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Analysis of Boundary Element Methods

Organised by
Martin Costabel, Rennes
Ernst P. Stephan, Hannover

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Abstract. At this workshop challenging problems in boundary element research were addressed. Diverse topics from high frequency acoustics/electromagnetic scattering to time domain problems describing wave propagation were covered. The central themes were the numerical analysis of the boundary element method and its fast and efficient implementation.

Mathematics Subject Classification (2000): 65N38, 74S15, 80M15, 78M15, 31B10, 76M15.

Introduction by the Organisers

The workshop *Analysis of Boundary Element Methods*, organized by Martin Costabel (Rennes) and Ernst P. Stephan (Hannover). This meeting brought together 46 experts in numerical analysis.

Boundary element methods (BEM) are well established numerical methods with a wide range of applications. There are still many challenging problems, the aim of this workshop was to coordinate the efforts to tackle these problems.

A central theme of many talks was the error analysis. In the following we focus on some central aspects.

1. ***BEM for time dependent problems:*** Space-time boundary integral equations using integral representations with the fundamental solution of hyperbolic problems have become increasingly important in elastodynamics and electrodynamics. New results were presented concerning error analysis, stability and performance of numerical algorithms. Special attention was given to efficient quadrature of retarded potentials and also to the method of convolution quadrature for the computation of transient acoustic waves.

2. ***Stability at high frequencies:*** The numerical modeling of scattering problems at higher frequencies poses practical challenges due to the need to approximate highly oscillating functions. In addition, the analysis of many of the standard BEM loses usefulness in the high frequency range because the constants in the stability and error estimates either grow rapidly with the frequency of have unspecified dependency on the frequency. New boundary element methods have been presented whose stability deteriorates only slowly with frequency or are even unconditionally stable for all frequencies.
3. ***Fast algorithms:*** Multilevel approaches have produced several families of highly efficient algorithms. Several talks reported on the applications of the fast multipole method and on iterative techniques based on multigrid methods, H-matrices or on Schwarz preconditioners.
4. ***New concepts for BEM:*** Topics have been here: sparse p-version BEM with random loading, meshless methods for pseudodifferential equations, adaptive procedures, least squares FEM-BEM coupling, p-version applied to EFIE, BEM for eigenvalue problems.
5. ***Specific applications:*** Here the following have been addressed: Source identification and source reconstructions in acoustics with the help of inverse boundary elements, BEM for Maxwell's equations (currents and charges boundary integral equations, shape optimization of integrated lens antennas, electromagnetic scattering by a dielectric body), BEM for option pricing.

The list of participants included several of the founders of the discipline of mathematical analysis of BEM, but also some PhD students. This mix of both established and younger researchers created a particularly stimulating research atmosphere.

Workshop: Analysis of Boundary Element Methods

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Abstracts

Robust boundary integral methods in high frequency acoustic scattering

Ivan G. Graham

In this talk we gave a survey of robust numerical methods for computing the scattering of a high-frequency incident plane wave $u_I(\mathbf{x}) = \exp(\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{d}})$ by a bounded obstacle $\Omega \subset \mathbb{R}^m$ with boundary Γ (where $\widehat{\mathbf{d}}$ is a unit vector and $m = 2$ or 3). The scattered wave u_S satisfies the Helmholtz equation:

$$\Delta u_S + k^2 u_S = 0 \quad \text{in} \quad \Omega' = \mathbb{R}^d \backslash \Omega \ ,$$

subject to suitable boundary and far field conditions.

In many practical applications (where the physical wavelength is small compared to the diameter of the scatterer), the wave number k is very large and the solution is highly oscillatory. Conventional (piecewise polynomial based) numerical methods have complexity which grows polynomially in k and thus become unusable for large k.

In recent years there has been considerable interest in design of numerical methods which build in information about the asymptotic behaviour of these problems as $k \to \infty$. Some of these methods are (almost) uniformly accurate as the wave number k increases, and can be realised in a computational time which is robust in k also.

This talk gave a survey of methods of this type and their analysis, including important contributions of other authors e.g. [1–3, 7, 8], and those of the speaker and co-authors [5, 6, 9, 10].

The methods discussed are be based on boundary integral reformulations of the Helmholtz equation and the key components of the algorithm design and analysis which were discussed are:

(i) Estimates for the continuity and coercivity of the boundary integral operators explicitly in terms of k.
(ii) A proper description of the asymptotic behaviour of the solution in a format suitable for numerical analysis, by further development of the classical asymptotics results (e.g. [4, 11]) for this problem.
(iii) Design of suitable ansatz spaces for use in the Galerkin method and the analysis of their consistency error.
(iv) Construction of quadrature methods for the highly oscillatory Galerkin integrals.

In the talk we described recent results in this field and some remaining open problems.

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Efficient solution of elliptic BIEs with random loading or on a random boundary

Alexey Chernov

(joint work with Christoph Schwab)

Many problems in computational science can be written in an abstract way as problems of finding a unique solution $u \in X$ of an implicit possibly nonlinear equation

$$J(u, f) = 0 \quad \text{in } Z, \tag{1}$$

for some fixed $f \in Y$, where X, Y and Z are suitable Banach spaces. The operator J represents the system behavior and f stands for a priori known system parameters. As a model problem we consider Symm's integral equation. Let $D \Subset \mathbb{R}^2$ be an open, simply connected, bounded domain with smooth boundary $\Gamma := \partial D \in C^\infty$

and outer normal $\mathbf{n}$. Given a function $g \in H^{s+1}_{\mathrm{loc}}(\mathbb{R}^2), s \in \mathbb{R}_{\geq 0}$, find $u \in H^{-1/2}(\Gamma)$:

$$(2)\quad \int_\Gamma G(\cdot,\mathbf{y})u(\mathbf{y})\,ds_y =: \mathcal{V}_\Gamma u = \gamma_\Gamma g \quad \text{in } H^{1/2}(\Gamma), \quad G(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log\|\mathbf{x}-\mathbf{y}\|$$

for the single layer operator $\mathcal{V}_\Gamma$, and the Dirichlet trace operator $\gamma_\Gamma g := g|_\Gamma$. We rewrite (2) in the general form (1) by setting $J(u,g,\Gamma) := \mathcal{V}_\Gamma u - \gamma_\Gamma g$ and $f := \{g,\Gamma\}$. Assume $\mathrm{diam}(D) < 1$ to ensure unique solvability of (2) [7].

Nowadays, efficient numerical methods are well developed for many kinds of linear and nonlinear problems, which together with permanently growing computer power allows to compute a numerical approximation of u with high accuracy. However, in many practical applications the data f are not known precisely, e.g. due to inexact measurements or parameter calibration, errors in experiments. This makes highly accurate deterministic computations based on some mean approximation of the data f_0 less meaningful. The remedy is given by the probabilistic modeling

$$f(\omega) = f_0 + r(\omega), \quad \omega \in \Omega,$$

for a suitable probability space (Ω,Σ,P) over X consisting of the space of "events" Ω, σ-algebra of its subsets Σ and the probability measure P on Σ. The random field $r(\omega)$ models deviation of the system parameters from the mean field $f_0 = \mathbb{E}[f]$. Then $u(\omega)$ becomes also a random field. Note that for every fixed $\omega \in \Omega$ the functions $u(\omega)$ and $f(\omega)$ belong to X and Y respectively, cf. [1,5,6,8,9].

Usually, a particular solution $u(\omega)$ of (1) is of less interest than its statistical moments: mean field $\mathbb{E}[u]$, covariance and higher order centered moments $\bar{\mathcal{M}}^k[u] := \mathbb{E}[\bigotimes_{i=1}^k (u - \mathbb{E}u)]$. The simplest and the most widely used approach for solving (1) with random parameters $f = f(\omega)$ is the Monte Carlo method, consisting of sampling particular realizations of $f(\omega)$, finding (possibly in parallel) corresponding solutions $u(\omega)$ and post-processing computation of the statistical moments $\bar{\mathcal{M}}^k[u]$. The major drawback of the method is the low convergence rate $\mathcal{O}(M^{-1/2})$ w.r.t. the number of samples M [9].

We develop an alternative deterministic technique, which allows the direct computation of $\bar{\mathcal{M}}^k[u]$ or its approximation, if $\bar{\mathcal{M}}^k[f]$ is known. We abbreviate $X^{(k)} := \bigotimes_{i=1}^k X$ for some Banach space X and $\mathcal{A}^{(k)} := \bigotimes_{i=1}^k \mathcal{A}$ for some linear operator $X \to Y$.

Theorem 1. *[3, 5] Assume $J \in C^1(X \times Y, Z)$ and $r \in L^k(\Omega, Y)$. Then the solution u of (1) is Fréchet differentiable, $\bar{\mathcal{M}}^k[du_0[r]] \in X^{(k)}$ exists and satisfies*

$$(3)\qquad \mathcal{A}^{(k)}\bar{\mathcal{M}}^k[du_0[r]] = \mathcal{B}^{(k)}\bar{\mathcal{M}}^k[r],$$

with Fréchet derivatives $\mathcal{A} := d_u J_0$, $\mathcal{B} := -d_r J_0$. Furthermore, there holds

$$\begin{aligned} \mathbb{E}[u_r] &= u_0 + o(\|r\|_{L^1(\Omega,Y)}), \\ \bar{\mathcal{M}}^k[u_r] &= \bar{\mathcal{M}}^k[du_0[r]] + o(\|r\|^k_{L^k(\Omega,Y)}), \qquad k \geq 2. \end{aligned}$$

Considering the model problem (2) on a deterministic curve Γ with random loading $\gamma_\Gamma g(\omega) = g_0 + r(\omega) \in L^k(\Omega, H^{1/2}(\Gamma))$, we note that u_r depends linearly on

r, and the centered moments of u_r and $du_0[r]$ are identical. Thus, $\bar{\mathcal{M}}^k[u_r]$ solves (3) with $\mathcal{A} := \mathcal{V}_\Gamma$ and $\mathcal{B} = \mathrm{Id}$ [4, 9]. In the case of deterministic loading g and randomly perturbed Γ the solution of (2) depends nonlinearly on perturbations. In detail, let $r \in L^k(\Omega, C^\alpha(\Gamma))$ with α large enough and

$$\|r(\omega)\|_{C^\alpha(\Gamma)} \ll 1 \qquad P\text{-almost sure.} \tag{4}$$

Define $\Gamma_r := \{\mathbf{x}+r(\mathbf{x},\omega)\mathbf{n}(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ the family of randomly perturbed domains. Every sufficiently small (in the sense of (4)) perturbation r defines a bijective mapping between Γ and Γ_r. The coordinate transformation in (2) gives

$$J(u_r, r) := \int_\Gamma G(\mathbf{x}_r, \mathbf{y}_r) u_r(\mathbf{y}_r) \mathcal{J}_r(\mathbf{y})\, ds_{\mathbf{y}} - g(\mathbf{x}_r) = 0,$$

where $\mathbf{x}_r = \mathbf{x} + r(\mathbf{x},\omega)\mathbf{n}(\mathbf{x})$, $\mathbf{y}_r = \mathbf{y} + r(\mathbf{y},\omega)\mathbf{n}(\mathbf{y})$.

Theorem 2. *[3,5] (cf. [10]) There holds $J \in C^1(H^{-1/2}(\Gamma)\times C^\alpha(\Gamma), H^{1/2}(\Gamma))$ with*

$$\mathcal{A} := d_u J = \mathcal{V}_\Gamma, \quad \mathcal{B}r := -d_r J[r] = \left(\frac{\partial g}{\partial \mathbf{n}} - \mathcal{K}'_\Gamma u_0\right) r - \mathcal{K}_\Gamma(u_0 r) - \mathcal{V}_\Gamma(u_0 r \mathrm{div}_\Gamma \mathbf{n}),$$

the double layer operator $\mathcal{K}_\Gamma u := \displaystyle\int_\Gamma \frac{\partial}{\partial \mathbf{n}_y} G(\cdot, \mathbf{y}) u(\mathbf{y})\, ds_y$ and its adjoint $\mathcal{K}'_\Gamma$.

The above approach generalizes with similar arguments to other equations with strongly elliptic Fréchet derivative $d_u J$ in two and three spatial dimensions and to the case of several random parameters [3, 5, 9].

Both, the case of random loading g and the case of randomly perturbed boundary Γ lead to the following problem: given $b \in (H^{1/2}(\Gamma))^{(k)}$, find $\mu \in (H^{-1/2}(\Gamma))^{(k)}$:

$$\mathcal{V}_\Gamma^{(k)} \mu = b \qquad \text{on } (H^{1/2}(\Gamma))^{(k)}. \tag{5}$$

Mapping properties of $\mathcal{V}_\Gamma^{(k)}$ yield unique solvability of (5). We refer to [4, 9] for more details. Note that the dimension of the computational domain Γ^k grows linearly with k and a naive full tensor product discretization would lead to a prohibitive number of unknowns already for the second moment problem (curse of dimensionality). Sparse Grid methods [2, 9] are known to overcome this difficulty. In [4] we develop a deterministic p-version Sparse Grid approach, which exploits the tensor product structure of (5) and allows to leave out the most of the unknowns preserving the same up to the logarithmic factor convergence rate. As a result we obtain that the problem of finding the kth statistical moment has up to logarithms the same complexity as the problem of finding the mean field.

For $p \in \mathbb{N}_0$ we define the index set $\gamma_p^k := \{\mathbf{l} \in \mathbb{N}_0^k : \prod_{i=1}^k (l_i + 1) \le p + 1\}$, which relates to the *hyperbolic cross*, cf. [11]. Define $S_p^\gamma := \mathrm{span}\{\prod_{i=1}^k L_{l_i}(x_i) : \mathbf{l} \in \gamma_p^k, \mathbf{x} \in [-1,1]^k\}$ with Legendre polynomials $\{L_j\}$ and let $P_p^\gamma : L^2(I^k) \to S_p^\gamma$ be the L^2-orthogonal projection. In [4] we prove that P_p^γ has the same approximation properties as the projection onto the full tensor product space, when applied to functions from $(H^s(-1,1))^{(k)}$, $s \ge 1$. Let $S_{hp}^\gamma(\Gamma^k)$ be the natural conforming

sparse discretization of $(H^{-1/2}(\Gamma))^{(k)}$, cf. [4]. Consider the Galerkin formulation

$$(6)\quad \text{Find } \mu_{hp} \in S^{\gamma}_{hp}(\Gamma^k): \quad \int_\Gamma \mathcal{V}^{(k)}\mu_{hp} v_{hp}\, ds_y = \int_\Gamma b v_{hp}\, ds_y, \quad \forall v_{hp} \in S^{\gamma}_{hp}(\Gamma^k).$$

Theorem 3. *[4] Given $b \in (H^s(\Gamma))^{(k)}$, $s \geq 1/2$. Let $\mu \in (H^{-1/2}(\Gamma))^{(k)}$ and $\mu_{hp} \in S^{\gamma}_{hp}(\Gamma^k)$ be the unique solutions of (5) and (6) resp. Then $\exists C(s,k) > 0$:*

$$\|\mu - \mu_{hp}\|_{(H^{-1/2}(\Gamma))^{(k)}} \leq C(s,k)\left(\frac{h}{p+1}\right)^{s-1/2} \|b\|_{(H^s(\Gamma))^{(k)}}.$$

Let N be number of unknowns needed for discretization of the mean field problem, i.e. (6) with $k = 1$. Then for fixed mesh size h and varying p there holds $|S^{\gamma}_{hp}(\Gamma^k)| = \mathcal{O}(N(\log N)^{k-1}) \ll \mathcal{O}(N^k)$, there the last term corresponds to the number of unknowns needed for the full tensor product discretization of (5).

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Meshless methods for pseudodifferential equations on the sphere

T. Tran
(joint work with Q.-T. Le Gia, T.-D. Pham, I.H. Sloan, E.P. Stephan)

Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^3$. We consider pseudodifferential equations on $\mathbb{S}$ of the form

$$Lu = g, \tag{1}$$

where

$$Lu = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{L}(\ell)\hat{u}_{\ell,m} Y_{\ell,m},$$

and g is given on a finite set of scattered data points $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathbb{S}$. Here for $\ell = 0, 1, 2, \ldots$ and $m = -\ell, \ldots, \ell$, the functions $Y_{\ell,m}$ are spherical harmonics of degree ℓ, and

$$\hat{u}_{\ell,m} = \langle u, Y_{\ell,m} \rangle_{L_2(\mathbb{S})}.$$

Equation (1) arises in many areas of earth sciences; see, e.g., [1, 10].

It is assumed that L is a pseudodifferential operator of order $\alpha \in \mathbb{R}$, i.e., there exists positive constants C_1 and C_2 such that the symbol $\hat{L}(\ell)$ satisfies

$$C_1(\ell+1)^\alpha \le \hat{L}(\ell) \le C_2(\ell+1)^\alpha \tag{2}$$

for sufficiently large ℓ. Examples are the Laplace-Beltrami operator ($\alpha = 2$), the hypersingular integral operator ($\alpha = 1$), and the weakly singular integral operator ($\alpha = -1$), whose symbols are successively

$$-\ell(\ell+1), \quad -\frac{\ell(\ell+1)}{2\ell+1}, \quad \text{and} \quad \frac{1}{2\ell+1}.$$

It is well-known that $\hat{L}(\ell)$ is an eigenvalue of L with associated eigenfunction $Y_{\ell,m}$, i.e.,

$$LY_{\ell,m} = \hat{L}(\ell) Y_{\ell,m}, \quad \ell = 0, 1, 2, \ldots \quad \text{and} \quad m = -\ell, \ldots, \ell.$$

If L is of order α then L maps $H^{s+\alpha/2}$ to $H^{s-\alpha/2}$ for all $s \in \mathbb{R}$, where the Sobolev space H^s is defined for any $s \in \mathbb{R}$ by

$$H^s = \{v : \mathbb{S} \to \mathbb{R} \mid \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell+1)^{2s} |\hat{v}_{\ell,m}|^2 < \infty\}.$$

Let $\mathcal{K}(L) = \{\ell : \hat{L}(\ell) = 0\}$. Then

$$\mathrm{Ker}(L) = \mathrm{span}\{Y_{\ell,m} : m = -\ell, \ldots, \ell, \quad \ell \in \mathcal{K}(L)\},$$

and L is not one-to-one if $\mathcal{K}(L) \neq \emptyset$. In the case that $\mathcal{K}(L) \neq \emptyset$ and $\mathrm{card}\, \mathcal{K}(L) < \infty$, side conditions will be imposed to ensure unique solvability of (1), namely,

$$\mu^j u = a_j, \quad j = 1, \ldots, M, \tag{3}$$

where $M := \dim(\mathrm{Ker}(L))$, $a_1, \ldots, a_M \in \mathbb{R}$ are given, and $\mu^1, \ldots, \mu^M$ are linear functionals.

We assume that $\{\mu^1, \ldots, \mu^M\}$ is unisolvent with respect to $\mathrm{Ker}(L)$, i.e., for any $v \in \mathrm{Ker}(L)$, if $\mu^j v = 0$ for all $j = 1, \ldots, M$, then $v = 0$. Under the unisolvency assumption, equations (1) and (3) have a unique solution; see [5].

Solutions to equations (1) and (3) are approximated by radial spherical basis functions defined as follows. Let $\phi : [-1, 1] \to \mathbb{R}$ be a univariate function whose Fourier-Legendre expansion is of the form

$$\phi(t) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) \widehat{\phi}(\ell) P_\ell(t),$$

where P_ℓ is the Legendre polynomial of degree ℓ, and

$$\widehat{\phi}(\ell) = 2\pi \int_{-1}^{+1} \phi(t) P_\ell(t)\, dt.$$

By noting the addition formula for spherical harmonics (see e.g., [6]), we associate to each data point $\boldsymbol{x}_i \in X$ a radial spherical basis function Φ_i defined by

$$\Phi_i(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}) = \phi(\boldsymbol{x} \cdot \boldsymbol{x}_i), \quad \boldsymbol{x} \in \mathbb{S}, \quad i = 1, \ldots, N.$$

An essential feature of these functions is the positive definiteness of the matrix $A = (\Phi_i(\boldsymbol{x}_j))_{i,j=1,\ldots,N}$ provided that $\widehat{\phi}(\ell) > 0$ for an infinite number of odd ℓ and an infinite number of even ℓ; see [8, 14]. In the implementation, we choose ϕ to be the Wendland function [13] satisfying

$$c_1(\ell + 1)^{-\tau} \le \widehat{\phi}(\ell) \le c_2(\ell + 1)^{-\tau} \quad \forall \ell \ge 0, \tag{4}$$

where c_1 and c_2 are positive constants, and $\tau > 1/2$.

Letting $V_X = \mathrm{span}\{\Phi_1, \ldots, \Phi_N\}$, we seek an approximate solution $\tilde{u}$ to (1) and (3) in the form $\tilde{u} = \tilde{u}_0 + \tilde{u}_1$ where $\tilde{u}_1 \in V_X$ satisfies

$$\langle L\tilde{u}_1, v \rangle_* = \langle g, v \rangle_* \quad \forall v \in V_X \tag{5}$$

and $\tilde{u}_0 \in \mathrm{Ker}(L)$ satisfies (cf. (3))

$$\mu^j \tilde{u}_0 = a_j - \mu^j \tilde{u}_1, \quad j = 1, \ldots, M. \tag{6}$$

Here the inner product $\langle \cdot, \cdot \rangle_*$ can be the L_2-inner product (which results in the usual Galerkin method) or the inner product in the native space (see, e.g., [3, 4])

$$\langle v, w \rangle_* = \langle v, w \rangle_\phi := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{v}_{\ell,m} \widehat{w}_{\ell,m}}{\widehat{\phi}(\ell)}. \tag{7}$$

Since

$$v(\boldsymbol{x}_j) = \langle v, \Phi_j \rangle_\phi \quad \forall j = 1, \ldots, N,$$

equation (5) in this case is in fact the collocation equation

$$L\tilde{u}_1(\boldsymbol{x}_j) = g(\boldsymbol{x}_j), \quad j = 1, \ldots, N.$$

We note that the unisolvency assumption implies the unique existence of $\tilde{u}_0$ satisfying (6).

Equation (5) with the inner product (7) can be rewritten as

$$\langle L_* \tilde{u}_1, v\rangle_{L_2(\mathbb{S})} = \langle g, v\rangle_{L_2(\mathbb{S})} \quad \forall v \in V_X,$$

where L_* is a pseudodifferential operator with symbol $\hat{L}_*(\ell) = \hat{L}(\ell)/\hat{\phi}(\ell)$. In other words, the collocation equation for $Lu = g$ (with L being a pseudodifferential operator of order α) is in fact the Galerkin equation for $L_* u = g$ (with L_* being a pseudodifferential operator of order $\alpha + \tau$); see (2) and (4). Therefore, a unified error analysis can be performed for both the Galerkin and collocation methods; see [12].

It is well-known [7] that the system of equations arising from (5) is very ill-conditioned. We study [11] a preconditioner to this system in the form of an overlapping additive Schwarz method. A bound for the condition number of the additive Schwarz operator is obtained by using the method of alternating projections (see e.g., [9]). This method is used in [2] which studies the Galerkin method for elliptic equations.

Numerical experiments using data points collected by MAGSAT satellite illustrate our theoretical results; see [11].

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Boundary Element Methods for Eigenvalue Problems

Olaf Steinbach
(joint work with Gerhard Unger)

We consider the use of boundary integral equation methods to solve eigenvalue problems for partial differential equations. While finite element methods lead to linear algebraic eigenvalue problems, the use of boundary element methods may be an interesting alternative since only a surface discretization is needed instead of a volume mesh. As a model problem we consider the Dirichlet eigenvalue problem for the Laplace operator,

$$-\Delta u(x) = \lambda u(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^3, \quad u(x) = 0 \quad \text{for } x \in \Gamma = \partial\Omega, \tag{1}$$

As a first boundary integral equation approach one may consider the eigenvalue problem (1) as a Poisson equation with a given right hand side. Then, by using the fundamental solution of the Laplace operator one obtains a coupled boundary–domain integral equation system to be solved. However, when using integration by parts recursively, the volume potential can be reformulated by using surface potentials only. In the limiting case we then obtain an equivalent nonlinear eigenvalue problem for the related real valued Helmholtz single layer potential to find $\kappa = \sqrt{\lambda}$ and $t \in H^{-1/2}(\Gamma)$ such that

$$(V_\kappa t)(x) \frac{1}{4\pi} \int_\Gamma \frac{\cos \kappa |x-y|}{|x-y|} t(y) ds_y = 0 \quad \text{for } x \in \Gamma, \quad \|t\|_{V_0} = 1. \tag{2}$$

In order to solve the nonlinear eigenvalue problem (2) one may first consider a Galerkin discretization of (2) where the unknown eigenvalue κ enters the stiffness matrix. Then one may use a Newton scheme to find all zeros of the associated characeteristic polynomial. Instead, we will use a Newton scheme to solve the nonlinear eigenvalue problem (2) by using boundary element discretizations of the linearized problems [6]. In this case, the convergence proof follows from the mapping properties of all boundary integral operators involved. However, this approach is restricted to simple eigenvalues only.

By introducing the Riesz operator $J : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ we can formulate the eigenvalue problem in $H^{-1/2}(\Gamma)$, i.e.

$$L(\kappa)t = JV_\kappa t = 0 \quad \text{in } H^{-1/2}(\Gamma). \tag{3}$$

Then we can consider the Galerkin discretization of the related variational problem and we can derive related quasi–optimal error estimates for both the generalized eigenspace and the corresponding eigenvalues, see also [1–3].

It is possible to characterize the eigenvalues of the boundary integral operator V_κ in (2) as poles of the resolvent. In particular, let κ^* be an eigenvalue of $L(\kappa)t = 0$ with

$$L^{-1}(\kappa) = \sum_{k=-n}^{\infty} L_k (\kappa - \kappa^*)^k, \quad L_{-n} \neq 0.$$

Further, let $z \in H^{-1/2}(\Gamma)$ be chosen such that $(L^{-1}(\kappa)z, z) \neq 0$. Then we can define

$$\psi(\kappa) \;=\; \frac{1}{(L^{-1}(\kappa)z, z)}\,.$$

Now, for κ^* being a zero of $\psi(\kappa)$ with multiplicity n we find the representation

$$\psi(\kappa) = \frac{(\kappa-\kappa^*)^n}{(L_{-n}(\kappa)z, z)} - \frac{(L_{-n+1}(\kappa), z, z)}{(L_{-n}(\kappa)z, z)}(\kappa-\kappa^*)^{n+1} + O((\kappa-\kappa^*)^{n+2})\,.$$

In order to find the zeros of the polynomial $\psi(\kappa)$ we apply a Newton scheme to obtain Kummer's method [4, 5]

$$\kappa_{i+1} \;=\; \kappa_i - \frac{\psi(\kappa_i)}{\psi'(\kappa_i)} \;=\; \kappa_i - \frac{(x_i, z)}{(L'(\kappa_i)x_i, x_i)}, \quad L(\kappa_i)x_i = z\,. \tag{4}$$

Note that the convergence of the Newton scheme (4) is quadratic if κ^* is a simple pole, and linear if κ^* is a multiple pole.

When combining the convergence results of the Galerkin approximation with the convergence results of Kummer's method we finally obtain quasi–optimal error estimates for both the approximate eigenvalue and the related approximate eigenfunction. Numerical results confirm the theoretical estimates.

The proposed approach can be used to determine eigenvalues of certain partial differential equations with boundary conditions of different type. Typical applications are the Laplace equation, and the system of linear elastostatics, both with either Dirichlet or Neumann boundary conditions. In general, such an approach can also be generalized to the Maxwell eigenvalue problems, but there a careful study of the related function spaces is needed.

For an efficient realization this approach can be used within the framework of fast boundary element methods. However, efficient and robust iterative solvers are required for the linearized problems.

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On domain derivatives of the solution to the electromagnetic transmission problem

Frédérique Le Louër
(joint work with Martin Costabel)

1. Introduction

This work inserts itself as a part of a thesis bearing upon the shape optimization of integrated lens antennas [12]. The recent numerical methods to solve this problem require the survey of the dependence of the solution to the lens antennas radiation problem with respect to the lens. The radiation problem in question here is an interface scattering problem described by time-harmonic Maxwell's equations in the sense of distributions:

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_1 - \kappa_1^2\mathbf{E}_1 = 0 \text{ in } \Omega, \tag{1}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{E}_2 - \kappa_2^2\mathbf{E}_2 = \mathbf{J} \text{ in } \Omega^c \tag{2}$$

with transmission conditions on the surface Γ of the domain Ω (the lens):

$$\mathbf{n}\times\mathbf{E}_1 = \mathbf{n}\times(\mathbf{E}_2) \tag{3}$$

$$\mu_1^{-1}(\mathbf{n}\times\operatorname{curl}\mathbf{E}_1) = \mu_2^{-1}(\mathbf{n}\times\operatorname{curl}\mathbf{E}_2) \tag{4}$$

The exterior field $\mathbf{E}_2$ has to verify the Silver-Müller condition :

$$\lim_{\mathbf{r}\to\infty}(\operatorname{curl}\mathbf{E}_2\times x - i\kappa\mathbf{r}\mathbf{E}_2) = 0. \tag{5}$$

The domain Ω is assumed to be bounded and simply connected with a sufficiently smooth boundary Γ, the support of the current density $\mathbf{J}$ is disjoint of Ω and is of vanishing divergence, the constants κ_1 and κ_2 are the wavenumbers, μ_1 and μ_2 are the interior and the exterior magnetic permeabilities and $\mathbf{n}$ is the unit outer normal vector to the boundary Γ.
In order to study the dependence of the solution with respect to the domain Ω, we follow [14], [15] and [16] where Potthast has already considered the question in the case of Helmoltz equations and Maxwell's equations in Hölder spaces, with Dirichet or Neumann boundary conditions. Let the operator $\mathcal{S}$ map the boundary Γ onto the solution to the transmission problem. Firstly, we show a new boundary integral equation method to solve the transmission problem following [10]. Thanks to it, we can derive an integral representation of $\mathcal{S}(\Gamma)$ consisting of the electric and magnetic potentials, namely

$$\Psi_{E_\kappa} j = \kappa\psi_\kappa j + \kappa^{-1}\nabla\psi_\kappa \operatorname{div}_\Gamma j$$

and

$$\Psi_{M_\kappa} m = \operatorname{curl}\psi_\kappa m$$

where

$$\psi_\kappa u(x) = \int_\Gamma \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} u(y)d\sigma(y),$$

and their tangential traces, weakly and strongly singular boundary integral operators [5]. Secondly, we prove the Fréchet differentiability of the operator $\mathcal{S}$.

2. Boundary integral equation method

If $Im(\kappa) \geq 0$, the potential operators Ψ_{E_κ} and Ψ_{M_κ} are linear and continuous from $H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ into $H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ and their tangential traces are linear and continuous from $H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ into itself (see [4], [5], [13] and [7]). The boundary integral operator C_0^* defined by :

$$C_0^* = \mathbf{n}(x) \times \psi_0 + \overrightarrow{\mathrm{curl}}_\Gamma \, \psi_0 \mathrm{div}_\Gamma$$

is continuous and invertible from $H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ into itself [17].
We assume that $\mathbf{E}_2$ admits the following integral representation :

$$\mathbf{E}_2(x) = \int_{\mathbb{R}^3} \frac{e^{\kappa_2|x-y|}}{4\pi|x-y|} \mathbf{J}(y) ds(y) - a(\Psi_{E_{\kappa_2}} j)(x) - b(\Psi_{M_{\kappa_2}} C_0^* j)(x)) \text{ in } \bar{\Omega}^c$$

where a and b are arbitrary constants and $j \in H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Using the transmission conditions, we derive a boundary integral equation of unknown j which is uniquely solvable in $H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ for all values κ_2^2 provided some restrictions on the constants a, b, κ_1, κ_2, μ_1 and μ_2. This method yields an integral representation of the solution to the transmission problem.

3. Fréchet differentiability

To study the depence of the solution with respect to the domain (or the interface) we are proceeding step by step. In a first time, we define some variations [9] of the domain Ω generated by the local transformations in $\mathbb{R}^3$ of the form $x \mapsto x + r(x)$ where r is a vector function defined at least in the neighbourhood of the boundary Γ. Provided some hypothesis on r, this transformation deforms the domain Ω into the domain Ω_r with smooth boundary Γ_r again. This leads us back to the study of the Fréchet differentiability of the operator $\mathcal{S}$ which maps r onto the solution of the scattering problem by the interface Γ_r. The solution $\mathcal{S}(r)$ is composed of potentials and boundary integral operators defined on $H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. It is natural to ask ourselves how to differentiate a mapping $r \mapsto f(r)$ with f belonging to a space of functions on Γ when Γ varies. In a second time, we get around this difficulty using the Helmholtz decomposition of $H_\times^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ (see [3] and [6]). Since Γ_r is smooth, the space $H_\times^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ admits the decomposition :

$$H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) = \nabla_{\Gamma_r}(H^{\frac{3}{2}}(\Gamma_r)/\mathbb{R}) \bigoplus \overrightarrow{\mathrm{curl}}_{\Gamma_r}(H^{\frac{1}{2}}(\Gamma_r)/\mathbb{R}).$$

Let the operator $\mathbf{P}_r$ map $\nabla_{\Gamma_r} p_r + \overrightarrow{\mathrm{curl}}_{\Gamma_r} q_r$ onto $\nabla_\Gamma \tau_r(p_r) + \overrightarrow{\mathrm{curl}}_\Gamma \tau_r(q_r)$ where $\tau_r(p_r)(x) = p_r(x_r)$. This operator is continuous and invertible from $H_\times^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ into $H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. We modify the integral representation of $\mathcal{S}(r)$ by inserting the

identity $\mathbf{P}_r^{-1}\mathbf{P}_r = I_{H_\times^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r},\Gamma_r)}$. Note that if the operator A is linear and continuous from $H_\times^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r},\Gamma_r)$ into itself then $\mathbf{P}_r A \mathbf{P}_r^{-1}$ is linear and continuous from $H_\times^{-\frac{1}{2}}(\mathrm{div}_{\Gamma},\Gamma)$ into itself. In a third time, we prove the Fréchet differentiabilty [1] of a class of boundary integral operators of pseudo-homogeneous kernels (see [13]). Finally we prove the Fréchet differentiability of the operator $\mathcal{S}$ and give the transmission problem associated to the first derivative.

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Boundary Integral Integral Equations and Pseudodifferential Operators

Wolfgang L. Wendland
(joint work with George C. Hsiao)

It is well known that boundary integral equations for regular elliptic boundary value problems can be understood as pseudodifferential equations acting on the boundary manifold provided it is smooth enough. Based on volume potentials which for systems of elliptic differential equations are given by classical pseudodifferential operators with symbols of rational type, this can be shown with the classical results by Boutet de Monvel and Gerd Grubb [1–4] every regular elliptic boundary value problem allows the reduction to a coupled system of domain and boundary equations. For the solution in the domain and the unknown parts of the Cauchy data on the boundary, the operators are pseudodifferential operators in the domain, their traces on the boundary, potentials with charges on the boundary, and pseudodifferential operators on the boundary. If one uses asymptotic expansions of the classical symbols into series of symbols which are homogeneous in the Fourier variable $\xi \in \mathbb{R}^n$,

$$a \sim \sum_{j\in\mathbb{N}_0} a_{m-j}\ ,\ a_{m-j}(x,\xi) = a^0_{m-j}(x,\xi) \quad \text{for } |\xi| \geq 1\,,$$
$$a^0_{m-j}(x,t\xi) = t^{m-j} a^0_{m-j}(x,\xi) \quad \text{for } 0<t\ ,\ 0 \neq \xi \in \mathbb{R}^n\,,$$

where the pseudodifferential operator of order $m \in \mathbb{R}$ is given by

$$A(x,D)u = (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} e^{i(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi\,;$$

and also for the kernel functions of Hadamard finite part integral operators,

$$Au(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u(x) + \text{ p.f.} \int\limits_{\Omega} k(x,x-y)u(y)dy\,,$$

i.e., with Seeley's pseudohomogeneous asymptotic expansions [6],

$$k(x,x-y) \sim \sum_{\kappa}^{\infty} k_{\kappa+j}(x,x-y) \quad \text{where with } \kappa = -m-n\,,$$
$$k_{\kappa+j}(x,z) = \begin{cases} f_{\kappa+j}(x,z) & \text{for } \kappa+j \notin \mathbb{N}_0, \\ f_{\kappa+j}(x,z) + \log|z| \wp_{\kappa+j}(x,z) & \text{for } \kappa+j \in \mathbb{N}_0, \end{cases}$$
$$f_{\kappa+j}(x,tz) = t^{\kappa+j} f_{\kappa+j}(x,z) \quad \text{for } 0<t\,;\ \wp_{\kappa+j}(x,z) = \sum_{|\alpha|=\kappa+j} c_\alpha(x) z^\alpha\,,$$

then it turns out that there is a beautiful correlation between the pseudohomogeneous terms $a_{m-j}(x,\xi)$, $j = 0,1,\ldots$, of the symbol expansion and the homogeneous term $k_{-n-m+j}(x,z)$ of the integral operator essentially given by the Fourier

transform (see [7] and [5, Section 7.1.4]). If the integral operator is given, then

$$a^0_{m-j}(x,\xi) = \begin{cases} \sum_{|\alpha|=m-j} c_\alpha(x)(i\xi)^\alpha + \text{ p.f.} \int\limits_{\mathbb{R}^n} k_{\kappa+j}(x,z)e^{-i\xi\cdot z}dz & \text{for } 0 \le m-j, \\ \lim\limits_{t\to\infty} \int\limits_{\mathbb{R}^n} k_{\kappa+j}(x,z)\psi\left(\frac{z}{t}\right) e^{-i\xi\cdot z}dz & \text{for } m-j<0, \end{cases}$$

with a C^∞_0 cut–off function ψ, which is identical 1 near the origin. If the symbol is given, then the kernels can be evaluated by

$$k_{\kappa+j}(x,z) = \begin{cases} (2\pi)^{-n} \text{ p.f.} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} a^0_{m-j}(x,\xi)d\xi & \text{for } m-j<0\,, \\ (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} a^0_{m-j}(x,\xi)\psi(\xi)d\xi + & \text{for } m-j\ge 0\,, \\ +|z|^{-2\ell}(2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}(-\Delta_\xi)^\ell \big\{a^0_{m-j}(x,\xi)\big(1-\psi(\xi)\big)\big\}d\xi\,. \end{cases}$$

If $m \in \mathbb{N}_0$, then the coefficients of the differential operator can be computed as

$$c_\alpha(x) = (2\pi)^{-n}\frac{1}{\alpha!}\Big\{ \int\limits_{\mathbb{R}^n}\int\limits_{\Omega} (y-x)^\alpha \psi(y) e^{i(x-y)\cdot\xi} dy a^0_{m-j}(x,\xi)d\xi$$

$$- \text{ p.f.} \int\limits_{\Omega} k_{\kappa+j}(x,x-y)(y-x)^\alpha\psi(y)dy\Big\} \quad \text{for } |\alpha| = m-j\ ,\ 0\le j\le m\in\mathbb{N}_0\,.$$

Therefore, every pseudodifferntial operator is also an Hadamard finite part operator (plus possibly a differential operator) whose pseudohomogeneous kernel expansion is explicitly given. For $m \in \mathbb{N}_0$ and $-n-m+j\le 0$, the finitely many terms $k_{\kappa+j}(x,z)$ satisfy the Tricomi conditions

$$\int\limits_{|\Theta|=1} \Theta^\alpha k_{\kappa+j} d\omega(\Theta) = 0\ ,\ |\alpha| = m-j\ ,\ 0\le j\le m\in\mathbb{N}_0\,.$$

Conersely, every Hadamard finite part operator whose kernel admits an asymptotic pseudohomogeneous expansion satisfying the Tricomi conditions, defines a pseudodifferential operator.

Since the pseudodifferential operators with symbols of rational type satisfy Boutet de Monvel's transmission conditions, the boundary traces of volume potentials in the domain Ω, and their traces on Γ define continuous mappings between appropriate Sobolev–Slobodecki spaces in the domain and on the boundary.

For a simple example we demonstrate the conversion of a strongly elliptic Dirichlet problem into a coupled strongly elliptic system of pseudodifferential equations of domain - and boundary integral equations.

The presentation is based on some chapters of our new book [5].

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Least Squares methods for FEM-BEM coupling

Matthias Maischak

We are investigating a non-linear transmission problem for which we will derive a fem-bem coupling formulation, based on a least squares approach, extending ideas in [2, 3].

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $\Gamma = \partial\Omega$. $\Omega_c := \mathbb{R}^d \backslash \bar{\Omega}$. Given $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, $t_0 \in \tilde{H}^{-1/2}(\Gamma)$, $\varrho \in C^1[0,\infty)$, the non-linear model problem reads:
Find $u_1 \in H^1(\Omega)$, $u_2 \in H^1_{\rm loc}(\Omega_c)$ with

$$
\begin{aligned}
-\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1) = f \text{ in } \Omega, \qquad & \Delta u_2 = 0 \text{ in } \Omega_c \\
u_1 = u_2 + u_0, \qquad & \varrho(|\nabla u_1|)\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} + t_0 \text{ on } \Gamma \\
u_2(x) = \begin{cases} A\log|x| + o(1), & d = 2 \\ \mathcal{O}(|x|^{2-d}), & d \geq 3 \end{cases} & , |x| \to \infty
\end{aligned}
$$

Let u_1 be replaced by u^*. Introducing the flux $\theta^* = \varrho(|\nabla u^*|)\nabla u^*$, the co-normal derivative $\sigma^* = \varrho(|\nabla u^*|)\frac{\partial u^*}{\partial n}$ and using boundary integral operators we can rewrite the pde as a non-linear first order system:

Find $(\theta^*, u^*, \sigma^*) \in X := L^2(\Omega)^d \times H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$ such that

$$
\mathcal{L}^{NL}(\theta^*, u^*, \sigma^*) = \mathcal{F} \in X' = L^2(\Omega)^d \times \tilde{H}^{-1}(\Omega) \times H^{1/2}(\Gamma). \tag{1}
$$

$\mathcal{L}^{NL} : X \to X'$ and $\mathcal{F} \in X'$ are given by

$$
\mathcal{L}^{NL}(\zeta, v, \tau) = \begin{pmatrix} \varrho(|\nabla v|)\nabla v - \zeta \\ \operatorname{div}\zeta - \frac{1}{2}\delta_\Gamma \otimes (Wv + 2\zeta\cdot n - (I - K')\tau) \\ (I-K)v + V\tau \end{pmatrix}
$$

$$
\mathcal{F} = \begin{pmatrix} 0 \\ -f - \frac{1}{2}\delta_\Gamma \otimes (Wu_0 + 2t_0 - (I-K')t_0) \\ (I-K)u_0 + Vt_0 \end{pmatrix}
$$

Under appropriate assumptions on ϱ the problem is uniquely solvable [6].

Then $\mathcal{L}_u : X \to X'$, the Fréchet derivative of $\mathcal{L}^{NL}$, is given by

$$
\mathcal{L}_u(\zeta, v, \tau) = \begin{pmatrix} \tilde{\varrho}(\nabla u)\nabla v - \zeta \\ \operatorname{div}\zeta - \frac{1}{2}\delta_\Gamma \otimes (Wv + 2\zeta\cdot n - (I - K')\tau) \\ (I-K)v + V\tau \end{pmatrix}
$$

with $\tilde{\varrho}(x) = \varrho(|x|)I_{d\times d} + \varrho'(|x|)\frac{xx'}{|x|}$. The Newton scheme for solving (1) reads:

Let $(\theta^{(0)}, u^{(0)}, \sigma^{(0)}) \in X$. For $n = 1, 2, 3, \ldots$ find $(\theta, u, \sigma) \in X$ such that

$$\mathcal{L}_{u^{(n-1)}}(\theta, u, \sigma) = \mathcal{F} - \mathcal{L}^{NL}(\theta^{(n-1)}, u^{(n-1)}, \sigma^{(n-1)}) \tag{2}$$

and

$$(\theta^{(n)}, u^{(n)}, \sigma^{(n)}) = (\theta^{(n-1)}, u^{(n-1)}, \sigma^{(n-1)}) + (\theta, u, \sigma).$$

We introduce the least squares functional $J_{(\vartheta,w,\xi)}$ by

$$J_{(\vartheta,w,\xi)}(\zeta, v, \tau) := \|\mathcal{L}_w(\zeta, v, \tau) - \mathcal{F} + \mathcal{L}^{NL}(\vartheta, w, \xi)\|^2_{X'}.$$

Now we can rewrite the linearized problem (2) using a least squares formulation:

Find $(\theta, u, \sigma) \in X$ with

$$J_{(\vartheta,w,\xi)}(\theta, u, \sigma) = \min_{(\zeta,v,\tau)\in X} J_{(\vartheta,w,\xi)}(\zeta, v, \tau) \text{ for } (\vartheta, w, \xi) \in X. \tag{3}$$

The variational formulation of (3) reads: Find $(\theta, u, \sigma) \in X$, s.t.

$$B_w((\theta, u, \sigma), (\zeta, v, \tau)) = G_{(\vartheta,w,\xi)}(\zeta, v, \tau) \quad \forall (\zeta, v, \tau) \in X,$$

$$\text{with} \quad B_w((\theta, u, \sigma), (\zeta, v, \tau)) = [\mathcal{L}_w(\zeta, v, \tau), \mathcal{L}_w(\theta, u, \sigma)]_{X'\times X'}$$

$$G_{(\vartheta,w,\xi)}(\zeta, v, \tau) = [\mathcal{L}_w(\zeta, v, \tau), \mathcal{F} - \mathcal{L}^{NL}(\vartheta, w, \xi)]_{X'\times X'}$$

Let X_h be a discrete subspace of X with the canonical imbedding $P_h : X \to X_h$ and its adjoint $P_h^* : X' \to X_h^*$. Analogously to [1] we replace now the inner product of the dual space X' by a discrete inner product on X_h^*, which is given by standard preconditioners I_h, B_h, C_h on the discrete subspaces $H_h \times V_h \times S_h$ of $L^2(\Omega)^d \times H^1(\Omega) \times \tilde{H}^{-1/2}(\Gamma)$. I.e. $E_h : X_h^* \to X_h$ is defined by $E_h(\theta_h, u_h, \sigma_h) = (I_h\theta_h, B_h u_h, C_h\sigma_h)$.

The discretized Newton scheme reads now: Let $(\theta_h^{(0)}, u_h^{(0)}, \sigma_h^{(0)}) \in X_h$. For $n = 1, 2, 3, \ldots$ find $(\theta_h, u_h, \sigma_h) \in X_h$ such that

$$B^{(h)}_{u_h^{(n-1)}}((\theta_h, u_h, \sigma_h), (\zeta_h, v_h, \tau_h)) = G^{(h)}_{(\theta_h^{(n-1)}, u_h^{(n-1)}, \sigma_h^{(n-1)})}(\zeta_h, v_h, \tau_h) \tag{4}$$

for all $(\zeta_h, v_h, \tau_h) \in X_h$ and

$$(\theta_h^{(n)}, u_h^{(n)}, \sigma_h^{(n)}) = (\theta_h^{(n-1)}, u_h^{(n-1)}, \sigma_h^{(n-1)}) + (\theta_h, u_h, \sigma_h)$$

$$\text{with} \quad B^{(h)}_w((\theta, u, \sigma), (\zeta, v, \tau)) = (E_h P_h^* \mathcal{L}_w(\zeta, v, \tau), P_h^* \mathcal{L}_w(\theta, u, \sigma))_{X_h\times X_h^*}$$

$$G^{(h)}_{(\vartheta,w,\xi)}(\zeta, v, \tau) = (E_h P_h^* \mathcal{L}_w(\zeta, v, \tau), P_h^*(\mathcal{F} - \mathcal{L}^{NL}(\vartheta, w, \xi)))_{X_h\times X_h^*}$$

Theorem (A priori estimate) Let $X_h \subset X$ satisfying the usual approximation properties. Then the unique solution $(\theta_h^*, u_h^*, \sigma_h^*) \in X_h$ of the discretized formulation (4) exists and there holds the following a priori estimate

$$\|\theta^* - \theta_h^*\|_{[L^2(\Omega)]^d} + \|u^* - u_h^*\|_{H^1(\Omega)} + \|\sigma^* - \sigma_h^*\|_{\tilde{H}^{-1/2}(\Gamma)} \le C\, h^{r-1} p^{r-1} \|u^*\|_{H^r(\Omega)}$$

with C depending on I_h, B_h, C_h and the continuity and ellipticity constants of $B_w(\cdot,\cdot)$.

Example Let Ω be a L-Shaped domain with vertices $(-\frac{1}{4}, -\frac{1}{4})$, $(\frac{1}{4}, -\frac{1}{4})$, $(\frac{1}{4}, 0)$, $(0, 0)$, $(0, \frac{1}{4})$, $(-\frac{1}{4}, \frac{1}{4})$. We set $f = 0$ in Ω, $u_0(r, \phi) = r^{2/3}\sin[\frac{2}{3}(2\pi - \varphi)] - \log \bar{r}$, and

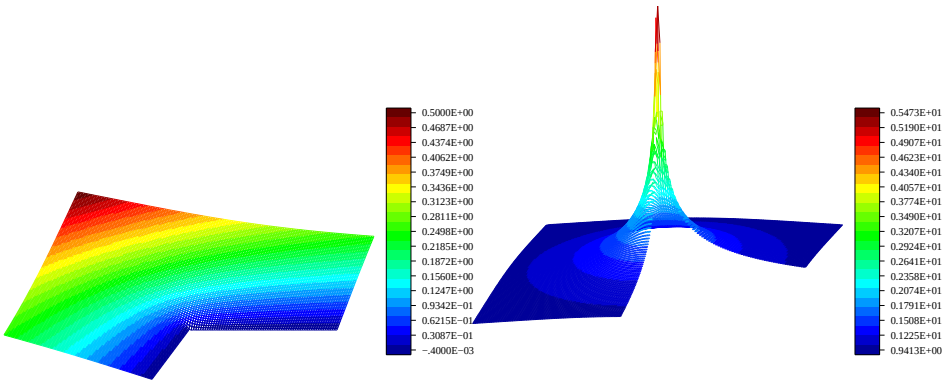

FIGURE 1. Solution u_h (left) and flux $|\theta_h|$ (right) of (4).

#total	δ_θ	α_θ	δ_u	α_u	δ_σ	α_σ	It_{New}
209	0.0645078	—	0.0031188	—	0.4409924	—	6
705	0.0374566	0.784	0.0009215	1.758	0.0833435	2.403	6
2561	0.0234335	0.676	0.0002988	1.624	0.0635836	0.390	6
9729	0.0148209	0.660	0.0001010	1.564	0.0544765	0.223	6
37889	0.0093813	0.659	.3450E-04	1.549	0.0480243	0.181	5
149505	0.0059336	0.660	.1171E-04	1.558	0.0426451	0.171	5

TABLE 1. Non-linear transmission problem. Multigrid preconditioner. L^2-errors of θ, u, σ. Convergence rates.

$t_0(r,\phi) = \frac{\partial u_0}{\partial n}$ on Γ, with $\bar{r} = |4(x,y) - (-0.5,-0.5)|$. The non-linear coefficient function is given by $\varrho(t) = \frac{1}{6}(1 + \frac{5}{1+5t})$.

Consequently, the exact solution is $u_1 = r^{2/3}\sin[\frac{2}{3}(2\pi - \varphi)]$ in Ω and $u_2 = \log \bar{r}$ in Ω_c, e.g. see Figure 1.

For the h-version we choose continuous and piecewise linear functions on Ω for u_h, piecewise constant functions on Γ for σ_h and $H(\mathrm{div};\Omega)$-conforming Raviart-Thomas elements of lowest order for θ_h. For preconditioning (and discrete inner products) we use the Multigrid V-cycle algorithm for B_h and C_h, and the identity for I_h. The CG method is choosen as iterative solver.

In Table 1 we observe that the number of Newton steps is bounded. Also the convergence rates for θ, u and σ show the expected values.

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Efficient quadrature of retarded potentials

Elke Ostermann
(joint work with M. Maischak, E. P. Stephan)

Using the retarded single layer potential, we give an outline of the discretization scheme of the Galerkin formulation of a retarded integral operator. Let $\Omega \subset \mathbb{R}^3$ bounded open domain with boundary $\Gamma := \partial\Omega$. Study the retarded integral equation

$$Vp(t,x) := \frac{1}{4\pi} \int_\Gamma \frac{p(t-|x-y|)}{|x-y|} ds_y = f(t,x) \tag{1}$$

with the retarded single layer potential V. The Galerkin method for (1) reads:

Find $p(t,x)$ for appropriate test functions $\eta(t,x)$, $x \in \Gamma$, $t \in \mathbb{R}_+$

$$\int_0^\infty \int_\Gamma Vp(t,x)\eta(t,x) ds_x \, dt = \int_0^\infty \int_\Gamma f(t,x)\eta(t,x) ds_x \, dt. \tag{2}$$

In [2] and the references therein the solvability and unconditional stability of the Galerkin scheme is discussed. Here, we study the properties of the discrete retarded potential. We triangulate Γ into triangles T_i $(i = 1, \dots N_s)$ and subdivide the time interval into equidistant subintervals with $t_i = i\Delta t$ $(i = 0, \dots)$. Choosing constant basis functions in space and time, $\gamma^m(t) = \chi_{(t_m, t_{m+1}]}(t)$ and $\varphi_i(x) = \chi_{T_i}(x)$, we obtain

$$p_h(t,x) = \sum_{i=1}^{N_s} \beta_i(t)\varphi_i(x) = \sum_{m=0}^{\infty} \sum_{i=1}^{N_s} b_i^m \gamma^m(t)\varphi_i(x) = \sum_{m=0}^{\infty} \gamma^m(t)\phi^m(x)$$

and thus the discrete formulation of (2) reads:

Find $p_h(t,x) \in S^0(\Delta t) \otimes S^0(h)$ such that

$$\int_0^\infty \int_\Gamma Vp_h(t,x)\gamma^n(t)\varphi_j(x) ds_x \, dt = \int_0^\infty \int_\Gamma f(t,x)\gamma^n(t)\varphi_j(x) ds_x \, dt \tag{3}$$

for all $\gamma^n \in S^0(\Delta t)$ and $\varphi_j \in S^0(h)$ $(n = 0, \dots, j = 1, \dots N_s)$.

One easily sees as e. g. pointed out in [1] that the Galerkin elements of (3) only depend on the time difference and read

$$
\begin{aligned}
V_{ij}^{n-m} &:= \int_0^\infty \int_\Gamma \int_\Gamma \frac{1}{|x-y|} \gamma^m(t-|x-y|)\varphi_i(y)\gamma^n(t)\varphi_j(x)\, ds_y\, ds_x\, dt \\
&= \iint\limits_{E_{n-m-1}} \varphi_i(y)\varphi_j(x)\, ds_y\, ds_x - t_{n-m-1} \iint\limits_{E_{n-m-1}} \frac{\varphi_i(y)\varphi_j(x)}{|x-y|}\, ds_y\, ds_x \\
&\quad + t_{n-m+1} \iint\limits_{E_{n-m}} \frac{\varphi_i(y)\varphi_j(x)}{|x-y|}\, ds_y\, ds_x - \iint\limits_{E_{n-m}} \varphi_i(y)\varphi_j(x)\, ds_y\, ds_x.
\end{aligned} \tag{4}
$$

The light cone integration domain is defined by

$$E_k := \{(x,y) \in \Gamma \times \Gamma : t_k \le |x-y| \le t_{k+1}\}$$

and is also known as domain of influence consisting of spheres with time dependent radii intersected with the mesh. Hence, a time stepping method is obtained. We will concentrate on the analysis of the discrete retarded potential and fix the usually time dependent radii to $r_{\max} > r_{\min} \ge 0$. For at most weakly singular kernels $G(x-y) = |x-y|^\nu$, $(\nu \ge -1)$, the Galerkin elements consist of sums of integrals of the type (compare (4))

$$G_{ij} = \iint\limits_{E_{T_i,T_j}} G(x-y)\, ds_y\, ds_x = \int\limits_{T_i} \underbrace{\int\limits_{E_{T_j}(x)} G(x-y)\, ds_y}_{:=P(x)}\, ds_x,$$

with integration domains

$$
\begin{aligned}
E_{T_i,T_j} &= \{x \in T_i, y \in T_j : r_{\min} \le |x-y| \le r_{\max}\} \\
E_T(x) &= \{y \in T : r_{\min} \le |x-y| \le r_{\max}\} = T \cap (B_{r_{\max}}(x) \setminus B_{r_{\min}}(x)).
\end{aligned}
$$

As we can rewrite the potential integral P in such a way that

$$P(x) = \int\limits_{T \cap B_{r_{\max}}(x)} G(x-y)\, ds_y - \int\limits_{T \cap B_{r_{\min}}(x)} G(x-y)\, ds_y, \tag{5}$$

and thus reduce the analysis to integrals of the type

$$I(x) = \int\limits_{T \cap B_R(x)} G(x-y)\, ds_y$$

with fixed radius R.

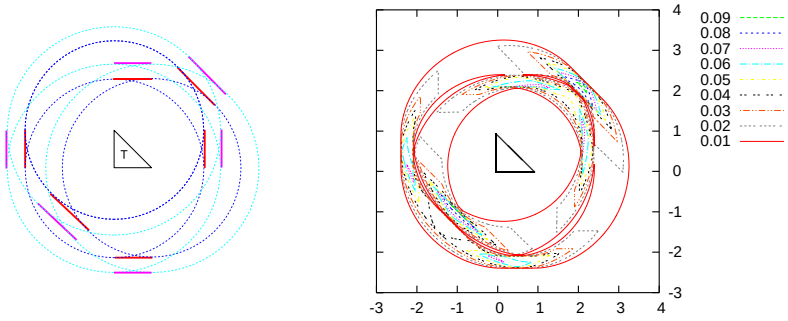
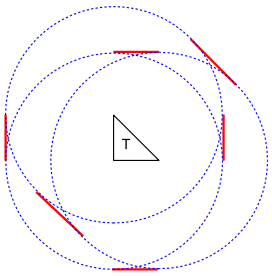


FIGURE 1. Distribution of 'geometrical light cone singularities': sketch (left), numerical result for $|\nabla P(x)|$ with $r_{\min} = 2.1$, $r_{\max} = 2.4$, $G(x, y) = 1/|x - y|$



In addition to 'classical' edge and corner-edge singularities of boundary integral operators as discussed in [4] and [5], one observes a new kind of geometrical singularity in the second derivative. As sketched in the left figure for these 'geometrical light cone singularities' also occuring for regular kernels G, we can distinguish propagated corner (dashed circles) and edge singularities (solid lines). These geometrical singularities are singular in the sense, that one observes jumps or onesided poles in the second derivative, respectively.

This means, that the retarded potentials do not only posses the classical so-called near and far field due to a singular kernel function, but geometrical singularities are distributed in each time step over the domain, although their strength decreases for growing time steps. If we now return to (5), we understand, that the whole retarded integral posses double singularities due to the minimal and maximal radius, see Fig. 1.

The decomposition of the test triangle in the light cone of the ansatz triangle with respect to the deduced 'geometrical light cone singularities' using appropriate grading strategies and a rigorous error analysis the used quadrature scheme are currently under investigation. For details on the implementation see [3].

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Why Time-Domain BEM has not had the operational deployment we could expect 10 years ago? And how to progress...

Isabelle Terrasse
(joint work with Toufic Abboud, Guillaume Sylvand)

Time marching schemes have been considered for a long time very delicate when applied to time domain integral equations arising from various wave problems. Stability issue has been and seems to remain at the centre of investigations [1], [6]. Attempts to reach stability include use of implicit schemes, spatial and temporal averaging procedures during the time stepping, or try to extrapolate the beginning of the response signals using different techniques as Prony's or autoregressive models. Other authors get round the difficulty by working on MFIE or CFIE rather than EFIE. This type of approach only delays the onset of the instability and/or seriously compromise the precision of the results. Recent studies are dedicated to improve the precision by using high order approximation [4] and leapfrogging [5], based on extrapolation techniques for band-limited signals. Thus complete time-space variational approximation leads to unconditionally stable and precise schemes. Several PhD thesis have been dedicated to study of this theory and its application to different situations in electromagnetism, acoustics and elastodynamics, under supervision of J.C. Nédélec and A. Bachelot. A rigorous functional framework and the first 3D electromagnetic EFIE stable computation without any averaging trick have been published in the thesis of Isabelle Terrasse [2]. Since then, delayed potentials have been applied and studied in many situations, and accelaration techniques have been transposed from previous frequency domain studies [3]. We present a unified presentation of the full variational theory. The resolution process for this system of equations involve at each time-step the resolution of a sparse system (easily done) for computing the current time step, and the computation of a convolution product for taking into account the influence of this new time step on the future time steps. This convolution product is the most time consuming part of the algorithm. It can be speeded-up through the use of a multipole algorithm, the time-domain equivalent of the widely accepted frequency domain fast multipole algorithm. The harmonic FMM is used to compute matrix vector product in order to solve BEM through an iterative solver, the transient FMM computes the convolution product at each time step : in both cases, it replaces an $O(n^2)$ algorithm by an $O(nlog(n))$ method.

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The *p*-version for the electric field integral equation

Norbert Heuer

(joint work with Alexei Bespalov)

We analyse the p-version of the boundary element method for the electric field integral equation on a plane open surface with polygonal boundary. We prove convergence of the p-version with Raviart-Thomas parallelogram elements and derive an a priori error estimate which takes into account the strong singular behaviour of the solution at edges and corners of the surface. Key ingredient of our analysis is the orthogonality of discrete Helmholtz decompositions in a Sobolev space of order $-1/2$.

The EFIE models the scattering of time-harmonic electro-magnetic waves at a perfect conductor, and its solution is the induced electric surface current on Γ, see, e.g., [12]. The basis of our BEM is a variational formulation of the EFIE, called Rumsey's formulation. For a given wave number $k > 0$ and a (tangential or scalar) vector field v we define the single layer operator Ψ_k by

$$\Psi_k v(x) = \frac{1}{4\pi} \int_\Gamma v(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y, \qquad x \in \Gamma.$$

Also, denoting by div and ∇ the two-dimensional divergence and gradient operators on Γ, respectively, we consider the space

$$\begin{aligned} \mathbf{X} = \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}, \Gamma) \quad := \quad & \{\mathbf{u} \in \tilde{\mathbf{H}}^{-1/2}(\Gamma);\ \mathrm{div}\, \mathbf{u} \in \tilde{H}^{-1/2}(\Gamma) \text{ and} \\ & \langle \mathbf{u}, \nabla v \rangle + \langle \mathrm{div}\, \mathbf{u}, v \rangle = 0 \quad \text{for all} \ \ v \in C^\infty(\bar{\Gamma})\}. \end{aligned}$$

The dual space of $\mathbf{X}$ (with $\mathbf{L}^2(\Gamma)$ as pivot space) is denoted by $\mathbf{X}'$ and $\langle \cdot, \cdot \rangle$ denotes the extension of the $\mathbf{L}^2(\Gamma)$-inner product by duality between $\mathbf{X}$ and $\mathbf{X}'$. Moreover, $\tilde{H}^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$.

Now, for a given tangential vector field $\mathbf{f} \in \mathbf{X}'$ ($\mathbf{f}$ represents the excitation by an incident wave), Rumsey's formulation reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \Psi_k \mathrm{div}\, \mathbf{u}, \mathrm{div}\, \mathbf{v} \rangle - k^2 \langle \Psi_k \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}.$$

For smooth surfaces, its boundary element discretisation has been studied by Bendali [2, 3]. With the study of traces of spaces that govern Maxwell's equations in Lipschitz domains [8] there has been some recent progress in the numerical analysis of the EFIE on Lipschitz surfaces. For polyhedral surfaces, Buffa *et al.* and Hiptmair and Schwab [9,11] studied BEM discretisations of the EFIE with Raviart-Thomas elements of fixed order on refined meshes, i.e., in the framework of the h-version. In particular, the solvability and quasi-optimal convergence of these discretisations have been proved. Moreover, considering lowest order Raviart-Thomas elements and assuming standard Sobolev regularity, Hiptmair and Schwab [11] derived an a priori error estimate in terms of the mesh parameter h. The issues of solvability and convergence of the h-BEM for the EFIE on open Lipschitz surfaces were addressed by Buffa and Christiansen [7]. We note that in [7, 9, 11] the authors focused on conforming discretisations of Rumsey's formulation, called natural boundary element method for the EFIE (the approach we follow in this presentation).

In general, there are two main advantages of high order methods, namely their less vulnerability to numerical dispersion errors and better approximation properties even in the presence of singularities. The influence of the order of basis functions on numerical dispersion has been analysed by Ainsworth [1], and the properties of polynomial approximations of singular functions inherent to first kind integral equations have been studied in [4, 5].

In the p-version of the BEM the mesh is fixed and approximations are improved by increasing polynomial degrees. To the best of our knowledge there have been no proofs of convergence for the p-version applied to the EFIE. The analysis of high order approximations for the EFIE on open or closed polyhedral surfaces poses two particular challenges.

First, in order to prove convergence of the method, one usually relies on properties of the continuous and discrete Helmholtz decompositions, and on the proximity in some sense of the discrete decompositions to the continuous one, see [7, 9]. Known techniques are inherently designed towards low order approximations, as it turns out when trying to generalise them to high order methods. For instance, the equivalence of norms in finite-dimensional spaces is usually used. This argument is not available for the p-version. Also, related with appearing singularities (which is the second challenge described below), the proofs of proximity of low-order discrete Helmholtz decompositions to the continuous decomposition utilise an error estimate for the standard Raviart-Thomas interpolation operator in $\mathbf{H}(\mathrm{div}, \Gamma)$ For the p-version, stability of this operator is guaranteed when the interpolated function is in $\mathbf{H}^s(\mathrm{div}, \Gamma)$ with $s > 1/2$, whereas on polyhedral surfaces less regularity has to be accounted for.

Second, the solution to the EFIE on polyhedral surfaces suffers from singular behaviour at edges and corners. This can be deduced from the behaviour of solutions to the Maxwell problem on polyhedral domains as studied by Costabel and Dauge in [10]. Open surfaces represent the least regular case, and there have been no high order approximation results for them whatsoever.

In this talk we deal with both issues. In particular, to prove convergence of the p-version of the BEM for the EFIE we follow the framework presented in [9]. However, rather than considering $\mathbf{L}^2$-orthogonal discrete Helmholtz decompositions, we consistently employ the $\tilde{\mathbf{H}}^{-1/2}$-inner product and orthogonality. This turns out to be crucial for the p-version. As for the approximation analysis of singularities, we partly rely on our previous results for the Laplacian, see [4, 5], by using continuity properties of the surface curl operator. The exception is a particular kind of vertex singularity, which does not have a vanishing tangential component on the boundary of Γ and which needs to be treated in a vector fashion (i.e., component-wise approximations are not sufficient for it). For detailed results we refere to [6].

We restrict ourselves to plane open surfaces which can be discretised by parallelogram meshes. A generalisation to smooth curved surfaces is possible by mapping techniques. The case of triangular elements, however, is not an easy generalisation as, for instance, standard p-version approximation results for Raviart-Thomas triangular elements are unknown. The approach presented in this talk is, in principle, applicable to polyhedral surfaces and we expect that all the results can be extended to that case. However, this extension is not straightforward as some technical details make use of the smoothness of Γ, except for its boundary. More general hp-methods, which increase polynomial degrees in combination with mesh refinements, are desirable but are not covered in this presentation. The corresponding analysis is a non-trivial extension of our results.

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Theoretical challenges in the application of convolution quadrature and BEM for transient waves

Francisco–Javier Sayas
(joint work with A. Laliena)

Convolution quadrature has been applied for the discretization of very different evolutionary problems that can be written as convolution equations. CQ appeared in two articles by Christian Lubich [18], [19], extending classical ideas of discrete operator calculus and thinking in very abstract levels of problems related to the approximation of convolutions and convolution equations. Unlike some methods based on the inversion of the Laplace transform [26], [13], CQ uses data in the time–domain and produces directly approximations in time steps, but the Laplace transform of the operator is used to generate the discrete sequence of operators that discretizes the convolution process.

The hypotheses on the operator that has to be approximated are given through its Laplace transform. In [18], this Laplace transform had to be defined in the complementary set of a sector of angle less than $\pi/2$ around the negative real axis. These original hypotheses did not therefore include any operator related to the wave equation, although new work of the same author [20] opened a way in this direction, giving at the same time the first application of CQ to wave scattering problems. Let it be mentioned though that most of the literature on CQ (for instance, the most recent general convergence results [21]) has been done with parabolic–type equations in mind and not so many general results are available when the method is applied to hyperbolic problems. Some years before the combined use of BEM, waves and CQ, [23] marks the first application of CQ to boundary integral equations, namely to the single–layer potential for the heat equation in two dimensions.

The original family of multistep–based CQ schemes offers only methods up to order two for scattering problems, because of a crucial A–stability requirement. Fortunately, another class of methods originated in [22], based on Runge–Kutta discretizations, and the order barrier can be thus overcome by using vector–valued approximations of the convolution. A more general analysis of this class of methods can be found in [5] and in [15] we have tried to clarify some practical algorithmic issues, especially when scattering problems are at hand.

In the realm of wave propagation with boundary integral equations, only the most elementary exterior problems have been approached with this method and its real power it is still to be exploited. The mathematical literature on the subject (see [11] or [14]) has concentrated on the single–layer retarded integral equation, that can be used for a simple model of scattering of transient waves by a sound–soft obstacle. Engineering experience is much richer in this direction (see the compilation of results and references in [25]), with the treatment of elastic, viscoelastic and poroelastic waves, dealing again with boundary value problems.

The literature of numerical treatment of time–domain boundary integral equations is wide. Since the pioneering work of [2], [3], much has been done in the numerical treatment of this kind of problems, undertaking elasticity problems [4], more complicated boundary behaviors [10] or even coupled models with finite elements for electromagnetism [1]. Although the engineering literature uses mainly the collocation method, not much have been done from the mathematical point of view (see [8]). A good review of numerical methods for time–domain integral equations is given in [9].

In this talk we mention some interesting theoretical questions that arise in the numerical analysis of convolution quadrature when applied to more complicated situations. The problem can be easily set as the need for the Laplace transform of the operator, its Galerkin space discretization and the inverses of both belong to a class of holomorphic functions in the half complex plane. By means of an example, arising from wave transmission in free space with a non-homogeneous penetrable inclusion, we illustrate a novel analytical technique that can be used to prove the required properties for a wide class of discretizations. We apply this idea to a coupled system of finite and boundary elements using a symmetric formulations and three spaces (see [6] for an example of the use of this simple matching formulation) and show how it equally works for the symmetric formulations with only one boundary element space [7], [12]. These results will appear in [16].

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From the Boundary Element DDM to new local Trefftz Finite Element Methods on Arbitrary Polyhedral Meshes

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(joint work with Dylan Copeland, David Pusch)

1. Introduction

We introduce new finite element methods based on the symmetric boundary element domain decomposition method presented in [3], which can be applied with general polygonal or polyhedral meshes. Each element of the mesh may be any polygon or polyhedron, as we treat the elements as subdomains. There are many important practical applications where one wants to discretize PDEs on such kinds of meshes without further decomposition of the polyhedra. Boundary integral operators are utilized to obtain a method which solves for traces of the solution on the element surfaces, from which the solution may be obtained via a representation formula. Simple, low-order boundary element spaces are used to approximate traces on the element surfaces, yielding a finite element method with PDE-harmonic basis functions.

Since boundary integral operators are used only locally, piecewise constant coefficients are admissible, and the coupling of boundary element functions is local. Consequently, sparse linear systems are obtained, which can be solved by Krylov iterative methods. In the case of the potential equation, the resulting system is symmetric and positive definite, and algebraic multigrid is a very effective preconditioner in the conjugate gradient solver.

2. The Potential Equation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a polygonal ($d = 2$) or polyhedral ($d = 3$) Lipschitz boundary $\Gamma = \partial\Omega$. As a model problem, we consider the potential equation

$$-\mathrm{div}(a(x)\nabla u(x)) = f(x) \quad \text{for } x \in \Omega \tag{1}$$

with the Dirichlet boundary condition $u = g$ on Γ. We assume that the coefficient a is piecewise constant, $f \in L_2(\Omega)$, and $g \in H^{1/2}(\Gamma)$. Further, we suppose that there is a non–overlapping decomposition of our domain Ω into e_h shape-regular polygonal elements Ω_i such that $\overline{\Omega} = \cup_{i=1}^{e_h} \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $\Gamma_i = \partial\Omega_i$, $\overline{\Gamma}_{ij} = \overline{\Gamma}_i \cap \overline{\Gamma}_j$ and $a(x) = a_i > 0$ for $x \in \Omega_i$, $i = 1, \ldots, e_h$. Under these assumptions, there obviously exists a unique weak solution $u \in H^1(\Omega)$ of the BVP (1).

Using the local Dirichlet-to-Neumann map

$$a_i \partial u/\partial \nu_i = a_i S_i u_{|\Gamma_i} - N_i f \quad \text{on } \Gamma_i, \tag{2}$$

we observe that the variational formulation of (1) is equivalent to the associated variational formulation on the skeleton $\Gamma_S = \cup_{i=1}^{e_h} \Gamma_i$ (see, e.g., [4]):

find $u \in H^{1/2}(\Gamma_S)$ with $u = g$ on Γ such that

$$\sum_{i=1}^{e_h} \int_{\Gamma_i} a_i(S_i u_i)\, v_i ds_x \;=\; \sum_{i=1}^{e_h} \int_{\Gamma_i} (N_i f) v_i ds_x \tag{3}$$

for all $v \in H_0^{1/2}(\Gamma_S)$, where u_i and v_i denote the traces of u and v on Γ_i, respectively. The Steklov–Poincaré operator S_i and the Newton potential operator N_i have different representations (see again [4]). Here we are using the symmetric representation

$$S_i \;=\; D_i + (0.5I + K_i')V_i^{-1}(0.5I + K_i) \tag{4}$$

of the local Steklov–Poincaré operator S_i via the local single layer potential integral operator V_i, the local double layer potential operator K_i, its adjoint K_i', and the local hypersingular boundary integral operator D_i, see, e.g., [5] for the definition and properties of these boundary integral operators. The operator $N_i = V_i^{-1}\widetilde{N}_{i,0}$ is defined by the Newton potential operator

$$(\widetilde{N}_{i,0} f)(x) = \int_{\Omega_i} U^*(x-y) f(y) dy, \quad x \in \Gamma_i, \tag{5}$$

where $U^*(x) = 1/(4\pi|x|)$ denotes the fundamental solution of the Laplace operator $-\Delta$ for $d = 3$.

For simplicity we use continuous piecewise linear boundary element functions for approximating the potential u on the skeleton Γ_S and piecewise constant boundary element functions for approximating the normal derivatives $t_i = \partial u/\partial \nu_i$ on the boundary Γ_i of the polygonal element Ω_i. This yields the element stiffness matrices $\mathbf{S}_{i,h} = a_i \mathbf{D}_{i,h} + a_i \big(0.5\,\mathbf{I}_{i,h}^\top + \mathbf{K}_{i,h}^\top\big)\big(\mathbf{V}_{i,h}\big)^{-1}\big(0.5\,\mathbf{I}_{i,h} + \mathbf{K}_{i,h}\big)$ and the element vectors $\mathbf{f}_{i,h} = \mathbf{I}_{i,h}^\top \big(\mathbf{V}_{i,h}\big)^{-1} \mathbf{f}_{i,h}^N$, where the matrices $\mathbf{V}_{i,h}$, $\mathbf{K}_{i,h}$, $\mathbf{D}_{i,h}$ and $\mathbf{I}_{i,h}$ arise from the BE Galerkin approximation to the local boundary integral operators V_i, K_i, D_i, and to the identity operator I_i living on Γ_i, respectively. $\mathbf{I}_{i,h}$ is nothing but the mass matrix. The vector $\mathbf{f}_{i,h}^N$ is defined by the Newton potential identity

$$(\mathbf{f}_{i,h}^N, \mathbf{t}_{i,h}) = \int_{\Gamma_i} \int_{\Omega_i} U^*(x-y) f(y) dy\, t_{h,i}(x) ds_x \tag{6}$$

for all vectors $\mathbf{t}_{i,h}$ corresponding to the piecewise constant functions $t_{h,i}$ on Γ_i. Now, we obtain the BE-based FE system

$$\mathbf{S}_h \mathbf{u}_h = \mathbf{f}_h \tag{7}$$

by assembling the stiffness matrix $\mathbf{S}_h$ and the load vector $\mathbf{f}_h$ from the element stiffness matrices $\mathbf{S}_{i,h}$ and the element load vectors $\mathbf{f}_{i,h}$, respectively, and by incorporating the Dirichlet boundary condition as usual.

The solution of (7) provides an approximation to the Dirichlet trace of the solution to (1) on the boundary $\partial\Omega_i$ of all elements Ω_i. Applying the Dirichlet-to-Neumann map locally (i.e. element-wise), we may obtain an approximate solution $\tilde{u}_h$to u in each element Ω_i via the representation formula (see, e.g., [4] or [5]).

Following [3] we immediately obtain the discretization error estimate $O(h^{3/2})$ in the mesh-dependent norm $\|v\|_h^2 := \sum_{i=1}^{e_h} \|v|_{\Gamma_i}\|^2_{H^{1/2}(\Gamma_i)}$ for a sufficiently (piecewise) smooth solution u, where u_h is the continuous piecewise linear function on the skeleton $\Gamma_{S,h}$ corresponding to the Dirichlet nodal values and to the nodal values from the solution vector $\mathbf{u}_h$ of (7). This yields the usual $O(h)$ estimate of the discretization error $u - \tilde{u}_h$ in the $H^1(\Omega)$-norm.

We refer the reader to [2] for more detailed description and for the results of our numerical experiments.

3. Generalization to the Helmholtz and Maxwell Eqautions

Consider the interior Dirichlet problem for the Helmholtz equation

$$-\Delta u(x) - \kappa^2 u(x) = f \quad \text{for } x \in \Omega, \tag{8}$$

with $u = g$ on Γ. We assume that the wavenumber $\kappa > 0$ is constant, or piecewise constant and not an interior eigenvalue, and $g \in H^{1/2}(\Gamma)$.

The BE-based FE method for the Helmholtz equation is formally identical to the method presented in the previous section for the potential equation (with $a = 1$). One only needs to use different operators D_i, K_i, and V_i based on the fundamental solution $U^*(x) = e^{i\kappa|x|}/(4\pi|x|)$ for the Helmholtz operator, see, e.g., [5].

We also study a similar method for the time-harmonic Maxwell equation

$$\mathbf{curl\,curl}\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{0} \qquad \text{in } \Omega, \tag{9}$$

with the Dirichlet boundary condition $\gamma_t\mathbf{u} := \mathbf{u} \times \mathbf{n} = \mathbf{g}$ on Γ, where $\mathbf{n}$ is the outward unit normal. The BE-based FE method for (9) involves quite technical trace spaces and boundary integral operators. For details and results, see [1].

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Volume and surface integral equations for electromagnetic scattering by a dielectric body

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(joint work with Martin Costabel, Eric Darrigrand)

We first analyze two integral formulations for the electromagnetic scattering by a dielectric object. The one is a volume integral equation (VIE) having a strongly singular kernel and the other is a coupled surface-volume system of integral equations with weakly singular kernels. There are more details in [4], on results and their proofs. We give afterwards, in the last part, a brief outline on the computation of the strongly singular integrals elements performed on tetrahedra, for the numerical implementation of the VIE. More details for both these parts can be found in the PhD thesis [8].

The problem

Let Ω^- be a bounded domain in $\mathbb{R}^3$ representing the dielectric scatterer. We use the notation $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega^-}$ and $\Gamma = \partial\Omega^-$, and we assume that the boundary Γ is regular (at least C^2). $\boldsymbol{n}$ is the unit outward normal vector to Ω^-.

The electric permittivity ε is a function of the space variable satisfying $\varepsilon(x) > 0$, $x \in \mathbb{R}^3$; $\varepsilon_{|_{\Omega^-}} \in C^1(\Omega^-) \cap C^0(\overline{\Omega^-})$; $\varepsilon_{|_{\Omega^+}} = \varepsilon_0$; and ε is discontinuous across Γ, in general. The vacuum permittivity ε_0 is a positive constant. We will denote the relative permittivity by $\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0}$. We will also use the notation $\eta = 1 - \varepsilon_r$. The electric conductivity σ vanishes everywhere. We assume for simplicity that the magnetic permeability μ is constant ($\mu \equiv \mu_0 > 0$). With the frequency ω, the wave number is $\kappa = \omega\sqrt{\varepsilon_0\mu_0} > 0$.

Let $\boldsymbol{F} \in H(\mathrm{div}, \Omega^+)$ be a vector field with a compact support contained in Ω^+, representing a current density that serves as source for the incident field scattered by the dielectric body Ω^-.

The scattering problem $(\mathcal{P})$ we want to solve can be written as follows:

Find $\boldsymbol{E}$, $\boldsymbol{H}$ such that $\boldsymbol{E}_i \in H(\mathrm{curl}, \mathrm{div}, \Omega^-)$, $\boldsymbol{E}_e \in H_{loc}(\mathrm{curl}, \mathrm{div}, \overline{\Omega^+})$, $\boldsymbol{H}_i \in H(\mathrm{curl}, \Omega^-)$, $\boldsymbol{H}_e \in H_{loc}(\mathrm{curl}, \overline{\Omega^+})$, with $\boldsymbol{E}_i = \boldsymbol{E}_{|_{\Omega^-}}$, $\boldsymbol{H}_i = \boldsymbol{H}_{|_{\Omega^-}}$, $\boldsymbol{E}_e = \boldsymbol{E}_{|_{\Omega^+}}$ and $\boldsymbol{H}_e = \boldsymbol{H}_{|_{\Omega^+}}$, satisfying the equations

$$
(\mathcal{P}) \quad \begin{cases} \nabla \times \boldsymbol{E}_i - i\kappa\, \boldsymbol{H}_i = 0 \quad \text{and} \quad \nabla \times \boldsymbol{H}_i + i\kappa\varepsilon_r\, \boldsymbol{E}_i = 0 & \text{in } \Omega^-, \\ \nabla \times \boldsymbol{E}_e - i\kappa\, \boldsymbol{H}_e = 0 \quad \text{and} \quad \nabla \times \boldsymbol{H}_e + i\kappa\, \boldsymbol{E}_e = \boldsymbol{F} & \text{in } \Omega^+, \\ \boldsymbol{n} \cdot \boldsymbol{E}_e = \boldsymbol{n} \cdot \varepsilon_r\, \boldsymbol{E}_i \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{H}_e = \boldsymbol{n} \cdot \boldsymbol{H}_i & \text{on } \Gamma, \\ \boldsymbol{H}_e \times \frac{\boldsymbol{x}}{r} - \boldsymbol{E}_e = \mathcal{O}\left(\frac{1}{r^2}\right),\ r = |\boldsymbol{x}| \to +\infty. \end{cases}
$$

Integral formulations

As a first step in the derivation of the integral equations, we extend the well-known Stratton-Chu integral representation to fields $(\boldsymbol{E}, \boldsymbol{H})$ in $H(\mathrm{curl}, \mathrm{div}, D) \times H(\mathrm{curl}, D)$, where D is a regular (at least of class C^2) bounded domain. Using this extended Stratton-Chu formula for $D = \Omega^-$ and for $D = \Omega^+ \cap B_R$, where the radius R of the ball B_R tends to infinity, together with the Maxwell equations of the problem $(\mathcal{P})$ and the radiation condition, we establish some integral representations formulas and derive the following equations:

Denoting by $\boldsymbol{\tau} = -\dfrac{1}{\varepsilon_r}\nabla\varepsilon_r$, the logarithmic gradient of ε_r, the coupled surface-volume system of integral equations is given by the problem $(\boldsymbol{\mathcal{E}_1})$ defined as follows:

$$(\boldsymbol{\mathcal{E}_1}) \qquad \begin{cases} \text{Find } (\boldsymbol{E}_*, e_*) \in (L^2(\Omega^-))^3 \times H^{-\frac{1}{2}}(\Gamma), \text{ such that} \\ \begin{pmatrix} 1 - \nabla\mathcal{N}_{\boldsymbol{\tau}} + \kappa^2\mathcal{N}_\eta & -\nabla\mathcal{S}_\eta \\ \kappa^2\gamma_n^-\mathcal{N}_\eta - \gamma_1^-\mathcal{N}_{\boldsymbol{\tau}} & 1 - \gamma_1^-\mathcal{S}_\eta \end{pmatrix} \begin{pmatrix} \boldsymbol{E}_* \\ e_* \end{pmatrix} = \begin{pmatrix} \boldsymbol{D} \\ \gamma_n^-\boldsymbol{D} \end{pmatrix} \end{cases}$$

and the VIE is given by the problem $(\boldsymbol{\mathcal{E}_2})$ defined as follows:

$$(\boldsymbol{\mathcal{E}_2}) \qquad \begin{cases} \text{Find } \boldsymbol{E}_\circ \in (L^2(\Omega^-))^3, \text{ such that} \\ \left(1 - \nabla\mathcal{M}_\eta + \kappa^2\mathcal{N}_\eta\right)\boldsymbol{E}_\circ = \boldsymbol{D}\ . \end{cases}$$

where we have used the following integral operators:

$$\begin{aligned} \mathcal{N}_\eta\boldsymbol{u}(x) &= \int_{\Omega^-} \eta(y)\boldsymbol{u}(y)G_\kappa(x-y)\,dy\ , \\ \mathcal{N}_{\boldsymbol{\tau}}\boldsymbol{u}(x) &= \int_{\Omega^-} \boldsymbol{\tau}(y)\cdot\boldsymbol{u}(y)G_\kappa(x-y)\,dy\ , \\ \mathcal{M}_\eta\boldsymbol{u}(x) &= \int_{\Omega^-} \eta(y)\nabla_y G_\kappa(x-y)\cdot\boldsymbol{u}(y)\,dy\ , \\ \mathcal{S}_\eta f(x) &= \int_{\partial\Omega^-} \eta(y)f(y)G_\kappa(x-y)\,ds(y). \end{aligned}$$

for f and $\boldsymbol{u}$ respectively scalar and vector fields defined on Γ and on Ω^-.
We have also used the one-sided traces

$$\gamma_0^\pm g := g^\pm_{|\Gamma}, \quad \gamma_1^\pm g := (\boldsymbol{n}\cdot\nabla g^\pm)_{|\Gamma} \text{ and } \gamma_n^\pm\boldsymbol{v} := \gamma_n\boldsymbol{v}^\pm,$$

for g and $\boldsymbol{v}$ respectively scalar and vector fields defined on $\mathbb{R}^3$, with $g^\pm := g_{|\Omega^\pm}$ and $\boldsymbol{v}^\pm := \boldsymbol{v}_{|\Omega^\pm}$.

Equivalence results and mapping properties

As a first result, we justify equivalence between these integral equations and the scattering problem and use the well known unicity for the scattering problem,

together with a Fredholm property for the operator in $(\mathcal{E}_1)$ to establish the well-posedness of all the problems $(\mathcal{P})$, $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$.

A more important motivation for the analysis of the integral operators in $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ is the question of their suitability for numerical computations. More general questions of mapping properties of the strongly singular integral operator of the VIE $(\mathcal{E}_2)$ in L^2 or in $H(\mathrm{div})$, in particular its spectral theory, remain largely open. However, we establish some results as for instance, the VIE operator is Fredholm of index zero and strongly elliptic in L^2 (satisfying a Gårding inequality). This ensures then stability for Galerkin schemes.

Treatment of singularities

Here, in order to solve the VIE, we point out some techniques to compute the strongly singular integrals on tetrahedra (K and L) :

$$I_{KL} = \int_K \int_L \mathbb{G}_\kappa(x-y)\varphi_c(y)\varphi_r(x)dydx; \qquad \mathbb{G}_\kappa(x) = \left(\mathbf{1} + \frac{\nabla\nabla}{\kappa^2}\right) G_\kappa(x).$$

These techniques depend on the relative positions of tetrahedra. When two tetrahedra share one vertex or one edge, we introduce a method similar to the well-know Duffy transformation [5], based on adapted changes of variables. The method is valid for more general kernels and leads to integration over simple domains such as the square or the cube and removes the singularity using factorization and reduction with Jacobians of the singular transformations. Otherwise, when the tetrahedra share one face or the integral is performed in the same tetrahedron, this method cannot remove the singularity, and then we use the regularization scheme introduced in [9] for the same kernel. However when they share one face, the first method can also be used to weaken the singularity without removing it, and therefore a quadrature formula can be applied to compute the integral.

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Adaptive boundary element method: Simple error estimators and convergence

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(joint work with Samuel Ferraz-Leite and Christoph Ortner)

Introduction. We consider Symm's integral equation in 2D with weakly singular integral operator

$$Vu(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,u(y)\,ds_y. \tag{1}$$

Here, $\Gamma \subseteq \partial\Omega$ is an open piece of the boundary $\partial\Omega$ of a Lipschitz domain $\Omega \subset \mathbb{R}^2$. Provided $\operatorname{diam}(\Omega) < 1$, $\langle\!\langle u\,,\,v\rangle\!\rangle := \int_\Gamma Vu(x)v(x)\,ds_x$ defines an equivalent scalar product on $\mathcal{H} = \widetilde{H}^{-1/2}(\Gamma)$. For a given linear and continuous functional $\Phi \in \mathcal{H}^*$, the Lax-Milgram lemma thus proves the unique existence of (some unknown) $u \in \mathcal{H}$ with

$$\langle\!\langle u\,,\,v\rangle\!\rangle = \Phi(v) \quad \text{for all } v \in \mathcal{H}. \tag{2}$$

To approximate u by the lowest-order Galerkin scheme, let $\mathcal{T}_\ell$ be a triangulation of Γ and $X_\ell = \mathcal{P}^0(\mathcal{T}_\ell) := \big\{v_\ell : \Gamma \to \mathbb{R} : \forall T \in \mathcal{T}_\ell \quad v_\ell|_T \text{ is constant}\big\} \subset \mathcal{H}$. The (numerically computable) Galerkin solution $u_\ell \in X_\ell$ is the unique solution of

$$\langle\!\langle u_\ell\,,\,v_\ell\rangle\!\rangle = \Phi(v_\ell) \quad \text{for all } v_\ell \in X_\ell. \tag{3}$$

In a posteriori error analysis, one aims to provide a computable quantity η_ℓ which only depends on known and computed data, for instance, on u_ℓ and Φ such that

$$C_{\text{eff}}^{-1}\eta_\ell \le |\!|\!|u-u_\ell|\!|\!| \le C_{\text{rel}}\eta_\ell. \tag{4}$$

Here, $|\!|\!|\cdot|\!|\!|$ denotes the energy norm induced by $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle$. The lower and upper estimate are referred to as *efficiency* and *reliability* of η_ℓ, respectively, and local information of η_ℓ will be used to improve the mesh by local mesh-refinement.

***h*-*h*/2-based error estimators for Symm's integral equation.** The h-$h/2$-based strategy is one very basic and well-known technique for the a posteriori error estimation for Galerkin discretizations of energy minimization problems. Let $u_\ell \in X_\ell$ and $\widehat{u}_\ell \in \widehat{X}_\ell = \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ be Galerkin solutions, where $\widehat{\mathcal{T}}_\ell$ is obtained by uniform refinement of $\mathcal{T}_\ell$. One then considers

$$\eta_\ell := |||\widehat{u}_\ell - u_\ell||| \tag{5}$$

to estimate the error $|||u - u_\ell|||$. By Galerkin orthogonality, η_ℓ is always efficient with known constant $C_{\rm eff} = 1$. Reliability of η_ℓ with $C_{\rm rel} = (1-\sigma^2)^{-1/2}$ follows from the saturation assumption

$$|||u - \widehat{u}_\ell||| \le \sigma\, |||u - u_\ell||| \quad \text{with some uniform constant} \quad \sigma \in (0,1). \tag{6}$$

Unlike to FEM, where (6) is proven for a sufficiently small mesh-size [11], the saturation assumption is open in the context of BEM but observed in practice [17].

Since the energy norm $|||\cdot|||$ is nonlocal, the error estimator η_ℓ does not provide information for a local mesh-refinement. Using a local inverse estimate from [18] and a local approximation result from [6], one may prove estimator equivalence [17]

$$C_{\rm apx}^{-1}\eta_\ell \le \mu_\ell := \|h_\ell^{1/2}(\widehat{u}_\ell - u_\ell)\|_{L^2(\Gamma)} \le C_{\rm inv}\eta_\ell, \tag{7}$$

where $h_\ell \in L^\infty(\Gamma)$ denotes the local mesh-width $h_\ell|_T = \mathrm{diam(T)}$ for $T \in \mathcal{T}_\ell$. The local contributions $\mu_\ell(T) := \mathrm{diam(T)}^{1/2}\|\widehat{\mathrm{u}}_\ell - \mathrm{u}_\ell\|_{\mathrm{L^2(T)}}$ of μ_ℓ are then used for the marking strategy in an adaptive mesh-refining algorithm.

Convergence of adaptive Galerkin BEM. Based on the error estimators η_ℓ and μ_ℓ from the previous section and based on a fixed parameter $\theta \in (0,1)$, the usual adaptive algorithm reads as follows: Until η_ℓ is sufficiently small, do:

(i) Refine $\mathcal{T}_\ell$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$.
(ii) Compute discrete solutions u_ℓ and $\widehat{u}_\ell$.
(iii) Find minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that

$$\theta \sum_{T\in\mathcal{T}_\ell} \mu_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \mu_\ell(T)^2. \tag{8}$$

(iv) Refine at least marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.
(v) Increase counter $\ell \mapsto \ell + 1$ and iterate.

Convergence of this type of algorithms has first been proven in [10], where also the marking criterion (8) is introduced. The latter work considered the residual error estimator for a P1-FEM discretization of the Poisson problem, and it is assumed that data oscillations on the initial mesh are sufficiently small. In [21], the resolution of the data oscillations is included into the adaptive algorithm. The convergence analysis is based on reliability and the so-called *discrete local efficiency* of the residual error estimator, which relies on an *interior node property* for the local refinement. The main idea of the convergence proof then is to show that the error is contractive up to the data oscillations. In [9], this has been weakened in the sense that it is proven that a weighted sum of error and error estimator yields a contraction property without requiring (discrete local) efficiency.

Only recently, analogous results for adaptive BEM could be derived, and a first convergence result reads as follows [16]: Provided that μ_ℓ is reliable and that marked elements are halved, there are constants $\kappa, \gamma \in (0,1)$ such that

$$\Delta_\ell^2 := |||u - u_\ell|||^2 + |||u - \widehat{u}_\ell|||^2 + \gamma\, \mu_\ell^2 \quad \text{satisfies} \quad \Delta_{\ell+1} \leq \kappa\, \Delta_\ell. \tag{9}$$

In particular, this implies convergence $u_\ell \to u$ as $\ell \to \infty$. The proof of (9) requires that the local contributions of μ_ℓ used for marking, have an h-weighting factor. Therefore, the analysis might carry over to adaptive algorithms steered by h-weighted residual error estimators [2, 4, 5] or averaging error estimators [6–8], whereas the two-level error estimators [12, 13, 19, 20, 22] and the Faermann error estimator [3, 14, 15] seem to need further arguments.

Concluding Remarks. The convergence proof of [16] for adaptive Galerkin BEM also applies to hypersingular integral equations and mixed formulations in 2D and 3D. For 3D, however, our proof — as well as the available a posteriori error analysis from [4–8,13,15,17,19,22] — is restricted to the case of isotropic mesh-refinement, whereas anisotropic mesh-refinement is needed to resolve edge singularities efficiently.

Despite of convergence, even the question of optimal convergence rates of the adaptive FEM based on residual error estimators is well-understood. Whereas prior works [1, 23] used an additional coarsening step to prove optimality, recent works [9, 24] prove optimality for the standard algorithm steered by the residual error estimator. The latter analysis relies on a *discrete local reliability* of the error estimator, which remains open for adaptive Galerkin BEM. This will be a major topic for future research.

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Wave Number Dependence of Condition Numbers in Boundary Integral Methods for Acoustic Problems

Simon Neil Chandler-Wilde
(joint work with Ivan G. Graham, Stephen Langdon, Marko Lindner, Peter Monk)

In this talk we discussed the classical problem of scattering of a time-harmonic acoustic wave by a bounded, sound soft obstacle occupying a compact set $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz boundary Γ. The wave propagates in the exterior domain $\Omega_e = \mathbb{R}^d \setminus \Omega$ and we suppose that the medium of propagation in Ω_e is homogeneous and that a time harmonic ($\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence) plane wave $u^i(x) = \exp(\mathrm{i}kx \cdot \hat{d})$, travelling in the direction given by the unit vector $\hat{d}$, is incident on Ω. Here $k > 0$ is the *wave number*. Then the problem we consider

is to find the resulting time-harmonic acoustic pressure field u which satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \Omega_e \tag{1}$$

and the sound soft boundary condition

$$u = 0 \qquad \text{on} \qquad \Gamma := \partial\Omega_e, \tag{2}$$

and is such that the scattered part of the field, $u^s := u - u^i$, satisfies the standard Sommerfeld radiation condition. It is well known that this problem has exactly one solution under the constraint that u and ∇u be locally square integrable.

The talk was concerned with the behaviour, in the important but difficult high frequency limit $k \to \infty$, of standard reformulations of this problem in terms of second kind boundary integral equations. Let $\Phi(x, y)$ denote the standard free-space fundamental solution of the Helmholtz equation, given, in the 2D and 3D cases, by

$$\Phi(x,y) := \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), & d = 2, \\ \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & d = 3, \end{cases} \tag{3}$$

for $x, y \in \mathbb{R}^d$, $x \neq y$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. It was proposed in the 60s (see e.g. [6]), as a means to obtain an integral equation uniquely solvable at all wave numbers, to look for a solution to the scattering problem in the form of the *combined single- and double-layer potential*

$$u^s(x) \;:=\; \int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)\, ds(y) - \mathrm{i}\eta \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Omega_e, \tag{4}$$

for some non-zero value of the *coupling parameter* $\eta \in \mathbb{R}$. (In this equation $\partial/\partial\nu(y)$ is the derivative in the normal direction, the unit normal $\nu(y)$ directed into Ω_e.) It follows from standard boundary trace results for single- and double-layer potentials that u^s, given by (4), satisfies the scattering problem if and only if φ satisfies a second kind boundary integral equation on Γ; see [6] and, for the Lipschitz case, [5] and the references therein. This integral equation, in operator form, is

$$(I + D_k - \mathrm{i}\eta S_k)\;\varphi \;=\; g, \tag{5}$$

where I is the identity operator, S_k and D_k are *single-* and *double-layer potential operators*, defined by

$$S_k\varphi(x) \;:=\; 2\int_\Gamma \Phi(x,y)\;\varphi(y)\; ds(y), \quad x \in \Gamma, \tag{6}$$

and

$$D_k\varphi(x) \;:=\; 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial\nu(y)}\;\varphi(y)\; ds(y), \quad x \in \Gamma, \tag{7}$$

and $g := -2u^i|_\Gamma$ is twice the Dirichlet data for the scattered field on Γ.

In this talk we studied (5) as an operator equation on the space $L^2(\Gamma)$. For every $\varphi \in L^2(\Gamma)$, the right hand sides of (6) and (7) are well-defined almost everywhere on Γ, with $D_k\varphi(x)$ understood as a Cauchy principal value, and both S_k and D_k are bounded operators on $L^2(\Gamma)$; see e.g. [10]. Choosing $\eta \neq 0$ ensures that (5) is uniquely solvable. Precisely, $A_{k,\eta} := I + D_k - i\eta S_k$ is invertible as an operator on $L^2(\Gamma)$. We note further that, generalising this result, it is shown in [5] that $A_{k,\eta}$ is invertible as an operator on the Sobolev space $H^s(\Gamma)$, for $0 \leq s \leq 1$.

The main aim of the talk was to study the conditioning of the standard integral equation formulation (5). Specifically we were interested in upper and lower bounds on the condition number of $A_{k,\eta}$, given by $\operatorname{cond} A_{k,\eta} = \|A_{k,\eta}\| \, \|A_{k,\eta}^{-1}\|$, and so we were interested in upper and lower bounds on the norms $\|A_{k,\eta}\|$ and $\|A_{k,\eta}^{-1}\|$. Our emphasis was on understanding the dependence on the wave number k, especially in the limit $k \to \infty$, and on the coupling parameter η, and on exploring the influence of the shape of Γ.

These questions have had some previous attention, starting with the work of Kress and Spassov [8] and Kress [9] (for more historical details see the recent review paper [4]). But we note that, with the exception of recent bounds in [1,3,7], rigorous estimates valid in the limit as $k \to \infty$ have not been obtained previously. Moreover, research to date has focussed almost entirely on the case when Γ is a circle or sphere where Fourier analysis methods are possible.

In this talk we summarised recent results in [2, 3]. The paper [3] provides explicit bounds on $\|A_{k,\eta}^{-1}\|$, for the case when Γ is piecewise smooth and starlike, for example a starlike polyhedron. In particular, assuming that Γ is starlike with respect to an origin which is inside Γ, and if a usual choice of η is made, specifically $\eta = R_0^{-1} + k$ (cf. [9]), where $R_0 := \sup_{x\in\Gamma} |x|$, then

$$\|A_{k,\eta}^{-1}\| \leq \frac{1}{2} + \theta \left[(1+4\theta)\,(3+\theta) + 2\right]^{1/2},$$

where $\theta := R_0/\delta_-$ and $\delta_- := \operatorname{ess.inf}_{x\in\Gamma} \; x \cdot n(x)$, which is just the distance from the origin to the closest side in the case when Γ is a starlike polyhedron. In particular, the above bound holds with $\theta = 1$ for a sphere and with $\theta = \sqrt{3}$ for a cube, giving $\|A_{k,\eta}^{-1}\| \leq 6$ and $\|A_{k,\eta}^{-1}\| \leq 12$, respectively. The methods used to obtain such bounds include Rellich-type identities and new subtle properties of radiating solutions of the Helmholtz equation.

The paper [2] obtains upper and lower bounds on $\|A_{k,\eta}\|$. In particular, crude bounds, ignoring the oscillation of the kernel of the integral operator, show that $\|A_{k,\eta}\| \lesssim 1 + k + \eta$ in 3D and that $\|A_{k,\eta}\| \lesssim 1 + k^{1/2} + \eta k^{-1/2}$ in 2D, which compares to the known growth for large k that $\|A_{k,\eta}\| \lesssim k^{1/3}$ for both a circle and sphere for the choice $\eta = k$ [7]. It is shown in [2] that, at least in 2D, this crude upper bound can be sharp in its dependence on k for large k, for example it is attained when Γ is a polygon. Further lower bounds on $\|A_{k,\eta}\|$ for the 2D case, which appear to be sharp, and investigate the subtle dependence on the geometry of Γ, are also contained in [2]. Finally [2] investigates the behaviour of $\|A_{k,\eta}^{-1}\|$ in a

2D case when Γ is not starlike, specifically the non-starlike case when Γ contains two parallel sides separated by a part of the exterior domain Ω_e.

A main message of [2] is that the conditioning of $A_{k,\eta}$ as $k \to \infty$ depends strongly on the geometry of Γ, in particular, in 2D for the choice $\eta \approx k$, cond $A_{k,\eta}$ grows like $k^{1/3}$ for a circle, like $k^{1/2}$ for a starlike polygon, and at least as fast as $k^{7/5}$ for the type of non-starlike domain just described.

The survey paper [4] contains a more detailed review of these results, indeed a review more generally of recent work on boundary integral equation methods for high frequency scattering problems.

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Currents and Charges Boundary Integral Equations

Abderrahmane Bendali

(joint work with Francis Collino, M'Barek Fares)

Recently, Taskinen and Ylä-Oijala [1] proposed to work with Boundary Integral Equations (BIEs) involving both the currents and charges for solving the scattering problems related to time-harmonic Maxwell's equations. Their main objective was to design BIEs that are robust for both moderate and low frequencies. Our aim in this abstract is to present some preliminary results establishing that the so-called Combined Currents and Charges Integral Equation (C3IE), when applied to the scattering of an electromagnetic wave by a perfectly conducting obstacle,

is well-posed at all frequencies, in the L^2 framework, that is, with no continuity requirement on the electric current. The C3IE is obtained along the same principle than the Combined Field Integral Equation (CFIE) but in the framework of currents and charges BIEs.

We start with the Boundary-Value Problem (BVP) related to the above mentionned scattering problem,

$$(1) \qquad \begin{cases} \nabla \times \mathbf{E} - ikZ\mathbf{H} = 0 \text{ and } \nabla \times \mathbf{H} + ikZ^{-1}\mathbf{E} = 0 \text{ in } D^+ \\ \mathbf{E}_{\mathrm{T}}^+ = 0 \text{ on } \Gamma, \ \mathrm{RC}(\mathbf{E} - \mathbf{E}^{\mathrm{inc}}, \mathbf{H} - \mathbf{H}^{\mathrm{inc}}) = 0 \end{cases}$$

where Γ is a closed surface, assumed to be $\mathcal{C}^\infty$ for simplicity, enclosing a bounded domain D^- and being the boundary of the unbounded domain D^+. The real positive numbers k and Z respectively stand for the wave number and the impedance of vacuum. The boundary condition on Γ is set in terms of the tangential component $\mathbf{E}_{\mathrm{T}}^+$. An exponent $+$ or $-$ indicates that the trace is taken from the values of $\mathbf{E}$ in respectively D^+ or D^-. We have denoted by $\mathbf{E}^{\mathrm{inc}}$ and $\mathbf{H}^{\mathrm{inc}}$ the electric and magnetic fields related to the given incident wave and by RC() one of the several ways to set the Silver-Müller radiation condition.

The construction of the C3IE starts from the well-known Stratton-Chu formula specialized to the present context since here the magnetic currents and charges are both equal to 0

$$(2) \qquad \mathbf{E} = \mathbf{E}^{\mathrm{inc}} - \nabla S\varrho + ikZS\mathbf{J}, \ \mathbf{H} = \mathbf{H}^{\mathrm{inc}} + \nabla \times S\mathbf{J} \text{ in } D^+$$

with $Sv(x) = \int_\Gamma (\exp(ik\,|x-y|)/4\pi\,|x-y|)v(y)d\Gamma_y$. The construction of a boundary integral formulation for the determination of the currents $\mathbf{J}$ and the charges ϱ can be seen as an attempt to nullify the right hand sides of (2) in D^- from conditions set on Γ. As for the classical CFIE, Taskinen and Ylä-Oijala [1] use the condition $Z^{-1}\mathbf{E}_{\mathrm{T}}^- + \mathbf{n} \times \mathbf{H}^- = 0$, which yields

$$(3) \qquad -ikS\mathbf{J} + \nabla_\Gamma S\varrho + \tfrac{1}{2}\mathbf{J} + \mathbf{n} \times K\mathbf{J} = Z^{-1}\mathbf{E}_{\mathrm{T}}^{\mathrm{inc}} + \mathbf{n} \times \mathbf{H}^{\mathrm{inc}}$$

with $K\mathbf{J} = \int_\Gamma \nabla_y \left(\exp(ik\,|x-y|\,/4\pi\,|x-y|) \times \mathbf{J}(y)d\Gamma_y\right.$. But instead of the equation relative to the conservation of charges $\nabla_\Gamma \cdot \mathbf{J} - ikZ^{-1}\varrho = 0$ on Γ, they take the equation on the normal componant $\mathbf{E}^- \cdot \mathbf{n} = 0$ augmented by the relation $S\left(\nabla_\Gamma \cdot \mathbf{J} - ikZ^{-1}\varrho\right) = 0$, i.e., $\mathbf{E}^- \cdot \mathbf{n} + S\left(\nabla_\Gamma \cdot \mathbf{J} - ikZ^{-1}\varrho\right) = 0$. We depart from this procedure and make use of a more straightforward approach based on the link of the C3IE and Picard's system recently given by Tsakinen and Vänskä [2]. In fact, Picard's system can be stated as a special instance of Maxwell's system, with no assumption on the conservation of charges, instead of introducing it a priori as this is done in [2]. It is enough to take the curl of $\mathbf{E}$ and $\mathbf{H}$ in (2)

$$(4) \qquad \nabla \times \mathbf{E} = ikZ\mathbf{H}^{\mathrm{inc}} + ikZ\nabla \times S\mathbf{J}, \ \nabla \times \mathbf{H} = -ikZ^{-1}\mathbf{E}^{\mathrm{inc}} + \nabla \times \nabla \times S\mathbf{J}$$

to get the system which is actually solved when the equation of conservation of charges is not satisfied

$$(5) \qquad \nabla \times \mathbf{E} - ikZ\mathbf{H} = 0, \ \nabla \times \mathbf{H} + ikZ^{-1}\mathbf{E} = -\nabla S(ikZ^{-1}\varrho - \nabla_\Gamma \cdot \mathbf{J}).$$

Introducing the auxiliary unknown $\varphi = S(ikZ^{-1}\varrho - \nabla_\Gamma \cdot \mathbf{J})$, we are led to the second equation of Taskinen and Ylä-Oijala C3IE system by simply posing $\mathbf{E}^- \cdot \mathbf{n} + \varphi = 0$

$$(6)\quad \tfrac{1}{2}\rho_e + \int_\Gamma \partial_{\mathbf{n}_x} \tfrac{\exp(ik|x-y|)}{4\pi|x-y|}\varrho(y)d\Gamma_y - ikZ\mathbf{n}\cdot S\mathbf{J} + S\left(\nabla_\Gamma \cdot \mathbf{J} - ikZ^{-1}\varrho\right) = \mathbf{E}^{\mathrm{inc}}\cdot\mathbf{n}$$

In order to prove that the above system of BIEs (3) and (6) is well-posed, we need the following two lemmas.

Lemma. *If* $\mathbf{J}$ *and* ϱ *are in* $L^2(\Gamma)$ *and are solution to (3) and (6), then* $\mathbf{E}$, $\mathbf{H}$ *and* φ *are in* $\mathcal{C}^\infty(\overline{D^-})$.

The main ingredient of the proof is to show that $\mathbf{E}$ and φ are solution in an appropriate functional setting of the following regular elliptic BVP (see, for instance, [3])

$$\begin{cases} \Delta\mathbf{E} + k^2\mathbf{E} = 0,\ \Delta\varphi + k^2\varphi = 0 \text{ in } D^- \\ (\nabla\times\mathbf{E}\times\mathbf{n})^- - ik\mathbf{E}_{\mathrm{T}}^- = 0,\ \mathbf{E}^- + \varphi^- = 0, \\ \qquad \partial_{\mathbf{n}}\varphi^- - ikZ^{-1}\varphi^- + Z^{-1}\nabla_\Gamma\cdot\mathbf{E}_{\mathrm{T}}^- = 0 \text{ on } \Gamma \end{cases}$$

Lemma. *Any solution to (3) and (6) yields that* $\mathbf{E}$, $\mathbf{H}$ *and* φ *are equal to* 0 *in* D^-.

The main ingredients of the proof are the following. Proceeding as for the classical CFIE, we first obtain from a Green formula

$$Z^{-1}\int_\Gamma \left|\mathbf{E}_{\mathrm{T}}^-\right|^2 d\Gamma + \mathrm{Re}\int_{D^-}\left(\mathbf{E}^-\cdot\nabla\times\overline{\mathbf{H}} - \nabla\times\mathbf{E}\cdot\overline{\mathbf{H}}\right)dx = 0$$

It is enough next to note that $\nabla\cdot\mathbf{E} + ikZ\varphi = 0$ in D^- to get from another application of Green formula that

$$\int_\Gamma \left(Z^{-1}\left|\mathbf{E}_{\mathrm{T}}^-\right|^2 + |\varphi|^2\right)d\Gamma = 0$$

thus leading to the result stated in the lemma.

We can then state the main result reported in this communication.

Theorem. *The system of BIEs (3) and (6) is well-posed in* $L^2(\Gamma)$.

The main argument of the proof is to write (3) and (6) in a variational form and to note that the left-hand side can be written as follows

$$\tfrac{1}{2}\int_\Gamma\left(\mathbf{J}\cdot\mathbf{J}' + Z^{-1}\varrho\varrho'\right)d\Gamma + \int_\Gamma\left(\varrho' S\nabla_\Gamma\cdot\mathbf{J} - \varrho S\nabla_\Gamma\cdot\mathbf{J}'\right)d\Gamma + \text{compact.}$$

The term "compact" stands for some compact bilinear form. The theorem results then from Fredholm alternative and a straightforward application of the Lax-Milgram lemma.

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Analysis of boundary element methods for high-frequency scattering simulations

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In the context of acoustic or electromagnetic waves, the classical issues that arise in connection with numerical simulations for other applications are additionally augmented with the intrinsic complexities (i.e. oscillations) of the quantities themselves. Still, very efficient methodologies (based on, for instance, finite elements, finite differences or boundary integral equations) have been devised to simulate the propagation of acoustic and electromagnetic waves in rather complicated settings. The very nature of these classical approaches, however, limits their applicability at high frequencies since the numerical resolution of field oscillations translates in a commensurately higher number of degrees of freedom and this, in turn, can easily lead to impractical computational times. For higher frequencies, accordingly, the only practical recourse is to resort to asymptotic methods (e.g. ray tracing) as these by-pass the need for frequency-dependent discretizations. These methods, on the other hand, are not error-controllable since they solve an approximate model instead of the original equations (e.g. the eikonal equation instead of the Helmholtz equation or the Maxwell system).

In this report, we survey a class of recently developed numerical schemes that *combine* the advantages of rigorous solvers (*error controllability*) with those of asymptotic methods (*frequency-independent discretizations*), and that therefore result in efficient and accurate simulators applicable throughout the frequency spectrum.

These algorithms pioneered by Bruno et. al. [4] in the context of single-scattering configurations (later extended by Bruno et. al. [5] to allow for the treatment of multiple scattering effects) are based on the solution of suitably chosen integral-equation formulations of the scattering problem, and they rely on three main elements, namely: 1) the use of an "ansatz" for the unknown surface currents which reduces the integral equation to one for a slowly varying modulation; 2) specialized quadrature rules for the new integral equation that take advantage of the highly-oscillatory nature of the kernel, and 3) full resolution of shadowing

transitions with discretizations that are adapted to their boundary-layer structure. The results in [4] clearly demonstrate the attainability of solutions within a prescribed error-tolerance in times that do not depend on the wavenumber k. An actual proof that provides a rigorous upper bound for the operation count of $\mathcal{O}(k^{1/9})$ in the case of circular/spherical boundaries was recently established by Dominguez et. al. [7] for a p-version boundary element implementation of a similar approach where, using the exponential decay (with increasing wavenumber k) of the surface current in the deep shadow region, they approximate this quantity by zero there as in [4]. Our contribution in this direction has been the design of two new Galerkin schemes [6] where we have shown that the error in best approximation of the surface current grows at most at $\mathcal{O}(k^{\epsilon})$ (for any $\epsilon > 0$) for the first algorithm, and at $\mathcal{O}(\log k)$ for the second one (based on a novel change of variables around the *transition regions*) over the *entire* boundary.

Returning to the treatment of multiple scattering effects, as we have found out, a fundamental step in understanding the high-frequency features of multiple scattering iterations is the derivation of accurate asymptotic expansions for the densities that are sequentially induced on the surface of the scatterers. Indeed, when a multiple scattering orbit is considered, the field diffracted from the surface of the m–th obstacle acts as an incidence impinging on the $(m+1)$–st surface and, thus, it generates a current therein. As we have shown in [1, 10], this allows one to recover the *symbolic classes* (in the sense of Hörmander) of the multiple scattering iterates; and this, in turn, enables one to derive their high-frequency asymptotic expansions that turn out to be *uniform* perturbations of order $\mathcal{O}(k^{-1})$ of a *discrete dynamical system* determined by the *open billiard flow* in the region exterior to the obstacles.

In two-dimensions [9–11], these expansions show that if an optical ray arrives at a point on the boundary of a scatterer after m transverse bounces, then (asymptotically) the current at that point equals the current at the $(m-1)$–st reflection-point times a continued fraction determined by geometric properties of the corresponding ray path; consequently, the current at that point is a perturbation of order $\mathcal{O}(k^{-1})$ of the *product* of m (recursively defined) *continued fractions* determined by the *entire* ray path. In three-dimensional settings, on the other hand, and for the *scalar acoustic* case [1], these continued fractions are replaced by expressions in the form of *two-dimensional continued fractions*; a distinctive property of these expressions, when compared to their two-dimensional counterparts, is that they depend smoothly on the *relative angle of rotation* between the principal axes of the successive reflection points of the optical rays. The fully three-dimensional *vector electromagnetic* expansions in [8], in turn, show that at each reflection the *asymptotic* currents are, as they ought to be, tangential to the surfaces and, most importantly, that they undergo a rotation and a projection onto the surface perpendicular to the reflection vector, followed by a second rotation and a projection onto the tangent space at the point of arrival.

To analyze these asymptotic expansions for a collection of convex structures, a fundamental observation relates to the convexity of wavefront sets corresponding

to successive wave reflections. This has resulted in a rather technical analysis yielding a proof that the ratios of high-frequency asymptotic expansions of multiple scattering iterates on a periodic orbit converge to an *explicitly computable* complex number (vector) $\mathcal{R}_k$ in the form of a wavenumber dependent phase term modulated by a (real) amplitude. Moreover, we have shown that this latter convergence is *exponential* in the number of reflections, *uniform* over the *entire* boundaries, and that the analysis is *optimal* with regards to the length of the periodic orbits.

Even though, as our work has shown, the multiple-scattering series converges spectrally, it is clearly desirable to design mechanisms to accelerate its convergence. In this connection, an essential consequence of our analysis is that the ratio of iterated currents differing by one period stabilizes after a *frequency dependent* number of reflections which grows only *logarithmically* with increasing frequency. Accordingly, once stabilized, the behavior of the series resembles an $\mathcal{O}(k^{-1})$ perturbation of a geometric series which, in turn, can be well approximated by rational functions. This, as we have shown, completely *clarifies* the enhanced convergence properties of the Padé approximation procedure when applied to the multiple-scattering series [5].

Moreover, based on the stabilization properties of the series, we have further devised two alternative acceleration algorithms that, as opposed to Padé approximants, *do not* require the solution of a linear system. The first algorithm makes explicit use of the derived rate of convergence formulas and provides an $\mathcal{O}(k^{-1})$ improvement once the series stabilizes. The second, in contrast, is based on the fact that the series itself is a perturbation of a geometric series and it provides a further significant reduction in the number of single-scattering problems necessary to solve the overall problem within a desired accuracy. Finally, for cases wherein the aforementioned convergence is slow, we have shown that utilization of a new post-processing algorithm based on a novel use of Krylov-subspaces provides a further significant reduction in the number of iterations while still retaining the frequency-independent computational cost [2, 3].

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Fast Methods for Option Pricing

Tobias von Petersdorff
(joint work with C. Schwab, R. Nochetto, C. Zhang)

Traditionally option pricing is based on Brownian motion which leads to a parabolic problem with the space operator $A = -\Delta$. In finance jump processes are now popular which lead to a parabolic problem $u_t + Au = 0$ with an integral operator A of order $p \in (0, 2)$. Using wavelet compresson and hp discretization in time we can achieve $\mathcal{O}(N(\log N)^c)$ complexity. For multidimensional problems in $\mathbb{R}^d$ with $N = n^d$ one can use sparse grids to achieve $\mathcal{O}(n(\log n)^c)$. For American options, however, many open questions remain.

Integral equations for open surfaces

Oscar P. Bruno

This note concerns recent progress in the numerical solution of problems of diffraction by infinitely thin open surfaces. These problems, which can be treated numerically by means of either, discretized boundary integral equations or volumetric formulations, present a number of difficulties—stemming, mainly, from the singular field behavior in a vicinity of the edges. Although approaches based on boundary integral methods require a lower dimensionality and, thus, smaller discretizations than volumetric PDE solvers, open-surfaces integral approaches do require, at least in their classical formulations, use of first-kind integral equations. Since, for efficiency, high-frequency three-dimensional scattering problems ought to be treated by means of iterative linear-algebra solvers (to take advantage of accelerated scattering solvers [3,4,14]), such formulations can prove computationally expensive: the eigenvalues of the associated first kind equations accumulate at zero and infinity and thus, Krylov subspace solutions of these equations require large numbers of iterations. In the contributions [5,6] we show that use of compositions of certain operators S_ω and N_ω (that arise as singular integration weights are used in conjunction with the classical integral operators S and N associated with the

Dirichlet and Neumann problems), gives rise to second-kind integral formulations requiring small numbers of Krylov subspace iterations.

The difficulties that arise as integral equations are used to treat open surface scattering problems are of course well known, and a significant number of efforts have been devoted to their treatment. A first class of approaches in these regards can be grouped around the idea of generalizing the classical closed-surface Calderon relations [9, 12] to the open-surface context. In this class we can include the contributions [7,13]. The work [13] gives rise to equations of the form $I+T_K$ where the kernel $K(x,y)$ of the operator T_K has local singularity of at most $O\left(\frac{1}{|x-y|}\right)$. This early result however does not take into account the singular edge behavior; the resulting operator T_K is not compact and $I+T_K$ is not a second-kind integral operator. The contribution [7], in turn, shows that, for open surface problems, use of the combination NS in conjunction with boundary elements that vanish on the edges results in low numbers of iterations for a given residual, at least for low frequencies, but it does not provide details on accuracy and, in fact, it suggests that, in contrast with the equations presented discussed here [5, 6], the condition number of the finite element approximation for NS actually tends to infinity as the discretizations are refined. The combination NS is also considered in [1]; again low iteration numbers are demonstrated for low frequency problems, but the edge behavior is not taken into account and accuracy studies are not presented.

In a second class of methods we can include the algorithms presented in [2, 10, 11]. These contributions rely on use of the cosine basis to produce spectral accuracy (an aspect that is incorporated in our work [5, 6] as well). The approach [2] treats the Dirichlet problem for Laplace's equation by means of second kind equations; the basis of this approach lies in the observation that the cosine basis has the dual positive effect of removing the singular edge behavior *and diagonalizing the logarithmic potential for a straight arc*, whose inverse can thus be easily computed and used to produce a second kind operator *for a general arc*. We have shown that a direct generalization of this approach to Helmholtz' acoustic equations gives rise to very large numbers of Krylov-subspace iterations, even larger than those required by the single-layer equation itself. Like [2], reference [10] considers the Laplace equation and uses a cosine basis and the inverse of the straight-arc problem to produce a second kind integral equation. The approach [11], finally, treats the Neumann problem for the non-zero frequency Helmholtz equation with spectral accuracy by means of first kind equations and direct solvers. Once again, our experiments have shown that this equation gives rise to very large numbers of Krylov subspace iterations.

A third class of approaches to our problem is well exemplified by the contributions [8, 15]; here the hp version of the finite element method is used to produce exponential convergence to the solution of first kind equations associated with open surface problems. Further, multilevel approaches are used to produce formulations whose condition number grows slowly as discretizations are refined. While effective for the Laplace and low-frequency Helmholtz problems, multilevel formulations are

limited at high-frequencies by the requirement of resolution of the wavelength by the coarsest discretization—for which a direct solve must be produced.

The main new results presented in [5, 6], which are concerned with two-dimensional configurations, may briefly be stated as follows. Defining $\omega \sim d^{1/2}$ (where d is the distance to the edge) and letting

$$S_\omega[\varphi] = S(\frac{\varphi}{\omega}) \tag{1}$$

and

$$N_\omega[\psi] = N(\omega\psi), \tag{2}$$

(where S and N denote the regular single- and double-layer potential), we have

$$N_\omega S_\omega = J_0 + K \tag{3}$$

where, *after appropriate changes of independent variables and in suitable function spaces of even periodic functions, the operator J_0 becomes a continuous operator that admits a continuous inverse and the operator K becomes a compact operator.* Further, high-order accurate numerical implementations of these equations show that 1) The eigenvalues for these equations are highly clustered, and that, 2) These equations give rise to solutions for open-arc Dirichlet and Neumann problems in significantly smaller number of iterations (improvements by factors of 30 were obtained for some realistic geometries) than those arising from direct use of the first kind equations.

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Rapid solution of the wave equation in unbounded domains

Stefan A. Sauter
(joint work with Lehel Banjai)

1. Integral Formulation of the Wave Equation

The efficient numerical solution of the wave equation plays a key role in the simulation of many physical applications such as electromagnetic wave propagation or the computation of transient acoustic waves. The homogeneous wave equation reads

$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \Omega \times (0,T) \tag{1a}$$

with initial conditions

$$u(\cdot,0) = \partial_t u(\cdot,0) = 0 \qquad \text{in } \Omega \tag{1b}$$

and boundary conditions

$$u = g \quad \text{on } \Gamma \times (0,T) \tag{1c}$$

on a time interval $(0,T)$ for some $T > 0$. Here, $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain with boundary Γ. For its solution, we employ an ansatz as a *single layer potential*

$$u(x,t) = \int_0^t \int_\Gamma k(x-y, t-\tau)\phi(y,\tau) d\Gamma_y d\tau\,, \quad (x,t) \in \Omega \times (0,T)\,, \tag{2}$$

where $k(z,t)$ is the fundamental solution of the wave equation,

$$k(z,t) = \frac{\delta(t - \|z\|)}{4\pi\|z\|}, \tag{3}$$

$\delta(t)$ being the Dirac delta distribution. The ansatz (2) satisfies the homogeneous equation (1a) and the initial conditions (1b). Hence, the unknown density ϕ in (2) is determined via the boundary conditions (1c), $u(x,t) = g(x,t)$. This results in the boundary integral equation for ϕ,

$$\int_0^t \int_\Gamma k(x-y, t-\tau)\phi(y,\tau) d\Gamma_y d\tau = g(x,t) \qquad \forall (x,t) \in \Gamma \times (0,T)\,. \tag{4}$$

Existence and uniqueness results for the solution of the continuous problem are proved in [24] and [1, Prop. 3].

Although this approach goes back to the early 1960s (cf. [14]), the development of fast numerical methods for integral equations in the field of hyperbolic problems is still in its infancies compared to the multitude of fast methods for elliptic boundary integral equations (cf. [28] and references therein). Existing numerical discretization methods include collocation methods with some stabilization techniques (cf. [5], [6], [9], [10], [11], [26], [27]) and Laplace-Fourier methods coupled with Galerkin boundary elements in space ([1], [8], [12], [15]). Numerical experiments can be found, e.g., in [16].

In [13] a fast version of the *marching-on-in-time* (MOT) method is presented which is based on a suitable plane wave expansion of the arising potential which reduces the storage and computational costs.

We here employ the convolution quadrature method for the time discretization and a Galerkin boundary element method in space. The convolution quadrature method for the time discretization has been developed in [22], [23], [24], [25]. It provides a straightforward way to obtain a stable time stepping scheme using the Laplace transform of the kernel function.

The coefficient matrix in the arising linear system is a block-triangular Toeplitz matrix consisting of N blocks of dimension $M \times M$, where N denotes the number of time steps and M is the number of spatial degrees of freedom. Due to the nonlocalness of the arising boundary integral operators, the $M \times M$ matrix blocks are densely populated.

In the literature, there exist (at least) two alternatives to solve the system arising by the convolution quadrature approach efficiently. In [19], FFT-techniques are employed which make use of the Toeplitz structure of the system matrix and the computational complexity is reduced to $\mathcal{O}\left(\left(N \log^2 N\right) M^2\right)$, while the storage complexity stays at $\mathcal{O}\left(NM^2\right)$. In [18], [17], [21], the $M \times M$ block matrices are approximated by data sparse representations based on a cutoff and panel-clustering strategy. This leads to a significant reduction of the storage complexity while the computational complexity is reduced compared to the naive approach (cost: $\mathcal{O}\left(N^2M^2\right)$) but increased compared to the FFT approach.

Here, we propose a third approach which combines the advantages of the FFT-technique with the sparse approximation. We transfer the block Toeplitz system to the Fourier image by the discrete Fourier transform and then face the problem of computing approximate solutions of Helmholtz problems at different (complex) wave numbers. These Helmholtz problems are fully decoupled and can hence be efficiently solved on parallel computers. Relatively standard, fast methods (e.g. fast multipole method, hierarchical matrices) for the solution of frequency domain scattering can effectively be applied to these problems; see [7, 30] and [2]. It may also be possible to further reduce the computational cost of assembling the matrices by using the techniques for multifrequency analysis described in [20, 29]. Further, we also show that if the boundary data is sufficiently smooth and compatible and of limited time duration, instead of N, only $\mathcal{O}(N^\epsilon)$, for any fixed $\epsilon > 0$, Helmholtz systems need to be solved.

This paper is based on results which have been published in [3], [4].

We summarize briefly the main results of these papers.

(1) A perturbation theory has been developed which allows to replace time-consuming parts of the algorithm such as quadrature, the generation of the full matrix, etc. by efficient approximations while preserving the asymptotic convergence rates.
(2) Let N denote the number of time steps and let M be the number of space points. The cost of our algorithm for computing time-space discrete approximations $\phi_{\Delta t,h}$ is dominated (essentially) by solving N decoupled Helmholtz problems with complex wave numbers by using multipole techniques. Hence, the computational and storage costs are of order NM up to logarithmic terms. (If the signal is sufficiently smooth and of limited time duration the number of Helmholtz solves is even reduced to N^ε for any $\varepsilon > 0$.)
(3) We have performed numerical experiments in two dimensions which show that the asymptotic estimates are already visible also for moderate problem sizes.

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Calderón-enhanced Time Domain Integral Equation Solvers: Preconditioning, Stability, and Nullspaces for Toroidal Objects

Eric Michielssen

(joint work with Francesco Andriulli, Hakan Bağcı, Annalisa Buffa, Snorre Christiansen, Kristof Cools, and Femke Olyslager)

Marching on in time (MOT)-based time domain integral equation (TDIE) solvers represent an increasingly appealing avenue for analyzing transient electromagnetic interactions with large and complex structures. MOT-TDIE solvers

for analyzing electromagnetic scattering from perfect electrically conducting objects, obtained by enforcing electric field boundary conditions, implicitly time advance electric surface current densities by iteratively solving sparse (MOT) systems of equations at all time steps. Here we report on three aspects relating to the recent development of Calderón-enhanced MOT-TDIE solvers: preconditioning, low-frequency stability, and the occurrence of nullspaces for toroidal objects.

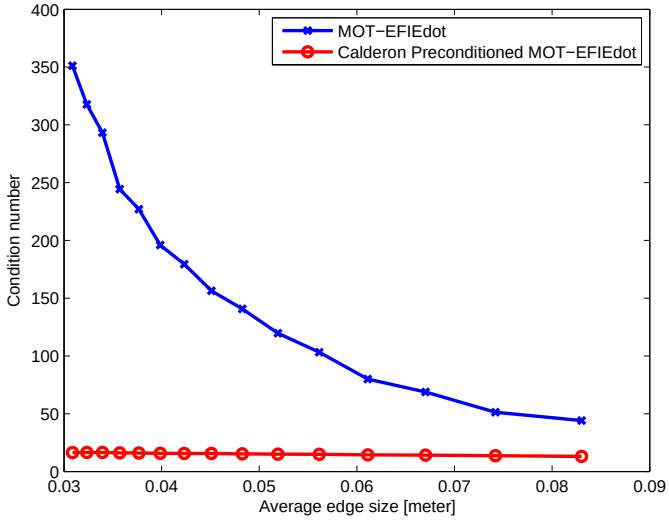


FIGURE 1. Condition number of the MOT-TDEFIE and Calderón preconditioned MOT-TDEFIE systems versus the average edge size for the sphere with a radius of one lightmeter.

1. Preconditioning. Classical MOT-TDIE solvers do not suffer from a CFL time step constraint and operate on unstructured meshes [2]. In other words, their time step sizes δ_t are dictated solely by excitation bandwidths, never by the spatial resolution δ_r of the meshes on which they operate. Meshes used by MOT-TDIE solvers only need to abide by two constraints: they must resolve (i) the shortest wavelength in the excitation and (ii) the structure's geometric features. When constraint (i) drives mesh construction – this situation is the norm when analyzing electromagnetic scattering from relatively smooth surfaces subject to high-frequency excitations - the MOT-TDIE system matrix tends to be well-conditioned and the number of iterations per time step small. In contrast, when constraint (ii) dictates surface mesh resolution – this often happens when analyzing electromagnetic radiation from, and guidance on, geometrically intricate and mixed-scale surfaces subject to low- to medium-frequency excitations – the MOT-TDIE system matrix tends to be ill-conditioned and the number of iterations per time step large. In this scenario, the MOT-TDIE system matrix condition number grows without bound as the mesh

resolution $\delta_r \to 0$. Recently, this dense mesh breakdown phenomenon was effectively cured by using semi-analytical and multiplicative Calderón preconditioned MOT-TDIE solvers [1]. The new Calderón-preconditioned MOT-TDIE solvers give rise to well-conditioned MOT systems, irrespective of the mesh resolution. Figure 1 shows the condition numbers of Calderón preconditioned MOT-TDIE matrices versus $1/\delta_r$ along with those obtained using of the non-preconditioned MOT-TDIE solver. The Calderón preconditioned formulation compares favorably to the standard MOT-TDIE approach.

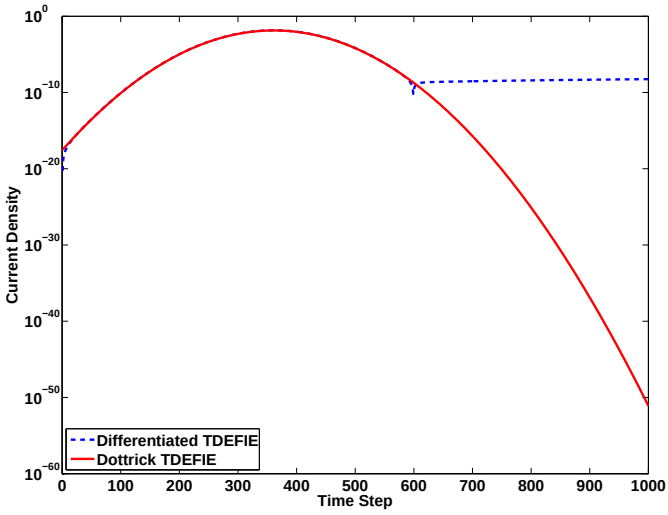
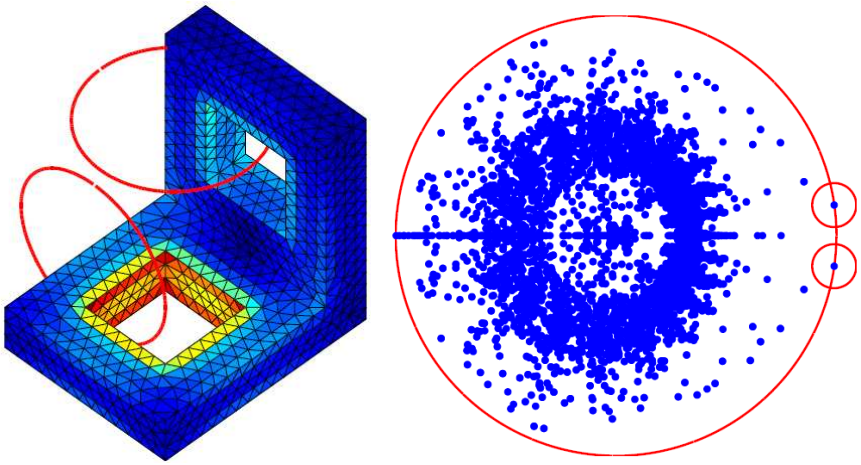


FIGURE 2. Currents on a sphere obtained with the dottrick TDEFIE and with the differentiated TDEFIE versus the time step.

2. Low-frequency stability. Classical MOT-TDIE solvers are prone to low-frequency (quasi-DC) instabilities arising from the fact that they possess a null-space comprising magnetostatic currents. Recently, we showed that this nullspace can be effectively removed by leveraging Calderón identities to construct the so-called "dot-trick" [5] time domain electric field integral equation, comprising products of "singular" and "hypersingular" time domain electric field operators with judiciously placed temporal integrators/differentiators. The "dot-trick" time domain electric field integral equation was shown to effectively resolve solenoidal functions of any time signature, including "DC", and applies to open as well as closed structures. In Figure 2 transient currents obtained with the standard MOT-TDIE and with the "dot-trick" MOT-TDEFIE are plotted versus $1/\delta_r$. It is evident that no low-frequency instabilities are present in the solution of the "dot-trick" MOT-TDEFIE.



(a) (b)

FIGURE 3. Kernel of the static outer MFIE operator due to the two cycles (a) , evidenced by a polynomial eigenvalue plot (b).

3. Nullspaces for toroidal scatterers. Unfortunately, Calderón-enhanced TDIEs are no panacea. These equations possess a nullspace when applied to the analysis of scattering from toroidal objects and the dimension of the nullspace is twice the genus of the object analyzed (Figure 3(a)). Its origins are quite different from those discussed above and relate to the nullspaces that plague static magnetic field equations applied to toroidal objects. We have not only elucidated the construction of a basis for this nullspace, but also investigated the effect of its existence on the numerical treatment of non-static problems [6]. Unfortunately, to date, no effective method for removing the nullspaces from the equations or annihilating its contribution from the solution vector has been found.

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New Boundary Element Convergence Results for Unilateral Boundary Value Problems

Joachim Gwinner

In this talk we address elliptic free boundary problems where the nonlinearity lies at the boundary and so the boundary element method is the method of choice. We focus on the boundary element method in its p-version to treat a scalar variational inequality of the second kind that simultaneously models unilateral contact and Coulomb friction in elasticity [10, 12]. Thus we complement recent work of Maischak and Stephan [13, 14] on adaptive hp-versions of the bem for unilateral Signorini problems, respectively on fem-bem coupling in its h-version for a nonlinear transmission problem modelling Coulomb friction contact.

Since the unilateral constraint can only be controlled in a finite set, polynomial approximation of higher order leads to a nonconforming discretization scheme. In contrast to [14] and to a related paper of Guediri [7] on a boundary variational inequality of the second kind with the Helmholtz operator modelling friction and on the convergence of the h-boundary element method we take also the quadrature error of the nonsmooth friction functional into account of the error analysis.

Our convergence results consist of two parts. Firstly without any regularity assumptions, we prove convergence of the $p-$bem Galerkin solution in the energy norm. To this end we adapt the discretization theory of Glowinski [6] and apply its extension to the more difficult semicoercive case in [8], see also [9] for $h-$boundary element convergence for variational inequalities of the second kind. The key to such a norm convergence result for the $p-$bem is the used Gauss-Lobatto integration rule with its high exactness order and its positive weights together with duality arguments in the sense of convex analysis [4].

Secondly we present a Céa-Falk lemma that extends the basic error estimate for abstract variational inequalities in [3, Theorem 23.1], [5, Theorem 1], see also [2, Theorem 6.1], to abstract variational inequalities of the second kind. This lemma permits to split the total discretization error into three different parts: the approximation error due to the approximation of the Steklov-Poincaré operator by its discrete counterpart, the distance of the continuous solution to the convex set of approximations in the trial space, and the consistency error caused by the nonconforming approximation. Here as in [13] we use the well-known approximation theory of spectral methods [1], the cutting technique of Falk [5], and interpolation arguments to obtain estimates in Sobolev norms of fractional order. Moreover, we exploit the special structure of the friction functional. Thus for our more general variational problem we arrive under mild regularity assumptions at an a priori error estimate of the same convergence order as in [13] which is suboptimal

because of the appearance of the consistency error in the nonconforming approximation scheme and because of the well-known regularity threshold in unilateral problems [11].

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New Elements for Inverse Boundary Element Methods

Roland Potthast

(joint work with Filippo Fazi, Martin Wannert, Jijun Liu, Phil Nelson, Fahmi ben Hassen)

We present some new techniques for the solution of *source identification* and *source reconstruction* in acoustics and magnetic tomography which are based on boundary integral representations and which employ boundary element techniques for their numerical realization.

Source splitting in acoustics. First, we study the recognition of the sound field of different sources which are located in n separate domains G_1, ..., G_n in space. Define $G := G_1 \cup ... \cup G_n$. We prove that the sum $u = u_1 + ... + u_n$ of the fields

measured on an open subset of a plane Γ with $\Gamma \cap \overline{G} = \emptyset$ uniquely deterines the fields u_1, ..., u_n. Then, we describe two new splitting algorithms to calculate u_j, $j = 1, ..., n$ from u. The first algorithms is based on the potential method of Kirsch and Kress (1886). The second algorithm employs the the point source method of the author (1996). We provide a convergence prove for field reconstruction. Further, we give a demonstration based on real data measurements separating two voices from microphone array measurements carried out at the Institute of Sound and Vibration Research ISVR at the University of Southampton, UK[1]. Details can be found in [1], [2].

Uniqueness of magnetic tomography for multilayer devices. The second part of the talk employs the splitting technique to prove uniqueness of magnetic tomography for a multi-layer device [3]. This applies in particular to magnetic tomography for fuel cells. A multi-layer device is defined by several layers Λ_k, $k = 1, ..., n$ in which the current can flow only in z-direction. In the planes Γ_ℓ between the layers Λ_ℓ and $\Lambda_{\ell+1}$ the currents flow in the $x-y$ plane. A uniqueness prove for the reconstruction of a current density j from its magnetic field H is give in five steps.

1. First, we argue that the magnetic field H of the currents in the layers Λ_ℓ has no H_3 component. Thus, the field H_3 is a sum of fields $H_{3,\ell}$ arising from the currents j_ℓ in Γ_ℓ for $\ell = 0, ..., n$. Analogously to the acoustic case above we prove that this field can be uniquely split into its parts $H_{3,\ell}$, $\ell = 0, ..., n$.

2. Now, we prove that H_ℓ can be reconstructed from $H_{3,\ell}$ measured on some closed surface containing the device. This is carried out by first showing that $H_{3,\ell}$, which solves the Laplace equation in the exterior domain $\mathbb{R}^3 \setminus \overline{\Gamma_\ell}$, is uniquely determined on the boundary of a half-space X with boundary parallel to the $x-y$ plane and with Γ_ℓ in its open exterior. Then, the full field H_ℓ is determined by its normal values $e_3 \cdot H_\ell = H_{3,\ell}$ on the boundary ∂X of X. By analyticity this holds on $\mathbb{R}^3 \setminus \Gamma_\ell$.

3. Then, we reconstruct H_ℓ supported on Γ_ℓ in $\mathbb{R}^3 \setminus \Gamma_\ell$. This is carried out either by the potential approach of Kirsch-Kress or by the point source method of the author.

4. By explicit use of the jump relations of the curl of a vectorial single-layer potential on Γ_ℓ we can reconstruct j_ℓ in Γ_ℓ from the knowledge of H_ℓ in $\mathbb{R}^3 \setminus \Gamma_\ell$.

5. Finally, we use $\mathrm{div}(j) = 0$ in the layers $\Gamma_\ell \cup \Lambda_\ell$ to reconstruct the currents j in Λ_ℓ from the knowledge of j in Γ_ℓ for $\ell = 0, ..., n-1$.

Duality principles for inversion methods. Part three of the talk studies the relation of the potential approach and the point source method introduced above. We show a duality principle for the methods [4] and use it to prove that the methods provide identical reconstructions when used with the same geometrical setup and Tikhonov regularization for stabilization.

[1]This research has been promoted by a *springboard fellowship* of the Engineering and Physical Sciences Research Council (EPSRC) in the United Kingdom under grant number EP/E032419/1.

Orthogonality Sampling for object identification. In the forth part of the talk we present recent results for inverse acoustic scattering. We study some new ideas for object identification from the far field pattern u^∞ of scattered waves u^s which have been called *orthogonality sampling* [5], [6]. Given the far field pattern $u^\infty(\hat{x}, k)$ for all $\hat{x} \in \mathbb{S}$ and an interval $k \in [k_1, k_2]$ of wave numbers, the method calculated the functional

$$\mu(y) := \int_{k_1}^{k_2} \left| \int_{\mathbb{S}} e^{ik\hat{x}\cdot y} u^\infty(\hat{x}, k)\, ds(\hat{x}) \right| dk, \quad y \in \mathbb{R}^m \tag{1}$$

with $m = 2, 3$.

We need to remark that the modulus of the interior integral in (1) is crucial and does not appear in Fourier arguments. Without the modulus we have not been able to obtain satisfactory reconstructions! We consider the functional (1) as an orthogonality test for the far field of a point source with the given far field pattern. Note that the philosophy and realization of the functional (1) is different from other schemes like the Linear Sampling Method and also different from minimization techniques which are widely used. In particular, the functional (1) is well-posed.

Via the Funck-Hecke formula for the case of a Dirichlet boundary condition $u|_{\partial D} = 0$ for the total field $u = u^i + u^s$ on the boundary ∂D of a scatterer D we prove that $\mu(y)$ is identical

$$\int_{k_1}^{k_2} \left| \int_{\partial D} j_0(k|y - z|) \frac{\partial u}{\partial \nu(z)}(z, k)\, ds(z) \right| dk, \quad y \in \mathbb{R}^m, \tag{2}$$

with the Bessel function j_0 of order 0. Here, we call

$$u^s_{red}(y) := \int_{\partial D} j_0(k|y - z|) \frac{\partial u}{\partial \nu(z)}(z, k)\, ds(z) \tag{3}$$

the *reduced scattered field*. By numerical examples we show that the sum of the modulus of the reduced scattered field provides information about the location and shape of an unknown scatterer. The numerical results show that we obtain a resolution of $\lambda/2$ where $\lambda = 2\pi/k_2$ is the minimal wavelength of the fields under consideration. Examples are shown for scattering by impenetrable scatterers with Dirichlet or Neumann boundary condition and for scattering by inhomogeneous media in two dimensions.

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Optimal panel-clustering for anisotropic mesh refinement

Wolfgang Hackbusch
(joint work with Ivan G. Graham, Lars Grasedyck, Stefan A. Sauter)

The given lecture is based on the paper [7] by mentioned authors.

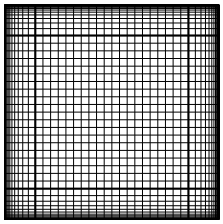

Fig. 1: graded mesh for the screen

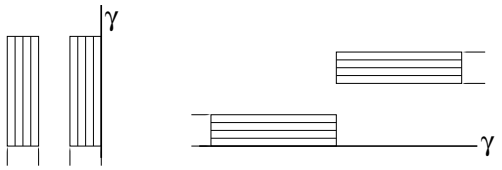

Fig. 2: two cases of anisotropic cluster pairs

BEM for polyhedra and graded meshes. We consider the boundary element methods for polyhedra. Their solutions contain certain edge singularities. Correspondingly, the optimal mesh is *algebraically graded* towards edges. Since the screen has the same features, we restrict to this case (cf. Fig. 1). The square contains strips of width $\delta = O(1)$ in which the edge-normal grid points are defined by $\delta * (i/n)^g$, $0 \le i \le n$ ($g > 1$: grading exponent). In the middle of the screen the step size $h = 1/n$ is used. The number of grid points is $N \sim n^2$. The rectangles along the edges have an aspect ration $\sim N^{(g-1)/2}$ (note that $(g-1)/2 > 1$ for $g > 3$). For the analysis for such graded meshes see, e.g., v. Petersdorff [20], v. Petersdorff, Stephan [21], Elschner [5], Ervin, Stephan, Abou El-Seoud [6] and Graham, Sauter, Hackbusch [10]. In particular (cf. [6, Theorem 1.4]) it is known that the (piecewise constant) Galerkin solution $U \in \mathcal{S}$ to the single layer potential problem on a Lipschitz polyhedral surface (for sufficiently smooth right-hand side) satisfies the optimal estimate in terms of number of degrees of freedom N:

$$\|u - U\|_{H^{-1/2}(\Gamma)} \lesssim N^{-3/4} \qquad \text{when } g > 3\,. \tag{1}$$

Panel clustering and hierarchical matrices for regular meshes. Techniques like panel clustering (Hackbusch, Nowak [17, 18]), hierarchical matrices (Hackbusch [14], Grasedyck, Hackbusch [12], Hackbusch, Khoromskij, Sauter [16]), multipole methods (Greengard, Rokhlin [13]) or wavelet compression techniques (Schneider [24]) approximate the kernel $\varkappa$ (here $\varkappa = 1/\|x-y\|$) by a separable expression of the form[1] $\tilde{\varkappa}(x,y) = \sum_{\nu=1}^{k} \varphi_\nu(x)\psi_\nu(y)$, provided that x, y belong to

[1]For hierarchical matrices any functions φ_ν, ψ_ν are allowed, multipole requires special functions, wavelet compression needs polynomials for either φ_ν or ψ_ν.

subsets Ω_1 and Ω_2 (i.e., $x \in \Omega_1$, $y \in \Omega_2$) which admit the *admissibility condition*

$$\min\{\operatorname{diam}(\Omega_1), \operatorname{diam}(\Omega_2)\} \;\le\; \eta \operatorname{dist}(\Omega_1, \Omega_2) \tag{2}$$

with a fixed positive constant η. Then all entries $\tilde{A}_{ij} = \iint \tilde{\varkappa}(x,y) b_i(x) b_j(y) \mathrm{d}x\mathrm{d}y$ of the system matrix with basis functions b_i, b_j supported in the respective set Ω_1 and Ω_2 lead to a rank-k-matrix. Let I be the index set of the indices i, j. This set is recursively divided into suitable subsets leading to a (binary) cluster tree. All elements τ of the cluster tree are subsets of I. The domain Ω_τ is the union of all supports of basis functions b_i, $i \in \tau$. Then two clusters τ, τ' are called admissible, if Ω_τ and $\Omega_{\tau'}$ satisfy the admissibility condition (2).

The matrix A can be written as $A \approx A_{\text{near}} + A_{\text{far}}$, where the far-field part A_{far} consists of the contribution of suitable admissible cluster pairs. The near-field part contained in A_{near} are the original matrix entries A_{ij} for which the supports of b_i, b_j are non-admissible. In the case of a regular mesh, A_{near} is a convenient sparse matrix. As a consequence, the storage amount of $A_{\text{near}}, A_{\text{far}}$ as well as the cost of the matrix-vector operation is linear in the dimension N up to logarithmic factors.

Standard panel clustering and graded meshes. Consider the situation of Fig. 2. Obviously the distance of the clusters is much smaller than the diameter. Accordingly, the cluster pairs are non-admissible and all involved entries A_{ij} must be represented by the near-field matrix which now has a number of non-zeros per row which is growing with N. A precise analysis shows the storage cost $O(N^{\frac{3}{2}-\frac{1}{g}})$. Hence for $g = 4$ (note that $g > 3$ is required), the cost is proportional to $N^{5/4}$ and therefore *non-optimal*.

Remedy by exact integration. Consider the right case of Fig. 2. Let $\tau = [A, B] \times [\alpha, \beta]$ and $\tau' = [A', B'] \times [\alpha', \beta']$ a pair of elements contained in the cluster pair. Note that all elements in the first cluster have the identical interval $[A, B]$, only the bounds $[\alpha, \beta]$ differ. Similarly, all elements in the second cluster have the first variable in $[A', B']$. The (τ, τ')-matrix entry is

$$\iint\limits_{\tau\times\tau'} \frac{\mathrm{d}x\mathrm{d}y}{\|x-y\|} = \int_\alpha^\beta \int_{\alpha'}^{\beta'} \underbrace{\left(\int_A^B \int_{A'}^{B'} \frac{\mathrm{d}x_1 \mathrm{d}y_1}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}} \right)}_{=:I(x_2,y_2)} \mathrm{d}x_2 \mathrm{d}y_2. \tag{3}$$

The inner integral depends only on x_2, y_2. It can easily be computed (for antiderivatives of the Newton potential compare Hackbusch [15]). We can consider (3) as the evaluation of a one-dimensional integral equation with the new kernel $I(x,y)$. Since $I(x,y)$ is asymptotically smooth, we can use the standard admissibility condition for the 1D-situation which means

$$\min\{\operatorname{diam}([\alpha,\beta]), \operatorname{diam}([\alpha',\beta'])\} \;\le\; \eta \operatorname{dist}([\alpha,\beta],[\alpha',\beta']).$$

Since now the long stretched direction is eliminated we get new contributions for the far field, while the near-field is sparse again. The overall cost is the same as for the case of a regular grid. In fact the situation is even better, since 1) 1D

kernel lead to a smaller rank k compared with 2D for the same accuracy and 2) for the graded case one interval (e.g., $[\alpha, \beta]$) is much smaller than the other and the minimum leads to a very small η.

The solution of the linear system uses the approximate LU-decomposition computed by the hierarchical matrix technique. The required time is usually clearly smaller than the time for the matrix set-up.

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Boundary Integral Equations for Multi-dielectric Electromagnetic Scattering

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Introduction. We consider the problem of time-harmonic electromagnetic scattering at a bounded object occupying the domain $\Omega \subset \mathbb{R}^3$. The object is composed of several homogeneous linear materials with different permittivities ϵ. More precisely, $\epsilon(\boldsymbol{x}) \equiv \epsilon_i$ in $\Omega_i \subset \Omega$, $i = 1, \ldots, P$, where all the Ω_i are curvilinear Lipschitz polygons, mutually disjoint, and $\overline{\Omega} = \bigcup_{i=1}^{P} \overline{\Omega}_i$. This also applies to $\Omega_0 := \mathbb{R}^3 \setminus \overline{\Omega}$.

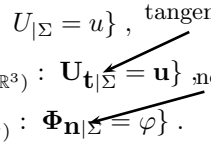
The resulting (scaled) electric field solves the electric wave equation $\mathbf{curlcurlE} - \kappa^2\epsilon(\boldsymbol{x})\mathbf{E} = 0$ with wave number $\kappa > 0$. In Ω_0, $\mathbf{E}$ represents the scattered field, which satisfies Silver-Müller radiation conditions at ∞ [5]. Across $\Gamma := \partial\Omega$ jumps $[\mathbf{E_t}]_\Gamma$ (tangential component) and $[\mathbf{curl}\,\mathbf{E} \times \mathbf{n}]_\Gamma$ ($\mathbf{n}$ the unit normal of Γ) are imposed.

Simpler, but closely related, is the problem of acoustic scattering governed by the scalar Helmholtz equation $-\Delta U - \kappa^2\epsilon(\boldsymbol{x})U = 0$. Here we have to prescribe the jumps $[U]_\Gamma$ and $[\mathbf{grad}\,U \cdot \mathbf{n}]_\Gamma$, and Sommerfeld radiation conditions at ∞. Many ideas will be elaborated below for both the scalar and vectorial case, in order to emphasize parallel structure.

Skeleton trace spaces. Recall from [9, Ch. 3] the point trace space $H^{\frac{1}{2}}(\partial\Omega)$ for $H^1(\Omega)$, from [1–3] the tangential trace space $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \partial\Omega)$ for $\boldsymbol{H}(\mathbf{curl}\,, \Omega)$, and from [7, Ch. I,§2] the normal trace space $H^{-\frac{1}{2}}(\partial\Omega)$ of $\boldsymbol{H}(\mathrm{div}, \Omega)$, $\Omega \subset \mathbb{R}^3$ a generic Lipschitz domain in each case. Next, we introduce the interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$ and their union, the *skeleton* Σ. Every Γ_{ij} is equipped with an *intrinsic orientation*, which can be described by an oriented unit normal vector field $\mathbf{n}_{ij}$.

The skeleton trace spaces $H^{\frac{1}{2}}(\Sigma)$, $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Sigma)$, and $H^{-\frac{1}{2}}(\Sigma)$ are closed subspaces of $\bigotimes_{i=0}^{P} H^{\frac{1}{2}}(\partial\Omega_i)$, $\bigotimes_{i=0}^{P} H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \partial\Omega_i)$, and $\bigotimes_{i=0}^{P} H^{-\frac{1}{2}}(\partial\Omega_i)$, respectively. These *Dirichlet trace spaces* can be endowed with the equivalent norms

$$\begin{aligned}
\|u\|_{H^{\frac{1}{2}}(\Sigma)} &:= \inf\{\|U\|_{H^1(\mathbb{R}^3)} : U_{|\Sigma} = u\}\,, \\
\|\mathbf{u}\|_{H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Sigma)} &:= \inf\{\|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}\,,\mathbb{R}^3)} : \mathbf{U_t}_{|\Sigma} = \mathbf{u}\}\,, \\
\|\varphi\|_{H^{-\frac{1}{2}}(\Sigma)} &:= \inf\{\|\boldsymbol{\Phi}\|_{\boldsymbol{H}(\mathrm{div},\mathbb{R}^3)} : \boldsymbol{\Phi}\mathbf{n}_{|\Sigma} = \varphi\}\,.
\end{aligned}$$



Denote by Υ the "wire-basket" of Σ, that is the union of points in the boundary of more than two subdomains. It is known [6, Sect. 2] that C^∞-functions supported away from Υ are dense in the function spaces $H^1(\mathbb{R}^3)$, $\boldsymbol{H}(\mathbf{curl}\,,\mathbb{R}^3)$, and $\boldsymbol{H}(\mathrm{div},\mathbb{R}^3)$. Thus, L^2-functions compactly supported inside Γ_{ij} will be dense in the Dirichlet trace spaces (inherent localization).

The concept of Neumann data arises from integration by parts formulas A look at them reveals that an *induced orientation* of the boundary is required. This can be concluded from the presence of the exterior unit normal $\mathbf{n}$. Thus, to obtain *Neumann trace spaces* the local orientations have to be imposed. We rely on orientation conversion functions $\mu_i : \partial\Omega_i \mapsto \{\pm 1\}$, $\mu_i(\boldsymbol{x}) := \mathbf{n}_{ij} \cdot \mathbf{n}_i$ for $\boldsymbol{x} \in \Gamma_{ij}$. Thus, in the scalar case, the Neumann trace space is

$$Y_N := \left(\mu_i H^{-\frac{1}{2}}(\Sigma)_{|\partial\Omega_i}\right)_{i=0}^{P} \subset \bigotimes H^{-\frac{1}{2}}(\partial\Omega_i)\ . \tag{1}$$

In the case of the electric wave equation, the relevant Neumann trace space reads

$$\boldsymbol{Y}_N := \left(\mu_i \bigotimes\nolimits_j H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Sigma)_{|\Gamma_{ij}} \times \mathbf{n}_{ij}\right)_{i=0}^{P} \subset \bigotimes H^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\partial\Omega_i)\ . \tag{2}$$

We retain the notation μ_i for the mappings $H^{-\frac{1}{2}}(\Sigma) \mapsto H^{-\frac{1}{2}}(\partial\Omega_i)$ and $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Sigma) \mapsto H^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\partial\Omega_i)$ enclosed in brackets in (1) and (2).

Lemma. *The respective Dirichlet and Neumann trace spaces are dual to each other (w.r.t. to pivot space $\bigotimes L^2(\partial\Omega_i)$)*

Direct boundary integral equations. We adopt an abstract treatment encompassing both the scalar and vectorial case. Writing γ_D for the Dirichlet trace operator (relies on intrinsic orientation) and $\gamma_{N,i}$ for the Neumann trace operator on Ω_i (relies on induced orientation), every solution U of the dielectric scattering problem satisfies the transmission conditions ($U_i := U_{|\Omega_i}$)

$$\gamma_D U_i = \gamma_D U_j \quad , \quad \gamma_{N,i} U_i = -\gamma_{N,j} U_j \quad \text{on } \Gamma_{ij}\ . \tag{3}$$

The potentially non-zero jumps across $\partial\Omega_0$ are suppressed.

Next, write X_0 for the space of skeleton Dirichlet traces of smooth functions supported away from Υ. Thanks to inherent localization, the transmission conditions (3) can equivalently be stated as ($\rho_i \mathrel{\hat{=}}$ restriction to $\partial\Omega_i$)

$$\begin{array}{rcrcl} \langle\gamma_D U_i, \mu_i\varphi\rangle_{\Gamma_{ij}} & + & \langle\gamma_D U_j, \mu_j\varphi\rangle_{\Gamma_{ij}} & = & 0 \quad \forall\varphi \in X_0\ , \\ \langle\gamma_{N,i} U_i, \rho_i v\rangle_{\Gamma_{ij}} & + & \langle\gamma_{N,j} U_j, \rho_j v\rangle_{\Gamma_{ij}} & = & 0 \quad \forall v \in \widetilde{X}_0\ . \end{array} \tag{4}$$

Note that for the scalar case $X_0 \subset H^{-\frac{1}{2}}(\Sigma)$, $\widetilde{X}_0 \subset H^{\frac{1}{2}}(\Sigma)$, and that for the electric wave equation $X_0, \widetilde{X}_0 \subset H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Sigma)$.

Now we follow the pioneering approach of [11] to derive direct boundary integral equations. Similar equations have been proposed for electromagnetics [8]. We start from the local Calderón projectors, see [3, Sect. 5] and [10, Sect. 3.6] for the definitions of the boundary integral operators $\mathsf{V}_i, \mathsf{K}_i, \mathsf{W}_i$,

$$\begin{array}{rcrcl} \gamma_D U_i & = & \mathsf{V}_i(\mu_i\varphi) & - & (\mathsf{K}_i - \frac{1}{2}\mathsf{I})\rho_i v\ , \\ \gamma_{N,i} U_i & = & (\mathsf{K}_i' + \frac{1}{2})\mu_i\varphi & + & \mathsf{W}_i(\rho_i v)\ , \end{array} \quad \text{on } \partial\Omega_i\ , \tag{5}$$

where v is the skeleton Dirichlet trace of U, and φ the skeleton Dirichlet trace of the associated dual quantity. For the Helmholtz equation this is the flux $\mathbf{grad}\, U \in \boldsymbol{H}(\mathrm{div}, \mathbb{R}^3)$, for the electric wave equation the magnetic field $\mathbf{curlE} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$. Combining (5) and (4), and summing over all interfaces yields the final boundary integral equation

$$\begin{pmatrix} \sum_{i=1}^P \mu_i^* \mathsf{V}_i \mu_i & -\sum_{i=1}^P \mu_i^* \mathsf{K}_i \rho_i \\ \sum_{i=1}^P \rho_i^* \mathsf{K}_i \mu_i & \sum_{i=1}^P \rho_i^* \mathsf{W}_i \rho_i \end{pmatrix} \begin{pmatrix} \varphi \\ v \end{pmatrix} = \text{r.h.s.} \tag{6}$$

It has a unique solution provided that there is no non-trivial solution for excitation free scattering problem on the foliated union of the complements of the Ω_i, see [11] for details.

Hodge-type decomposition. In the scalar case coercivity of the variational problem associated with (6) is immediate from its block skew-symmetric structure. However, the electromagnetic boundary integral operators V_i and W_i fail to be coercive; they merely satisfy a generalized Gårding inequality with respect to a Hodge-type decomposition of the trace space $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Sigma)$, see [4] and [3, Sect. 6]. Its recursive construction hinges on the following assumption

Assumption. *The subdomains $\{\Omega_i\}_i$ can be colored black and white such that*

- *the interface separating the unions of subdomains of the same color is Lipschitz and orientable,*
- *the sets of black and white subdomains again satisfy this assumption.*

Lemma. *Under the above assumption, there is a bounded projection*

$$\mathsf{P} : H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Sigma) \mapsto H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Sigma)$$

such that

$$\mathrm{curl}_\Gamma\, \mathsf{P}\mathbf{u} = \mathrm{curl}_\Gamma\, \mathbf{u}\,, \quad \mathrm{Ker}(\mathsf{P}) = \mathrm{Ker}(\mathrm{curl}_\Gamma)\,, \ \textit{and} \quad \mathrm{Im}(\mathsf{P}) \subset L^2_{\mathbf{t}}(\Sigma) \ \textit{compact}\,.$$

This Lemma guarantees that the operator of (6) (in weak form) satisfies a generalized Gårding inequality also for the case of electromagnetic scattering.

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Direct and inverse problems in fluid-solid interaction

Johannes Elschner
(joint work with George C. Hsiao, Andreas Rathsfeld)

If an elastic body is surrounded by a fluid and if an acoustic plane wave is incident, then an elastic wave is incited inside of the body, and the acoustic wave in the fluid is scattered. This phenomenon is modelled by a transmission problem for the displacement amplitude and the acoustic pressure. The displacement amplitude satisfies the reduced elastodynamic equations inside the body, and the acoustic pressure is a solution of the Helmholtz equation in the domain exterior to it. On the boundary of the body the traction of the displacement amplitude points into the normal direction and is equal to the acoustic pressure from the outside. Moreover, the normal component of the displacement is proportional to the normal derivative of the pressure over the boundary of the body. Finally, the scattered field satisfies the Sommerfeld radiation condition at infinity.

Similarly to inverse problems for the scattering of acoustic waves, inverse problems for the fluid-solid interaction can be formulated. Suppose the shape of an elastic obstacle is unknown, but the far field patterns of scattered waves resulting from certain incident plane waves are known. The inverse problem is to recover the shape of the elastic scatterer from the measured far field patterns. For the uniqueness in the case of known far fields for all incident directions we refer to [6]. Treating the numerical solution, the number of given far field patterns is finite, and all the methods developed for the reconstruction in the case of inverse acoustic obstacle scattering (cf. e.g. [1]) should have counterparts for the case of elastic scatterers.

We consider the variational formulation of the transmission problem for the reduced elastodynamic and the Helmholtz equations. The unbounded exterior domain is truncated by a boundary integral equation method. Following [3], we prove that the sesqui-linear variational form satisfies a Gårding inequality. If a technical condition for the boundary integral operator is satisfied, then the variational equation has a unique acoustic field solution. This is true even in the exceptional case where the elastic wave is not unique. Note that the homogeneous transmission problem may have eigensolutions for special shapes of the elastic obstacle and for special values of the frequency (cf. e.g. [4,7]). In this case, the variational equation cannot be solved directly. Instead, the variational equation should be solved for a

slightly modified frequency. We prove that this commonly known approximation (cf. [5]) is correct, i.e., that the acoustic field solutions for the modified frequencies converge to the true solution if the perturbed frequencies tend to the correct value.

To solve the inverse problem, we restrict the class of obstacles to starlike domains with boundary parametrizations from a Sobolev space. Our inverse problem is ill-posed, and its solution requires a regularization. We propose a reformulation of the inverse problem as an optimization problem, where the cost functional is the least squares deviation of the measured far field patterns from those corresponding to the obstacle which is to be optimized. Of course, we add the scaled square norm of the boundary parametrizations for regularization.

We prove that the scattered acoustic field depends continuously on the shape of the obstacles even if the incited elastic wave is not unique. Consequently, the cost functional of the reformulated inverse problem is continuous, and, for any regularization parameter, there exists a regularized solution, i.e. a minimizer of the regularized cost functional. In case the involved far field patterns determine the obstacle uniquely, the regularized solutions converge to this unique obstacle whenever the regularization parameter tends to zero. For this convergence, we can even admit measurement errors in the size of the regularization parameter.

For the numerical solution of the optimization problem, we derive a formula for the directional derivative of the scattered acoustic field with respect to the parametrization of the obstacle boundary. The gradient computation is based on the solution of the variational equation of the transmission problem with modified right-hand sides. Thus this gradient formula is efficient if the discretized variational equation, i.e. the finite element system, is solved inverting its matrix by a direct solver which may be adapted to sparse systems (cf. [9]).

The parametrizations of the obstacle boundaries are discretized by finite sums of trigonometric functions resp. spherical harmonics. Having formulas for the gradients at our disposal, we suggest the Gauß-Newton method (cf. [8]) for the numerical computation of the minimizers. Numerical computations for a simple two-dimensional scatterer confirm our theoretical results, and we refer to [2] for the details.

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