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Classical Algebraic Geometry

Organised by
David Eisenbud, Berkeley
Joe Harris, Harvard
Frank-Olaf Schreyer, Saarbrücken
Ravi Vakil, Stanford

June 8th – June 14th, 2008

Abstract. Algebraic geometry studies properties of specific algebraic varieties, on the one hand, and moduli spaces of all varieties of fixed topological type on the other hand. Of special importance is the moduli space of curves, whose properties are subject of ongoing research. The rationality versus general type question of these spaces is of classical and also very modern interest with recent progress presented in the conference. Certain different birational models of the moduli space of curves have an interpretation as moduli spaces of singular curves. The moduli spaces in a more general setting are algebraic stacks. In the conference we learned about a surprisingly simple characterization under which circumstances a stack can be regarded as a scheme. For specific varieties a wide range of questions was addressed, such as normal generation and regularity of ideal sheaves, generalized inequalities of Castelnuovo-de Franchis type, tropical mirror symmetry constructions for Calabi-Yau manifolds, Riemann-Roch theorems for Gromov-Witten theory in the virtual setting, cone of effective cycles and the Hodge conjecture, Frobenius splitting, ampleness criteria on holomorphic symplectic manifolds, and more.

Mathematics Subject Classification (2000): 14xx.

Introduction by the Organisers

The Workshop on Classical Algebraic Geometry, organized by David Eisenbud (Berkeley), Joe Harris (Harvard), Frank-Olaf Schreyer (Saarbrücken) and Ravi Vakil (Stanford), was held June 8th to June 14th. It was attended by about 45 participants from USA, Canada, Japan, Norway, Sweden, UK, Italy, France and Germany, among of them a large number of strong young mathematicians. There were 18 talks with a maximum of four talks per day, which left plenty of time for lots of informal discussions among smaller groups.

A widespread variety of themes in classical algebraic geometry were discussed, many of them with a very modern point of view. Although it is tempting to summarize each of the talks, we limit ourselves to four highlights:

- Mihnea Popa reported on ongoing work with Lazarsfeld and Pareschi, in which they use the Bernstein-Gel'fand-Gel'fand correspondence to obtain very geometric results. The BGG correspondence is between an exterior module and a linear complex of vector bundles on projective space. Popa considers the cohomology algebra of the sheaf of functions, which is naturally an exterior module over the exterior algebra generated by the first cohomology group. Using recent results of Eisenbud, Floystad, and Schreyer, they translate results from the linear complex of vector bundles back to the geometric setting. For example, if X is a compact Kähler manifold with generically finite Albanese map, then the canonical module is generated in degree 0, and has a linear free resolution over the exterior algebra. In the case when X has no higher irrational pencils, the complex on $\mathbb{P}^n$ is actually a linear free resolution of a vector bundle. Its short length imposes, via vector bundle techniques, bounds on the Betti numbers of such manifolds, extending and improving well-known results in the case of surfaces.
- In ideal circumstances, objects in algebraic geometry are parametrized by geometric spaces as we are used to working with them: varieties and schemes. However, even in very simple circumstances, such as curves, this is too much to hope for. Mumford's Geometric Invariant Theory is a good approximation in a precise sense. Then, in 1969, Deligne and Mumford developed what are now known as Deligne-Mumford stacks, a new class of geometric objects that include many important moduli spaces. They have an underlying geometric space, known as the coarse moduli space. However, objects with infinitely many automorphisms (such as vector bundles) don't admit such moduli spaces, and the more general notion of Artin stack is often needed. This is seen by much of the community as a difficult, non-geometric concept. But in Jarod Alper's talk, he introduced the notion of a *good moduli space* for Artin stacks, which is an honest scheme. The definition is remarkably short: a map $\mathcal{X} \to X$ from an Artin stack is a *good moduli space* if it is cohomologically affine (pushforward of quasicoherent sheaves is exact) and if the pushforward of the sheaf of functions on $\mathcal{X}$ is the sheaf of functions on Y. From this simple definition, a number of wonderful properties readily follow, and indeed the theory of GIT comes out as a special case.
- In the last decade, Gavril Farkas of Berlin has revolutionized our understanding of the space of curves of "intermediate" dimension, between ten (where the space ceases to be easilly seen to be unirational) to 24 (where it is known to be general type by work of Eisenbud, Harris, and Mazur). Farkas has turned his attention to the space of covers, the theory of Prym

varieties. Farkas explained his argument that this moduli space is of general type in genus at least 13. His argument is surprisingly classical in nature, by explicitly computing the class of a cleverly chosen effective divisor on the moduli space.

- Sam Payne, a Clay Research Fellow and one of the leaders in the theory of toric varieties, described applications of Frobenius splittings to the geometry of these varieties. Frobenius splittings originated as a positive characteristic construction, but Payne's work is characteristic-independent, and indeed he is motivated by classical questions about Schubert and toric varieties over the complex numbers. Suppose X is a projective toric variety, with an ample line bundle L and a nef line bundle L'. In the 1980's, Oda asked, if X is smooth, is the natural multiplication map $H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L')$ surjective. For example, if the answer is affirmative in the special case where L' is a power of L, then the section ring $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is normal. In the 1990's, Sturmfels conjecture that if X is smooth and $R(X, L)$ is normal then the ideal of relations $I(X, L)$ is generated in degree two. In 1995, Bøgvad, conjectured that if X is smooth then the section ring $R(X, L)$ is normal and Koszul. These conjectures are wide open, starting in dimension three. Nevertheless, a central problem in toric geometry is to understand under what hypotheses $R(X, L)$ is normal or Koszul. Payne explained how Frobenius splitting can give progress on such problems. For example, he showed that if X is diagonally split at some prime p and $L_1, \ldots, L_r$ are nef line bundles on X, then $R(X; L_1, \ldots, L_r) = \bigoplus_{a_1, \ldots, a_r} H^0(X, L_1^{a_1} \otimes \cdots \otimes L_r^{a_r})$ is normal and Koszul. In particular, the analogues of Oda's problem and both conjectures are true in this case, even if X is singular. In order to prove such results and apply them in practice, Payne proved a beautiful combinatorial characterization of diagonally split toric varieties. Using this criterion, he settles a longstanding question in the negative: There is a Schubert variety (of type G_2) that is not diagonally split.

Workshop: Classical Algebraic Geometry

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Abstracts

The Kodaira dimension of the moduli space of Prym varieties

Gavril Farkas

We consider the moduli stack $\mathcal{R}_g$ parametrizing pairs (C, η) where $[C] \in \mathcal{M}_g$ is a smooth curve and $\eta \in \mathrm{Pic}^0(C)[2]$ is a torsion point of order 2 giving rise to an étale double cover of C. We denote by $\pi : \mathcal{R}_g \to \mathcal{M}_g$ the natural projection forgetting the point of order 2 and by $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$ the Prym map given by

$$P(C, \eta) := \mathrm{Ker}\{f_* : \mathrm{Pic}^0(\widetilde{C}) \to \mathrm{Pic}^0(C)\}^0,$$

where $f : \widetilde{C} \to C$ is the étale double covering determined by η. It is known that P is generically injective for $g \geq 7$ (cf. [FS]), hence one can view $\mathcal{R}_g$ as a birational model for the moduli stack of Prym varieties of dimension $g-1$. If $\overline{\mathcal{R}}_g$ denores the normalization of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_g$), then it is known that $\overline{\mathcal{R}}_g$ is isomorphic to the stack of Beauville admissible double covers (cf. [B]), and also to the stack of Prym curves in the sense of [BCF]. It is known that the space $\mathcal{R}_g$ is unirational for $g \leq 6$ (cf. [D]). Verra has recently announced a proof of the unirationality of $\mathcal{R}_7$. The main result (obtained jointly with K. Ludwig) is the following:

Theorem 1. *The moduli space $\overline{\mathcal{R}}_g$ is of general type for all $g > 13, g \neq 15$.*

The strategy of the proof is similar to the one used by Harris and Mumford for proving that $\overline{\mathcal{M}}_g$ is of general type for large g (cf. [HM]). One first computes the canonical class $K_{\overline{\mathcal{R}}_g}$ in terms of the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)$ and then shows that $K_{\overline{R}_g}$ is effective for $g > 13$ by explicitly computing the class of a specific effective divisor on $\overline{\mathcal{R}}_g$ and comparing it to $K_{\overline{\mathcal{R}}_g}$. The divisors we construct are of two types, depending on whether g is even or odd. We also show that for $g \geq 4$ any pluricanonical form on $\overline{\mathcal{R}}_{\mathrm{g,reg}}$ automatically extends to any desingularization. This is a key ingredient in carrying out the program of computing the Kodaira dimension of $\overline{\mathcal{R}}_g$.

In the odd genus case we set $g = 2i + 1$ and consider the vector bundle Q_C defined by the exact sequence

$$0 \longrightarrow Q_C^{\vee} \longrightarrow H^0(K_C) \otimes \mathcal{O}_C \to K_C \longrightarrow 0.$$

(If other words, Q_C is the normal bundle of C embedded in its Jacobian). It is well-known that Q_C is a semi-stable vector bundle of rank $g-1$ on C of slope $\nu(Q_C) = 2 \in \mathbb{Z}$, so it makes sense to look at the theta divisors of its exterior powers. Recall that

$$\Theta_{\wedge^i Q_C} = \{\xi \in \mathrm{Pic}^{g-2i-1}(C) : h^0(C, \wedge^i Q_C \otimes \xi) \geq 1\},$$

and the main result from [FMP] identifies this locus with the difference variety $C_i - C_i \subset \mathrm{Pic}^0(C)$.

Theorem 2. *For $g+2i+1$, the locus E_i consisting of those points $[C,\eta]\in\mathcal{R}_{2i+1}$ such that $\eta\in\Theta_{\wedge^i Q_C}$, is an effective divisor on $\mathcal{R}_{2i+1}$. Its class on $\overline{\mathcal{R}}_{2i+1}$ is given by the formula*

$$E_i \equiv \frac{2}{i}\binom{2i-2}{i-1}\cdot\Big((3i+1)\lambda-\frac{i}{2}\delta_0^u-\frac{2i+1}{4}\delta_0^r-(\text{ higher boundary divisors})\Big).$$

This proves our main result in the odd genus case. The diviors we consider for even genus are of Koszul type in the sense of [F].

Theorem 3. *For $g=2i+6$, the locus D_i of those $[C,\eta]\in\mathcal{R}_{2i+6}$ such that the Koszul cohomology group $K_{i,2}(C,K_C+\eta)$ does not vanish (or equivalently, $(C,K_C+\eta)$ fails the Green-Lazarsfeld property (N_i)), is a virtual divisor on $\mathcal{R}_{2i+6}$. Its class on $\overline{\mathcal{R}}_{2i+6}$ is given by the formula:*

$$D_i\equiv\frac{1}{2}\binom{2i+2}{i}\Big(\frac{6(2i+7)}{i+3}\lambda-2\delta_0^u-3\delta_0^r-\cdots\Big).$$

In both Theorems 2 and 3, $\lambda\in\mathrm{Pic}(\overline{\mathcal{R}}_g)$ denotes the Hodge class and $\pi^*(\delta_0)=\delta_0^u+2\delta_0^r$ (that is δ_0^r is the ramification divisor of π whereas δ_0^u is the unramified part of the pull-back of the boundary divisor δ_0 from $\overline{\mathcal{M}}_g$.). The boundary divisors δ_0^u and δ_0^r have clear modular description in terms of Prym curves and the same holds for the higher boundary divisors.

We have similar results for moduli spaces of spin curves. We mention the following theorem cf. [F1]:

Theorem 4. *The compact moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves of genus g is of general type for $g>8$.*

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Coniveau 2 complete intersections and cones of effective cycles

Claire Voisin

A rational Hodge structure of weight k is given by a $\mathbb{Q}$-vector space L together with a Hodge decomposition

$$L_{\mathbb{C}} = \bigoplus_{p+q=k} L^{p,q}$$

satisfying Hodge symmetry

$$\overline{L^{p,q}} = L^{q,p}.$$

The coniveau of such a Hodge structure is the smallest integer c such that $L^{k-c,c} \neq 0$.

When the Hodge structure comes from geometry, the notion of coniveau is conjecturally related to codimension by the generalized Hodge-Grothendieck conjecture. Suppose X is a smooth complex projective variety and $L \subset H^k(X,\mathbb{Q})$ is a sub-Hodge structure of coniveau c.

Conjecture 1. *(GHC) There exists a closed algebraic subset $Z \subset X$ of codimension c such that L vanishes under the restriction map $H^k(X,\mathbb{Q}) \to H^k(U,\mathbb{Q})$, where $U := X \setminus Z$.*

Notice that it is a non trivial fact that the kernel of this restriction map is indeed a sub-Hodge structure of coniveau $\geq c$. This needs some arguments from mixed Hodge theory.

Consider a complete intersection $X \subset \mathbb{P}^n$ of r hypersurfaces of degree $d_1 \leq d_2 \ldots \leq d_r$. By Lefschetz hyperplane theorem, the only interesting Hodge structure is the Hodge structure on $H^{n-r}(X,\mathbb{Q})$. We will say that X has coniveau c if the Hodge structure on $H^{n-r}(X,\mathbb{Q})$ has coniveau c.

The coniveau of a complete intersection can be computed using Griffiths residues and the comparison of pole order and Hodge filtration. The result is as follows:

Theorem 1. *X has coniveau $\geq c$ if and only if*

$$n \geq \sum_i d_i + (c-1)d_r.$$

For $c = 1$, the result is obvious, as *coniveau*$(X) \geq 1$ is equivalent to $H^{n-r,0}(X) = H^0(X, K_X) = 0$, that is X is a Fano complete intersection. In this case, the generalized Hodge-Grothendieck conjecture is known to be true, using the correspondence between X and its Fano variety of lines F. Denoting by

$$\begin{array}{ccc} P & \stackrel{q}{\to} & X \\ p\downarrow & & \\ F & & \end{array}$$

the incidence correspondence, where p is the tautological $\mathbb{P}^1$-bundle on F, one can show that taking a $n-3$-dimensional complete intersection $F_{n-3} \subset F$ and restricting P to it, the resulting morphism of Hodge structures

$$q'_* \circ p'^* : H^{n-3}(F_{n-3},\mathbb{Q}) \to H^{n-1}(X,\mathbb{Q})$$

is surjective, where $P' = p^{-1}(F_{n-3})$ and p', q' are the restrictions of p, q to P'. It follows that $H^{n-1}(X, \mathbb{Q})$ vanishes on the complement of the hypersurface $q(P') \subset X$.

In the case of coniveau 2, the numerical range predicted by theorem 1 is not easy to understand geometrically. Furthermore the generalized Hodge conjecture is not known to hold in this range. For hypersurfaces, it is known only when the dimension becomes much larger than the degree, so that X becomes covered by a family of planes (a slightly weaker condition has been obtained by Otwinowska). The numerical range in which the result is known looks like

$$n \geq \frac{d^2}{4} + O(d)$$

which is very different from the bound $n \geq 2d$ of theorem 1.

In this talk, we give a geometric proof of the bound

$$n \geq \sum_i d_i + d_r \Rightarrow coniveau(X) \geq 2$$

(the coniveau 2 case of theorem 1) involving the geometry of the varieties of lines in X. Assume to simplify that X is an hypersurface of degree d, so that the bound becomes $n \geq 2d$. The variety F of lines in X has dimension $2n - 3 - d$. Choose a generic polynomial G of degree $n - d - 1$ on $\mathbb{P}^n$, which defines a smooth hypersurface $X_G \subset X$. The variety of lines $F_G \subset F$ contained in X_G is smooth of codimension $n - d$. We show the following (an analogous result holds in the complete intersection case):

Proposition 1. *For $n \geq 2d$, the family of subvarieties $F_G \subset F$ has the following "very moving" property:*

For a generic $l \in F$, and generic vector subspace $K \subset T_{F,l}$ of codimension $n-d$, there exists a G with X_G smooth, and

$$l \in F_G,\ K = T_{F_G,l}.$$

This proposition allows to give a rather direct and geometric proof of the fact that $H^{n-2,1}(X) = 0$, that is *coniveau*$(X) \geq 2$.

Let us now make the following conjecture concerning cones of effective cycles.

Given a smooth projective variety Y and an integer k, consider the $\mathbb{R}$-vector space

$$A^{2k}(Y) \subset H^{2k}(Y, \mathbb{R})$$

generated by classes of codimension k algebraic cycles. Let $E^k(Y) \subset A^{2k}(Y)$ be the cone generated by classes of effective cycles. We will say that a class $\alpha \in A^{2k}(Y)$ is big if it belongs to the interior of the cone $E^k(Y)$.

Conjecture 2. *Let $W \subset Y$ be a codimension k subvariety which is very moving in the previous sense, that is, we can impose to deformations of W to pass through a generic point of Y with a generically given tangent space. Then the class $[W]$ is big.*

This conjecture is easy to prove in the case of divisors and slightly less easy to prove but still true in the case of curves. We show the following:

Theorem 2. *Assume the varieties F_G of proposition 1 satisfy the conjecture, that is $[F_G]$ is big. Then the generalized Grothendieck-Hodge conjecture is satisfied by complete intersections of coniveau* 2*.*

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Bernstein-Gel'fand-Gel'fand correspondence and the cohomology algebra

Mihnea Popa

(joint work with Robert Lazarsfeld, Giuseppe Pareschi)

This is a report on ongoing projects with R. Lazarsfeld and G. Pareschi. To any compact Kähler manifold X one can associate the cohomology algebra of the structure sheaf $P := \bigoplus H^i(X, \mathcal{O}_X)$. This is naturally a module over the exterior algebra $E := \bigoplus \bigwedge^i V$ of $V := H^1(X, \mathcal{O}_X)$, via the Albanese map. To any such exterior module, one can associate via the Bernstein-Gel'fand-Gel'fand correspondence a linear complex of vector bundles on projective space $\mathbf{P}^n$ (with $n = q(X)-1$ in our case). We use results of Green-Lazarsfeld [2], Hacon [3] and Pareschi-Popa [4] providing a description of the $\mathbf{P}^n$ side of the story in order to describe the structure of the dual $P^\vee = \bigoplus_i H^i(X, \omega_X)$, i.e. the canonical cohomology module, over the exterior algebra. This translation uses recent algebraic results of Eisenbud-Floystad-Schreyer [1]. For instance, we show that the canonical module is 0-regular over the exterior algebra.

Theorem. *Let X be a compact Kähler manifold with generically finite Albanese map. Then the canonical module $P^\vee = \bigoplus_{i=0}^{d} H^i(X, \omega_X)$ is generated in degree* 0 *and has a linear free resolution over the exterior algebra E.*

We also recover various geometric information, as for example the normal cone of the canonical linear series inside paracanonical space, directly from the cohomology algebra.

In the case when X has no higher irrational pencils (i.e. higher dimensional analogues of fibrations over curves of genus at least 2), the complex constructed on

$\mathbf{P}^n$ is in fact a linear free resolution of a vector bundle. Its short length imposes, via vector bundle techniques, bounds on the Betti numbers of such manifolds, extending (and sometimes improving) well-known results in the case of surfaces. Inequalities for the number of sections of the canonical bundle can be deduced from the Linear Syzygy Theorem. It also allows us, by using a consequence of the Evans-Griffith Syzygy Theorem, to recover the higher dimensional Castelnuovo-de Franchis inequality obtained recently by Pareschi and the author.

Theorem. *Let X be a smooth projective variety X with no higher irrational pencils. Then $\chi(\omega_X) \geq q(X) - \dim X$.*

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Pencils of plane curves with completely reducible members

Igor Dolgachev

Let $\lambda F(x, y, z) + \mu G(X, y, z) = 0$ be an irreducible pencil of plane curves of degree d. A member of the pencil is called *completely reducible* if it is equal to the union of lines. If the number of such members is equal to k, then the union of such members is called a (d, k)-net. It is a multi-arrangement of lines in the projective planes, i.e. the union of lines taken with multiplicities. In the following we assume that all members are reduced, so we are dealing with an arrangement of lines.

The first problem discussed in the talk is to find a bound on k. We assume that the ground field is the field of complex numbers. Considering the blowup of the base points of the pencil and computing the Euler-Poincaré characteristic, one can show that $k \leq 5$.

It is easy to construct a $(d, 3)$-net for any d. For example, $\lambda(x^d - y^d) + \mu(y^d - z^d) = 0$ will do. The corresponding arrangement of lines is a Ceva arrangement. The only known example of $(d, 4)$-net is the Hesse pencil of cubic curves $\lambda(x^3 + y^3 + z^3) + \mu xyz = 0$. The union of its reducible fibres is given by $xyz = 0$, $(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) = 0$, $(x+\omega y+z)(x+\omega^2 y+\omega^2 z)(x+y+\omega z) = 0$, $(x+\omega^2 y+z)(x+\omega y+\omega z)(x+y+\omega^2 z) = 0$, where $\omega = e^{2\pi i/3}$.

A $(d, 3)$-net defines a *Latin square* of size d. Order the set of lines in each of the there completely reducible member. A line component L_i of the first member meets a line component M_j of the second member at some base point. There exists a unique line N_k of the third member passing through this point; we put the number

k at the intersection of the ith row and jth column. A Latin square defines (and is defined) by a finite quasigroup (a group without the axiom of associativity and the unity). By reordering lines, we may assume that the quasigroup is a loop, i.e. has the unity. For example, the Ceva arrangement, is defined by a cyclic group of order d.

The following is an example of Sergey Yuzvinsky [6]. Let G be a fine subgroup of an elliptic curve E. Realize E as a plane cubic in the dual plane $\check{\mathbb{P}}^2$. Let $a, b, c \in E$ be three distinct collinear points in E and $a+G, b+G, c+G$ be the three cosets in E. Consider the dual arrangement of $3|G|$ lines. It is a $(3, |G|)$-net that realizes the quasigroup defined by G.

The first example of a realization of a non-abelian group was given by Giancarlo Urzua [5]. The group is the quaternion group Q_8. In his thesis one can find a classification of $(d, 3)$-nets for $d \leq 6$.

The main part of the talk is devoted to explaining a result of my other former student, Janis Stipins, that (d, k) nets do not exist for $k \geq 5$ and $(4, q)$ nets may exist only for $q \geq 8$. The following conjecture is still open.

Conjecture. The only $(d, 4)$-net is the Hesse arrangement of 12 lines.

The main original idea of Stipins is to consider the Hesse curve

$$H(\lambda, \mu) = \mathrm{He}(\lambda F + \mu G) = 0$$

of any member of the pencil. Recall that the Hessian of a homogeneous polynomial is the homogeneous polynomial Φ equal to the determinant of the matrix of the second partial derivatives of Φ. It vanishes at any inflection point of the curve $\Phi = 0$. Note that $H(\lambda, \mu) = 0$ defines a family of curves of degree $3d-6$ depending on the parameters as polynomials of degree 3. Since any point of a completely reducible member $\lambda_i F + \mu_i G = 0$ is an inflection point we obtain that $H(\lambda, \mu) = C_i(x, y, z)(\lambda_i F + \mu_i G)$ for each $i = 1, \ldots, k$. Thus $H(\lambda_i, \mu_i)$ vanishes at any base point of the pencil and hence, if $k = 4$, $H(\lambda, \mu)$ vanishes at any base point for any (λ, μ). By Max Noether Theorem $H(\lambda, \mu) = AF + BG$, where A, B are polynomials of degree $2d - 6$ in (x, y, z) that depend polynomially on (λ, μ) of degree 3. A clever argument shows that this implies that $k \leq 4$ and if the equality holds $C_1 = \ldots = C_4 = C$ and $H(\lambda, \mu) = C(x, y, z)(P(\lambda, \mu)F + Q(\lambda, \mu)G$ for some cubic polynomials P, Q .

Let $\mathcal{A}$ be an arrangement of hyperplanes $L_i = 0, i = 1, \ldots, N$ in $\mathbb{P}_n$. Then $Hesse(L_1 \cdots L_N) = C(\mathcal{A})(L_1 \cdots L_N)^{n-1}$, where

$$C(\mathcal{A}) = (-1)^n (N-1) \sum_{I \subset [1,\ldots,N], |I| = n+1} |a_I|^2 \prod_{j \notin I} L_i^2,$$

where a_I is the determinant of the maximal minor of the $(n+1) \times N$ matrix whose columns are the coefficients of the linear forms L_i. This fact can be found on p. 660 in [3] (Exercise 10).

Assume $p = 4, k \geq 4$. Now we use Brill's theorem that the Hesse curve has multiplicity $3m - 4$ at each singular point of the curve of multiplicity m.

The formula for the Euler characteristic of the blow-up of the base points shows that the set of singular points of the members of the pencil consists of $2k(k-1)$ singular points on the union of lines and $(q-3)(k-1)$ singular points on other members (counting with multiplicities). Our curve C must have singular points at $(k-3)(k-1)$ points. Since $\deg C = 2d-6$, we get a contradiction for $k < 8$.

A new interest to the classical geometry of (d,k)-nets was inspired by the work on the topology of the complements of line arrangements and the characteristic varieties that they define, see [1], [2].

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Good moduli spaces for Artin stacks

Jarod Alper

I wish to provide a generalization of geometric invariant theory using the theory of algebraic stacks. I hope to convey that these results are both of interest and accessible to the non-stacky reader.

In [5], Mumford systematically develops the theory of quotients by reductive groups as a means to construct moduli spaces. For instance, one can construct the projective moduli scheme $M^{\mathrm{ss}}(X)$ parameterizing Gieseker-semistable sheaves on a connected projective variety X up to S-equivalence by realizing it as a quotient of an open locus of a Quot scheme. However, there is significant work involved in showing that GIT-semistability agrees with Gieseker-semistability.

It is often easy to construct moduli spaces as Deligne-Mumford or Artin stacks. One simply can define a stack by specifying what families of objects and their morphisms are (as an analogue of writing down a functor). Then to show that the stack is Deligne-Mumford or Artin (as an analogue of showing that a functor is representable) one can either explicitly give a presentation of the stack (e.g. $[X/G]$) or one can verify Artin's criteria for representability (which involves checking deformation-theoretic properties). This gives a construction of the moduli space as an Artin stack which has the advantage of having a universal family as well as often being smooth (while the coarse moduli scheme may not be).

Since there are more tools for studying schemes and algebraic spaces than stacks (and even more to study projective varieties), it is natural to try to associate to the stack an algebraic space that retains most of the geometry. If the moduli problem is a separated Deligne-Mumford stack (which is a condition on the finiteness of the automorphism groups), then the Keel-Mori theorem ([3]) implies existence of a coarse moduli space. We introduce the notion of a *good moduli space* which characterizes morphisms from Artin stacks to algebraic spaces generalizing both the geometric invariant theory of linearly reductive group actions and the notion of tame stacks ([2]). A good moduli space generalizes the existing notion of a good GIT quotient and parameterizes points up to closure equivalence.

Definition 1. *Let $\mathcal{X}$ be an Artin stack. A quasi-compact morphism $\phi : \mathcal{X} \to Y$ is a good moduli space where Y is an algebraic space if:*

(i) $\phi_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(Y)$ *is exact.*
(ii) $\mathcal{O}_Y \to \phi_* \mathcal{O}_{\mathcal{X}}$ *is an isomorphism.*

Theorem 2. *If $\phi : \mathcal{X} \to Y$ is a good moduli space, then:*

(i) *ϕ is surjective, universally closed, universally submersive and has geometrically connected fibers.*
(ii) *If Z_1, Z_2 are closed substacks of $\mathcal{X}$, then $\phi(Z_1) \cap \phi(Z_2) = \phi(Z_1 \cap Z_2)$.*
(iii) *ϕ is universal for maps to algebraic spaces (that is, for any morphism $\psi : \mathcal{X} \to Z$, there exists a unique map $\xi : Y \to Z$ such that $\xi \circ \phi = \psi$).*
(iv) *If $\mathcal{X} \to S$ is flat, then $Y \to S$ is flat.*
(v) *If S is an excellent scheme, then $\mathcal{X} \to S$ finite type implies that $Y \to S$ is finite type.*

Properties (i) and (ii) above characterize the geometric properties inherited by the good moduli space. Property (iii) gives uniqueness and implies the new result that GIT quotients are unique in the category of algebraic spaces. Property (iv) implies that GIT quotients behave well in flat families and (v) provides a generalization of Hilbert's Fourteenth Problem on finite generation of invariants. The property of being a good moduli space is stable under arbitrary base change and descends in the fpqc topology.

If $\mathcal{X} = [\mathrm{Spec}\, A/G]$ where G is a linearly reductive group scheme over an affine base, then $\mathcal{X} \to \mathrm{Spec}\, A^G$ is a good moduli space. The theorem above recovers the affine case of classical GIT. We point out that the theory works over an arbitrary base. By interpreting linearly reductive groups as groups whose classifying stack has no higher coherent cohomology, we recover a quick proof and generalization of Matsushima's theorem ([4]).

One cannot expect that every Artin stack admits a good moduli space. Mumford introduced linearized line bundles to give invariant open subschemes which have good quotients. In our setting, one chooses a line bundle $\mathcal{L}$ on the stack $\mathcal{X}$ and after defining natural analogues of the semi-stable locus $\mathcal{X}^{\mathrm{ss}}_{\mathcal{L}}$ and the stable locus $\mathcal{X}^{\mathrm{s}}_{\mathcal{L}}$, one obtains:

Theorem 3. *If $\mathcal{X}$ is a noetherian Artin stack over S and $\mathcal{L}$ is line bundle on $\mathcal{X}$, there exists a good moduli space $\phi : \mathcal{X}^{\mathrm{ss}}_{\mathcal{L}} \to Y$ where Y is a scheme admitting an S-ample line bundle $\mathcal{M}$ such that $\phi^* \mathcal{M} \cong \mathcal{L}^N$ for some N. There is an open subscheme $V \subseteq Y$ such that $\phi^{-1}(V) = \mathcal{X}^{\mathrm{s}}_{\mathcal{L}}$ and $\phi|_{\mathcal{X}^{\mathrm{s}}_{\mathcal{L}}} : \mathcal{X}^{\mathrm{s}}_{\mathcal{L}} \to V$ is a coarse moduli space.*

While the above discussion holds in arbitrary characteristic, the notion of linearly reductive is very strong in characteristic p. However, the definitions and theorems can be generalized to develop an analogue of the theory of good moduli spaces that characterizes quotients of *reductive* groups.

Interesting questions:

(1) *Does there exist a generalization of the Hilbert-Mumford criterion to good moduli spaces?*
(2) *Does there exist topological criteria on an Artin stack (e.g. a weak valuative criterion) which would imply existence of a good moduli space?*
(3) *Can good moduli spaces be constructed locally?*

A satisfactory answer to one of the questions above could provide an approach to construct moduli spaces without GIT. There are many examples of non-separated Artin stacks where one expects the existence of a separated (or even proper) good moduli space based on studying the behavior of one-parameter families.

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Fibers of General Projections of Smooth Varieties

Roya Beheshti
(joint work with David Eisenbud)

We will work over an algebraically closed field of characteristic zero. Let $X \subset \mathbf{P}^r$ be a smooth projective variety of dimension n, and denote by $\pi : X \to \mathbf{P}^{n+c}$ a general projection of X into $\mathbf{P}^{n+c}$, $c \geq 1$. Then π is a finite map which is generically injective, and we are interested in how large and complex the fibers of π can be.

General projections have been a classical tool for studying projective varieties in low dimensions. For example, a general smooth projective curve cannot be always embedded in $\mathbf{P}^2$, but it can always be embedded in $\mathbf{P}^3$, and then projected into the plane in a such a way that it acquires only ordinary double singularities. In particular the length of every fiber is at most 2. Results of John Matther [4] show that a similar pattern holds when n is small. He proves that if $n \leq 6c + 7$, or if $n \leq 6c + 8$ and $c \leq 3$, then the length of any fiber of π is at most $n/c + 1$. When n is large, however, an argument due to Lazarsfeld shows that the fibers of π can have points of high coranks, and this implies that the length of the fibers can grow exponentially with n.

Although there is no nice bound on the length of the fibers, one might still hope to bound the complexity of the fibers in terms of other invariants. One such candidate is the regularity of the fibers.

Conjecture 1 (Eisenbud). *If $X \subset \mathbf{P}^r$ is a smooth projective variety of dimension n, and if $\pi : X \to \mathbf{P}^{n+c}$ is a general projection, then for every $y \in \mathbf{P}^{n+c}$,*

$$\operatorname{reg}(\pi^{-1}(y)) \leq n/c + 1$$

If this conjecture is true, it would improve the existing bounds on the regularity of smooth projective varieties: it implies that for any n and r, there exists a constant $c_{n,r}$ such that for any smooth non-degenerate projective variety of dimension n in $\mathbf{P}^r$, its Castelnuovo-Mumford regularity is bounded by $\deg(X) - \operatorname{codim}(X) + c_{n,r}$. The conjecture of Eisenbud and Goto [1] predicts that the regularity of any integral non-degenerate subscheme of $\mathbf{P}^r$ is bounded by $\deg(X) - \operatorname{codim}(\mathrm{X}) + 1$. This conjecture is only known to be true for curves [2] and smooth surfaces [3].

In this work, we introduce an invariant of the fibers which takes values $\leq n/c+1$, and in many cases, for example when the fiber is a local complete intersection, agrees with the length of the fiber. A fiber of $\pi : X \to \mathbf{P}^{n+c}$ over $y \in \mathbf{P}^{n+c}$ can be written as $X \cap \Lambda$, where Λ is a linear subvariety of $\mathbf{P}^r$ containing the center of the projection in codimension 1. We define a sheaf Q_y whose support is contained in $X \cap \Lambda$, and can be regarded as the space of obstructions to deforming Λ while keeping the length of the intersection with X fixed. We show that $\frac{1}{c}\operatorname{length} Q_y$ is bounded from above by $n/c + 1$, and we conjecture that $\frac{1}{c}\operatorname{length} Q_y$ is bounded from below by the regularity of the fiber.

Using this invariant we prove the following

Theorem 2. *Suppose that X is a smooth n dimensional variety in $\mathbf{P}^r$ and that π is a general projection of X into $\mathbf{P}^{n+c}$, with $c \geq 1$. If we write $\pi^{-1}(p) = Y \cup Y'$ where Y is the union of all licci components, then*

$$\operatorname{length} Y + (1 + \frac{3}{c}) \operatorname{length} Y'_{\mathrm{red}} \leq n/c + 1.$$

A subscheme of $\mathbf{P}^r$ is called licci if it is in the linkage class of a local complete intersection subscheme. In particular, local complete intersection fibers have length $\leq n/c + 1$. The special case of this result, when the fiber is curvilinear, follows from the results of Mather, and it was also proved by Ziv Ran [5], and in a weaker form, by Joel Roberts [6] using more elementary methods.

We can also use our invariant to give bounds on the dimension of the subvariety of $\mathbf{P}^r$ swept out by l-secant lines of X. An l-secant line of X is a line in $\mathbf{P}^r$ whose intersection with X has length at least l or is contained in X. With X as above, let S_l be the subvariety of $\mathbf{P}^r$ swept out by the l-secant lines of X. Ziv Ran's "Dimension+2 Secant Lemma" [5] shows that the dimension of S_{n+2} is at most $n + 1$. In the following theorem, we generalize this to get a sharp bound on the dimension of the subvariety swept out by the l-secant lines of X for an arbitrary l.

Theorem 3. *Let X be a smooth projective variety of dimension n in $\mathbf{P}^r$, and let S_l be the subvariety of $\mathbf{P}^r$ swept out by all the l-secant lines of X. Then*

$$\dim S_l \leq \frac{nl}{l-1} + 1.$$

It is also possible to give a similar bound for the subvariety of $\mathbf{P}^r$ swept out by the l-secant planes of X for any l.

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Regularities of projective subvarieties of complex projective spaces

Lawrence Ein
(joint work with Tommaso de Fernex)

Let V be a closed subscheme of $\mathbb{P}^n$ the complex projective $n-$space. We denote by $\mathcal{I}_V$ the coherent ideal sheaf of V. A basic invariant that controls the complexity in computing the syzygies of I_V, the homogenous ideal of V, is the regularity of V. Recall that $\mathcal{I}_V$ is said to be $k-$regular in the sense of Castelnuovo and Mumford, if $H^i(\mathcal{I}_V(k-i)) = 0$ for all $i > 0$. reg(V), the regularity of V is defined to be the smallest integer k such that $\mathcal{I}_V$ is $k-$regular. It is well known that if X is $k-$regular, then I_V is generated by elements of degree less than or equal to k. The first syzygies of I_V are generated by relations of degree less than or equal to $k+1$ etc.

Let S be the polynomial ring of $\mathbb{P}^n$. Suppose that I_V is an homogenous ideal in S generated by elements of degree less than or equal to d. A theorem of Giusti [G] says that if $n \geq 3$, then

$$\mathrm{reg}(I_V) \leq 6^{2^{n-3}} \cdot d^{3 \cdot 2^{n-3}} \cdot 2^{n-1} + 2^{n-1}.$$

There are examples due to Mayr-Meyer [MM] which show that indeed for an arbitrary closed subscheme the regularity bound needs to be doubly exponential. In practice, we often only deal with closed subschemes with only mild singularities and we expect that we would obtain a much sharper bound in this situation. For instance, a famous conjecture of Eisenbud and Goto says that if V is a codimension e irreducible variety of degree d in $\mathbb{P}^n$, then $\mathrm{reg}(V) \leq d-e+1$. The conjecture is known to be true for curves and smooth surfaces [GLP] and [L]. In the following we'll describe some of my recent joint work with Tommaso de Fernex [DE] showing that if the pair $(\mathbb{P}^n, eV)$ is log-canonical and V is defined by degree d equations, then one obtain a sharp bound for the regularity of V generalizing the old results in [BEL] when V is smooth and [CU] when V is a local complete intersection with only rational singularities.

The basic strategy is very simple. Bounding the regularity of V is equivalent to show that $\mathcal{I}_V$ satisfies certain vanishing theorems. From higher dimensional birational geometry, we have the Nadel vanishing theorem for multiplier ideal sheaves. If we can show that $\mathcal{I}_V$ is a multiplier ideal sheaf, then we can apply the Nadel vanishing theorem to obtain a bound on regularity. It should be noted that by a recent result of Lazarsfeld and Lee, not every integrally closed ideal can be realized as multiplier ideals. This implies that in general there will be some obstructions in employing this strategy. In our situation, our key technical tool is the inversion of adjunction for local complete intersection varieties [EM1] and [EM2]. The following is the general vanishing theorem that will imply the regularity bound for varieties in $\mathbb{P}^n$.

Theorem 1. (de Fernex and Ein) Let X be a locally complete intersection projective variety with rational singularities, and let $V \subset X$ be a pure-dimensional proper subscheme with no embedded components. Suppose that V is scheme-theoretically given by

$$V = H_1 \cap \cdots \cap H_t$$

for some divisors $H_i \in |\mathcal{L}^{\otimes d_i}|$, where $\mathcal{L}$ is a globally generated line bundle on X and $d_1 \geq \cdots \geq d_t$. If the pair (X, eV) is log canonical where $e = \mathrm{codim}_X V$, and $\mathcal{A}$ is a nef and big line bundle on X, then

$$H^i(X, \omega_X \otimes \mathcal{L}^{\otimes k} \otimes \mathcal{A} \otimes \mathcal{I}_V) = 0 > 0,\ k \geq d_1 + \cdots + d_e.$$

Theorem 2 (de Fernex and Ein) Let $V \subset \mathbb{P}^n$ be a pure-dimensional proper subscheme with no embedded components. Suppose that V is scheme-theoretically given by

$$V = H_1 \cap \cdots \cap H_t$$

where H_i is a hypersurface of degree d_i in $\mathbb{P}^n$ and we assume that $d_1 \geq d_2 \cdots \geq d_t$. If $(\mathbb{P}^n, eV)$ is log-canonical where $e = \mathrm{codim}_{\mathbb{P}^n} V$, then $\mathrm{reg}(V) \leq d_1 + \cdots + d_e - e + 1$.

Furthermore, the inequality is an equality if and only if V is a complete intersection of type $(d_1, \cdots, d_e)$.

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Notes on π_1 of Smooth Loci of Log Del Pezzo Surfaces

Chenyang Xu

A projective surface R over $\mathbb{C}$ is called *log Del Pezzo*, if it contains only quotient singularities, and K_R is an anti-ample $\mathbb{Q}$-divisor. Although the fundamental group of R is always trivial, the fundamental group of the smooth locus $\pi_1(R^{sm})$ is in general not zero, but it is known such a group is always finite (cf. [GZ95], [KM99]). The aim of this paper is to determine these groups.

Our approach to this problem is as follows. Given a log Del Pezzo surface R, we take the universal cover of its smooth locus R^{sm}. Thanks to the above theorem of [GZ95], [KM99] and the Riemann Existence Theorem (cf. [SGA1]), we know it is an algebraic variety, so we can take the normal closure S of R in the function field of this covering space. Then we get a pair $(S, \pi_1(R^{sm}))$, where S is also a log Del Pezzo surface, and $\pi_1(R^{sm})$ is a finite group acting on it, such that for every nontrivial element of $g \in \pi_1(R^{sm})$, the fixed locus S^g is isolated. We can also equivariantly resolve S to get a smooth rational surface with the same finite group action. This motivates the following definitions,

Definition 1. We call a finite group G acting on a normal surface S *an action with isolated fixed points* (IFP), if S has only (at worst) quotient singularities, and for every nonunit element g of G, the fixed locus S^g consists of finite points. Similarly, we call (S, G) *birational to an action with* IFP if there is a G-equivariant birational proper model S' of S, such that (S', G) is an action with IFP. In particular, S' contains only quotient singularities.

Now we divide the question into 3 problems.

(1) Finding all the birational classes (S, G) containing a representative $(\tilde{S}, G)$ with IFP.
(2) Determining those groups G, for which we can choose $(\tilde{S}, G)$ as in (1) with the additional property that $K_{\tilde{S}}$ is anti-ample.
(3) For any G appearing in the step(2), checking the existence of $(\tilde{S}, G)$ satisfying $\pi_1(\tilde{S}^{sm}) = e$.

In a recent paper [DI06], all finite subgroups of the Cremona group are classified. Based on their table, we can solve the Problem (1).

Theorem 2. *Assume a finite group G acts on a rational surface S such that (S, G) is birational to an action with* IFP*, then G is one of the following groups:*

(1) a finite subgroup G of $\mathrm{GL}_2(\mathbb{C})$ whose abelian subgroups are all cyclic,
(2) a finite subgroup G of $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ whose subgroups as $G_1 \times G_2$ have the property that $|G_1|$ and $|G_2|$ are coprime,
(3) $\mathbb{Z}/n : \mathbb{Z}/3$ or $\mathbb{Z}/2 \times (\mathbb{Z}/n : \mathbb{Z}/3)$, where n is an odd integer and $\mathbb{Z}/n : \mathbb{Z}/3$ means the group generated by $u : (x_0, x_1, x_2) \to (x_1, x_2, x_0)$ and $v : (x_0, x_1, x_2) \to (\mu^n x_0, \mu^s x_1, x_2)(s^2 - s + 1 \equiv 0 \mod n)$,
(4) for $n \equiv 2 \mod 4$, F_{4n} or G_{4n}, which are groups of order $4n$. Here F_{4n} is the group generated by $(x, y) \to (-\frac{1}{x}, -\frac{1}{y})$, $(x, y) \to (e^{\frac{2i\pi}{n}} x, e^{\frac{2i\pi}{n}} y)$ and $(x, y) \to (e^{\frac{i\pi}{n}} y, -e^{\frac{i\pi}{n}} x)$, and G_{4n} is the group generated by $(x, y) \to (-\frac{1}{x}, -\frac{1}{y})$, $(x, y) \to (e^{\frac{2i\pi}{n}} x, e^{\frac{2i\pi}{n}} y)$ and $(x, y) \to (-y, x)$, or
(5) $(\mathbb{Z}/3)^2 : \mathbb{Z}/2$, $(\mathbb{Z}/3)^2 : \mathbb{Z}/4$ and $(\mathbb{Z}/3)^2 : Q_8$.

Conversely, every group G in the above list can act on some rational surface S' with (at most) quotient singularities, such that the action is with IFP.

Although Theorem(2) is the strongest statement, we emphasize that there is a more conceptual version as follows:

Theorem 3. *Given a finite group G which acts on a smooth projective rational surface S, it gives an action birationally with IFP if and only if it satisfies the following conditions:*

(1) for any point $x \in S$, every abelian subgroup of the stabilier G_x is cyclic, and
(2) for any element $g \in G$, every curve $C \subset S^g$ satisfies genus $g(C) = 0$.

Moreover, by the case study, it shows that for any minimal action G on a rational surface S, if S

(1) S is a nonminimal ruled surface, or
(2) S is a smooth Del Pezzo surface of degree less or equal to 4,

G always contains nontrivial element g, such that S^g contains a smooth curve of positive genus.

In [Ko06b], a similar method is used to study the case when G is abelian, yielding a list of possible first homology groups of log Del Pezzo surfaces. By listing the abelian groups in the above table (notice that $D_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$), we can refine the results there (cf. [Ko06b], 11).

Corollary 4. *Let S be a log Del Pezzo surface. Then $H_1(S^{sm}, \mathbb{Z})$ is one of the following groups: $(\mathbb{Z}/3)^2$, $\mathbb{Z}/3 \times \mathbb{Z}/6$, $\mathbb{Z}/2 \times \mathbb{Z}/n$ (n is 4 or $4k+2$) or $\mathbb{Z}/m$ for any m.*

Then apply the equivariant minimal model program, we can also answer the question (2), namely

Theorem 5. *If we can choose $(\tilde{S}, G)$ in Theorem(2) satisfying the additional property: $\tilde{S}$ is a log Del Pezzo surface. Then G is precisely one of the groups in (1)-(4) there.*

Finally, we aim to solve the third problem. We construct models $(\tilde{S}, G)$ which satisfies the property $\pi_1(\tilde{S}^{sm}) = e$ for most groups G in (2). Unfortunately, we leave three series of groups undetermined.

Remark 6. A work in progress of the author will show that the smooth loci of log Del Pezzo surfaces have a stronger geometric property than rational connectedness, namely strong rational connectedness, which will give an affirmative answer to the Conjecture 20 in [BH06].

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Theta dualities on moduli spaces of sheaves

Alina Marian

(joint work with Dragos Oprea)

Let S be a smooth complex projective surface, with a generic polarization H. For v a class in the topological K-theory $K_{\mathrm{top}}(S)$, denote by $\mathfrak{M}_v$ the moduli space of Gieseker H-semistable sheaves on S of class v. For any w in the Grothendieck group $K_{\mathrm{coh}}(S)$ of coherent sheaves on S, satisfying $\chi(v \otimes w) = 0$, there is a determinant line bundle Θ_w on $\mathfrak{M}_v$. The phenomenon of *theta duality* appears when one considers two moduli spaces $\mathfrak{M}_v$ and $\mathfrak{M}_w$ such that

$$\chi(v \otimes w) = 0, \text{ and } H^2(E \otimes F) = 0 \text{ for all } (E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w.$$

In this case, the Brill-Noether locus

$$(E,F) \in \mathfrak{M}_v \times \mathfrak{M}_w \text{ such that } h^0(E \otimes F) \neq 0$$

should correspond to a divisor Δ in the product of the two moduli spaces of sheaves of type v and w. When Δ actually has the expected dimension, and the corresponding line bundle splits as

$$\mathcal{O}(\Delta) = \Theta_w \boxtimes \Theta_v \text{ on } \mathfrak{M}_v \times \mathfrak{M}_w,$$

there is an induced map (defined up to scalars)

$$D : H^0(\mathfrak{M}_v, \Theta_w)^\vee \to H^0(\mathfrak{M}_w, \Theta_v). \tag{1}$$

The issue, first proposed in this context by Le Potier, is to investigate firstly when the numerical duality $\chi(\mathfrak{M}_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v)$ holds, and secondly, when the geometric map D is an isomorphism.

It is straightforward [2] that both the numerical and the geometric duality hold when $\mathfrak{M}_v$ and $\mathfrak{M}_w$ are moduli spaces of rank 1 sheaves on S with fixed determinant *i.e.,* they are isomorphic to Hilbert schemes of points on S. For arbitrary topological types, the easiest setting for studying at least the numerical duality is when S is an abelian or a $K3$ surface. In other contexts, for example when S is a rational surface, establishing even the numerical duality is much harder.

When S is a $K3$ surface and the K-class v corresponds to a primitive Mukai vector, classic work of Mukai, O'Grady, and Yoshioka implies that the moduli space $\mathfrak{M}_v$ is irreducible holomorphic symplectic, and is moreover deformation equivalent to the Hilbert scheme of points on S of the right dimension. Further exploiting the Beauville-Bogomolov form that generally accompanies an irreducible holomorphic symplectic structure, one can in fact calculate the holomorphic Euler characteristics $\chi(\mathfrak{M}_v, \Theta_w)$ on the Hilbert scheme. The argument is presented in [4], [1], yielding the answer

$$\chi(\mathfrak{M}_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \binom{d_v + d_w}{d_v}, \tag{2}$$

where $\mathfrak{M}_v$ is deformation equivalent to $S^{[d_v]}$.

When S is an abelian surface, the moduli space $\mathfrak{M}_v$ itself does not carry an irreducible holomorphic symplectic structure. Following Yoshioka [5], one considers in this case at the Albanese map

$$\alpha = (\alpha^+, \alpha^-) : \mathfrak{M}_v \to \widehat{S} \times S,$$

where

$$\alpha^+(E) = \det E, \ \ \alpha^-(E) = \det \mathbf{R}\mathcal{S}(E),$$

and

$$\mathbf{R}\mathcal{S} : \mathbf{D}(S) \to \mathbf{D}(\widehat{S})$$

is the Fourier-Mukai functor on S. (Here identifications of the images of the determinant maps α^+ and α^- with $\widehat{S}$ and S are made using appropriate reference line bundles on the abelian surface and its dual.) Yoshioka proved that the fiber K_v of the Albanese map α is irreducible holomorphic symplectic, and is moreover deformation equivalent to the generalized Kummer variety of S in the suitable dimension, in other words to the fiber of the addition morphism from the Hilbert scheme of points on S to S. As in the case of $K3$ surfaces, one can then calculate holomorphic Euler characteristics of determinant line bundles on the Albanese fiber K_v. It is shown in [3] that upon fixing suitable Mukai vectors v and w, it is the Albanese fiber K_v and the full moduli space $\mathfrak{M}_w$ that should be considered as theta duality partners. Indeed, numerically one has

$$\chi(K_v, \Theta_w) = \chi(\mathfrak{M}_w, \Theta_v) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}, \tag{3}$$

where $d_v = \frac{1}{2}\dim K_v + 1$. While the Euler characteristic calculation on K_v is similar to the $K3$ case, the calculation on the full moduli space $\mathfrak{M}_w$ uses the Cartesian diagram

$$\begin{array}{ccc} K_w \times \widehat{S} \times S & \xrightarrow{\Phi} & \mathfrak{M}_w \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \alpha} \\ \widehat{S} \times S & \xrightarrow{\Psi} & \widehat{S} \times S \end{array},$$

where $\Phi : K_w \times \widehat{S} \times S \to \mathfrak{M}_w$ is defined as

$$\Phi(E, x, y) = t_y^\star E \otimes x,$$

and the explicit expression of Ψ is given in [3]. Determining $\chi(\mathfrak{M}_w, \Theta_v)$ is then of course conceptually straightforward by pullback under the étale morphism Φ, but is in practice quite involved.

Two other flavors of theta duality are obtained in this context, if one considers the fibers $\mathfrak{M}_v^+$ and $\mathfrak{M}_v^-$ of the two component maps α^+ and α^- of α. Suitable étale morphisms yield in this situation the equalities

$$\chi(\mathfrak{M}_v^+, \Theta_w) = \chi(\mathfrak{M}_w^+, \Theta_v) = \frac{c_1(v \otimes w)^2}{2(d_v + d_w)}\binom{d_v + d_w}{d_v}, \tag{4}$$

$$\chi(\mathfrak{M}_v^-, \Theta_w) = \chi(\mathfrak{M}_w^-, \Theta_v) = \frac{c_1(\widehat{v} \otimes \widehat{w})^2}{2(d_v + d_w)}\binom{d_v + d_w}{d_v}, \tag{5}$$

where $\widehat{v}$ and $\widehat{w}$ denote the cohomological Fourier-Mukai transforms of v and w.

This concludes the analysis of the numerical theta duality in the setting of $K3$ and abelian surfaces. The question of the *geometric* duality (1) has not yet been explored in a satisfactory way, but partial progress exists in the situation when the theta-dual moduli spaces are in fact birational to Hilbert schemes of points.

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Compact Moduli of Singular Curves: A case study in genus one

David Ishii Smyth

The moduli space of stable curves is not the only modular compactification of the space of smooth curves [1, 4], and recent work of Hassett and Hyeon [2, 3] suggests the relevance of alternate modular compactifications for understanding the minimal model program, as applied to $\mathcal{M}_g$. Thus, we are led to consider the following question:

Question. Given a deformation-open collection of isolated curves singularities, is there a corresponding stability condition which yields a proper compactification of the moduli space of smooth curves?

We obtain a good theory [5] when we consider the special case of compactifying $M_{1,n}$, the moduli space of smooth elliptic curves with n distinct marked points. The only Gorenstein singularities which can occur on a reduced curve of arithmetic genus one are the following:

Definition (The elliptic m-fold point)**.** If C is a curve over an algebraically closed field k, then C has an elliptic m-fold point $p \in C$ if

$$\hat{O}_{C,p} \simeq \begin{cases} k[[x,y]]/(y^2-x^3) & m=1 \quad \text{(ordinary cusp)} \\ k[[x,y]]/(y^2-yx^2) & m=2 \quad \text{(ordinary tacnode)} \\ k[[x,y]]/(x^2y-xy^2) & m=3 \quad \text{(planar triple-point)} \\ k[[x_1,\dots,x_{m-1}]]/I_m & m\geq 4, \quad \begin{array}{l}(m \text{ general lines through} \\ \text{the origin in } \mathbb{A}^{m-1}),\end{array} \end{cases}$$

$$I_m := (x_hx_i - x_hx_j \; : \; i,j,h \in \{1,\dots,m-1\} \text{ distinct}).$$

For a fixed integer m, the set {nodes, cusps, tacnodes ..., elliptic m-fold points} is a deformation-open collection of singularities, and we can formulate an associated stability condition.

Definition (m-stability)**.** Let $(C, p_1, \dots, p_n)$ be a connected, reduced curve of arithmetic genus one, with n distinct smooth marked points. For any positive integer $m < n$, we say that C is m-stable if

1. C has only nodes, cusps, ..., elliptic m-fold points as singularities.
2. $(C, p_1, \ldots, p_n)$ has finite automorphism group (as a pointed curve).
3. If $Z \subset C$ is any connected subcurve of arithmetic genus one, then

$$|Z \cap \overline{C \backslash Z}| + |\{p_i \,|\, p_i \in Z\}| > m.$$

This definition extends to a moduli functor in the usual way: Over an arbitrary scheme, an m-stable curve is a flat, proper, finitely-presented morphism with n sections, whose geometric fibers are m-stable. Then one has

Theorem. *$\overline{\mathcal{M}}_{1,n}(m)$, the moduli stack of m-stable curves, is a proper irreducible Deligne-Mumford stack over Spec $\mathbb{Z}[1/6]$.*

The restriction to Spec $\mathbb{Z}[1/6]$ is due to the existence of 'extra' automorphisms of cuspidal curves in characteristics 2 and 3.

Following the usual recipes on $\overline{\mathcal{M}}_{1,n}$, we define invertible sheaves on $\overline{\mathcal{M}}_{1,n}(m)$ by

$$\lambda : = \det(\pi_* \omega_{\mathcal{C}/\overline{\mathcal{M}}_{1,n}(m)})$$
$$\psi : = \otimes_{i=1}^{n} \sigma_i^* \omega_{\mathcal{C}/\overline{\mathcal{M}}_{1,n}(m)},$$

and we define δ_0 to be the invertible sheaf associated to the Cartier divisor $\Delta_0 \subset \overline{\mathcal{M}}_{1,n}(m)$ parametrizing curves that contain a disconnecting node. If B is a complete curve, $(\mathcal{C} \to B, \sigma_1, \ldots, \sigma_n)$ is an m-stable curve over B, and $\phi : B \to \overline{\mathcal{M}}_{1,n}(m)$ is the natural classifying map, then we may consider the intersection number $\deg_B \phi^* \mathcal{L}$ for any invertible sheaf $\mathcal{L} \in \mathrm{Pic}(\overline{\mathcal{M}}_{1,n}(m))$. We develop methods for evaluating the degrees of λ and $\psi - \delta_0$ on complete 1-parameter families of m-stable curves, and prove

Theorem. *The $\mathbb{Q}$-divisor $s\lambda + (\psi - \delta_0)$ is positive on an arbitrary complete 1-parameter family of m-stable curves iff $s \in \mathbb{Q} \cap [-m-1, -m]$.*

A strengthening of this result allows us to verify Nakai's criterion for ampleness and we obtain

Corollary. *$\overline{\mathcal{M}}_{1,n}(m)$ has projective coarse moduli space $\overline{M}_{1,n}(m)$, with ample $\mathbb{Q}$-divisor $s\lambda + (\psi - \delta_0)$.*

Finally, we consider the problem of interpreting the spaces $\overline{M}_{1,n}(m)$ as the projective models associated to section-rings of certain divisors on $\overline{M}_{1,n}$. Here, the main obstacle is that we do not have a good description of the local geometry of the stack $\overline{\mathcal{M}}_{1,n}(m)$. At present, we cannot even preclude the possibility that it has embedded components. We work around this by setting

$$\widetilde{\mathcal{M}}_{1,n}(m) := \text{The normalization of } \overline{\mathcal{M}}_{1,n}^{red},$$

and then we obtain

Corollary. *For* $s \in \mathbb{Q} \cap [-m-1, -m]$,

$$\widetilde{M}_{1,n}(m) := \operatorname{Proj}\left(\oplus_{m>>0} H^0(\overline{M}_{1,n}, m(s\lambda + \psi - \Delta_0))\right)$$

where the sum is taken over m sufficiently divisible so that $m(s\lambda+\psi-\Delta_0)$ *descends to an integral line-bundle on* $\overline{M}_{1,n}$.

This gives a complete decomposition of the effective cone

$$\operatorname{Eff}(\overline{M}_{1,n}) \cap \mathbb{Q}\{\lambda, \psi - \delta\}$$

into chambers corresponding to the ample cones of our alternate birational models. We remark that it is unknown whether $\overline{\mathcal{M}}_{1,n}(m) = \widetilde{\mathcal{M}}_{1,n}(m)$ for all m. We do know that the stacks $\overline{\mathcal{M}}_{1,n}(m)$ are smooth iff $m \leq 5$. (Smoothness follows from the fact that the elliptic m-fold point has unobstructed deformations for $m \leq 5$. In the opposite direction, we show that the image of the exceptional locus of the birational morphism $\overline{M}_{1,7}(5) \to \overline{M}_{1,7}(6)$ is a collection of isolated points, and that this morphism extracts exceptional divisors with discrepancy less than $\dim \overline{\mathcal{M}}_{1,7}(6) - 1$.)

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Frobenius splittings of toric varieties

Sam Payne

The following problems continue to stimulate significant active research in toric geometry. Let X be a projective toric variety, with an ample line bundle L and a nef line bundle L'.

Problem (Oda, 1980s)**.** *If* X *is smooth, is the natural multiplication map*

$$H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L')$$

necessarily surjective?

If the answer is affirmative in the special case where L' is a power of L, then the section ring $R(X, L) = \bigoplus_{m \geq 0} H^0(X, L^m)$ is normal. However, this special case remains completely open.

Conjecture (Sturmfels, 1990s)**.** *If* X *is smooth and* $R(X, L)$ *is normal then the ideal of relations* $I(X, L)$ *is generated in degree two.*

Conjecture (Bøgvad, 1995)**.** *If X is smooth then the section ring $R(X, L)$ is normal and Koszul.*

These conjectures are wide open, starting in dimension three. In dimension two, Oda's Problem and both conjectures have affirmative solutions, and these solutions generalize to arbitrary singular toric surfaces [Fak, HNPS]. However, there are easy examples of polarized singular toric threefolds for which $R(X, L)$ is not normal, and it is unclear what role smoothness should play in these problems. Nevertheless, a central problem in toric geometry is to understand under what hypotheses $R(X, L)$ is normal or Koszul. The following results are from [Pay].

Theorem 1. *If X is diagonally split at some prime p and $L_1, \ldots, L_r$ are nef line bundles on X, then*

$$R(X; L_1, \ldots, L_r) = \bigoplus_{a_1,\ldots,a_r} H^0(X, L_1^{a_1} \otimes \cdots \otimes L_r^{a_r})$$

is normal and Koszul.

In particular, the analogues of Oda's problem and both conjectures are true in this case, even if X is singular.

Corollary 2. *Under the hypotheses of Theorem 1, if $V \subset X$ is an invariant subvariety then $R(V; L_1, \ldots, L_r)$ is normal and Koszul.*

Theorem 3. *Every toric variety V can be embedded as an invariant subvariety of a diagonally split toric variety X.*

In view of Corollary 2, it should be interesting to understand when an ample line bundle on V can be extended to a nef line bundle on X.

The key to proving the above results, and to applying them in practice, is the following combinatorical characterization of diagonally split toric varieties. Say Δ is the fan corresponding to X, and let v_ρ denote the primitive generator of a ray $\rho \in \Delta$. Let M be the character lattice of the dense torus, with $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$.

Theorem 4. *The toric variety X is diagonally split at p if and only if the open polytope*

$$\mathbb{F}_X = \{u \in M_\mathbb{R} \mid -1 < \langle u, v_\rho \rangle < 1 \text{ for all } \rho \in \Delta\}$$

contains representatives of every equivalence class in $\frac{1}{p}M/M$.

This criterion for diagonal splitting can be used to check that the Hirzebruch surface F_a is diagonally split at p if and only if a is equal to zero or one, or a is equal to two and p is odd. In particular, the diagonal in F_3 is not split for any p. Since F_3 occurs as a Schubert variety in the G_2 flag variety, this gives a negative answer to a problem that has been open for more than two decades, since the introduction of Frobenius splittings as a tool for studying linear series on Schubert varieties [MR, Ram].

Corollary 5. *There is a Schubert variety in the G_2 flag variety that is not diagonally split.*

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Hypersurfaces cutting out a projective variety

Atushi Noma

Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension n, degree d, and codimension $e = N - n \geq 2$, not contained any hyperplane. For a given integer m and X, we consider the condition (B_m): X is equal to, as set or as scheme, the intersection $E_m(X)$ of all hypersurfaces of degree $\leq m$, containing X. The purpose here is to study (B_{d-e+1}) and give a partial answer. (For detail, see [7]; some parts of this talk are in progress.) The reason for this study is that we naturally expect (B_{d-e+1}) since, by Castelnuovo-Mumford theory, it follows from the regularity conjecture([3], [4]), which is still open and asserts that X is $(d-e+1)$-regular, i.e., $H^i(\mathbb{P}^N, \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(d-e+1-i)) = 0$ for all $i > 0$ and the ideal sheaf $\mathcal{I}_X$ of X (see [5] and [2] for the current situation of the conjecture).

To study (B_{d-e+1}), our basic idea comes from a result of Mumford.

Theorem 1 (Mumford [6])**.** *(B_d) holds as set. Moreover, if X is smooth, (B_d) holds as scheme.*

To prove this result, Mumford considered the projection $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{n+1}$ from an $(e-2)$-plane $\Lambda \subseteq \mathbb{P}^N$ with $\Lambda \cap X = \emptyset$, and obtained a hypersurface F_Λ of $\mathbb{P}^N$ of degree $\leq d$ containing X, by pulling-back the image $\overline{X} := \pi_\Lambda(X)$ that is a hypersurface of degree $\leq d$ in $\mathbb{P}^{n+1}$. In other words, F_Λ is the cone $\mathrm{Cone}(\Lambda, X)$ over X with vertex Λ. In this case, Mumford observed that (1) given $v \in \mathbb{P}^N \setminus X$, if $\Lambda \cap \mathrm{Cone}(v, X) = \emptyset$, then $F_\Lambda \not\ni v$; and that (2) given $x \in \mathrm{Sm}\, X := X \setminus \mathrm{Sing}\, X$ and $w \in \mathbb{P}^N \setminus T_x(X)$ (i.e., w is a point of $\mathbb{P}^N$ outside of the embedded tangent space $T_x(X)$ to X at x), if $\Lambda \cap (\mathrm{Cone}(x, X) \cup \langle w, T_x(X)\rangle) = \emptyset$, then F_Λ is smooth at x and $w \notin T_x(F_\Lambda)$. Here $\langle\ \rangle$ denotes the linear span. If Λ varies depending on v, x and w, this observation clearly implies the above result on (B_d).

To study (B_{d-e+1}), we modify this argument: we consider the projection from an $(e-2)$-plane $\langle x_1, \ldots, x_{e-1}\rangle$ spanned by general $(e-1)$-points of $\mathrm{Sm}\, X$. Consequently the hypersurface $F_{\langle x_1,\ldots,x_{e-1}\rangle}$ obtained in the same way is of degree $\leq d - e + 1$. Such hypersurfaces cannot separate X away from the following sets:

(1) $\mathcal{B}(X) := \{v \in \mathbb{P}^N \setminus X \,|\, \mathrm{length}(\langle v, x\rangle \cap X) \geq 2 \text{ for general } x \in X\}$,

(2) $\mathcal{C}(X) := \{u \in \mathrm{Sm}\, X \,|\, \mathrm{length}(\langle u, x\rangle \cap X) \geq 3 \text{ for general } x \in X\}$.

Here we remark that $\mathcal{B}(X)$ and $\mathcal{C}(X)$ are closed in $\mathbb{P}^N \setminus X$ and $\operatorname{Sm} X$ respectively.

Now our first result is stated as follows.

Theorem 2. *(1) $X \subseteq E_{d-e+1}(X) \subseteq X \cup \mathcal{B}(X)$ holds as set, i.e., on $\mathbb{P}^N \setminus \mathcal{B}(X)$, $X = E_{d-e+1}(X)$ holds as set.* (cf. Calabri-Ciliberto [1], Sommese-Verscherde-Wampler [10]).

(2) On $\mathbb{P}^N \setminus (\mathcal{B}(X) \cup \mathcal{C}(X) \cup \operatorname{Sing}(X))$, $X = E_{d-e+1}(X)$ holds as scheme .

Before going to look at the structure of $\mathcal{B}(X)$ and $\mathcal{C}(X)$, we give some examples of X with empty $\mathcal{B}(X)$ and $\mathcal{C}(X)$.

Example 3. *$\mathcal{B}(X)$ and $\mathcal{C}(X)$ are empty in the following cases.*

(1) X is contained in the image $v_l(\mathbb{P}^m) \subseteq \mathbb{P}^M (\supseteq \mathbb{P}^N)$ $(M = \binom{m+l}{l} - 1)$ of the lth $(l \geq 2)$ Veronese embedding of the projective space $\mathbb{P}^m$ for some $m > 0$.

(2) X is a scroll over a curve C that is not a cone over C and is regular in codimension 1.

Next we study the structure of $\mathcal{B}(X)$ and $\mathcal{C}(X)$. The structure of the closure $\overline{\mathcal{B}(X)}$ is already studied by B.Segre ([8],[9]) and Calabri and Ciliberto ([1]).

Theorem 4 (Segre [8], Calabri-Ciliberto [1])**.** *Every irreducible component Λ of $\overline{\mathcal{B}(X)}$ is linear of dimension $\leq n-1$. Moreover $\dim\langle T_x(X), \Lambda\rangle = n+1$ for a general point $x \in \operatorname{Sm} X$.*

Here we look at the dimension of $\mathcal{B}(X)$ and $\mathcal{C}(X)$ in terms of $\dim \operatorname{Sing} X$.

Theorem 5. *(1) If Λ is an irreducible component of $\overline{\mathcal{B}(X)}$, then $\dim(X \cap \Lambda) = \dim \Lambda - 1$ and $X \cap \Lambda \subseteq \operatorname{Sing} X$. In particular, $\dim \Lambda \leq \min\{n-1, \dim \operatorname{Sing} X + 1\}$. Here we mean $\dim \emptyset = -1$.*

(2) If Z is an irreducible component of $\overline{\mathcal{C}(X)}$, then Z is linear of dimension $\leq \min\{n-1, \dim \operatorname{Sing} X + 2\}$ and $\dim\langle T_x(X), Z\rangle = n+1$ for a general point $x \in \operatorname{Sm} X$.

As a corollary, we have the following.

Corollary 6. *Assume X is smooth of dimension $n \geq 2$. Then $\dim \mathcal{B}(X) \leq 0$ and $\dim \mathcal{C}(X) \leq 1$. Hence on $\mathbb{P}^N \setminus \{$finite points$\}$, $X = E_{d-e+1}(X)$ holds as sets, and on $\mathbb{P}^N \setminus \{$finite points$\} \cup \{$finite lines on $X\}$, $X = E_{d-e+1}(X)$ holds as schemes.*

In general, we cannot expect that $\mathcal{B}(X)$ and $\mathcal{C}(X)$ are empty. Thus, to study (B_{d-e+1}) further, we have to look at the structure of X with nonempty $\mathcal{B}(X)$ and $\mathcal{C}(X)$. Roughly speaking, if Λ is an l-dimensional irreducible component of $\overline{\mathcal{B}(X)}$ or $\overline{\mathcal{C}(X)}$, then X is a divisor of an $(n+1)$-dimensional cone over an $(n-l)$-dimensional variety with vertex Λ. If we resolve this cone so that it will be a smooth projective space bundle $\mathbb{P}$ over a smooth $(n-l)$-dimensional variety Y, then we see that X is the image of a prime Cartier divisor on $\mathbb{P}$ which is defined one equation on $\mathbb{P}$.

Theorem 7. *Assume that there exists an l-dimensional irreducible component Λ of $\overline{\mathcal{B}(X)}$ or $\overline{\mathcal{C}(X)}$. Let $\pi_\Lambda : \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{N-l-1}$ be the projection from Λ. Then there exist a smooth projective variety Y of dimension $n-l$ with morphism $\nu : Y \to \mathbb{P}^{N-l-1}$ birational onto image and with $\mathcal{O}_Y(1) = \nu^*\mathcal{O}_{\mathbb{P}^{N-l-1}}(1)$, the projective space bundle $\mathbb{P} := \mathbb{P}_Y(\mathcal{O}_Y^{\oplus l+1} \oplus \mathcal{O}_Y(1))$ with projection $\tau : \mathbb{P} \to Y$ and with the subbundle $\tilde{\Lambda} := \mathbb{P}_Y(\mathcal{O}_Y^{\oplus l+1})$, and a prime divisor $\tilde{X} \in |\mathcal{O}_{\mathbb{P}}(\mu) \otimes \tau^*\mathcal{L}|$ for some $\mu \geq 2$ and some $\mathcal{L} \in \operatorname{Pic} Y$ such that X and $\operatorname{Cone}(\Lambda, X)$ are the birational images of $\tilde{X}$ and $\mathbb{P}$ by the morphism $\varphi : \mathbb{P} \to \mathbb{P}^N$ defined by ν and $\tilde{\Lambda} \to \Lambda$. Moreover*

1. *If $\Lambda \subseteq \overline{\mathcal{B}(X)}$, then $\mathcal{L} \cong \mathcal{O}_X$.*
2. *If $\Lambda \subseteq \overline{\mathcal{C}(X)}$ and if the restricted Gauss map $\gamma|\Lambda : \Lambda \cap \operatorname{Sm} X \to \mathbb{G}(n, \mathbb{P}^N)$ mapping $x \in \Lambda \cap \operatorname{Sm} X$ to $T_x(X) \subseteq \mathbb{P}^N$ is non-constant, then Y is a rational scroll and $\mathcal{L}$ is the line bundle associated with its fibre.*
3. *If $\Lambda \subseteq \overline{\mathcal{C}(X)}$ and if the restricted Gauss map $\gamma|\Lambda$ is constant, then $\mathcal{L}$ has a global section and $(\mathcal{L}, \mathcal{O}_Y(1)^{n-l-1}) = 1$.*

This structure theorem tells us how to construct the variety with nonempty $\mathcal{B}(X)$ and $\mathcal{C}(X)$, as well as how to obtain other hypersurfaces cutting out X.

Corollary 8. *Under the same assumption as in Theorem 3, if $\tilde{X}$ is defined by $G \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu) \otimes \tau^*\mathcal{L})$ as $\tilde{X} = (G)_0$, then the homogeneous ideal I_X of X in $\mathbb{P}^N$ is $\oplus_{m \geq 0}(\varphi^*)^{-1}(G \cdot H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m-\mu) \otimes \tau^*\mathcal{L}^\vee))$, where $\varphi^* : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m))$ is the homomorphism induced by $\varphi : \mathbb{P} \to \mathbb{P}^N$.*

Roughly speaking, I_X is generated by the equations of $\nu(Y)$ and those obtained from G by elimination. The latter equations are necessary to separate Λ. At this stage, I cannot control the degree of such equations based on d and e in general.

Example 9. *Let $\mathbb{P}$ be the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{Q})$ over $Y := \mathbb{P}^1$, associated with $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^1}z_0 \oplus \mathcal{O}_{\mathbb{P}^1}z_1 \oplus \mathcal{O}_{\mathbb{P}^1}(4)z_2$, with projection $\tau : \mathbb{P} \to \mathbb{P}^1$ and the tautological bundle $\mathcal{O}_{\mathbb{P}}(1)$. Here z_i are formal basis. Let s, t be the homogeneous coordinates of $\mathbb{P}^1$ and let $T_0, \ldots, T_5$ be those of $\mathbb{P}^5$. Let $\varphi : \mathbb{P} \to \mathbb{P}^5$ be the morphism defined by $T_0 = z_0$, $T_1 = z_1$, $T_2 = s^4z_2$, $T_3 = s^3tz_2$, $T_4 = st^3z_2$, $T_5 = t^4z_2$ for $z_0, \ldots, t^4z_2 \in H^0(\mathcal{O}_{\mathbb{P}}(1))$. Let $\tilde{X}_i$ $(i = 1, 2)$ be divisors defined by $G_1 = z_1^2 - s^2t^2z_0z_2$ and $G_2 = z_1^2 - s^3tz_0z_2 \in H^0(\mathcal{O}_{\mathbb{P}}(2))$. Then $X_i := \varphi(\tilde{X}_i)$ are nondegenerate projective surfaces of degree 8 such that the line $\Lambda := \varphi(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}z_0 \oplus \mathcal{O}_{\mathbb{P}^1}z_1))$ is contained in $\overline{\mathcal{B}(X_i)}$. In this case, $G_2 = T_1^2 - T_0T_3$ but G_1 cannot be expressed by a polynomial of $T_0, \ldots, T_5$. If we set $\tilde{G}_1 := G_1 \cdot (z_1^2 + s^2t^2z_0z_2)$, then $\tilde{G}_1 = T_1^4 - T_0^2T_2T_5$. By these equations, Λ can be separated from X_i, i.e., as sets, $X_1 \cap \Lambda = (T_1^4 - T_0^2T_2T_5 = 0) \cap \Lambda$ and $X_2 \cap \Lambda = (T_1^2 - T_0T_3 = 0) \cap \Lambda$. This seems to relate the fact* $\operatorname{length}(X_1 \cap \Lambda) = 4$ *and* $\operatorname{length}(X_2 \cap \Lambda) = 2$.

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Mirror symmetry and tropical geometry

Janko Böhm

1. Introduction

Mirror symmetry is a phenomenon postulated by theoretical physics in the context of superstring theory, which has the goal to unify quantum mechanics and general relativity. In two out of 5 types of superstring theories the world is represented as the product of a Minkowski 4-space and a Calabi-Yau 3-fold. Mirror symmetry is the prediction that one type of string theory on a Calabi-Yau 3-fold X is equivalent to the second type on some other Calabi-Yau X° identifying the complex moduli of X with the Kähler moduli of X° and vice versa. We develop a framework for mirror symmetry via tropical and toric geometry, which leads to an explicit mirror construction.

2. Tropical mirror construction

The setup we consider is a degeneration of Calabi-Yau varieties $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ with fibers in a toric Fano variety $Y = X(\Sigma)$, $\Sigma \subset N_{\mathbb{R}}$, $N = \mathbb{Z}^n$, $M = \operatorname{Hom}(N, \mathbb{Z})$ with Cox ring

$$S = \mathbb{C}[y_r \mid r \in \Sigma(1)] \qquad 0 \to M \overset{A}{\to} \mathbb{Z}^{\Sigma(1)} \overset{\deg}{\to} A_{n-1}(Y) \to 0$$

$\mathfrak{X}$ defined by $I \subset \mathbb{C}[t] \otimes S$ and monomial special fiber X_0 given by $I_0 \subset S$. Let Y be represented by the Fano polytope Δ^* via the one-to-one correspondence of Fano polytopes and toric Fano varieties.

We associate to $\mathfrak{X}$ polyhedral objects representing a new toric Fano variety Y°, a new monomial Calabi-Yau $X_0^\circ \subset Y^\circ$ and the deformations of X_0° building the mirror degeneration $\mathfrak{X}^\circ$.

2.1. **Embedding step of the tropical mirror construction.** We identify the space of homogeneous weight vectors on S as

$$\frac{\mathrm{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)},\mathbb{R}\right)}{\mathrm{Hom}_{\mathbb{R}}\left(A_{n-1}\left(Y\right)\otimes\mathbb{R},\mathbb{R}\right)}\underset{\varphi}{\overset{\cong}{\rightleftarrows}} N_{\mathbb{R}}$$

and associate to $\mathfrak{X}$ the special fiber Gröbner cone

$$C_{I_0}\left(I\right)=\left\{-\left(w_t,w_y\right)\in\mathbb{R}\oplus N_{\mathbb{R}}\mid L_{>_{\left(w_t,\varphi\left(w_y\right)\right)}}\left(I\right)=I_0\right\}$$

and special fiber polytope

$$\nabla=C_{I_0}\left(I\right)\cap\left\{w_t=1\right\}\subset N_{\mathbb{R}}$$

∇^* is a Fano polytope, hence defines a toric Fano variety Y° with Cox ring S°.

2.2. **Special fiber step of the tropical mirror construction.**
Let $L=\mathbb{C}\left\{\left\{s\right\}\right\}$ be the field of Puiseux series. Extending the original definition from [Bergman, 1971] we introduce a non-Archimedian type definition of the Bergman fan of I as the closure of the image of

$$V_L\left(I\right)\subset L^*\times\left(L^*\right)^n\overset{\mathrm{val}}{\to}\mathbb{R}\oplus N_{\mathbb{R}}$$

under the component-wise valuation map. Here we identify the tori

$$\left(L^*\right)^{\Sigma(1)}/\,\mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}\left(Y\right),L^*\right)\cong\left(L^*\right)^n$$

The special fiber Bergman complex

$$B\left(I\right)=BF\left(I\right)\cap C_{I_0}\left(I\right)\cap\left\{w_t=1\right\}\subset\nabla$$

represents the strata of a monomial Calabi-Yau $X_0^\circ\subset Y^\circ$ defined by $I_0^\circ\subset S^\circ$.

2.3. **Deformation step of the tropical mirror construction.** The lattice points m in the support of $B\left(I\right)^*\subset\nabla^*$ correspond to degree 0 Cox Laurent monomials $A\left(m\right)$ and represent the first order deformations of X_0 contributing to $\mathfrak{X}$. In the same way the first order deformations of X_0° contributing to the mirror degeneration $\mathfrak{X}^\circ$ with fibers in Y° are represented by the lattice points of the support of $\mathrm{Strata}\left(X_0\right)^*\subset\Delta^*$ and $\mathfrak{X}^\circ$ is defined by

$$I^\circ=\left\langle m^\circ+t\cdot\sum_{\alpha\in\mathrm{Strata}(X_0)^*\cap N}c_\alpha\cdot\alpha(m^\circ)\mid m^\circ\in I_0^\circ\right\rangle\subset\mathbb{C}[t]/\langle t^2\rangle\otimes S^\circ$$

The construction comes with a natural mirror map via the interpretation of lattice points as deformations and divisors. We obtain a computer implemented algorithm to compute the mirror family.

3. Results

Theorem. *Let Δ be reflexive, $Y = \mathbb{P}(\Delta)$ the corresponding Gorenstein toric Fano variety with Cox ring S, $\Sigma = \mathrm{NF}(\Delta)$ the normal fan of Δ, $\Sigma(1) = \bigcup_{j=1}^{c} I_j$ a partition of the Cox variables such that $E_j = \sum_{r \in I_j} D_r$ are Cartier, spanned by global sections. Let $\mathfrak{X}$ be the natural monomial degeneration, defined by the ideal*

$$\langle m_j + t \cdot \sum_{m \in S_{[E_j]}} c_m \cdot m \mid j = 1, ..., c \rangle \subset S \otimes \mathbb{C}[t]$$

$$m_j = \prod_{r \in I_j} y_r$$

associated the complete intersection X given by general sections of $\mathcal{O}_Y(E_j)$. Then the tropical mirror construction applied to $\mathfrak{X}$ gives the natural degeneration associated to the Batyrev-Borisov mirror of X.

- Introducing the notion of a subset of Fermat deformations of $\mathrm{supp}\left(B(I)^*\right)$ the twofold interpretation of lattice points as divisors and deformations gives new insight into the concept of orbifolding mirrors.
- We reproduce the mirror given in [Rødland, 1998] for the Pfaffian non complete intersection Calabi-Yau 3-fold of degree 14 in $\mathbb{P}^6$ given by the Pfaffians of a general skew-symmetric map $7\mathcal{O}(-1) \to 7\mathcal{O}$.
- We obtain a new mirror by applying the tropical mirror construction to a monomial degeneration of the Pfaffian Calabi-Yau 3-fold of degree 13 in $\mathbb{P}^6$ defined by the Pfaffians of a general skew-symmetric map $\mathcal{O}(-2) \oplus 4\mathcal{O}(-1) \to \mathcal{O}(1) \oplus 4\mathcal{O}$.

We expect the tropical mirror algorithm to work for a large class of explicit Gorenstein Calabi-Yau examples, e.g., toric Pfaffians and deformations of Stanley-Reisner ideals.

4. Perspectives

The polyhedral data involved in the tropical mirror construction naturally relates to various concepts from mirror symmetry, in particular, one may ask how to compute from the data

(1) *stringy E-functions,*
(2) *SYZ-fibrations,*
(3) *integrally affine structures and ask for the relation to the mirror construction by Gross and Siebert,*
(4) *instanton numbers and the A-model correlation functions*
(5) *periods and Picard-Fuchs equations via GKZ-hypergeometric systems.*

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Virtual Riemann-Roch-Theorem

Barbara Fantechi

(joint work with Lothar Göttsche)

The Grothendieck Riemann-Roch-Theorem applies to proper morphisms of smooth varieties. Ciocan-Fontanine and Kapranov have proven a “virtual” version for proper morphisms $f : X \longrightarrow Y$ of schemes where Y is smooth, and X is the π_0 of a 1-dg-manifold.

Jointly with Lothar Göttsche, we extend their result to the weaker (and easier to verify) assumption that X admits a one-perfect obstruction theory and is globally embeddable in a smooth scheme. As a corollary one obtains a virtual Hirzebruch Riemann-Roch (i.e., case where Y is a point) and a notion of virtual holomorphic Euler characteristic; the latter can be used to define virtual analogues of, e.g., the χ_{-y} genus and the elliptic genus.

The desire to extend this result to the case where Y is also only virtually smooth (i.e., has a 1-perfect-obstruction theory and is embeddable) leads us to introduce a (2-)category of virtually smooth schemes.

A virtual pullback theorem by C. Manolache reduces Grothendieck Riemann-Roch for virtually smooth schemes to the construction of a virtual pushforward among the K^0 groups satisfying a compatibility relation with the virtual pullback.

We expect that we will be able to define a virtual blow up in the category of virtually smooth schemes, and use it to extend Grothendick Riemann-Roch to Deligne-Mumfort stacks.

We also conjecture that the virtual pushforward can be naturally defined for morphisms of 1-dg-manifolds.

What are tropical counterparts of algebraic varieties?

Grigory Mikhalkin

The talk was a survey of basic definitions in Tropical Geometry from [6] along with some examples and considerations justifying these definitions.

The tropical *affine space* $\mathbb{T}^n$ is defined to be the nth power of the tropical line $\mathbb{T} = [-\infty, +\infty)$ enhanced with the Euclidean topology. The *structure sheaf* $\mathcal{O}$ on $\mathbb{T}^n$ is defined in the following way.

Let $U \subset \mathbb{T}^n$ be an open set. We say that $f : U \to \mathbb{T}$ is a *regular function* and write $f \in \mathcal{O}(U)$ if

- $f|_{U\cap\mathbb{R}^n}$ coincides with a tropical Laurent polynomial

$$f(x_1, \dots, x_n) = \text{“}\sum_{j_1,\dots,j_n} a_{j_1\dots j_n} x_1^{j_1} \dots x_n^{j_n}\text{”} = \max_{j_1,\dots,j_n} \{a_{j_1\dots j_n} + j_1 x_1 + \dots + j_n x_n\},$$

  where $a_{j_1\dots j_n} \in \mathbb{T}$ is a finite collection of coefficients with $j_k \in \mathbb{Z}$
- f is a continuous function on U.

Let us consider the framework where we have a topological space X with the structure sheaf $\mathcal{O}$ of $\mathbb{T}$-algebras (see [6]). Note that in such case we also have a natural structure sheaf on the product space $X \times \mathbb{T}$. Each regular function defines a *Cartier divisor* also called a *hypersurface* V_f along with a *tropical modification* $\tau_f : \tilde{U} \to U$, cf. [5]. The hypersurface V_f is defined as the locus where “$\frac{1_{\mathbb{T}}}{f}$” $= -f$ is not locally regular. The result $\tilde{U}$ of the modification τ_f is defined as $V_{\text{“}y+f(x)\text{”}} \subset U \times \mathbb{T}$ and τ_f is defined as the projection onto the first coordinate. Note that $V_{\text{“}y+f(x)\text{”}}$ comes with the structure sheaf induced from $U \times \mathbb{T}$.

We say that a map $\tau_Z : (\tilde{X}, \mathcal{O}_{\tilde{X}}) \to (X, \mathcal{O}_X)$ is a tropical modification along a hypersurface $Z \subset X$ if locally τ_Z and Z are given by τ_f and V_f for suitable f. The hypersurface Z is called the *center* of τ_Z.

We define *tropical manifolds* inductively by their dimension. For simplicity in this extended Oberwolfach abstract we restrict our attention to the compact case, i.e. we define here *closed* tropical manifolds. The 0-dimensional tropical manifolds are finite collection of points (with the discrete topology). Any function on such a manifold is regular. Once we get the class of $(k-1)$-dimensional tropical manifolds it defines for us the following equivalence relation. We say that the pairs $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ are k-equivalent if they can be connected with the sequence of tropical modifications (or their converses) so that all the centers of modifications in this process are tropical $(k-1)$-manifolds.

Definition 1. *A pair $(X, \mathcal{O})$ is called a* tropical n-manifold *if*

- *X is a compact Hausdorff topological space and $\mathcal{O}$ is a sheaf of $\mathbb{T}$-valued functions there.*
- *$(X, \mathcal{O})$ is locally n-equivalent to the tropical affine space $(\mathbb{T}^n, \mathcal{O})$.*

Note that if two n-manifolds are k-equivalent then n must be equal to k, so we may speak just of tropical equivalence of tropical manifolds. This equivalence relation is much finer than birational equivalence as only smooth hypersurfaces are allowed to be the centers. This is a tropical analogue of biregular equivalence (isomorphism) in classical geometry as in classical geometry smooth blowups in codimension 1 do not change the isomorphism type. We say that a Hausdorff compact topological space with a sheaf of $\mathbb{T}$-valued functions is a *tropical variety* if it locally can be presented as an image of a tropical manifold under a tropical morphism.

Note that this definition postulates the resolution of singularities in tropical geometry. Note also that the current definitions are within the $(X, \mathcal{O})$-paradigm and thus exclude such elegant examples as the tropical K3-surfaces modeled on the 2-sphere (see [4], [3]).

The following theorem is a rather straightforward corollary of the definitions.

Theorem 1. *Any tropical manifold as well as any tropical variety can* locally *be presented as a limit of a 1-parametric family of complex algebraic varieties,*

Clearly any tropical variety X can locally be embedded to $\mathbb{T}^N$ for some large $N >> 0$. But in general detecting whether $X \subset \mathbb{T}^N$ is a tropical n-variety is a difficult task. Some easy *necessary* combinatorial conditions to be a variety are the structure of a n-dimensional polyhedral complex, the integrality of the slope of all facets (n-cells) and the *balancing condition* at each $(n-1)$-dimensional cell E of X (see [5] and [7]). The last condition takes into account only the facets adjacent to E and can be adjusted to accommodate integer weights on the facets of X.

A balanced weighted n-dimensional subcomplex of $\mathbb{T}^N$ (or, more general, of a tropical manifold X) is called a *tropical n-cycle*, see [5] (or [1] for a detailed version of Chapter 4 of [5]). A tropical cycle is called *effective* if all the weights on its facets are non-negative.

For a point x in an effective n-cycle in $\mathbb{R}^N$ one can define a *local multiplicity* $m(x)$ by taking the local intersection number with all *standard* cycles of dimension $N-n$ centered at x associated to an integer basis of $\mathbb{R}^N$ and minimizing the result. If X is a tropical manifold then $m_X(x) = 1$ for any $x \in X$, but for tropical varieties the multiplicity can be larger.

The theorem below is an observation made jointly by G. Mikhalkin, B. Sturmfels and G. Ziegler. It is based on the works [2] and [8]. These papers associate a tropical cycle in $\mathbb{R}^N$ to each matroid. It is easy to check that all cycles arising from matroids have $m(x) = 1$ at all their points x.

Theorem 2. *Let $Z \subset \mathbb{R}^N$ be a tropical cycle and $x \in Z$ be a point with $m_Z(x) = 1$. Then the cycle Z is locally given by a matroid near x.*

Effective cycles with $m(x) = 1$ at all points may behave like smooth submanifolds, but they form a much wider class of subspaces. Since the only condition (balancing) is required at codimension 1 they are tropical counterparts of pseudomanifols (or homological manifolds), they carry the fundamental class but do not possess finer properties of smooth submanifolds such as Poincaré duality. Yet as they do carry the fundamental class they do encode a homology class in the ambient manifold. Furthermore, such classes are would-be degenerations of Hodge classes in the complex world: they are integer and of (p,p)-types (where p is the dimension of the tropical cycle).

Note that in classical geometry any effective p-dimensional degree 1 algebraic cycle in $\mathbb{P}^N$ must be a (smooth) projective space $\mathbb{P}^p$ itself. The $N+1$ boundary hyperplanes of $\mathbb{P}^N$ cut an arrangement of $N+1$ hyperplanes in $\mathbb{P}^p$. This arrangement corresponds to a (classically realizable) matroid.

However tropically passing from a smooth submanifold of $\mathbb{TP}^N$ to an effective degree 1 cycle in $\mathbb{TP}^N$ actually yields a weaker concept of realizability. It makes any matroid tautologically realizable in the tropical world.

Corollary 3. *Any matroid is realizable by hyperplane arrangement in a degree 1 effective tropical cycle in* $\mathbb{TP}^N$.

Corollary 4. *If* $Z \subset \mathbb{R}^N$ *is a tropical cycle corresponding to a non-realizable (over* $\mathbb{C}$*) matroid then* Z *is not representable as the limit of amoebas of any family of effective algebraic cycles in* $(\mathbb{C}^\times)^N$.

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On the Canonical Covers of Surfaces of Minimal Degree

B. P. Purnaprajna

(joint work with Francisco J. Gallego)

Let X be a surface of general type with at worst canonical singularities and with K_X ample and base point free. Let $\varphi : X \to W$ be the morphism induced by the complete linear series $|K_X|$ such that W is a surface of minimal degree. We say φ is a canonical cover of a surface of minimal degree.

Canonical covers of surfaces of minimal degree have a ubiquitous presence in a variety of contexts. We mention a few here: the surfaces of general type for which Noether's inequality $K_X^2 \geq 2p_g - 4$ is an equality are precisely double canonical covers of surfaces of minimal degree [Hor]. They appear in the study of linear series on Calabi-Yau threefolds as illustrated in the works [OP], [GP4] and [BS].

They have an unavoidable presence as extremal cases in the determination of ring generators of the canonical ring of a variety of general type as seen in the results of Catanese, Ciliberto and more generally in the work of M. Green [Gr] (see [GP3] for the complimentary and converse results.) They also have a compelling presence in the so-called mapping geography of surfaces of general type.

It follows from a result of Beauville that if $\chi(\mathcal{O}_X) \geq 31$, then $\deg \varphi \leq 9$. As mentioned above, double covers were classified by Horikawa and degree three covers were classified by Konno [Ko].

In [GP1, 2] we have classified degree four Galois canonical covers. If W is smooth the classification shows that these covers are either non-simple cyclic covers or bidouble covers. If they are bidouble, then they are fiber product of double covers. We construct examples to show that all the possibilities of the classification do exist.

In [GP2] the quadruple Galois canonical covers of singular W are classified. To briefly describe these we need the following:

Let $q : Y \to W$ be the minimal desingularization of W. Let $\varphi : X \to W$ be the canonical morphism. There exists the following diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\bar{q}} & X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{q} & W \end{array}$$

where $\overline{X}$ is the normalization of the reduced part of $X \times_W Y$, which is irreducible, and the left vertical map $p : \overline{X} \to Y$ and $\bar{q}$ are induced by the projection from the fiber product onto each factor.

The classification shows four distinct possibilities for $\bar{q}$: for two of which $\bar{q}$ is crepant and for the other two $\bar{q}$ is non-crepant. When $\bar{q}$ is crepant, after quite a bit of work it can be shown that $\overline{X}$ behaves much like X when W was smooth. The non-crepant case is distinctly different. For example, when $\bar{q}$ is non-crepant and $G = \mathbf{Z}_2 \times \mathbf{Z}_2$, the surface $\overline{X}$ is not a fiber product but normalization of a fiber product and when $G = \mathbf{Z}_4$ the morphism p is still a composition of two double covers but unlike in the smooth case the branch divisors are not pull backs. The main philosophical reason why these two exceptions occur is because the canonical divisor of $\overline{X}$ has a fixed part that contracts to w, the unique singular point of W.

Some consequences of the classification:

The classification shows that the quadruple covers are unique among covers of *all* other degrees from various perspectives.

Unboundedness of geometric genus p_g and irregularity q: one of the consequences of the classification is the existence of families of quadruple canonical covers with unbounded geometric genus which is in sharp contrast with triple covers, and existence of families with unbounded irregularity which is in sharp contrast with covers of *all* other degrees. We show the existence of families for each possible value of p_g and for each possible value of q.

Geography of surfaces: For double covers the Chern ratio $\frac{c_1^2}{c_2} \to \frac{1}{5}$ as $p_g \to \infty$, but $\frac{c_1^2}{c_2} < \frac{1}{5}$. For triple covers, $\frac{c_1^2}{c_2}$ takes on three values highest of which is $\frac{1}{7}$. For both double and triple covers the Chern pair (c_1^2, c_2) lies well below the line $c_1^2 = 2c_2$. By contrast the quadruple covers traverse much richer geography than covers of other degrees due to existence of families with unbounded p_g and q. For quadruple covers (c_1^2, c_2) approaches two different lines $c_1^2 = \frac{1}{2}c_2$ and $c_1^2 = 2c_2$, the latter line is actually attained.

Existence of higher degree canoncial covers: The classification results of quadruple Galois canonical covers show that if W is singular then $p_g \leq 4$ and $q = 0$. Together with the results of Horikawa for double covers and Konno for triple covers, the following striking numerology emerges for surfaces of general type that are canonical covers of singular targets: if $\deg \varphi = 2$, then $p_g \leq 6$ and $q = 0$, if $\deg \varphi = 3$, then $p_g \leq 5$ and $q = 0$, for degree four Galois, $p_g \leq 4$ and $q = 0$.

A natural question to ask is: Let $\varphi : X \to W$ be a canonical cover of a singular surface of minimal degree W. Is $\deg \varphi \leq 4$.

If the answer to the above question is positive, then together with the results in [GP3] (which says that there are no odd degree canonical covers of smooth scrolls), there will be no canonical covers of degree odd bigger than 3 of surfaces of minimal degree, except perhaps for covers of $\mathbf{P}^2$ thus showing that the situation is much closer to the case of curves.. There are some strong hints towards a positive solution to the question above: in [GP3, Corollary 7.3], we prove that there are no regular Galois canonical covers of degree prime $p \geq 5$ of a surface of minimal degree W, either smooth or singular.

Bicanonical maps: Bicanonical map of a surface of general type is a much studied concept. Quadruple covers [GP5] exhibit diverse behavior regarding bicanonical maps: there are families for which bi-canonical maps are finite, families for which they are very ample and embeds X as a projectively normal variety and families for which the bi-canonical maps are birational (in two different ways) but not very ample.

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Unliftable Calabi-Yau threefolds

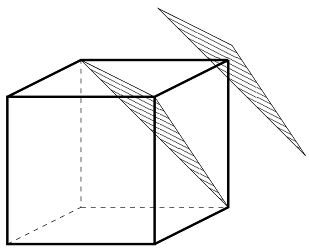
Duco van Straten
(joint work with Sławomir Cynk)

R. Vakil paraphrased *Murphys Law* in algebraic geometry as the statement that deformation spaces can and will be arbitrarily bad, unless there is a good reason that prevents this, [9]. It is well-known that curves, abelian varieties and K3 surfaces in characteristic p can be lifted to characteristic zero and in fact in each of these cases there are good cohomological reasons for the result to hold. The question arises what is the case for Calabi-Yau threefolds. By the theorem of Bogomolov-Tian-Todorov, a Calabi-Yau threefold in characteristic zero has unobstructed deformations. Somewhat surprisingly, M. Hirokado [5] and S. Schroer [8] have constructed examples of Calabi-Yau 3-folds in characteristic 2 and 3 that do not lift to characteristic zero. It is an interesting problem to construct further examples, especially in higher characteristic, [4].
We propose a general strategy to obtain examples using the peculiar properties of small resolutions of rigid Calabi-Yau spaces with nodes, [3]. The following theorem from [2] follows from general deformation theoretical considerations.

Theorem
Let $\mathcal{X}$ be a scheme over $S = Spec(A)$, A a complete domain with residue field k and fraction field $K = Q(A)$. Assume that:
1) The generic fibre $X_\eta := \mathcal{X} \otimes_A K$ is smooth.
2) The special fibre $X := \mathcal{X} \otimes_A k$ is rigid with nodes as singularities.
Let $\pi : Y \longrightarrow X$ be a small resolution. Then Y does not lift to S.

Many interesting examples can be constructed by considering so called *double octics*, two-fold coverings of $\mathbb{P}^3$ ramified over a surface of degree 8. If the singularities of the octic are not too bad, the resulting singular double cover will still have a Calabi-Yau resolution, whose infinitesimal deformations can be identified with the infinitesimal *equisingular* deformations of the ramification octic, [1].
As an example, consider the double octic ramified over a union of eight planes.

$$u^2 = (x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x+y+z-t)(x+y+z-3t)$$

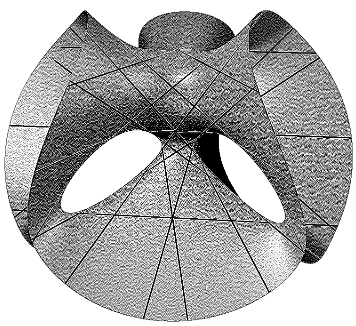


In characteristic zero, the configuration has exactly 10 fourfold points. The Hodge numbers of a small resolution can be seen to be $h^{11} = 42$, $h^{12} = 0$. However, in characteristic 3 something special happens: the plane $x+y+z-3t$ passes through the points $(1,1,1,1)$ and $(-1,-1,-1,1)$ of the cube and the configuration has 11 fourfold points. In characteristic 3 the double cover thus has two extra nodes, which can be resolved to give a non-liftable smooth projective Calabi-Yau threefold with $h^{1,1} = 41$, $h^{1,2} = 0$.
In a similar way, an example can be given of a smooth projective Calabi-Yau space over $\mathbb{F}_5$ which has an obstructed deformation, but which lifts to characteristic zero. In fact, it has a smooth model over $Spec(\mathbb{Z}[x]/(x^2+x-1))$.
A further nice example arises from the resolution of a double octic with equation

$$v^2 = (x^3+y^3+z^3+t^3+u^3)(x+y)(y+z)(z+t)(t+u)(u+x)$$

where

$$x+y+z+t+u=0$$



The branch-octic consists of the Clebsch cubic, together with tangent planes at 5 of its 10 Eckart points. This configuration is rigid and has a Calabi-Yau resolution whose Hodge numbers are $h^{1,1} = 52$, $h^{1,2} = 0$. In characteristic 5 the Clebsch cubic aquires an extra node, as to be expected from its interpretation as Hilbert-modular surface for $\mathbb{Q}(\sqrt{5})$, [6]. The resulting Calabi-Yau space has an extra node in characteristic 5. Upon taking a small resolution of this node, we obtain a non-liftable Calabi-Yau in characteristic 5, which however only exists in the category of algebraic spaces and cannot be realised as a projective variety.

In fact, it is quite easy to obtain such non-projective examples. Consider for example the fibre product of $E_1 \times_{\mathbb{P}^1} E_2 \longrightarrow \mathbb{P}^1$ of two semi-stable rational elliptic fibred surfaces. For example, one can pair up the singular fibres as follows:

Value	$8/9$	0	1	∞	$\frac{1}{2}(55\sqrt{5}-123)$
E_1	I_1	I_3	I_6	I_2	–
E_2	–	I_5	I_1	I_5	I_1

This singular threefold admits a Calabi-Yau resolution. As

$$\frac{8}{9} = \frac{1}{2}(55\sqrt{5} - 123) \mod 9001,$$

in characteristic 9001 the two 'free' I_1 fibres match up and so the space aquires an extra A_1 singularities, the small resolution of which does not have a lifting to characteristic zero. These examples were also found independently by C. Schoen, [7].

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Rational curves on holomorphic symplectic varieties

Brendan Hassett

(joint work with Yuri Tschinkel)

Let F be a *holomorphic symplectic variety*, by which we mean a smooth projective complex simply-connected variety, with $\Gamma(F, \Omega^2_F)$ generated by a nondegenerate holomorphic two-form ω. Examples include [1, 6,7]

- projective K3 surfaces;
- punctual Hilbert schemes of projective K3 surfaces S, i.e., $F = S^{[n]}$ the Hilbert scheme parametrizing zero-dimensional length-n subschemes of S;

- generalized Kummer varieties, arising the fibers of addition map
$$\Sigma : A^{[n]} \to A$$
where A is an abelian surface.

The symplectic form ω can be used to define a nondegenerate integral quadratic form $(,)$ on $H^2(F, \mathbb{Z})$ [1, 8], called the *Beauville–Bogomolov form.* This extends via duality to a $\mathbb{Q}$-valued quadratic form on $H_2(F, \mathbb{Z})$. For a K3 surface S, $(,)$ is the intersection form which equals

$$H^2(S, \mathbb{Z})_{(,)} \simeq U^{\oplus 3} \oplus_{\perp} (-E_8)^{\oplus 2},$$

where U is the hyperbolic plane lattice and E_8 is the positive-definite lattice associated with the corresponding Dynkin diagram. When $F = S^{[n]}$, the Hilbert scheme of a K3 surface S, we have

$$H^2(F, \mathbb{Z})_{(,)} = H^2(S, \mathbb{Z})_{(,)} \oplus_{\perp} \mathbb{Z}\delta, \quad (\delta, \delta) = -2(n-1). \tag{1}$$

The non-reduced subschemes form a divisor with class 2δ; each homology class of S determines a class on $S^{[n]}$, corresponding to subschemes with support along the corresponding cycle. (We abuse notation by using the same symbol for the class on S and the resulting class on $S^{[n]}$.) The induced form on homology can be written

$$H_2(F, \mathbb{Z})_{(,)} = H_2(S, \mathbb{Z})_{(,)} \oplus_{\perp} \mathbb{Z}\delta^{\vee}, \quad (\delta^{\vee}, \delta^{\vee}) = -\frac{1}{2(n-1)}, \delta.\delta^{\vee} = 1, \tag{2}$$

where $\delta^{\vee}$ parametrizes subschemes supported at $n-1$ fixed distinct points of S. Decompositions (1) and (2) are compatible with the Hodge structures on the second homology/cohomology groups.

Our point of departure is the following characterization of the ample cone of a K3 surface (see [5, §2], for example):

Theorem 1. *Let (S, g) be a polarized K3 surface. A divisor class h is ample if and only if $h.C > 0$ for each curve class C with $g.C > 0$ and $(C, C) \geq -2$.*

This characterization depends only on the Hodge structure on $H^2(S, \mathbb{C})$ and the datum of a single polarization. Note that extremal effective curve classes R with $(R, R) = -2$ represent smooth rational curves $\mathbb{P}^1 \subset S$.

Our main goal is to generalize this statement to higher-dimensional holomorphic symplectic varieties. Such a result would allow us to read off the ample cone of the Hilbert scheme $S^{[n]}$ of a polarized K3 surface (S, f) from its Hodge structure. Indeed, for a sufficiently small rational $\epsilon > 0$ the divisor $f - \epsilon\delta \in \mathrm{Pic}(S^{[n]})$ is ample. While some qualitative statements have been obtained [2, 4], these do not completely determine the ample cone.

We prove the following generalization to dimension four

Theorem 2. *Let (F, g) be a polarized holomorphic symplectic variety deformation equivalent to $S^{[2]}$. A divisor class h is ample if $h.R > 0$ for each curve class R with $g.R > 0$ and $(R, R) \geq -5/2$.*

Previously [3], we conjectured this condition is necessary and sufficient for h to be ample; the necessity remains open.

The proof of the theorem uses classification results for extremal rays R and the associated contractions $\beta : F \to F'$:

(1) If β is divisorial then the exceptional locus is fibered in $\mathbb{P}^1$'s over a holomorphic symplectic surface; here $(R,R) = -1/2$ or -2 and R is the class of a fiber.

(2) If β is small then it is the contraction of a $\mathbb{P}^2 \subset F$; here $(R,R) = -5/2$ and R is the class of a line in the $\mathbb{P}^2$.

We propose the following generalization:

Conjecture 3. *Let (F,g) be a polarized holomorphic symplectic manifold deformation equivalent to $S^{[n]}$, where S is a K3 surface. A divisor class h is ample if and only if $h.R > 0$ for each curve class R with $g.R > 0$ and $(R,R) \geq -(n+3)/2$.*

Furthermore, extremal rays R with

$$-(r+3)/2 \leq (R,R), \quad 1 \leq r \leq n,$$

correspond to contractions $\beta : F \to F'$ with codimension $j \leq r$ exceptional locus birationally fibered in $\mathbb{P}^j$'s

$$\begin{array}{ccccccc} \mathbb{P}^j & \rightarrow & E^\circ & \subset & E & \subset & F \\ & & \downarrow & & \downarrow & & \\ & & \Sigma^\circ & \subset & \Sigma & & \end{array}.$$

Here Σ is the base of the fibration, a (possibly singular) variety of dimension $2(n-j)$ with a holomorphic symplectic form on its smooth locus, and $\Sigma^\circ \subset \Sigma$ is an open subset over which the fibration is smooth. The extremal ray R is the class of a line in a fiber.

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Reporter: Janko Böhm

Participants

Prof. Dr. Dan Abramovich
Department of Mathematics
Brown University
Box 1917
Providence, RI 02912
USA

Jarod Alper
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Dr. Roya Beheshti
Dept. of Mathematics
Washington University
Campus Box 1146
One Brookings Drive
St. Louis, MO 63130-4899
USA

Janko Böhm
FR 6.1 - Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken

Prof. Dr.
Hans-Christian von Bothmer
Institut für Algebraische Geometrie
Gottfried Wilh. Leibniz Universität
Welfengarten 1
30167 Hannover

Prof. Dr. Lucia Caporaso
Dipartimento di Matematica
Universita degli Studi Roma Tre
Largo S. L. Murialdo, 1
I-00146 Roma

Dawei Chen
Department of Mathematics
Harvard University
One Oxford Street
Cambridge, MA 02138
USA

Dr. Ethan Cotterill
Dept. of Mathematics and Statistics
Queen's University
Jeffery Hall
Kingston ONT K7L 3N6
CANADA

Prof. Dr. Olivier Debarre
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Prof. Dr. Wolfram Decker
Fachrichtung - Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken

Prof. Dr. Igor Dolgachev
Department of Mathematics
University of Michigan
East Hall, 525 E. University
Ann Arbor, MI 48109-1109
USA

Robert Easton
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
USA

Prof. Dr. Lawrence Ein
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
USA

Prof. Dr. David Eisenbud
Mathematical Science Research
Institute
17 Gauss Way
Berkeley CA 94720-5070
USA

Prof. Dr. Carel Faber
Matematiska Institutionen
Kungl. Tekniska Högskolan
Lindstedtsvägen 25
S-10044 Stockholm

Prof. Dr. Barbara Fantechi
S.I.S.S.A.
Via Beirut 2 - 4
I-34014 Trieste

Prof. Dr. Gavril Farkas
Institut für Mathematik
Humboldt-Universität
10099 Berlin

Maksym Fedorchuk
Dept. of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Dr. Lothar Göttsche
International Centre for
Theoretical Physics, International
Atomic Energy Agency, UNESCO
P. O. B. 586 Miramare
I-34100 Trieste

Prof. Dr. Joseph Harris
Dept. of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Prof. Dr. Brendan Hassett
Dept. of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
USA

Prof. Dr. Alina Marian
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
USA

Prof. Dr. Grigory Mikhalkin
Department of Mathematics
University of Toronto
40 St.George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Atsushi Noma
Department of Mathematics
Yokohama National University
79-2 Tokiwadai, Hodogaya-ku
Yokohama 240-8501
JAPAN

Prof. Dr. Dragos Oprea
Department of Mathematics
Stanford University
Building 380
Serra Mall
Stanford CA 94305-2125
USA

Dr. Brian Osserman
Department of Mathematics
University of California, Davis
1, Shields Avenue
Davis, CA 95616-8633
USA

Prof. Dr. Giorgio Ottaviani
Dipartimento Matematica "U.Dini"
Universita degli Studi
Viale Morgagni, 67/A
I-50134 Firenze

Prof. Dr. Stavros Papadakis
Departamento de Matematica
Instituto Superior Tecnico
Avenida Rovisco Pais, 1
Lisboa 1049-001
PORTUGAL

Prof. Dr. Sam Payne
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Prof. Dr. Mihnea Popa
Department of Mathematics
University of Illinois at Chicago
Chicago, IL 60607
USA

Prof. Dr. Bangere P. Purnaprajna
Department of Mathematics
University of Kansas
405 Snow Hall
Lawrence, KS 66045-7567
USA

Prof. Dr. Kristian Ranestad
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
N-0316 Oslo

Dr. Mike A. Roth
Dept. of Mathematics and Statistics
Queen's University
Jeffery Hall
Kingston ONT K7L 3N6
CANADA

Prof. Dr. Frank-Olaf Schreyer
FB Mathematik und Informatik
GEb. E 2 4
Universität des Saarlandes
66123 Saarbrücken

Prof. Dr. Gregory G. Smith
Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston, Ontario K7L 3N6
CANADA

David Smyth
Department of Mathematics
Harvard University
One Oxford Street
Cambridge, MA 02138
USA

Prof. Dr. Jason Starr
Department of Mathematics
Stony Brook University
Stony Brook, NY 11794-3651
USA

Prof. Dr. Duco van Straten
Fachbereich Mathematik
Universität Mainz
Saarstr. 21
55122 Mainz

Prof. Dr. Tomasz Szemberg
Fachbereich Mathematik
Universität Duisburg-Essen
45117 Essen

Prof. Dr. Jenia Tevelev
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst MA 01003-9305
USA

Prof. Dr. Michael Thaddeus
Department of Mathematics
Columbia University
2990 Broadway, Math. Building 509
MC 4406
New York NY 10027
USA

Prof. Dr. Ravi Vakil
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
USA

Prof. Dr. Claire Voisin
Inst. de Mathematiques de Jussieu
Universite Paris VI
175 rue du Chevaleret
F-75013 Paris

Prof. Dr. Keiichi Watanabe
Department of Mathematics
Nihon University
Sakura Josui
Setagaya
Tokyo 156-8550
Japan

Prof. Dr. Chenyang Xu
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton, NJ 08544
USA

Prof. Dr. Stephanie Yang
Matematiska Institutionen
Kungl. Tekniska Högskolan
Lindstedtsvägen 25
S-10044 Stockholm