{E} \cdot \mathbf{E})\mathbf{I}, \tag{1}$$

where $\mathbf{I}$ is the identity tensor. The effect of the 'traction' on the body boundary produced by this stress is equivalent to a body force (per unit volume) given by $(\mathbf{P} \cdot \nabla)\mathbf{E}_a$, where $\mathbf{E}_a$ is the applied contribution to $\mathbf{E}$. Note that this is not a traction in the conventional mechanical contact sense.

The equation of equilibrium for the body can be written in the pointwise form

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{f} + \mathbf{f}_{ea} = \mathbf{0}, \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor that matches the applied mechanical traction per unit area of the body boundary, $\mathbf{t}_a$ say, via the connection $\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n} = \mathbf{t}_a$, ρ is the mass density of the material, $\mathbf{f}$ is the mechanical body force per unit mass, $\mathbf{f}_{ea} = (\mathbf{P} \cdot \nabla)\mathbf{E}_a$, $\mathbf{n}$ is the unit outward normal to the boundary of the body and

$^{\mathrm{T}}$ signifies the transpose of a second-order tensor. Note that, in general, $\boldsymbol{\sigma}$ is not symmetric.

An alternative and equivalent formulation of the equilibrium equation is

$$\operatorname{div}\bar{\boldsymbol{\sigma}} + \rho\mathbf{f} + \mathbf{f}_e = \mathbf{0}, \tag{3}$$

where $\mathbf{f}_e \equiv (\mathbf{P}\cdot\nabla)\mathbf{E}$ is the electric body force based on the total field $\mathbf{E}$ and $\bar{\boldsymbol{\sigma}}$ is another stress tensor, which is related to $\boldsymbol{\sigma}$ by $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\tau}_{ms}$, $\boldsymbol{\tau}_{ms}$ being the Maxwell stress corresponding to the self field, i.e.

$$\boldsymbol{\tau}_{ms} = \mathbf{D}_s \otimes \mathbf{E}_s - \tfrac{1}{2}\epsilon_0(\mathbf{E}_s\cdot\mathbf{E}_s)\mathbf{I}, \tag{4}$$

with $\mathbf{E}_s = \mathbf{E} - \mathbf{E}_a$, $\mathbf{D}_s = \mathbf{D} - \mathbf{D}_a$. The 'traction' boundary condition associated with this stress tensor, however, involves a term related to the polarization in addition to $\mathbf{t}_a$ and does not have a clean interpretation.

A third option is the equilibrium equation

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0}, \tag{5}$$

where $\boldsymbol{\tau}$ is a symmetric 'total' stress tensor related to $\bar{\boldsymbol{\sigma}}$ by $\boldsymbol{\tau} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\tau}_m$, with $\boldsymbol{\tau}_m$ given by the formula (1) inside the material (where it is not in general symmetric). In this case $\boldsymbol{\tau}$ matches the traction boundary condition that incorporates the Maxwell traction, say $\mathbf{t}_m$, where $\mathbf{t}_m = \boldsymbol{\tau}_m\mathbf{n}$ is calculated from the exterior of the material, so that $\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_a + \mathbf{t}_m$ on the body boundary. Note that this formulation involves neither a Maxwell stress nor an electric body force within the material. The symmetry of $\boldsymbol{\tau}$ ensures that the rotational balance equation is satisfied and, consequently, no intrinsic body couple is associated with this formulation. On the other hand, with $\boldsymbol{\varepsilon}$ denoting the alternating tensor, $\boldsymbol{\varepsilon}\bar{\boldsymbol{\sigma}} = -\boldsymbol{\varepsilon}\boldsymbol{\tau}_m = -\mathbf{D}\times\mathbf{E} = -\mathbf{P}\times\mathbf{E}$, which corresponds to an intrinsic body couple.

Energy balance equations associated with the above formulations may be given in the form of virtual work identities for the current configuration of the body, which we denote by $\mathcal{B}$ with boundary $\partial\mathcal{B}$. Energy balance has its simplest expression in terms of the stress tensor $\boldsymbol{\tau}$. If we denote by $\mathbf{v}$ a virtual displacement from the configuration $\mathcal{B}$ and its gradient by $\mathbf{L} = \nabla\mathbf{v}$ then we have

$$\int_{\mathcal{B}} \rho\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}v + \int_{\partial\mathcal{B}} (\mathbf{t}_a + \mathbf{t}_e)\cdot\mathbf{v}\,\mathrm{d}a = \int_{\mathcal{B}} \operatorname{tr}(\boldsymbol{\tau}\mathbf{L})\,\mathrm{d}v. \tag{6}$$

This enables contact to be made with the constitutive formulation in [2] and with the variational formulation in [1]. In terms of the density function Ω per unit reference volume introduced in [2], which is a function of the deformation gradient tensor $\mathbf{F}$ and the Lagrangian electric field vector $\mathbf{E}_l = \mathbf{F}^{\mathrm{T}}\mathbf{E}$ we have simply

$$\operatorname{tr}(\boldsymbol{\tau}\mathbf{L})\,\mathrm{d}v = (\dot{\Omega} + \mathbf{D}_l\cdot\dot{\mathbf{E}}_l)\,\mathrm{d}V, \tag{7}$$

where $\mathrm{d}V$ is the volume element in the reference configuration and the superposed dot signifies a variation, which allows for independent variations in $\mathbf{F}$ and φ, where $\mathbf{E}_l = -\operatorname{Grad}\varphi$, φ being the scalar electric potential arising from the equation $\operatorname{curl}\mathbf{E} = \mathbf{0}$, or, equivalently, its Lagrangian counterpart $\operatorname{Curl}\mathbf{E}_l = \mathbf{0}$, in the static specialization of Maxwell's equations.

By identifying the virtual displacement $\mathbf{v}$ with a variation in the displacement we write equation (6) as

$$\dot{E} - \int_{\mathcal{B}} \rho \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}v - \int_{\partial \mathcal{B}} \mathbf{t}_a \cdot \mathbf{v} \mathrm{d}a = 0, \tag{8}$$

where

$$\dot{E} = \int_{\mathcal{B}} \mathrm{tr}\,(\boldsymbol{\tau}\mathbf{L}) \, \mathrm{d}v - \int_{\partial \mathcal{B}} \mathbf{t}_m \cdot \mathbf{v} \, \mathrm{d}a, \tag{9}$$

and the terms following $\dot{E}$ in (8) are purely mechanical. Then, after some manipulations using the divergence theorem and transformations between the reference and current configurations we find that $\dot{E}$ is the variation of the functional

$$E = \int_{\mathcal{B}_r} \Omega \, \mathrm{d}V - \frac{1}{2}\varepsilon_0 \int_{\mathcal{B}'} \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}v - \epsilon_0 \int_{\partial \mathcal{B}^\infty} \varphi \mathbf{E}_a \cdot \mathbf{n} \, \mathrm{d}a, \tag{10}$$

with respect to independent variations in the deformation function and the electric potential, which recovers a functional derived in [1]. In (10), $\mathcal{B}_r$ is the reference configuration of the body, $\mathcal{B}'$ is the region of free space exterior to the body in its deformed configuration $\mathcal{B}$, $\partial \mathcal{B}^\infty$ is the boundary of free space at infinity and $\mathbf{E}_a$ is the applied electric field at infinity.

At this point the energy density Ω remains general but must be subject to both mathematical restrictions and physical considerations. These include requirements that ensure boundary-value problems are well posed and lead to existence of solutions, and that particular forms of Ω are consistent with the results of experiments. At the present time there are not sufficient data available from systematic experiments in which different electric fields are controlled for a wide range of deformations. There is a pressing need for such data as the basis for characterizing the form of Ω for particular materials, which can then be used in the solution of boundary-value problems. Very few such problems have been solved and prospects for obtaining closed-form solutions are slim, even for very simple prototype forms of Ω. Moreover, solution of boundary-value problems with realistic geometries that underpin practical applications requires the application of computational methods, and variational or virtual work formulations such as that discussed above provide a basis for their development.

References

[1] R. Bustamante, A. Dorfmann and R. W. Ogden, *Nonlinear electroelastostatics: a variational framework*, Zeitschrift für angewandte Mathematik und Physik, in press (2008).

[2] A. Dorfmann and R. W. Ogden, *Nonlinear electroelasticity*, Acta Mechanica **174** (2005), 167–183.

[3] R. Bustamante, A. Dorfmann and R. W. Ogden, *On electric body forces and Maxwell stresses in an electroelastic solid*, in preparation (2008).

Damage localization and stability in electro-active polymers

Giuseppe Puglisi

The technological evolution on the manufacturing of polymers indicates electro-active polymers as very promising materials for several applications, ranging from robotic, to medical, and biological technologies. These materials are characterized by important qualities such as lightweight, small size, low-cost, flexibility, fast response. The drawbacks are related to low actuation force, low mechanical energy density, low robustness and, principally, the requirement of high electric fields.

From a theoretical point of view the description of the electro-mechanical behavior of these materials leads to several complication due to the observed strain and damage localization and complex history dependence. In this work we propose, based on a recent damage theory for amorphous materials [2, 3], a model for the analysis of these effects. By considering a first simple example of application we show that the model is suitable to describe known experimental effects observed under electromechanical cyclic loading such as a damage induced softening, strain and damage localization, and the phenomenon of pull-in instability (see [1, 2, 6, 7] and reference therein).

1. Constitutive assumption

Let $\mathbf{F}$ be the strain gradient, $\mathbf{D}$ and $\mathbf{E}$ the electric displacement and the electric field in the current configurations, respectively, and $\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}$ and $\mathbf{E}_l = \mathbf{F}^{\mathrm{T}}\mathbf{E}$ their lagrangian counterpart (see [4, 5]), where $J = \det \mathbf{F}$. Moreover we indicate with ρ_0 and ρ the mass density in the reference and current configurations, respectively, $J\rho = \rho_0$, and with ε the electric permittivity. In a variational formulation [4], we may introduce an electromechanical energy density $\Psi = \rho\Phi(\mathbf{F}) + \frac{\mathbf{D}\cdot\mathbf{D}}{2\varepsilon}$, additively decomposed in a strain dependent part Φ and in an electrostatic component. This energy per unit reference volume can be rewritten as

$$\bar{\Psi} = \rho_0\Phi(\mathbf{F}) + \frac{1}{2\,\varepsilon J}\mathbf{F}\,\mathbf{D}_l \cdot \mathbf{F}\,\mathbf{D}_l. \tag{1}$$

Based on experimental observations (see e.g. [7] and references therein) we neglect the dependence of ε on the deformation. Moreover we consider an incompressibility assumption $J = 1$.

To take care of damage effects, we consider for the polymeric material a strain energy density Φ of the type proposed in [2]. Specifically, we suppose that at each material point there exists a fraction α of elastic material and a fraction $1-\alpha$ of damageable material, with $\Phi = \alpha\Phi_e + (1-\alpha)\Phi_d$. The α fraction, assumed neo-Hookean with modulus μ_e, represents an elastic foundation delivering finite strength even in the state of damage saturation. The remaining fraction takes care of the complex scission effects at the scale of the polymeric network. This energy depends on an activation and breaking criterion. In the simplest case, we may assume that the damageable material is activated when $\mathbf{F} = \mathbf{F}_a$, such that the first invariant $I = \mathrm{tr}\mathbf{B}$, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, reaches the activation threshold $I = I_a$. Similarly the material is broken when I reaches a fracture value $I_b = I_a + \Delta$

where Δ represents a material parameter measuring the elastic range. Moreover, we assume that the stress elastically depends on the deformation measured from the activation state: $\hat{\mathbf{F}} := \mathbf{F}\mathbf{F}_a^{-1}$, $\hat{\mathbf{B}} = \hat{\mathbf{F}}\hat{\mathbf{F}}^T$, $\hat{I} = \mathrm{tr}\hat{\mathbf{B}}$. Thus, based again on a neo-Hookean assumption, we take an energy Φ_d and a Cauchy stress $\mathbf{T}_d$

$$(2)\quad \Phi_d = \begin{cases} 0 & \text{if} \quad I < I_a \\ \mu_d(\hat{I}-3) & \text{if} \quad I_a < I < I_b \\ \mu_d(\hat{I}_b-3) & \text{if} \quad I > I_b \end{cases}, \quad \mathbf{T}_d = \begin{cases} \mathbf{0} & \text{if} \quad I < I_a \\ \mu_d\hat{\mathbf{B}} & \text{if} \quad I_a < I < I_b \\ \mathbf{0} & \text{if} \quad I > I_b \end{cases},$$

where μ_d is an elastic modulus. To describe the properties of the polymeric network, constituted by chains with different reference and fracture strains, we assume that the damageable fraction is assigned by a distribution p of materials with different activation thresholds: $p = p(I_a)$. Moreover, we assume that rupture events are irreversible, so that $\mathbf{T}_d = \mathbf{0}$ for any strain history after $I = I_b$.

With all this standing (see [2] for details), the stress $\mathbf{T} = d\Psi/d\mathbf{F}$ is given by

$$\mathbf{T} = -\pi\mathbf{I} + \alpha\mu_e\mathbf{B} + (1-\alpha)\int_{I_{max}-\Delta}^{I} \mu_b\,\hat{\mathbf{B}}\,p(I_a)\;dI_a + \frac{1}{\varepsilon}\mathbf{D}\otimes\mathbf{D}$$

where π is a pressure taking care of the incompressibility hypothesis and I_{max} is the maximum past value of I attained in the strain history.

To describe the behavior of the obtained model, we consider a simple example of application, represented in Fig.1_a, of a capacitor under an applied electric field $\mathbf{E}_l$. Under the hypothesis of uniaxial deformation $x = \lambda X, y = Y/\sqrt{\lambda}, z = Z/\sqrt{\lambda}$, with $\lambda \le 1$, we obtain $I = \lambda^2 + 2/\lambda, I_a = \lambda_a^2 + 2/\lambda_a, \widehat{I} = \lambda^2/\lambda_a^2 + 2\lambda_a/\lambda$. By imposing a zero traction condition $\mathbf{T}_{11} = \mathbf{T}_{22} = \mathbf{T}_{33} = 0$ we obtain

$$\frac{D_l^2}{\varepsilon} = \alpha\mu_e(\frac{1}{\lambda^3}-1) + 2(1-\alpha)\mu_b\int_{\lambda(I_{max}-\Delta)}^{\lambda}(\frac{\lambda_a}{\lambda^3}-\frac{1}{\lambda_a^2})(\lambda_a-\frac{1}{\lambda_a^2})p(I(\lambda_a))d\lambda_a.$$

Correspondingly, the total potential energy can be evaluated as

$$\begin{aligned}\frac{G}{LA} &= \alpha\frac{\mu_e}{2}(\hat{I}(D_l)-3) + (1-\alpha)\frac{\mu_b}{2}\int_{\hat{I}(D_l)-\Delta}^{\hat{I}(D_l)}(\hat{I}(D_l)-3)p(I_a)dI_a \\ &\quad + (1-\alpha)\frac{\mu_b}{2}\int_0^{\hat{I}(D_l)-\Delta}(\hat{I}_b-3)p(I_a)dI_a) - E_lD_l.\end{aligned}$$

This energy is represented in Fig.1_b for different assigned E_l. Interestingly for some choice of α and $p(I_a)$, G can be a two-wells energy with the possibility, experimentally observed, of deformation and damage localization (see [3, 1, 7]).

Moreover, by minimizing $\bar{\Psi}$ with respect to D_l, we obtain $D_l = \varepsilon\frac{E_l}{\lambda^2(D_l)}$. In Fig.$1_c$ we show the resulting electric field-electric displacement curves under a cyclic experiment of assigned E_l with increasing cycles size. The different curves $1, 2, 3, 4$ correspond to increasing size and show a damage induced variation of the capacity of the conductor reproducing an electromechanical Mullins effect [3].

The influence of the material distribution properties on the electromechanical properties is described in Fig.$1_{d,e}$ where we show the behaviors for different fractions of damageable material. In the case of $\alpha = 1$ the polymer behaves as an

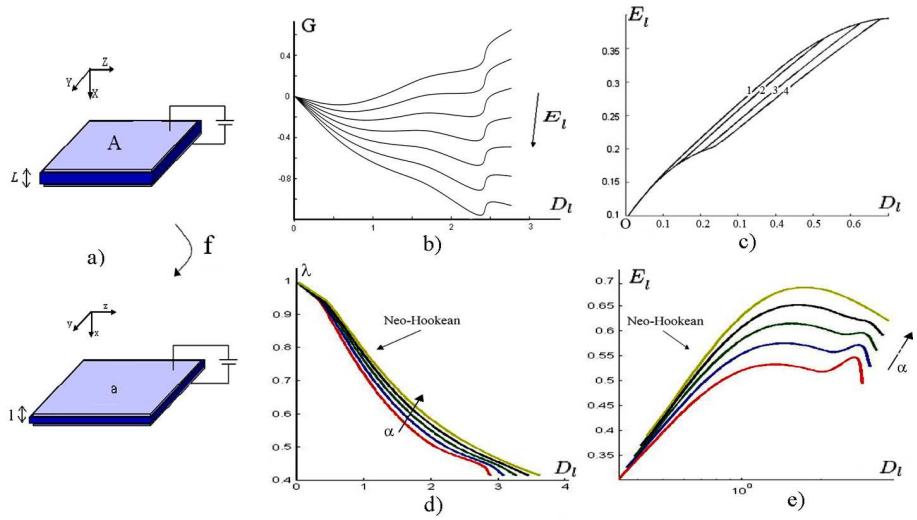

FIGURE 1. a) scheme of a capacitor under an applied electric field $\mathbf{E}_l$, b) non convex total energy for different assigned $\mathbf{E}_l$, c) Mullins effect for electromechanical cycles, d), e) different electromechanical response corresponding to different fractions $1-\alpha$ of damageable material.

elastic neo-Hookean material and the capacitor shows a single stable equilibrium solution for any given value of E_l until a limit threshold is reached leading to a pull-in instability rupture (see [7]). For smaller α two metastable solutions may be available at given E_l, corresponding to possible coexistent states with different strain and damage as experimentally observed ([1, 6, 7] and reference therein). The description of the effective geometries of damage and deformation requires the solution of appropriate boundary value problems that will be the subject of our future work. Observe that after the transition to the higher deformation configuration the system reaches again the rupture through a pull-in instability.

REFERENCES

[1] J. Block, D.G. LeGrand. *Dielectric breakdown of polymer films*, J. Appl. Phys. **40** (1969), 288–293.

[2] D. De Tommasi, G. Puglisi, G. Saccomandi, *A micromechanics based model for the Mullins effect*, J. Rheology **50** (2006), 495–512.

[3] D. De Tommasi, G. Puglisi, G. Saccomandi, *Localized versus Diffuse Damage in Amorphous Materials*, Phys. Rev. Lett. **100** (2008), 085502.

[4] R. Bustamente, A. Dorfman, R.W. Ogden. *Nonlinear electroelastostatics: a variational framework*, Z. angew. Math. Phys. **59** (2008), 1–24.

[5] G.A. Maugin. Continuum mechanics of electromagnetic solids, Elsevier (1988).

[6] S. Plante, S. Dubowsky. *Large-scale failure modes of dielectric elastomer actuators*, Int. J. Sol. Struct. **43** (2006), 7727-51.

[7] X. Zhao, W. Hong and Z. Suo. *Electromechanical hysteresis and coexistent states in dielectric elastomers*, Phys. Rev. B **76** (2007), 134113.

Mathematical modeling and analysis of electrorheological fluids

Michael Ruzicka

We have discussed two models for the flow of ERFs. Both lead to interesting questions in non-standard function spaces. Thus we will also discuss the underlying mathematical theory of these spaces and the consequences for the flow of ERfs.

References

[1] K. R. Rajagopal K.R. and M. Ruzicka. *On the modeling of electrorheological materials*, Mech. Research Comm. **23** (1996), 401–407.
[2] M. Ruzicka. *Flow of shear dependent electrorheological fluids: unsteady space periodic problem*, 485–504 in Applied Nonlinear Analysis (1999), ed. A. Sequeira, Plenum Press.
[3] M. Ruzicka. *Flow of shear dependent electrorheological fluids*, CRAS **329** (1999), 393–398.
[4] M. Ruzicka. *Electrorheological Fluids: Modeling and Mathematical Theory* (2000), Lecture Notes in Mathematics **1748** , Springer.
[5] K.R. Rajagopal and M. Ruzicka. *Mathematical modeling of electrorheological materials*, Cont. Mech. and Thermodyn. **13** (2001), 59–78.
[6] M. Ruzicka, *Modeling, mathematical and numerical analysis of electrorheological fluids*, Applications of Mathematics , **49** (2004), 565–609.
[7] W. Eckart and M. Ruzicka. *Modeling micropolar electrorheological fluids*, Int. J. Appl. Mech. Eng., **11** (2006), 813–844.
[8] F. Ettwein and M. Ruzicka. *Existence of local strong solutions for motions of electrorheological fluids in three dimensions* , Comp. Appl. Math., **53** (2007), 595–604.

Non smooth solutions in fiber reinforced and electroactive materials

Giuseppe Saccomandi

It has been known for some time that certain radial anisotropies in some linear elasticity problems can give rise to stress singularities which are absent in the corresponding isotropic problems. Recently Roger Fosdick [1] has pointed out that those classical solution supports a subregion of material interpenetration and they must be rejected because in clash with the fundamental axioms of continuum mechanics. Similar singular solutions are possible in linear piezolectric materials [2].

Here we show that in the case of incompressible fiber reinforced elastic materials, depending on the reinforcement strength and the fiber orientation in the undeformed configuration, weak solutions, i.e. solutions for which the smoothness required by the differential equations is relaxed, are to be expected. The fact that we are considering only isochoric deformation ensures that material interpenetration cannot occurs and therefore our solutions fulfill all the necessary axioms. These deformations are are true weak solutions because although the deformation field is continuous the strain jumps. These non smooth solutions are interesting because they are related to instability phenomena (like micro-buckling of fibers) and they are non-unique. The mathematics and mechanics of these solutions have been extensively studied in [3] and [4] where the rectilinear shear and the azimuthal shear deformations are investigated.

Similar weak solution are possible in electro-active materials (both solids and fluids). It is interesting that in this case the possibility of of non smooth solutions is driven not only by the intensity of the load, but also by the intensity of the electric field. As illustrative example we consider the Poisseuile flow of an electro-active fluid. We use the same model proposed in [5] and the same arrangement of the electric field. Although this problem have been considered previously in details (see for example [6]) it seems that the possibility of the existence of weak solutions as the ones here presented has been skipped. By pointing out the similarities between the governing equations of the various problems we show that the same mathematical techniques may be used both to study the existence of singular solution in the case of fiber reinforced materials and electro-active materials.

References

[1] R. Fosdick, G. Royer-Carfagni, *The constraint of local injectivity in linear elasticity theory*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), 2167–2187.
[2] C. Horgan, D. Galic, *Internally pressurized radially polarized piezoelectric cylinders*, Journal of Elasticity **66** (2002), 257–272.
[3] J. Merodio, G. Saccomandi,I. Sgura, *The rectilinear shear of fiber-reinforced incompressible non-linearly elastic solids*, International Journal of Non-Linear Mechanics **42** (2007), 342–354.
[4] F. Kassianidis, J. Merodio, R. W. Ogden, T. J. Pence, *Azimuthal shear of a transversely isotropic elastic solid*, Mathematics and Mechanics of Solids, to appear, 2008.
[5] S. L. Ceccio, A.S. Wineman, *Influence of Orientation of Electric Field on Shear Flow of Electrorheological Fluids*, Journal Rheology **38** (1994), 453–463.
[6] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory* (2000), Lecture Notes in Mathematics 1748 , Springer.

Dynamic equations for smectic A liquid crystals

Iain W. Stewart

Smectic liquid crystals are anisotropic fluids that consist of rod-like molecules which are arranged in equidistant layers. In smectic A (SmA) liquid crystals these molecules have a common preferred local average direction that is usually assumed to be parallel to the local layer normal. The average orientation of the long molecular axes is described by the unit vector **n**, called the director. Further details on the physical and mathematical descriptions of SmA liquid crystals can be found in de Gennes and Prost [1] and Stewart [2]. It is common to denote the unit smectic layer normal by **a** and to make no distinction between **n** and **a** in static configurations. Nevertheless, the dynamics of SmA materials are quite complex and recent work by Auernhammer *et al.* [3] and Soddemann *et al.* [4] indicates that samples of SmA under simple shear may exhibit a decoupling between **n** and **a**. This indicates a need for a mathematical model that will allow for this type of effect and its consequences. A theory that does not necessarily require **n** and **a** to coincide has recently been developed by Stewart [5]. It is based in part upon many of the ideas used in the classical formulation of dynamics for nematic liquid crystals by Ericksen and Leslie and dynamics for SmA by Ahmadi [6] and E [7],

but with some differences in approach. The Oseen constraint for smectic liquid crystals, which has been used extensively in earlier static theories [1, 2], states that $\nabla \times \mathbf{a} = \mathbf{0}$ in the absence of defects. This constraint cannot be expected to hold for general disturbances to the smectic layers and, unlike previous theories, the dynamic theory summarised below from [5] does not impose this requirement. The smectic layers can be described via a scalar function Φ so that $\mathbf{a} = \nabla\Phi/|\nabla\Phi|$. The motion of fluid across the smectic layers in the direction of the layer normal is called permeation. This effect is also incorporated and is intricately linked to Φ.

The dynamic equations of SmA can be summarised as follows [5]. The smectic layers are described by Φ, a scalar function that depends on the spatial coordinates and time. The layer normal $\mathbf{a}$ is as defined above and therefore satisfies $\mathbf{a} \cdot \mathbf{a} = 1$. The director $\mathbf{n}$ must fulfil the constraint

$$n_i n_i = 1\,, \tag{1}$$

while the usual incompressibility condition is given by

$$v_{i,i} = 0\,, \tag{2}$$

where $\mathbf{v}$ is the velocity. The suffix i equals 1, 2 or 3 and repeated suffixes are summed from one to three. The equations that arise from the balance law for linear momentum are

$$\rho\dot{v}_i = \rho F_i - \tilde{p}_{,i} + \tilde{g}_j n_{j,i} + G_j n_{j,i} + |\nabla\Phi| a_i J_{j,j} + \tilde{t}_{ij,j}\,, \tag{3}$$

where ρ is the density, a superposed dot represents the material time derivative, F_i is the external body force per unit mass, G_i is the generalised external body force that is related to the external body moment per unit mass (e.g. that which may arise from the application of an external field), $\tilde{p} = p + w_A$ where p is the pressure and w_A is the energy density, and $\mathbf{J}$ is defined by

$$J_i = -\frac{\partial w_A}{\partial\,\Phi_{,i}} + \frac{1}{|\nabla\Phi|}\left[\left(\frac{\partial w_A}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w_A}{\partial a_p}\right](\delta_{pi} - a_p a_i)\,. \tag{4}$$

Despite the full viscous stress $\tilde{t}_{ij}$ having thirteen viscosity coefficients, many simple models require only a few viscosities (see (11) below). The balance of angular momentum leads to the equations

$$\left(\frac{\partial w_A}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w_A}{\partial n_i} + \tilde{g}_i + G_i = \lambda n_i\,, \tag{5}$$

where the scalar function λ is a Lagrange multiplier that arises from the constraint (1) and can usually be eliminated from these equations or evaluated by taking the scalar product of (5) with $\mathbf{n}$. The dynamic term $\tilde{g}_i$ is given by

$$\tilde{g}_i = -(\alpha_3 - \alpha_2)N_i - (\alpha_2 + \alpha_3)A_{ip}n_p - 2\kappa_1 A_{ip}a_p\,, \tag{6}$$

where the rate of strain tensor A and vorticity tensor W are defined in the usual way by

$$A_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i})\,, \qquad W_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i})\,. \tag{7}$$

Following the standard procedure for nematics, the co-rotational time flux $\mathbf{N}$ of the director $\mathbf{n}$ is introduced as $\mathbf{N} = \dot{\mathbf{n}} - \mathsf{W}\mathbf{n}$. The coefficients α_2, α_3 and κ_1 are dynamic viscosity coefficients. The permeation equation is

$$\dot{\Phi} = -\lambda_p J_{i,i}\,, \tag{8}$$

where $\lambda_p \geq 0$ is the permeation coefficient. The stress tensor is given by

$$t_{ij} \;=\; -p\,\delta_{ij} + |\nabla\Phi| a_i J_j - \frac{\partial w_A}{\partial n_{p,j}} n_{p,i} - \frac{\partial w_A}{\partial a_{p,j}} a_{p,i} + \tilde{t}_{ij}\,. \tag{9}$$

An expression for the couple stress tensor is also available [5]. Equations (1), (2), (3), (5) and (8) provide nine equations in the nine unknowns Φ, n_i, v_i, p and λ; the smectic layer normal $\mathbf{a}$ is, of course, determined from the solution for Φ.

An elementary energy density that has been suggested is [5]

$$w_A = \tfrac{1}{2}K_1^n(\nabla\cdot\mathbf{n})^2 + \tfrac{1}{2}K_1^a(\nabla\cdot\mathbf{a})^2 + \tfrac{1}{2}B_0(\,|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2\,)^2 + \tfrac{1}{2}B_1\big\{1 - (\mathbf{n}\cdot\mathbf{a})^2\big\}\,. \tag{10}$$

Here K_1^n and K_1^a are elastic constants related to the distortions of $\mathbf{n}$ and $\mathbf{a}$, respectively, B_0 is a layer compression constant and B_1 is a constant that reflects the strength of the coupling between $\mathbf{n}$ and $\mathbf{a}$. The viscous stress $\tilde{t}_{ij}$ in the simplest model for SmA is

$$\tilde{t}_{ij} = \alpha_4 A_{ij} + \tau_1(a_k A_{kp} a_p) a_i a_j + \tau_2(a_i A_{jp} a_p + a_j A_{ip} a_p)\,, \tag{11}$$

where α_4, τ_1 and τ_2 are dynamic viscosity coefficients. It has been shown in this elementary model that planar layered SmA samples are linearly stable to small periodic disturbances [5]. This is accomplished by setting

$$\mathbf{n} = (\theta(x,z,t), 0, 1)\,, \quad \Phi = z - u(x,z,t)\,, \quad \mathbf{v} = (v_1(x,z,t), 0, v_3(x,z,t))\,, \tag{12}$$

where $|\theta| \ll 1$ represents a small perturbation to the director $\mathbf{n}$, $|u| \ll 1$ represents a displacement of the smectic layers from their initial planar arrangement and $\mathbf{v}$ is the velocity with $|v_1| \ll 1$ and $|v_3| \ll 1$. This represents a perturbation to a static configuration having $\mathbf{a} = \mathbf{n} = \hat{\mathbf{z}}$ when flow is initially absent. When the linearised versions of equations (1) to (8) are employed and the above small perturbations θ, u, v_1 and v_3 are assumed proportional to $\exp\{\omega t + i(q_x x + q_z z)\}$ it follows that $\Re(\omega)$ is always negative for all material parameters and wave numbers q_x and q_z under the assumption that $\tau_1 \geq -(\alpha_4 + \tau_2)$, which indicates linear stability.

An instability arises in the above example when an electric field is applied across a finite sample of SmA material. This occurs once the magnitude of the field increases through a critical threshold value E_c. The onset of an instability can be detected with the aforementioned theory and this leads to a novel analysis of the classical Helfrich–Hurault effect in SmA in which the smectic layers begin to undulate. It can be shown that the critical value E_c of the electric field (at which the onset of the smectic layer undulations occurs) predicted by this theory is lower than that obtained from previous theories [8]: it is known from experiments that previous models for such effects in cholesteric and lamellar liquid crystals have overestimated the experimentally observed E_c. The theory presented here determines a value for E_c that is substantially reduced from that obtained in previous theories and this lower threshold is more consistent with experimental

results. Smectic A liquid crystals are very sensitive to electrically induced disturbances and it is this sensitivity that is of interest to those modeling bio-sensors and related devices.

References

[1] P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, 2nd Edition, Oxford University Press, Oxford, 1993.

[2] I.W. Stewart, *The Static and Dynamic Continuum Theory of Liquid Crystals*, Taylor and Francis, London and New York, 2004.

[3] G.K. Auernhammer, H.R. Brand and H. Pleiner, *The undulation instability in layered systems under shear flow – a simple model*, Rheol. Acta **39** (2000), 215–222.

[4] T. Soddemann, G.K. Auernhammer, H. Guo, B. Dünweg and K. Kremer, *Shear-induced undulation of smectic-A: Molecular dynamics simulations vs. analytical theory*, Eur. Phys. J. E **13** (2004), 141–151.

[5] I.W. Stewart, *Dynamic theory for smectic A liquid crystals*, Continuum Mech. Thermodyn. **18** (2007), 343–360.

[6] G. Ahmadi, *A Continuum Theory of Smectic A Liquid Crystals*, J. Rheol. **26** (1982), 535–556.

[7] W. E, *Nonlinear Continuum Theory of Smectic-A Liquid Crystals*, Arch. Rat. Mech. Anal. **137** (1997), 159–175.

[8] F. Stewart and I.W. Stewart, *A Novel Method for Measuring Compression Constants in Smectics*, Mol. Cryst. Liq. Cryst. **478** (2007), 23–32.

Modeling active contractions in soft living tissues

Luciano Teresi

(joint work with Paola Nardinocchi)

Key points to be found in the biomechanical literature about muscles modeling are the following: at the macroscopic level, the presence of muscle fiber enters the model through the tension generated by the fiber itself (the active stress); in addition, this tension has a preferred direction, which is defined by the orientation of the fiber. When activable soft tissues are considered as a whole, the overall tension state is described by adding up the passive and active stress [1].

Here, we introduce a novel point of view to modeling the mechanics of muscle contraction borrowed from [2], in which the notion of active deformation is presented and discussed. We assume that the contraction experienced by a muscle fiber under stimulus is described at the macroscopic scale by a change in the rest length of the fiber, a change that we call active deformation. As is standard for elasticity, the stress state in a body is due to the difference between its actual configuration and the rest state. In our model, the rest state is active, in other words, can vary in time, driven by electrophysiological variables.

We regard a muscular tissue as a smooth region $\mathcal{B}$ of the three-dimensional Euclidean space $\mathcal{E}$; a placement is a map $p : \mathcal{B} \times T \to \mathcal{E}$, and we denote with $\mathbf{F} = \nabla p$ the (visible) deformation gradient. At the macroscopic level, we describe the activation of the muscle fibers of the tissue as a variation in the rest length of the body elements measured by a distortion $\mathbf{F}_o$, a tensor field on $\mathcal{B} \times T$ to be

known as *active deformation*. Moreover, the distribution of muscle fibers enters the model by assigning specific active deformation fields.

Let us present a simple example. We consider isotropic distribution of the muscle fibers represented by $\mathbf{F}_o = \gamma_o \mathbf{I}$, with $\mathbf{I}$ the identity tensor; thus, it is the scalar field $\gamma_o : \mathcal{B} \times T \to \mathcal{R}$ that must be related through suitable kinetic laws to the changes in intracellular calcium concentration.

The elastic deformation $\mathbf{F}_e$ of the body elements turns out to be the difference between the active and the visible deformation, defined through the multiplicative decomposition:

$$\mathbf{F}_e = \mathbf{F}\mathbf{F}_o^{-1}\,; \tag{1}$$

the corresponding elastic strain measure $\mathbf{E}_e = \frac{1}{2}(\mathbf{F}_e^T\mathbf{F}_e - \mathbf{I})$ is introduced. The material response of the body is described through an isotropic elasticity tensor $\mathbf{C}$, relating linearly the strain $\mathbf{E}_e$ to the corresponding stress measure $\mathbf{S}$ in such a way to realize an isotropic material response: $\mathbf{S} = \mathbf{C}[\mathbf{E}_e]$. Granted for the multiplicative decomposition (1), the active deformation $\mathbf{F}_o$ and the active strain $\mathbf{E}_o = \frac{1}{2}(\mathbf{F}_o^T\mathbf{F}_o - \mathbf{I})$ enter the material response as

$$\mathbf{S} = \mathbf{C}[\mathbf{F}_o^{-T}(\mathbf{E} - \mathbf{E}_o)\mathbf{F}_o^{-1}]\,. \tag{2}$$

with $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$. The balance equations of mechanics are naturally written in integral form on the body $\mathcal{B}$ in terms of the reference stress measure $\mathbf{S}_R = \mathbf{F}_e\,\mathbf{S}\,\mathbf{F}_o^*$; denoted with $\tilde{\mathbf{u}}$ the test velocity, we write

$$0 = \int_{\mathcal{B}} -\mathbf{S}_R \cdot \nabla\tilde{\mathbf{u}} + \int_{\partial_o\mathcal{B}} \mathbf{t}\cdot\tilde{\mathbf{u}}\,, \tag{3}$$

with $\mathbf{t}$ a traction field acting on the part $\partial_o\mathcal{B}$ of the boundary. Standard localization arguments turns out the balance equations of forces:

$$\operatorname{div}\mathbf{S}_R = 0\,, \quad \text{in} \quad \mathcal{B}\,; \quad \mathbf{S}_R\mathbf{n} = \mathbf{t}\,, \quad \text{on} \quad \partial_o\mathcal{B}\,, \tag{4}$$

with $\mathbf{n}$ the unit normal field to $\partial_o\mathcal{B}$.

The approach of active contraction has been recently applied to model the activation-contraction mechanism in cardiac muscles [3].

References

[1] F.B., Sachse (2004): *Computational Cardiology.* Springer-Verlag, Berlin Heidelberg.

[2] P., Nardinocchi, L., Teresi (2007): On the active response of soft living tissues. J. Elasticity 88, 27–39.

[3] C. Cherubini, S. Filippi, P. Nardinocchi, L.Teresi: An Electromechanical Model of Cardiac Tissue: Constitutive Issues and Electrophysiological Effects, Progress on Biophysics & Molecular Biology, (2008) doi:10.1016/ j.pbiomolbio.2008.02.001.

Fully coupled modeling and numerical implementation of electromagnetic forming processes in solids

Jesse D. Thomas
(joint work with Nicolas Triantafyllidis)

Electromagnetic forming (EMF) is a cost-effective and flexible manufacturing technique for sheet metal forming. It consists of connecting an actuator (typically a copper wire solenoid) to a high energy capacitor equipped with fast action switches. When the capacitor is discharged, the large transient current that goes through the actuator generates by induction strong eddy currents in the nearby metallic workpiece. The presence of these induced currents, inside the magnetic field generated by the currents of the actuator, results in Lorentz body forces in the workpiece which are responsible for its plastic deformation. The EMF techniques are popular in the aerospace and automotive industries because of several advantages they hold over conventional forming techniques. These advantages are: process repeatability and flexibility (due to its electric nature, energy input can be easily and accurately adjusted), low cost single side tooling (thus reducing need for lubrication and tool marks) and high speed (typical process duration is on the order of 50 μsec). The most important advantage – and the main reason for the recent interest in EMF – is the resulting significant increase in ductility observed in certain metals, with aluminum featuring preeminently among them.

There are two methods for deriving the fully coupled governing equations and interface conditions of an electromagnetic-thermal-mechanical process such as electromagnetic forming. First is the direct method, which uses conservation principles in the current configuration to derive the governing equations and boundary conditions. This approach essentially follows Kovetz [1], where the interested reader is addressed for additional details. Subsequently, kinematic relations from continuum mechanics are applied to transform these conservation principles from the current to the reference configuration, which is similar to the work of Lax and Nelson [2]. This provides the relations between current and reference configuration fields, and with these relations one may employ the second, variational method, namely the classical least-action principle [3] in the reference configuration. This method gives the reference configuration governing equations and interface conditions, and the two methods agree. The second method has been discussed by others, for the electromagnetic equations by Lax and Nelson [2] and for the electromagnetic and mechanical equations by Trimarco and Maugin [4, 5].

For the EMF processes of interest, the problem formulation may be simplified considerably by applying the eddy current approximation. Detailed discussion of this approximate formulation may be found in the literature, e.g. in Hiptmair and Ostrowski [6] and references cited therein. The aspects typical of EMF processes that make this simplification appropriate are: the material velocities are much less than the speed of light, the effective electric current frequencies are on the order of 10 kHz, the geometry is on the order of 1 cm, and the material electrical conductivities are large. The eddy current approximation is a result of neglecting

electric energy, and imposing the eddy current approximation implies displacement currents are neglected, which means volumetric charges are not accounted for and charge conservation must be imposed separately from the variational principle. In addition, for simplicity one may restrict the numerical implementation of the general theory to axisymmetric processes. Implementing the aforementioned theoretical formulation for axisymmetric problems significantly simplifies the resulting formulation by reducing the independent variables.

Our numerical implementation of the general theory employs a variational integration approach [7, 8]. According to this method space and time interpolation schemes are concurrently applied to the Lagrangian, followed by the variational principle applied on the discrete nodal variables for each time step. This provides the time stepping routine to solve for the degrees of freedom at each time step.

With the numerical implementation of the general theory as discussed above, it remains only to provide the constitutive behavior to complete the simulation. The electromagnetic constitutive response for both the coil and workpiece is an isotropic Ohm's law with constant conductivity. The mechanical constitutive response for the deforming workpiece is taken as one of two materials. One is a compressible Mooney-Rivlin hyperelastic material, where the material parameters are chosen to resemble, but not match, the response of the actual workpiece material. The second is a hyperelastic formulation based on J_2 deformation theory of plasticity (see Abeyaratne and Triantafyllidis [9] and references cited therein), where the material parameters match the actual strain hardening behavior of the workpiece material under loading.

The complete electromagnetic forming simulation can then be verified with two types of simulations. First is a comparison with a one dimensional semi-analytical ring expansion simulation, similar to that in Triantafyllidis and Waldenmyer [10], using the Mooney-Rivlin material. Convergence of the two dimensional (axisymmetric) solution is shown, along with agreement between the one and and two dimensional simulations. Second, two representative tube expansion experiments are modeled, again using the Mooney-Rivlin material. The two experiments are: a tube much taller than the forming coil and a tube with height equal to that of the forming coil. These simulations show qualitatively the deformed tube shapes observed in corresponding experiments from Thomas et al. [11].

The final result is a comparison with an experiment from Thomas et al. [11] whereby a tube of equal height with a coil is expanded. The mechanical constitutive response is the hyperelastic formulation that matches the actual strain hardening behavior of the tube material. In the experiments tubes were deformed up to the onset of necking or failure, and strains were measured after deformation far from necking or failure. Comparing these strains to those at max deformation in the simulation shows good agreement.

References

[1] A. Kovetz. *Electromagnetic Theory*, Oxford, UK: Oxford University Press; 2000.
[2] M. Lax and D.F. Nelson, *Maxwell equations in material form*, Physical Review B, **13** (1976), 1777–1784.

[3] B. Lazzari and R. Nibbi, *Variational principles in electromagnetism*, IMA Journal of Applied Mathematics, **65** (2000), 45–95.
[4] C. Trimarco and G.A. Maugin. Material mechanics of electromagnetic solids, In: *Configurational Mechanics of Materials. CISM Courses and Lectures, no. 427.* Wien, NY: Springer-Verlag; 2001:129–171.
[5] C. Trimarco, *Material electromagnetic fields and material forces*, Archive of Applied Mechanics, **77** (2007), 177–184.
[6] R. Hiptmair and J. Ostrowski, *Coupled boundary-element scheme for eddy-current computation*, Journal of Engineering Mathematics, **51** (2005), 231–250.
[7] C. Kane, J.E. Marsden, and M. Ortiz, *Symplectic-energy-momentum preserving variational integrators*, Journal of Mathematical Physics, **40** (1999), 3353–3371.
[8] A. Sanyal, J. Shen, and N.H. McClamroch. Variational integrators for mechanical systems with configuration dependent inertia. 2005. Preprint, available at `http://math.asu.edu/∼sanyal/research/dELmech_pap.pdf`
[9] R. Abeyaratne and N. Triantafyllidis, *On the emergence of shear bands in plane strain*, International Journal of Solids and Structures, **17** (1981), 1113–1134.
[10] N. Triantafyllidis and J. Waldenmyer, *Onset of necking in electro-magnetically formed rings*, Journal of the Mechanics and Physics of Solids, **52** (2004), 2127–2148.
[11] J.D. Thomas, M. Seth, G.S. Daehn, J.R. Bradley, N. Triantafyllidis, *Forming limits for electromagnetically expanded aluminum alloy tubes: Theory and experiment*, Acta Materialia, **55** (2007), 2863–2873.

On magnetorheological elastomers: general theory and application to the stability of a rectangular block under a transverse magnetic field

Nicolas Triantafyllidis
(joint work with Sundeep Kankanala)

Magnetorheological elastomers (MREs) are ferromagnetic particle impregnated rubbers whose mechanical properties are altered by the application of external magnetic fields. Due to their strong magnetoelastic coupling response MREs are finding an increasing number of engineering applications in aerospace, automotive, civil and electrical engineering fields as vibration damping devices, variable stiffness mounts and so on.In the general theory part of the presentation we give the continuum mechanics formulation for these solids using two different approaches: a) a direct approach formulated in the current configuration and based on the thermodynamics approach of [1] and b) a variational approach formulated in the reference configuration and based on energy. The main advantage of the direct approach presented is that no *a priori* assumptions are made for the expressions of electromagnetic body forces and electromagnetic part of the stress. Unlike in earlier works, an energy minimizer is used in the latter energetic approach to derive Ampère's law, equilibrium equation and traction condition. It is also shown that both approaches result in the same governing equations and boundary conditions. In discussing the properties of the solid's free energy we pay particular attention to the quasiconvexity of the potential energy and derive sufficient, for the quasiconvexity, pointwise conditions on the free energy.

The proposed theory can be easily fine tuned to account for anisotropic MREs, as is the case of MREs that are cured in the presence of strong magnetic fields.

Moreover, the proposed new variational formulation intended to be used in numerical calculations of application devices made of MREs, where its Lagrangian formulation and energy minimization features plus the absence of constraints in the variables used, make for an efficient numerical algorithm. In addition,the proposed theory, because of its energy minimization feature, is ideally suited for the study of stability problems in MREs, as in the application presented here. A detailed account of the theoretical part of the presentation can be found in [2]. In the application part of the talk we address the stability of a rectangular block subjected to a uniform magnetic field in the direction of its minor (x) axis. The two y=const. faces of the block are frictionless and kept parallel to each other. This boundary value problem is motivated by the classical problem of magnetoelastic buckling of a thin beam. The benefits of the continuum approach over traditionally employed structural models are in (a) the ability to assess effect of different nonlinear material responses and (b) the validity of the formulation for a wide range of block aspect ratios.

Critical magnetic fields, i.e. those corresponding to the onset of a bifurcation buckling, in the form of symmetric and anti-symmetric modes, are obtained for three different constitutive laws. In general, the critical magnetic field is shown to increase monotonically with the block's aspect ratio for each material and mode type. For most aspect ratios, antisymmetric modes are always the critical buckling modes for stress saturated and neo-hookean materials. In the narrow range of moderate aspect ratios (about 0.25) symmetric modes become critical. For strain saturated solids no buckling is possible above a maximum aspect ratio. As expected, the results for stubby blocks are found to be very sensitive to the nonlinearity of the governing constitutive laws.

Furthermore, an asymptotic solution is obtained for slender beams that shows a linear relationship between the critical buckling field and the block's slenderness ratio. This result is found to agree with the formula obtained from structural models. A detailed account of the application part of the presentation can be found in [3].

The general methodology described here covers only the onset of magnetoelastic instability (analogous to the works of, for example, [4] for hyperelastic materials). Of particular interest would be the study of the post-buckling behavior in magnetoelastic solids corresponding to the recent article [5], also for hyperelastic materials.

References

[1] B. D. Coleman and W. Noll *The Thermodynamics of Elastic Materials with Heat conduction and Viscosity*,Archive for Rational Mechanics and Analysis, **13**, (1963) 167–178.

[2] S. Kankanala and N. Triantafyllidis *On Finitely Strained Magneto-Rheological Elastomers*, Journal of the Mechanics and Physics of Solids, **52** (2004), 2869–2908.

[3] S. Kankanala and N. Triantafyllidis *Magnetoelastic Buckling of a Rectangular Block in Plane Strain*, Journal of the Mechanics and Physics of Solids, **56** (2008), 1147–1169.

[4] R. Ogden *Non-linear Elastic Deformations*Chichester: Ellis Horwood; 1984

[5] N. Triantafyllidis, W. H. Scherzinger and H. J. Huang *Post-Bifurcation Equilibria in the Plane Strain Test for a Hyperelastic Rectangular Block*, International Journal of Solids and Structures, **44** (2007), 3700–3719.

Magnetic and electric field responsive soft and smart materials

Miklós Zrínyi

Many useful engineering materials, as well as living organisms have a heterogeneous composition. The components of composite materials often have contradictory, but complementary properties. Fillers are usually solid additives that are incorporated into the polymer to modify the physical properties. Fillers can be divided into three categories: those that reinforce the polymer system and improve its mechanical performance, those that are used to take-up space, and thus reduce material cost. The third, less common category is when filler particles are incorporated into the material to improve its responsive properties. Composite materials consisting of rather rigid polymeric matrices filled with magnetic particles are long time known and called magnetic elastomers or magnetoelasts. These materials are successfully used as permanent magnets, magnetic cores, connecting and fixing elements in many areas. These traditional magnetic elastomers have low flexibility and practically do not change their size, shape and elastic properties in the presence of external magnetic field.

The new generation of magnetic gels and elastomers represent a new type of composites, consisting of small (mainly nano-sized) magnetic particles dispersed in a high elastic polymeric matrix. The particles couple the shape of the elastomer to the external magnetic fields. Since the particles cannot leave the polymer matrix, so that all of the forces acting on the particles are transmitted directly to the polymer chains resulting in either locomotion or deformation. Shape distortion occurs instantaneously and disappears abruptly when external fields are applied or removed, respectively. Combination of magnetic and elastic properties leads to a number of striking phenomena that are exhibited in response to impressed magnetic fields. Giant deformational effect, tunable elastic modulus, non-homogeneous deformation and quick response to magnetic field (1) open new opportunities for using such materials for various applications.

FIGURE 1. Snapshots of shape change of a magnetite load PDMS elastomer due to modulated magnetic field. The frequency of the fiedl is 40Hz.

Synthesis of elastomers in uniform magnetic field can be used to prepare anisotropic samples. In uniform field there are no attractive or repulsive field-particle interactions therefore particle-particle interactions become dominant. In monomer solutions, containing dispersed magnetic particles, the imposed field orients the

magnetic dipoles. If the particles are spaced closely enough, so that their field can reach their neighbors, mutual particle interactions present. These interactions can be very strong leading to significant change in the structure of the particle ensemble. The particles attract each other when aligned end to end, and repel each other when placed side by side. Due to the attractive forces pearl chain structure develops. The ordered structure can be fixed by cross-linking polymerisation. The anisotropy manifest itself in both direction dependent elastic modulus as well as direction dependent swelling (Fig. 2).

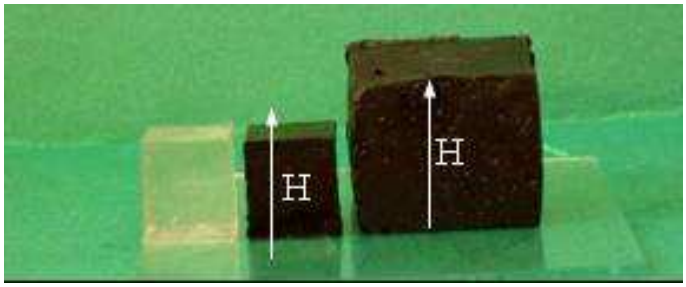

FIGURE 2. Anisotropic swelling behaviour as seen by the naked eye. The arrow indicates the direction of the magnetic field during the preparation.

The magnetic field responsive elastomers have shown a change in compressive modulus under uniform magnetic field. Depending on the orientation of particle chains in the network and that of the magnetic field, the induced temporary reinforcement may exceed one order of magnitude. These results suggest that magnetic field responsive gels and elastomers have several potential applications as tuned vibration absorbers, stiffness tunable mounts and suspensions. Since the magnetic fields are convenient stimuli from the point of signal control, therefore it is of great importance to develop and study such soft and flexible magnetic systems. Quincke rotation is the rotation of non-conducting objects immersed in liquid dielectrics and subjected to a strong homogeneous DC electric field. The rotation is spontaneous when the field exceeds a threshold value. Wide range of applications (e.g. microscopic motor) motivates researchers to find materials with micro-fabrication possibilities. Polymer composites that fulfill these requirements have been developed for the first time. Electro-rotation of disk shaped polymer composites is studied as a function of electric field intensity. Magnetic and electric field induced deformation, locomotion and rotation, as well as on/off switching control of magnetic polymeric membranes will be the subject of the oral presentation.

REFERENCES

[1] M. Zrínyi, L. Barsi, D. Szabó,H.-G.Kilian, *Direct observation of abrupt shape transition in ferrogels induced by nonuniform magnetic field*, J.Chem Phys, **108** (1997) 5685–5692.

[2] D. Szabó, G.Szeghy and M.Zrínyi, *Shape transition of magnetic field sensitive polymer gels Macromolecules*, **31** (1998) 6541–6548.

[3] M. Zrínyi, D. Szabó and L. Barsi, *Magnetic Field Sensitive Polymeric* in Actuators Polymer Sensors and Actuators, Y. Osada, D. E. Rossi (ed.), Springer Verlag Berlin (1999) 385–408
[4] M. Zrínyi, D. Szabóó, G. Filipcsei and J. Fehér, *Electric and Magnetic Field -Sensitive Smart Polymer Gels* in Polymer Gels and Networks, Y. Osada, A. Khokhlov (ed.), Marcel Dekker Inc., New York (2001) 309–355.
[5] G. Filipcsei, I. Csetneki, A. Szilágyi and M. Zrínyi, *Magnetic Field-Responsive Smart Polymer Composites*, Advances in Polymer Science Springer **206** (2007) 137–189.

Reporter: Giuseppe Saccomandi

Participants

Prof. Dr. Gal deBotton
Dept. of Mechanical Engineering
Ben-Gurion University of the Negev
P.O.Box 653
84105 Beer-Sheva
ISRAEL

Prof. Dr. Antonio DeSimone
SISSA
International School for Advanced Studies
Via Beirut n. 2-4
I-34014 Trieste

Prof. Dr. Luis Dorfmann
Department of Civil & Environmental Engineering
Tufts University
Anderson Hall
Medford , MA 02155
USA

Prof. Dr. Gerard A. Maugin
Universite Pierre et Marie Curie
Institut Jean le Rond d'Alembert
CNRS, Tour 55-65
4, Place Jussieu
F-75252 Paris Cedex 05

Prof. Dr. Robert McMeeking
Department of Mechanical Engineering
University of California
Santa Barbara CA 93106-5070
USA

Dr. Andrea Nobili
Dep. Meachnical and Civil Engineering
University of Modena and Reggio Emilia
Via Vignolese 904
I-41100 Modena

Prof. Dr. Ray W. Ogden
Department of Mathematics
University of Glasgow
University Gardens
GB-Glasgow G12 8QW

Dr. Giuseppe Puglisi
Dipartimento di Ingegneria ed Ambientale
Politecnico di Bari
Via Orabona, 4
I-70125 Bari

Prof. Dr. Michael Ruzicka
Abteilung f. Angewandte Mathematik
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Dr. Giuseppe Saccomandi
Dipartimento di Ingegneria Industriale
Universita degli Studi di Perugia
Via G. Duranti, 1
I-06125 Perugia

Prof. Dr. Iain W. Stewart
Department of Mathematics
University of Strathclyde
Livingstone Tower
26, Richmond Street
GB-Glasgow G1 1XH

Dr. Luciano Teresi
Dept. Studies on Structures
Universita di Roma Tre
Via Corrado Segre, 6
I-00146 Roma

Jesse Thomas
Department of Aerospace Engineering
The University of Michigan
Aerospace Engineering Building
Ann Arbor , MI 48109-2140
USA

Prof. Dr. Nicolas Triantafyllidis
Department of Aerospace Engineering
The University of Michigan
Aerospace Engineering Building
Ann Arbor , MI 48109-2140
USA

Prof. Dr. Miklos Zrinyi
Department of Physical Chemistry
Budapest University of Technology and Economics
Laboratory of Soft Matters
H-1521 Budapest