Mathematisches Forschungsinstitut Oberwolfach

Report No. 17/2008

Arbeitsgemeinschaft: Julia Sets of Positive Measure

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March 30th – April 5th, 2008

Abstract. In this Arbeitsgemeinschaft, we presented the proof that there exist quadratic polynomials having a non-linearizable fixed point and a Julia set of positive Lebesgue measure.

Mathematics Subject Classification (2000): 37Fxx.

Introduction by the Organisers

A polynomial $P : \mathbb{C} \to \mathbb{C}$ can be considered as a dynamical system. We are interested in the sequences (z_n) defined by induction:

$$z_0 \in \mathbb{C} \quad \text{and} \quad z_{n+1} = P(z_n).$$

The filled-in Julia set K_P is the set of points $z_0 \in \mathbb{C}$ for which the sequence (z_n) is bounded. This set is compact. The Julia set J_P is the boundary of K_P. In particular, it has empty interior.

There is a small collection of polynomials, for instance

$$P(z) = z^d \quad , \quad P(z) = z^2 - 2,$$

for which the Julia set can be fairly easily understood, but most exhibit "fractal" geometry and "chaotic" behavior, the analysis of which requires serious tools from complex analysis, dynamical systems, topology, combinatorics, ...

This subject has a fairly long history, with contributions by Koenigs, Schröder, Böttcher in the late 19th century, and the great memoirs of Fatou and Julia around 1920.

There followed a dormant period, with notable contributions by Cremer (1936) and Siegel (1942), and a rebirth in the 1960's (Brolin, Guckenheimer, Jakobson). Since the early 1980's, partly under the impetus of computer graphics, the subject has grown vigorously, with major contributions by Douady, Hubbard, Sullivan, Thurston, and more recently Lyubich, McMullen, Milnor, Shishikura, Yoccoz ...

Fatou found sufficient conditions for the boundary of the basin of an attracting fixed point to be a Cantor set with Lebesgue measure equal to 0. He could not tell whether or not the measure could be positive.

For some time and until the 1990's, the conjecture, reinforced by the analogy with Ahlfors's conjecture on the area of limit sets of Kleinian groups, was that no Julia set of a polynomial could have positive area.

Results in this direction were obtained by Douady and Hubbard in the case of hyperbolic or subhyperbolic maps, by Branner, Hubbard and McMullen in the case of non-renormalizable cubic polynomials with an escaping critical point, by Lyubich and Shishikura in the case of finitely renormalizable quadratic polynomials without indifferent cycles, by Petersen in the case of quadratic polynomials having a Siegel disk with bounded type rotation number.

In the 1990's, Douady began to catch a glimpse of a method for Julia sets of positive area: in the family of degree 2 polynomials with an indifferent Cremer fixed point. Recently, we brought Douady's method to completion.

The Arbeitsgemeinschaft *Julia sets of positive measure* focused on the proof of existence of quadratic polynomials having a Julia set of positive area. It was held March 30th–April 5th, 2008. It was attended by 36 participants.

Arbeitsgemeinschaft: Julia Sets of Positive Measure

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Abstracts

Talk 1—Fatou and Julia sets

Michèle Audin

Motivation: in order to find the roots of a polynomial $P \in \mathbb{C}[z]$, one may study the iterates of the rational map:

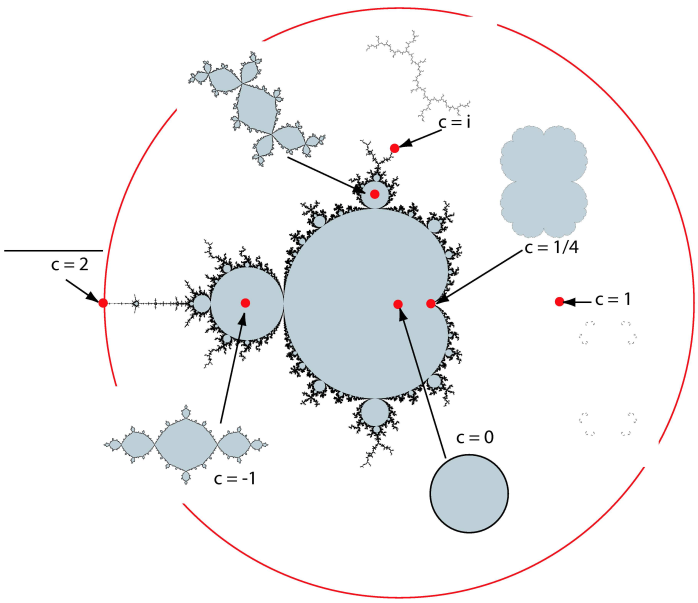
$$N_P(z) = z - \frac{P(z)}{P'(z)}.$$

Main tool: Montel's theorem on normal families (1903) gives a criteria for a subset of the space of holomorphic functions to be compact.
Definition: if $R \in \mathbb{C}(z)$, the Fatou set $F(R)$ is the set of points in a neighborhood of which the sequence of iterates $R^{\circ n}$ is normal; the Julia set $J(R)$ is the complement of the Fatou set. See [1] and [2].

I presented the first properties of these subspaces, examples and pictures (I showed the first Julia set ever hand-drawn, from a draft of Gaston Julia). Then, I showed that the Julia set contains the repelling and parabolic periodic points while the Fatou set contains the attracting ones. In the case of polynomials, I showed that

- either the critical orbits are bounded and $J(R)$ is connected,
- or at least one critical orbit goes to infinity and $J(R)$ has uncountably many connected components.

In the second case, if the polynomial has degree 2, then $J(R)$ is a Cantor set.



I then described the moduli space of polynomials of degree 2. Such a polynomial is conjugate to a polynomial z^2+c with $c \in \mathbb{C}$, and this c is unique. It is conjugate to a polynomial $\lambda z + z^2$ with $\lambda \in \mathbb{C}$, but this λ is not unique. The correspondence between λ and c is given by the relation $c = \lambda/2 - \lambda^2/4$. The Mandelbrot sets of c (or λ) is the set of parameters for which the Julia set is connected.

I then showed a few examples to illustrate some properties of the Julia sets according to the position of c in or out of the Mandelbrot set.

References

[1] N. Steinmetz, *Rational iteration*, de Gruyter Studies in Mathematics, Vol. 16, Walter de Gruyter & Co, Berlin, 1993, Complex analytic dynamical systems.

[2] J. Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, Introductory lectures.

Talk 2—Periodic Fatou components

Norbert Schappacher

Let $f \in \mathbb{C}[z]$. According to Sullivan's *non-wandering domain* theorem, every connected component of the Fatou set of f is eventually periodic under iteration of the map f. Studying periodic components easily reduces to studying *fixed* ones: U is a connected component of the Fatou set of f and $f(U) = U$.

It turns out that there are precisely 3 types of such U:

- immediate attracting basin of an attracting fixed point $\hat{p}$ of f,
- immediate basin of an attracting petal of a parabolic fixed point and
- Siegel disks.

These three types of components were first presented via their normal forms (after suitable change of coordinates). Then, the proof of this classification was indicated, passing through the classification of all possible holomorphic maps $f : S \to S$, for a hyperbolic Riemann surface S. This latter classification theorem is proved as Theorem 5.2 in [1]; its application to the Fatou components (whose types are presented in the chapters preceding this application) is given in Chapter 16 of [1].

References

[1] J. Milnor, *Dynamics in One Complex Variable: Third Edition*, Annals of Mathematical Studies 160, Princeton University Press (2006).

Talk 3—Does the Julia set depend continuously on the polynomial?

Federico Coglitore

In this talk we report on Douady's results about continuity of the Julia set and the filled-in Julia set with respect to the coefficients of the polynomial [1].

1. Preliminaries

For a complex polynomial f of degree $d \geq 2$, viewed as a map $f : \mathbb{C} \to \mathbb{C}$, we denote by $K(f)$ the *filled-in Julia set* and by $J(f)$ the *Julia set* of f.

Considering a sequence $\{f_n\}_{n\in\mathbb{N}}$ of complex polynomials of degree d, $f_n \longrightarrow f$, a natural question is whether (and in which sense)

$$K(f_n) \overset{?}{\longrightarrow} K(f) \qquad \text{or} \qquad J(f_n) \overset{?}{\longrightarrow} J(f).$$

First, to give a precise formulation to the problem, we define the *Hausdorff metric* on the set $\mathrm{Comp}^*(\mathbb{C})$ of non-empty compact subsets of $\mathbb{C}$. For $X, Y \in \mathrm{Comp}^*(\mathbb{C})$ we say that X *is contained in* Y *up to* r if $\mathrm{dist}\,(x, Y) \leq r$ for any $x \in X$, and we denote by $\partial(X, Y)$ the smallest r such that $X \subset Y$ up to r. Then, we define the Hausdorff distance on $\mathrm{Comp}^*(\mathbb{C})$ as follows.

Definition (Hausdorff distance)**.**

$$d_H(X, Y) := \max\bigl\{\partial(X, Y), \partial(Y, X)\bigr\}.$$

Since the Hausdorff distance is defined as the maximum of two semi-distances, continuity for map into $\mathrm{Comp}^*(\mathbb{C})$ can be decomposed into *upper* and *lower semi-continuity.*

If Λ is a topological space and φ is a map $\varphi : \Lambda \longrightarrow \mathrm{Comp}^*(\mathbb{C})$, $\varphi : \lambda \longmapsto X(\lambda)$, we consider the following notions:

Definition (Upper semi-continuity)**.**
φ is said to be upper semi-continuous at $\lambda \in \Lambda$ if

$$\forall \{\lambda_n\} \subset \Lambda \ s.t.\ \lambda_n \to \lambda, \quad \partial\bigl(X(\lambda_n), X(\lambda)\bigr) \to 0.$$

Definition (Lower semi-continuity)**.**
φ is said to be lower semi-continuous at $\lambda \in \Lambda$ if

$$\forall \{\lambda_n\} \subset \Lambda \ s.t.\ \lambda_n \to \lambda, \quad \partial\bigl(X(\lambda), X(\lambda_n)\bigr) \to 0.$$

As in the usual case of real function, a function is continuous at λ if and only if it is both upper and lower semi-continuous there.

2. Results

Let $\mathcal{P}_d$ denote the set of complex polynomials of degree d, identified with the metric space $\mathbb{C}^\star \times \mathbb{C}^d$. The following two results hold:

Theorem (Douady)**.**

(A) *The map $f \mapsto K(f)$ from $\mathcal{P}_d$ to* $\mathrm{Comp}^*(\mathbb{C})$ *is upper semi-continuous.*
(B) *The map $f \mapsto J(f)$ from $\mathcal{P}_d$ to* $\mathrm{Comp}^*(\mathbb{C})$ *is lower semi-continuous.*

As a corollary, joining (A) and (B), one obtains that if f_0 is a polynomial such that $\mathring{K}(f_0) = \emptyset$ (that is $J(f_0) = K(f_0)$), then the two maps $f \mapsto K(f)$ and $f \mapsto J(f)$ are continuous at f_0.

By Sullivan's non-wandering theorem (see Talk n. 11) and former results by Fatou, we know that $\mathring{K}(f_0) \neq \emptyset$ if and only if f_0 has attracting cycles, parabolic cycles, or Siegel disks.

Douady has also examined the effect on continuity of this three types of cycles, proving that attracting cycles cause no discontinuity, Siegel disks cause discontinuity for $f \mapsto J(f)$ but not for $f \mapsto K(f)$.

The discontinuity at polynomials having parabolic cycles is not treated in this talk and will be proved later (Talk n. 14).

References

[1] A. Douady, *Does a Julia set depend continuously on the Polynomial?*, Complex dynamical systems, Proc. Sympos. Appl. Math., vol. **49**, Amer. Math. Soc., Providence, RI, (1994), 91–138.

Talk 4—The dynamics is controlled by the behaviour of critical points

Jörn Peter

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 2$. A *critical point* of f is a point $z \in \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with $f'(z) = 0$ (where $f'(\infty) = \frac{d}{dz}|_{z=0} \frac{1}{f(1/z)}$). By $\text{sing}(f^{-1})$, we denote the set of *critical values* of f, which are the images of the critical points. $\text{sing}(f^{-1})$ is exactly the set of points w where at least one branch of f^{-1} cannot be defined near w.

We show that critical values (and therefore critical points) play an important role in the dynamics of f:

(1) Every attracting basin contains a critical point and
(2) The boundary of a Siegel disc is contained in the postcritical set $P(f)$, which is defined as

$$P(f) := \overline{\bigcup_{n=1}^{\infty} \text{sing}(f^n)^{-1}}.$$

Both results can be found in [1]. For the proof of (1), we will need Kœnigs' Theorem:

Theorem 1 (Kœnigs)**.** *Let* $z_0 \in \mathbb{C}$ *be a fixed point of* f *such that* $\lambda := f'(z_0)$ *satisfies* $0 \neq |\lambda| < 1$. *Then there exist open neighborhoods* U *of 0 and* V *of* z_0 *and a function* $S : U(0) \to V(z_0)$ *such that*

$$f(S(z)) = S(\lambda z) \text{ for all } z \text{ s.t. } z, \lambda z \in U.$$

S is uniquely determined by the condition $S(0) = z_0, S'(0) = 1$.

We now show that every attracting basin contains a critical point (the attracting basin $A(z_0)$ of an attracting fixed point z_0 is the set of all points z s.t. $f^n(z) \to z_0$ as $n \to \infty$).

Theorem 2. *Let z_0 be an attracting fixed point of f, i.e. $\lambda := f'(z_0)$ satisfies $|\lambda| < 1$. Then $A(z_0)$ contains a critical point of f.*

Proof. Without loss of generality, we may assume that $z_0 = 0$ and $0 < |\lambda| < 1$ (the case where $\lambda = 0$ is trivial). Let S be as in Kœnigs' Theorem and

$$R := \sup\{r > 0 | S \text{ has an analytic continuitation to } D(0, r)\}.$$

Because

$$f^n(S(z)) = S(\lambda^n z) \to 0 \text{ as } n \to \infty,$$

we have $S(D(0, R)) \subseteq A(0)$. It follows by Liouville's Theorem that $R < \infty$, because S is non-constant. So there exists v such that $|v| = R$ and S can not be continued analytically along $[0, v]$. Let φ be the branch of f^{-1}, defined on a neighborhood of 0, such that $\varphi(0) = 0$. It follows that

$$\varphi(S(\lambda z)) = S(z)$$

on a neighborhood of 0, which implies that $z \mapsto \varphi(S(\lambda z))$ cannot be continued analytically along $[0, v]$. But the function $z \mapsto S(\lambda z)$ can be continued analytically along $[0, v]$ (because $|\lambda| < 1$), which means that $w := S(\lambda v) \in \text{sing}(f^{-1})$. Hence $A(z_0)$ contains a critical value of f, and since $A(z_0)$ is completely invariant, it also contains a critical point of f. □

It can also be shown that every parabolic basin contains a critical point, which implies that f has at most $d - 1$ attracting or parabolic cycles. This result was improved by Douady: He showed that f has at most $d - 1$ non-repelling cycles.
Now we show that the postcritical set always contains the boundary of a Siegel disc:

Theorem 3. *If f has a Siegel disc D, then $\partial D \subseteq P(f)$.*

Proof. Suppose that this is not the case. Then we can find $z_0 \in \partial D$ and a small disc $U := D(z_0, R)$ such that all branches of the inverses of all iterates can be defined on U. We consider the branches φ_n of $(f^n)^{-1}$ that map $D \cap U$ into D. Because φ_n is conjugate to an irrational rotation on $D \cap U$, we can find a subsequence (φ_{n_k}) that converges to the identity on $D \cap U$. But by Montel's Theorem, $\{\varphi_n\}$ is a normal family on U, which implies that (φ_{n_k}) actually converges to the identity on all of U. If we choose $r < R$, then $\varphi_{n_k}(D(z_0, r))$ contains $D(z_0, r - \varepsilon)$ if k is large enough. It follows that

$$f^{n_k}(D(z_0, r - \varepsilon)) \subseteq D(z_0, r)$$

if k is large enough, which implies that $\{f^{n_k}\}$ is normal in a neighborhood of z_0. Hence $\{f^{n_k}\}$ is equicontinuous, from which it follows easily that f^{n_k} also converges to the identity on a neighborhood V of z_0. But this is impossible, since $z_0 \in J(f) = \partial K(f)$, which implies that there are points in V that converge to ∞ under iteration. □

References

[1] John Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, Introductory lectures.

Talk 5—Existence of Cremer Points and Siegel Discs

Norbert Steinmetz

Given any irrational $\alpha \in (0,1)$, any rational map f of degree d with $f(0) = 0$ and $f'(0) = e^{2\pi i\alpha}$ either has a Cremer point or else a Siegel fixed point at the origin; which case actually occurs may and will depend on α, and *a priori* also on f. This talk will discuss several results due to Cremer, Siegel, Rüssmann, Bryuno, and Yoccoz on existence in either direction; we distinguish between results *with* and *without number theory.* Let $[a_1, a_2, a_3, \ldots]$ denote the continued fraction expansion of α, and set $[a_1, a_2, \ldots, a_n] = p_n/q_n$. Then the following *number theoretical* results hold:

- $\sup \dfrac{\log q_{n+1}}{d^{q_n}} = \infty$ *implies that $z = 0$ is a Cremer point* (Cremer [6]).
- $|\alpha - p_n/q_n| \geq c q_n^{-\mu}$ *for some $c > 0$, $\mu > 2$, and all n implies that the origin is a Siegel point* (Siegel [10]).
- $(*)$ $\displaystyle\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty$ *also implies that the origin is a Siegel point* (Rüssmann [9] and Bryuno [3] independently).

It is remarkable that Bryuno utilised a refined version of Siegel's method (based on Cauchy's *calcul des limites, method of majorants*), while Rüssmann used KAM theory (named after Kolmogorov, Arnold, and Moser). Condition $(*)$ is coined after *Bryuno*, although *Bryuno-Rüssmann* or even *Bryuno-Cherry-Rüssmann condition* would be much more appropriate.

It was Yoccoz who observed that existence of Siegel discs can be proved *without number theory*, and it is an irony of history that his existence proof combines two well-known results of Fatou in function theory and integration theory. Actually the question whether or not Siegel discs do exist could have been decided already by Fatou (or his contemporaries), at least in the special case of P_α.

Suppose $0 < |\lambda| < 1$; then $P_\lambda(z) = \lambda z + z^2$ (we slightly change notation) has an attracting fixed point at $z = 0$, Schröder's functional equation

$$\Phi_\lambda \circ P_\lambda = \lambda \Phi_\lambda$$

has a solution in the basin of attraction A_λ, normalised by $\Phi_\lambda(0) = 0$ and $\Phi'_\lambda(0) = 1$. The local inverse $\Psi_\lambda = \Phi_\lambda^{-1}$ fixing the origin extends analytically to the disc $|\zeta| < R_\lambda$, but not to any larger disc, and R_λ is called the *conformal radius* of the domain $U_\lambda = \Psi_\lambda(\{\zeta : |\zeta| < R_\lambda\})$, which is occasionally called *domain of univalence* (of Φ_λ). Since $A_\lambda \subset \{z : |z| < 2\}$, the Lemma of Schwarz applied to $\zeta \mapsto \Psi_\lambda(R_\lambda \zeta)/2$ yields $R_\lambda \leq 2$.

Observing that Φ_λ depends holomorphically on λ, Yoccoz defined the map

$$\eta(\lambda) = \Phi_\lambda(-\lambda/2)$$

which is holomorphic and bounded on $\mathbb{D} \setminus \{0\}$, hence $\lambda = 0$ is a removable singularity, and η belongs to the *Hardy space* H^∞. By a theorem of Fatou, η has radial and nonzero boundary values a.e.–the latter follows from *Fatou's Lemma* in integration theory; one may also apply the theorem of Herglotz on positive harmonic functions like $u(\lambda) = \log 2 + \log|\lambda| - \log|\eta(\lambda)|$. A simple normal family argument shows that P_{λ^*} has a Siegel disc whenever $\rho(\lambda^*) = \limsup_{\lambda\to\lambda^*} |\eta(\lambda)| > 0$, hence

- a.e. *polynomial* P_{λ^*} *has a Siegel disc about* $z = 0$ *with conformal radius* $\geq \rho(\lambda^*)$ (Yoccoz [11]).

The Bryuno-Rüssmann condition is supposed to be also necessary for a rational function (with $f(0) = 0$ and $f'(0) = \lambda^*$) to have a Siegel disc; this was confirmed by Yoccoz [11] for P_{λ^*}, but remains open in the general case. Yoccoz also proved that the radial limit $R_{\lambda^*} = \lim_{t\uparrow 1} R_{t\lambda^*}$ exists *everywhere*, and coincides with the conformal radius of the corresponding Siegel disc (hence also with $\rho(\lambda^*)$), if there is any, and vanishes else. This deep result was the key for Avila, Buff, Chéritat, and Geyer (in various subsequently written papers starting with [4]) to prove the striking result that there exists a dense subset of $\partial\mathbb{D}$ such that P_{λ^*} has a Siegel disc with C^∞-smooth boundary[1]. This set has to be small, since by a result of Carleson and Jones [5], for almost all λ^* the Siegel disc contains the critical point on its boundary, which is not consistent with a smooth boundary curve.

References

[1] A. Avila, *Smooth Siegel disks via semicontinuity: A remark on a proof by Buff and Chéritat,* Preprint `arXiv:math.DS/0305272`

[2] A. Avila, X. Buff, and A. Chéritat, A., *Siegel disks with smooth boundaries*, Preprint.

[3] A. D. Bryuno *Analytic forms of differential equations,* Trans. Mosc. Math. Soc. **25** (1971), 131-288.

[4] X. Buff and A. Chéritat, *Quadratic Siegel disks with smooth boundaries*, Preprint 242, Université Toulouse.

[5] L. Carleson and P. Jones, *On coefficient problems for univalent functions and conformal dimension*, Duke Math. J. **66** (1992), 169-206.

[6] H. Cremer, *Über die Iteration rationaler Funktionen*, Jahresbericht DMV **33**, (1925), 185-210.

[7] L. Geyer, *Smooth Siegel discs without number theory: a remark on a proof by Buff and Chéritat*, Preprint.

[8] R. Perez-Marco, *Siegel disks with smooth boundary,* (1997), Preprint.

[9] H. Rüssmann, *Über die Iteration analytischer Funktionen,* Journ. Math. Mech. **17** (1967), 523-532.

[10] C.L. Siegel, *Iteration of analytic functions,* Ann. of Math. **43** (1942), 607-616.

[11] J.-C. Yoccoz, *Théorème de Siegel, nombres de Brjuno et polynômes quadratiques*, Astérisque **231** (1995), 3-88.

[1] Perez-Marco had already proved the existence of Siegel disks with smooth boundary by a different method, if one allows non-polynomial maps. See [8].

Talk 6—Douady-Ghys's renormalization and Yoccoz's theorem on the Brjuno function

Dzmitry Dudko

Let $\mathbb{D}$ be the unit disk in the complex plane, $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$ and $\{x\}$ be a fractional part of x. Assume $\alpha \in]0,1[$ and let $f_0 : \mathbb{D} \to \mathbb{C}$ be a univalent holomorphic map fixing 0 with derivative $e^{2\pi i\alpha}$. We would like to make the following construction: take a sector $\mathcal{U}_0$ between the segment $[0,1]$ and its image by f_0 (the one with angle α at the vertex 0). The Riemann surface $\mathcal{V}_0$ obtained as the quotient of $\mathcal{U}_0$ with $]0,1]$ identified with its image by f_0 is a punctured disk. The first-return map to $\mathcal{U}_0$ associated to f_0 induces a holomorphic map $g : \mathcal{V}_0' \to \mathcal{V}_0$ with $\mathcal{V}_0' \subseteq \mathcal{V}_0$. We can identify $\mathcal{V}_0$ with $B(0, S_0)\backslash\{0\}$ where S_0 is chosen so that $\mathbb{D}^* \subseteq \mathcal{V}_0'$. Then g is univalent and extends at the origin by $g(0) = 0$ and $g'(0) = e^{-2\pi i\alpha_1}$ with $\alpha_1 = \{1/\alpha\}$. The Douady-Ghys's renormalized map f_1 is defined as the restriction to $\mathbb{D}$ of $\overline{g(\overline{z})}$, which has derivative $e^{2\pi i\alpha_1}$ at the origin.

By restricting f_0 to a disk with sufficiently small radius we can always guarantee existence of the Douady-Ghys's renormalization. The main technical problem is to control the size of the renormalization domain $\mathcal{U}_0$.

The second renormalization f_2 of f_0 is just the renormalization of f_1. This process can be continued. It is by construction that f_n has derivative $e^{2\pi i\alpha_n}$ at the origin, where $\alpha_n = \{1/\alpha_{n-1}\}$.

Let $\alpha \notin \mathbb{Q}$, then 0 is called irrational indifferent fixed point which is either a Siegel disk or a Cremer point. Let $\alpha = [a_0; a_1, a_2, \ldots]$ be a continued fraction for α, $\alpha_0 = \{\alpha\}$, $\alpha_{i+1} = \{1/\alpha_i\} = [0; a_{i+1}, a_{i+2}, \ldots]$. Then Brjuno function is by definition

$$\Phi(\alpha) = \sum_{i=0}^{\infty} \alpha_0\alpha_1 \ldots \alpha_{i-1} \log \frac{1}{\alpha_i}.$$

The theorem of Brjuno and Rüssmann says that if $\Phi(\alpha) < \infty$, then any indifferent fixed point with multiplier $e^{2\pi i\alpha}$ is linearizable. Yoccoz refined this statement [2] by showing that

$$\Phi(\alpha) + \log(r) > -\mathbf{C},$$

where r is the conformal radius of the Siegel disk of f_0 (if f_0 has a Cremer point then by assumption $r = 0$). The last inequality gives the estimate for the size of the Siegel disk.

The purpose of the proof of Yoccoz's inequality is to show that exist a neighborhood of 0 with radius $\mathbf{c}\alpha_0\alpha_1^{\alpha_0}\alpha_2^{\alpha_0\alpha_1}\ldots$ which will not escape too far away from 0. It would mean that this neighborhood belongs to the Fatou set and so it is a part of the Siegel disc. See [2, 1].

The idea of the proof is to construct the infinite sequence $f_0, f_1, f_2, \ldots$ of renormalizations for f_0, such that the desired neighborhood translates to the appropriate neighborhood of 0 for the map f_1, which translates further to the neighborhood of 0 for the map f_2 and so on. To do this it is important to control the size of the renormalization's domain and the closeness of f_n to the rotation $R_{\alpha_n} = e^{2\pi i\alpha_n}$. It is easy to see that if f_n is close enough to R_{α_n} then it is possible to take $\mathcal{U}_n$ close to the sector with radius ρ_n close to 1, the canonical map from $\mathcal{U}_0$ to $\mathcal{V}_0$ close to $z \to (\frac{z}{\rho_n})^{\frac{1}{\alpha_n}}$ and the renormalized map f_{n+1} close to $R_{\alpha_{n+1}}$. The crucial technical point is to show that we can take $\rho_n = \mathbf{c}\alpha_n$ for a universal constant $\mathbf{c}$. After this it is possible to show that f_0 can be iterated infinitely many times on the disk $B(0,\sigma)$, where $\sigma = \rho_0\rho_1^{\alpha_0}\rho_2^{\alpha_0\alpha_1}\dots$. The rest is trivial:

$$\log\sigma = -\sum_{i=0}^{\infty}\alpha_0\alpha_1\dots\alpha_{i-1}\log\frac{1}{\alpha_i} + \log\mathbf{c}(1+\alpha_0+\alpha_0\alpha_1+\dots) \geq -\Phi_{\alpha_0} + 4\log\mathbf{c},$$

where $\alpha_i\alpha_{i+1} < 0.5$ is a property of continued fractions.

References

[1] Xavier Buff and Arnaud Chéritat, The Yoccoz function continuously estimates the size of Siegel disks, Annals of Math 164 (2006), 265–312.

[2] Jean-Christophe Yoccoz, Théoreme de Siegel, nombres de Bruno et polynômes quadratiques, Astérisque (1995), no. 231, 3–88, Petits diviseurs en dimension 1.

Talk 7—The Yoccoz inequality

Peter J. Grabner

Throughout this exposition we assume that P is a monic polynomial of degree $d > 1$ with connected Julia set $J(P)$. Then also the filled-in Julia set $K(P)$ is connected. This talk was based on material covered in [2, 3, 7].

Proposition 1 (Böttcher [1])**.** *There exists a neighbourhood $U \subset \mathbb{P}^1$ of ∞ with $P(U) \subseteq U$ and analytic map $\Phi_P : U \to \mathbb{P}^1$, such that $\Phi_P(z) = z + \mathcal{O}(1)$ for $z \to \infty$ and*

$$\Phi_P(P(z)) = \Phi_P(z)^d. \tag{1}$$

This function can be obtained by considering a convergent subsequence of the sequence of functions $(P^{\circ n}(z))^{1/d^n}$ (choosing the branch of the root so that $(P^{\circ n}(z))^{1/d^n} = z + \mathcal{O}(1)$). If $K(P)$ is connected, there are no critical points of P in $\mathbb{C} \setminus K(P)$. Thus the functional equation (1) can be used to find an analytic continuation of Φ_P to $\mathbb{C} \setminus K(P)$.

This gives

Proposition 2. *If $K(P)$ is connected, the Böttcher function Φ_P extends to an isomorphism*

$$\Phi_P : \mathbb{C} \setminus K(P) \to \mathbb{C} \setminus \overline{\mathbb{D}},$$

where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Furthermore, $g_P(z) = \log|\Phi_P(z)|$ is the Green function with pole at infinity for the set $K(P)$ (see for instance [10]).

Definition 1. The external ray at angle $\theta \in \mathbb{R}/\mathbb{Z}$ is defined as

$$R_P(\theta) = \Phi_P^{-1}\left(\left\{re^{2\pi i\theta} \mid r > 1\right\}\right).$$

The ray $R_P(\theta)$ is said to land at z_0, if

$$\lim_{r\downarrow 1} \Phi_P^{-1}(re^{2\pi i\theta}) = z_0.$$

A ray $R_P(\theta)$ is said to be (eventually) periodic, if the sequence of rays

$$(P^{\circ n}(R_P(\theta)))_{n\in\mathbb{N}}$$

is (eventually) periodic.

As a consequence of the simple fact that $P(R_P(\theta)) = R_P(d\theta)$, the periodic rays are exactly the rays $R_P(\theta)$ with $\theta = \frac{p}{q}$, $(p,q) = 1$ and $(d,q) = 1$. The eventually periodic rays are those with $\theta = \frac{p}{q}$, $(p,q) = 1$ and $(d,q) > 1$.

Theorem 1 (Douady's landing theorem (cf .[2, 3]))**.** *Let P be a monic polynomial and $\alpha \in J(P)$ a repelling fixed point. There are finitely many external rays that land at α. They are all periodic with the same period.*

The proof uses the linearisation map ψ_P given as the entire solution of the functional equation (cf. [4, 5, 11])

$$P(\psi_P(z)) = \psi_P(\lambda z) \quad \psi_P(0) = \alpha, \quad \psi_P'(0) = 1, \quad \lambda = P'(\alpha). \tag{2}$$

The crucial fact used is that $\mathbb{C} \setminus \psi_P^{-1}(K(P))$ has only finitely many components $U_0, \ldots, U_{q'-1}$ (ordered counter-clockwise). This is proved by a length-area argument. By (2) these components are permuted cyclically by multiplication by λ and $U_j = \lambda^q U_j$ for some q dividing q' (we set $q' = mq$). Then we have $\lambda U_j = U_{j+mp \pmod{q'}}$. Furthermore, each of these components contains exactly one preimage of an external ray which is closed in the torus $\mathbb{C}^*/\lambda^q$.

Proposition 3. *The annulus U_j/λ^q has modulus $\pi/(q\log d)$.*

Let $R_0, \ldots, R_{q'-1}$ be the external rays landing at the repelling fixed point α ordered counter-clockwise. These rays are permuted cyclically by an application of P by conformity (in the same way as the components U_j). Thus setting $p' = mp$ we have $P(R_j) = R_{j+p' \pmod{q'}}$.

Definition 2. The rational number p'/q' is called the combinatorial rotation number of the fixed point α. The greatest common divisor $m = (p', q')$ is called the cycle number of α. We set $p = p'/m$ and $q = q'/m$.

The Yoccoz inequality now bounds the multiplier λ in terms of the rotation number of the fixed point for connected Julia sets.

Theorem 2 (The Pommerenke-Levin-Yoccoz inequality (cf. [6, 8, 9]))**.** *Let P be a polynomial of degree d with connected Julia set $J(P)$ and let $\alpha \in J(P)$ be a repelling fixed point of P with multiplier λ, combinatorial rotation number p/q, and cycle number m. Then there is a branch τ of $\log\lambda$ such that*

$$\frac{\mathrm{Re}(\tau)}{|\tau - 2\pi i p/q|^2} \geq \frac{mq}{2\log d}. \tag{3}$$

The proof is based on the fact that a curve connecting z and $\lambda^q z$ in U_j has Poincaré length $\geq |q\tau - 2\pi ip|$ on the torus $\mathbb{T} = \mathbb{C}^*/\lambda$ of area $2\pi\,\mathrm{Re}(\tau)$. The m cylinders U_j/λ^q $(j = 0, \ldots, m-1)$ are disjoint and all have modulus $\pi/(q\log d)$. Thus by an application of Grötzsch's and Bers' inequalities (cf. [7, Appendix B]) we get

$$\frac{m\pi}{q\log d} = \sum_{j=0}^{m-1} \mathrm{mod}(U_j/\lambda^q) \leq \frac{\mathrm{area}(\mathbb{T})}{|q\tau - 2\pi ip|^2} = \frac{2\pi\,\mathrm{Re}(\tau)}{|q\tau - 2\pi ip|^2}.$$

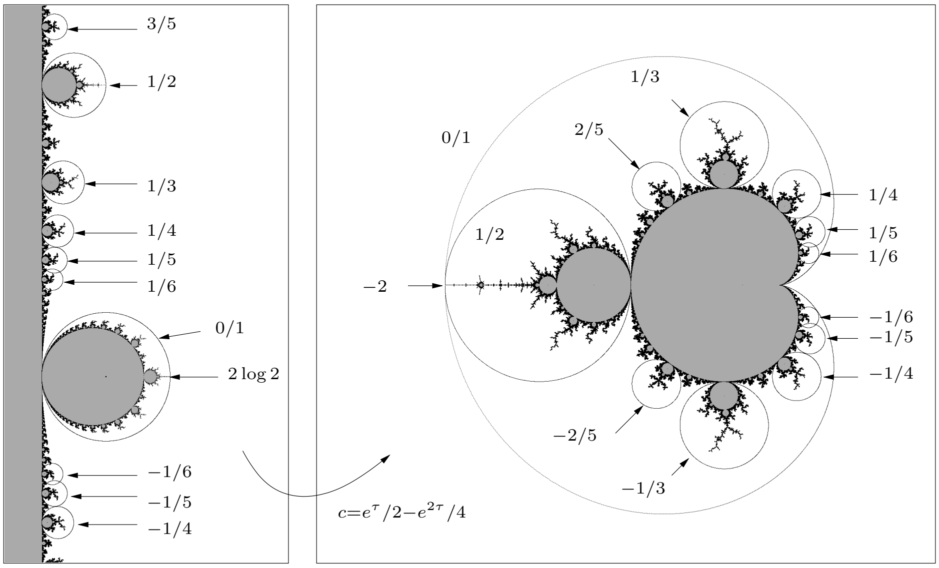


Figure 1. Geometric interpretation of the Pommerenke-Levin-Yoccoz inequality.

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Talk 8—Parabolic explosions in the quadratic family (via Puiseux's theorem)

Małgorzata Stawiska

Consider the family of quadratic polynomials in one complex variable z: $P_\alpha(z) = e^{2\pi i\alpha}z + z^2, \quad \alpha \in \mathbb{C}$. For each rational number p/q, $P_{p/q}$ has a parabolic periodic point at $z = 0$. When α is sufficiently close to p/q, this fixed point "splits" into a fixed point at 0 and a periodic cycle of order q, close to 0 (this motivates the name "explosion"). A theorem of Buff and Chéritat (cf. [1], [2]) says that this cycle depends holomorphically on the parameter α if it does not collide with another periodic cycle. More precisely, the following holds:
For p/q rational, let $R(p/q)$ denote the largest real number such that the iterate $P_\alpha^{\circ q}$ has no multiple fixed point for $\alpha \in B(p/q; R(p/q)) \setminus \{p/q\}$ and $r(p/q) := (R(p/q))^{1/q}$.

Theorem 1. *Let p/q be rational and let $\zeta = e^{2\pi i p/q}$. Then there exists an analytic function $\chi = \chi_{p/q} : B(0, r(p/q)) \setminus \{p/q\} \mapsto \mathbb{C}$ such that $\chi(0) = 0$, $\forall \delta \in B(0, r(p/q)) \setminus \{0\} \quad \chi(\delta) \neq 0$ and $\{\chi(\delta), \chi(\zeta\delta), ..., \chi(\zeta^{q-1}\delta)\}$ forms a cycle of period q for $P_{p/q+\delta^q}$.*

The original proof considers geometry of the germ of the analytic surface

$$\{P_\alpha^{\circ q}(z) - z = 0, \quad \alpha \in B(p/q; R(p/q))\}.$$

It is also possible to consider $P_\alpha^{\circ q}$ as a monic polynomial in the variable z (of degree 2^q) with coefficients that are functions of α holomorphic in a neighborhood of $p/q \in \mathbb{C}$, to argue as follows:

Proof. According to Douady and Hubbard ([3], Chapter IX), $P_{p/q}^{\circ q}(z) = z + Az^{q+1} + \mathcal{O}(z^{q+2})$ and $P_{p/q+\delta}^{\circ q}(z) = z + z(2\pi i q\delta + Az^q + \mathcal{O}(\delta z) + \mathcal{O}(z^{q+1}))$. We have $P_{p/q}^{\circ q}(z) - z = z^{q+1}(z - c_1)...(z - c_n)$, with all roots $c_1, ..., c_n \neq 0$ simple (since the z-derivative of $P_{p/q+\delta}^{\circ q}$ at $(c_j, 0)$ is nonzero). By Hensel's Lemma ([4], 23.10), there exist monic polynomials $F_0, F_1, ..., F_n$ with coefficients holomorphic in δ in a neighborhood of 0 of degrees respectively $q + 1, 1, ..., 1$ such that $P_{p/q+\delta}^{\circ q} - z = F_0 F_1 \ldots F_n$, $F_0(z, 0) = z^{q+1}, F_j(z, 0) = (z - c_j), j = 1, ..., n$. Comparing with the above form for $P_{p/q+\delta}^{\circ q}$ we see that $F_0(z, \delta) = zF(z, \delta)$, where F is of degree q in z, irreducible in the ring of polynomials with coefficients holomorphic in a neighborhood of 0 in the δ-plane. (If F factored into a product of two nonconstant polynomials of lower degrees, both would have a value at $z = 0$ of order at least 1 in δ, so the expansion for $(P^{\circ q}_{p/q+\delta}(z) - z)/z$ near $\delta = 0$ would have have a value at $z = 0$ of order at least 2 in δ, contrary to the form above where it is $2i\pi q\delta$.) The Puiseux theorem (cf. [5], II.6.1) says that there exists a function χ defined in a neighborhood of 0, holomorphic, with $\chi(0) = 0$ such that $F(z, \delta^q) = \prod_{j=0}^{q-1}(z - \chi(e^{2\pi ij/q}\delta))$. When $\delta \in B^* := B(0, r(p/q)) \setminus \{0\}$ each root of $P^{\circ q}_{p/q+\delta^q}(z) - z$ is simple. Thus by the implicit function theorem, locally, they depend holomorphically on δ. Hence since B is simply connected, χ extends to a holomorphic function on B such that $\chi(e^{2\pi ij/q}\delta)$ is a root of $P^{\circ q}_{p/q+\delta^q}(z) - z$ for all $\delta \in B$ and all j. Note that $P_\alpha(\chi(\delta)) = \chi(\zeta\delta)$ for $\alpha = p/q + \delta^q$: both are solutions to $P^{\circ q}_\alpha(z) - z = 0$ and have the same derivative at $\delta = 0$. Hence $\{\chi(\delta), ..., \chi(\zeta^{q-1}\delta)\}$ form a periodic cycle of order q for $P^{\circ q}_\alpha$ as claimed. □

Using Yoccoz's inequality and properties of external rays one can estimate $R(p/q)$ to be at least $1/q^3$ for any rational p/q in a reduced form. When α is a Bryuno number and p_k/q_k is the k-th approximant of α in the continued fraction expansion, the functions $\psi(\delta) = \chi(\delta/\chi'(0))$ play an important role (cf. proposition 6 in [1]): If $\rho_{p/q} := (2\pi R(p/q)/A(p/q))^{1/q}$, then the sequence $\{\psi_{p_k/q_k}\} : B(0, \rho_{p_k/q_k}) \mapsto \mathbb{C}$ converges uniformly on compact subsets to the linearization map $\varphi_\alpha : B(0, r_\alpha) \mapsto \Delta_\alpha$ which fixes 0 with derivative 1, where Δ_α is the Siegel disk for P_α. The proof uses a normal family argument and certain convergence results by H. Jellouli.

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Talk 9—Digitated Siegel Disks

Nikita Selinger

Let $P_\alpha(z) = e^{2\pi i\alpha}z + z^2$. If U and X are two measurable subsets of $\mathbb{C}$ and $0 < \text{area}(U) < +\infty$ then denote

$$\text{dens}_U(X) = \frac{\text{area}(U \cap X)}{\text{area}(U)}.$$

In this talk we give a sketch of the proof of the following theorem by Buff and Cheritat (see [1]):

Theorem 1. *Assume* $\alpha := [a_0, a_1, \ldots]$ *and* $\theta := [0, t_1, \ldots]$ *are Brjuno numbers and let* p_n/q_n *be the approximants to* α*. Assume*

$$\alpha := [a_0, a_1, \ldots, a_n, A_n, t_1, t_2, \ldots]$$

with (A_n) *a sequence of positive integers such that*

$$\limsup_{n\to\infty} \sqrt[q_n]{1 + \log A_n} = 1.$$

Let Δ *be the Siegel disk of* P_α *and* Δ'_n *the Siegel disk of the restriction of* P_{α_n} *to* Δ*. For all non-empty open set* $U \subset \Delta$*,*

$$\liminf_{n\to\infty} \mathrm{dens}_U(\Delta'_n) \geq \frac{1}{2}.$$

This theorem is a key to proving the existence of the Julia sets with positive measure.

The idea of the proof is to compare the dynamics with a vector field and introduce sets $X_n(\rho) \subset \Delta$, invariant by the flow of the vector field and prove that:

(1) for any open U compactly contained in Δ there exist ρ with

$$\lim \inf_{n\to\infty} \mathrm{dens}_U(X_n(\rho)) \geq \frac{1}{2}$$

and

(2) $X_n(\rho) \subset \Delta'_n$ for n large enough.

To prove the second statement authors show by using sector renormalization that the P_{α_n} orbit of any point in $X_n(\rho)$ remains bounded for large values of n, therefore $X_n(\rho)$ lies in the corresponding Siegel disk. We refer the reader to [1] for a detailed proof of the theorem.

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Talk 10—The Measurable Riemann Mapping Theorem

Thilo Kuessner

An orientation-preserving diffeomorphism $f : R^2 \to R^2$ is K-quasiconformal if the dilatation

$$D_f := \frac{\mid \frac{\partial f}{\partial z} \mid + \mid \frac{\partial f}{\partial \bar{z}} \mid}{\mid \frac{\partial f}{\partial z} \mid - \mid \frac{\partial f}{\partial \bar{z}} \mid}$$

satisfies

$$D_f \leq K,$$

or equivalently, if the inequality $Jac(f) \geq \frac{1}{K} \parallel Df \parallel^2$ is satisfied with $Jac(f) = \mid \frac{\partial f}{\partial z} \mid^2 - \mid \frac{\partial f}{\partial \bar{z}} \mid^2$ and $\parallel Df \parallel = \mid \frac{\partial f}{\partial z} \mid + \mid \frac{\partial f}{\partial \bar{z}} \mid$. Yet another formulation is to consider the Beltrami coefficient

$$\mu := \frac{\frac{\partial f}{\partial \bar{z}}}{\frac{\partial f}{\partial z}}$$

and to require $\mid \mu \mid \leq k$ with $k := \frac{K-1}{K+1}$.

In dynamical systems one uses quasiconformal mappings to conjugate conformal mappings one to another. However the quasiconformal mappings arising in this context are usually not smooth. This makes it necessary to consider a wider class of quasiconformal homeomorphisms.

Definition 1. An orientation-preserving homeomorphism $f : R^2 \to R^2$ is K-quasiconformal if $f \in W^{1,2}_{loc}$ and its distributional derivatives satisfy

$$\mid \frac{\partial f}{\partial \bar{z}} \mid \leq k \mid \frac{\partial f}{\partial z} \mid$$

almost everywhere, for $k := \frac{K-1}{K+1}$.

A geometric characterization of K-quasiconformal homeomorphisms is given by the Theorem of Grötzsch: An orientation-preserving homeomorphism $f : R^2 \to R^2$ is K-quasiconformal if and only if the inequality

$$\frac{1}{K} Mod(A) \leq Mod(fA) \leq K Mod(A)$$

holds for each annulus A.

Here, the modulus $Mod(A)$ is defined as follows:
- if A is the annulus between two concentric circles of radius r_1 and r_2, then

$$Mod(A) := \frac{1}{2\pi} log\left(\frac{r_2}{r_1}\right),$$

- the modulus of annuli is invariant under conformal mappings.

The Theorem of Grötzsch can be used to show that K-quasiconformal homeomorphisms of the disk (for fixed K) are equicontinuous and (hence) form a compact set.

Theorem 1. *(Measurable Mapping Theorem) :*
Let $U \subset R^2$ open, $\mu \in L^\infty(U), \parallel \mu \parallel_\infty < 1$.
Then there exists a quasiconformal homeomorphism $f : U \to f(U)$ with $\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$ almost everywhere,
unique up to composition with some conformal map.

The proof consists of two steps. First, if μ were R-analytic, then the problem can be reduced to solving a certain partial differential equation. In the second step, μ is approximated in L^1-norm by R-analytic functions μ_ε, for which a solution f_ε exists by the first step. By compactness, (a subsequence of) f_ε converges to some quasiconformal map f, and it is easily checked that f satisfies the Beltrami equation for μ. See [1].

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Talk 11—Sullivan's non-wandering Theorem

Clément Hongler

In this talk, we prove, by means of quasiconformal mappings theory, the non-wandering theorem from Sullivan which can be stated as follows:

Theorem 1. *Let $P : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map. Then every connected component of the Fatou set U is eventually periodic, that is, we have $P^{n+k}(U) = P^n(U)$ for some $n \geq 0, k > 0$. In other words it is impossible that U is wandering, i.e. that the iterates $P(U), P^2(U), P^3(U), \dots$ of U are all disjoint.*

In our talk we restrict ourselves to the polynomial case which is a little bit simpler. We proceed by contradiction, assuming that there exists a wandering component U. The fact that P is assumed polynomial allows us to suppose that U is a simply connected domain (in the rational case we may use Baker's lemma to make such an assumption and this is in fact the only difference with the polynomial case). By iterating U if necessary we can also suppose that none of the forward images of U contains a critical point.

The technique used by Sullivan goes as follows: let G be the group of the diffeomorphisms of the unit circle $\mathbb{S}^1$ that fix the points $1, -1$ and i (such a diffeomorphism is orientation-preserving). For any element $g \in G$, we can extend g to a diffeomorphism $\hat{g} : D(0,1) \to D(0,1)$ in such a way that the application $g \mapsto \hat{g}$ is smooth. Thus we get a quasiconformal map $\hat{g}$ that defines a conformal structure (i.e. an ellipse field) on the unit disc for each g.

Fixing ϕ a conformal mapping from our domain U to $D(0,1)$ we can pull this conformal structure back to the domain U. Call this structure μ_g. Notice that μ_g depends smoothly on g. Now we can extend μ_g to the grand orbits of U, i.e. the set

$$\bigcup_{k \geq 1} \bigcup_{j \geq 0} P^{-k}\big(P^j(U)\big)$$

(negative exponents of P denote preimages). This can be done on the forward iterates for U since P is a local diffeomorphism there, and on the backwards iterates almost everywhere (except at the precritical points, but they form a subset of $\mathbb{P}^1$ of zero measure). Finally we extend μ_g on the rest of the sphere by letting $\mu = 0$ on $\mathbb{P}^1 \setminus \bigcup_{k \geq 1} \bigcup_{j \geq 0} P^{-k}\big(P^j(U)\big)$.

It is easy to see that our conformal structure depends smoothly on g. Integrating the ellipse field (using the measurable uniformization theorem) we obtain a unique quasiconformal homeomorphism $h_g : \mathbb{P}^1 \to \mathbb{P}^1$ that has distortion μ_g, fixes $0, 1$ and ∞ and depends smoothly on g. It is easy to show that since our polynomial P preserves μ_g, the conjugated map $P_g := h_g^{-1} \circ P \circ h_g$ is also a polynomial of the same degree as P . By a dimension argument (our group G is infinite-dimensional as a manifold and the space of polynomials of fixed degree is finite-dimensional) and since P_g depends smoothly on g, we can find a non-constant curve $[0,1] \to G$, $t \mapsto g(t)$ such that $P_{g(t)}$ is constant. Let $f_t := h_{g(t)} \circ h_{g(0)}^{-1}$. We have that $f_t^{-1} \circ P \circ f_t = P$ for all $t \in [0,1]$. Since f_t commutes with P, it has to preserve its periodic points of given period. But since f_0 is the identity and the points of a given period form a discrete set, it must actually fix all the periodic points (act like the identity on them), by a continuity argument. The Julia set of P must hence be fixed since the periodic points are dense in it. We have therefore that the boundary of our wandering domain U (which is contained in the Julia set of P) is fixed by f_t for every $t \in [0,1]$.

Using a topological lemma, one can conclude that f_t acts like the identity on the prime ends of U. So the maps $h_{g(t)}$ act the same way on the prime ends of U for all t and this contradicts the fact that $g(t)$ is non-constant (if it were non-constant, this would appear on the boundary of the unit disc which is by Carathéodory's theorem homeomorphic to the set of prime ends of U). So our construction is impossible and we get a contradiction with the fact that U is wandering.

Eventually we give an application to the study of the Julia set of Cremer quadratic polynomials. If a quadratic polynomial has a Cremer fixed (or periodic) point, then its filled-in Julia set is equal to its Julia set. Suppose by contradiction that there exists a bounded Fatou component V. Then it is eventually periodic and we have that $P^\ell(V) = V$ for some $\ell > 0$. In the second talk of this Arbeitsgemeinschaft (by Norbert Schappacher), it has been shown that this implies the existence of a non-Cremer non-repelling periodic point. But by Fatou-Shishikura theorem there cannot be more than $\deg(P) - 1 = 1$ non-repelling periodic points of our polynomial in $\mathbb{C}$ (and there is already the Cremer point by assumption). So the filled-in Julia set has empty interior and is hence equal to the Julia set.

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Talk 12—Siegel disks with Jordan Boundary and the Herman-Swiatek Theorem

Dierk Schleicher

A Siegel disk of a holomorphic function f is an open and simply connected domain U so that $f : U \to U$ is conformally conjugate to an irrational rotation of the complex unit disk $\mathbb{D}$ (and so that U is maximal with respect to inclusion). For polynomials, Siegel disks are one of three possible types of Fatou components. We consider quadratic polynomials $P : z \mapsto \lambda z + z^2$, $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}/\mathbb{Z}$, so that P has a Siegel disk U around the origin. We prove that if θ is of bounded type, (all

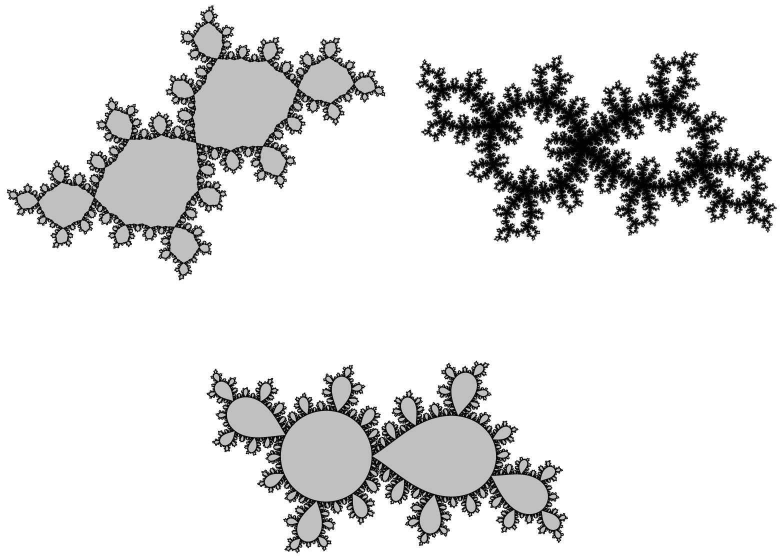

Figure 2. The quadratic Julia set with the Golden mean Siegel disk. The Julia set of the Blaschke fraction, before the surgery. The Julia set of the modified Blaschke fraction: it is quasiconformally equivalent to the first one.

entries in the continued fraction expansion are bounded) then ∂U is a quasi-circle, that is, the image of a Euclidean circle by a quasiconformal homeomorphism of the complex plane. We prove this by using a Blaschke fraction $B : z \mapsto e^{2\pi i\alpha} z^2 \frac{z-3}{1-3z}$: this has a Julia set containing $\mathbb{S}^1$, so that $B : \mathbb{S}^1 \to \mathbb{S}^1$ is an analytic circle homeomorphism. By the Herman-Swiatek theorem [2], when $\alpha \in \mathbb{R}$ chosen so that the rotation number of B on $\mathbb{S}^1$ equals the bounded type number θ, then this circle homeomorphism is quasisymmetrically conjugated to the rotation $z \mapsto e^{2i\pi\theta} z$ on $\mathbb{S}^1$. We then specify a quasiconformal surgery [1, 2, 3], due to Douady and Ghys, which sends $\mathbb{S}^1$ to ∂U, thus proving the claim.

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Talk 13—Bounded type Siegel disks and Lebesgue measure

Philipp Meerkamp

Suppose that θ is an irrational number of bounded type, and let

$$f : \mathbb{C} \to \mathbb{C}, z \mapsto e^{2\pi i\theta} z + z^2.$$

We call f a quadratic polynomial of bounded type. The map f has a unique critical point c_0, a Siegel disk Δ, and the postcritical set $P(f) := \overline{\{f^{\circ n}(c_0) \mid n \geq 0\}}$ is a dense subset of $\partial\Delta$. For $\varepsilon > 0$, define $K_\varepsilon(f) = \{z \in \mathbb{C} \mid \forall\ n > 0,\ d(z, \Delta) < \varepsilon\}$. Denote the set of points $z \in \mathbb{C}$ with bounded orbits under iteration by f by $K(f)$, and note that $K_\varepsilon(f) = K(f)$ for ε large enough. McMullen proved the following theorem in [2]:

Theorem 1. Every point $p \in P(f)$ is a point of Lebesgue density of $K_\varepsilon(f)$. In particular, every $p \in P(f)$ is a point of Lebesgue density of the filled-in Julia set $K(f)$.

We sketched McMullen's proof of those statements, which Arnaud Chéritat and Xavier Buff used as a tool to show the existence of Julia sets of positive measure.[1]

Definition 2. A compact set $\Lambda \subset \mathbb{C}$ is *shallow* if there exists $c > 0$ such that for every $z \in \Lambda, 0 < r < 1$, there exists a ball B disjoint from Λ that satisfies $r/c < \operatorname{diam} B$, and $d(z, B) < c\, r$.

Definition 3. Let $\Lambda \subset \mathbb{C}$ be closed and $x \in \Lambda$. Then x is a *deep point* of Λ if there exist $c, \delta > 0$ such that for all $z \in \mathbb{C}$ and $r, s > 0$ satisfying $B(z, s) \subset B(x, r) - \Lambda$, we have that $s \leq cr^{1+\delta}$.

McMullen first proves three other statements:

Theorem 4. $J(f)$ is shallow.

Theorem 5. Every point $p \in P(f)$ is a deep point of $K_\varepsilon(f)$.

Lemma 6. Let A be compact. If $x \in A$ is deep and ∂A is shallow, then x is a point of Lebesgue density of A.

While lemma 6 is not difficult, the proofs of theorems 4 and 5 require a considerable amount of work. For subsets A of the Riemann sphere, shallowness implies that the box-dimension of A is smaller than 2. It is thus an easy consequence of lemma 4 that the Hausdorff dimension of $J(f)$ is strictly smaller than 2, an interesting result in its own right.

The proof of lemma 5 requires two high-caliber statements: the first is a theorem by Herman, Świątek, Douady and Ghys [4], which will provide a change of

[1]Note however that the Julia sets of the bounded-type quadratics $f(z) = e^{2\pi i\theta} z + z^2$ considered in this paper do not have have positive measure. This was shown by Petersen in 1996, see [3].

coordinates φ, conjugating f restricted to the Siegel disk to a rotation around the origin. The second statement is proved in McMullen's paper, and it says that the behaviour of f close to the Siegel disk is – up to a change of coordinates by φ – almost that of an irrational rotation around the origin. In particular, points cannot drift away too quickly from the Siegel disk.

By theorem 4.9.14 from [1], the statement by Herman, Świątek, Douady and Ghys may be presented as follows:

Theorem 7. Suppose that θ is of bounded type and $f(z) = e^{2\pi i\theta}z + z^2$. Then there exists a map $\varphi : \mathbb{C} \to \mathbb{C}$ that is k-quasiconformal and satisfies that $\varphi(\Delta) = \mathbb{D}$ and that $\varphi \circ f \circ \varphi^{-1}|_{\mathbb{D}}$ is a rotation around the origin.

Let $R > 0$ be a large constant. In particular, D_R contains $K(f)$. Let $\Omega = \mathbb{C} - \overline{\Delta}$, let d_Ω denote the distance with respect to the Poincaré metric on Ω and let $\Delta' := f^{-1}(\Delta) - \Delta$ be the pre-Siegel disk.

Lemma 8 (Approximate Rotations). There exists $c > 0$ such that the following holds: let $z \in \Omega \cap D_R$. Then there exist $y \in K(f), i_0 \geq 0$ and quasidisks $U, V \subset \mathbb{C}$ open such that $z \in U, c_o \in V$,

$$f^{i_0} : (U, y) \to (V, c_0)$$

is univalent, and that for all $z \in U \cap \Omega$

$$\log(|\varphi(f^{i_0}(z))| - 1) \leq \log(|\varphi(z)| - 1) + c$$

and such that

$$d_\Omega(f^{i_0}(z), \Delta') \leq c.$$

In the proof of theorem 5, McMullen shows that there exist $c, \delta > 0$ such that given any $z \in D_R$, we have that $d(z, K_\varepsilon(f)) \leq c(d(z, P(f)))^{1+\delta}$. Deepness of every $p \in P(f)$ in $K_\varepsilon(f)$ follows immediately. For the case $K(f)$, choose ε large enough.

To prove the inequality, we use the approximate rotation lemma to construct a sequence of iterates $f^{i_1}(z), \ldots, f^{i_N}(z) \in K_\varepsilon(f)$ that lie within a bounded distance (with respect to the Poincaré metric d_Ω) of Δ'. Thus $f^{i_N}(z)$ can be joined to a point $y' \in \partial\Delta'$ by an arc of bounded length. Taking preimages and a hyperbolic contraction argument show that there exists a point $y \in K_\varepsilon(f)$ such that $d_\Omega(z, y) \leq c'(d(z, \Delta))^{\delta'}$ for some c', δ' not depending on z. Using the fact that the Poincaré metric d_Ω and the $1/d$ metric on Ω are comparable (see [1], chapters 2.2 and 3.3), the desired inequality is obtained in a standard calculation.

References

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Talk 14—Parabolic implosion (after Douady)

Walter Bergweiler

We discuss results of Douady [1] concerning the behavior of the Julia set $J(f_\varepsilon)$ of the function $f_\varepsilon(z) := z + z^2 + \varepsilon$ as $\varepsilon \to 0$.

We first note that a compactness argument shows that for suitable sequences (ε_n) tending to 0 the limit $L := \lim_{n\to\infty} J(f_{\varepsilon_n})$ exists. It was shown in the third talk of this Arbeitsgemeinschaft that the Julia set $J(f)$ is a lower semicontinuous function of the polynomial f while the filled Julia set $K(f)$ is an upper semicontinuous function of f. Since $J(f_\varepsilon) = K(f_\varepsilon)$ for $\varepsilon > 0$, this implies that $J(f_0) \subset L \subset K(f_0)$. One purpose of this talk is to show that these inclusions are strict. Moreover, it is explained how the sequence (ε_n) has to be chosen in order that the limit exists.

The main tool used are the (extended) Fatou coordinates

$$\phi_{\text{att}} : \text{int}(K(f_0)) \to \mathbb{C} \quad \text{and} \quad \psi_{\text{rep}} : \mathbb{C} \to \mathbb{C},$$

where $\text{int}(K(f_0))$ denotes the interior of $K(f_0)$. These Fatou coordinates are (suitably normalized) functions satisfying

$$\phi_{\text{att}}(f_0(z)) = \phi_{\text{att}}(z) + 1 \quad \text{and} \quad f_0(\psi_{\text{rep}}(z)) = \psi_{\text{rep}}(z+1).$$

One way to obtain the attracting Fatou coordinate ϕ_{att} is as follows: First one considers with $w = M(z) := -1/z$ the function

$$F_0(w) := M(f_0(M^{-1}(w))) = w + 1 + \frac{1}{w-1}.$$

It can then be shown that for fixed w_0 the limit

$$\alpha(w) := \lim_{n\to\infty} (F_0^n(w) - F_0^n(w_0))$$

exists in the halfplane $\{z : \text{Re } z > R\}$ if R is sufficiently large. It follows that $\alpha(F_0(w)) = \alpha(w) + 1$ and this implies that $\phi_{\text{att}} := \alpha \circ M$ satisfies the above functional equation in the disk of radius $1/(2R)$ around $-1/(2R)$. The functional equation can then be used to extend ϕ_{att} to the interior of $K(f_0)$. Similarly one obtains a map β defined in a left halfplane satisfying $\beta(F^{-1}(w)) = \beta(w) - 1$ and then $\psi_{\text{rep}} := M^{-1} \circ \beta^{-1}$ satisfies the above functional equation. Again one can use the functional equation to obtain an analytic continuation. What is actually needed are "perturbed" Fatou coordinates $\phi_{\text{att},\varepsilon}$ and $\psi_{\text{rep},\varepsilon}$ satisfying the above functional equations in suitable domains with f_0 replaced by f_ε.

Let now $a := -1/8$ and $b := 1/8$. For each $\varepsilon > 0$ we have $f_\varepsilon^k(a) \to \infty$ as $k \to \infty$. Thus there exists a unique $t(\varepsilon)$ with $f_\varepsilon^{t(\varepsilon)-1}(a) < b \le f_\varepsilon^{t(\varepsilon)}(a)$. It is shown that if (ε_n) is chosen such that the sequence $\left(f_{\varepsilon_n}^{t(\varepsilon_n)}(a)\right)$ converges, then $\left(f_{\varepsilon_n}^{t(\varepsilon_n)}\right)$ converges locally uniformly in $\text{int}(K(f))$. Moreover, the limit $g : \text{int}(K(f)) \to \mathbb{C}$ satisfies $g \circ f_0 = f_0 \circ g$ and has the form $g = \psi_{\text{rep}} \circ T_\sigma \circ \phi_{\text{att}}$ for some $\sigma \in [0,1]$, where $T_\sigma(z) = z + \sigma$ is the translation.

The behavior of $J(f_{\varepsilon_n})$ as $n \to \infty$ can now be studied by iterating g. More precisely, one has to consider the dynamics of the semigroup generated by f and g.

If $g^k(z) \notin K(f)$ for some $z \in \mathbb{C}$ and some $k \geq 0$, then there exists a disk around z which does not intersect $J(f_{\varepsilon_n})$ if n is large. If $g^k(z) \in J(f)$, then there exists points $z_n \in J(f_{\varepsilon_n})$ satisfying $z_n \to z$. Finally it can be shown that the set of all z for which $g^k(z) \in \mathrm{int}(K(f))$ for all $k \geq 0$ has empty interior. Combining these results one can deduce that the limit $L := \lim_{n\to\infty} J(f_{\varepsilon_n})$ exists and is equal to the closure of the set of all z for which there exists $k \geq 0$ such that $g^k(z) \in J(f)$. The limit L can be interpreted as the Julia set of the semigroup generated by f and g.

References

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Talk 15—The near parabolic renormalization of Inou and Shishikura

Arnaud Chéritat

After giving a short survey on renormalization in dynamics, illustrated by the Douady-Hubbard renormalization of quadratic-like maps, the circle maps renormalization, and the renormalization of logistic maps associated to the Feigenbaum bifurcation cascade, the talk focused on the work of Inou and Shishikura concerning the *parabolic renormalization* and its perturbation, the *near parabolic renormalization* [1].

1. The parabolic renormalization

Start from an analytic map fixing 0 with multiplier 1:

$$f(z) = z + a_2 z^2 + \dots$$

Assume $a_2 \neq 0$, i.e. there is only one repelling and one attracting petal in a Leau flower for the parabolic point at the origin. Let h be its horn map[1] and $h_\sigma = T_\sigma \circ h$, to be considered as a map from the repelling cylinder to itself. Conjugate h_σ by the isomorphism $z \mapsto e^{2i\pi z}$ from $\mathbb{C}/\mathbb{Z}$ to $\mathbb{C}^*$. This yields an analytic map g_σ, defined in a neighborhood of 0 and ∞, fixing both, with multipliers $\neq 0$. Since $g_\sigma = e^{2i\pi\sigma} g_0$, there is a unique value of σ such that $g'_\sigma(0) = 1$. For this σ we get the *parabolic renormalization*[2] of f:

$$\mathcal{R}(f) \stackrel{\text{def}}{=} g_\sigma.$$

[1] The quotient of a petal by the equivalence relation $z \sim f(z)$ is isomorphic, via the Fatou coordinates, to the cylinder $\mathbb{C}/\mathbb{Z}$. This quotient is referred to as the *attracting/repelling cylinder*. Now take a fundamental domain D_{rep} in the repelling petal. Take a point in the repelling cylinder. Consider the corresponding point w in D_{rep}. Iterate w until it falls in the attracting petal. To such an iterate corresponds a uniquely defined point in the attracting cylinder. Identifying the repelling and attracting cylinders to $\mathbb{C}/\mathbb{Z}$ via the Fatou coordinates, this gives the *horn map h*.

[2] Note that this puts the emphasis on the upper end of the cylinder. If one prefers the lower end, replace the conjugacy $z \mapsto e^{2i\pi z}$ by $z \mapsto e^{-2i\pi z}$.

Since the domain of definition of g_σ depends on choices, the map $\mathcal{R}$ is only defined at the level of *germs*[3] of functions, taken up to linear conjugacy.

An invariant class has been known since around 1990 (Shishikura). It consists in all holomorphic functions $f: U \to \mathbb{C}$ with:

- U is a connected open set,
- $0 \in U$ and $f(z) = z + a_2 z^2 + \dots$ with $a_2 \neq 0$,
- f is a ramified covering from $U \setminus \{0\}$ to $\mathbb{C}^*$,
- all critical points have local degree 2,
- there is exactly one critical value.

For instance, the polynomial $z + z^2$ belongs to this family (take $U = \mathbb{C} \setminus \{-1\}$. Let us call $\mathcal{C}_0$ the set of maps satisfying these conditions and *normalized* as follows: the critical value is equal to $-1/4$ (same as for $z + z^2$). We have a well-defined parabolic renormalization operator

$$\mathcal{R}: \ \mathcal{C}_0 \to \mathcal{C}_0$$

that acts on functions and not only on germs (but still agrees with the previous operator). This $\mathcal{R}: \ \mathcal{C}_0 \to \mathcal{C}_0$ is neither injective, nor surjective.

2. Covering properties

The image $\mathcal{C}_1 = \mathcal{R}(\mathcal{C}_0)$ has the following property: any two maps $f_1, f_2 \in \mathcal{C}_1$ are equivalent covers over $\mathbb{C}$, i.e. $\exists \phi$ an isomorphism between their sets of definition such that $f_2 = f_1 \circ \phi$. Why? Because for all map in $\mathcal{C}_0$, the immediate parabolic basin U contains exactly one critical point and moreover, f is conjugated on U to a universal map: the degree 2 Blachke product $\frac{3z^2+1}{3+z^2}$ on $\mathbb{D}$. Figure 3 illustrates the covering properties of $\mathcal{R}(z + z^2) \in \mathcal{C}_1$. On the rightmost picture, the sequence of dots materializes the image by Φ_- of the grand orbit of the critical point $z = -1/2$ of $z \mapsto z + z^2$. We colored the plane with two colors, darker on the lower half plane below the dots, lighter above. There is also a texture effect allowing to visualize the verticals through the dots. On the middle picture, we pulled-back this color scheme by the map Φ_-. On the leftmost picture, we pulled once again by Ψ_+. Figure 4 is a closeup on the upper part of the left frame of the previous one. These pictures are useful for understanding the structure of the horn maps as infinite degree *ramified covers* over $\mathbb{C}$.

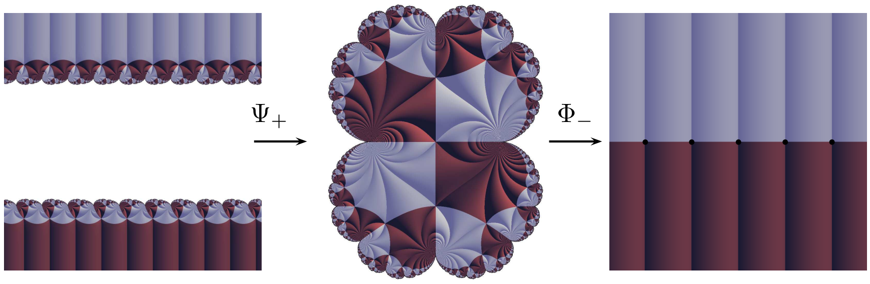
3. Near parabolic renormalization: introducing some flexibility

The class $\mathcal{C}_1$ is of the form

$$\mathcal{C}_1 = \left\{ f_0 \circ \phi^{-1} \,\middle|\, \begin{array}{l} \phi: \ \mathrm{Def}(f_0) \to \mathbb{C} \text{ is a univalent analytic} \\ \text{map with } \phi(0) = 1, \ \phi'(0) = 1 \end{array} \right\}.$$

We view this as a ramified covering over $\mathbb{C}$, with a given "Covering structure", which is a mix of topological data (homotopy) and analytic data (moduli). Since $\mathcal{R}$ maps $\mathcal{C}_1$ to a strict subset of $\mathcal{C}_1$, it is tempting to deduce from this a non-expansion

[3]equivalence class for the relation $f \sim g \iff \exists V$, a neighborhood of 0, on which f and g coincide



$\Phi_- : \mathring{K} \to \mathbb{C}$ attracting Fatou coordinates, extended

$\Psi_+ : \mathbb{C} \to \mathbb{C}$ repelling Fatou parameterization, extended

$h = \Phi_- \circ \Psi_+$ is the horn map

FIGURE 3.

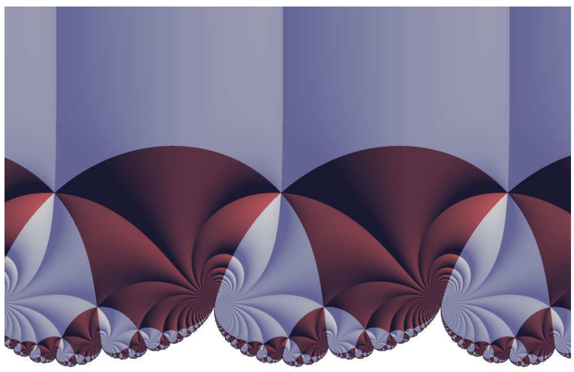


FIGURE 4.

statement, like in Schwarz's lemma; or even better, a strict contraction and the existence of a unique fixed point of $\mathcal{R}$ in $\mathcal{C}_1$. However, it is not obvious how to put a complex structure on the space of univalent maps. One way to solve this problem is to try to loosen the invariant class. Fix f_0 in $\mathcal{C}_1$, fix an open subset $V \subset \mathrm{Def}(f_0)$ and let

$$\mathcal{C}_1(V) = \left\{ f_0 \circ \phi^{-1} \;\middle|\; \begin{array}{l} \phi : V \to \mathbb{C} \text{ is a univalent analytic} \\ \text{map with } \phi(0) = 1,\ \phi'(0) = 1 \\ \text{and } \phi(V) \text{ is a quasidisk} \end{array} \right\},$$

This consists in retaining only part of the covering structure. Note that "$V' \subset V \implies \mathcal{C}_1(V) \subset \mathcal{C}_1(V')$" at the level of germs. The requirement that $\phi(V)$ be a quasidisk allows to put on $\mathcal{C}(V)$ the complex structure inherited from Teichmüller

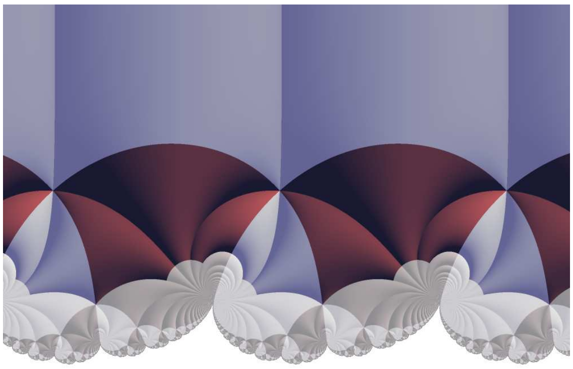


FIGURE 5.

theory. It would not work for $\mathcal{C}_1$ because for $f \in \mathcal{C}_1$ even if $\mathrm{Def}(f)$ is a quasidisk, $\mathrm{Def}(\mathcal{R}(f))$ is not anymore.

Theorem (Inou, Shishikura)**.** *For the domain V corresponding to what is illustrated in figure 5 and for some domain $V' \subset\subset V$, one can still define a parabolic renormalization $\mathcal{R}$ (which agrees with the previously defined $\mathcal{R}$ at the level of germs) such that $\mathcal{R}(\mathcal{C}_1(V')) \subset \mathcal{C}_1(V)$.*

In particular $\mathcal{R}(\mathcal{C}_1(V')) \subset \mathcal{C}_1(V')$ at the level of germs. The benefits of leaving some flexibility are manifold:

- Contraction can be proved (c.f. Inou and Shishikura, using the Teichmüller distance between quasidisks).
- Perturbations can be done, easily: the compactness of the set of univalent maps yields uniform lower bounds on how big the perturbations can be.

Theorem (Inou, Shishikura)**.** *Let $\mathcal{C}_2 \overset{def}{=} \mathcal{C}_1(V')$. There exists some*[4] *$\varepsilon > 0$ such that: If $f = e^{2i\pi\alpha}g$ with $g \in \mathcal{C}_2$ then one can define a (cylinder/near-parabolic) renormalization of f, $\mathcal{R}(f)$ which belongs to $e^{-2i\pi/\alpha}\mathcal{C}_2$* provided *$\alpha \in]0, \varepsilon[$, and corresponds to a return map*[5]*.*

Note that if $f = e^{2i\pi\alpha}g$ with $\alpha \in]0, \varepsilon[$ and $g \in \mathcal{C}_2$ then $\mathcal{R}(f) = e^{2i\pi\beta}h$ with $h \in \mathcal{C}_2$ and $\beta = \frac{-1}{\alpha} \bmod \mathbb{Z}$. Moreover, the class $\mathcal{C}_2$ is invariant under conjugation by $z \mapsto \overline{z}$, which transforms α into $-\alpha$. Hence, a map whose continued fraction entries are all $> 1/\varepsilon$ is *infinitely* cylinder renormalizable[6]. Since cylinder renormalization corresponds to return maps, the dynamics of $g = \mathcal{R}(f)$ is related to that of f. In particular iterating g once corresponds to iterating f many times (roughly $1/\alpha$).

[4] $\varepsilon = 1/23$ seems to work, c.f. numerical experiments by Inou.

[5] one can still define a perturbed fundamental domain, D_{rep} whose quotient is still isomorphic to $\mathbb{C}/\mathbb{Z}$, and $\mathcal{R}(f)$ is still the conjugated by $z \mapsto e^{2i\pi z}$ of a return map, defined on some subset of D_{rep}

[6] for the modified renormalization $\mathcal{R}' : f \mapsto s \circ \mathcal{R}(f) \circ s$ where $s(z) = \overline{z}$

In the talk of X. Buff, one will see how this can be used to control the post-critical set of polynomials P_α for some values of α.

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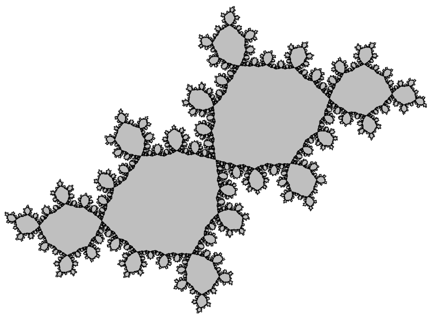
Talk 16—The control of the postcritical set

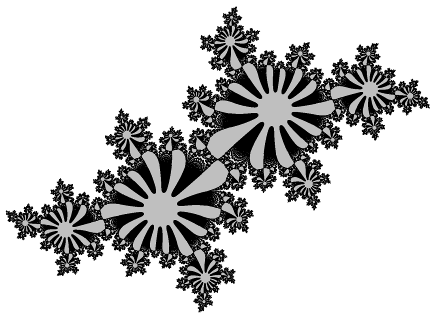
Xavier Buff

Our goal is to find a set $\mathcal{S}$ of irrational numbers such that

- for all $\alpha \in \mathcal{S}$, the polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$ has a Siegel disk Δ_α and
- for any $\varepsilon > 0$ and any $\alpha \in \mathcal{S}$, there is an $\alpha' \in \mathcal{S}$ with

$$\text{Area}(K_{\alpha'}) > (1-\varepsilon)\text{Area}(K_\alpha).$$





We saw in Talk n. 9 that if α is any Bruno number, then for suitably chosen perturbations α' of α, we have

$$\text{Area}(\Delta_{\alpha'} \cap \Delta_\alpha) > \left(\frac{1}{2} - \varepsilon\right)\text{Area}(\Delta_\alpha).$$

We easily deduce that if α is any Bruno number, then for all $\varepsilon > 0$ and for suitably chosen perturbations α' of α, we have

$$\text{Area}(K_{\alpha'}) > \left(\frac{1}{2} - \varepsilon\right)\text{Area}(K_\alpha).$$

Our goal is now to promote the coefficient $1/2 - \varepsilon$ to $1 - \varepsilon$. This will be done in Talk n. 17.

The tools will be

- McMullen's result on the Lebesgue density of K_α at the boundary points of Δ_α when α is of bounded type (see Talk n. 13) and
- a control of the postcritical set of $P_{\alpha'}$.

Definition 1. ∂ is the Hausdorff semi-distance:

$$\partial(X,Y) = \sup_{x\in X} d(x,Y).$$

Definition 2. $\mathcal{PC}(P_\alpha)$ is the post-critical set of $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$:

$$\mathcal{PC}(P_\alpha) := \bigcup_{k\geq 1} P_\alpha^{\circ k}(\omega_\alpha) \quad \text{with} \quad \omega_\alpha := -\frac{e^{2i\pi\alpha}}{2}.$$

Definition 3. $\mathcal{S}_N$ is the set of irrational numbers of *bounded type* whose continued fractions have entries $\geq N$.

According to Herman-Swiatek-Douady-Ghys, when α is of bounded type, the closure of $\mathcal{PC}(P_\alpha)$ is equal to the boundary of the Siegel disk Δ_α. The aim of the talk was to present the following result.

Theorem 1. *There exists N such that as $\alpha' \in \mathcal{S}_N$ tends to $\alpha \in \mathcal{S}_N$ we have*

$$\partial\big(\mathcal{PC}(P_{\alpha'}), \overline{\Delta}_\alpha\big) \to 0,$$

with Δ_α the Siegel disk of P_α.

In other words, there is an N such that for all $\varepsilon > 0$ and all $\alpha \in \mathcal{S}_N$, if $\alpha' \in \mathcal{S}_N$ is close enough to α, then the Siegel disk $\Delta_{\alpha'}$ is contained in the ε-neighborhood of Δ_α.

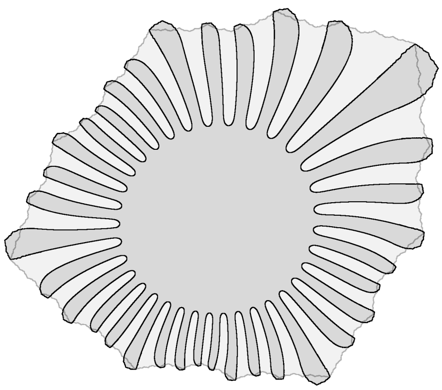


FIGURE 6. Illustration of Theorem 1 for $\alpha = [0,1,1,1,\ldots]$ and $\alpha' = [0,1,1,1,1,1,1,1,1,1,10^{10},1,1,1,\ldots]$. Light gray: Δ_α. Dark gray: Δ'_α.

In order to sktech the proof of Theorem 1, we first recalled a few things regarding renormalization.

In order to prove that $\text{Area}(\Delta_{\alpha'}) > (1/2-\varepsilon)\text{Area}(\Delta_\alpha)$, we used Douady-Ghys renormalization; this was the aim of Talks n. 6 and n. 9.

- An advantage of Douady-Ghys renormalization is that it works for all rotation numbers α.
- An inconvenient is that we have to restrict to a neighborhood of the fixed point where the map is univalent. In the process, we lose the control of the postcritical set.
- An alternative is to use the parabolic renormalization. Douady used it for proving that the Julia set does not always depend continuously on the polynomial, as explained in Talk n. 14. Shishikura used it for proving that the boundary of the Mandelbrot set has Hausdorff dimension 2.
- An inconvenient of the parabolic renormalization is that α' has to be close to a parabolic parameter, not to bounded type irrational number.

In order to control the postcritical set, we use the techniques of near-parabolic renormalization of Inou and Shishikura presented in Talk n. 15.

Definition 4. We denote by $\mathrm{Irrat}_{\geq N}$ the set of irrational numbers in $(0,1)$ whose continued fraction has all its entries $\geq N$.

- If $\alpha \in \mathrm{Irrat}_{\geq N}$, then $f_0{:=}P_\alpha$ is renormalizable in the sense of Inou and Shishikura and the renormalization $f_1{:=}\mathcal{R}(f_0)$ has rotation number $-1/\alpha$ which, modulo 1 is in $\mathrm{Irrat}_{\geq N}$.
- We can thus define an infinite sequence of renormalizations $f_{j+1}{:=}\mathcal{R}(f_j)$.

If $\alpha \in \mathrm{Irrat}_N$, this allows us to construct a nested sequence of open set $\big\{U_j(\alpha)\big\}_{j\geq 1}$ such that $\mathcal{PC}(P_\alpha) \Subset U_j(\alpha)$ for all $j \geq 1$.

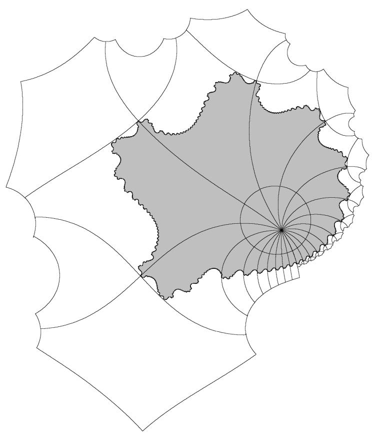


FIGURE 7. The open set $U_1(\alpha)$ for $\alpha = [10, 10, 10, \ldots]$.

We then show that

- for all $j \geq 1$, the set $U_j(\alpha)$ depends continuously on α,
- if α is of bounded type, then $\Delta(\alpha) \subset U_j(\alpha)$ and

$$\partial\big(U_j(\alpha), \Delta_\alpha\big) \underset{j\to\infty}{\longrightarrow} 0.$$

The result follows easily.

Talk 17—The proof

Matías Carrasco

The main theorem of the talk is the following,

Theorem 1. *There exist quadratic polynomials which have a Cremer fixed point and a Julia set with positive Lebesgue measure.*

For $\alpha \in \mathbb{C}$ we denote by P_α the quadratic polynomial $z \mapsto e^{2\pi i\alpha}z + z^2$. K_α denotes the filled-in Julia set (which is the complement of the basin of infinity) of P_α and J_α denotes the Julia set.

The proof of the theorem is based on the following idea: construct inductively a sequence of real parameters θ_n such that the sequence area(K_{θ_n}) is bounded bellow by some positive constat c. Then provided that the sequence θ_n converges to some limit parameter θ whose corresponding polynomial P_θ is not linearizable, the Julia set of this limit polynomial will have positive measure. To carry out this construction we need an induction step that enables us to control the existence of some cycle arbitrarily close to the origin and to control the loss of measure when we pass from one parameter to the next. More precisely

Proposition 1 (Induction Step)**.** *There exists a non-empty set S of bounded type irrationals such that: for all $\alpha \in S$ and all $\varepsilon > 0$ there exists $\alpha' \in S$:*

- $|\alpha - \alpha'| < \varepsilon$
- *$P_{\alpha'}$ has a cycle contained in $D^*(0, \varepsilon)$ and*
- *area$(K_{\alpha'}) \geq (1-\varepsilon)$area$(K_\alpha)$*

The set S consists of bounded type irrationals, so the corresponding polynomial P_α associated to a parameter $\alpha \in S$ has a Siegel disk at the origin Δ. In particular K_α has positive area. The proposition states that we can take a special perturbation α' of α (also in the set S) with the mentioned properties.

Indeed we can specify a possible set S and possible perturbtions α' as follows: Let N be the Inou-Shishikura constant. Then we can take

$$S_N = \{\alpha = [a_0, a_1, \ldots] \in \mathbb{R} - \mathbb{Q} : \sup a_i < +\infty, \inf a_i \geq N\}$$

and $\alpha' = \alpha_n = [a_0, a_1, \ldots, a_n, A, N, N, \ldots]$, for A and n sufficiently large.

The control of the existence of some cycle near the origin is proved using the techniques of explosion functions. We denote by p_n/q_n the rational approximations of α given by the continued fraction algorithm. Then the cycle is obtained taking the image under the explosion function at p_n/q_n of the q_n-th roots of $\alpha_n - p_n/q_n$.

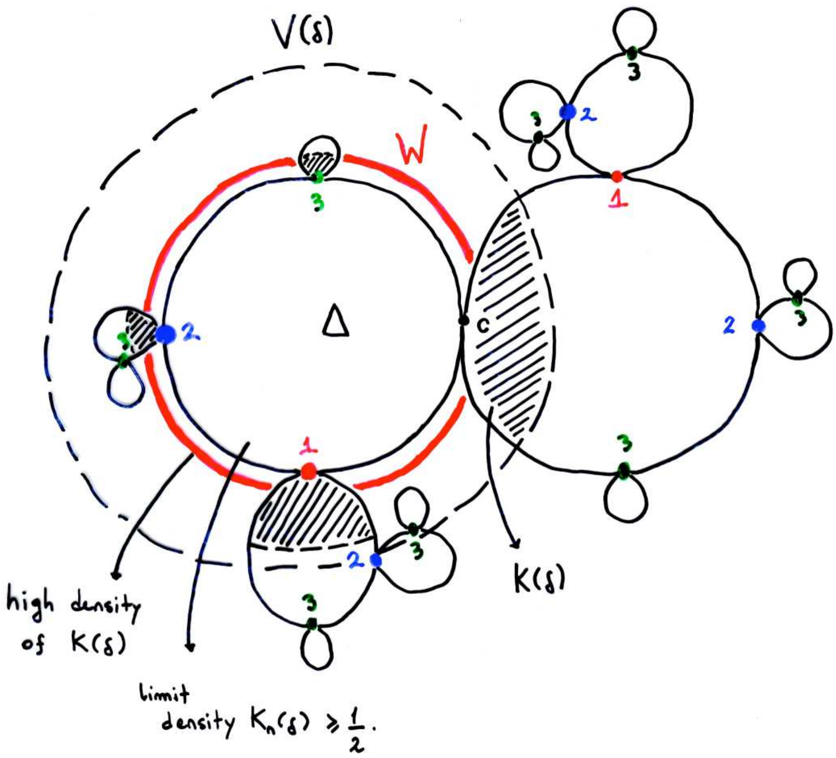


Figure 8.

So the distance of the cycle to the origin is small if A and n are large enough. But the control of the loss of measure is more difficult.

One central result is the following: the Siegel disk Δ_n of P_{α_n} (restricted to Δ) fills allmost 1/2 of the Siegel disk Δ. So the work concentrates on the promotion of this coefficient 1/2 to a coefficient 1. We cannot do this promotion working only with Δ_n. We have to look at points that escape Δ under iteration of P_{α_n} but still remain close or eventually belong to the Siegel disk. More precisely we look at the sets

$$\begin{aligned} K(\delta) &= \{z \in \mathbb{C} : \forall k \geq 0, P_\alpha^k(z) \in V(\delta)\} \quad \text{and} \\ K_n(\delta) &= \{z \in \mathbb{C} : \forall k \geq 0, P_{\alpha_n}^k(z) \in V(\delta)\} \end{aligned}$$

where $V(\delta) = \delta$-neighbourhood of Δ, and $\delta > 0$. See figure 8.

The main tools used are the following.

Theorem 2 (McMullen)**.** *Let α be a bounded type irrational and $\delta > 0$. Then (using the same notation as above) every point of $\partial\Delta$ is a Lebesgue density point of $K(\delta)$.*

The promotion is carried out using a bounded distortion pull-back argument, made possible thanks to the control on the postcritical set. Here it is important the fact that the entries of the continued fraction expansion of both α and α_n are greater than N. Finally Vitali's covering lemma enables us to prove the following lemma:

Lemma 1. *Define $\rho_n : (0, +\infty) \to [0, 1]$ by*

$$\rho_n(\delta) = dens_\Delta(\mathbb{C} - K_n(\delta)) = area\big((\mathbb{C} - K_n(\delta)) \cap \Delta\big)/area(\Delta).$$

Then for all $\delta > 0$, there exist $0 < \delta' < \delta$ and a sequence $c_n > 0$ converging to 0 such that:

$$\rho_n(\delta) \le \frac{3}{4}\rho_n(\delta') + c_n$$

If we set $\rho(\delta) = \limsup_{n\to+\infty} \rho_n(\delta)$, the lemma implies that the function ρ is constant equal 0. Since $K_n(\delta) \subset K_{\alpha_n}$ this implies

$$\mathrm{dens}_\Delta K_{\alpha_n} \to 1$$

Pulling back this situation to the preimages of Δ one obtains the desired result.

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