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Invariants in Low-Dimensional Topology

Organised by
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May 4th – May 10th, 2008

Abstract. The complex set of topics in that workshop originates from the classical algebraic low–dimensional topology and the influx of new invariants that began with the Jones polynomial in the 1980's, that continued with the recent discovery of Khovanov and Ozsvath-Szabo homology.

Mathematics Subject Classification (2000): 57M25, 57M27.

Introduction by the Organisers

The workshop *Invariants in low-dimensional topology*, organised by Louis H. Kauffman (Chicago), Simon A. King (Jena), Vassily O. Manturov (Moscow) and Jozef Przytycki (Washington) was held May 4th–May 10th, 2008. This meeting was attended by 46 participants, including various researchers with recent PhD, one PhD student and one graduate student.

The main objective of this workshop was to look intensively at the present state of the art of invariants in low dimensional topolgy, particularly invariants of knots and three dimensional manifolds. This field is rapidly growing with a remarkable influx of new ideas and techniques. This activity was originally inspired in the 1980's by the discovery of the Jones polynomial. Vaughan Jones formulated his invariant in terms of braids and von Neumann algebras and also gave a formula for it via a skein relation (a relation among knot and link diagrams) that was a clear generalization of the Conway skein relation for the classical Alexander polynomial. This led at once to the discovery by many people of a generalization called the Homflypt polynomial, and then sometime later another skein theoretic generalization — the Kauffman polynomial — and to a general skein theory for knots in three-manifolds (skein modules and skein algebras) due to Jozef Przytycki. The Jones polynomial is related to statistical mechanics. A key notion from statistical mechanics that was imported here was the idea to write an invariant in analogy to a partition function. This means that the invariant is expressed as a sum over combinatorial states related either to a link diagram or a triangulation of a three-manifold. The first example of this is the state sum for the Jones polynomial called Kauffman bracket. The state sums generalize to sums involving solutions to the Yang-Baxter Equation (originally used in statistical mechanics) and this in turn led to constructions of invariants from quantum groups and Hopf algebras.

Then in the latter part of the 1980's ideas from quantum field theory (and the work of Witten) crystallized in the notion of Topological Quantum Field Theories (due to Witten and Atiyah) and corresponding invariants of knots and three-manifolds. At about the same time Vassiliev and Goussarov introduced new ideas about the space of knots and associated invariants that were quickly reformulated through the work of Birman, Lin, Kontsevich and Bar-Natan yielding deep relationships with Lie algebras and new ways to understand the previous constructions involving quantum groups. In 1996 Kauffman introduced a generalization of knot theory called virtual knot theory, which is given diagrammatically, and is equivalent to a stabilized theory of knots in thickened surfaces.

At the same time there was an evolution in the concept of the fundamental group of the knot complement with Joyce's introduction of the quandle in 1979 and Matveev's independent discovery of this structure (distributive groupoids). After 2000 Fenn, Kauffman, Manturov and others generalized the quandle to a stronger invariant called the biquandle which is very important for virtual knots.

In 1998 Khovanov introduced a categorification of the Jones polynomial. This means that he constructed a homology theory based on link diagrams whose graded Euler characteristic equals the original Jones polynomial. This led eventually to an explosion of significant results and new constructions. Khovanov and Rozansky found ways to categorify the Homflypt polynomial, and at this conference Rozansky announced the categorification of the Kauffman 2-variable polynomial. After Khovanov's original discovery, a categorification of the Alexander polynonmial (Knot Floer Homology) was found by Ozsvath and Szabo (2002) and eventually made purely combinatorial by Manolescu, Ozsvath and Sarkar. This categorification of the Alexander polynomial has led one of the most startling results in knot theory: The minimal genus of an orientable spanning surface for a knot, can be calculated from the Knot Floer homology, and this is a finite combinatorial matter in terms of a diagram for the knot.

This conference collected many active researchers in this field with talks on all of the topics mentioned above. We shall now mention the main themes of the workshop and the participants who were influential or gave talks on these themes:

- **TQFT and skein theory** Marta Asaeda, Christian Blanchet, Charles Frohman, Patrick Gilmer, Uwe Kaiser, Joanna Kania-Bartoszyńska, Thomas Kerler, Gregor Masbaum, Michael Müger, Jozef Przytycki and Adam Sikora.

- **Knots, braids, and geometric topology** Simon King, Elena Kudryavtseva, Sofia Lambropoulou, Sóstenes Lins, Sergei Matveev, Cameron McA. Gordon, Wolfgang Metzler, Hugh Morton, Carlo Petronio, Michael Polyak, Nikolai Saveliev, Radmila Sazdanovic, Pawel Traczyk, Vladimir Vershinin and Oleg Viro.
- **Categorified invariants and link homology** Marta Asaeda, Dror Bar-Natan, Anna Beliakova, Christian Blanchet, Mikhael Chmutov, Karl-Magnus Jacobsson, Vassily Manturov, Jozef Przytycki, Yongwu Rong, Lev Rozansky, Radmila Sazdanovic and Alexander Shumakovitch.
- **Quandles, Biquandles and Algebraic Categorification** Dror Bar-Natan, Scott Carter, Alissa Crans, Roger Fenn, Sergei Matveev, Maciej Niebrzydowski, Jozef Przytycki and Masahico Saito.
- **Virtual and Welded Knots** Dror Bar-Natan, Heather Dye, Roger Fenn, Louis Kauffman and Lev Rozansky.
- **State Sums** Sergei Chmutov, Heather Dye, Louis Kauffman, Simon King, Sostenes Lins, Sergei Matveev and Maciej Mroczkowski.
- **Hopf and Frobenius Algebras and three-manifold invariants** Uwe Kaiser, Louis Kauffman, Thomas Kerler, Michael Müger and David Radford.
- **Vassiliev Invariants** Dror Bar-Natan, Sergei Chmutov, Sergei Duzhin and Michael Polyak.

In our opinion, many of the results in the conference were new and significant. Having the conference at the Mathematisches Forschungsinstitute enabled the participants not only to listen to new results but also to the opportunity to converse and do mathematics together.

The conference was dedicated to the 60th birthday of Oleg Yannovich Viro.

Workshop: Invariants in Low-Dimensional Topology

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Abstracts

Projectivization, Welded Knots and Alekseev-Torossian

Dror Bar-Natan

Summary

My talk had two parts:

- In the first part I described the (tentative and speculative) "Projectivization Paradigm", which says, roughly speaking, that everything graded and interesting is the associated graded of something plain ("ungraded", "global") and even more interesting. The paradigm is absolutely general, encompassing practically every algebraic structure that might exist, and there is a diverse base of interesting examples and candidates for future examples.
- In the second part I described my latest example of an instance of the Projectivization Paradigm: I showed that the projectivization of "the circuit algebra of welded tangles" describes a good part (and maybe, in the future, all) of the recent work by Alekseev and Torossian on Drinfel'd associators and the Kashiwara-Vergne conjecture. This is cool: it leads to a nice conceptual construction of tree-level associators which might even be brought to a closed form, and it seems like a step towards a better understanding of quantum universal enveloping algebras and the work of Etingof and Kazhdan.

The work is very new. I'm quite confident of the overall picture but the details are subject to change.

To a very large extent my talk followed the two-page handout which is available at `http://www.math.toronto.edu/~drorbn/Talks/Oberwolfach-0805/`

1. The Projectivization Speculative Paradigm

I started by reminding the conference about the "Categorification Speculative Paradigm", which says, in very rough terms, that all of mathematics, or at least all of integer-coefficient mathematics, is the "Euler shadow" of vector-space, homological, mathematics. This, of course, is merely a speculative paradigm. One cannot expect it to be literally true, yet it is an excellent guiding principle for research. A lot of interesting mathematics arises as one tries to explore the extent to which this speculative paradigm holds true.

In a similar manner I proposed the "Projectivization Tentative[1] Speculative Paradigm", which says, in very rough terms, that all of graded mathematics is the projectivization of "plain", "ungraded" or "global" mathematics: all graded

[1] "Tentative" because I'm not even sure if the name "projectivization" (meant to be catchy and convey a "graded" feeling) is appropriate.

algebraic structures are the projectivizations of global ones, and all graded equations are the equations for "homomorphic expansions", or for "automorphisms" of homomorphic expansions.

I then proceeded to explain most of the terms appearing in the above paragraph. For a start, I gave a few examples of "graded equations" (these are the entities the projectivization paradigm is supposed to explain):

- The exponential equation $e(x + y) = e(x)e(y)$ [BN4].
- The pentagon and hexagon equations for Drinfel'd associators [Dr2, Dr3, BN1, BN2].
- The equations defining a quantized universal enveloping algebra in the sense of Drinfel'd [Dr1] and Etingof-Kazhdan [EK]. For the long term, these are the equations I care about the most, and my dream is to eventually incorporate them to within the projectivization paradigm.
- The equations appearing in the Alekseev-Torossian work [AT] on Drinfel'd associators and the Kashiwara-Vergne Conjecture [KV]. These equations are the main concern of the second part of this talk. One wonderful feature of these equations is that (in suitable quotients) they have explicit solutions, that will likely lead to explicit formulas for tree-level associators.

I then moved on to explain what is "the projectivization of an algebraic structure". For this purpose, an "algebraic structure" $\mathcal{O}$ is practically anything that is made of "spaces" and "operations". Allowing for formal linear combinations and extending all operations in a multi-linear manner, we can always define an "augmentation ideal" I along with its powers I^n, and then we can set

$$\operatorname{proj}\mathcal{O} := \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

One can see that $\operatorname{proj}\mathcal{O}$ is endowed with the same operations as $\mathcal{O}$, though they need not satisfy the same "axioms" that the operations of $\mathcal{O}$ may satisfy. We noted that if $\mathcal{O}$ is an appropriate space of knotted objects, then $\operatorname{proj}\mathcal{O}$ is the corresponding space of "chord diagrams".

Some warm up examples followed. We noted that the projectivization of a group is a graded associative algebra, and that the projectivization of a quandle is a graded Lie algebra.

I then moved on to discuss the central notion in the statement of the projectivization paradigm — the notion of an "expansion" [Li], and more importantly, of a "homomorphic expansion" — a "homomorphism" $Z\colon \mathcal{O} \to \operatorname{proj}\mathcal{O}$ which "covers" the identity map on $\operatorname{proj}\mathcal{O}$. When $\mathcal{O}$ is "finitely presented", finding an expansion involves finding values for $Z(g_i)$ (where the g_i's are the generators of $\mathcal{O}$), where these values must satisfy the equations corresponding to the "defining relations" of $\mathcal{O}$. Hence as promised[2] in the statement of the projectivization paradigm, finding a homomorphic expansion is a matter of solving equations in a graded space, $\operatorname{proj}\mathcal{O}$.

[2] The other source of graded equations, "automorphisms of homomorphic expansions", was not discussed in my talk.

A pretty example involves "knotted trivalent graphs" [BN3]. Here the relevant algebraic structure $\mathcal{O} = \mathrm{KTG}$ has a "space" for each trivalent graph — the space of "knottings" of that graph, and the operations are "delete", "unzip" and "connected sum". With these operations KTG is finitely generated, with the most interesting generator being the unknotted tetrahedron T. The interesting relations that T satisfies turn out to be (after appropriate language changes) the pentagon and the hexagons, and therefore it turns out that the equations for a "homomorphic expansion" for KTG are equivalent[3] to the equations for an associator.

I then explained how homomorphic expansions may be used — they convert certain kinds of "global" problems into problems that can be addressed "degree by degree". In the case of knotted trivalent graphs we arrive at what one may call "Algebraic Knot Theory" [BN3]. Certain knot theoretic properties, such as the knot genus and the property of being a ribbon, are "definable" using "delete", "unzip" and "connected sum", and hence they are in principle susceptible to study using homomorphic expansions.

2. Welded Knots and Alekseev-Torossian

Due to time constraints, the second half of my talk had to be sketchy. Following a talk Lou Kauffman gave in 2001, I recalled virtual knots [Ka], welded knots [FRR], and the relationship between welded knots and tori in $\mathbb{R}^4$ [Sa].

Welded knots form a "circuit algebra", and as a circuit algebra, their projectivization turns out to contain all the spaces (most notably tder_n, sder_n and tr_n) considered by Alekseev and Torossian [AT]. As a circuit algebra, the related space of "welded trivalent graphs" is generated by the "Y-vertex" and by crossings. Calling the images of these generators via a homomorphic expansion F and R, we find that F and R need to satisfy some equations — precisely the equations studied by Alekseev and Torossian. Finally, as welded trivalent graphs contain a quotient of knotted trivalent graphs, the Alekseev-Torossian theory contains a quotient of the Drinfel'd theory, which turns out to be the theory of tree-level associators.

3. Propaganda

"God created the knots,
all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

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A simplification of combinatorial link Floer homology

Anna Beliakova

Heegaard Floer homology provides a new powerful invariant of knots discovered independently by Ozsváth–Szabo and Rasmussen. This invariant, called knot Floer homology, detects the knot genus and its fiberedness. Knot Floer homology assigns to a knot a chain complex, whose graded Euler characteristic is the Alexander polynomial and whose homology is a new knot invariant. This procedure is known as categorification.

The quantum $sl(2)$ link invariant – the Jones polynomial – was categorified in 1998 by Khovanov [6]. In [7], Khovanov and Rozansky categorified the quantum $sl(N)$ link invariant. The last two constructions are purely algebraic.

In 2004, Rasmussen [10] gave a combinatorial proof of the Milnor conjecture, which was previously accessible only via gauge theory. In his proof, Rasmussen uses a new invariant, now known as Rasmussen invariant, extracted from the Khovanov homology.

In 2006 Manolescu, Ozsváth, Szabó and D. Thurston (MOST) provided a combinatorial construction of link invariants arising from Heegaard Floer homology [9]. The MOST construction uses rectangular link diagrams, where counting of holomorphic discs reduces to applying the Riemann mapping theorem. Unfortunately, the MOST complex is quite big. Already in the simplest case of the trefoil, it has 120 generators, while the knot Floer homology has rank 3. Existing programs realizing MOST algorithm allows to compute knot Floer homology over $\mathbb{Z}/2\mathbb{Z}$ for knots up to 12 crossings.

In Heegaard Floer theory, the differential is given by counting holomorphic discs (or domains), alternatively bounded by two Lagrangian submanifolds. The main difficulty is to decide which domains count and which do not. In the MOST complex, all domains are of the same simple form (rectangles), and therefore all of them count. The price for that is that the complex is very big.

In my talk, I explained how the MOST complex can be simplified [1]. For this, by using a well known lemma from homological algebra I construct a recursive algorithm, which decides the countability for any domain. In my setting, a choice of the complex structure is replaced by an additional combinatorial choice (the order in which ovals are shortened). As a byproduct, I defined a big class of domains which always count (independently of the order).

My chain complex is homotopy equivalent to the MOST complex, but it turns out to be simpler. For all knots with less than 6 crossings, my complex has the same rank as its homology. My student Jean–Marie Droz extended my construction over $\mathbb{Z}$ [5] and wrote a program computing the homology of this complex [4]. His program allows to compute knot Floer homology over $\mathbb{Z}$ for knots with 16 crossings and detect fiberedness and Seifert genus for knots up to 18 crossings. Moreover, Droz checked that for knots with up to 12 crossings, knot Floer homology is torsion free.

There exist homological constructions for the Jones and HOMFLY polynomials, which count graded intersection points between two submanifolds of a configuration space of points in a punctured disc (compare [2], [3]). A categorification of these constructions will clarify the relationship between Seidel–Smith [11] and Khovanov homologies, as well as between Manolescu [8] and Khovanov–Rozansky ones. My next goal is to provide combinatorial approaches to Seidel–Smith and Manolescu Lagrangian Floer homologies. By applying methods developed in [1], we were able to construct a differential between Bigelow generators. The complex with this differential is conjectured to be homotopy equivalent to the Seidel–Smith and Khovanov complexes.

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Link homology and trivalent TQFT

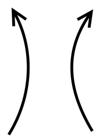
Christian Blanchet

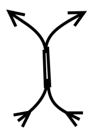
1. Introduction

Our purpose is to consider $sl(n)$ link homology in relation with trivalent TQFT. Here a trivalent TQFT is a TQFT functor on the cobordism category whose objects are colored trivalent graphs, and whose morphisms are represented by colored trivalent surfaces or more generally by so called foams. A trivalent surface has a 1-dimensional singular locus (a binding) which is modeled on a colored Y times an interval; a foam may contain singular points whose link is a colored tetrahedron. We first revisit $sl(2)$ link homology and present a slight variant which is defined over the integers and is strictly functorial.

2. $sl(2)$ functorial link homology revisited

A state s of a link diagram D associates to a positive (resp. negative) crossing either 0 or 1 (resp. -1 or 0). D_s is a planar trivalent graph defined by the rule below. Here in a trivalent graph edges are labeled by either 1 or 2; a 2-labeled edge is depicted by a thick edge.

if $s(c) = 0$, then c is replaced by 

if $|s(c)| = 1$, then c is replaced by 

Let $d_s = \sum s(c)$, and let Δ_s be the free abelian group generated by crossings c with $|s(c)| = 1$. Suppose that we have a functor:

$$\text{trivalent graph } G \mapsto \text{ module } V(G)\ .$$

$$\text{cobordism } \Sigma \mapsto \text{ linear map } V(\Sigma)\ .$$

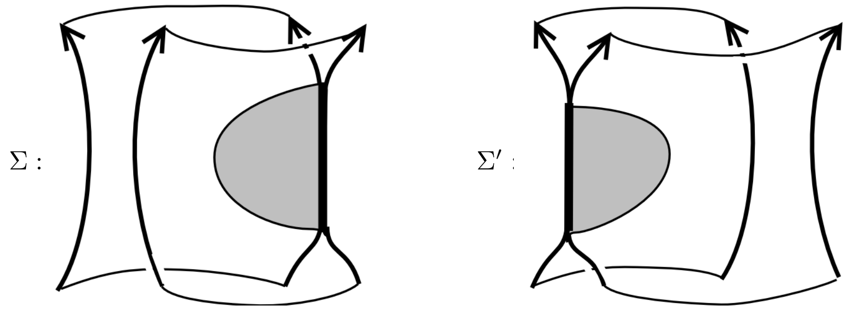


FIGURE 1.

(Here Σ is a *trivalent* surface or more generally a foam). Then we define a complex whose underlying abelian group is

$$K(D) = \bigoplus_s V(D_s) \otimes \wedge^{d_s} \Delta_s \tag{1}$$

The cohomological degree is d_s; if the functor V is graded then the complex is bigraded. The boundary operator δ between summands indexed by states s and s' is zero unless s and s' are different only in one crossing c where $s'(c) = s(c) + 1$. It is then defined using the TQFT map associated with the cobordisms Σ, Σ' which are identity outside a neighbourhood of the crossing, and are depicted in figure 1 around the crossing c. For a positive crossing c:

$$\delta = V(\Sigma) \otimes (\bullet \wedge c) : V(D_s) \otimes \wedge^{d_s} \Delta_s \to V(D_{s'}) \otimes \wedge^{d_{s'}} \Delta_{s'}$$

For a negative crossing c,

$$\delta = V(\Sigma') \otimes < \bullet, c >: V(D_{s'}) \otimes \wedge^{d_{s'}} \Delta_{s'} \to V(D_s) \otimes \wedge^{d_s} \Delta_s$$

A TQFT functor for trivalent surfaces can be obtained from the Frobenius algebras $A = H^*(\mathbb{C}P^1) \approx \mathbb{Z}[X]/X^2$, and $B = \mathbb{Z}$ with non standard trace $\epsilon_B(1) = -1$. We obtain this way a categorification of the Jones polynomial. Following Lee-Rasmussen construction [Lee, Ras] we can start with the deformed Frobenius algebra $A' = \mathbb{Z}[X]/X^2 - 1$ and obtain a filtrated complex and a corresponding spectral sequence which can be used to prove strict functoriality of the categorification.

3. $sl(n)$ LINK HOMOLOGY VIA TQFT AND FOAMS

The categorification of the $sl(n)$ specialisation of Homflypt polynomial was first obtained by Khovanov-Rozansky [KR] using matrix factorisations. We obtain a categorification over integers using the trivalent cobordism category. A similar approach using Kapustin-Li formula was carried over by Mackaay-Stosic-Vaz [MSV]. Our TQFT construction uses cohomology of partial flag manifolds as Frobenius algebras, may be with some non standard signs in the definition of traces. Cohomology of grassmanian are associated resto colored faces; cohomology of flags of length 2 are associated to bindings, and cohomology of flags of length 3 are associated to vertices. In this situation we have a whole bunch of structural maps which can be used for evaluating closed foams. The TQFT functor is then obtained by a universal construction [BHMV]. It is shown to satisfy the required properties needed for Reidemeister moves invariance.

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Diagrammatic Cohomology and Knot Invariants

J. Scott Carter

A *quandle* is a set with a self-distributive binary operation for which all elements are idempotent and for which right multiplication is a bijection. In [12] a cohomology theory for (the slightly weaker structure of) a rack [11] was defined. This was modified in [9] to take into consideration the idempotence of the elements, and to define invariants of classical knots and knotted surfaces. These invariants have a number of applications such as:

(1) proving that many 2-twist spun knots are non-invertible [2, 9];
(2) providing an orientation class for classical knots [10];
(3) giving lower bounds for the number of triple points of projections of knotted surfaces [15, 13];
(4) measuring colored chirality of classical knots [7];
(5) determining the minimal number of Reidemeister type III moves in isotopies that connect two diagrams [8];
(6) determining the minimal number of sheets needed to represent spun knots [14];
(7) determining if certain tangles embed in the unknot [1].

In a series of papers [3], [4],[5], and [6], my collaborators and I have developed a cohomology theory in which quandle cocycles and other cocycles co-exist and are non-trivial. The cohomology theory is defined diagrammatically by a process of "infiltrating algebraic identities." This talk develops the idea in the case of a Hopf algebra. For the rest of this abstract, the Hopf algebra case will be discussed in more detail.

A *Hopf Algebra* is a vector space, H, with an associative unital multiplication, μ, a coassociative counital comultiplication, Δ, a compatibility between these two operations, and an antipode map, $S : H \to H$, whose properties will not be listed here. The *adjoint map* $A : H \otimes H \to H$ in a Hopf algebra is defined by the equation $A(x \otimes y) = S(y_{(1)})xy_{(2)}$ where the notation suppresses a sum and the Sweedler notation is used for the coproduct of two elements. The adjoint satisfies two important properties:

$$A(A \otimes |) = A(| \otimes \mu)$$

and

$$(A \otimes \mu)(| \otimes X \otimes |)(\Delta \otimes \Delta) = (| \otimes \mu)(X \otimes |)(| \otimes \Delta)(| \otimes A)(X \otimes |)(| \otimes \Delta)$$

where X denotes the transposition of adjacent tensor factors, and $|$ denotes the identity map on the appropriate tensor factor.

In this talk, I indicate a diagrammatic proof of the second identity qand demonstrate (via diagrams) Woronowicz's proof [16] that these two properties give that $R = (|\otimes A)(X \otimes |)(| \otimes \Delta)$ satisfies the Yang-Baxter equation: $(R \otimes |)(| \otimes R)(R \otimes |) = (| \otimes R)(R \otimes |)(| \otimes R)$. The differentials in the "adjoint cohomology" theory for Hopf algebras are described in dimension 1, 2, and 3. The talk sketches the proofs that the composition of a pair of successive differentials (d^2) is 0. by using diagrammatic techniques.

Let H denote a Hopf algebra. Lower dimensional chain groups are given as:

$$C^2_A(H; H) = \mathrm{Hom}(H^{\otimes 2}, H), \quad C^3_A(H; H) = \mathrm{Hom}(H^{\otimes 3}, H) \oplus \mathrm{Hom}(H^{\otimes 2}, H^{\otimes 2}),$$

and for $n = 1$,

$$C^1_A(H; H) = \{f \in \mathrm{Hom}_k(H, H) \mid f\mu = \mu(f \otimes 1) + \mu(1 \otimes f), \\ \Delta f = (f \otimes 1)\Delta + (1 \otimes f)\Delta \ \}.$$

Differentials are defined as

$$D_1 = d^{1,1} : C^1_A(H; H) \to C^2_A(H; H), \quad D_2 = d^{2,1} + d^{2,2} : C^2_A(H; H) \to C^3_A(H; H),$$

$$D_3 = d^{3,1} + d^{3,2} + d^{3,3} : C^3_A(H; H) \to C^3_A(H; H),$$

where the maps $d^{i,j}$ are given as follows. For $f \in C^1_A(H; H)$, define $d^{1,1}(f) = A(| \otimes f) - fA + A(f \otimes |)$. For $\phi \in C^2_A(H; H)$, the second differentials are given by

$$\begin{aligned} d^{2,1}_A(\phi) &= A(\phi \otimes |) + \phi(A \otimes |) - \phi(| \otimes \mu), \\ d^{2,2}_A(\phi) &= (\phi \otimes \mu)(| \otimes X \otimes |)(\Delta \otimes \Delta) \\ &\quad - (| \otimes \mu)(X \otimes |)(| \otimes \Delta)(| \otimes \phi)(X \otimes |)(| \otimes \Delta). \end{aligned}$$

Let $\xi_i \in C^{3,i}(H;H)$ for $i = 1, 2$. Then

$$\begin{aligned}
d_A^{3,1}(\xi_1,\xi_2) &= A(\xi_1 \otimes |) + \xi_1(| \otimes \mu \otimes |) - \xi_1(A \otimes |^{\otimes 2} + |^{\otimes 2} \otimes \mu), \\
d_A^{3,2}(\xi_1,\xi_2) &= (A \otimes \mu)(| \otimes X \otimes |)(|^{\otimes 2} \otimes \Delta)(\xi_2 \otimes |) \\
&\quad + (| \otimes \mu)(X \otimes |)(| \otimes \xi_2)(R_A \otimes |) \\
&\quad + (1 \otimes \mu)(|^{\otimes 2} \otimes \mu)(X \otimes 1^2)(1 \otimes | \otimes 1)(|^{\otimes 2} \otimes \Delta)((|^{\otimes 2} \otimes \xi_1) \\
&\qquad \cdot (| \otimes X \otimes |^{\otimes 2})(X \otimes |^{\otimes 3})(|^{\otimes 3} \otimes X \otimes |)(| \otimes \Delta \otimes \Delta)) \\
&\quad - (\xi_1 \otimes \mu)(|^{\otimes 2} \otimes X \otimes |)(|^{\otimes 2} \otimes \mu \otimes |^{\otimes 2}) \\
&\qquad \cdot (1 \otimes X \otimes 1^{\otimes 3})(\Delta \otimes \Delta \otimes \Delta) \\
&\quad - \xi_2(| \otimes \mu), \\
d_A^{3,3}(\xi_1,\xi_2) &= (| \otimes \mu \otimes |)(X \otimes |^{\otimes 2})(| \otimes \Delta \otimes |)(| \otimes \xi_2)(X \otimes |)(| \otimes \Delta) \\
&\quad + (\xi_2 \otimes \mu)(| \otimes X \otimes 1)(\Delta \otimes \Delta) - (| \otimes \Delta)\xi_2.
\end{aligned}$$

In the case of the group algebra of a finite group, where $\Delta(g) = g \otimes g$, and $S(g) = g^{-1}$, the adjoint map is given by conjugation. In this case, the differentials give cocycle conditions that are essentially those for a groupoid. A discussion of the distinct cohomologies of the group algebra and the function algebra for a finite group are given. The talk ends with a discussion for further directions of this research.

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Polyak–Viro formulas for coefficients of the Conway polynomial

Sergei Chmutov
(joint work with Michael (Cap) Khoury, Alfred Rossi)

This work has been done by students Michael (Cap) Khoury and Alfred Rossi during the Summer 2006 VIGRE working group "Knots and Graphs": (`http://www.math.ohio-state.edu/~chmutov/wor-gr-su06/wor-gr.htm`) at the Ohio State University, funded by NSF grant DMS-0135308. The problem was to describe Polyak-Viro arrow diagram formulas for coefficients of the Conway polynomial. Later we realized that this description is equivalent to F. Jaeger's state model [Ja].

1. Jaeger's state model for the Conway polynomial

Let us first reformulate the Jaeger model on a language suitable for our purpose.

A subset S of the crossings of a knot diagram K is said to be *one-component* if the curve obtained from K by smoothing all crossings of S according to orientation has one component.

Assume that the diagram K has a base point and S is a one-component subset of crossing. Let us travel along K starting with the base point. Suppose we approach to the first crossing of the subset S along an overpass. Let us jump down to the underpass and continue to travel along K (more precisely along the oriented smoothing of K at the crossing). Repeat the procedure for the remaining crossings of S. If it is possible to trace the whole curve K like this, always jumping down at the first approaching to a crossing of S and jumping up at the second approaching to it, then the subset S is said to be *jump down.*

Define the *down* polynomial, in variable t, as

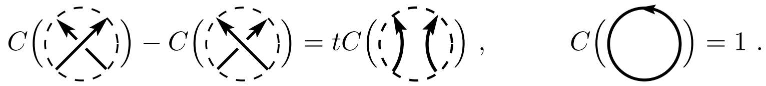
$$C_{\text{down}}(K) \quad := \sum_{\substack{S \ \text{jump down} \\ \text{one-component}}} \Big(\prod_{\mathsf{x} \in S} \text{wr}(\mathsf{x}) \Big) \, t^{|S|} \ ,$$

where $\text{wr}(\mathsf{x})$ is the local writhe of the crossing x. If S is the empty set, then we set the product to be equal 1 by definition. Therefore the free term of $C_{\text{down}}(K)$ always equals 1.

Corollary of the main Theorem. *The Conway polynomial $C(K)$ a knot K is equal to the down polynomial of its diagram,*

$$C(K) = C_{down}(K) \ .$$

Let us remind that the Conway polynomial is defined by the equations



Similarly to $C_{\mathrm{down}}(K)$, one can define the *up* polynomial $C_{\mathrm{up}}(K)$. It turns out that for all classical knots $C_{\mathrm{down}}(K) = C_{\mathrm{up}}(K)$. However this fails for virtual knots.

2. Gauss diagrams and Polyak-Viro formulas

Definition 1. A *Gauss diagram* is a chord diagram with oriented chords and with numbers $+1$ or -1 assigned to each chord.

With a knot diagram we associate a Gauss diagram whose outer circle is the parameterizing circle S^1 of our knot, a chord is drawn for each double point of the diagram, each chord is oriented from the overpass to the underpass and the local writhe number is assigned to each double point (chord). M. Polyak and O. Viro suggested [PV] the following approach to represent knot invariants in terms of Gauss diagrams.

Definition 2. An *arrow diagram* is a based chord diagram with oriented chords.

Definition 3. Let A be an arrow diagram and let G be a Gauss diagram, both with base points. A homomorphism φ from A to G, $\varphi \in \mathrm{Hom}(A, G)$, is an injective map of chords of A to chords of G which respects the orientation of the chords and their positions to the base points.

Definition 4. The pairing between a based arrow diagram and a based Gauss diagram is defined by

$$\langle A, G\rangle := \sum_{\varphi \in \mathrm{Hom}(A,G)} \quad \prod_{c \text{ chord in } A} \mathrm{sign}(\varphi(c)) \ .$$

In general, if you take an arbitrary arrow diagram A, the value $\langle A, G(K)\rangle$ is not uniquely defined by the knot K. Nevertheless, for we can extend the pairing to a linear combination of arrow diagrams

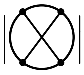
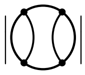
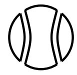
$$\langle \sum_i \lambda_i A_i, G\rangle := \sum_i \lambda_i \langle A_i, G\rangle$$

by linearity. Then some linear combinations of arrow diagrams may yield knot invariants by this construction. Moreover, with a slight generalization of arrow diagrams pairing, there is a general theorem due to M. Goussarov [G, GPV] stating that any Vassiliev invariant can be obtained from a suitable linear combination of arrow diagrams (possibly with signed chords).

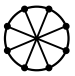
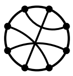
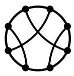
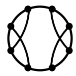
3. Main Theorem

Definition 5. A chord diagram D is said to be *one-component* if after parallel doubling of each chord the resulting curve will have one component, $|D| = 1$.

Example 6. There is only one one-component chord diagram with two chords:

 $= 1 \Longleftarrow$  ,  $= 3 \Longleftarrow$  .

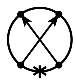
With four chords, there are four one-component diagrams:

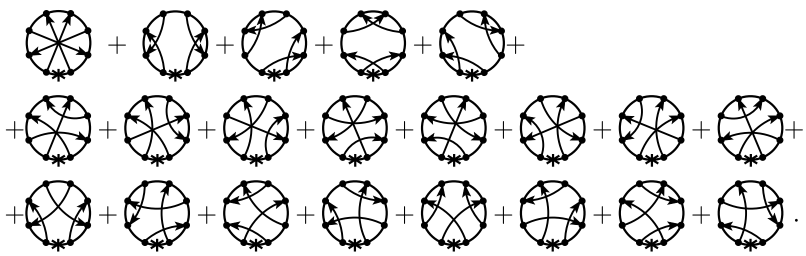
$d_1^4 =$  , $d_5^4 =$  , $d_6^4 =$  , and $d_7^4 =$  .

Definition 7. Choosing a base point we can turn a one-component chord diagram into an arrow diagram according to the following rule. *Starting from the base point we travel along the diagram with doubled chords. In this journey we pass both copies of each chord in opposite directions. Choose an arrow on a chord which correspond to the direction of the first passage of the copies of the chord.* Here is an example.

 .

Definition 8. Let us define the *Conway combination* $\mathfrak{C}_{2n}$ of arrow diagrams as a sum of all based arrow diagrams with $2n$ arrows obtained from one-component chord diagrams by the rule above. For example,

$\mathfrak{C}_2 :=$  ,

$\mathfrak{C}_4 :=$  .

Note that for a given one-component chord diagram we have to consider all possible choices for the base point. However, some choices may lead to the same arrow diagram. In $\mathfrak{C}_{2n}$ we list them without repetitions.

Main Theorem. *For $n \geqslant 1$, the coefficient c_{2n} of t^{2n} in the Conway polynomial of a knot K with the Gauss diangram G is equal to*

$$c_{2n} = \langle \mathfrak{C}_{2n}, G \rangle \ .$$

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Algebraic Categorification

Alissa S. Crans

In the past several decades, operations satisfying self-distributivity:

$$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$$

have secured an important role in knot theory. Such operations not only provide solutions of the Yang–Baxter equation and satisfy a law that is an algebraic distillation of the type (III) Reidemeister move, but they also capture one of the essential properties of group conjugation.

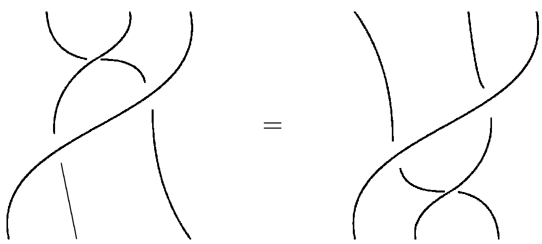
Let X be a set equipped with a binary operation $\triangleleft : X \times X \to X$. The map $\triangleleft$ satisfies the **self-distributive law** if

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z).$$

The primordial example of a self-distributive operation comes from group conjugation:

$$x \triangleleft y = y^{-1}xy.$$

The Yang–Baxter equation arises in many contexts in mathematics and physics. All these concepts are related by the fact that this equation is an algebraic distillation of the 'third Reidemeister move' in knot theory:



Originally, mathematical physicists concentrated on solutions to the Yang–Baxter equation in the category of vector spaces with the tensor product, obtaining solutions from quantum groups. More recently, there has been interest in set-theoretic solutions to the Yang–Baxter equation, and a set with a self-distributive operation provides such a solution.

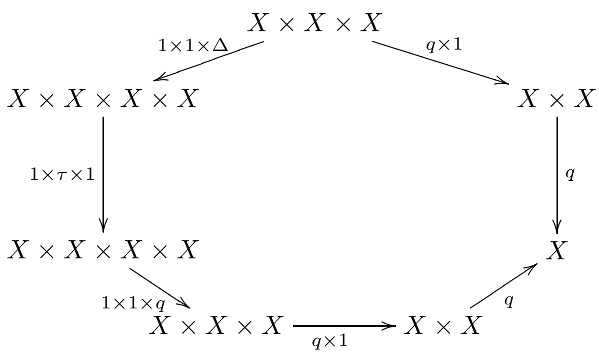
It turns out that there is a relationship between Lie algebras and solutions to the Yang–Baxter equation. It is, perhaps, not so surprising that Lie algebras are related to braidings since the bracket in a Lie algebra is all about switching the order of two Lie algebra elements. Moreover, the interesting feature of a Lie algebra, the Jacobi identity, involves three Lie algebra elements and the Yang–Baxter equation, or third Reidemeister move, involves three strands. It turns out that the the two are equivalent in a suitable context:

Proposition 1 ([2]). Let L be a vector space over k equipped with a skew-symmetric bilinear operation $[\cdot,\cdot] : L \times L \to L$. Let $L' = k \oplus L$ and define the isomorphism $B : L' \otimes L' \to L' \otimes L'$ by $B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y])$. Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

At this point, several questions arise: What is so special about the space L' that enabled us to define a solution to the Yang–Baxter equation on it? What is the relationship between self-distributive operations and solutions to the Yang–Baxter equation? Are we able to define a self-distributive operation on the space L'?

All of these questions can be answered using the category-theoretic language of ‘internalization. [3]’ All familiar mathematical concepts were defined in the category of sets, but most of these can live in other categories as well. This idea, known as internalization, is actually very familiar. For example, the notion of a group can be enhanced by looking at groups in categories other than Set, the category of sets and functions between them. We have the notions of topological groups, which are groups in the category of topological spaces, Lie groups, groups in the category of smooth manifolds, and so on. Internalizing a concept consists of first expressing it completely in terms of commutative diagrams and then interpreting those diagrams in some sufficiently nice ambient category, K.

Definition 2. Let X be an object in a category K with finite products. A map $q : X \times X \to X$ is a **self-distributive map in** K if the following diagram commutes:



where $\Delta : X \to X \times X$ is the diagonal morphism in K and $\tau : X \times X \to X \times X$ is the transposition. We also say that a map q satisfies the **self-distributive law**.

Returning to the Lie algebra example considered above, we will denote elements of $L' = k \oplus L$ as either (a, x) or $a + x$, depending on clarity, where $a \in k$ and $x \in L$. In fact, L' is a cocommutative coalgebra with counit. Recall that a **coalgebra** is a vector space C over a field k together with a **comultiplication** $\Delta : C \to C \otimes C$ that is linear and **coassociative**: $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. A coalgebra is **cocommutative** if the comultiplication satisfies $\tau\Delta = \Delta$, where $\tau : C \otimes C \to C \otimes C$ is the transposition $\tau(x \otimes y) = y \otimes x$. A **coalgebra with counit** is a coalgebra with a linear map called the **counit** $\epsilon : C \to k$ such that $(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta$ via $k \otimes C \cong C$.

The space L' is a cocommutative coalgebra with comultiplication and counit given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$ and $\Delta(1) = 1 \otimes 1$, $\epsilon(1) = 1$, $\epsilon(x) = 0$ for $x \in L$. We can extend these to elements of the form $(a, x) \in L'$. Since the solution to the classical YBE follows from the Jacobi identity, and the YBE is related to self-distributivity via the third Reidemeister move, it makes sense to expect that there is a relation between the Lie bracket and the self-distributivity axiom.

Proposition 3 ([1]). The map $q : L' \otimes L' \to L'$ defined by

$$q((a, x) \otimes (b, y)) = q((a + x) \otimes (b + y)) = ab + bx + [x, y] = (ab, bx + [x, y])$$

satisfies the self-distributive law in the category of cocommutative coalgebras.

This Lie algebra example suggests that self-distributive operations deserve study independent of their knot theory applications and in the context of other well-known algebraic structures. We therefore consider self-distributive maps in the category of coalgebras and their relationships to solutions of the Yang–Baxter equation.

Let X be a coalgebra and $q : X \otimes X \to X$ a linear map. The linear map $B_q : X \otimes X \to X \otimes X$ defined by

$$B_q = (1 \otimes q)(\tau \otimes 1)(1 \otimes \Delta)$$

is said to be **induced from** q. Conversely, let $B : X \otimes X \to X \otimes X$ be a linear map. The linear map $q_B : X \otimes X \to X$ defined by

$$q_B : (\epsilon \otimes 1)B$$

is said to be **induced from** B.

Our goal is to relate solutions of the YBE and self-distributive maps in the category of coalgebras via these induced maps, which we do in the following results:

Theorem 4 ([1]). Let X be a coalgebra with counit. Let $B : X \otimes X \to X \otimes X$ be a solution of the Yang–Baxter equation. Suppose B also satisfies

$$(\epsilon \otimes \epsilon)B = \epsilon \otimes \epsilon$$

$$B_{q_B} = B$$

Then q_B satisfies the self-distributive law and is a coalgebra morphism.

Theorem 5 ([1]). Let X be a cocommutative coalgebra equipped with a linear map $q : X \otimes X \to X$ that preserves comultiplication and satisfies the self-distributive law. Then B_q is a solution of the Yang–Baxter equation.

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Polynomial Invariants and Virtual Crossing Number

Heather A. Dye
(joint work with Louis H. Kauffman)

The virtual crossing number of a virtual knot or link, L, (denoted $v(L)$) is the minimum number of virtual crossings in any diagram equivalent to L. If $v(L) \geq 0$ then L is non-classical and not equivalent to the unknot. We introduce a polynomial invariant that determines a lower bound on the virtual crossing number of the virtual link and is computed from a skein relation.

Virtual knot theory is a generalization of classical knots theory introduced by Louis Kauffman in [10]. Virtual knot and link diagrams incorporate virtual crossings (indicated by a solid, circled crossing) in addition to classical crossings which are marked with under and over passing information. Two diagrams are equivalent if the are related by a sequence of classical Reidemeister moves and virtual Reidemeister moves. The virtual Reidemeister moves are illustrated in figure 1. Classical invariants such a the bracket polynomial can be extended to virtual knots.

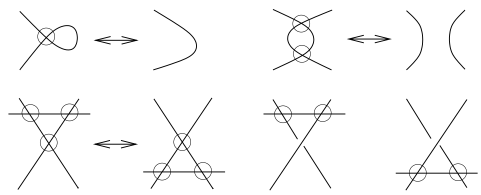

Figure 1. Virtual Reidemeister moves

However, the bracket polynomial is also invariant under virtualization; resulting in larges classes of virtual knots that have bracket polynomial equivalent to that of the unknot. Other classical invariants, such as the fundamental group and the

Witten-Reshetikhin-Turaev invariant [3] have also been extended to virtual knots and links. Kishino's knot [11] is undetected by the bracket polynomial and is typically used to test the efficacy of any invariant of virtual knots and links.

Virtual links can be viewed as stable equivalence classes of knots and links embedded in thickened two dimensional, oriented surfaces. Equivalence classes are determined by isotopy within the thickened surfaces, Dehn twists of the surface, and handle addition and cancelation. The virtual Reidemeister moves can be described in this context [1], [9]. Kuperberg [12] demonstrated that a virtual knot or link (viewed as a stable equivalence class of a knot or link embedded in a thickened surface) has a unique minimal genus surface. As a result, virtual knots are non-classical if the minimal genus surface has genus greater than zero. This was utilized in the the surface polynomial [2]. These results suggest several approaches to determining if a knot is non-classical or not equivalent to the unknot: polynomials obtained from skein relations, detecting the minimal genus, and determining the virtual crossing number. The problem of determining the virtual crossing number is introduced in [4]. For example, it is possible to have a virtual knot with minimal genus one but with an arbitrarily high virtual crossing number.

We present a simplified version of the extended bracket polynomial [8] that also generalizes the Miyazawa polynomial [5]. The polynomial is invariant under the Reidemeister II and III moves, as well as the virtual Reidemeister moves. The polynomial can also be normalized to obtain invariance under the Reidemeister I move. The skein relation, shown in figure 2, incoroporates nodal arrows that

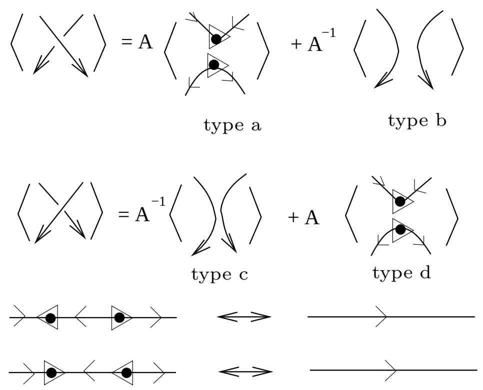

FIGURE 2. Skein relation and arrow reduction

associate a set of numbers, the arrow set of a link L, with the polynomial. States, determined by selecting a smoothing for each crossings, are collections of closed curves with nodal arrows. Reducing the states under the Reidemeister moves and the move shown in figure 2 results in a disjoint collection of closed curves marked with an even number of nodal arrows. The arrow number of a state is determined by the number of nodal arrows remaining in the state after reduction. The arrow number of a state is the total number of arrows in a state divided by two and the arrow set is the set of arrow numbers obtained from all possible states of the diagram. The maximum value of the arrow set is a lower bound on the virtual crossing number. This information is encapsulated in the polynomial; a closed loop with arrow number zero is evaluated as $-A^2 - A^{-2}$ and a loop with arrow number n ($n \geq 0$) is evaluated as A_n. The polynomial invariant is a polynomial in the variables $\mathbb{Z}[A, A^{-1}, A_1, A_2, \ldots]$. Individual arrow numbers are determined by summing the subscripts of a summand. With Naoko Kamada's argument [6],developed for the Miyazawa polynomial, and some combinatorial arguments, we show that the maximum value of the arrow set is a lower bound on the crossing number. The simiplicity of the argument suggests that this technique may be applied to other skein theoretic invariants. This invariant also suggests new approaches to the Khovanov homology of virtual knots and links [13].

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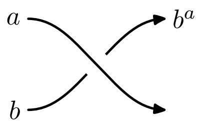
Biquandles

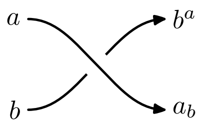
Roger Fenn

Many invariants of knots and links have been found by labelling the arcs of a diagram and introducing a relation at every crossing.



The binary operation $(a, b) \longrightarrow b^a$ means b is acted upon by a above and is chosen to respect the Reidemeister moves. This defines a **quandle** and allows us to calculate the determinant or the Alexander polynomial or the fundamental group or indeed the fundamental quandle [FR, J, M]. It turns out that this is almost a complete invariant for knots. The only ambiguity is orientation. Even this can be taken care of by considering a homology class in the classifying space of the quandle [FRS].

An obvious question is "what if we change the label of the overcrossing as well"?



We get a new binary operation $(a, b) \longrightarrow a_b$ meaning that a is acted upon by b below and now both operations have to respect the Reidemeister moves. This defines a **biquandle**.[FJK]

The two operations can be bundled together by $S(a, b) = (b^a, a_b)$ giving a map $S : X^2 \longrightarrow X^2$ where X is the labelling space. A consequence of the third Reidemeister move is the equation

$$(S \times 1)(1 \times S)(S \times 1) = (1 \times S)(S \times 1)(1 \times S)$$

a set version of the Yang Baxter equation.

Many examples of biquandles are known [FJK]. One of the most useful is the **linear** $S = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B satisfy

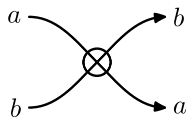
$$\mathcal{F} : A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A$$

and C, D are defined by $C = A^{-1}B^{-1}A(1 - A)$ and $D = 1 - A^{-1}B^{-1}AB$.

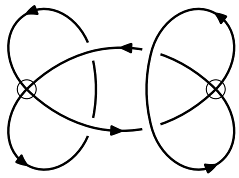
The commutative case $AB = BA$ defines the Alexander biquandle. A non-commutative example is the Budapest $A = 1 + i, B = j, C = -i, D = 1 + i$ where i, j is the usual notation for quaternions. All quaternion and 2×2 matrix in addition to families of $n \times n$ solutions to $\mathcal{F}$ have been found [F, FT].

Why do we do this? After all the fundamental quandle is a complete invariant of knots so what is the point of a fundamental biquandle?

The answer is to consider **virtual** knots where there have been many useful applications [BF, BuF, F, FT]. Diagrams of virtual knots have additional crossings of the form



The Budapest solution to $\mathcal{F}$ can be used to show that the Kishino knot illustrated below is not classical.



Two questions with the expected answer yes can be posed.

(1) Is there a non-trivial virtual knot which cannot be seen by the linear biquandle? One candidate for this could be defined as follows. Let κ be a non-trivial braid in the kernel of the Burau map. [B, SB]. Let τ be a virtual braid. Now consider the closure of $\kappa\tau\kappa^{-1}\tau^{-1}$.
(2) Is the fundamental biquandle a complete invariant of virtual knots (up to orientation)?

For the second question it may be necessary to make the operator T at a virtual crossing change by something stronger than $T(a,b) = (b,a)$. If we put $q = A^{-1}B^{-1}AB - B^{-1}AB$ then q commuting with A, B implies the following.

(1) A, B satisfy $\mathcal{F}$
(2) $S^2 = (1-q)S + q$
(3) If $u = B$ and $v = B^{-1}A^{-1}$ then $uv - qvu = 1$.

The last equation is sometimes called the quantum harmonic oscillator or quantum Weyl algebra.

In general $S^n = \alpha_n S + \beta_n$ where

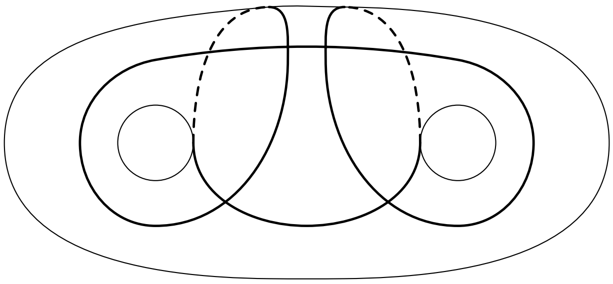
$$\alpha_n = 1 - q + q^2 - \cdots (-q)^{n-1}, \quad \beta_n = q(1 - q + q^2 - \cdots (-q)^{n-2}) = 1 - \alpha_n.$$

The soubriquet quantum, is dropped when $q = 1$. This case corresponds to **flat** knots when an overcrossing cannot be distinguished from an undercrossing and $S^2 = 1$ eg [FT].

For example if

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ v = \begin{pmatrix} y & 0 & 0 \\ 1 & y & 0 \\ 0 & 2 & y \end{pmatrix}$$

working mod 3 we find that the second ideal of the presentation matrix of the flat Kishino knot is generated by $\Delta_1 = 2 + 2y$ [As].



This shows that it is non-trivial.

In general $S^n = 1$ when q is a root of unity. This may be useful for finding invariants of n-moves when n twists are ignored.

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Khovanov homology vs. Representation Varieties

Magnus Jacobsson

We report on work in progress which explores a relation between Khovanov homology and $SU(2)$-representation varieties of knot groups. The work is joint with Ryszard Rubinsztein at Uppsala University [2].

Let L be a link and π the fundamental group of the complement of L in S^3. Let C be the conjugacy class of traceless matrices in the Lie group $SU(2)$.

We consider the space $J_C(L)$ of representations of π in $SU(2)$ which send each meridian of the link into C. Typically $J_C(L)$ has several connected components $J_C^r(L)$. Let

$$H^*(J_C(L);\mathbb{Z}) = \bigoplus_r H^*(J_C^r(L);\mathbb{Z})$$

be the singular integral cohomology of this space. On the other hand, let $Kh^{i,j}(L)$ denote the integral Khovanov homology of L. This is a bigraded collection of abelian groups. Let $Kh^k(L)$ denote the singly graded homology theory obtained by collapsing the bigrading along $k = i - j$.

We make the following observation.

Observation 1. For every knot L with seven crossings or less, and for every $(2,n)$-torus link, there are integers $N_r = N_r(L)$ such that

$$Kh^*(L) = \bigoplus_r H^*(J_C^r(L);\mathbb{Z})\{N_r\}.$$

In other words, the singly graded Khovanov homology consists of pieces, which are isomorphic to the cohomology of the components of the representation variety $J_C(L)$, but with each such component shifted by some integer.

Example 2. A $(2,n)$-torus link $T_{2,n}$ has two components if n is even and one if n is odd. For n even, the representation variety for $T(2,n)$ is

$$J_C(T_{2,n}) = S^2 \cup S^2 \cup \bigcup_{i=1}^{\frac{n-1}{2}} \mathbb{R}P^3.$$

For n odd, the formula is the same, except that only one of the copies of S^2 appears.

The Khovanov homology of $(2,n)$-torus links was computed in [3]. Collapsing the bigrading in Khovanov's formula gives

$$Kh^*(T_{2,n}) \cong H^*(S^2)\{n-2\} \oplus H^*(S^2)\{2n-2\} \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} H^*(\mathbb{R}P^3)\{2i-2+n\}$$

where the second term only appears when n is even, and we see that the observation holds in this case.

The representation variety $J_C(L)$ fibres over C. One may consider the cohomology of its fibre over a point in C. In all the examples of the Observation above this reproduces the singly graded *reduced* Khovanov homology [4].

Example 3. In the case of $(2,n)$-torus links, the fibre projection is the identity on the S^2-components and the natural projection on the $\mathbb{R}P^3$:s. Thus the fibre is

$$\{*\} \cup \{*\} \cup \bigcup_{i=0}^{\frac{n-1}{2}} S^1,$$

where the second term only appears if n is even. Reduced Khovanov homology is

$$H^*(*)\{n-1\} \oplus H^*(*)\{2n-1\} \bigoplus_{i=0}^{\frac{n-1}{2}} H^*(S^1)\{2i-1+n\}.$$

In this talk we describe an enhancement of the link invariant $J_C(L)$ in the form of a grading on its connected components. This grading has the form of a Maslov index. To define it we need to put $J_C(L)$ in a symplectic context. This is achieved as follows.

In [1], Guruprasad, Huebschmann, Jeffreys and Weinstein study the moduli space $\mathcal{M}$ of flat $SU(2)$-connections on a surface with n marked points. (The space $\mathcal{M}$ plays a central role in quantum Chern-Simons theory.) They express $\mathcal{M}$ by symplectic reduction of a certain symplectic manifold $\mathscr{M}$. We put $\mathscr{M}$ to use in the following way. We show that $\mathscr{M}$ admits a braid action preserving the symplectic structure. Furthermore, every braid defines a Lagrangian submanifold Γ_σ in $\mathscr{M}$. In particular the identity braid defines a Lagrangian submanifold Λ, and $\Gamma_\sigma = \sigma(\Lambda)$. Finally, Λ and Γ_σ intersect in a space homeomorphic to $J_C(L)$. When the Lagrangian intersection is clean we can use the two Lagrangian submanifolds to define a Maslov index associated to the components of $J_C(L)$. This requires care since the second homotopy group of $\mathscr{M}$ is non-trivial. At least for $(2,n)$-torus links the grading coincides with the shifts in the above examples. In future work we aim to extend our new invariant to a Lagrangian Floer homology theory, which most likely will be related to Khovanov homology by a spectral sequence.

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Skein modules and skein algebras of surfaces in 3-manifolds

Uwe Kaiser

Frobenius algebras, respectively 2-dimensional topological quantum field theories (see [1]), can be used to define skein modules of surfaces in 3-manifolds in a very obvious way. For details and proofs see [5]. This is motivated by recent work on Khovanov homology [3], [6], [9], [4], [7] and the work of Asaeda and Frohman [2].

Throughout R is a commutative ring with 1 and $\otimes = \otimes_R$. A Frobenius algebra over R is a 3-tuple $\mathcal{F} = (A, \Delta, \varepsilon)$, where A is a commutative R-algebra (multiplication μ) with a fixed inclusion $R \subset A$ (unit) and a cocommutative and coassociative coproduct $\Delta : A \to A \otimes A$ (an A-bialgebra map) with counit $\varepsilon : A \to R$ (an R-module map) satisfying the Frobenius identity: $\mu(\varepsilon \otimes Id)\Delta = Id$ (see [6] for further details and references).

Example 1. (i) The universal rank 2 Frobenius algebra is defined over $R = \mathbb{Z}[h,t]$ with $A = R[x]/(x^2 - hx - t)$ and $\Delta(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1$, $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$.
(ii) For G a finite group the group algebra RG has the natural structure of a Frobenius algebra with $\Delta(1) = \sum_{g \in G} g \otimes g^{-1}$, $\varepsilon(1) = 1$ and $\varepsilon(g) = 0$ for $g \neq 1$.

We prove in [5] that the natural functor from the cobordism category of surfaces and their boundaries into the category of R-modules extends naturally to a functor where the morphism sets of the cobordism category are extended to R-linear combinations of surfaces with components colored by elements of A. Moreover, the kernels are generated by three types of relations: (i) R *multi-lineariy* in the colors of the components of the surfaces, (ii) *sphere-relations*: a sphere colored by $a \in A$ can be replaced by multiplication by $\varepsilon(a)$, (iii) *neck-cutting*: If $\Delta(1) = \sum_{i=1}^{r} u_i \otimes v_i$ then a surface with a simple closed curve on it can be replaced by a sum of r surfaces which are all topologically given by cutting the neck (replace the annulus neighborhood by two disks) and with the colors u_i, v_i distributed to the *left* respectively *right* hand side of the neck. The original color of the components and the u_i, v_i will be multiplied in A to define morphisms of the linearized category.

The above *kernel relations* are *local* and extend eaasily to embedded surfaces in 3-manifolds: Just require in (ii) that 2-spheres bound 3-balls, and in (iii) that the simple closed curve is compressible (such that the neck cutting will produce an embedded surface). Then let (M, α) be a 3-manifold with a closed 1-manifold α in the boundary. We consider R-linear combinations of isotopy classes of surfaces, properly embedded in M and bounding α, and with the components colored by elements of A. The quotient of this free R-module by the submodule generated by the embedded versions of (i)-(iii) is denoted $\mathcal{F}(M, \alpha)$. Let $\mathcal{F}(M)$ denote the skein module for $\alpha = \emptyset$. Here is a list of some easily proved properties:

- $\mathcal{F}(M, \alpha \cup u^r) = \mathcal{F}(M, \alpha) \otimes A^{\otimes r}$, where u^r is a union of r inessential circles in $\partial M \setminus \alpha$, in particular $\mathcal{F}(D^3, u^r) = A^{\otimes r}$, for all $r \geq 0$.
- The image of a connected surface of genus g, colored by $a \in A$, is $\varepsilon((\mu\Delta(1))^g a) \in R \cong \mathcal{F}(D^3)$, and the images of components are multiplied in R.
- If A is R-generated by $\mathfrak{b} \subset A$ then $\mathcal{F}(M, \alpha)$ is R-generated by *incompressible* $\mathfrak{b}$-colored surfaces. (This extends the result of Asaeda and Frohman [2].)
- $(M, \alpha) \to \mathcal{F}(M, \alpha)$ is a functor from the category of pairs (M, α) where morphisms $(M, \alpha) \to (N, \beta)$ are embeddings $i : M \hookrightarrow N$ such that $i(\alpha) = \beta \subset \partial N$.

Example 2. (i) If $A = R$ is the trivial Frobenius algebra then

$$\mathcal{F}(M) \cong RH_2(M; \mathbb{Z}_2),$$

and there is a similar result in the relative case [5]. The skein modules of group algebras will R-map onto $RH_2(M; \mathbb{Z}_2)$ and thus can be considered as natural deformations of the 2-dimensional homology of M with $\mathbb{Z}_2$-coefficients.
(ii) For $h = t = 0$ in Example 1 (i) above the module $\mathcal{F}(M)$ is the Bar-Natan module as discussed in [2]. If an inverse of 2 is adjoined then this module is equivalent to the *geometric* Bar-Natan module defined in [3], see also [9].

The following result is discussed in detail and proved in [5]:

Theorem 3. *Suppose $\mathfrak{b} \subset A$ is an R-basis of an irreducible 3-manifold M and $\alpha \subset \partial M$. Let $\mathcal{B}(M, \alpha)$ denote the set of incompressible $\mathfrak{b}$-colored surfaces in M bounding α. Then*

$$\mathcal{F}(M, \alpha) \cong R\mathcal{B}(M, \alpha)/\textit{tunneling relations}$$

Tunneling relations can be described in general [5] but the following example gives a good hint at the idea:

Example 4. Let $K \subset S^3$ be a nontrivial knot, and M be the closed knot complement. Let S be a torus parallel to ∂M, which is incompressible. If the tunnel number of K is g then in $\mathcal{F}(M)$ the following relation holds: $\varepsilon((\mu\Delta(1))^{g+1})$ multiplied by the empty surface is equal to S colored by $(\mu\Delta(1))^g$.

In [8] the Bar-Natan skein modules of the torus for $2n$ longitudes in the boundary are computed, using the above representation theorem.

Of particular interest are the skein modules $\mathcal{F}(\Sigma \times I, \alpha \times 0 \cup \beta \times 1)$ for Σ an oriented surface, because they are related to Bar-Natan's tautological functor in the surface case [3]. In fact, using naturality of the skein modules with respect to mapping class group actions one can see that the most important cases to understand are the *algebras* $\mathcal{F}(M)$ and $\mathcal{F}(M, \eta \times 0 \cup \eta \times 1)$, where η is a simple closed non-separating curve on Σ.

Theorem 5. *For each closed oriented genus g surface Σ_g there is a natural epimorphism of algebras:*

$$T(A)/((\varepsilon((\mu\Delta(1))^{2g}a) = \Delta(a), a \in A)) \to \mathcal{F}(\Sigma_g \times I)$$

where $T(A)$ is the tensor algebra of A and $((\ ))$ denotes the two-sided ideal.

It is a conjecture that this is an isomorphism.

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A Extended Bracket Polynomial for Virtual Knots and Links

Louis H. Kauffman

We define a new invariant of virtual knots and flat virtual knots that we call the *extended bracket invariant* [3]. Virtual knot theory is an extension of classical knot theory to stabilized embeddings of circles into thickened orientable surfaces of genus possibly greater than zero. Classical knot theory is the case of genus zero. There is a diagrammatic theory for studying virtual knots and links, and this diagrammatic theory lends itself to the construction of numerous new invariants of virtual knots as well as extensions of known invariants. In the bibliography of this announcement we list a number of papers on virtual knot theory and background material that will be of interest to the reader. This papers include the initial paper [11] by the author, and an independent initial paper [1] on the subject by Goussarov, Polyak and Viro.

Figure 1 illustrates the moves for virtual knots and the extra virtual crossing that is used to extend classical knot theory in this way. Figure 2 gives the oriented expansion on which the exteneded bracket invariant is based.

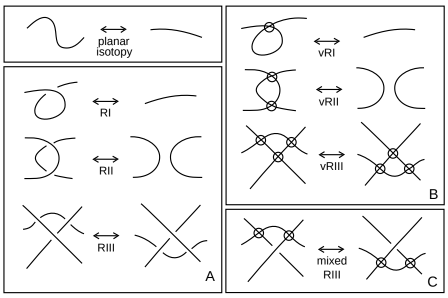

Figure 1. **Moves for Virtual Knot Theory**

For a given virtual link K, the extended bracket invariant is denoted by $\langle\langle K\rangle\rangle$ and takes values in the module generated by isotopy classes of virtual 4-regular

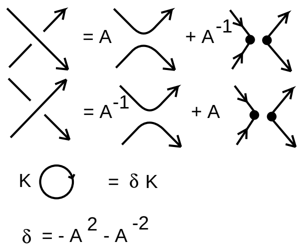

FIGURE 2. **Oriented Bracket Expansion**

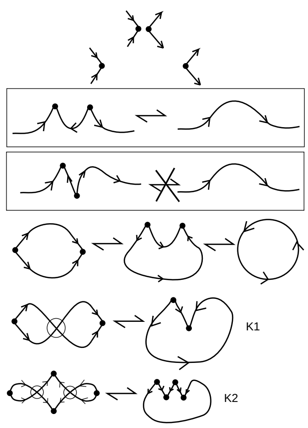

FIGURE 3. **Reduction Relation for Simple Extended Bracket.**

graphs over the ring of Laurent polynomials $Z[A, A^{-1}]$ where Z denotes the integers. A *virtual graph* is represented in the plane via a choice of cyclic orders at its nodes. The virtual crossings in a virtual graph are artifacts of the choice of placement in the plane, and we allow detour moves for consecutive sequences of virtual crossings just as in the virtual knot theory. Two virtual graphs are *isotopic* if there is a combination of planar graph isotopies and detour moves that connect them. The extended bracket is defined by a state summation with a new reduction relation on the states of the original bracket state sum.

Examples of calculation of the extended bracket include a verification of the non-classicality of the simplest example of virtualization of a classical knot diagram, a verification that the Kishino diagram and the flat Kishino diagram are non-trivial, and a verification that a particular flat diagrams are non-trivial. We use the extended bracket state sum to prove that an infinite family of single crossing virtualizations of classical diagrams are non-trivial and non-classical. The extended bracket is an invariant of flat diagrams by taking the specialization of its parameters so that $A = 1$ and $\delta = -2$. We give an example showing that the extended bracket can detect a long virtual knot whose closure is trivial. This is a capability that is beyond the reach of the Jones polynomial.

We prove two estimates for the virtual crossing number $VC(K)$ for a virtual link K. The virtual crossing number is the least number of virtual crossings in any planar diagram that represents the virtual link. Our estimates are based on the virtual crossing numbers of the graphs that appear in the extended bracket state sum. We combine use the fact an adequate or semi-adequate link (one whose A or A^{-1} states do not have any self-touching sites) has highest or lowest degree terms that can be pinpointed without calculating the entire state sum. This means that one can, for such links, estimate the virtual crossing number without calculating the entire extended bracket state sum. We use this method to calculate virtual crossing numbers and give an infinite collection of virtual links $L(n)$ whose virtual crossing number is $n = 1, 2, 3, \cdots$ and such that each $L(n)$ can be represented by an embedding in a thickened torus. Finally, we give an infinite collection of virtual knots $K(n, m)$ with minimal genus $n + 1$ and virtual crossing number $n + m$.

We construct a *simple extended bracket invariant* that we denote by $B[K]$. This invariant of regular isotopy of virtual knots and links is obtained by the same method as the extended bracket, but we weaken the state reductions so that the reverse oriented smoothings each become two individual graphical vertices. Figure 3 illustrates the reduction rules for the simple extended bracket invariant. The separated vertices do not all disappear in the reduction process and one obtains reduced states that are disjoint unions of decorated circle graphs. These non-trival circle graphs are denoted by commuting algebraic variables K_n so that $B[K]$ is a polynomial in the variables A, A^{-1} and K_n, representing the reduced circle graphs as shown in Figure 3. The invariant $B[K]$ is quite strong and can be determined by a computer program. The simple extended bracket is studied further by Heather Dye and the author in joint work under preparation. In this work, we show that, weighting each K_n with degree n, the maximal degree in these extra variables is a lower bound for the virtual crossing number.

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Invariants of framed links and p-adic framed links

Sofia Lambropoulou
(joint work with J. Juyumaya)

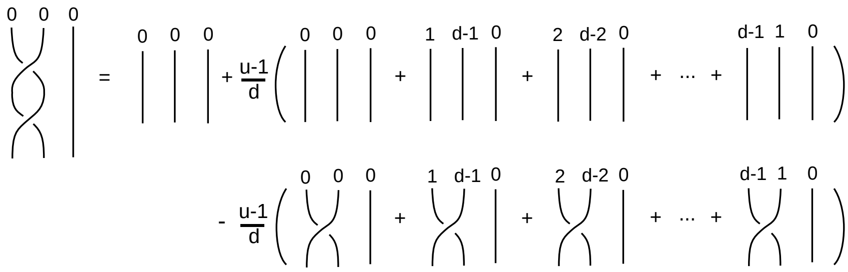
The *framed braid group* on n strands is defined as $\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$ and it is generated by the elementary braids $\sigma_1, \ldots, \sigma_{n-1}$ and the 'elementary framings' $h_i := (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith position. Then, an element of $\mathcal{F}_n$ can be written as $h_1^{a_1} h_2^{a_2} \ldots h_n^{a_n}\, \sigma$ where $\sigma \in B_n$. Geometrically, a framed braid is a classical braid with an integer, its framing, attached to each strand. Further, for $d \in \mathbb{N}$, the *d-modular framed braid group* on n strands is defined as $\mathcal{F}_{d,n} = (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$. Closure of framed braids gives rise to oriented framed links and there is an obvious analogue of the classical Markov theorem for oriented framed link isotopy in terms of equivalence classes of framed braids.

Passing now to the group algebra $\mathbb{C}\mathcal{F}_{d,n}$, we have the following idempotents:

$$e_{d,i} := \frac{1}{d}\sum_{s=0}^{d-1} h_i^s h_{i+1}^{-s} \qquad (i = 1, \ldots, n-1)$$

The *Yokonuma-Hecke algebra* $\mathrm{Y}_{d,n}(u)$, for $u \in \mathbb{C}\backslash\{0\}$ fixed, is then defined as the quotient of the group algebra $\mathbb{C}\mathcal{F}_{d,n}$ over the quadratic relations:

$$g_i^2 = 1 + (u-1)\, e_{d,i} - (u-1)\, e_{d,i}\, g_i \tag{1}$$



In [2] Juyumaya constructed inductively a unique linear Markov trace:

$$\mathrm{tr}_d : \ \mathrm{Y}_{d,n+1}(u) \longrightarrow \mathbb{C}[z, x_1, \ldots, x_{d-1}]$$

with the main two rules: $\mathrm{tr_d}(a g_n b) = z\, \mathrm{tr_d}(ab)$ and $\mathrm{tr_d}(a h_{n+1}^m) = x_m \mathrm{tr_d}(a)$ for $a, b \in \mathrm{Y}_{d,n}(u)$. In order that $\mathrm{tr_d}$ yields a framed link invariant, say Γ_d, it should have the property that it factors through $\mathrm{tr_d}(\alpha)$:

$$\mathrm{tr_d}(\alpha g_n^{-1}) = \mathrm{tr_d}(g_n^{-1})\, \mathrm{tr_d}(\alpha)$$

for any $\alpha \in \mathrm{Y}_{d,n}(u)$, so that after an appropriate re-scaling one obtains $\Gamma_d(\widehat{\alpha\sigma_n^{-1}}) = \Gamma_d(\widehat{\alpha\sigma_n})$. By $g_i^{-1} = g_i - (u^{-1}-1)\, e_{d,i} + (u^{-1}-1)\, e_{d,i}\, g_i$, linearity of $\mathrm{tr_d}$ and by the fact that $\mathrm{tr_d}(\alpha\, e_{d,n}\, g_n) = z\, \mathrm{tr_d}(\alpha)$, the property reduces to requiring: $\mathrm{tr_d}(\alpha e_{d,n}) = \mathrm{tr_d}(e_{d,n})\, \mathrm{tr_d}(\alpha)$. To have this, we must impose conditions on the set of variables $\{x_1, \ldots, x_{d-1}\}$. Indeed, for $0 \le k \le d-1$ we define the more general elements:

$$e_{d,i}^{(k)} := \frac{1}{d}\sum_{s=0}^{d-1} h_i^{k+s} h_{i+1}^{-s} \qquad \text{and also,} \qquad E_d^{(k)} := \mathrm{tr}_d\left(e_{d,i}^{(k)}\right) = \frac{1}{d}\sum_{s=0}^{d-1} x_{k+s} x_{-s}$$

where $x_0 := 1$ and the sub-indices are regarded modulo d. With the above notation $e_{d,i} = e_{d,i}^{(0)}$ and we set $E_d^{(0)} := E_d$. We shall say that the set of variables $\{x_1, \ldots, x_{d-1}\}$ has the *E–condition* if it satisfies the following *E–system* of equations in $\mathbb{C}$:

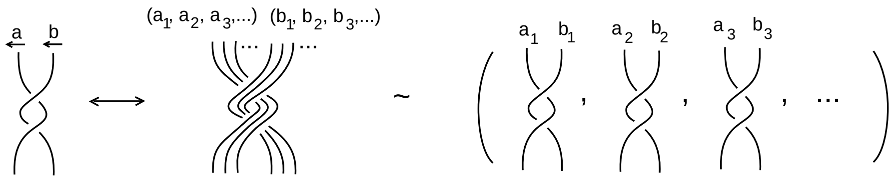
$$E_d^{(m)} = x_m E_d \qquad (1 \le m \le d-1) \tag{2}$$

The E–system (2) has non-trivial solutions for any d. Then, given the E–condition, the following is an isotopy invariant of the oriented framed link $\widehat{\alpha}$ for any $\alpha \in \mathcal{F}_n$:

$$\Gamma_d(\alpha) := \left(\frac{1-\omega u}{\sqrt{\omega}(1-u)E_d}\right)^{n-1} (\mathrm{tr}_d \circ \Omega_{d,\omega})\,(\alpha)$$

where $\omega = (uz)^{-1}[(u-1)E_d + z]$ and the representation $\Omega_{d,\omega} : \mathcal{F}_n \longrightarrow \mathrm{Y}_{d,n}(u)$ is defined by: $\sigma_i \mapsto \sqrt{\omega} g_i$, $h_j \mapsto h_j$.

In [3] the authors introduced, for a prime p, the notion of a *p–adic framed braid*. This is an infinite duplication of the same classical braid, such that the framings of the corresponding strands form a p–adic integer. Alternatively, a p–adic framed braid can be viewed as a classical braid, each strand of which is assigned a p–adic integer, or as an infinite cabling of a braid in B_n, such that the framings of each infinite cable form a p-adic integer.



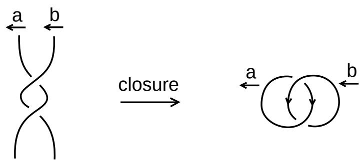
The *p-adic framed braid group on n strands* $\mathcal{F}_{\infty,n}$ is defined as:

$$\mathcal{F}_{\infty,n} := \mathbb{Z}_p^n \rtimes B_n \cong \varprojlim \mathcal{F}_{p^r,n}$$

and it contains $\mathcal{F}_n$ as a dense subgroup. This means that any p-adic framed braid can be approximated by a sequence of classical framed braids. Further, the natural algebra epimorphisms: $\phi_s^r : \mathbb{C}\mathcal{F}_{p^r,n} \longrightarrow \mathbb{C}\mathcal{F}_{p^s,n} \quad (r \geq s)$, induce the algebra epimorphisms: $\varphi_s^r : \mathrm{Y}_{p^r,n}(u) \longrightarrow \mathrm{Y}_{p^s,n}(u) \quad (r \geq s)$, through which we define the *p-adic Yokonuma–Hecke algebra* as the inverse limit: $\mathrm{Y}_{\infty,n}(u) := \varprojlim \mathrm{Y}_{p^r,n}(u)$. In $\mathrm{Y}_{\infty,n}(u)$ quadratic relations analogous to (1) hold. For the algebra $\mathrm{Y}_{\infty,n+1}(u)$ we then have that there exists *a unique p-adic linear Markov trace*:

$$\tau := \varprojlim \tau_r : \mathrm{Y}_{\infty,n+1}(u) \longrightarrow \varprojlim \mathbb{C}[\mathrm{X}_r]$$

where $\mathrm{X}_r = \{z, x_1, x_2, \ldots, x_{p^r-1}\}$ is a set of indeterminates and τ_r is the trace tr_{p^r}. Now, the closure of a p-adic framed braid defines a p-adic oriented framed link:



Moreover, solutions of the E–condition for some d lift to solutions for d', when d is a factor of d'. This means that the framed link invariant Γ_d lifts to an invariant of p-adic oriented framed links, that uses the p-adic trace τ. We hope that this construction may lead to new 3-manifold invariants ([4], [7]).

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A state sum regular isotopy link invariant with only $18n + 1$ states and even though detecting mutants

Sóstenes Lins

Summary

The main state sum regular isotopy invariant for links on $\mathbb{S}^3$ (given by plane diagrams) which I introduce in this work is a generalization of the Kauffman bracket and so of the Jones Polynomial. Thus it distinguishes any pair of links which are distinguishable by the latter. For a link diagram $\mathcal{L}$ with n crossings it is called the *VSE-invariant* (or the *ν_∞-invariant*) of $\mathcal{L}$, it has 3^n states and is denoted by $\nu_\infty(\mathcal{L})$. For each positive integer ℓ there is an specialization of ν_∞, denoted ν_ℓ and named the *truncation* of the VSE-invariant at the level ℓ. The value of $\nu_\ell(\mathcal{L})$ is also a regular isotopy invariant. The number of states of $\nu_\ell(\mathcal{L})$ is $\max\{\sum_{j=0}^{\ell} \binom{n}{j}\ 2^j,\ 3^n\}$. The invariant $\nu_\ell(\mathcal{L})$ is a normal form of a polynomial in 8 variables relative to some fixed Gröbner basis B_ℓ. VSE stands for *Virtual-Shaded-Exterior*.

If $\mathcal{L}$ is an n-crossing link diagram and k is a positive integer, denote by $\mathcal{L} * k$ the link diagram obtained from $\mathcal{L}$ by replacing each component of it by k parallel copies of it. The link diagram $\mathcal{L} * k$ is called the *k-parallel cabling* of $\mathcal{L}$ and it has $k^2\, n$ crossings. For a link diagram $\mathcal{L}$, $\ell \in \{\infty, 1, 2, \ldots\}$ and $k \in \{1, 2, \ldots\}$ define $\eta_{k,\ell}(\mathcal{L}) = \nu_\ell(\mathcal{L} * k)$. It follows that $\eta_{k,\ell}$ is a regular isotopy invariant of $\mathcal{L}$ and that it has $\max\{\sum_{j=0}^{\ell} \binom{n\,k^2}{j}\ 2^j,\ 3^{nk^2}\}$ states. For $k = 3$ and $\ell = 1$ this formula yields only $18n + 1$ states for $\eta_{3,1}$.

Surprisingly enough the $\eta_{3,1}$-invariant proves that mutants *Conway knot* ($K11n34$) and *Kinoshita-Terasaka knot* ($K11n42$) are distinct. The state sums of these knots involve the addition of only 199 monomials. The normal forms of the resulting pair of polynomials are relative to a fixed Gröbner basis, B_1, which has only 14 polynomials whose maximum absolute value of a coefficient is 3. The maximum number of monomials of a polynomial in B_1 is 16.

The name-codes for the knots are from the tables in Bar-Natan page [1]. The Gröbner basis B_1 is given at the end of Section 1 and the values of $\eta_{3,1}(Conway)$ and of $\eta_{3,1}(KinoshitaTerasaka)$ in Section 2. The $\eta_{3,1}$-invariant is a subtle invariant which deserves to be more investigated. It can be computed for links with thousand of crossings. A glimpse of its behavior is given in Section 2.

1. Overview of the strategy

The Jones polynomial, [2] or its equivalent non-oriented counterpart, Kauffman's bracket [3] does a superb job of distinguishing inequivalent knots and links. However, computations are limited to links with a few crossing because there are 2^n states to be enumerated and evaluated for a link diagram having n crossings. Here I present a practical strategy to overcome exponentiability. The result of the strategy yields the discovery of regular isotopy invariants with a very small number of states and it consists in a 4-step strengthening of Kauffman's expansion for the bracket [3]. The state sum of the VSE-invariant lives in the ring

$$\mathcal{R} = \mathbb{Z}[A, B, F, X, Y, Z, M, o].$$

The *first strengthening* relative to Kauffman's bracket is to use the 2-coloration (shaded and white faces) of the link diagram. This permits the distinctions of two kinds of crossing $X1$ and $X2$: the crossing of type $X1$ is the one that going counterclockwise from an overpass to an underpass the sweeped region is shaded; otherwise, if this region is white, the crossing is of type $X2$. The two types of crossings enable the definition of 4 variables $A, B, X, Y)$ instead of the usual 2 variables A, B, of the bracket. The *second strengthening* is that the virtual term of the expansion is included, by means of new variables F and Z. The *third strengthening* is to introduce a new variable M to control the level: to obtain the ℓ-specialization, this variable is declared to satisfy $M^{\ell+1} = 0$. Crossings of both types are expanded according to the two rules of Fig. 1.

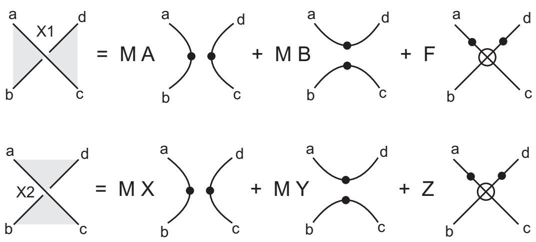

Figure 1. **The virtual shaded 3-fold expansion producing the VSE-invariant**

Note that the bracket expansion corresponds to the particular case $M = 1$, $X = A$, $Y = B$ and $F = Z = 0$. Each monomial of a full expansion is the coefficient of a set of m (maybe crossing) closed curves in the plane which are replaced by o^m. Because of the shading, there are two types of Reidemeister moves 2 and two types of Reidemeister moves 3.

The *fourth strengthening* relative to Kauffman's bracket is to consider each type of exterior in the above moves to be a *pesudo-variable* V_{ext}, where *ext* is an encoding of the particular transitions relative to the corresponding exterior. Equality must hold for all values of the exterior variables. In each monomial of the state sum, each exterior variable has degree at most 1. So, if I take the partial derivatives of the state sum relative to each of these variables, the exterior pseudo-variables disappear. But to have invariance, I must impose each such derivative to be zero, thus obtaining a polynomial equation for each exterior pseudo-variables and each move. This scheme using exterior pseudo-variables (which disappear) is clearly stronger than the usual one which does not make use of these variables: any solution of the old scheme is a solution for the new scheme but not vice-versa. Details about similar external pseudo-variables can be found in the simpler expansion of [6].

To obtain invariance under the 4 moves, a set of 27 polynomial equations are produced. Instead of trying to solve a system of 27 polynomial equations $pol_i = 0$, in the spirit of King, [5], I take the ideal generated by the left hand side of the system of equations. These polynomials generate an ideal, named $\mathcal{I}_\infty$. I compute a Gröbner basis B_∞ for the ideal $\mathcal{I}_\infty$ relative to a fixed monomial ordering. The VSE-invariant is the normal form $\nu_\infty(p)$ of the classes of polynomials $\overline{p} \in \mathcal{R}/\mathcal{I}_\infty$. It follows that if p and q are VSE-state sums of two links diagrams L_p and L_q which can be transformed one into the other by Reidemeister moves 2 and 3, then $\nu_\infty(p) = \nu_\infty(q)$.

I have written a subroutine to obtain automatically the polynomials relative to a given set of moves. The ideal $\mathcal{I}_\infty$ of $\mathcal{R} = \mathbb{Z}[A, B, F, X, Y, Z, M, o]$ corresponding to Reidemeister moves 2 and 3 is generated by the 27 polynomials. A specific Gröbner basis B_∞ with the previous variable ordering and lexicographical monomial ordering is found and has 15 polynomials. The *VSE-invariant of a link* is defined to be the normal form ν_∞ relative to the Gröbner basis B_∞ applied to the VSE-state sum of the link. Define

$$\mathcal{I}_\ell = \langle \mathcal{I}_\infty \cup \{M^{\ell+1}\} \rangle$$

and let B_ℓ be a Gröbner basis for $\mathcal{I}_\ell$ with the same monomial order. I have computed explicitly specific Gröbner basis $B_1, B_2, \ldots, B_{10}, B_{11}$. They have respectively

$$14, 25, 30, 37, 44, 53, 62, 73, 84, 97, 110$$

(not horrendous) polynomials. They permit the definition of ν_ℓ, $\ell \in \{1, \ldots, 11\}$, along similar lines as ν_∞ was defined. Below I present B_1:

$$B_1 = \{M^2,\ Z^2o^3 - o^3 + Z^2o^2 - o^2 - 2Z^2o + 2o,\ MoZ^2 - Mo,\ oZ^4 - 2oZ^2 + o,\ o^2Z^3 - o^2Z + 2MoX + 2MoY,\ Fo^3 - Zo^3 + Fo^2 - Zo^2 - 2Fo + 2Zo,\ FMo - MoZ,\ -oZ^3 + FoZ^2 + oZ - Fo,\ oF^2 - 2oZF + oZ^2,\ -Fo^2 + BMo^2 - MYo^2 + Zo^2 + Fo - BMo + MYo - Zo,\ BZ^2o^2 + XZ^2o^2 - Bo^2 + Yo^2 - FXZo^2 - FYZo^2 - BZ^2o - XZ^2o + Bo - Yo + FXZo + FYZo,\ o^2Z^3 - 3o^2Z + 2Fo^2 + 2AMo + 2BMo,\ AoZ^2 + BoZ^2 + oXZ^2 + oYZ^2 - 2FoXZ - 2FoYZ - Ao -$$

$$Bo + oX + oY,\ AFo^2 - BFo^2 - FXo^2 + FYo^2 - AZo^2 + BZo^2 + XZo^2 - YZo^2 - AFo + BFo + FXo - FYo + AZo - BZo - XZo + YZo\}.$$

2. Remarks: a glimpse of $\eta_{3,1}$

The $\eta_{3,1}$-invariant also distinguishes the Thistlethwaite first link in [7] from the unlink. It does not distinguishes 9_{42} from its mirror. It does distinguish 4_1 and $K11n19$. It does not distinguish 8_8 from 10_{129}. Repeating once more, it does distinguish the mutants Conway ($K11n34$) and Kinoshita-Terasaka ($K11n42$). Here are the value of the $\eta_{3,1}$-invariants of these knots:

$$\eta_{3,1}(Conway) = \frac{1}{2}o\left(2M(o-1)Yo^2 + Z\left(2o^2 - 3Z^2o + 3o + 2Z^2 - 2\right)\right),$$

$$\begin{aligned}\eta_{3,1}(KinoshitaTerasaka) \quad = \quad & -\frac{1}{2}o\Big(6M(o-1)Yo^2 \\ & + Z\left(-2o^2 - 13\left(Z^2-1\right)o + 14\left(Z^2-1\right)\right)\Big).\end{aligned}$$

The above computations give a glimpse of $\eta_{3,1}$. It is a subtle invariant which deserves to be better understood. If not otherwise, because of the speed in which it can be computed for links with thousands of crossings. (This awaits a proper implementation, in a non-interpreted language.)

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Atoms, Khovanov Homology, and Heegaard-Floer Homology

Vassily O. Manturov

The Main Goal. We argue that for links and virtual links there is a genus called *atom genus*, [Ma1] (Turaev genus [Tu]) which plays the same role for the Kauffman bracket [Ka1] and Khovanov homology [Kh] as Seifert genus plays for the Alexander polynomial and Heegaard-Floer homology. From the point of view of atoms, classical knots are seen as an essential part of virtual knots [Ka2], [Ka3].

The main construction. An *atom* (first defined by Fomenko [Fom]) is a pair (M, Γ) where M is a 2-manifold, and Γ is a four-valent graph embedded in M such that $M\backslash\Gamma$ is a disjoint collection of checkerboard coloured 2-cells with fixed colouring. For an *alternating* diagram of a classical link K, one constructs an atom as follows: one takes all circles of the A-state of the Kauffman bracket $\langle K\rangle$ and all circles of the B-state. They naturally correspond to the 2 cells of $S^2\backslash$shadow K. Thus, for an alternating link, one gets an atom with $S^2 = M$. In general, the *atom genus* measures the non-alternatibility of the diagram. Starting with a diagram K of a (possibly, virtual) link, we take its shadow to be the frame of the atom. For each classical vertex, we fix how the edges are split into two pairs of *opposite* ones. Now, for each vertex of the atom, we have two pairs of *opposite angles*. The angle swept by passing from an overcrossing half-edge to the undercrossing half-edge in the clockwise direction is decreed to be white as well as the opposite angle; the remaining two angles are black. Paste black and white cells to it as follows. The boundary of each cell is a cycle on the shadow of the knot passing from any edge to an adjacent one in such a way that two adjacent edges locally form the angle of the corresponding colour. An atom can be reconstructed from the corresponding knot diagram in a unique way. The inverse operation is not unique: our knot diagram is defined up to *virtualization*, a move which flanks a classical crossing by two virtual moves with writhe number preserved.

White cells of the atom are in 1-1 correspondence with the circles of the A-state. One can restore the whole state cube from the atom, that is, to recover the Kauffman bracket of the initial knot, which agrees with the fact that the Kauffman bracket is invariant under virtualization.

Main Theorems. Recall the formulation of the celebrated Kauffman-Murasugi Theorem.

Theorem 1 ([Mur]). *For a non-split connected diagram K of a classical knot, we have $span\langle K\rangle \leq 4n$, where n is the number of crossings of K. Moreover, if K is alternating, then the equality holds.*

This yields the positive solution of the *Tait conjecture* saying that alternating diagram are *minimal* with respect to the number of classical crossings. Analysing the Kauffman bracket of a link diagram, one gets the following far-reaching generalization of Theorem 1.

Theorem 2 (see, e.g.[Ma1]). *For a non-split connected virtual diagram K we have $span\langle K\rangle \leq 4n-4g$, where n is the number of classical crossings of K and g is the genus of the corresponding atom.*

This theorem means that for a link K where the equality $span\langle K\rangle = 4n - 4g$ holds we can decrease the number of crossings only at the expense of decreasing the genus g. A combination of this result with *cabling* leads to a new [Ma1] proof of Thistlethwaite's celebrate theorem [Th] saying that *adequate diagrams are minimal* where a classical knot diagram is adequate if the corresponding atom has no vertex where one black cell touches itself. On the other hand, the genus of the atom can be estimated by using Khovanov homology.

Theorem 3 ([Ma1])**.** *For a non-split connected virtual link diagram K we have $Th(Kh(K)) \leq 2 + g$, where Th means thickness, i.e., measures the number of diagonals with slope 2 supporting Khovanov homology.*

This theorem follows from understanding the notion of atom in the context of the Wehrli-Kofman-Champanerkar [Weh, ChK] spanning tree (a Khovanov complex has the same homology as a complex with chains corresponding to single circle states of the Kauffman bracket). The same estimate $Th(Kh(K)) \leq 2 + g$ holds for *odd Khovanov homology*, [ORS]. For reduced (and reduced odd) Khovanov homology one has $Th(Kh(K)) \leq 1 + 2g$. The set of 1-circle states not only support the usual Khovanov homology, but they are also in 1-1 correspondence with the knot Heegaard-Floear complex, [OSz]. This leads to a theorem of Lowrance, [Low] saying that $Th(\widehat{HF}(K)) \leq 1 + g$, where by Th we consider the number of diagonals of slope -1 supporting the Heeggaard-Floer homology. This yields

Theorem 4. *Assume for a reduced non-split virtual diagram K we have $span\langle K\rangle = 4n - 4g$ and either $Th(Kh(K)) = 2 + g$, where Kh means usual or odd Khovanov homology or $Th(Kh_R(K)) = 1 + g$, where Kh_R means the reduced Khovanov homology, or $Th(\widehat{HF}(K)) = 1 + g$ holds. Then K is minimal with respect to the number of classical crossings.*

Khovanov's homology also estimates the Seifert genus of a link. In Rasmussen's paper [Ras], it is shown how to estimate the Seifert genus by using a single invariant s coming from a spectral sequence starting from usual Khovanov homology and having differentials with boundary slopes $0, 4, 8$ etc. For instance, for alternating link diagrams (for which the complex is supported in 2 adjacent diagonals) all differentials except the first one equal zero. Now we see that the atom genus essentially estimates the term when the Rasmussen spectral sequence converges. Namely, for $g = 0, 1$ we have $E_2 = E_\infty$ for $g = 2, 3$ we have $E_3 = E_\infty$ etc. A construction similar to Rasmussen's goes straightforwardly if we deal with the category of knots and cobordisms corresponding to orientable atoms (see ahead).

On Khovanov Homology for Virtual Knots. Atoms play a key role not only for estimating crossing number, atom genus, knot genus etc., but or *constructing* the Khovanov homology itself. The Khovanov complex [Kh, BN, Viro, Ma1] is a bigraded complex constructed out of a link diagram in such a way that its Euler characteristic is equal to Kauffman bracket (in a slightly different normalization). Going along the lines of [Kh], we associate with each circle a 2-dimensional graded vector space V (thus, for a state with m circles we have $V^{\otimes m}$), and set the homological grading to be the number of B-smoothings of the state. Thus, from each state we have partial differentials going along edges of the cube, and (being endowed with $\pm$ signs, together they compose the differential of the Khovanov complex. The second grading comes from the grading of $V^{\otimes \cdots}$ shifted by the first grading.

In the classical construction, there are two possibilities when we change the smoothing at a classical crossing: two circles merge to a single circle or one circle splits into two circles. At the level of partial differentials, it is expressed in terms of multiplication $V \times V \longrightarrow V$ and comultiplication $V \longrightarrow V \times V$. For commutativity of 2-faces of the cube the multiplication and the comultiplication should satisfy the axioms of the Frobenius algebra. An appropriate edge labeling (by ± 1) makes faces anticommutative which guarantees $\partial^2 = 0$. For virtual knots, a bifuraction of type $1 \longrightarrow 1$ occurs which means that one should introduce a new operation $V \longrightarrow V$ compatible with all possible 2-faces of the bifurcation cube. All possible 2-faces are atoms with 2-vertices. By grading reasons, this map $V \longrightarrow V$ can not be anything but zero, which does not fit into the original Khovanov setup. We have

Lemma 1 ([Ma2]). For a diagram K, $1 \longrightarrow 1$-bifurcations occur if and only if the corresponding atom is non-orientable.

So, the original Khovanov homology works in the category of orientable atoms. For the case of non-orientable atoms, I have introduced twisted coefficients in the Frobenius algebra and the exterior of different V's corresponding to circles of the same state (instead of just tensor product). This leads to Khovanov homology theory for all virtual knots [Ma2]. An extension of that theory is described in [Ma3].

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On representations of mapping class groups in Integral TQFT

Gregor Masbaum

In this talk, I discussed the theory of integral TQFT which I have developed in joint work with Patrick Gilmer [6, 7]. In usual Reshetikhin-Turaev TQFT, the mapping class group of a compact orientable surface Σ is represented on a finite-dimensional vector space, say $V_p(\Sigma)$, over a cyclotomic field, say $\mathbb{Q}(\zeta_p)$ (here ζ_p is a primitive pth root of unity). For integral TQFT, the vector space should be replaced by a free lattice $\mathcal{S}_p(\Sigma)$ over $\mathbb{Z}(\zeta_p)$ = the ring of algebraic integers in $\mathbb{Q}(\zeta_p)$. In particular, it means that mapping classes are now represented by matrices with integral coefficients.

In [7], we have shown how to construct such an integral TQFT refinement for the Reshetikhin-Turaev $SO(3)$ TQFT at $q = \zeta_p$, p an odd prime, starting from the skein-theoretical approach to this TQFT as in [2]. The integral lattice $\mathcal{S}_p(\Sigma)$ is contained in the vector space $V_p(\Sigma)$ and has a natural definition in terms of the vector-valued quantum $SO(3)$-invariants for 3-manifolds with boundary (see below). (If $p \equiv 1 \bmod 4$, the coefficient ring considered in [7] is actually a quadratic extension of $\mathbb{Z}(\zeta_p)$, but for simplicity of exposition I will ignore this and similar details in this talk.)

The mapping class group representation on the lattice $\mathcal{S}_p(\Sigma)$ preserves a natural non-degenerate hermitian form with values in $\mathbb{Z}(\zeta_p)$. One may ask whether the image of the mapping class group under this representation coincides with the automorphism group of this form. Note that the analogous statement for the U(1)-TQFT is the well-known fact that the image of the mapping class group acting in homology is the symplectic group $Sp(2g, \mathbb{Z})$, that is, the group of automorphisms of the integral homology lattice of Σ which preserve the intersection form.

Another question about the image of the mapping class group concerns its group theoretic structure. It is known that Dehn twists are represented by matrices of order p; are there any other relations in the image that don't already hold in the mapping class group? For the torus without boundary, there must be more relations, because the image is known to be a finite group (Gilmer [4]). But for the torus with one boundary component, I can show that there are no other relations. One may wonder whether this is a general fact for hyperbolic surfaces, and if so, what is its geometric meaning?

Bases of the vector space $V_p(\Sigma)$ are well understood in terms of admissible colorings of uni-trivalent graphs. But the $\mathbb{Z}(\zeta_p)$-span of such a *graph basis* is almost never invariant under the mapping class group, and hence cannot be equal to the integral lattice $\mathcal{S}_p(\Sigma)$. In [7], we show that $\mathcal{S}_p(\Sigma)$ admits what we call *graph-like* bases associated to a special kind of uni-trivalent graph which we call a *lollipop tree*. Roughly speaking, a graph-like basis is obtained from the usual graph basis associated to the lollipop tree by the composition of two operations: a certain triangular base change, and some rescaling depending on the colors. For precise definitions, see [7].

Integral TQFT contains more topological information than the usual TQFT over a field. For example, it allows to study embedding questions as follows. Consider the following problem. Given an orientable compact connected 3-manifold N with boundary $\partial N = \Sigma$, does it embed into the 3-sphere? This translates in TQFT to a condition on the vector $v = v_p(N)$ in $V_p(\Sigma)$ associated to N: since

$$N \cup (S^3 - N) = S^3$$

there must be a vector v' (namely $v' = v_p(S^3 - N)$) such that

$$\langle v, v' \rangle = 1$$

(since the quantum invariant of S^3 is 1 in the normalization which is relevant here). In usual TQFT, this condition just requires v to be non-zero (since the form $\langle\ ,\ \rangle$ is non-degenerate). But in integral TQFT, both v and v' must lie in the integral lattice $\mathcal{S}_p(\Sigma)$. This puts lots of restrictions on v, and they may be used to show in some cases that N does not embed into S^3. An example is given at the end of our paper [7]. More examples can be found in Gilmer [5].

To understand how this works in practice, one needs to know that the integral lattice $\mathcal{S}_p(\Sigma)$ is exactly the span, over $\mathbb{Z}(\zeta_p)$, of the vectors $v_p(N')$ where N' has boundary Σ and no closed components. The numbers $\langle v, v' \rangle$ where $v' \in \mathcal{S}_p(\Sigma)$ span an ideal in $\mathbb{Z}(\zeta_p)$ which we call the FKB-ideal since Frohman and Kania-Bartoszynska were the first to consider this kind of quantum obstruction to embedding one manifold into another [3]. Clearly, if N embeds into S^3, then there is a v' in $\mathcal{S}_p(\Sigma)$ such that $\langle v, v' \rangle = 1$, so the FKB-ideal is trivial (i.e., contains 1). But to decide effectively whether such a v' exist, we need a basis (or at least, a finite generating set) of $\mathcal{S}_p(\Sigma)$. Frohman and Kania-Bartoszynska could not compute the ideal except in rather trivial situations. But our integral TQFT-bases from [7] make their idea into an effective tool. I like to think that this shows at the same time that integral TQFT, which is defined over a ring of algebraic integers, represents the actual topological information much more closely than the usual TQFTs defined over a field.

I would like to close this short report with two more results about TQFT representations of mapping class groups.

The first one concerns the relationship between TQFT and the Nielsen-Thurston classification of mapping classes of surfaces. In my paper [1] with J. E. Andersen and K. Ueno, we make the following

Conjecture. *Let Σ be a compact orientable surface with negative Euler characteristic and let ρ_k be the TQFT representation of the mapping class group of Σ at level k* (say for the Reshetikhin-Turaev TQFT associated to some quantum group). *Then a mapping class φ has a pseudo-Anosov piece if and only if there exists $k_0 = k_0(\varphi)$ such that the matrix $\rho_k(\varphi)$ has infinite order for all $k \geq k_0$.*

Note that it is easy to see that if φ has no pseudo-Anosov piece, then the matrix $\rho_k(\varphi)$ has finite order for all k (although φ itself may have infinite order as a mapping class). For more discussion of this conjecture, see [1].

In [1], we prove the conjecture in the $SU(n)$-case for the mapping class group $M(0,4)$ (i.e. when Σ is a four-holed sphere). In the $SU(2)$-case, we can even show that the stretching factor of a pseudo-Anosov mapping class φ is the limit, as $k \to \infty$, of the maximal eigenvalue of the TQFT-matrix $\rho_k(\varphi)$. As already mentioned in [1], I also know how to prove this for $M(1,1)$ (i.e. Σ is now a torus with one boundary component), but the proof in this case involves integral bases [8].

The second result about TQFT representations I would like to mention is unpublished work of mine from 2005 [9]. It affirms the existence of a limit representation (at least on the Torelli group) as the order of the quantum parameter $q = \zeta_p$ goes to infinity. For this result integral TQFT is crucial and I consider again the integral $SO(3)$-TQFT lattices constructed with Gilmer in [7].

Theorem. *There exist ordered bases of the integral lattices $\mathcal{S}_p(\Sigma)$ (p an odd prime), such that for every mapping class φ in the Torelli subgroup of the mapping class group of Σ, and for every (i,j), the matrix entries $(\rho_p(\varphi))_{ij}$ converge in Ohtsuki's sense as $p \to \infty$.*

Note that since the rank of $\mathcal{S}_p(\Sigma)$ goes to infinity as $p \to \infty$, for every (i,j) the matrix entry $(\rho_p(\varphi))_{ij}$ is defined for all big enough p. This matrix entry lies in $\mathbb{Z}(\zeta_p)$. The limit in Ohtsuki's sense of a sequence of algebraic integers $I_p \in \mathbb{Z}(\zeta_p)$ is defined as follows. Write

$$I_p = \sum_{n=0}^{p-2} a_{n,p}(\zeta_p - 1)^n$$

where $a_{n,p} \in \mathbb{Z}$. We say that the sequence I_p converges to a power series

$$\tau = \sum_{n=0}^{\infty} a_n h^n \in \mathbb{Q}[[h]]$$

if for every n and every prime $p >> n$, the integer $a_{n,p}$ and the rational number a_n are congruent modulo p (note that this makes sense for p bigger than the denominator of a_n).

This definition goes back to Ohtsuki. If $I_p(M)$ denotes the Reshetikhin-Turaev invariant of an integral homology sphere M, it is known by H. Murakami [11] that $I_p(M) \in \mathbb{Z}(\zeta_p)$ (a skein-theoretical proof of this result was given in my paper

[10] with J. Roberts; it was the beginning of my interest in integrality questions in TQFT). Then Ohtsuki showed that $I_p(M)$ converges in the above sense to a power series $\tau(M) \in \mathbb{Q}[[h]]$ called the Ohtsuki series of M [12].

My theorem stated above generalizes Ohtsuki's result to the TQFT representation of the Torelli group. If the integral homology sphere M is obtained in the usual way from a Torelli mapping class φ, we may choose the basis of the lattice $\mathcal{S}_p(\Sigma)$ such that the invariant $I_p(M)$ is one of the entries of the matrix $\rho_p(\varphi)$ (in fact, the entry in the upper left corner of the matrix). While Ohtsuki's theorem says that this matrix entry converges as $p \to \infty$, my theorem says the same thing for *all* matrix entries. Observe that the truth of a statement of this kind will depend crucially on what basis one chooses. In fact, this convergence result would not be true without using the integral TQFT bases I found in my work with Gilmer in [7].

The limit representation can be explicitly described using skein theory, and as a corollary I obtain a purely skein-theoretical construction of the Ohtsuki series $\tau(M)$. I made some more comments in my talk about this limit representation, but for lack of space I will not reproduce them here. Hopefully a written account of this matter will soon appear elsewhere.

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Invariants of genus 2 mutants

Hugh R. Morton
(joint work with Nathan Ryder)

This is a report of joint work with Nathan Ryder, [6].

Genus 2 mutation of knots was introduced by Ruberman [8] in a general 3-manifold. Cooper and Lickorish [1] give a nice account of an equivalent construction for knots in S^3, using genus 2 handlebodies, and we use their construction.

Genus 2 mutant knots provide a test-bed for comparing knot invariants, in the sense that they can be shown to share a certain collection of invariants, and so any invariant on which some mutant pair differs must be completely independent of the shared collection. This procedure can be refined by restricting further the class of genus 2 mutants under consideration, so as to increase the shared collection, and then looking for invariants which differ on some restricted mutants.

In a recent paper [2] Dunfield, Garoufalidis, Shumakovitch and Thistlethwaite survey some of the known results about shared invariants for genus 2 mutants, and show that Khovanov homology is not shared in general. They also give an example of a pair of genus 2 mutants with 75 crossings which differ on their Homfly polynomial. These are smaller examples than the known satellites of the Conway and Kinoshita-Teresaka knots [4]. They ask for examples of genus 2 mutants which don't share the 2-variable Kauffman polynomial, in the expectation that their 75 crossing knots, which are out of range of current programs for calculating the Kauffman polynomial, will indeed give such an example.

We have found a number of smaller genus 2 mutant pairs with different Homfly polynomials, and can show that they also have different 2-variable Kauffman polynomials. The smallest examples to date have 55 crossings.

The fact that their Kauffman polynomials are different can be detected without having to make a complete calculation. When their Homfly polynomials are compared as polynomials in z with coefficients in $\mathbf{Z}[v^{\pm 1}]$ they differ in their constant term $P_0(v)$. Since the constant terms in the Homfly and Kauffman polynomials for a knot are always the same this establishes quickly that the knots found have different Kauffman polynomials. This technique does not work for the 75 crossing knots in [2], since the polynomials $P_0(v)$ agree in this case. This is also the case for the recent examples of Stoimenow and Tanaka [9].

The difference in their Homfly polynomials persists in our 55 crossing examples, and in some but not all of the other examples, after making the substitution $v = s^3$. This substitution calculates their quantum $sl(3)$ invariant when coloured by the fundamental 3-dimensional module. Work of Morton and Ryder [5] on the Kuperberg skein of the twice punctured disc, which can be used in comparing the quantum $sl(3)$ invariant of genus 2 mutant knots, in fact pointed us in the direction of the 55 crossing examples.

Our examples also allow us to make a distinction between the invariants for general genus 2 mutants and those arising as satellites of Conway mutant knots. Our examples include a pair of genus 2 mutants which differ on a degree 7 Vassiliev invariant, while work of Duzhin [3] ensures that satellites of Conway mutants share all Vassiliev invariants of degree ≤ 8, extended to degree 10 more recently by Jun Murakami [7].

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Simple formulas for 3-manifold invariants

Michael Polyak

1. Casson-Walker Invariant

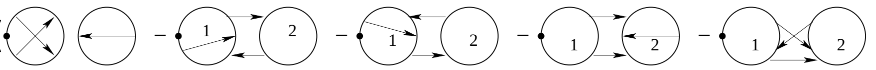
The Casson-Walker invariant $\lambda(M)$ of rational homology 3-spheres is one of the fundamental invariants in 3-manifold topology [1, 6]. It is an integer extension of the Rokhlin invariant. In the theory of finite type invariants of 3-manifolds it is the simplest $\mathbb{Z}$-valued invariant after $|H_1(M)|$.

For a manifold $M = M_L$ obtained from S^3 by surgery on a framed link L $\lambda(M)$ remains, however, in general quite difficult to compute. While it is easy to calculate if L is a knot [2], the same question for links remains quite complicated. In particular, for 2-component links only some special cases were studied (see e.g. [3]).

We present a simple diagrammatic formula (in the spirit of [4, 5]) for calculating $\lambda(M_L)$ for spheres presented by 2-component framed links. The formula helps us to understand/separate the dependence of the invariant on the link L and on the surgery coefficients.

Theorem 1. *Let L be an oriented, framed 2-component link with the linking matrix $\mathbb{L} = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ and $\det(\mathbb{L}) \neq 0$. Denote by signature$(\mathbb{L})$ the signature of the bilinear form defined by $\mathbb{L}$. Let G be a Gauss diagram of L with the blackboard framing. Then*

$$\det(\mathbb{L}) \left(\frac{\lambda(M_L)}{2} + \frac{signature(\mathbb{L})}{8} \right) =$$

$\langle$  $, G\rangle$

$$+ \frac{b^3 - b}{12} - \frac{(a+d)(2b^2 - ad - 2)}{24}$$

Here the pairing $\langle A, G\rangle$ denotes the algebraic number of subdiagrams of G isomorphic to A. Each such subdiagram of G is counted with its sign, which equals to the product of signs of all its arrows.

2. Surgery on graphs and extension of invariants

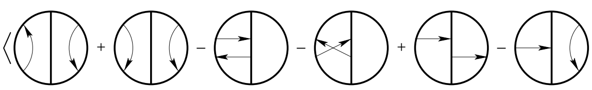
We also introduce a notion of surgery on framed knotted graphs. This allows us to reduce the operation of handle slide to a simple local move. Finally, we extend Gauss diagram formulas for invariants from links to graphs.

To prove that an invariant of framed links gives an invariant of 3-manifolds, one should check that it is preserved under (de)stabilization and handle slides. Unfortunately, handle slide is not a local move, and is therefore difficult to check. We reduce it to a simple local "fusing/unfusing" move, fusing two link components together into a ribbon Θ-graph, pulling one of the 3-valent vertices along by isotopy, and unfusing the link components. For this purpose we introduce surgery on framed knotted trivalent graphs similarly to surgery on links – cutting out a tubular neighborhood of each component, gluing discs in an appropriate way detrmined by the framing, and finally gluing in 3-balls to the remaining boundary components.

It turns out that Gauss diagram invariants of framed links which are invariant under handle slides naturally extend to knotted graphs. A problem of extension is of an independent interest and has various applications.

For example, the simplest finite type invariant of 3-manifolds is $|H_1(M)|$. For a manifold otained by surgery on a 2-component framed link L in S^3, it equals $|\det(\mathbb{L})|$. It turns out that the determinant is easy to extend to Θ-graphs:

Theorem 2. *For a knotted Θ-graph with a Gauss diagram G one has*

$\det(\Theta) = \langle$  $, G\rangle$

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Some algebraic structures, and representations of Hopf algebras, leading to invariants of knots and links

David E. Radford

There are very basic algebraic structures, generalizations of quasi-triangular Hopf algebras, which account for regular isotopy invariants of 1-1 tangles or knots and links [7, 9, 11, 12, 13]. Twist quantum algebras give rise to invariants of unoriented knots and links. Invariants of oriented knots and links come from their twist oriented counterparts. See [15] for a survey. There is a corresponding structure for virtual links [8].

Let k be a field. A basic structure for us is a Yang-Baxter algebra, a pair (A, ρ), where A is an algebra over k and $\rho \in A{\otimes}A$ is an invertible solution to the quantum Yang-Baxter equation. A twist quantum algebra over k is a tuple (A, ρ, s, G), where (A, ρ) is a Yang–Baxter algebra, $s : A \longrightarrow A^{op}$ is an algebra isomorphism such that $\rho = (s{\otimes}s)(\rho)$ and $\rho^{-1} = (s{\otimes}I)(\rho)$, and $G \in A$ is invertible and satisfies $s(G) = G^{-1}$ and $s^2(a) = GaG^{-1}$ for all $a \in A$. If (A, ρ, v) is a ribbon Hopf algebra with antipode s then (A, ρ, s, G) is a twist quantum algebra, where $G = uv^{-1}$ and u is the Drinfel'd element of A. Basic examples of quasditriangular Hopf algebras are $(D(H), \rho)$, where H is any finite-dimensional Hopf algebra over k. A necessary and sufficient condition is found for $(D(H), \rho)$ to have a ribbon element in [10].

A final ingredient is needed to construct regular isotopy invariants of unoriented knots and links from a twist quantum algebra (A, ρ, s, G); namely a tracelike functional $\mathrm{tr} : A \longrightarrow k$ which satisfies $\mathrm{tr}{\circ}s = \mathrm{tr}$. Tracelike means $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$. Generally there are many possibilities for tracelike functionals [18]. Regular isotopy invariants are computed via the "bead sliding" formalism of [7]. See [9, 13] also. That the Jones polynomial can be derived from such a twist quantum algebra structure on $A = \mathrm{M}_2(k)$ is shown in [7].

A twist oriented quantum algebra is a tuple (A, ρ, D, U, G), where (A, ρ) is a Yang–Baxter algebra, D and U are commuting algebra automorphisms of A such that ρ is invariant under $D{\otimes}D$, $U{\otimes}U$, and $(I{\otimes}U)(\rho)$, $(D{\otimes}I)(\rho^{-1})$ are inverses in the algebra $A{\otimes}A^{op}$, and finally $G \in A$ is invertible, invariant under D and U, and $(D{\circ}U)(a) = GaG^{-1}$ for all $a \in A$. A basic example: if (A, ρ, s, G) is a twist quantum algebra over k then $(A, \rho, I, s^{-2}, G^{-1})$ is a twist oriented quantum algebra over k. If (A, ρ) is a quasitriangular Hopf algbebra with antipode s over k then $(A, \rho, I, s^{-2}, u^{-1})$ is a twist oriented quantum algebra over k.

Regular isotopy invariants of oriented knots and links are computed from a twist oriented quantum algebra via bead sliding formalism and the additional ingredient of a tracelike functional $\mathrm{tr} : A \longrightarrow k$ which satisfies $\mathrm{tr}\circ D = \mathrm{tr}$ and $\mathrm{tr}\circ U = \mathrm{tr}$. To compute invariants there is no loss of generality in assuming $D = I$. Such a twist oriented quantum algebra is called standard. Note that (A, ρ, I, U, G) is determined by the triple (A, ρ, G) and the conditions on a tracelike function are automatically satisfied. The triple is (A, ρ, G) described by very simple axioms.

There is a rich algebraic theory of these objects which involves their representations [11, 12]. Representations have a similar algebraic structure and thus account for invariants as well. The reason is quite general.

Suppose that (A, ρ, G) determines a standard twist oriented quantum algebra as above and $f : A \longrightarrow B$ is an algebra homomorphism. Then $(B, (f\otimes f)(\rho), f(G))$ determines a strandard oriented quantum algebra. Suppose further that $\mathrm{tr}_A : A \longrightarrow k$ and $\mathrm{tr} : B \longrightarrow k$ are tracelike functions. If $\mathrm{tr}_A = \mathrm{tr}_B\circ f$ then the invariants derived from (A, ρ, G) and $(B, (f\otimes f)(\rho), f(G))$ and their respective tracelike functions are the same. Strict oriented quantum algebra structures on A give rise to strict oriented quantum algebra structures on representations of A, and via these structures invariants derived from representations are invariants arising from strict twist oriented algebra structures on A.

The classification of finite-dimensional pointed Hopf algebras over an algebraically closed field of characteristic zero whose coradical is commutative has been completed to a very satisfactory degree [1, 2, 3, 4, 5, 6]. These Hopf algebras include the very important "small quantum groups" and thus form a fundamental class of finite-dimensional Hopf algebras [14]. Hopf algebras of special interest are $A = u(\mathcal{D}, \lambda, 0)$, where $\mathcal{D} = (\Gamma, (g_i)_{1\le i\le\theta}, (\chi_i)_{1\le i\le\theta}, (a_{i,j})_{1\le i,j\le\theta})$, where Γ is a finite abelian group, $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ is a k-valued character, and $(a_{i,j})$ is a matrix of finite Cartan type. $\lambda = (\lambda_{i,j})_{1\le i,j\le\theta}$ is a family of scalars called linking parameters. The conditions

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{i,j}} \quad \text{and} \quad \chi_i(g_i) \neq 1$$

are satisfied for all $1 \le i, j \le \theta$.

We do not write down the generators and relations for $u(\mathcal{D}, \lambda, 0)$ here. The reader is referred to [2]. In our examples the root vector relations are zero; generally they are not.

Work on the general representation theory of these algebras is well under way [16, 17]. As we have seen combinatorial data determines these Hopf algebras. Characterization of data which gives rise to quasi-triangular Hopf algebras, and nature of the resulting invariants, especially those arising through their representations, are problems whose time has come.

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Clock Moves and Combinatorial Knot Homology

Yongwu Rong
(joint work with Kerry Luse)

This talk is motivated by an attempt to reconstruct the combinatorial knot homology by Manolescu, Ozsvath, and Sarkar [MOS] using the clock moves introduced by Kauffman [K83]. In the past several years, there has been a great deal of developments in various homology theories for knots and 3-manifolds. In [K00], Khovanov introduced a graded homology theory for knots, and proved that its graded Euler characteristic is the Jones polynomial. In a series of papers, Ozsvath and Szabo developed what they call the "Heegaard Floer homology" theory. The corresponding theory for knots was also independently developed by Rasmussen. In this case, the graded Euler characteristic is the Alexander polynomial.

The Heegaard Floer homology is considered as a milestone towards an easier approach to the gauge theory type invariants over the past two decades. It is almost entirely combinatorial, except for the Maslov grading that is still needed. This was finally resolved in a 2006 paper by Manolescu, Ozsvath, and Sarkar [MOS], where a combinatorial description of the Heegaard Floer homology for links in S^3 is given. Their chain complex, while entirely combinatorial and certainly beautiful, is quite complicated. It is generated by n! generators where n is the size of the "grid diagram" of the link. An interesting problem is to find a simpler combinatorial description for the Heegaard Floer homology.

In this talk, we explain a graded homology theory, the *clock homology*, using a state sum model of the Alexander polynomial and the clock moves due to Kauffman. For each link diagram D represented by a (1,1) tangle, its states are assignments, for each crossing, of a dot to a region adjacent to that crossing such that each of the bounded regions of D contains exactly one dot. One defines the Alexander grading and the Maslov grading appropriately. The boundary map is then defined using a restricted version of Kauffman's clock moves. This does yield a graded chain complex whose Euler characteristic is the Alexander polynomial of the link. Unfortunately, the homology groups are not always preserved under Reidemeister moves. Therefore, we are unable to achieve our goal of reconstructing the knot Floer homology. Nonetheless, we do have a homology theory for link diagrams that categorifies the Alexander polynomial.

The idea of applying clock moves to knot homologies is natural, and several participants indicated that they have looked into this. One may want to modify the differentials to get topological invariance. Presumably this can be done by translating the counting of gradient flows of pseudoholomorphic disks in the original definition by Ozsvath-Szabo and Rasmussen. It is also natural to see whether one can extract topological invariants, beyond the Alexander polynomial, from these groups. Some specific questions are as follows.

(1) Study how the clock homology behaves under Reidemeister moves. This may help to modify the construction here to yield a topological invariant.
(2) Explore the possibility of extracting a topological invariant from the clock homology. Certainly the Alexander polynomial is an invariant extracted from these groups. We would like to know if one can obtain information beyond the Alexander polynomial.
(3) Understand relationships between the clock homology and the knot Floer homology. Peter Ozsvath has pointed out the existence of a spectral sequence starting at the clock homology and converging to knot Floer homology. It would be interesting to use this connection to determine inductively all the differentials needed to get topological invariance.
(4) Study the clock homology as an invariant of link diagrams, to see if it could give an estimate on the number of Reidemeister moves between two given diagrams of the same knot. This certainly depends on a good

understanding of the behavior of the clock homology under Reidemeister moves (see Problem 1).

We wish to thank Marta Asaeda, Charlie Frohman, Vassily Manturov, and others for their comments during the workshop.

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Categorification and virtual knots

Lev Rozansky

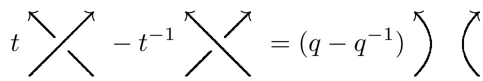
(joint work with Mikhail Khovanov)

1. A 2-variable HOMFLY-PT polynomial and categorification program

In [4] we categorify the 2-variable HOMFLY-PT polynomial, thus extending our $SU(N)$ HOMFLY-PT categorification [3] as well as the previous results of Soergel [6] and Rouquier [5], who worked out the commutative algebra categorification of the Hecke algebra.

The 2-variable HOMFLY-PT polynomial $P_L(q,t)$ is an invariant of an oriented link L defined by the famous skein relation

(1)



and the normalization condition

$$P_{\text{unknot}}(q,t) = \frac{t - t^{-1}}{q - q^{-1}}. \tag{2}$$

Let α be a diagram of a braid. Its circular closure creates an oriented link diagram L_α. To a diagram L_α we associate a complex of bigraded vector spaces

$$C_\bullet(L_\alpha) = \big(\cdots \xrightarrow{\partial} C_i(L_\alpha) \xrightarrow{\partial} C_{i-1}(L_\alpha) \xrightarrow{\partial} \cdots\big), \quad C_i(L_\alpha) = \bigoplus_{j,k} C_{i,j,k}(L_\alpha) \tag{3}$$

in such a way that, first, if two diagrams L_α and L_β represent the same link (that is, the braids represented by α and β are related by a sequence of Markov moves) then the corresponding complexes are homotopy equivalent: $C_\bullet(L_\alpha) \simeq C_\bullet(L_\beta)$

and, second, the bi-graded Euler characteristic of the complex $C_\bullet(L_\alpha)$ is equal to the HOMFLY-PT polynomial:

$$P_{L_\alpha}(q,t) = \sum_{i,j,k} (-1)^{i+j}\, t^{2j} q^k \dim C_{i,j,k}(L_\alpha). \tag{4}$$

2. A categorification of the braid group algebra

Let us sketch the construction of $C_\bullet(L_\alpha)$ in terms used in [2] in order to establish the relation between the results of [4] and those of Soergel [6] and Rouquier [5]. Fix a set of 'incoming' variables $\mathbf{x} = x_1, \ldots, x_n$ and a set of outgoing variables $\mathbf{y} = y_1, \ldots, y_n$. To an n-strand braid diagram α Rouquier associates a complex of $\mathbb{Q}[\mathbf{x},\mathbf{y}]$-modules

$$\hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}} = \big(\cdots \xrightarrow{\partial} \hat{\alpha}_{i;\mathbf{x},\mathbf{y}} \xrightarrow{\partial} \hat{\alpha}_{i-1;\mathbf{x},\mathbf{y}} \xrightarrow{\partial} \cdots \big). \tag{5}$$

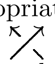
The modules $\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}$ are 'q-graded': $\deg_q x_i = \deg_q y_i = 2$.

The assignment $\alpha \mapsto \hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}}$ satisfies three properties. First, if the diagrams α and β define the same braids, then their complexes are homotopy equivalent: $\hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}} \simeq \hat{\beta}_{\bullet;\mathbf{x},\mathbf{y}}$. Second, to a composition of braids $\alpha\beta$ one associates a $\mathbb{Q}[\mathbf{y}]$ tensor product of modules:

$$\widehat{\alpha\beta}_{\bullet;\mathbf{x},\mathbf{z}} = \hat{\alpha}_{\bullet;\mathbf{y},\mathbf{z}} \otimes_{\mathbb{Q}[\mathbf{y}]} \hat{\beta}_{\bullet;\mathbf{x},\mathbf{y}}. \tag{6}$$

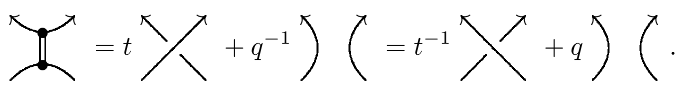
Third, for an n-strand braid α and an m-strand braid β let $\alpha\times\beta$ denote an $(n+m)$-strand braid constructed by placing α and β 'side-by-side'. Then to $\alpha\times\beta$ one associates a $\mathbb{Q}$ tensor product of modules:

$$\widehat{\alpha\times\beta}_{\bullet;\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}'} = \hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}} \otimes_{\mathbb{Q}} \hat{\beta}_{\bullet;\mathbf{x}',\mathbf{y}'}. \tag{7}$$

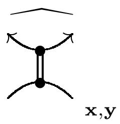
The relations (6) and (7) indicate that in order to construct the complex $\hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}}$, it is sufficient to define it appropriately for the 'identity' 1-strand braid and for two elementary 2-strand braids  and . The choice for the 1-strand braid module complex is determined by the rule (6): it is just the module

$$\widehat{x \longrightarrow y} = \mathbb{Q}[x,y]/(y-x). \tag{8}$$

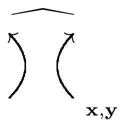
In order to construct the complexes  and  one has to extend the notion of a braid by allowing braided graphs of a special kind. Following Soergel [6], one allows the element  to be a part of a braid diagram. The HOMFLY-PT polynomial of a graph-link diagram, which includes the elements , is defined through the relations

(9) 

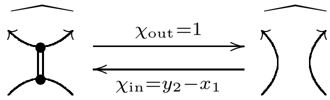
Soergel chose the associated $\mathbb{Q}[\mathbf{x},\mathbf{y}]$-module as

(10)  $= \mathbb{Q}[\mathbf{x},\mathbf{y}]/(y_1+y_2-x_1-x_2, y_1y_2-x_1x_2)\,\{1\}\,,$

where $\{1\}$ denotes the q-degree shift. Since, according to eqs. (8) and (7),

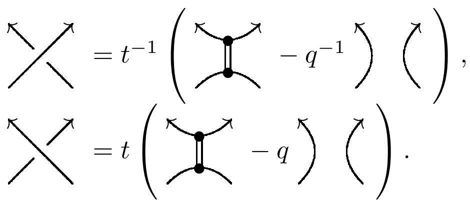
(11)  $= \mathbb{Q}[\mathbf{x},\mathbf{y}]/(y_1-x_1, y_2-x_2),$

there exist two homomorphisms (of lowest possible q-degree)

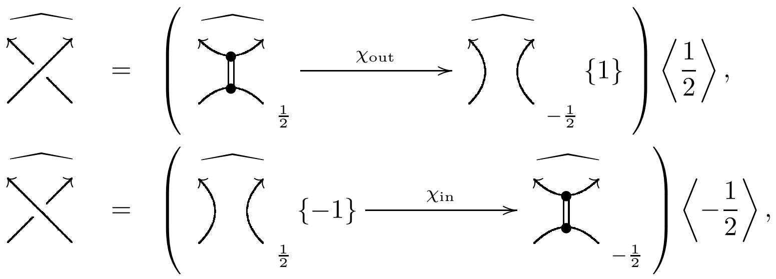
(12) 

$, \quad \deg_q \chi_{\rm in} = \deg_q \chi_{\rm out} = 1.$

The formulas (9) can be used to express the crossings  and  in terms of the elementary graph :

(13) 

Motivated by these relations, Rouquier chose the categorification complexes for elementary braids as

(14), (15) 

where $\langle 1\rangle$ denotes a shift of the t-degree. Note that the formula (4) indicates that t-degree is of homological nature, hence total homological degrees of the modules in eqs. (14) and (15) are integer despite half-integer degree shifts.

3. Braid closure and Hochschild homology

Now it remains to turn the braid complex $\hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}}$ (constructed by taking the tensor products (6) and (7) of the elementary complexes (14) and (15)) into a link complex $C_\bullet(L_\alpha)$. The idea is to apply the Hochschild homology to individual modules of the complex (5) (considered as bimodules over $\mathbb{Q}[\mathbf{x}]$). For an individual module $\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}$, its Hochschild homology is defined as torsion

$$\mathrm{HH}_\bullet(\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}) = \mathrm{Tor}_\bullet(\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}, \mathbb{1}_{\mathbf{x},\mathbf{y}}), \quad \text{where} \quad \mathbb{1}_{\mathbf{x},\mathbf{y}} = \mathbb{Q}[\mathbf{x},\mathbf{y}]/(y_1 - x_1, \ldots, y_n - x_n). \tag{16}$$

In other words, one can start with a Koszul resolution of the module $\mathbb{1}_{\mathbf{x},\mathbf{y}}$

$$\mathbb{1}^{\mathrm{free}}_{\mathbf{x},\mathbf{y}} = \bigotimes_{i=1}^{n} \left(\mathbb{Q}[\mathbf{x},\mathbf{y}]_1 \xrightarrow{y_i - x_i} \mathbb{Q}[\mathbf{x},\mathbf{y}]_0 \right) \tag{17}$$

and take the δ-homology of its tensor product with $\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}$:

$$\mathrm{HH}_\bullet(\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}) = \mathrm{H}^\delta_\bullet\left(\hat{\alpha}_{i;\mathbf{x},\mathbf{y}} \otimes_{\mathbb{Q}[\mathbf{x},\mathbf{y}]} \mathbb{1}^{\mathrm{free}}_{\mathbf{x},\mathbf{y}}\right), \tag{18}$$

where δ is the total differential of the Koszul complex (17). The q-degree shifts of the resolution modules have to be arranged in such a way that $\deg_q \delta = 2$. The Hochschild homology degree translates into t-degree (that is, it becomes the index j of $C_{i,j,k}(L_\alpha)$).

Now we define

$$\begin{aligned} C_\bullet(L_\alpha) &= \mathrm{HH}(\hat{\alpha}_{\bullet;\mathbf{x},\mathbf{y}})[-n/2]\langle n/2\rangle\{-n\} \\ &= \left(\cdots \xrightarrow{\partial} \mathrm{HH}(\hat{\alpha}_{i;\mathbf{x},\mathbf{y}}) \xrightarrow{\partial} \mathrm{HH}(\hat{\alpha}_{i-1;\mathbf{x},\mathbf{y}}) \xrightarrow{\partial} \cdots\right), \end{aligned} \tag{19}$$

where [1] denotes the shift of the homological degree of the categorification complex (3). The simplest case of this formula is a categorification complex for the unknot which, up to a degree shift, is the Hochschild homology of the algebra $\mathbb{Q}[x]$. It is easy to verify that its graded Euler characteristic is precisely (2).

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Cohomology Theories of Frobenius Algebras and Applications

Masahico Saito

Frobenius algebras are used for Khovanov homology [7] in relation to $2D$-TQFT. The relation between categorifications of knot polynomials to Hochschild cohomology of associative algebras was pointed out in [11], and further used in [8]. This is one of the situations that make it interesting to explore cohomology theories of Frobenius algebras in analogy with Hochschild cohomology of algebras and bialgebras.

A series of work [2, 3, 4] gave a unified view on self-distributive operations and their cohomology theories in coalgebra category, that include quandles, Lie algebras and adjoint maps of Hopf algebras. These are related by constructions of low dimensional cohomology theories by deformations and graph diagrams. Following these ideas, we define cohomology theories for Frobenius algebras.

We construct chain complexes [1] for Frobenius algebras from deformations of associativity, coassociativity and compatibility of multiplications and comultiplications of Frobenius algebras, also using graph diagrams and polyhedrons. The 2-cocycles are explicitly computed, and used to find deformations of R-matrices. Variations of differentials will be discussed.

We expand this view of self-distributivity to state-sum invariants of 3-manifolds constructed by triangulations, Heegaard splittings and framed links, including those found in [9, 10] for example, and make direct connections between manifold structures and self-distributive structures. For example, Hopf algebra invariants [10] assign the adjoint map to each crossing of a knot diagram. On the other hand, the multiplication of quantum double is constructed from the adjoint map and gives rise to the R-matrix used in [9]. Self-distributive operations are visualized from manifold structures in each context. Methods used for quandle cocycle invariants are expected to be applied to other cocycle invariants, for example those for 4-manifolds studied in [6].

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A link invariant from SO(3) representations

Nikolai Saveliev
(joint work with Eric Harper)

In 1992, Xiao–Song Lin [2] introduced an invariant $h(K)$ of knots $K \subset S^3$ by counting irreducible representations $\pi_1(S^3 - K) \longrightarrow SU(2)$ sending the meridians of K to trace zero matrices. He computed his invariant using the skein model and discovered that

$$h(K) = \frac{1}{2}\,\mathrm{sign}(K).$$

This somewhat mysterious equality has since been generalized in several directions, see for instance Herald [1], and found a number of applications in equivariant gauge theory and other areas.

In the spirit of Lin's construction, we introduce an invariant $h(L)$ of two–component links $L \subset S^3$, in any of the following three equivalent ways, compare with [4]:

- count representations $\pi_1(S^3 - L) \longrightarrow SO(3)$ that do not lift to $SU(2)$ representations;
- let T be a tunnel in $S^3 - L$ connecting the two link components, then count representations $\pi_1(S^3 - (L \cup T)) \longrightarrow SU(2)$ that send the meridian of T to -1;
- count gauge equivalence classes of flat connections in the $SO(3)$ bundle over $S^3 - L$ with the non–trivial second Stiefel–Whitney class.

We prove that the invariant $h(L)$ is well defined, and compute it using the skein model and Milnor's work [3] on link homotopy. It turns out that, for all links $L = L_1 \cup L_2$ in S^3,

$$h(L) = \pm\,\mathrm{lk}\,(L_1, L_2).$$

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Patterns in odd Khovanov homology

Alexander Shumakovitch

In this talk we discuss properties of an odd version of the Khovanov homology [2] that was recently introduced by Ozsváth, Rasmussen and Szabó [6]. The odd Khovanov homology equals the original (even) one modulo 2 and, in particular, categorifies the same Jones polynomial. In fact, the corresponding chain complexes are isomorphic as free bigraded $\mathbb{Z}$-modules and their differentials are only different by signs. On the other hand, the resulting homology theories often have drastically different properties.

The definition of the odd Khovanov homology is motivated by a paper by Ozsváth and Szabó [7] where they showed that for every link L there exists a spectral sequence that has the reduced Khovanov homology of L over $\mathbb{Z}_2$ as its second term and converges to the Heegaard-Floer homology (over $\mathbb{Z}_2$) of the double branched cover of S^3 along L. It is conjectured that a similar spectral sequence exists for the odd Khovanov homology over $\mathbb{Z}$.

Most of the experimental observations about the odd Khovanov homology that are discussed below, were obtained using KhoHo, a program by the author to compute and study the Khovanov homology.

We start by listing properties that are the same for both even and odd Khovanov homology. First of all, for a non-split alternating link both of them are completely determined by the Jones polynomial and signature of the link. They both satisfy the same long exact sequence. The odd Khovanov homology over $\mathbb{Z}$ often behaves similar to the even one over $\mathbb{Z}_2$. It was proved by the author [8] that the Khovanov homology over $\mathbb{Z}_2$ equals two copies of the reduced Khovanov homology with an appropriate grading shift. The same is true [6] for the odd Khovanov homology over $\mathbb{Z}$. This implies that it is enough to consider the reduced version of the odd Khovanov homology only.

This is where the similarities end though. While the non-reduced even Khovanov homology has many torsion factors, most of which have even order (the first known example of a knot with odd torsion in the original Khovanov homology, the (5,6)-torus knot, has 24 crossings), and the first knot with torsion in the reduced even Khovanov homology has 13 crossings, the odd Khovanov homology has plenty of torsion of all orders. The most common one is 2- and 3-torsion, but other torsion orders appear frequently as well.

Odd Khovanov homology is much better at detecting quasi-alternating knots [4] than the even one. The class of quasi-alternating knots and links is defined recurrently as the smallest one that contains the unknot and has a property that if a link L has a plane diagram D such that two Kauffman resolutions of this diagram at one crossings represent two links L_0 and L_1 that are both quasi-alternating and, moreover, $\det(L) = \det(L_0) + \det(L_1)$, then L is quasi-alternating as well. Because of the recurrent definition, it is often highly non-trivial to show that a given link is quasi-alternating. It is equally challenging to show that it is not.

To proof the latter, one usually employs the fact that quasi-alternating knots have homologically thin Khovanov homology over $\mathbb{Z}$ and Knot Floer homology over $\mathbb{Z}_2$ (see [4]). The same also can be shown for the odd Khovanov homology.

After the work of Champanerkar and Kofman [1], there were only two knots, 9_{46} and 10_{140} in Rolfsen notation, for which it was not known whether they are quasi-alternating or not. Both of them have homologically thin Khovanov and Knot Floer homology. On the other hands, computations with KhoHo have shown that these knots have homologically thick odd Khovanov homology. More precisely, the free parts of their homology are located along one diagonal, while there is a single finite cyclic homology group of order 3 that is located outside of that diagonal. It is worth mentioning that the knots 9_{46} and 10_{140} are $(3, 3, -3)$- and $(3, 4, -3)$-pretzel knots respectively. As it turns out, $(n, n, -n)$- and $(n, n+1, -n)$-pretzel links for $n \leq 6$ all have torsion of order n. This suggest a certain n-fold symmetry on the chain complex for the odd Khovanov homology for these pretzel links that cannot be explained by the construction.

Another application of the odd Khovanov homology is in finding bounds for the Thurston-Bennequin number of knots. It was shown by Ng [3] that the Khovanov homology can be used to provide an upper bound for this number. This bound is often better than those that were known before. There are only two knots with up to 13 crossings for which the Khovanov homology bound on the Thurston-Bennequin number is worse than the one coming from the Kauffman polynomial. As it turns out, the odd Khovanov homology bound is equally good than the Kauffman one for these two knots. This means that this bound is the best among all currently known ones for all knots with at most 13 crossings.

Finally, it turns out that the odd Khovanov homology might have only torsion in homological grading 0. The first such knot has 10 crossings. There is one more with 11 crossings and 8 more with 12. This is very surprising, since the even Khovanov homology must have rank of at least 2 in homological grading 0. What is even more interesting is that such knots appear to have special properties, namely being transversely non-simple.

A (topological) knot type is said to be transversely non-simple, if it has two transverse representatives with respect to the standard contact structure on $\mathbb{R}^3$ that are different as transverse knots but have the same self-linking number. The first example of a transversely non-simple knot, a (2,3)-cable of the trefoil, was found by Etnyre and Honda in 2005. More examples were found by Birman and Menasco in 2006 and later by Ng, Ozsváth and Thurston [5] using the Knot Floer homology. Comparison with [5] and personal communication with Lenhard Ng show that many if not all of the knots with only torsion in homological degree 0 are transversely non-simple. This observation has no explanation at the moment.

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Quantizations of Character Varieties and Quantum Knot Invariants

Adam S. Sikora

Witten introduced (in a mathematically non-rigorous way) a family of invariants of 3-manifolds defined by path integrals with Chern-Simons action. These invariants fit into the framework of the topological quantum field theory which associates with each semi-simple Lie group G, a positive integer r, and a closed orientable surface F, a Hilbert space $V_{G,r}(F)$, quantizing the moduli space, $X_G(F)$, of flat G-connections over F. (If G is a complex reductive algebraic group then $X_G(F)$ is the G-character variety of $\pi_1(F)$.)

Witten's quantum invariants of a 3-manifold M with boundary F are vectors $I_{G,r}(M) \in V_{G,r}(F)$, for $r = 2, 3, ...$ Although they have been rigorously defined by Reshetikhin and Turaev, [RT], a (mathematically rigorous) geometric interpretation of $I_{G,r}(M)$ in the context of moduli spaces of flat connections is still missing. Weitsman and Jeffrey, [W, JW], suggested that $I_{G,r}(M)$ can be defined via a quantization of $X_G(\partial M)$ with polarization given by a foliation of $M_G(\partial M)$ by Lagrangian submanifolds, one of which is the image of $X_G(M)$ in $X_G(\partial M)$. (This approach is based on the fact that $X_G(\partial M)$ is a symplectic manifold, [Go1], and the image of $X_G(M)$ in $X_G(\partial M)$ is an isotropic submanifold, [Go2].) Motivated by their approach we propose the following conjecture:

Conjecture 6. The Witten-Reshetikhin-Turaev quantum invariants, $I_{G,r}(M) \in V_{G,r}(F)$, for all r determine the image of $X_G(M)$ in $X_G(\partial M)$.

If true, the above conjecture provides a very strong connection between the topology of 3-manifolds and their quantum invariants. G-quantum invariants of 3-manifolds can be generalized to invariants of oriented framed links in 3-manifolds labeled by representations of G. It is easy to prove that the following statement similar to Garoufalidis' AJ-conjecture, [Ga2], is a special case of Conjecture 6.

Conjecture 7. For every complex reductive G, the quantum G-invariants of a knot $K \subset S^3$ determine the image of $X_G(\pi_1(S^3 \setminus K))$ in $X_G(torus)$. (Equivalently, if knots K_1, K_2 have the same G-quantum invariants then the images of $X_G(\pi_1(S^3 \setminus K_i))$ in $X_G(torus)$ coincide.)

As the relation between quantum invariants and character varieties in the above conjecture is implicit only, our goal is to make it explicit, by building upon the work of Garoufalidis and Le on q-holonomicity of quantum invariants.

Let $WRT_{\mathfrak{g},V}(L)$ be the Witten-Reshetikhin-Turaev $U_q(\mathfrak{g})$-quantum invariant of a link $L \subset S^3$ whose all components are labeled by a representation V of $\mathfrak{g}$. It is a polynomial in $q^{\pm 1/D(\mathfrak{g})}$, where $D(\mathfrak{g})$ is the determinant of the Cartan matrix of $\mathfrak{g}$. Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and fixed positive roots of $\mathfrak{g}$, each finite dimensional irreducible representation of $\mathfrak{g}$ is determined by its highest weight. Therefore, if we denote the $\mathfrak{g}$-representation with the highest weight λ by $V(\lambda)$ then $\lambda \longrightarrow WRT_{\mathfrak{g},V(\lambda)}(L)$ is a function defined on the set of all dominant weights. We extend this function to the entire weight lattice of $\mathfrak{g}$, $\Lambda_{\mathfrak{g}}$ as follows:

Each $\lambda \in \Lambda_{\mathfrak{g}}$ defines the Verma module $M(\lambda)$ which is an infinite-dimensional indecomposable $\mathfrak{g}$-module of highest weight λ such that each indecomposable $\mathfrak{g}$-module of highest weight λ is a quotient of $M(\lambda)$. Rozanski (for $sl(2)$) and Le (for all $\mathfrak{g}$) observed that Reshetikhin-Turaev construction of quantum invariants for knots (but not links) makes sense for all Verma modules of $\mathfrak{g}$. Furthermore,

$$WRT_{\mathfrak{g},M(\lambda)}(K) = WRT_{\mathfrak{g},V(\lambda)}(K), \tag{1}$$

for all dominant weights λ. Let

$$J_{\mathfrak{g},K} : \Lambda_{\mathfrak{g}} \longrightarrow \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}], \quad J_{\mathfrak{g},K}(\lambda) = WRT_{\mathfrak{g},M(\lambda-\rho)}(K), \tag{2}$$

where ρ is the half-sum of positive roots of $\mathfrak{g}$. We call it *the* $\mathfrak{g}$*-Witten-Reshetikhin-Turaev function of* K. Due to the shift by ρ in (2) $J_{\mathfrak{g},K}$ is equivariant with respect to the Weyl group action on the weight lattice of $\mathfrak{g}$:

Proposition 8. For every element w of the Weyl group,

$$J_{\mathfrak{g},K}(w \cdot \lambda) = sgn(w) \cdot J_{\mathfrak{g},K}(\lambda),$$

where $sgn(w) = \pm 1$ is the sign of w.

Example 9. For $\mathfrak{g} = sl(2)$, $\rho = 1 \in \Lambda_{\mathfrak{g}} = \mathbb{Z}$. $J_{\mathfrak{g},K}(0) = 0$, $J_{\mathfrak{g},K}(1) = 1$, and $J_{\mathfrak{g},K}(2)$ is the Jones polynomial of K. More generally, $J_{\mathfrak{g},K}(n)$ is the Jones polynomial of K colored by the n-dimensional representation for $n \geq 1$ and $J_{\mathfrak{g},K}(n) = -J_{\mathfrak{g},K}(-n)$ for negative n.

We are going to argue that $J_{\mathfrak{g},K}$ encodes the $\mathfrak{g}$-quantum invariants of K in a form which is very useful in the context of Conjecture 7. For that consider the $\mathbb{C}[q^{\pm 1/D(\mathfrak{g})}]$-vector space $F(\Lambda_g, \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}])$ of all $\mathbb{C}[q^{\pm 1/D(\mathfrak{g})}]$-valued functions on $\Lambda_{\mathfrak{g}}$ and two families of operators on it:

$$E_\alpha f(\beta) = f(\alpha + \beta), \quad Q_\alpha f(\beta) = q^{(\alpha,\beta)} f(\beta),$$

for all $\alpha, \beta \in \Lambda_{\mathfrak{g}}$. Let $\mathbb{A}_g$ be the algebra of $\mathbb{C}[q^{\pm 1/D(\mathfrak{g})}]$-linear endomorphisms of $F(\Lambda_g, \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}])$ generated by E_α's and Q_α's for $\alpha \in \Lambda_{\mathfrak{g}}$. $\mathbb{A}_{sl(2)}$ is the q-torus algebra of [GL].

Theorem 10. (1) For every complex reductive algebraic group G and its Lie algebra $\mathfrak{g}$, $\mathbb{A}_g^W$ is a deformation-quantization of $X_G(\text{torus})$.

(2) For every classical group, $G = GL(n,\mathbb{C}), SL(n,\mathbb{C}), O(n,\mathbb{C}), Sp(n,\mathbb{C})$, this deformation-quantization is in the direction of Goldman bracket.

For any $f : \Lambda_{\mathfrak{g}} \longrightarrow \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}]$ the set

$$I_f = \{P \in \mathbb{A}_{\mathfrak{g}} : Pf = 0\} \subset \mathbb{A}_{\mathfrak{g}}$$

is a left-sided ideal in $\mathbb{A}_{\mathfrak{g}}$ called the *recursive ideal* of f, c.f. [Ga1]. This term reflects the fact that each element of I_f represents a recursive relation for f. By adapting the definition of [GL], one can define the q-holonomicity of f and prove it for $J_{\mathfrak{g},K}$ for $\mathfrak{g} \neq G_2$. (Roughly speaking, function f is q-holonomic iff I_f is "as large as possible".)

The Weyl group of $\mathfrak{g}$ acts on $F(\Lambda_g, \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}])$ by $w \cdot f(\alpha) = f(w^{-1} \cdot \alpha)$. (The inverse is needed to make sure that this is a left action.) Additionally, W acts on $\mathbb{A}_g$ via $w \cdot E_\alpha = E_{w\cdot\alpha}$, $w \cdot Q_\alpha = Q_{w\cdot\alpha}$, and this action is compatible with the W-action on $F(\Lambda_g, \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}])$. We call the W-invariant part of the recursive ideal, $I^W_{\mathfrak{g},K} \triangleleft \mathbb{A}^W_{\mathfrak{g}}$, the invariant $\mathfrak{g}$-recursive ideal of K.

Conjecture 11. For every $\mathfrak{g}$ and K, $J_{\mathfrak{g},K}$ is uniquely determined among W-equivariant functions (i.e. functions satisfying the statement of Proposition 8) by a finite number of its values together with the recursive relations of $I^W_{\mathfrak{g},K}$.

In [S] we relate the ideal $I^W_{\mathfrak{g},K} \triangleleft \mathbb{A}^W_{\mathfrak{g}}$ to the topology of $S^3 \setminus K$ in two different ways: an algebraic one (using character varieties, related to Conjecture 7) and a "quantum topological" way (using skein modules of $S^3 \setminus K$). Here is a summary of the first approach:

By Theorem 10, there is a $\mathbb{C}$-algebra homomorphism

$$\varepsilon : \mathbb{A}^W_{\mathfrak{g}} \longrightarrow \mathbb{C}[X_G(\mathbb{Z}^2)] \tag{3}$$

given by evaluation $q = 1$.

Given a knot $K \subset S^3$, let M_K be the compactification of $S^3 \setminus K$ with boundary torus, $\partial M_k = T$. The embedding $\partial M_K \hookrightarrow M_K$ defines a homomorphism $\phi_K : \mathbb{C}[X_G(T)] \longrightarrow \mathbb{C}[X_G(M_K)]$ whose kernel we denote by $A_{G,K}$. We call it the A_G-ideal of K. The reason for this name is that the A_G-ideal of K for $G = SL(2,\mathbb{C})$ determines the A-polynomial of K of [CCGLS].

Conjecture 12. The zero set of $\varepsilon(I^W_{\mathfrak{g},K}) \triangleleft \mathbb{C}[X_G(T)]$ is the closure of the image of $X_G(M_K) \longrightarrow X_G(T)$. Equivalently,

$$\sqrt{\varepsilon(I^W_{\mathfrak{g},K})} = A_{G,K},$$

where $\sqrt{\cdot}$ denotes the nil-radical.

We can prove some simple special cases of this conjecture. We can also prove that Conjecture 12 for a given K and $\mathfrak{g}$ implies the AJ conjecture (Conjecture 2 and Question 1 of [Ga2]). Nonetheless, Conjecture 12 appears stronger than the AJ conjecture.

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Twisted acyclicity of circle and link signatures

Oleg Ya. Viro

Introduction The goal of this talk is to simplify and generalize a part of the classical link theory based on various signatures of links (Murasugi [5],[6], Tristram [11], Levine [2] [3] signatures). This part is known for its relations to topology of 4-dimensional manifolds, see [11], [1], and applications in topology of real algebraic curves [7].

Similarity of the signatures to the new invariants [9], [8], which were defined in the new frameworks of link homology theories and had spectacular applications [9], [4], [10] to problems on classical link cobordisms, gives a new reason to revisit the old theory.

There are two ways to introduce Murasugi-Tristram-Levine signatures: the original 3-dimensional, via Seifert surface and Seifert form, and 4-dimensional, via the intersection form of the cyclic coverings of 4-ball branched over surfaces. Here the latter approach is developed. Technically the work is based on a systematic use of intersection forms in the twisted homology of the link complements and other auxiliary spaces. Only the simplest kinds of twisted homology is used, the one with coefficients in $\mathbb{C}$.

This approach allows us to generalize the classical links to collections of transversal to each other oriented submanifolds of codimension two.

Twisted homology In this paper by twisted homology we mean homology with coefficients in local system, which is a $\mathbb{C}$-bundle with a fixed flat connection, that is an operation of parallel transport. A local coefficient system of this kind is defined by the monodromy representation $\pi_1(X) \longrightarrow \mathbb{C}^\times$. The theory is parallel to the untwisted homology theory, but H_0 may be *trivial*.

Example 1. $X = S^1$, with *non-trivial* monodromy $\pi_1(X) = \mathbb{Z} \longrightarrow \mathbb{C}^\times$, say $\mu : 1 \mapsto a \neq 1$. Then $\partial\sigma_1 = (a-1)\sigma_0 \neq 0$, and $H_1(X; \mathbb{C}_\mu) = H_0(X; \mathbb{C}_\mu) = 0$.

Generalization. $X = S^1 \times Y$, $\pi_1(X) = \mathbb{Z} \times \pi_1(Y)$. The monodromy is $\varphi \times \psi : \mathbb{Z} \times \pi_1(Y) \longrightarrow \mathbb{C}^\times$. Then $C_*(X; \mathbb{C}_{\varphi\times\psi}) = C_*(S^1; \mathbb{C}_\varphi) \otimes C_*(Y; \mathbb{C}_\psi)$ and $H_*(X; \mathbb{C}_{\varphi\times\psi}) = H_*(S^1; \mathbb{C}_\phi) \otimes H_*(Y; \mathbb{C}_\psi) = \mathbf{0} \otimes H_*(Y; \mathbb{C}_\psi) = \mathbf{0}$.

Furthermore, the same holds true for any locally trivial fibration with fiber S^1 and non-trivial monodromy along the fiber. Pieces of a space of this kind are **invisible for twisted homology.**

Duality Let X be a connected oriented compact manifold of dimension n. Then $H_n(X, \partial X) = \mathbb{Z}, H_n(X, \partial X; \mathbb{C}) = \mathbb{C}$, an orientation of X is a generator of $H_n(X, \partial X)$.

Pairing of local coefficient systems: $\mathbb{C}_\mu \otimes \mathbb{C}_{\mu^{-1}} = \mathbb{C}$ induces a non-singular bilinear intersection pairing $H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \longrightarrow \mathbb{C}$.

Representation $\mu : \pi_1(X) \longrightarrow \mathbb{C}^\times$ is unitary if $\mu^{-1} = \overline{\mu}$ pointwise: $(\mu(\alpha))^{-1} = \overline{\mu(\alpha)}$ for any $\alpha \in \pi_1(X)$. If μ is unitary, then the conjugation induces a semilinear bijection $H_q(X; \mathbb{C}_\mu) \longrightarrow H_q(X; \mathbb{C}_{\overline{\mu}}) = H_q(X; \mathbb{C}_{\mu^{-1}})$.

In the case of oriented compact n-dimensional manifold, it turns a non-singular bilinear intersection pairing $H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_{\mu^{-1}}) \longrightarrow \mathbb{C}$ into a non-singular sesqui-linear intersection pairing $H_p(X, \partial X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \longrightarrow \mathbb{C}$, composed with relativization, it gives $H_p(X; \mathbb{C}_\mu) \otimes H_{n-p}(X; \mathbb{C}_\mu) \longrightarrow \mathbb{C}$.

In the middle dimension this is a *Hermitian* or *skew-Hermitian* form. If $\partial X = \varnothing$, or ∂X is fibered with fibre S^1, then the intersection pairing is non-singular.

Signatures Let M be a compact oriented $2n$-dimensional manifold, $L_1, \dots, L_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L_i = L_i \cap \partial M$, let $L = \cup_i L_i$. Let $\mu \in \operatorname{Hom}(H_1(M \smallsetminus L), \mathbb{C}^\times)$, and $\mathbb{C}_\mu$ be the corresponding local coefficient system on $M \smallsetminus L$. If n is even, then denote by $\sigma_\mu(M \smallsetminus L)$ the signature of the Hermitian intersection form in $H_n(M \smallsetminus L; \mathbb{C}_\mu)$. If n is odd, then denote by $\sigma_\mu(M \smallsetminus L)$ the signature of the Hermitian form obtained from the skew-Hermitian intersection form in $H_n(M \smallsetminus L; \mathbb{C}_\mu)$ multiplied by $\sqrt{-1}$.

2. If W is an oriented compact manifold, $M = \partial W$, and $F_i \subset W$ are compact oriented transversal to each other, $L_i = \partial F_i$, then $\sigma_\mu(M \smallsetminus L) = 0$.

3. Let M' be another compact oriented $2n$-dimensional manifold, $L'_1, \dots, L'_k$ its oriented compact $(2n-2)$-dimensional submanifolds transversal to each other with $\partial L'_i = L'_i \cap \partial M'$, and $L' = \cup_i L'_i$. Let $M \cap M' = \partial M \cap \partial M'$ be a compact manifold of dimension $2n-1$ and the orientations induced on $M \cap M'$ from M and M' are opposite to each other.

Let $\mu' \in \operatorname{Hom}(H_1(M' \smallsetminus L'), \mathbb{C}^\times)$, and $\mathbb{C}_{\mu'}$ be the corresponding local coefficient system on $M' \smallsetminus L'$ and $\mathbb{C}_\mu|_{M\cap M'} = \mathbb{C}_{\mu'}|_{M\cap M'}$. Assume that $\partial(M \cap M')$ is fibered with fibers circles on which μ is non-trivial. Then $\sigma_{\mu\cup\mu'}((M\cup M')\smallsetminus(L\cup L'))=\sigma_\mu(M\smallsetminus L)+\sigma_{\mu'}(M'\smallsetminus L')$.

Corollary 4. *$\sigma_\mu(M\smallsetminus L)$ is invariant with respect to cobordisms of $(M;L_1,\dots,L_k;\mu)$.*

Link signatures Let $L=L_1\cup\dots\cup L_m\subset S^3$ be a classical link, $\zeta_i\in\mathbb{C}$, $|\zeta_i|=1$, $\zeta=(\zeta_1,\dots,\zeta_m)\in(S^1)^m$ and $\mu:\pi_1(S^3\smallsetminus L)\longrightarrow\mathbb{C}^\times$ takes a meridian of L_i to ζ_i. Let $F_i\subset D^4$ be smooth oriented surfaces transversal to each other with $\partial F_i=F_i\cap\partial D^4=L_i$. Extend μ to $D^4\smallsetminus\cup_i F_i$.

In $H_2(D^4\smallsetminus\cup_i F_i;\mathbb{C}_\mu)$ there is a Hermitian intersection form.

Theorem 5. *Its signature $\sigma_\zeta(L)$ does not depend on $F_1,\dots,F_m$.*

Proof. Any F_i' with $\partial F_i'=F_i'\cap\partial D^4=L_i$ is cobordant to F_i. The cobordisms $W_i\subset D^4\times I$ can be made pairwise transversal. They define a cobordism $D^4\times I\smallsetminus\cup_i N(W_i)$ between $D^4\smallsetminus\cup_i N(F_i)$ and $D^4\smallsetminus\cup_i N(F_i')$. The boundary of the cobordism consists of $D^4\smallsetminus\cup_i N(F_i)$, $D^4\smallsetminus\cup_i N(F_i')$ and a *homologically negligible* part $\partial(N(\cup_i W_i))$, the boundary of a regular neighborhood of the cobordism $\cup_i W_i$ between $\cup_i F_i$ and $\cup_i F_i$. Hence, $\sigma(D^4\smallsetminus\cup_i F_i)=\sigma(D^4\smallsetminus\cup_i F_i')$.

The same arguments work for $L=\cup_{i=1}^m L_i$, where L_i are oriented submanifolds of codimension 2 of S^{2n-1} transversal to each other, and F_i are submanifolds of D^{2n} transversal to each other.

If n is odd, then the intersection form in $H_n(D^{2n}\smallsetminus\cup_i F_i;\mathbb{C}_\mu)$ is skew-Hermitian. Multiply it by $i=\sqrt{-1}$ and denote the signature of the Hermitian form by $\sigma_\zeta(L)$.

Digression on higher dimensional links. There is a spectrum of objects considered generalizations of classical links. The closest higher-dimensional counterpart of classical links are pairs (S^n,L), where L is a collection of its disjoint smooth submanifolds diffeomorphic to S^{n-2}. Then the restrictions to submanifolds are weakened, but the submanifolds are usually required to be *disjoint*.

I suggest to *allow transversal intersections* of the submanifolds. Here are a few arguments in favor of this. In the classical dimension it is easy to be disjoint. Generic submanifolds of codimension 2 in a manifold of dimension >3 intersect. A link of an algebraic hypersurface $H\subset\mathbb{C}^n$ with $n\geq 3$ cannot be a union of disjoint submanifolds.

Span inequalities Let $L_1,\dots,L_m\subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L=L_1\cup\dots\cup L_m$. Let $\zeta_i\in\mathbb{C}$ be algebraic numbers with $|\zeta_i|=1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i)=0$. Suppose prime number p divides $f_i(1)$ for $i=1,\dots,m$. Let $\mu:\pi_1(S^{2n-1}\smallsetminus L)\longrightarrow\mathbb{C}^\times$ take a meridian of L_i to ζ_i.

Let $F_i\subset D^{2n}$ be oriented compact smooth submanifolds transversal to each other, with $\partial F_i=F_i\cap\partial D^{2n}=L_i$. Put $F=\cup_i F_i$. Extend $\mu:\pi_1(S^{2n-1}\smallsetminus L)\longrightarrow\mathbb{C}^\times$ to $\mu:\pi_1(D^{2n}\smallsetminus F)\longrightarrow\mathbb{C}^\times$.

Then $|\sigma_\zeta(L)|\leq\dim H_{n-1}(F;\mathbb{Z}/p)$. Indeed, $|\sigma_\zeta(L)|\leq\dim H_n(D^{2n}\smallsetminus F;\mathbb{C}_\mu)\leq H_n(D^{2n}\smallsetminus F;\mathbb{Z}/p)=\dim H_{n-1}(F;\mathbb{Z}/p)$. Similarly one can prove:

Theorem 6. *For any integer r with $0 \le r \le \frac{n}{2}$*
$|\sigma_\zeta(L)| + \sum_{s=0}^{2r}(-1)^s \dim H_{r-1-s}(S^{2n-1} \smallsetminus L; \mathbb{C}_\zeta)$
$\le \sum_{s=0}^{2r}(-1)^s \dim H_{n-1+s}(F, L; \mathbb{Z}/p) + \sum_{s=0}^{2r}(-1)^s \dim H_{n-2-s}(F, L; \mathbb{Z}/p)$

Put $n_\zeta^r(L) = \sum_{s=0}^{2r}(-1)^s \dim H_{n+s}(S^{2n-1} \smallsetminus \cup_{i=1}^m L_i; \mathbb{C}_\mu)$. This allows us to rewrite the inequlity of Theorem 6 as follows:
$|\sigma_\zeta(L)| + n_\zeta^r(L)$
$\le \sum_{s=0}^{2r}(-1)^s \dim H_{n-1+s}(F, L; \mathbb{Z}/p) + \sum_{s=0}^{2r}(-1)^s \dim H_{n-2-s}(F, L; \mathbb{Z}/p)$
In particular, $|\sigma_\zeta(L)| + n_\zeta^0(L) \le \dim H_n(F, L; \mathbb{Z}/p) + \dim H_{n-1}(F, L; \mathbb{Z}/p)$.

Slice inequalities Again, let $L_1, \dots, L_m \subset S^{2n-1}$ be smooth oriented transversal to each other submanifolds of codimension 2, $L = L_1 \cup \dots \cup L_m$. Let $\zeta_i \in \mathbb{C}$ be algebraic numbers with $|\zeta_i| = 1$, and f_i be irreducible integer polynomials with $f_i(\zeta_i) = 0$. Suppose prime number p divides $f_i(1)$ for $i = 1, \dots, m$. Let $\mu : \pi_1(S^{2n-1} \smallsetminus L) \longrightarrow \mathbb{C}^\times$ takes a meridian of L_i to ζ_i.

Theorem 7. *Let $\Lambda_i \subset S^{2n}$ be oriented closed smooth submanifolds transversal to each other and to S^{2n-1}, with $\partial\Lambda_i \cap S^{2n-1} = L_i$. Put $\Lambda = \cup_i \Lambda_i$. Extend $\mu : \pi_1(S^{2n-1} \smallsetminus L) \longrightarrow \mathbb{C}^\times$ to $\mu : \pi_1(S^{2n} \smallsetminus \Lambda) \longrightarrow \mathbb{C}^\times$. Then $|\sigma_\zeta(L)| \le \frac{1}{2} \dim H_{n-1}(\Lambda; \mathbb{Z}/p)$*
$|\sigma_\zeta(L)| + n_\zeta^0(L) \le \frac{1}{2} \dim H_{n-1}(\Lambda; \mathbb{Z}/p) + \dim H_{n-2}(\Lambda \smallsetminus L; \mathbb{Z}/p)$ $|\sigma_\zeta(L)| + n_\zeta^r(L) \le$
$\frac{1}{2} \sum_{s=-2r}^{2r}(-1)^s \dim H_{n-1+s}(\Lambda; \mathbb{Z}/p) + \sum_{s=0}^{2r}(-1)^s \dim H_{n-2-s}(\Lambda \smallsetminus L; \mathbb{Z}/p)$

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