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Stochastic Analysis

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Abstract. The talks reviewed several directions in which progress in the general field of stochastic analysis occurred. Several themes were covered in depth: Among these themes a prominent role was played by the SLE (Schramm-Loewner equation), that appeared in talks on scaling scaling limits of dynamical percolation, Ising models on (a wide class of) graphs, and scaling limits of triangulations. Two other main themes were also present: the zeroes of random analytic functions (including the eigenvalues of random matrices), and the long time behavior of stochastic flows motivated by fluid dynamics. In addition a broad overview of recent developments such as trapping models, combinatorics of random matrices, diffusions on fractals, Wasserstein diffusions, Ising models on tree-like graphs, large deviations, the exclusion speed process and rough paths.

Mathematics Subject Classification (2000): 60xx, 60Hxx.

Introduction by the Organisers

The meeting took place on June 2-6, 2008, with over 45 people in attendance. Each day had 5-6 talks, except for Wednesday: the traditional hike took place in the afternoon, despite somewhat gray skies.

The talks reviewed several directions in which progress in the general field of stochastic analysis occurred since the last meeting of this theme in Oberwolfach three years ago. As has become a tradition, several themes were covered in some depth, in addition to a broad overview of recent development. Among these themes a prominent role was played by the SLE (Schramm-Loewner equation), that appeared in talks on scaling scaling limits of dynamical percolation, Ising models on (a wide class of) graphs, and scaling limits of triangulations. Two other main themes were also present: the zeroes of random analytic functions (including the eigenvalues of random matrices), and the long time behavior of stochastic flows motivated by fluid dynamics. What follows is a brief description of the topics covered in the talks.

A cluster of talks discussed the zero set of random functions (including the eigenvalues of random matrices). In the first lecture of the week, B. Tsirelson discussed moderate deviations for random fields, motivated by the asymptotic normality of the process of zeroes of random analytic functions; the result presented is not restricted however to this application. M. Sodin reported on his work with F. Nazarov, where they studied the number of nodal domains for random spherical harmonics. Concentration played an important role, and many exciting open questions were discussed. Later in the week, M. Krishnapur described the determinantal process arising from considering the zeroes of the determinant of random analytic matrices, and managed to provide a full proof within 30 minutes! Following his work with B. Valko, B. Virag introduced the "Brownian caroussel", a diffusion process closely related to the study of eigenvalues in the bulk of the spectrum for the β-ensembles of (tri-diagonal) random matrices, and showed how asymptotics of gap probabilities can be computed using this process. N. O'Connell explained how a diffusion process studied by Matsumoto and Yor was used (in his work with Baudoin) to study the partition function of polymer measures.

Another cluster of talks discussed various scaling limits of processes in the plane. O. Schramm and C. Garban described their work (joint with G. Pete) concerning the scaling limit of dynamical percolation, which exhibits several nice properties, such as conformal invariance and Markovianity. Y. LeJan, motivated by Lawler-Werner's Loop Soup, discussed first on a base graph and then in an infinite dimensional situation (where a renormalization procedure is employed) measures on ensembles of loops, with Dynkin's isomorphism theorem and certain "α-permanents" playing a role. A cluster of 3 talks discussed the Ising model over finite (non periodic) graphs. C. Boutillier reviewed the notion of isoradial graphs and the associated statistical mechanics of dimer configurations on decorated graphs, and discussed a formula derived by him and B. de Tiliere for the free energy of the Ising model. D. Chelkak and S. Smirnov then described, in back to back talks, their work on convergence of harmonic measures, Poisson kernels, and observables, for the Ising model, when the mesh size goes to 0, to the SLE based limits, when no special periodic structure of the mesh approximation is assumed. This is the first time where conformal invariance of a model from statistical physics for a large class of graphs is derived. The talk of V. Beffara described conditions on sequences of iterations of a triangulation that lead to site percolation probabilities converging to conformally invariant limits. Finally, N. Makarov discussed conformal field theory from a complex analyst point of view, and D. Wilson described his joint work with Rick Kenyon deriving formulas for the probabilities of the different possible node connections in a grove (a grove is a planar spanning forest in which every component tree contains one or more of a specified set of vertices on the outer face) or dimer models. These results are related to multiple SLE crossing probabilities.

The last thematic cluster of talks related to infinite dimensional flows. M. Hairer described a condition (based on the strong Feller property), that ensures ergodicity of flows by a coupling approach. This was followed by J. Mattingly, who related infinite dimensional ergodicity to the study of the Navier-Stokes equation. S. Kuksin's talk dealt with 2D turbulence in the Eulerian limit.

The other talks were on other directions but together gave a broad view of current progress in stochastic analysis. E. Maurel-Segala described the combinatorics of matrix models and the rigorous evaluation of terms in the expansion of the free energy. S. Chatterjee described a new, Stein-method based, proof of the strong embedding of random walk into a Brownian motion (a result originally due to Komlos-Major-Tusjnadi). G. Ben Arous identified all possible limits of randomly trapped random walks, as appearing for example in the context of random walk on the incipient critical Galton-Watson tree. Anton Bovier explained us how ageing takes place at intermediate time scales for spin glass models, in particular showing the connection with the universality of random traps as in Ben Arous talk. T. Kumagai described joint work with M. Barlow in which devoted to construction and uniqueness of Brownian motions on the Sierpinski carpet. M. von Renesse presented joint work with K-T. Sturm where they construct a natural "canonical process" on the set of probability distributions on the unit circle, where the space of measures is equipped with the Wasserstein distance. A. Dembo in a joint work with A. Montanari described Ising models on locally tree-like graphs, in particular he gave a very explicit method in order to compute the free entropy density in this context. O. Angel discussed the speed process of various types of multi-class totally asymmetric exclusion process. Den Hollander explained a quenched large deviation result (joint work with Birkner and Greven) for the empirical process of words cut out from a letter sequence via a renewal process. Finally Terry Lyons talked about the signature of a path: this is a map from a path in $\mathbb{R}^d$ to the infinite vector of its iterated integrals/ Levy areas. The algebraic and topological properties of this map were discussed as well as applications for numerical computation of expected values of SDE.

Workshop: Stochastic Analysis

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Abstracts

Moderate deviations for random fields and random complex zeroes

Boris Tsirelson

What is the most remarkable stationary random point process on the plane? The Poisson process is *hors concours;* being invariant (in distribution) under all measure preserving transformations of the plane, it is unrelated to the geometry of the plane. Chaotic analytic zero points (CAZP), actively investigated the last 10 years, are a point process on the complex plane $\mathbb{C}$ invariant under isometries (shifts, rotations, reflections) of $\mathbb{C}$. It consists of zeroes of the random entire function $\psi : \mathbb{C} \to \mathbb{C}$ defined by

$$\psi(z) = \sum_{k=0}^{\infty} \frac{\zeta_k z^k}{\sqrt{k!}} \,,$$

where $\zeta_0, \zeta_1, \dots$ are independent standard complex Gaussian random variables. (The process ψ is not stationary, but its zeroes are.)

On small distances the zeroes repel each other, like particles of the one-component plasma (OCP; not to be confused with the two-component Coulomb gas). On large distances the three models are related as follows:

Poisson — OCP — CAZP

in the sense explained below, see (4).

We consider so-called smooth linear statistics $Z(h)$ and $Z_0(h)$, — random variables indexed by compactly supported C^2-smooth test functions $h : \mathbb{C} \to \mathbb{R}$, — defined by

$$Z(h) = \sum_{z:\psi(z)=0} h(z) \,; \quad Z_0(h) = Z(h) - \mathbb{E}\, Z(h) = Z(h) - \frac{1}{\pi} \int h \, \mathrm{d}m$$

(m being Lebesgue measure on $\mathbb{C}$). We introduce rescaled test functions h_r,

$$h_r(z) = h\Big(\frac{1}{r} z\Big) \quad \text{for } r \in (0,\infty), \ z \in \mathbb{C} \,,$$

and examine the asymptotic behavior of $Z_0(h_r)$ as $r \to \infty$.

Asymptotic normality [6]: there exists an absolute constant $\sigma \in (0,\infty)$ such that for every test function h and every $c \in \mathbb{R}$,

$$\text{(1)} \qquad \mathbb{P}\bigg(\frac{r}{\sigma \|\Delta h\|} Z_0(h_r) > c \bigg) \to \frac{1}{\sqrt{2\pi}} \int_c^\infty \mathrm{e}^{-x^2/2} \, \mathrm{d}x \quad \text{as } r \to \infty \,;$$

here Δ is the Laplace operator, and $\| \cdot \|$ is the norm in $L_2(\mathbb{C})$.

Note that $Z_0(h_r)$ is multiplied by r, not divided by r. This is not a mistake! Large-scale fluctuations in CAZP are much smaller than in the Poisson process.

Moderate deviations [7]:

$$\ln \mathbb{P}\left(\frac{r}{\sigma\|\Delta h\|} Z_0(h_r) > c \right) \sim \ln \left(\frac{1}{\sqrt{2\pi}} \int_c^\infty \mathrm{e}^{-x^2/2}\, \mathrm{d}x \right) \tag{2}$$

as $r \to \infty$, $\frac{c \log^2 r}{r} \to 0$.

That is, for every ε there exist R and d such that the ratio of the left-hand side to the right-hand side is ε-close to 1 for all $r \ge R$ and all c such that $|c| \frac{\log^2 r}{r} \le \mathrm{d}$.

The asymptotic normality is a special case, $c = \text{const}$. In the case $c \to \infty$ the right-hand side may be replaced with $(-c^2/2)$. (For the other tail, note that $Z_0(-h) = -Z_0(h)$.)

It is natural to expect that (2) holds whenever $\frac{c}{r} \to 0$; higher c could satisfy a large deviations principle, with a non-quadratic rate function.

Question. Does (2) hold when $\frac{r}{\log^2 r} \ll c \ll r$?

Linear response [7]:

$$\frac{1}{r^2 \varepsilon^2} \ln \mathbb{E} \exp\bigl(\varepsilon r^2 Z_0(h_r)\bigr) \to \frac{\sigma^2}{2} \|\Delta h\|^2 \tag{3}$$

as $r \to \infty$, $\varepsilon \log^2 r \to 0$.

By the Gärtner (-Ellis) theorem, (3) implies (2).

I like to call (3) 'linear response principle' for the following reason. In the spirit of equilibrium statistical physics (by analogy with Gibbs measures) we may treat the test function h as a (given, nonrandom) external field that multiplies all probabilities by $\frac{\exp Z_0(h)}{\mathbb{E} \exp Z_0(h)}$ (as if $Z_0(h)$ was subtracted from the Hamiltonian; the inverse temperature $\beta = 1$ is assumed). The response of the observable $Z_0(h)$ to this external field is the new average of $Z_0(h)$ (according to the new probabilities), equal to

$$\mathbb{E} \left(Z_0(h) \cdot \frac{\exp Z_0(h)}{\mathbb{E} \exp Z_0(h)} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=1} \ln \mathbb{E} \exp \lambda Z_0(h) \, .$$

Taking into account that the function $\lambda \mapsto \ln \mathbb{E} \exp \lambda Z_0(h)$ is convex, we see that the response is approximately linear if and only if $\ln \mathbb{E} \exp Z_0(h)$ is approximately quadratic (in h, when h is small in an appropriate sense).

Physical intuition suggests that the linear response principle should hold under quite general conditions. Accordingly, the moderate deviations principle (MDP) for random fields should hold under quite general conditions. In contrast, available general results on MDP are scanty, need restrictive assumptions and considerable effort [1], [2], [7]. What could it mean? Maybe my physical intuition is naive; in this case important counterexamples should be found.

Question. Does the moderate deviations principle (and moreover, the linear response principle) for stationary random fields hold under general conditions, such as exponential decay of correlations and finite exponential moments?

It follows from (3) that

$$\mathbb{E}\left(Z_0(g_r)\cdot\frac{\exp Z_0(h_r)}{\mathbb{E}\exp Z_0(h_r)}\right)\sim\sigma^2\int g_r\Delta^2 h_r\,\mathrm{d}m\quad\text{as } r\to\infty$$

for all test functions g, h. (Here $\Delta^2 h_r = \Delta(\Delta h_r)$.) This is the response of the observable $Z_0(g_r)$ to the external field h_r. In more physical language, the response of the macroscopic observable $\sum_{z:\psi(z)=0} g(z)$ to the macroscopic external field h is $\sigma^2\int g\Delta^2 h\,\mathrm{d}m$. It means that $\sigma^2\Delta^2 h$ is the response (to h) of the macroscopic density of the chaotic analytic zero points.

For the Poisson process the response is proportional to h. For OCP it is (believed to be) proportional to Δh, which is electrostatic screening of external charges [4]. We summarize:

(4)

Model	:	Poisson	OCP	CAZP
Response to h is proportional to	:	h	Δh	$\Delta^2 h$

See also three toy models discussed in [6] (the end of the introduction).

I restrict myself to smooth (C^2) test functions. The number of random points in the disk of large radius r is a different story. It satisfies the Jancovici-Lebowitz-Manificat law [5], like OCP [3]. Its moderate deviations are a boundary effect. In contrast, (2) is a bulk effect. What happens to *large* deviations in the bulk? I do not know.

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On the number of nodal domains of random spherical harmonics

Mikhail Sodin

This is a report on the joint work with Fedor Nazarov from the University of Wisconsin at Madison.

Let f be a spherical harmonic of degree n on the 2-dimensional sphere, and let $N(f)$ be the number of connected components of the zero set $\{f = 0\}$. The celebrated Courant nodal domain theorem yields that $N(f) \le (n+1)^2$. On the other hand, H. Lewy showed that no non-trivial lower bound is possible: one can find spherical harmonics f of arbitrarily large degree with $N(f) \leq 3$. The question we want to discuss here is: *What is the "typical" value of N(f) when the degree n is large?*

To give the word "typical" a precise meaning, let us consider the random spherical harmonic

$$f = \sum_{k=-n}^{n} \xi_k Y_k$$

where ξ_k are independent identically distributed standard Gaussian random variables, and $\{Y_k\}$ is an orthonormal basis of the $2n+1$-dimensional Hilbert space of spherical harmonics of degree n.

What can be said about $N(f)$ as $n \to \infty$? For instance, *whether the expectation* $\mathbb{E}N(f)$ *grows like* n^2*, or it has a more slow growth?* These questions are non-local, and apparently they cannot be handled by the classical Kac-Rice techniques. Several years ago, some non-rigorous predictions based on a percolation-like model were made by Bogomolny and Schmit.

We prove that $\mathbb{E}N(f) \geq \mathrm{const}\, n^2$. Moreover, we show that *there exists a positive numerical constant* $a > 0$ *such that, for every* $\epsilon > 0$,

$$\mathrm{Prob}\left\{ \left| \frac{N(f)}{n^2} - a \right| > \epsilon \right\} \leq C(\epsilon) e^{-c(\epsilon) n}$$

where $c(\epsilon)$ *and* $C(\epsilon)$ *are some positive constants depending on* ϵ *only.* Numerical evidences presented by Bogomolny and Schmit suggest that $a \approx 0.06$.

In the proofs we use only relatively simple tools from the classical analysis, which work in a more general setting of random functions of several real variables, while it seems that the Bogomolny-Schmit model is essentially a two-dimensional one.

The details can be found in `arXiv:0706.2409`.

Large gaps between random eigenvalues

Bálint Virág
(joint work with Benedek Valkó)

In the 1950s Wigner endeavored to set up a probabilistic model for the repulsion between energy levels in large atomic nuclei. His first models were random meromorphic functions related to random Schrödinger operators, [4] [5]. Later [6] he introduced standard models for random matrices, such as the Gaussian orthogonal ensemble (GOE). In this model one fills an $n \times n$ matrix M with independent standard normal random variables, then symmetrizes it to get

$$A = \frac{M + M^t}{\sqrt{2}}.$$

The Wigner semicircle law is the limit of the empirical distribution of the eigenvalues of the matrix A. However, Wigner's main interest was the local behavior of the eigenvalues, namely the repulsion between them. One mathematical manifestation of this repulsion is large gap probabilities. If we scale the eigenvalue point process of A so that the average spacing is 2π, then Wigner predicted that the probability that there is no eigenvalue in a fixed interval of length λ is given by

$$p_\lambda = \exp\left(-(c + o(1))\lambda^2\right)$$

where this is a $\lambda \to \infty$ limit, and we assume that the $n \to \infty$ limit has already been taken. Wigner estimate of the constant, $1/(16\pi)$, later turned out to be incorrect. Dyson [2] improved this estimate to

$$p_\lambda = (c_\beta + o(1))\lambda^{\gamma_\beta} \exp\left(-\frac{\beta}{64}\lambda^2 + \left(\frac{\beta}{4} - \frac{1}{8}\right)\lambda\right) \tag{1}$$

where β is a new parameter introduced by noting that the joint eigenvalue density of the GOE is the $\beta = 1$ case of

$$\frac{1}{Z_{n,\beta}}\, e^{-\beta \sum_{k=1}^n \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \tag{2}$$

Dyson's computation of the exponent γ_β, namely $\frac{1}{8}(\beta + \frac{4}{\beta} + 12)$, turned out to be slightly incorrect. Indeed, des Cloizeaux and Mehta [1], who showed that $\gamma_1 = -1/8$, rather than $17/8$. Our main theorem is

Theorem A. *The formula* (1) *holds with*

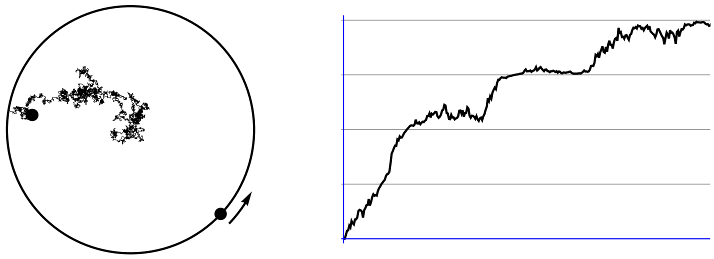
$$\gamma_\beta = \frac{1}{8}\left(\beta + \frac{4}{\beta} - 6\right).$$

The proof is based on the Brownian carousel, a geometric representation of the $n \to \infty$ limit of the eigenvalue process. We first introduce the **hyperbolic carousel**. Let

- b be a path in the hyperbolic plane
- z be a point on the boundary of the hyperbolic plane, and
- $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an integrable function.

To these three objects, the hyperbolic carousel associates a multi-set of points on the real line defined via its counting function $N(\lambda)$ taking values in $\mathbb{Z} \cup \{-\infty, \infty\}$. As time increases from 0 to ∞, the boundary point z is rotated about the center $b(t)$ at angular speed $\lambda f(t)$. $N(\lambda)$ is defined as the integer-valued total winding number of the point about the moving center of rotation.

The **Brownian carousel** is defined as the hyperbolic carousel driven by hyperbolic Brownian motion b (see figure). It is connected to random matrices via the following theorem [3]:



Theorem B. *Let* Λ_n *denote the point process given by* (2)*, and let* μ_n *be a sequence so that* $n^{\frac{1}{6}}(2\sqrt{n} - |\mu_n|) \to \infty$*. Then*

$$\sqrt{4n - \mu_n^2}\big(\Lambda_n - \mu_n\big) \Rightarrow \mathrm{Sine}_\beta,$$

where Sine_β *is the discrete point process given by the Brownian carousel with parameters* $f(t) = (\beta/4)e^{-\beta t/4}$ *and arbitrary* z*.*

The Brownian carousel description gives a simple way to analyze the limiting point process. The hyperbolic angle of the rotating boundary point as measured from $b(t)$ follows the following coupled one-parameter family of stochastic differential equations

$$d\alpha_\lambda = \lambda f\,dt + \Re((e^{-i\alpha_\lambda} - 1)dZ), \qquad \alpha_\lambda(0) = 0, \tag{3}$$

driven by a two-dimensional standard Brownian motion. For a single λ, this reduces to the one-dimensional stochastic differential equation

$$d\alpha_\lambda = \lambda f\,dt + 2\sin(\alpha_\lambda/2)dW, \qquad \alpha_\lambda(0) = 0, \tag{4}$$

which converges as $t \to \infty$ to an integer multiple $\alpha_\lambda(\infty)$ of 2π. In particular, the number of points of the point process Sine_β in $[0, \lambda]$ has the same distribution as $\alpha_\lambda(\infty)/(2\pi)$.

Theorem A is proved via the analysis of the equation (4). Here we provide heuristics for the leading term. There are no points in $[0, \lambda]$ if and only if α converges to 0 as $t \to \infty$. Note that α is always monotone at multiples of 2π, so it is not allowed to surpass 2π. The main difficulty with this is that for large λ, there is a large drift upwards. This drift has to be overcome by the Brownian term. This is easier to do if the variance of the Brownian term is higher. Indeed, for the leading term, we may consider the simpler equation

$$d\alpha_\lambda = \lambda f\,dt + 2dW, \qquad \alpha_\lambda(0) = 0,$$

where the variance is equal to its maximal possible value. The difficult part here is for α to stay in an interval, stay $[-2\pi, 2\pi]$ as long as the large drift is in effect, namely for time between 0 and $c\log\lambda$. This is now a standard large deviation event for Brownian motion, with probability

$$\exp\left(-(1+o(1))\lambda^2 \int f(t)^2/8\,dt\right) = \exp\left(-(\beta/64+o(1))\lambda^2\right)$$

and this agrees with the leading term in Theorem A.

It is interesting to note that the proof of Theorem B goes through random Schrödinger operators, which is how Wigner started out his original quest to understand energy levels.

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Scaling limits of dynamical and near-critical percolation

Christophe Garban, Oded Schramm

(joint work with Gábor Pete)

Following recent progress in the understanding of critical percolation in the plane, we proceed to study two closely related models. In **dynamical percolation** the bits defining the percolation configuration undergo independent Poisson updates with rate 1. The starting configuration is chosen to be critical percolation, and the Poisson updates are set in such a way that this is also the stationary measure.

We also consider a variant of this setup in which the updates are monotone. At time $t = 0$, take a critical percolation configuration ω_0. To each bit i defining the configuration take a Poisson point process sample $X_i \subset \mathbb{R}$. Now for $t > 0$ let ω_t denote the percolation configuration whose open (occupied) bits are the union of the bits that are open in ω_0 and any bits i such that $X_i \cap [0, t] \neq \emptyset$. Likewise, for $t < 0$ let ω_t denote the percolation configuration whose open bits are the intersection of the open bits in ω_0 and the set of bits i satisfying $X_i \cap [t, 0] = \emptyset$. This defines a coupling of Bernoulli(p) percolation for each $p \in (0, 1)$. We call this the **monotone percolation coupling**.

We prove that for site percolation of the triangular lattice each of these has a scaling limit when time is scaled appropriately. In this abstract, we omit the discussion of the topology in which this convergence takes place, but let us explain what the right scaling of time is. Let $\eta > 0$ be small, and consider the above systems on the triangular lattice of mesh η. Let the Poisson clocks have the rate

$\rho = \rho_\eta$, where ρ^{-1} is the expected number of sites in the unit square that have the alternating four arm event to distance 1 or greater, defined as follows. A site v has the alternating four arm event to distance 1 if there is a sequence $(\beta_1, \beta_2, \beta_3, \beta_4)$ of lattice paths starting from vertices adjacent to v and terminating at distance at least 1 away from v such that β_1 and β_3 are occupied, β_2 and β_4 are vacant, and the counterclockwise order of the starting vertices of these paths around v is the order $(\beta_1, \beta_2, \beta_3, \beta_4)$. For site percolation on the triangular lattice, $\rho_\eta = \eta^{-3/4+o(1)}$.

We prove a few properties of these limits. They are rotationally invariant, their time evolution is Markov, the dynamical percolation scaling limit is ergodic. They also satisfy a natural conformal co-variance property. If you apply a conformal homeomorphism to a domain, the image dynamical system is described by a similar scaling limit in the range, except that the rates of the Poisson updates are different from point to point. More precisely, if $f : D \to D^*$ is the conformal homeomorphism, then the rates in D^* which produce a scaling limit that is equal to the f-image of the scaling limit in D vary from point to point and are given by $\rho^*_\eta\big(f(z)\big) = |f'(z)|^{-3/4}\, \rho_\eta$.

It is well-known that the minimal spanning tree is closely related to critical and near-critical percolation. We use this connection to deduce statements about a variant of the minimal spanning tree that is related to site percolation on the triangular lattice. For this variant, we prove, for example, that the minimal spanning tree scaling limit is rotationally invariant.

Several of these results were conjectured by Camia, Fontes and Newman. Our proofs partially follow their heuristic narrative.

Markov paths, loops and fields

Yves Le Jan

The purpose of this talk is to explore some simple relations between Markovian path and loop measures, spanning trees, determinants, and Markov fields.such as the free field. Some of these relations involve graph theory and symbolic dynamics, but the main emphasis is put on the study of occupation fields defined by Poissonian ensembles of Markov loops. These were defined in [2] for planar Brownian motion in relation with SLE processes and in [3] for simple random walks. They appeared informally already in [5]. For half integral values $\frac{k}{2}$ of the intensity parameter α, these occupation fields can be identified with the sum of squares of k copies of the associated free field (i.e. the Gaussian field whose covariance is given by the Green function). This is related to Dynkin's isomorphism (cf. [1], [4]). We first present the results in the elementary framework of symmetric Markov chains on a finite space, proving also en passant several interesting results such as the relation between loop ensembles and spanning trees and the Markov property. Then we show some results can be extended to more general Markov processes. There are no essential difficulties when points are not polar but other cases are problematic. As for the square of the free field, cases for which the Green function is Hilbert Schmidt such as two and three dimensional Brownian motion can be dealt with through appropriate renormalization.

We can show that the renormalised powers of the occupation field (i.e. the self intersection local times of the loop ensemble) converge in the two dimensional case and that they can be identified with higher even Wick powers of the free field when α is a half integer.

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The critical Ising model on infinite isoradial graphs via dimers

Cédric Boutillier

(joint work with Béatrice de Tilière)

An *isoradial graph* G is a planar graph, together with an embedding on the plane, such that every (inner) face is inscribed in a circle of radius 1. The square lattice and the honeycomb lattice with their standard embeddings are isoradial. So is the graph represented on Figure 1.

Kenyon and Schlenker gave a topological condition for a graph to have an isoradial embedding [8]. The dual of an isoradial graph G is also isoradial, and all the faces of the *diamond graph* of G are rhombi with side length equal to 1. This last property allows the development of a theory of discrete holomorphic functions [4, 9]. Moreover there are a number of operations preserving isoradiality, such as the so-called *star-triangle* transformation that corresponds to Yang-Baxter equation for integrable systems from statistical mechanics [1]. These aspects of isoradial graphs make them particularly suitable for statistical mechanical models at criticality, that are supposed to have a conformally invariant scaling limit.

If the isoradial graph G is finite, we define the Ising model on G as follows. A *spin configuration* σ on the graph G is a function from the vertices of G to $\{-1, +1\}$. The *energy* of σ measures the disagreement between neighbouring vertices:

$$E(\sigma) = -\sum_{e=(v,w)} J_e \sigma_v \sigma_w,$$

where the interaction constants J_e, indexed by edges, are positive real numbers. Since an isoradial graph comes with a particular embedding, it is natural to ask J_e to depend on the geometry of the embedding, and namely on the half angle θ_e of the rhombus corresponding to the edge e in the diamond graph. There is a one-parameter family of such interaction constants $(J_e)_e$ satisfying the Yang-Baxter equation, corresponding to the so-called *Z-invariant Ising model* [3, 2]. Among this family, only one choice satisfies also the property of *self-duality.* This choice of interaction constants is thus referred to as the *critical weights*, given by the formula:

$$\sinh 2J_e = \tan \theta_e.$$

The Boltzmann probability measure for the critical Ising model on G assigns to each configuration a probability proportional to the exponential of the negative of its energy.

We study this model on an infinite isoradial graph, where the notion of Boltzmann measure does not make sense any more. The main goal is to construct and describe a *Gibbs measure* for the critical Ising model. For this, we use a correspondence discovered by Fisher [5] between the Ising model on a planar graph and a dimer model on a decorated version of the same graph (see Figure 2).

On a finite planar graph, the partition function of the dimer model is given by the Pfaffian of the *Kasteleyn matrix* K , which is a weighted adjacency matrix for a *clockwise odd* orientation of the graph [6]. The correlations are expressed in terms of Pfaffians of submatrices of K^{-1}.

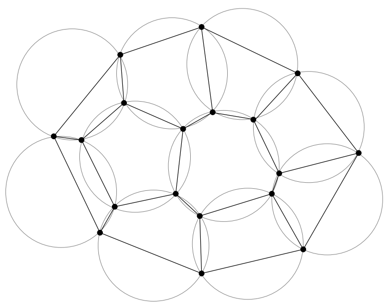

FIGURE 1. An isoradial graph, together with the circumscribing circles of the inner faces, having all the same radius.

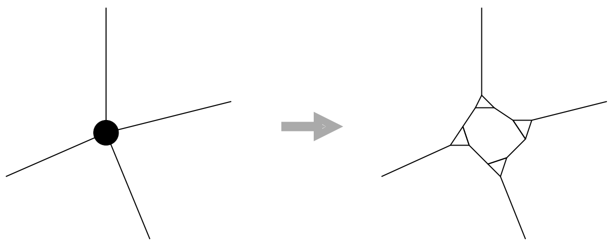

FIGURE 2. The decoration of a vertex of degree 4 in the Fisher correspondence. On the left, a vertex from the initial Ising graph. On the right, its replacement in the dimer graph.

On the dimer graph corresponding to the infinite isoradial graph, we give an explicit formula for any entry $K^{-1}_{v,w}$ of the inverse of the infinite Kasteleyn matrix, in the same spirit as [7], which has the surprising property of depending only on the geometry of the embedding along a path joining the two vertices v and w.

More precisely, if v and w do not project to the same vertex of the isoradial graph, the entry $K^{-1}_{v,w}$ has the following form:

$$K^{-1}_{v,w} = \frac{1}{(2\pi)^2} \oint_{C_{v,w}} f_{v,w}(\lambda) \log \lambda \mathrm{d}\lambda$$

where $f_{v,w}$ is a rational fraction, written as a product of local terms (homographies) collected along a path from v to w (and does not depend on the path) and $C_{v,w}$ is a positively oriented contour in the complex plane containing all the poles of $f_{v,w}$ but avoiding the half-line starting from the origin with direction $\overrightarrow{vw}$.

Using this property, we prove that the Pfaffian formula with kernel K^{-1} defines a Gibbs measure for the dimer model and thus for the critical Ising model on the infinite isoradial graph, with the property that the probability of a local event involving finitely many spins does not depend on the geometry of the embedding outside of a ball containing these vertices.

We also give an explicit expression for the free energy per site f_I of the Ising model for this Gibbs measure, in the case of a periodic isoradial graph with V_1 spins in the fundamental domain:

$$-f_I = \frac{\log 2}{2} + \frac{1}{V_1} \sum_e \left\{ \frac{\theta}{\pi} \log \theta + \frac{1}{\pi} \Big(L(\theta) + L(\frac{\pi}{2} - \theta) \Big) \right\}.$$

When comparing this expression with the free energy of the massless free field on isoradial graphs [7], we recover the fact that the central charge of the Ising model should be $\frac{1}{2}$.

For other aspects of the critical Ising model on isoradial graphs, we refer the reader to the abstracts by S. Smirnov and D. Chelkak in this volume.

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Universality and conformal invariance in the Ising model Part I. Conformally invariant observables on isoradial graphs Part II. Interfaces and Schramm's SLE curves

Dmitri Chelkak, Stanislav Smirnov

We describe in detail our proof that for a large family of planar graphs the Ising model at criticality has a conformally invariant scaling limit. In particular, we construct discrete holomorphic fermionic observables for the spin and random cluster representations, show that they converge to conformally covariant scaling limits, and deduce that interfaces converge to Schramm's SLE(3) and SLE(16/3) curves. One particular feature of our approach is that we use no deep results about the Ising models: all the required tools are deduced form our observables.

Isoradial graphs. The notion of isoradial graphs (or, equivalently, rhombic lattices) was introduced by R.J. Duffin [Duf65] in the context of electrical networks and, later on, by C. Mercat [Mer01] and R.Kenyon [Ken02], [Ken03] in connection with the Ising model and the dimer model, respectively. The planar graph (together with its embedding into the plane) is called *isoradial* iff it is drawn in such a way that each face is a cyclic polygon, that is, is inscribable in a circle, and all these circles have the same radii $\delta > 0$. In this case the dual graph is isoradial too. Let Γ, Γ^* and $\diamondsuit$ be the sets of vertices, faces and edges of some infinite isoradial graph. Note that $\Gamma \cup \Gamma^*$ are vertices of the planar graph (*rhombic lattice*) with rhombic faces (which are in one-to-one correspondence with $\diamondsuit$).

Let $\mathcal{F}(X)$ be the set of functions defined on X. Following [Ken02] (up to some normalizing factors), we introduce the natural difference operators

$$\overline{\partial}^\delta, \partial^\delta : \mathcal{F}(\diamondsuit) \to \mathcal{F}(\Gamma \cup \Gamma^*) \qquad \text{and} \qquad \Delta^\delta : \mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma), \quad \mathcal{F}(\Gamma^*) \to \mathcal{F}(\Gamma^*).$$

In particular,

$$[\Delta^\delta H](u) = \frac{1}{m^\delta(u)} \sum_{u_s \sim u} \tan\alpha_s \cdot (H(u_s) - H(u)), \qquad m^\delta(u) = \frac{\delta^2}{2} \sum_{u_s \sim u} \sin 2\alpha_s,$$

where $u_s \in \Gamma$ are all neighbors of $u \in \Gamma$ and $\alpha_s = \arccos\big[|u-u_s|/(2\delta)\big]$ are the half-angles (adjacent to u) of the corresponding rhombi.

Solutions of the equations $\overline{\partial}^\delta F = 0$ and $\Delta^\delta H = 0$ are called *discrete holomorphic* and *discrete harmonic* functions respectively. Let $u \in \Omega^\delta \subset \Gamma$, Ω^δ be some discrete simply connected domain and the arc $b^\delta a^\delta$ be some part of its boundary. We denote by $\omega^\delta(u, b^\delta a^\delta, \Omega^\delta)$ the **discrete harmonic measure** of $b^\delta a^\delta$ in Ω^δ from the point u. Clearly, $\omega^\delta(u, b^\delta a^\delta, \Omega^\delta)$ is the probability that the corresponding random walk on Γ started at u firstly hits the boundary of Ω^δ on $b^\delta a^\delta$. The most important feature of the discrete Laplacian Δ^δ (and, similarly, the discrete Cauchy-Riemann operators $\overline{\partial}^\delta, \partial^\delta$) on isoradial graphs is

Approximation property: Let $H:\mathbb{C}\to\mathbb{R}$ be some smooth function. Denote by H^δ its restriction on Γ (or Γ^*). Then,

$$(\Delta^\delta H^\delta)(u) = (\Delta H)^\delta(u) + O(\delta)$$

for all $u \in \Gamma$.

In particular, the restriction of the (continuous) harmonic function on Γ is approximately discrete harmonic. Using this property and the *asymptotics* of the free *Green function* and the free *Cauchy kernel* obtained by R.Kenyon [Ken02], we derive some other tools:
Harnack's estimate for the gradient $\partial^\delta H(z)$ of the discrete harmonic function H:

$$|\delta^\delta H(z)| \leq \frac{const}{\mathrm{dist}(z,\partial\Omega^\delta)} \cdot \max_{u\in\Omega^\delta}|H(u)|$$

and **weak Beurling-type estimate**

$$\omega^\delta(u,b^\delta a^\delta,\Omega^\delta) \leq const \cdot \left[\frac{\mathrm{d}ist(u,a^\delta b^\delta)}{\mathrm{d}ist(u,b^\delta a^\delta)}\right]^\beta,$$

where $\beta > 0$ is some absolute constant. Then, using the uniform Lipshitzness of functions $\omega^\delta(\cdot,b^\delta a^\delta,\Omega^\delta)$ inside Ω, compactness arguments and the weak Beurling-type estimate for the identification of boundary values, we prove

Theorem 1. *Let $(\Omega^\delta;a^\delta,b^\delta)\to(\Omega;a,b)$ as $\delta\to 0$ in the Carathéodory sense with respect to some point $u\in\Omega$. Then, $\omega^\delta(u,b^\delta a^\delta,\Omega^\delta)\to\omega(u,ba,\Omega)$ as $\delta\to 0$, where $\omega(u,ba,\Omega)$ is the continuous harmonic measure of the arc ba in the domain Ω from the point u.*

Remark. Carathéodory convergence is defined as convergence of normalized Riemann uniformization maps on compact subsets. Namely, let ϕ^δ (or ϕ) be conformal maps from the unit disk $\mathbb{D}$ to Ω^δ (or Ω) such that points $0,1,\zeta^\delta$ (or $0,1,\zeta$) are mapped to u,a^δ,b^δ (or u,a,b) or corresponding prime ends. We say that $(\Omega^\delta;a^\delta,b^\delta)$ converge to $(\Omega;a,b)$ if ϕ^δ converge to ϕ inside $\mathbb{D}$ and ζ^δ tends to ζ.

Further, using compactness arguments once more (now, for the Carathéodory topology on the set of domains), we obtain

Corollary 2 (uniform convergence for discrete harmonic measures). *Let $u\in\Omega^\delta$, $\mathrm{diam}(\Omega^\delta)\leq R<+\infty$ and $\mathrm{d}ist(u,\partial\Omega^\delta)\geq r>0$. Then, for each arc $b^\delta a^\delta\subset\partial\Omega^\delta$, the estimate*

$$|\omega^\delta(u,b^\delta a^\delta,\Omega^\delta)-\omega(u,b^\delta a^\delta,\Omega^\delta)|\leq\varepsilon(\delta)=\varepsilon(\delta;r,R)\to 0,\quad \delta\to 0,$$

holds true. Here $\varepsilon(\delta)$ doesn't depend on the particular shape of the discrete domain Ω^δ.

Remark. Similarly, we prove the uniform convergence of the discrete Poisson kernels to their continuous counterpart. In view of [LSW04], [Sm06], this gives the (universal) convergence of THE LOOP ERASED RANDOM WALK on isoradial graphs to SLE(2). Universality for other lattice models having the harmonic martingale observable also can be derived by this way.

Ising model. For the FK-Ising model on the square lattice, the holomorphic martingale observable (*"holomorphic fermion"*) was constructed and the convergence to its continuous counterpart was proved in [Sm06, Sm07] (the spin-Ising model was also considered in the similar way). We introduce Dobrushin boundary conditions: wired on the counterclockwise arc $b^\delta a^\delta$ (meaning that all edges along the arc are open) and dual-wired on the counterclockwise arc $a^\delta b^\delta$ (meaning that all dual edges along the arc are open, or equivalently all primal edges orthogonal to the arc are closed). For the loop representation this reduces to introducing two vertices with odd number of edges: a source a^δ and a sink b^δ. Then besides a number of closed loop interfaces there is a unique interface γ running from a^δ to b^δ, which separates the cluster containing the arc $a^\delta b^\delta$ from the dual cluster containing the arc $b^\delta a^\delta$. Let $F^\delta(z)$, $z \in \diamondsuit$, be the expectation that the interface γ passes through a point z taken with a complex weight:

$$F^\delta(z) \ := \ (2\delta)^{-1/2}\, \mathbb{E}\left[\chi_{z\in\gamma} \cdot \exp^{-\frac{i}{2}\,\mathrm{winding}(\gamma, b^\delta \to z)}\right].$$

Here $\mathrm{winding}(\gamma, b^\delta \to z)$ denotes the total turn of γ from b^δ to z, measured in radians.

It was known that $F^\delta(z)$ is a preholomorphic function for the FK-Ising model on an arbitrary isoradial graph (see also [RC06]). Nevertheless, some results (in particular, the estimate [KO49] for the spin-correlations) dependent on the precise square structure of the lattice were used in the proof of the convergence [Sm07]. Unfortunately, the known proof of this estimate strongly depends on the particular square structure of the lattice.

On the other hand, it was pointed out in [Sm07] that the holomorphic fermion is defined uniquely by its boundary conditions. Thus, there should be an Ising-independent proof, using only discrete analyticity and boundary conditions. Indeed, now we are able to prove (*without the appeal to the magnetization estimate*) the convergence of the observable $F^\delta(z)$ to its continuous counterpart universally on arbitrary isoradial graphs.

Theorem 3. *Let $(\Omega^\delta; a^\delta, b^\delta) \to (\Omega; a, b)$ as $\delta \to 0$ in the Carathéodory sense with respect to some point $z \in \Omega$. Then, $F^\delta(u, b^\delta a^\delta, \Omega^\delta) \to \sqrt{\Phi'(z)}$ as $\delta \to 0$, where Φ is the conformal map of Ω to an (infinite) horizontal strip of width 1, with a and b mapped to the ends $\mp\infty$.*

Remark. (i) As above, due to the compactness arguments (for the Carathéodory topology), we have the immediate corollary: this convergence is uniform inside Ω and don't depend neither on the particular (microscopic) structure of the lattice nor on the (macroscopic) shape of Ω.
(ii) The crucial observation is that the imaginary part of the discrete integral

$$H = \mathrm{Im} \int^\delta (F^\delta(z))^2 d^\delta z$$

is well-defined on $\Gamma \cup \Gamma^*$, subharmonic on Γ and superharmonic on Γ^*, the same as on the square lattice [Sm06, Sm07]. In order to identify the boundary values of H, we firstly estimate $F(z)$ inside and then derive some average estimate for $|F(z)|$

on the boundary using its special boundary conditions and the discrete contour integration.
(iii) As it was shown in [Sm06], this Theorem is a large part (via the Conformal Martingale Principle) of the proof that the interface in FK-Ising random model (on arbitrary isoradial graphs) converge to SLE(16/3).
(iv) Using similar arguments, we prove convergence of the observable for the spin-Ising model.

Schramm-Loewner Evolution. Loewner Evolution is a differential equation for a Riemann uniformization map for a domain with a growing slit. It was introduced by Charles Loewner in [L23] in his work on Bieberbach's conjecture. *Chordal Loewner Evolution* describes uniformization for the upper half-plane $\mathbb{C}_+$ with a slit growing from 0 to ∞ (one deals with a general domain Ω with boundary points a, b by mapping it to $\mathbb{C}_+$ so that $a \mapsto 0$, $b \mapsto \infty$). Loewner only considered slits given by smooth simple curves, but more generally one allows any set which grows continuously in conformal metric when viewed from ∞. We will omit the precise definition of *allowed slits* (more extensive discussion in this context can be found in [Law05]), only noting that all simple curves are included. The random curves arising from lattice models (e.g. cluster perimeters or interfaces) are simple (or can be made simple by altering them on the local scale). Their scaling limits are not necessarily simple, but they have no "transversal" self-intersections. For such a curve to be an allowed slit it is sufficient if it touches itself to never venture into the created loop. This property would follow if e.g. a curve visits no point thrice.

Parameterizing the slit γ in some way by time t, we denote by $g_t(z)$ the conformal map sending $\mathbb{C}_+ \setminus \gamma_t$ (or rather its component at ∞) to $\mathbb{C}_+$ normalized so that at infinity $g_t(z) = z + \alpha(t)/z + O(1/|z|^2)$, the so called *hydrodynamic normalization*. It turns out that $\alpha(t)$ is a continuous strictly increasing function (it is a sort of capacity-type parameter for γ_t), so one can change the time so that

$$g_t(z) \; = \; z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right).$$

Denote by $w(t)$ the image of the tip $\gamma(t)$. The family of maps g_t (also called a *Loewner chain*) is uniquely determined by the real-valued *"driving term"* $w(t)$. The general form of the Loewner theorem can be roughly stated as follows:

Loewner's Theorem *There is a bijection between allowed slits and continuous real valued functions* $w(t)$ *given by the ordinary differential equation*

$$\partial_t g_t(z) \; = \; \frac{2}{g_t(z) - w(t)} \; , \; g_0(z) \; = \; z \; .$$

While a deterministic curve γ corresponds to a deterministic driving term $w(t)$, a random γ corresponds to a random $w(t)$. One obtains SLE(κ) by taking $w(t)$ to be a Brownian motion with speed κ: *Schramm-Loewner Evolution*, or SLE(κ), is the Loewner chain one obtains by taking $w(t) = \sqrt{\kappa} B_t$, $\kappa \in [0, \infty)$. Here B_t denotes the standard (speed one) Brownian motion (Wiener process). The resulting slit will be almost surely a continuous curve. So we will also use the term SLE for the resulting random curve, i.e. a probability measure on the space of curves (to be rigorous one can think of a Borel measure on the space of curves with uniform norm). Different speeds κ produce different curves: we grow the slit with constant speed (measured by capacity), while the driving term "wiggles" faster. Naturally, the curves become more "fractal" as κ increases: for $\kappa \leq 4$ the curve is almost surely simple, for $4 < \kappa < 8$ it almost surely touches itself, and for $\kappa \geq 8$ it is almost surely space-filling (i.e. visits every point in $\mathbb{C}_+$) – see [Law05, RSch05] for these and other properties.

Conformally covariant martingales. Suppose that for every simply connected domain Ω with a boundary point a we have defined a random curve γ starting from a. Mark several points $b, c, \dots$ in Ω or on the boundary.

Definition. We say that a function (or rather a *differential*) $F(\Omega, a, b, c, \dots)$ is a *conformal (covariant) martingale* for a random curve γ if
(i) F is conformally covariant:

$$F\left(\Omega, a, b, c, \dots\right) = F\left(\phi(\Omega), \phi(a), \phi(b), \phi(c), \dots\right) \cdot \phi'(b)^{\alpha} \bar{\phi}'(b)^{\beta} \phi'(c)^{\gamma} \bar{\phi}'(c)^{\delta} \dots;$$

(ii) $F(\Omega \setminus \gamma[0,t], \gamma(t), b, c, \dots)$ is a martingale with respect to the random curve γ drawn from a (with Loewner parameterization).

Introducing covariance at $b, c, \dots$ we do not ask for covariance at a, since it always can be rewritten as covariance at other points. And applying factor at a would be troublesome: once we started drawing a curve the domain becomes non-smooth in its neighborhood, creating problems with the definition.

The properties (i), (ii) show that for the curve γ mapped to half-plane from any domain Ω so that $a \mapsto 0$, $b \mapsto \infty$, $c \mapsto x$ (note that the image curve in $\mathbb{C}_+$ might depend on Ω – we only know the conformal invariance of an observable, not of the curve itself) we have

$$F(\mathbb{C}_p, 0, \infty, g_t(x), \dots) \cdot g_t'(x)^{\gamma} \bar{g}_t'(x)^{\delta} \dots,$$

is a martingale with respect to the random Loewner evolution (covariance factor at $b = \infty$ is absent, since $g_t'(\infty) = 1$). Equating coefficients of Taylor's expansions near ∞, one extracts the information about random driving force $w(t)$ and arrives at

Martingale principle. *If a random curve γ admits a (non-trivial) conformal martingale F, then γ is given by SLE with κ (and drift depending on modulus of the configuration) derived from F.*

Remark. (i) See [Sm06] for details. In particular, the holomorphic fermions constructed in FK-Ising and spin-Ising models lead to SLE(16/3) and SLE(3), respectively.

A priori estimates. A priori estimates are necessary to show that collection of interface laws is precompact in weak-$*$ topology (on the space of measures on continuous curves which are allowed slits). It turns out that all needed properties of the discrete random interfaces (6 arms events and so on) can be derived from the following consideration.

Let Ω^δ be some discrete domain with four marked boundary points $a^\delta, b^\delta, c^\delta, d^\delta$ and Dobrushin-type boundary conditions: wired on the arcs $b^\delta c^\delta$ and $d^\delta a^\delta$ and dual-wired on the arcs $a^\delta b^\delta$ and $c^\delta d^\delta$. Then there are two possibility for interfaces: either there are two interfaces from a^δ to b^δ and from c^δ to d^δ, or from a^δ to d^δ and from c^δ to b^δ. Denote the probabilities of these events by $p^\delta = \mathbb{P}[a^\delta \to b^\delta, c^\delta \to d^\delta)$ and $q^\delta = 1 - p^\delta = \mathbb{P}[a^\delta \to d^\delta, c^\delta \to b^\delta)$.

Lemma. *Let Ω^δ be some discrete domains with four marked boundary points $a^\delta, b^\delta, c^\delta, d^\delta$ and Dobrushin-type boundary conditions as described above.*

Let $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta) \to (\Omega; a, b, c, d)$ as $\delta \to 0$ in the Carathéodory sense. Then $p^\delta \to p = p(\Omega; a, b, c, d)$ that depends only on the conformal modulus of the quadrilateral $p(\Omega; a, b, c, d)$.

Remark. (i) In particular, $p(\Omega; a, b, c, d)$ is bounded away from 0 and 1, if the conformal modulus of $(\Omega; a, b, c, d)$ is bounded away from 0 and ∞.
(ii) The proof is based on the construction of discrete holomorphic function that is the discrete analogue of $\sqrt{\Theta'(z)}$, where Θ conformally maps Ω onto the horizontal strip of total width 1 with horizontal slit from $+\infty$ to some point ih with a mapped to $-\infty$, b and d mapped to (two different prime ends) $+\infty$ and c mapped to ih. By the construction, h^δ is related with p^δ by some explicit formula. Then, due to the convergence of the discrete functions to the continuous counterpart, we obtain $p^\delta \to p$.

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Isotropic embeddings

Vincent Beffara

The principle of universality in two-dimensional critical models of statistical physics is the following: If one considers the same model in two different lattices, then they should behave similarly in the scaling limit. In particular, while their critical points will often differ, they will exhibit the same scaling behavior (same critical exponents). This phenomenon is well understood, at the physics level, through the formalism of the renormalization group.

One question that is not well-studied though, is that of uniqueness of the scaling limit itself, and in fact, it is easy to see that such uniqueness does not hold. Indeed, considering the same model on the square lattice and on a lattice that is combinatorially equivalent to it but that it "squeezed" in one direction (so that the faces of the lattice are rectangles instead of squares), and assuming a scaling limit can be defined for both, than the scaling limit of the second will be the image of the scaling limit of the first via the same deformation of space — and if it is non-trivial, then the two limits will differ.

In the particular case of percolation, and more precisely for critical site-percolation on a periodic triangulation of the plane, the scaling limit is actually expected to be conformally invariant, and in particular invariant under rotation. Conformal invariance was proved by Smirnov [2] in the case of the triangular lattice, but so far it is essentially the only known result in this direction.

Given a lattice (considered as a combinatorial object, without an embedding into the plane), one can ask the following question: Among all possible periodic embeddings of this lattice, at most two (at most one if orientation is specified) can correspond to a conformally invariant scaling limit — by the argument above, — so which embedding is it ?

In fact, it can be argued (see [1]) that giving a satisfactory answer to that question is exactly the missing link in the proof of universality for 2D critical percolation. Here, the word "satisfactory" has no formal meaning; however, the following answer is certainly *not* satisfactory: If the lattice has additional symmetry, then this symmetry needs to be preserved by the embedding, and this is enough to compute it in many cases. This answer, while certainly correct, actually says very little about the model ...

Given a planar lattice L and α in the upper-half plane, one can embed L into the plane in such a way that a fundamental domain is the parallelogram with vertices 0, 1, α and $1+\alpha$; let α_p be the unique value of α (if it exists) for which critical percolation is isotropic in the scaling limit. Besides, let α_w be the unique value of α for which the simple random walk on the embedded lattice is isotropic in the scaling limit.

A first surprising result is the following: In general, $\alpha_p \neq \alpha_w$. In other words, there is no "universally natural" embedding of the lattice L into the plane ... In fact, α_w can easily be computed explicitly, and it is not difficult to prove that α_p and α_w are invariant under different transformations of the lattice.

Define the *barycentric subdivision* T' of a triangulation T as follows: For each triangle of T, add a vertex at the midpoint of each of its edges, and split the triangle into four smaller triangles by connecting these three new vertices pairwise. Then, T' is itself a triangulation.

The main result I presented during my talk concerns the behavior of α_p under barycentric subdivisions. More precisely, starting again from a planar lattice L, define a sequence of lattices (L_n) inductively by successive barycentric subdivisions, letting $L_{n+1} = L'_n$.

Besides, let $\mathcal{S}_L$ be the Riemann surface obtained by gluing equilateral triangles together with the same combinatorics as a fundamental domain of L — it is a surface of genus 1, so it defines a modulus α_u in the upper-half plane in a natural way.

An informal statement of the result is then $\alpha_p(L_n) \to \alpha_u$ as $n \to \infty$. However, $\alpha_p(L_n)$ is not known to exist, so a formal statement is the following:

(1) For each α in the upper-half plane, critical site-percolation on the lattice L_n (embedded with periods 1 and α) has a "mesoscopic" scaling limit as $n \to \infty$. This scaling limit is not in general conformally invariant, as L is still visible in each of the L_n, but the usual observable (the probability that there is an open path separating a point from a boundary component in a domain) is quasi-conformal in the position of the point, with explicit and piecewise constant Beltrami differential.

(2) Consider the mesoscopic scaling limit above, and scale it by a factor $\delta > 0$: Then, as $\delta \to 0$, it converges to a continuum limit. Again, this scaling limit is not conformally invariant in general, but the limit observables are quasi-conformal with constant Beltrami differential, depending (though not quite explicitly) on L and α.

(3) The unique choice of α making this constant Beltrami differential equal to 0, and thus for which the scaling limit is conformally invariant, is α_u.

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Coulomb gas formalism

Nikolai Makarov

This talk is a joint project with Nam-Gyu Kang (Caltech). Coulomb gas formalism is a model of conformal field theory based on modifications of the free bosonic field. We give a random field version of this theory following Chapter 9 of the monograph "Conformal field theory" by P. Di Francesco, P. Mathieu, and D. Senechal, and a paper by I. Rushkin, E. Bettelheim, I. Gruzberg, and P. Wiegmann (J. Phys. A, 40, 2165-2195, 2007).

The fields are random generalized functions (but not necessarily distributions) with values in the Fock space. We'll be considering the simplest topological situation – the field theory in a simply connected domain D with one marked boundary point $q \in \partial D$. There are several steps in the construction.

(1) *Theory with central charge* $c = 1$. The starting point is the (conformally invariant) Gaussian free field Φ_0 in D with Dirichlet boundary conditions. Its current is a (1,0)-differential $J_0 = \partial\Phi_0$, and the "holomorphic" component of the stress energy tensor (S.E.T.) is defined by the formula

$$T_0 := -\frac{1}{2} J * J = -\frac{1}{2} J \odot J + \frac{1}{12} S_w.$$

Here $\odot$ denotes Wick's multiplication in the Fock space, and $*$ the OPE multiplication, the meaning of which will be explained in the talk. We will also give a more conceptual definition of S.E.T.

(2) *Feigin-Fuchs-Miura modification.* Let $w : (D, q) \to (\mathbb{H}, \infty)$ be a conformal map onto the halfplane $\mathbb{H}$. Given $b \in \mathbb{R}$, we define the bosonic field Φ by the formula

$$\Phi = \Phi_0 - 2b \arg w'.$$

Then $J := \partial\Phi$ is a PS-form (pre-schwarzian) of order ib, and

$$T := -\frac{1}{2} J * J + ib\partial J$$

is a Schwarzian form of order $(c/12)$, where

$$c = 1 - 12b^2 \leq 1.$$

(3) *Vertex fields.* For $\alpha \in \mathbb{C}$ we define

$$V^\alpha := e^{*\alpha\Phi} = \left(\frac{w'}{\bar{w}'}\right)^{\alpha b i} C^{\alpha^2} \, e^{\odot\alpha\Phi_0},$$

where $C(z)$ denotes conformal radius. V^α is a $(d, d^\#)$-differential with

$$d = -\frac{\alpha^2}{2} + i\alpha b, \qquad d^\# = -\frac{\alpha^2}{2} - i\alpha b.$$

(4) *Chiral fields.* The "holomorphic" bosonic field Φ^+ is a Gaussian field on paths:

$$\Phi^+(z;\gamma) = \int_\gamma J(\zeta) \, d\zeta;$$

it is a PPS-form (pre-pre-schwarzian) of order ib. "Holomorphic" vertex fields are then defined as follows:

$$V^{+\alpha}(z;\gamma) = e^{*\alpha\Phi^+(z;\gamma)} = (w')^{-\alpha^2/2 + \alpha b i} \, e^{\odot\alpha\Phi_0^+}.$$

$V^{+\alpha}$ is a $(d, 0)$- differential in z, and a d-boundary differential wrt q.

(5) *Operator product expansion.* We explain the meaning of the Virasoro OPE

$$T(\tilde{z})T(z) \sim \frac{c/2}{(\tilde{z}-z)^4} + \frac{2T(z)}{(\tilde{z}-z)^2} + \frac{\partial T(z)}{\tilde{z}-z},$$

and Ward's OPE

$$T(\tilde{z})A(z) \sim \frac{dA(z)}{(\tilde{z}-z)^2} + \frac{\partial A(z)}{\tilde{z}-z},$$

where A is a "primary" field, namely A is a vertex or a chiral vertex field.

The operation $*$ is defined in terms of the OPE; it is neither commutative nor associative, but for linear operators $L_A : X \mapsto A * X$ we have the identity

$$L_{[A,B]} = [L_A, L_B]\,.$$

Notation: $L_T = L_{-2}$; more generally ("Virasoro generators")

$$T(\tilde{z})X(z) \sim \cdots + \frac{L_{-1}X(z)}{\tilde{z}-z} + L_{-2}X + \ldots.$$

(6) *Elementary representation theory of Virasoro generators.* Lemma:

$$2a(a+b) = 1 \quad \Rightarrow \quad T * V^{+ia} = \frac{1}{2a^2}\partial^2 V^{+ia}.$$

Numerology: denote

$$\frac{a+b}{a} = \frac{\kappa}{4}.$$

Then $2a^2 = 4/\kappa$. Given b [or c] there are 2 corresponding values of κ,

$$c = 1 - \frac{3}{2}\frac{(\kappa-4)^2}{\kappa}, \quad \kappa_1\kappa_2 = 16.$$

(7) *Ward's identities in* $\mathbb{H}$ *and the nature of S.E.T.* Given $(d_j, d^{\#})$-differentials A_j, we define the field T by the formula

$$\mathbb{E}[T(z)A_1(z_1)\ldots] = \mathcal{L}_{-2}(z)\mathbb{E}[A_1(z_1)\ldots],$$

where

$$\mathcal{L}_{-2}(z) = \sum_j \left[\frac{d_j}{(z-z_j)^2} + \frac{d_j^{\#}}{(z-\bar{z}_j)^2}\right] + + \left[\frac{\partial_j}{z-z_j} + \frac{\bar{\partial}_j}{z-\bar{z}_j}\right].$$

If such a field exists, then A_j's are "primary" fields of the theory with T as S.E.T. The meaning of Ward's identities: S.E.T. characterizes the response of A_j's to the variation of the conformal structure in D. The identities are true for (chiral) vertex fields and T introduced above.

(8) *Insertion operators.* Inserting $A(z_0)$ means the change of probability measure

$$P \;\mapsto \hat{P} = \frac{A(z_0)\cdot P}{\mathbb{E}[A(z_0)]}$$

in the Fock space $L^2(\Omega, P)$, so

$$\frac{\mathbb{E}[A(z_0)X_1(z_1)\dots]}{\mathbb{E}[A(z_0)]} = \hat{\mathbb{E}}[\hat{X}_1(z_1)\dots].$$

Examples:

(i) if $z_0 \in D$ and $A = V^\alpha(z_0)$, then

$$\hat{\Phi} \stackrel{d}{=} \Phi + 2\alpha G(\cdot, z_0);$$

(ii) if $p \in \partial D$ and $A = V^{+ia}(p)$, then

$$\hat{\Phi} = \Phi + 2[a \arg w - b \arg w'] + \text{GFF}.$$

(9) *SLE theory.* Let p be a boundary point other than q. Insert $A = V^{+ia}(p)$ with a satisfying $2a(a+b) = 1$. Combining the lemma in (6) with Ward's identities, we get PDEs for correlation functions under insertion.

Corollary: correlation functions of primary fields are martingale-observables for SLE(κ).

(10) *SLE fields.* This corollary can be interpreted as the following statement due to O. Schramm and S. Sheffield. Given (D, p, q) consider

$$\hat{\Phi}_p(D, q) \equiv 2[\arg w - b \arg w'] + \Phi_0$$

with $a = a(\kappa), b = b(\kappa)$. There is a coupling of the Brownian motion and the Fock space such that

$$\mathbb{E}[\hat{\Phi}_p | \mathcal{F}_t] = \hat{\Phi}_{\gamma_t}$$

(Markov property). Now, let $\mathcal{F}$ denotes the sigma-algebra generated by SLE events, so we can consider "SLE-fields"

$$A_{\text{SLE}} = \mathbb{E}[\hat{A} | \mathcal{F}].$$

(For chiral fields we define them as local martingales $\mathbb{E}[\hat{A}|\mathcal{F}_t]$.) We get a version of non-Gaussian conformal fields theory, which preserves all basic features of CFT but has a somewhat complicated definition of the OPE multiplication.

Combinatorics of Boundary Connection Formulas for Trees and Dimers

David B. Wilson
(joint work with R. Kenyon)

Given a finite planar graph, a grove is a spanning forest in which every component tree contains one or more of a specified set of vertices (called nodes) on the outer face. For the uniform measure on groves, we compute the probabilities of the different possible node connections in a grove. These probabilities only depend on boundary measurements of the graph and not on the actual graph structure, i.e., the probabilities can be expressed as functions of the pairwise electrical resistances between the nodes, or equivalently, as functions of the Dirichlet-to-Neumann operator (or discrete Hilbert transform or response matrix) on the nodes. These formulae can be likened to generalizations (for spanning forests) of Cardy's percolation crossing probabilities, and generalize Kirchhoff's formula for the electrical resistance. Remarkably, when appropriately normalized, the connection probabilities are in fact integer-coefficient polynomials in the matrix entries, where the coefficients have a natural combinatorial interpretation. In particular, for "tripartite" pairings of the nodes, the probability can be computed as a Pfaffian in the entries of the response matrix. These formulas generalize the determinant formulas given by Curtis, Ingerman, and Morrow, and by Fomin, for parallel pairings. A similar phenomenon holds in the so-called double-dimer model: connection probabilities of boundary nodes are polynomial functions of certain boundary measurements, and as formal polynomials, they are specializations of the grove polynomials. Upon taking scaling limits, we show that the double-dimer connection probabilities coincide with those of the contour lines in the Gaussian free field with certain natural boundary conditions. These results have direct application to connection probabilities for multiple-strand SLE_2, SLE_8, and SLE_4.

Ergodic properties of a class of non-Markovian processes

Martin Hairer

Our aim is to study a fairly general class of time-homogeneous stochastic evolutions driven by noises that are not white in time, so that the resulting processes do not have the Markov property. In this setting, we would like to obtain constructive criteria for the uniqueness of stationary solutions that are very close in spirit to the existing criteria for Markov processes. If we denote by $\mathcal{X}$ the state space of the process and by $\mathcal{W}$ a sufficiently large space that incorporates all the required information about the past of the driving noise, we can realise such an evolution as a Markov process on $\mathcal{X} \times \mathcal{W}$ with a skew-product structure (that is the projection on $\mathcal{W}$ is again Markov).

Denote by $\mathcal{Q}_t$ the Markov semigroup for the process on this augmented phase space. Denote also by $\mathcal{S}_t \colon \mathcal{X} \times \mathcal{W} \to \mathcal{M}_1(\mathcal{C}([t,\infty), \mathcal{X}))$ the map that, given an initial condition $x \in \mathcal{X}$ and a realisation $w \in \mathcal{W}$ of the 'past' of the driving noise process, gives the law of the process in $\mathcal{X}$ for times greater than t. With this notation, we define an equivalence relation on probability measures on $\mathcal{X} \times \mathcal{W}$ by $\mu \sim \nu \leftrightarrow \mathcal{S}_0\mu = \mathcal{S}_0\nu$. We are then looking for criteria for the invariant measure of $\mathcal{Q}_t$ to be unique, modulo this equivalence relation, that is analogous to the celebrated Doob-Khashminskii criterion (strong Feller property + topological irreducibility).

A natural analogue to the strong Feller property is that the map $\mathcal{S}_t$ is continuous in the total variation distance. A natural analogue to topological irreducibility is that $\mathcal{Q}_t(x, w; A \times \mathcal{W}) > 0$ for every open set $A \subset \mathcal{X}$. It turns out that this is not sufficient in general to ensure uniqueness of the invariant measure for $\mathcal{Q}_t$, even modulo the aforementioned equivalence relation. It is however possible to introduce a type of 'quasi-Markov property' that ensures that the set

$$\Lambda = \Big\{(w, w') \in \mathcal{W} \times \mathcal{W} \,:\, \mathcal{S}_t(x, w) \approx \mathcal{S}_t(x, w') \,\forall x \in \mathcal{X}\Big\}$$

is sufficiently 'large' in the sense that one can construct couplings that charge Λ without altering the dynamic too much. Here, the symbol $\approx$ denotes the mutual equivalence of probability measures. With these notions at hand, our main result is

Theorem 1. If a non-Markovian process on $\mathcal{X}$ as described above has the strong Feller property, is topologically irreducible, and has the quasi-Markov property, then the corresponding semigroup $\mathcal{Q}_t$ can have at most one invariant measure, modulo the equivalence relation $\sim$.

This result can be applied for example to diffusions driven by fractional Brownian motion, as well as random dynamical systems driven by a stationary Gaussian sequence.

The Spread of Randomness: Ergodicity in Infinite Dimensions

Jonathan C. Mattingly

Consider a partial differential equation forced by an additive white noise,

$$du(x,t) = -Lu(x,t)\,dt + N(u(x,t))\,dt + \sum_k g_k(x)d\beta_k(t)$$

where L is a "nice" linear operator (positive, self-adjoint, compact resolvent), N is a "nice" polynomial nonlinearity and the $\{\beta_k(t) : k \in \mathbf{N}\}$ are a collection of mutually independent Brownian motions. Assuming that N is nice enough that the system has a sufficiently smooth global solution, it is interesting to ask how the randomness spreads from the directions in which it is injected (the $\{g_k\}$) to the rest of the phase space.

Interesting, motivating examples to which the theory applies include:

$$(2D-NS)\begin{cases} du + (Ku\cdot\nabla)u\,dt = \nu\Delta u\,dt + \displaystyle\sum_k g_k d\beta_k(t) \\ \displaystyle\int_{[0,2\pi]^2} u(x,t)dx = 0 \quad \text{and} \quad u(\,\cdot\,,t) \text{ is } 2\pi-\text{periodic} \end{cases}$$

$$(RD)\begin{cases} du = \Delta u\,dt + (u-u^3)\,dt + \displaystyle\sum_k g_k d\beta_k(t) \\ u(x,0) = u(x,2\pi) = 0 \end{cases}$$

In the first case $u(x,t)\in L^2([0,L]^2)$ is the vorticity (the curl of the velocity field) and K is the Biot-Savart integral operator which reconstructs the velocity field from the vorticity. In the second case $u(x,t)\in L^2([0,L])$ is a scalar field. We also consider the Navier-Stokes equations on the two dimentional sphere.

In all cases, we are mainly interested in the setting where the stochastic forcing only contains a finite number of terms. Hence if the randomness is to spread throughout the phase space this must be accomplished through the drift term ("dt" term) which couples all of the degree's of freedom. In the above examples this term is nonlinear however it could also be a linear term.

Associated with each of these Markov process is a transition semigroup $\{P_t : t\geq 0\}$ acting on an infinite dimensional space and defined by

$$(P_t\phi)(v) = \mathbf{E}_v\phi(u_t)$$

for real valued test functions $\phi : L^2 \to \mathbf{R}$. Here, and throughout, $\mathbf{E}_v$ means the expected value when the initial condition taken to be v. The local smoothing properties of such a semigroup and whether it is uniquely ergodic are intimately related.

Finite Dimensions: If the phase space is finite dimensional, then the simplest assumption giving such smoothing is uniform ellipticity which states that quadratic form Q defined by

$$\langle Qv, v\rangle = \sum_k \langle g_k, v\rangle^2$$

satisfies $\langle Qv,v\rangle \geq \alpha\|v\|^2$ for all v and for some fixed $\alpha\geq 0$. This the same as asking that the span$\{g_k\}$ is the whole space. With this assumptions and some reasonable assumptions on the smoothness of the lower order terms, one quickly obtains that

$$(P_t\phi)(v) = \int \rho_t(v,z)\phi(z)dz$$

for some smooth (even positive) "density" ρ_t.

If span$\{g_k\}$ does not the equal the entire space, then one is required to consider the more delicate condition of Hörmander which ensures that the system still has a smooth (though not necessarily positive) density ρ_t . The above quadratic form Q is replaced with a different one which is built from the Lie brackets (commutators) of the constant vector-fields of $\{g_k\}$ and the "drift" vector field.

In both cases for fixed t, one is ensured that the semi-group $P_t(v, \cdot)$ is continuous in v where the topology of total variation convergence is used. That is to say, if $v_n \to v$ then

$$\lim_{n\to\infty} \|P_t(v_n, \cdot) - P_t(v, \cdot)\|_{TV} = 0 \; . \tag{1}$$

Whether in finite or infinite dimensions, any continuous time strong-Feller semigroup satisfies (1). (See [Sei01, HM06].)

Infinite Dimensions: In infinite dimensions, what should be meant by the "uniformly elliptic" case is less clear. For example for $x \in [0, 2\pi]$ and $u(\cdot, t) \in L^2([0, 2\pi])$, consider

$$\text{(HEAT)} \begin{cases} du(x,t) = \nu\Delta u \, dt + \displaystyle\sum_{k=1}^{\infty} \sigma_k \sin(xk) d\beta_k(t) \\ u(0,t) = u(2\pi, t) = 0 \, . \end{cases}$$

If $\sigma_k = k^\alpha$ for $\alpha < 0$ then (1) does hold for $v_n, v \in L^2([0, 2\pi])$ while if $\sigma_k = \exp(-|k|^\alpha)$ with $\alpha > 2$ it does not for an arbitrary $v_n, v \in L^2([0, 2\pi])$. (It does of course hold for well chosen v_n and v.)

The problem is the use of the total variation norm. It is possible to replaces it with a weaker norm and have convergence of the form given in (1). For example, if one uses the 1-Wasserstien distance then (1) holds. In this setting, one can define the 1-Wasserstien between two probability measure μ_1 and μ_2 by

$$\|\mu_1 - \mu_2\|_1 = \sup_{f:\text{Lip}(f)\le 1} \int f(v)\mu_1(dv) - \int f(v)\mu_2(dv)$$

where $\text{Lip}(f) = \sup_{v\neq u} |f(v) - f(u)|/|v - u|$ and f is a real valued function.

A few ergodic results: The preceding considerations have lead to a number of results which do not rely on (1) or assumptions which imply it. (See [EMS01, KS00, BKL01, BM04, HM06, HM08a]) In particular, the presentation in [EMS01, BM04] simply gives the uniqueness of an invariant measure (if it exists) while the conditions in [Mat02, Hai02, HM06, HM08a] (with additional assumptions) can be used to show exponential convergence. it has proven powerful to exculpate these ideas in the notion of a *asymptotic strong Feller* diffusion which as introduced by the author and Martin Hairer in [HM06] along with many of the tools used in this note.

Theorem 1. (Hairer, M ('04-'08)) Let the g_k be analytic, the stochastic forcing finite dimensional and define $\Lambda_0 = \{g_k\}$,

$$\Lambda_k = \Lambda_{k-1} \cup \big\{[N, Q], [\cdots[Q, g_{j_1}], \cdots], g_{j_r}] : g_{j_i} \in \Lambda_0, Q \in \Lambda_{k-1}\big\} \, .$$

If constant vector fields in Λ_∞ are dense in the phase space in the setting of (2D-NS) or (RD) then the system is uniquely ergodic. (See [HM06, HM08b]).

The non-constant vector fields and g_k which are not analytic can be considered with further assumptions (See [HM08b]).

Furthermore one can show that the systems posses a spectral gap in the sense if that for any probability measure μ_1 and μ_2 one has

$$\|P_t^*\mu_1 - P_t^*\mu_2\|_{1^*} \le Ce^{-ct}\|\mu_1 - \mu_2\|_{1^*}$$

for some $C, c > 0$. Here the norm is the 1-Wasserstien metric induced by a (possibly) weighted norm on the phase space. In the case of the 2D Navier-Stokes equation, the norm is given by

$$\|\mu_1 - \mu_2\|_{1^*} = \sup_{f:f\in F} \int f(v)\mu_1(dv) - \int f(v)\mu_2(dv)$$

where

$$F = \Big\{ f : H^1([0,2\Pi]^2) \to \mathbf{R}, \sup_v e^{-\gamma\|v\|^2}\big(|f(v)| + \|DF(v)\|\big)\Big\}$$

for some $\gamma > 0$. See [HM08a] for the details in the 2D-NS setting. The result proven in the reaction diffusion setting is similar.

Smoothing Results: An important ingredient in all of the ergodic results is the local smoothing implied by an estimate of the form

$$\|DP_t\phi\|_\infty(v) \le C(v)\sqrt{P_t\|\phi\|^2_\infty(v)} + \alpha\sqrt{P_t\|D\phi\|^2_\infty(v)}$$

for any appropriate test function ϕ and a fixed $\alpha \in (0,1)$. The estimate implies that the system is *asymptotic strong Feller* (see [HM06]).

The ideas to prove such an estimate in the case when all of the "unstable" directions are forced date back to [EMS01]. The precise calculation in this setting can be found in [HM06]. There it is proven that under appropriate assumptions the case when all of the "unstable" directions are not directly forced, one must make essential use of the fact that the appropriate brackets are dense. To do so, one uses estimates on the non-degeneracy of smallest eigenvalue of the Malliavin matrix whose eigenfunction is concentrated in the unstable directions. This makes use of estimates developed in [MP06] and extended in [BM07, HM08b]. There it is proven that under appropriate assumptions, given any finite dimensional projection Π

$$\mathbf{P}\Big(\inf_{\xi:\|\Pi\xi\|\ge\frac{1}{2}\|\xi\|}\langle M_t\xi,\xi\rangle < \epsilon\|\xi\|^2\Big) = o(\epsilon^p)$$

for all $p \ge 1$. In words, "With high probability the eigenfunctions with large projection under Π do not have small eigenvalues." This estimate also leads quickly to the fact that finite dimensional marginal $P_t\Pi^{-1}$ has a C^∞ density with respect to Lebesgue measure.

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Periodic 2d turbulence and limiting properties of homogeneous measures on two-torus

Sergei B. Kuksin

Space-periodic 2D turbulence is described by solutions of the small-viscosity 2D Navier-Stokes equations (NSE) on the 2-torus $\mathbb{T}^2$, perturbed by a stationary random force. In this talk we assume that the force is Gaussian, is smooth and stationary in x, while as a function of t it is a white noise. It turns out that in this case the solutions remain of order one as the viscosity $\nu \to 0$ if and only if the force is proportional to $\sqrt{\nu}$. The goal of this talk is to discuss properties of solutions for this equation for small ν. It is known [1] that when time $t \to \infty$, solutions of NSE converge in distribution to the unique stationary measure μ_ν which is a homogeneous measure in the space $\mathcal{H}$ of divergence-free vector-fields on $\mathbb{T}^2$. When $\nu \to 0$ along a sequence $\{\nu_j\}$, we have the convergence $\mu_{\nu_j} \to \mu_0$, where μ_0 is an invariant measure for the free 2D Euler equation (we cannot rule out that it depends on the sequence ν_j). The measures μ_ν with $\nu \ll 1$ and the limiting measure μ_0 (rather the limiting *measures*) describe statistical properties of the space-periodic 2D turbulence. Some of them are discussed in the talk. Namely:

I) *Balance relations.* For $\nu > 0$ let $u_\nu(x) \in \mathcal{H}$ be a random field, distributed as μ_ν. Denote by $\xi_\nu(x)$ its vorticity, $\xi_\nu = \frac{\partial u_\nu^2}{\partial x^1} - \frac{\partial u_\nu^1}{\partial x^2}$. For any $\tau \in \mathbb{R}$ set

$$\Gamma = \Gamma_\nu^\tau = \{x \in \mathbb{T}^2 \mid \xi_\nu(x) = \tau\}.$$

Theorem 1 (see [1]). For any $\tau \in \mathbb{R}$ and $\nu > 0$ we have

$$\mathbf{E}\int_\Gamma |\nabla \xi_\nu|\, d\gamma = K\mathbf{E}\int_\Gamma |\nabla \xi_\nu|^{-1}\, d\gamma\,.$$

Here $d\gamma$ is the length element on Γ and K is an explicit constant, depending on the applied force.

II) *Properties of the measure μ_0.*
Theorem 2 (see [2]). For any $\delta > 0$

$$\mu_0\{\|u\| < \delta\} \leq C\sqrt{\delta}.$$

Consider the energy-map

$$E : \mathcal{H} \to \mathbb{R}, \qquad u(x) \mapsto \frac{1}{2}\,\|u\|^2.$$

Theorem 3 (see [2]). The measure $E \circ \mu_0$ is absolutely continuous with respect to the Lebesgue measure.

For any $N \geq 1$ let $g_1, \dots, g_N$ be analytic functions $g_j : \mathbb{R} \to \mathbb{R}$ which have at most a polynomial growth at infinity, are bounded from below and are independent modulo constant functions. Consider the map

$$u(x) \mapsto F(u) = \{\int g_j(\operatorname{rot} u(x))\, dx,\ 1 \leq j \leq N\} \in \mathbb{R}^N\,.$$

Theorem 4 (see [2]). The measure $F \circ \mu_0$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$.

In particular, the measure μ_0 is 'genuinely infinite-dimensional' in the sense that for any compact set K in $\mathcal{H}$ of finite Hausdorff dimension we have $\mu_0(K) = 0$.

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Zeros of a random matrix-valued analytic function

Manjunath Krishnapur

We proved the following theorem in the lecture.

Theorem 1. Fix a positive integer m and let G_k be i.i.d. $m \times m$ matrices with i.i.d. standard complex Gaussian entries. Then the zeros of $\det\left(G_0 + zG_1 + z^2G_2 + \dots\right)$ form a determinantal point process on d with kernel

$$\mathbb{K}(z,w) = \frac{1}{(1-z\bar{w})^{m+1}}$$

with respect to the background measure $d\mu_m(z) = \frac{m}{\pi}(1-|z|^2)^{m-1}dm(z)$. Equivalently, we may say that the defining Hilbert space is the subspace of *analytic* functions in $L^2(\mathrm{d}, \mu_m)$.

Some remarks.

(1) The case $m = 1$ was discovered by Peres and Virág [4] and appears to be the first case where the exact distribution of zeros of a random analytic function was found explicitly (unlike in random matrices, where there were many such results known).
(2) The determinantal processes appearing here were studied by Jancovici and Tellez [1] under the name "one component plasma", but without any connection to zeros of random analytic functions. The determinantal processes make sense for any postive m while the matrix-valued analytic functions make sense only for integer m.
(3) The point process appearing here is invariant in distribution under conformal automorphisms of the unit disk. This is easily proved, following the proof of Lebœuf [3] for the case $m = 1$.
(4) The above theorem was proved (for general m) by Krishnapur [2] which contains full details on this problem and more complete references. However a much simplified proof was presented in the lecture, and is jointly together with Victor Katsnelson and Bernd Kirstein. This is yet to appear.
(5) There is an analogous theorem for the whole plane (due to Ginibre) and on the sphere (due to Krishnapur. See [2] for details.

The proof relies on a theorem of Życzkowski and Sommers [5] who found the exact distribution of truncations of Haar distributed unitary random matrices. If m columns and rows are deleted from a unitary matrix, the resulting matrix has eigenvalues that form a determinantal process with the truncation of the kernel in the theorem above. By taking limits we arrive at our theorem. The proof goes through the 'characteristic function', a rational inner function associated to the unitary matrix. Another idea needed is this. Let e_i be standard co-ordinate vectors in $\mathbb{C}^n$. Then the spectral measures of an $n \times n$ Haar-unitary matrix at the pairs of vectors (e_i, e_j) converge to independent complex white noises on the unit circle.

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Asymptotics of measures on the unitary group

Édouard Maurel-Segala

(joint work with Benoît Collins, Alice Guionnet)

Since the work of Brézin Itzykson Parisi and Zuber [1], matrix integrals are known to be a very efficient tool to enumerate combinatorial structures (see Chapter 3 in the book of Lando and Zvonkin [5] for a very accessible introduction to that field). More precisely, the authors show that various generating functions for the enumeration of planar graphs appear as the limit of the free energy of some well chosen matrix integral. The simplest example is the equality between the two formal series in t:

$$\lim_{N\to\infty}\frac{1}{N^2}\ln\int_{\mathcal{H}_N(\mathcal{C})} e^{-Nt\mathrm{Tr}A^4+\frac{N}{2}\mathrm{Tr}A^2}d^NA=\sum_{k\in\mathbb{N}}(-t)^k\mathcal{M}_k.$$

On the left hand side we have the limit of an integral over the hermitian matrices and on the right hand side we have a combinatorial series: $\mathcal{M}_k$ is the number of planar graphs on the sphere with k faces of degree 4.

Such equalities were used to compute various combinatorial quantities since there are various ways to analyze such a matrix integral. The first step is often to change the space of integration. More precisely since any hermitian matrix has a real spectrum, it can be shown that the previous integral can be changed into an integral on the eigenvalues $\lambda_1\cdots\lambda_N$ of the matrix:

$$\int_{\mathcal{H}_N(\mathcal{C})} e^{-Nt\mathrm{Tr}A^4+\frac{N}{2}\mathrm{Tr}A^2}d^NA=c_N\int_{\mathbb{R}^N}(\Delta(\lambda))^2e^{\sum_i(-Nt\lambda_i^4+\frac{N}{2}\lambda_i^2)}\prod_i d\lambda_i$$

where Δ is the Van der Monde determinant. This allows to use many analytical tools such as the technology of orthogonal polynomials to understand combinatorial problems.

In the 80's, the observation that matrix integrals could be used to understand enumeration of graphs was generalized to the enumeration of graphs with some models of physical statistics on them. For example, one can find matrix integrals that enumerate configuration of the Ising models, Potts models or $O(n)$ models on finite graphs. The issue is that now the integrals is no longer on only one matrix but on several and furthermore the potential appearing in the integral is a non-commutative polynomial of those matrices. This make those integral hard to analyze and the reduction to the space of the eigenvalues of the matrices is not as simple. Still, this could be done by integrating on the "angle" between the matrices. More precisely, define,

$$\mathrm{IZ}\ _c(A,B) = \int_{\mathcal{U}_N(\mathcal{C})} e^{cN\mathrm{Tr}(U^*AUB)} d^N(U)$$

where the integration is done over the unitary group. It can be shown that this integral take care of the "angle" between the matrices and that integrals enumerating those models of physical statistic can be reduced to simple integral over the space of eigenvalues up to the understanding of this integral. The main issue is then to understand the large N asymptotic of such a quantity. It turns out to be a tricky challenge.

Even the convergence of this integral when the spectral measure of A and B are converging was unclear for a long time. Guionnet and Zeitouni [4] managed to prove under few conditions that the integral was converging. Collins [2] proved the existence of limit of this integrale seen as a formal series in c. In a recent work [3], we proved that the formal series has a positive radius of convergence and that for small c's it is equal to the real limit. Besides we proposed a combinatorial interpretation in term of planar graphs to the limit which explained why there are so much integer quantities in the limit.

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Uniqueness of Brownian motion on Sierpinski carpets

Takashi Kumagai

(joint work with M.T. Barlow, R.F. Bass and A. Teplyaev)

1. Brownian motion on Sierpinski carpets

Let $F_0 = [0,1]^d$, and consider the family of contraction maps $\{\Psi_i\}_{i=1}^N$ where $N = 3^d - 1$ with contraction rate 1/3. Let $F_1 = \cup_{i=1}^N \Psi_i(F_0) = F_0 \setminus (1/3, 2/3)^d$, and define inductively $F_{n+1} = \cup_{i=1}^N \Psi_i(F_n)$. Then, $F := \cap_{n=0}^\infty F_n$ is the d-dimensional

Sierpinski carpet we will consider. Natural diffusions have been constructed on F as follows.

i) The Barlow-Bass processes ([1, 2])

Let W_t^n be a reflected Brownian motion on F_n. Then, it is proved that there exists $t_F > 0$ such that $X_t^n := W^n_{(\frac{t_F}{9})^n t}$ are tight. Further, there exists a subsequence $\{n_j\}$ such that $U^\lambda_{n_j} f$ (λ-resolvent) converges uniformly on F for all $f \in C(F_0)$. The key point for proving these facts are to prove uniform elliptic Harnack inequalities on F_n, where uniform means that the constants are independent of n. By the above facts, $X_t^{n_j}$ converges to some non-trivial diffusion process B_t on F ($\mathbb{P}^x$ law converges weakly for all $x \in F$). Note that the limiting process may depend on the choice of $\{n_j\}$. We fix one of them; $\{B_t\}_{t\ge 0}$ is called Brownian motion on F.

It is shown that there exists a jointly continuous heat kernel $p_t(\cdot,\cdot)$ such that

$$c_1 t^{-\frac{d_f}{d_w}} \exp(-c_2(\frac{|x-y|^{d_w}}{t})^{\frac{1}{d_w-1}}) \le p_t(x,y) \le c_3 t^{-\frac{d_f}{d_w}} \exp(-c_4(\frac{|x-y|^{d_w}}{t})^{\frac{1}{d_w-1}}),$$

for all $x, y \in F, 0 < t \le 1$, where $d_f = \frac{\log N}{\log 3}$ is the Hausdorff dimension and $d_w = \frac{\log t_F}{\log 3} > 2$ is called a walk dimension. (Such estimates are sometimes called sub-Gaussian heat kernel estimates.)

Let $(\mathcal{E}_{BB}, \mathcal{F})$ be the corresponding Dirichlet form on $L^2(F,\mu)$, where μ is the Hausdorff measure. Then, one can prove that

$$\begin{aligned}\mathcal{F} &= \Lambda^{d_w/2}_{2,\infty}(F)\\ &:= \{f \in L^2 : \sup_m 3^{m(d_w+d_f)} \int\int_{|x-y|<c3^{-m}} |f(x)-f(y)|^2 d\mu(x)d\mu(y) < \infty\},\end{aligned}$$

namely, the domain of the form is a Besov-Lipschitz space.

ii) The Kusuoka-Zhou Dirichlet form ([5])

In [5], a non-trivial local regular Dirichlet form $(\mathcal{E}_{KZ}, \mathcal{F})$ on $L^2(F,\mu)$ is constructed as a limit of Dirichlet forms on the approximating graphs. Moreover, it has the following self-similarity:

$$\mathcal{E}_{KZ}(f,g) = \frac{t_F}{N}\sum_{i=1}^N \mathcal{E}_{KZ}(f\circ\Psi_i, g\circ\Psi_i), \qquad \forall f,g \in \mathcal{F}.$$

It is known that the domain $\mathcal{F}$ is the same Besov-Lipschitz space and the process enjoys the sub-Gaussian heat kernel estimates.

Remark: 1) In [5], such a form was constructed only for strongly recurrent case ($d = 2$ in this case). However, given the uniform Harnack inequalities proved in [2], one can construct the self-similar Dirichlet form for $d \ge 3$ as well.

2) It was not known whether $(\mathcal{E}_{BB}, \mathcal{F})$ satisfies the self-similarity or not.

Now the following natural question arises. Do these Dirichlet forms coincide up to constant? Is there a unique 'nice' Dirichlet form on F?

This question of uniqueness of Brownian motion on F has been one of the biggest open problem in this area. We solve it affirmatively.

To explain the difficulty of the problem, we refer the corresponding results for other fractals. First, uniqueness of the Brownian motion for the Sierpinski gasket was proved already long before ([4]). The statement is roughly as follows: any diffusion process on the gasket whose law is invariant under local translations and reflections of each small triangle is a constant time change of Brownian motion. This was later extended to several classes of finitely ramified fractals (i.e. fractals that can be disconnected by removing some finite number of points) including nested fractals (see [6, 8]). There the key point was to prove the uniqueness of non-degenerate fixed point for a renormalization map of the induced random walk. The map is non-linear, but it is finite dimensional because of the finitely ramified properties of the fractals. Sierpinski carpets are infinitely ramified and so the corresponding renormalization maps are infinite dimensional. This makes the problem hard. We note that on the carpet, 'exotic' diffusions were constructed in [7].

Our approach to prove the uniqueness is different from above; we do not analyze the renormalization map. – Imitating our proof, it should be possible to give an alternative proof of uniqueness for Brownian motion on the above mentioned fractals as well.

2. Main theorem

In order to state our main theorem, we first define a F-invariant Dirichlet form, which is a 'nice' Dirichlet form on F.

Let $\mathcal{S}_n$ be a set of cubes in F with length 3^{-n}. For $S \in \mathcal{S}_n$, define a folding map $\varphi_S : F \to S$ as follows. Let $\overline{\varphi}_0 : [-1, 1] \to \mathbb{R}$ be defined by $\overline{\varphi}_0(x) = |x|$ for $|x| \le 1$ and then extend it to all of $\mathbb{R}$ periodically. If y is the point of S closest to the origin, define $\varphi_S(x)$ for $x \in F$ to be the point whose i^{th} coordinate is $y_i + 3^{-n}\overline{\varphi}_0(3^n(x_i - y_i))$.

Given $S \in \mathcal{S}_n$, $f : S \to \mathbb{R}$ and $g : F \to \mathbb{R}$, let $U_S f := f \circ \varphi_S$, $R_S g = g|_S$.

Now, for a local regular Dirichlet form on $(\mathcal{E}, \mathcal{F})$ and $S \in \mathcal{S}_n$, define

$$\mathcal{E}^{(S)}(g, g) = \frac{1}{N^n}\mathcal{E}(U_S g, U_S g),\ \mathcal{F}^{(S)} = \{g : S \to \mathbb{R} : U_S g \in \mathcal{F}\}.$$

Definition 1. $(\mathcal{E}, \mathcal{F})$ is an *F-invariant Dirichlet form* if the following hold:

(1) If $S \in \mathcal{S}_n(F)$, then $U_S R_S f \in \mathcal{F}$ for all $f \in \mathcal{F}$.

(2) Let $S_1, S_2 \in \mathcal{S}_n$, and let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be any isometry of $\mathbb{R}^d$ such that $\Phi(S_1) = S_2$. If $f \in \mathcal{F}^{(S_2)}$, then $f \circ \Phi \in \mathcal{F}^{(S_1)}$ and

$$\mathcal{E}^{(S_1)}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{(S_2)}(f, f).$$

(3) $\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n} \mathcal{E}^{(S)}(R_S f, R_S f)$ for all $f \in \mathcal{F}$.

(4) $\mathcal{E}$ is local and regular, non-zero, $1 \in \mathcal{F}$ and $\mathcal{E}(1, 1) = 0$.

We write $\mathfrak{E}$ for the set of F-invariant Dirichlet forms.

Our main theorem is the following.

Theorem 2 ([3]). *1) Up to scalar multiples, $\mathfrak{E}$ consists of only one element. Further, this one element of $\mathfrak{E}$ satisfies the self-similarity.*
2) $\mathcal{E}_{BB}, \mathcal{E}_{KZ} \in \mathfrak{E}$.

Corollary 3 ([3]). *1) If X is a non-degenerate symmetric Feller diffusion on F, and if the corresponding Dirichlet form is F-invariant, then the law of X under $\mathbb{P}^x$ is uniquely defined, up to scalar multiples of the time parameter, for each $x \in F$.*
2) $\mathcal{E}_{BB}, \mathcal{E}_{KZ}$ are (up to a constant) the same. $\mathcal{E}_{BB}$ satisfies self-similarity.

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A new approach to strong embeddings

Sourav Chatterjee

Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(\varepsilon_1) = 0$ and $\mathbb{E}(\varepsilon_1^2) = 1$. For each k, let

$$S_k = \sum_{i=1}^{k} \varepsilon_i.$$

Suppose we want to construct a standard Brownian motion $(B_t)_{t \geq 0}$ on the same probability space so as to minimize the growth rate of

$$\max_{1 \leq k \leq n} |S_k - B_k|. \tag{1}$$

The study of such embeddings began with the works of Skorohod [10, 11] and Strassen [13], who showed that under the condition $\mathbb{E}(\varepsilon_1^4) < \infty$, it is possible to make (1) grow like $n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}$. In fact, under the finite fourth moment assumption, this was shown to be the best possible rate by Kiefer [5].

For a long time, this remained the best available result in spite of numerous efforts by a formidable list of authors to improve upon Skorohod's idea. (For a detailed account of these activities, let us refer to the comprehensive survey of Obłój [8].) Therefore it came as a great surprise when Komlós, Major, and Tusnády [6], almost fifteen years after Skorohod's original work, proved by a completely different argument that one can actually have

$$\max_{k \le n} |S_k - B_k| = O(\log n)$$

when ε_1 has a finite moment generating function in a neighborhood of zero. Moreover, they showed that this is the best possible result that one can hope for in this situation.

Theorem 1 (Komlós-Major-Tusnády [6])**.** *Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. random variables with $\mathbb{E}(\varepsilon_1) = 0$, $\mathbb{E}(\varepsilon_1^2) = 1$, and $\mathbb{E}\exp\theta|\varepsilon_1| < \infty$ for some $\theta > 0$. Let $S_k := \sum_{i=1}^k \varepsilon_i$, $k = 0, 1, \ldots$ be the corresponding random walk. It is possible to construct a version of the sequence $(S_k)_{k\ge 0}$ and a standard Brownian motion $(B_t)_{t\ge 0}$ on the same probability space such that for every n and every $t \ge 0$,*

$$\mathbb{P}\big(\max_{k \le n} |S_k - B_k| \ge C \log n + t\big) \le K e^{-\lambda t},$$

where C, K, and λ do not depend on n.

We should mention that the paper [6] also has another very important result, a similar embedding theorem for uniform empirical processes, that we shall not discuss here. Let us refer to the recent articles by Mason [7] and CsörgⅢo [3] for very nice and comprehensive expositions of the ideas and literature surrounding the KMT embedding theorem for empirical procceses.

One problem with the proof of Theorem 1, besides being rather hard to read, is that it is extremely difficult to generalize. Indeed, even the most basic extension to the case of non-identically distributed summands by Sakhanenko [9] is so complex that some researchers are hesitant to use it. A nearly optimal multivariate version of the theorem was proved by Einmahl [4]; the optimal result was obtained by Zaitsev [14] at the end of an extraordinary amount of hard work. For further details and references, let us refer to the survey article by Zaitsev [15] in the Proceedings of the ICM 2002.

Our investigation is targeted towards a more conceptual understanding of the problem that may allow one to go beyond sums of independent random variables. It begins with the following abstract method of coupling a random variable W with a Gaussian random variable Z so that $W - Z$ has exponentially decaying tails at the appropriate scale.

Lemma 1. Suppose W is a random variable with $\mathbb{E}(W) = 0$ and $\mathbb{E}(W^2) < \infty$. Let T be another random variable, defined on the same probability space as W, satisfying

$$\mathbb{E}(W\varphi(W)) = \mathbb{E}(\varphi(W)T)$$

for all Lipschitz functions φ. Suppose $|T|$ is almost surely bounded by a constant. Then, given any $\sigma^2 > 0$, we can construct $Z \sim N(0, \sigma^2)$ on the same probability space such that for any $\theta \in \mathbb{R}$,

$$\mathbb{E}\exp(\theta|W - Z|) \le 2\,\mathbb{E}\exp\bigg(\frac{2\theta^2(T - \sigma^2)^2}{\sigma^2}\bigg).$$

The key idea, inspired by Stein's method of normal approximation [12], is that if $T \simeq \sigma^2$ with high probability, then W is approximately $N(0,\sigma^2)$, since $N(0,\sigma^2)$ is the only distribution satisfying $\mathbb{E}(Z\varphi(Z)) = \sigma^2\mathbb{E}(\varphi'(Z))$ for all φ. (We say that T is a 'Stein coefficient' of W.) However, classical Stein's method can only give bounds on quantities like

$$\sup_{f\in\mathcal{F}} |\mathbb{E}f(W) - \mathbb{E}f(Z)|,$$

for various classes $\mathcal{F}$. The above result seems to be of a fundamentally different nature. To see how Stein coefficients can be constructed, let us consider a few examples.

Example 1. Suppose X is a random variable with $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) < \infty$, and density ρ. Let

$$h(x) = \frac{\int_x^\infty y\rho(y)dy}{\rho(x)}.$$

Then, by integration-by-parts, we have $\mathbb{E}(X\varphi(X)) = \mathbb{E}(\varphi(X)h(X))$ for all φ. Thus, $h(X)$ is a Stein coefficient for X.

Example 2. Suppose $X_1,\ldots,X_n$ are i.i.d. copies of X, and let $W = n^{-1/2}\sum_{i=1}^n X_i$. Then by Example 1,

$$\mathbb{E}(W\varphi(W)) = n^{-1/2}\sum_{i=1}^n \mathbb{E}(X_i\varphi(W)) = n^{-1}\sum_{i=1}^n \mathbb{E}(h(X_i)\varphi(W)).$$

Thus, $n^{-1}\sum_i h(X_i)$ is a Stein coefficient for W.

Example 3. Suppose $\varepsilon_1,\ldots,\varepsilon_n$ are i.i.d. symmetric ±1-valued r.v. Let $S_n = \sum_{i=1}^n \varepsilon_i$. Let $Y \sim \text{Uniform}[-1,1]$. Let $W = S_n + Y$. Let

$$T = n - S_nY + \frac{1-Y^2}{2}.$$

It follows from a calculation involving integration-by-parts that for all φ,

$$\mathbb{E}(W\varphi(W)) = \mathbb{E}(\varphi'(W)T),$$

that is, T is a Stein coefficient for W. Letting $\sigma^2 = n$, Lemma 1 tells us that it is possible to construct $Z \sim N(0,n)$ such that

$$\mathbb{E}\exp(\theta|W-Z|) \le 2\,\mathbb{E}\exp\left(\frac{2\theta^2(T-n)^2}{n}\right).$$

Since $T = n + O(\sqrt{n})$, this proves that one can construct W and Z on the same probability space such that irrespective of n,

$$\mathbb{E}\exp(\theta|W-Z|) \le C$$

for some fixed constants θ and C, and so $\mathbb{P}(|W-Z| \ge t) \le Ce^{-\theta t}$ for all t.

Example 4. Suppose $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of i.i.d. standard Gaussian random variables. Let $W = f(\mathbf{X})$, where f is absolutely continuous. Suppose $\mathbb{E}(W) = 0$. Let $\mathbf{X}' = (X'_1, \ldots, X'_n)$ be an independent copy of $\mathbf{X}$. Let

$$T = \int_0^1 \frac{1}{2\sqrt{t}} \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{X}) \frac{\partial f}{\partial x_i}(\sqrt{1-t}\mathbf{X} + \sqrt{t}\mathbf{X}')dt.$$

Then one can show that T is a Stein coefficient for W. This can be used, e.g., to prove CLTs for linear statistics of eigenvalues of random matrices [1].

Sketch of proofs. The KMT embedding theorem for the simple random walk can be proved using combination of Lemma 1, Example 3, and an induction argument. The details are worked out in [2]. Let us now give a very brief sketch of the proof of Lemma 1. Let $h(W) = \mathbb{E}(T \mid W)$. Then $h(W)$ is again a Stein coefficient for W, and one can show that it suffices to construct a coupling such that

$$\mathbb{E} \exp(\theta |W - Z|) \leq 2\, \mathbb{E} \exp(2\theta^2 (\sqrt{h(W)} - \sigma)^2).$$

Fix a function $r : \mathbb{R}^2 \to \mathbb{R}$. For $f \in C^2(\mathbb{R}^2)$, let

$$\mathcal{L}f(x, y) := h(x)\frac{\partial^2 f}{\partial x^2} + 2r(x, y)\frac{\partial^2 f}{\partial x \partial y} + \sigma^2 \frac{\partial^2 f}{\partial y^2} - x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}.$$

Suppose there exists a probability measure μ on $\mathbb{R}^2$ such that for all f,

$$\int_{\mathbb{R}^2} \mathcal{L}f \; d\mu = 0. \tag{2}$$

We observe that every choice of r that allows a μ satisfying (2), gives rise to a coupling of W and Z. Indeed, suppose $(X, Y) \sim \mu$. Take any $\Phi \in C^2(\mathbb{R})$, and let $\varphi = \Phi'$. Putting $f(x, y) = \Phi(x)$ in (2), we get $\mathbb{E}(h(X)\varphi(X)) = \mathbb{E}(X\varphi(X))$. Thus, X has the same law as W. Similarly, putting $f(x, y) = \Phi(y)$, we get $\mathbb{E}(Y\varphi(Y)) = \sigma^2 \mathbb{E}(\varphi(Y))$, and thus, $Y \sim N(0, \sigma^2)$.

One can show that with $r(x, y) := \sigma\sqrt{h(x)}$, such a measure μ exists. With this choice of r and $f(x, y) = \frac{1}{2k}(x - y)^{2k}$, we get

$$\mathcal{L}f(x, y) = (2k - 1)(x - y)^{2k-2}(\sqrt{h(x)} - \sigma)^2 - (x - y)^{2k}.$$

So, if $\mathbb{E}(\mathcal{L}f(X, Y)) = 0$ for all f, then

$$\begin{aligned}\mathbb{E}(X - Y)^{2k} &= (2k - 1)\mathbb{E}((X - Y)^{2k-2}(\sqrt{h(X)} - \sigma)^2) \\ &\leq (2k - 1)(\mathbb{E}(X - Y)^{2k})^{\frac{k-1}{k}} (\mathbb{E}(\sqrt{h(X)} - \sigma)^{2k})^{1/k}.\end{aligned}$$

This gives

$$\mathbb{E}(X - Y)^{2k} \leq (2k - 1)^k \mathbb{E}(\sqrt{h(X)} - \sigma)^{2k}.$$

The proof of the lemma is completed by summing over $k \geq 1$.

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Exponential functionals of Brownian motion and class one Whittaker functions

Neil O'Connell
(joint work with Fabrice Baudoin, Toulouse)

If $(B_t^{(\mu)}, t \geq 0)$ is a standard one-dimensional Brownian motion with drift μ,

$$\left(B_t^{(\mu)} + \log \left(\int_0^t e^{-2B_s^{(\mu)}} ds \right), t \geq 0 \right)$$

is a diffusion with generator given by

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{d}{dx} \log K_\mu(e^{-x}) \right) \frac{d}{dx}.$$

This is a theorem of Matsumoto and Yor [8] and can be regarded as an extension of Pitman's '$2M - X$' theorem; the latter can be recovered by Brownian scaling and the method of Laplace. An interpretation of the law of this process as that of the Brownian motion with drift conditioned on the law of the exponential functional $\int_0^\infty \exp(-2B_t^{(|\mu|)})dt$ is given in [2]. We identify a class of diffusions which should play a similar role in a *multi-dimensional* version of this theorem, analogous to the multi-dimensional versions of Pitman's '$2M-X$' theorem obtained in [6, 9, 5]. As a starting point, we consider exponential functionals of a multi-dimensional Brownian motion with drift, defined via a collection of linear functionals. We give a characterisation of the Laplace transform of their joint law as the unique bounded solution, up to a constant factor, to a certain partial differential equation. We then consider a family of diffusions which can be interpreted as having the law of the Brownian motion with drift conditioned on the joint law of these exponential functionals. In the case where the collection of linear functionals is a set of simple roots, the Laplace transform of the joint law of the corresponding exponential functionals can be expressed in terms of a class one Whittaker function associated with the corresponding semi-simple Lie group. In this case, we study in detail some properties of the associated diffusion processes, which we call *Whittaker processes*. Class one Whittaker functions associated with semi-simple Lie groups have been studied extensively in the literature. They are closely related to Whittaker models of class one principal series representations and play an important role in the study of automorphic forms associated with semi-simple Lie groups [7]. In the integrable systems literature, they arise as eigenfunctions of the quantum Toda lattice [1].

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Randomly Trapped Random Walks and incipient critical trees

Gérard Ben Arous
(joint work with R. Royfman, J. Černý)

We give a general model of trapping for random walks: RTRWs Randomly Trapped Random Walks on a graph. At every vertex is attached a probability measure on the positive real numbers, the "trapping-time distribution" at the vertex. At every visit of a vertex the random walk spends a time sampled anew from the trapping time distribution at the vertex, and then moves to one of the neighbors chosen uniformly at random. This model contains the simple and well studied case of CTRW (Continuous Time Random Walks) introudced in the 60's by Montroll and Weiss, where the trapping time distributions are all equal. It also contains the Markovian case where the trapping trap distributions are exponential, known as the Bouchaud trap model. We strudy the interesting case where the trapping time distributions are chosen themselves as random objects, as an i.i.d sample from a distribution Q on the set of probability measures on the positive real line. We study the interesting case where the expectation of the mean of the trapping time distributions is infinite, and heavy tailed. We study in full generality in dimension one, the scaling limits of these RTRWs and show that they are an interesting set of time change of Brownian motion, we call RTBMs (Randomly Trapped Brownian Motions). The usual scaling limit of the Bouchaud trap model, the Fontes-Isopi- Newman singular diffusion, is a particular case of RTBMs, the only Markovian one. The classical limit of heavy tailed CTRWs, i.e the Fractional Kinetics model, is also obtained, under a different scaling exponent, when the corresponding RTBM limit is trivial. We developped this machinery in order to understand the transition between Fractional Kinetics and the FIN singular diffusion, and models of geometric trapping. We show how this applies to simple geometric trapping models as the "comb model". More importantly we give the scaling limit of the projection of standard random walk on the backbone a critical incipient Galton Watson tree. This scaling limit is an RTBM, which is not FIN (nor FK). The proof relies on recent results by D.Croydon on the convergence of the RW on (finite) critical trees to the Brownian Motion on the Continuous Random Tree.

Entropic Measure and Wasserstein Diffusion

Max von Renesse

The classical quadratic Wasserstein metric d_W on the space of probability measures has an infinite dimensional Riemannian structure, where the Riemannian energy of a curve in the space of measures is given by the kinetic energy of the associated mass flow. We present an infinite dimensional drift-diffusion process on the space $\mathcal{P}([0,1])$ of probability measures on the unit interval which is adapted to this Riemannian structure and which yields a mathematical model of a diffusing fluid under the influence of a kinetically uniform random forcing. A crucial object for the construction of this process is a special symmetrizing ('Entropic') reference measure on $\mathcal{P}([0,1])$ with the formal Gibbs representation

$$\mathbb{P}^{\beta}(d\mu) = \frac{1}{Z} e^{-\beta \operatorname{Ent}(\mu)} \mathbb{P}^0(d\mu)$$

with some $\beta > 0$ where $\operatorname{Ent}(\mu) = \int_{[0,1]} \log(d\mu/dx) d\mu$ is the Boltzmann entropy and $\mathbb{P}^0$ a certain 'uniform measure' on $\mathcal{P}([0,1])$ and which is quasi-invariant under the action of smooth diffeomorphisms of $[0,1]$. The corresponding Malliavin-type calculus on $\mathcal{P}([0,1])$ allows to construct an associated probabability valued diffusion via Dirichlet form arguments. Its intrinsic metric coincides with the Wasserstein metric such that in particular Varadhan's formula holds true

$$\lim_{t \to 0} t \log p_t(A, B) = -\frac{d_W(A, B)^2}{2}.$$

In accordance with its underlying heuristics this process ('Wasserstein diffusion') solves a martingale problem which corresponds to an SPDE with the heat equation as a drift component and singular multiplicative white noise

$$d\mu_t = \beta \Delta \mu_t dt + \Gamma(\mu_t) dt + \operatorname{div}(\sqrt{2\mu} dB_t).$$

Finally we present an approximation of the Wasserstein diffusion by a sequence of finite dimensional interacting Bessel processes.

Based on joint works with Karl-Theodor Sturm, Marc Yor, Lorenzo Zambotti and Sebastian Andres.

Ising models on locally tree-like graphs

Amir Dembo
(joint work with A. Montanari)

An *Ising model on the finite graph G* (with vertex set V, and edge set E) has the Boltzmann distribution

$$\mu(\underline{x}) = \frac{1}{Z(\beta, B)} \exp\left\{ \beta \sum_{(i,j) \in E} x_i x_j + B \sum_{i \in V} x_i \right\}, \tag{1}$$

over $\underline{x} = \{x_i : i \in V\}$, with $x_i \in \{+1, -1\}$, parametrized by the 'magnetic field' B and 'inverse temperature' $\beta \geq 0$, where the partition function $Z(\beta, B)$ is fixed by the normalization condition $\sum_{\underline{x}} \mu(\underline{x}) = 1$.

For sequences of graphs $G_n = (V_n, E_n)$ of diverging size n, non-rigorous statistical mechanics techniques, such as the 'replica' and 'cavity methods,' make a number of predictions on this model when the graph G 'lacks any finite-dimensional structure.' The most basic quantity in this context is the asymptotic *free entropy density*

$$\phi(\beta, B) \equiv \lim_{n \to \infty} \frac{1}{n} \log Z_n(\beta, B) \,. \tag{2}$$

It is characterized in great detail [1] for the complete graph $G_n = K_n$ (scaling β by $1/n$ to get a non-trivial limit). Predictions exist for a much wider class of graphs. Most notably, sparse random graphs with bounded average degree that arise in a number of problems from combinatorics and theoretical computer science [2], where the uniform measure over solutions can be regarded as the Boltzmann distribution for multi-spin interactions. Such problems have been successfully attacked using non rigorous statistical mechanics techniques whose mathematical foundation is still lacking and of much interest.

The asymptotic free entropy density (2) was determined rigorously for random regular graphs (in [3]) and for random graphs with independent edges (in [4]), but only at high or zero temperature (and with vanishing magnetic field). We generalize these results by considering generic graph sequences that converge locally to trees and control the free entropy density by proving that the Boltzmann measure (1) converges locally to the Boltzmann measure of a model on the appropriate infinite random tree. Our results also have algorithmic interpretations, providing an efficient procedure for approximating the local marginals of the Boltzmann measure. The essence of this procedure consists in solving by iteration certain mean field (cavity) equations. Such an algorithm is known in artificial intelligence and computer science under the name of *belief propagation*.

Our emphasis is on the low-temperature regime where although uniform decorrelation does not hold, we show that belief propagation converges exponentially fast on any graph, and that the resulting estimates are asymptotically exact for large locally tree-like graphs. The main idea is to introduce a magnetic field to break explicitly the $+/-$ symmetry, and to carefully exploit the monotonicity properties of the model.

Locally tree-like graphs and free entropy. Let $P = \{P_k : k \geq 0\}$ be a probability distribution over the non-negative integers, with finite, positive first moment $\overline{P}$, and set $\rho_k = (k+1)P_{k+1}/\overline{P}$ of mean $\overline{\rho}$. We use $\mathbb{P}_\rho\{\,\cdot\,\}$ for the law of the random rooted Galton-Watson tree $\mathsf{T}(\rho, t)$ of $t \geq 0$ generations where independently of each other each node has offspring distribution $\{\rho_k\}$ and denote by $\mathsf{T}(P, \rho, t)$ the modified ensemble where the offspring distribution at the root is changed to P.

Definition 1. Let $\mathbb{P}_n$ denote the law of the ball $\mathsf{B}_i(t)$ (i.e. the subgraph induced by vertices of G_n whose distance from i is at most t), centered at a uniformly chosen random vertex $i \in V_n$. We say that $\{G_n\}$ *converges locally* to the random tree $\mathsf{T}(P, \rho, \infty)$ if, for any t, and any rooted tree T with t generations

$$\lim_{n\to\infty} \mathbb{P}_n\{\mathsf{B}_i(t) \simeq T\} = \mathbb{P}_\rho\{\mathsf{T}(t) \simeq T\} \tag{3}$$

(where $T_1 \simeq T_2$ if two trees T_1 and T_2 of same size are identical upon vertex labeling in a breadth first fashion following lexicographic order among siblings). We also say that $\{G_n\}$ is *uniformly sparse* if

$$\lim_{l\to\infty} \sup_n \frac{1}{|V_n|} \sum_{i\in V_n} |\partial i|\, \mathbb{I}(|\partial i| \geq l) = 0\,, \tag{4}$$

where ∂i denotes the set neighbors of $i \in V_n$ and $|\partial i|$ its size (i.e. the degree of i).

The model (1) has a line of first order phase transitions for $B = 0$ and $\beta > \beta_{\rm c}$ (that is, where the continuous function $B \mapsto \phi(\beta, B)$ exhibits a discontinuous derivative). The critical temperature is determined by the condition $\overline{\rho}\,(\tanh\beta_{\rm c}) = 1$. The asymptotic free-entropy density is given in terms of the following fixed point distribution.

Lemma 1. Consider the random variables $\{h^{(t)}\}$ where $h^{(0)} \equiv 0$ and for $t \geq 0$,

$$h^{(t+1)} \stackrel{\rm d}{=} B + \sum_{i=1}^{K} \text{atanh}[\tanh(\beta)\tanh(h_i^{(t)})]\,, \tag{5}$$

with $h_i^{(t)}$ i.i.d. copies of $h^{(t)}$ that are independent of the variable K of distribution ρ. If $B > 0$ and $\overline{\rho} < \infty$ then $t \mapsto h^{(t)}$ converges in law to the unique fixed point h^* of (5) that is supported on $[0, \infty)$.

Our main result confirms the statistical physics prediction for the free entropy density.

Theorem 1. *If $\overline{\rho}$ is finite then for any $B \in \mathbb{R}$, $\beta \geq 0$ and sequence $\{G_n\}_{n\in\mathbb{N}}$ of uniformly sparse graphs that converges locally to $\mathsf{T}(P, \rho, \infty)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta, B) = \phi(\beta, B)\,, \tag{6}$$

where taking L of distribution P independently of the 'cavity fields' h_i that are i.i.d. copies of the fixed point h^* of Lemma 1, $\phi(\beta, B) = \phi(\beta, -B)$ is given for $B > 0$ by

$$\begin{aligned}\phi(\beta, B) \equiv{}& \frac{\overline{P}}{2} \log\cosh(\beta) - \frac{\overline{P}}{2}\, \mathbb{E}\log[1 + \tanh(\beta)\tanh(h_1)\tanh(h_2)] \\ &+ \mathbb{E}\,\log\Big\{e^B \prod_{i=1}^{L}[1 + \tanh(\beta)\tanh(h_i)] + e^{-B}\prod_{i=1}^{L}[1 - \tanh(\beta)\tanh(h_i)]\Big\}\,,\end{aligned} \tag{7}$$

and $\phi(\beta, 0)$ is the limit of $\phi(\beta, B)$ as $B \to 0$.

The proof of Theorem 1 is based on two steps

(a) Reduce the computation of $\phi_n(\beta, B) = \frac{1}{n}\log Z_n(\beta, B)$ to computing expectations of local (in G_n) quantities with respect to the Boltzmann measure (1). This is achieved by noticing that the derivative of $\phi_n(\beta, B)$ with respect to β is a sum of such expectations.

(b) Show that expectations of local quantities on G_n are well approximated by the same expectations with respect to an Ising model on the associated tree $\mathsf{T}(P, \rho, t)$ (for t and n large.) This is proved by showing that, on such a tree, local expectations are insensitive to boundary conditions that dominate stochastically free boundaries.

The key step is of course the last one with the challenge to carry it out above β_c, when we no longer have uniqueness of the Gibbs measure on $\mathsf{T}(P, \rho, \infty)$. Indeed, insensitivity to positive boundary conditions is proved for a large family of 'conditionally independent' trees. Beyond the random tree $\mathsf{T}(P, \rho, \infty)$, these include deterministic trees with bounded degrees and multi-type branching processes (so it allows for generalizing Theorem 1 to other graph sequences, that converge locally to random trees different from $\mathsf{T}(P, \rho, \infty)$).

Algorithmic implications. The free entropy density is not the only quantity that can be characterized for Ising models on locally tree-like graphs. Indeed, local marginals can be efficiently computed with good accuracy. The basic idea is to solve a set of mean field equations iteratively. These are known as Bethe-Peierls or cavity equations and the corresponding algorithm is referred to as 'belief propagation' (BP).

More precisely, associate to each directed edge in the graph $i \to j$, with $(i, j) \in G$, a distribution (or 'message') $\nu_{i\to j}(x_i)$ over $x_i \in \{+1, -1\}$, using then the following update rule

$$\nu^{(t+1)}_{i\to j}(x_i) = \frac{1}{z^{(t)}_{i\to j}}\, e^{Bx_i} \prod_{l\in\partial i\setminus j} \sum_{x_l} e^{\beta x_i x_l} \nu^{(t)}_{l\to i}(x_l) \tag{8}$$

starting at a *positive* initial condition, namely where $\nu^{(0)}_{i\to j}(+1) \geq \nu^{(0)}_{i\to j}(-1)$ at each directed edge.

We establish uniform exponential convergence of the BP iteration to the same fixed point of (8), irrespective of its positive initial condition.

Theorem 2. *Assume $\beta \geq 0$, $B > 0$ and G is a graph of finite maximal degree Δ. Then, there exists $A = A(\beta, B, \Delta)$ and $c = c(\beta, B, \Delta)$ finite, $\lambda = \lambda(\beta, B, \Delta) > 0$ and a fixed point $\{\nu^*_{i\to j}\}$ of the BP iteration (8) such that for any positive initial condition $\{\nu^{(0)}_{l\to k}\}$ and all $t \geq 0$,*

$$\sup_{(i,j)\in E} \|\nu^{(t)}_{i\to j} - \nu^*_{i\to j}\|_{\mathrm{TV}} \leq A\exp(-\lambda t)\,. \tag{9}$$

Further, for any $i_* \in V$, if $\mathsf{B}_{i_*}(t)$ is a tree then for $U \equiv \mathsf{B}_{i_*}(r)$

$$\|\mu_U - \nu_U\|_{\mathrm{TV}} \leq \exp\left\{c^{r+1} - \lambda(t-r)\right\}, \tag{10}$$

where $\mu_U(\,\cdot\,)$ is the law of $\underline{x}_U \equiv \{x_i : i \in U\}$ under the Ising model (1) and ν_U the probability distribution

$$\nu_U(\underline{x}_U) = \frac{1}{z_U} \exp\Big\{\beta \sum_{(i,j)\in E_U} x_i x_j + B \sum_{i\in U\setminus\partial U} x_i\Big\} \prod_{i\in\partial U} \nu^*_{i\to j(i)}(x_i)\,, \tag{11}$$

with E_U the edge set of U whose border is ∂U and $j(i)$ is any fixed neighbor of $i \in \partial U$ in U.

Examples. Many common random graph ensembles naturally fit our framework.

Random regular graphs. Let G_n be a uniformly random graph with degree k. As $n \to \infty$, the sequence $\{G_n\}$ is obviously uniformly sparse, and converges locally almost surely to the random rooted Cayley tree of degree k. Therefore, in this case Theorem 1 applies with $P_k = 1$. The distributional recursion (5) then evolves with a deterministic sequence $h^{(t)}$ recovering the result of [3].

Erdös-Renyi graphs. Let G_n be a uniformly random graph with $m = n\gamma$ edges over n vertices. The sequence $\{G_n\}$ converges locally almost surely to a Galton-Watson tree with Poisson offspring distribution of mean 2γ. This corresponds to taking $P_k = (2\gamma)^k e^{-2\gamma}/k!$. The same happens to classical variants of this ensemble. For instance, one can add an edge independently for each pair (i, j) with probability $2\gamma/n$, or consider a multi-graph with Poisson$(2\gamma/n)$ edges between each pair (i, j). In all these cases $\{G_n\}$ is almost surely uniformly sparse. In particular, Theorem 1 extends the results of [4] to arbitrary non-zero temperature and magnetic field.

Arbitrary degree distribution. Let P be a distribution with finite second moment and G_n a uniformly random graph with degree distribution P (the number of vertices of degree k is obtained by rounding nP_k). Then $\{G_n\}$ is almost surely uniformly sparse and converges locally to $\mathsf{T}(P, \rho, \infty)$. The same happens if G_n is drawn according to the so-called configuration model.

Ising models on trees. We extend (1) by allowing for vertex-dependent magnetic fields B_i, namely, we consider

$$\mu(\underline{x}) = \frac{1}{Z(\beta, \underline{B})} \exp\Big\{\beta \sum_{(i,j)\in E} x_i x_j + \sum_{i\in V} B_i x_i\Big\} \tag{12}$$

and derive correlation decay results for Ising models on trees, which are of independent interest. More precisely, let T denote a *conditionally independent* infinite tree rooted at the vertex ø. That is, for each integer $k \geq 0$, conditional on the subtree $\mathsf{T}(k)$ of the first k generations of T, the number of offspring Δ_j for $j \in \partial\mathsf{T}(k)$ are independent of each other, where $\partial\mathsf{T}(k)$ denotes the set of vertices at generation k. We further assume that the (conditional on $\mathsf{T}(k)$) first moments of Δ_j are uniformly bounded by a given non-random finite constant Δ. In addition to $\mathsf{T} = \mathsf{T}(P, \rho, \infty)$ this flexible framework accommodates for example random bipartite trees, deterministic trees of bounded degree and percolation clusters on them.

For each $\ell \geq 1$, we denote by $\mu^{\ell,0}$ the Ising model (12) on $\mathsf{T}(\ell)$ with magnetic fields $\{B_i\}$ (also called free boundary conditions), and by $\mu^{\ell,+}$ the modified Ising model corresponding to the limit $B_i \uparrow +\infty$ for all $i \in \partial\mathsf{T}(\ell)$ (also called plus boundary conditions), using μ^ℓ for statements that apply to both free and plus boundary conditions.

Theorem 3. *Let $\langle \cdot \rangle$ denote the expectation with respect to the Ising distribution on a conditionally independent tree $\mathsf{T}(\ell)$ of average offspring numbers bounded by Δ. There exist A finite and λ positive, depending only on $0 < B_{\min} \le B_{\max}$, $\beta_{\max}$ and Δ finite, such that if $B_i \le B_{\max}$ for all $i \in \mathsf{T}(r-1)$ and $B_i \ge B_{\min}$ for all $i \in \mathsf{T}(\ell)$, then for any $r \le \ell$ and $\beta \le \beta_{\max}$,*

$$\mathbb{E}\Big\{ \sum_{i \in \partial \mathsf{T}(r)} [\langle x_\emptyset x_i \rangle - \langle x_\emptyset \rangle \langle x_i \rangle] \Big\} \le A\, e^{-\lambda r}\,. \tag{13}$$

If in addition $B_i \le B_{\max}$ for all $i \in \mathsf{T}(\ell-1)$ then for some $C = C(\beta_{\max}, B_{\max})$ finite

$$\mathbb{E}\, ||\mu^{\ell,+}_{\mathsf{T}(r)} - \mu^{\ell,0}_{\mathsf{T}(r)}||_{\mathrm{TV}} \le A e^{-\lambda(\ell - r)}\, \mathbb{E}\{C^{|\mathsf{T}(r)|}\}\,. \tag{14}$$

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The exclusion speed process

Omer Angel

(joint work with G. Amir, B. Valko)

Model definition. The multi-class TASEP is a process taking values in $\Omega = \mathbb{R}^{\mathbb{Z}}$ where a state Y implies that the particle at position k has class Y_k. The dynamics is that whenever $Y_k < Y_{k+1}$ the two particles swap at rate 1 (see e.g. [4] for details). Holes are particles of the maximal class present.

The following is a well known result by Mountford-Guiol [5], strengthening results of Rost and of Ferrari-Kipnis [6, 2]: Start a TASEP with first class particles at negative integers, a unique second class particle at 0 and third class particles (holes) at the positive integers. Let X_t be the location of the second class particle at time t, then $X_t/t \xrightarrow[t\to\infty]{a.s.} U$ where U is uniformly distributed on $[-1, 1]$.

Let Y_t be the TASEP started with $Y_{0,k} = k$, i.e. a particle of class k at position k. Let $X_{t,k}$ be the location of the particle of class k at time t. It follows easily from the above that $X_{t,k}/t \xrightarrow[t\to\infty]{a.s.} U_k$ where U_k is uniformly distributed on $[-1, 1]$. Our main objective is to study the joint distribution of the *speed process* $\{U_k\} \in [-1, 1]^{\mathbb{Z}}$.

A second natural object to consider is the stationary measures for the multi-type TASEP. It can be seen by standard techniques that the marginal of η_0 determines a unique translation invariant stationary measure on Ω. Since only the relative order of the classes affects the proess, these are all easily related to the stationary translation invariant measure with η_i uniform on $[-1,1]$. If the TASEP is started with i.i.d. uniform $[-1,1]$ random classes then it converges to this stationary measure (in the usual product topology).

Main results. Our first main result is an identifucation between these two measures on $[-1,1]^{\mathbb{Z}}$.

Theorem 1. *The law of the speed process $(U_k)_{k\in\mathbb{Z}}$ is the stationary translation invariant measure with uniform $[-1,1]$ marginals.*

The Second main result is also a key tool for our study of the speed process. Some of the above results can be derived directly from the following symmetry result. In the context of the TASEP this result was proved in Angel-Holroyd-Romik [1]. Remarkabley, it holds for the partially asymmetric case as well.

Theorem 2. *For either the TASEP or PASEP started with state $Y_{0,k} = k$ and for any t we have $X_t \stackrel{d}{=} Y_t$.*

This implies in particular that the class of the particle at position k at time t is equal in distribution to the location of particle k.

It appears that many of the results of [3] can be recovered using only this symmetry without using the stationarity of the speed process. In particular, the linear repulsion between U_0 and U_1 when $U_1 < U_1$ can be proved in this way.

Joint distribution. As an application we can extend recent results of Ferrari-Goncalves-Martin, and give explicitly the joint distribution of U_0, U_1:

Theorem 3. *The density of two consecutive speeds is given by*

$$\mu(U_0,U_1) = \begin{cases} \frac{1}{4} & U_0 > U_1 \\ \frac{U_1-U_0}{4} & U_0 < U_1 \\ \delta_{U_0-U_1}\frac{1-U_0^2}{4} & U_0 = U_1 \end{cases}$$

Thus when $U_0 > U_1$, the density is $1/4$, which is as if they were independent. Note however, that there is a positive probability (1/6 in fact) that $U_0 = U_1$. In light of Theorem 1 this shows that even when starting with all distinct classes, the process converges to a distribution where multiple particles have the same class (more on this below).

We are also able to compute the joint distribution of 3 consecutive speeds (and many of the possible configurations on 4 speeds). However, there are 13 cases to consider here corresponding to full weak orders on 3 elements. Some special cases that are of interest:

Theorem 4. *The density of $U_0,\dots,U_{k-1}$ on the simples where $U_0 > U_1 > \cdots > U_{k-1}$ is 2^{-k} (as if they were independent uniform $[-1,1]$).*

Theorem 5. *The density of* $U_0, \dots, U_{k-1}$ *on the simples where* $U_0 < U_1 < \dots < U_{k-1}$ *is* $2^{-k}k! \prod_{i<j}(U_j - U_i)$*.*

Convoys. Let $C_k = \{n : U_n = U_k\}$ be the set of all particles with the same speed as k.

Theorem 6. *Conditioned on* U_0*, the convoy* C_0 *is an infinite renewal process with 0 density.* C_0 *has the same law as the times of last visits to integers of a certain random walk* $(R_n)_n$ *conditioned to tend to stay positive for all* $n > 0$ *(i.e. a certain discrete Bessel process).*

In particular the cnvoys are all infinite. In fact the number of distinct speeds among $\{U_0, \dots, U_n\}$ is only of order $\sqrt{n}$. The convoys partition $\mathbb{Z}$ into an infinite number of infinite sets in a translation invariant way.

Extension to the PASEP. As noted above, the symmetry applies also for the general asymmetric exclusion process. However, anything concerning the speed process is less clear. For one thing, it is not at all clear that the speed process is well defined:

Conjecture 7. In the PASEP, $\lim_{t\to\infty} X_{0,t}/t$ exists a.s..

If we assume this conjecture then the speed process is still equal in law to the stationary distribution for the PASEP with uniform $[-1, 1]$ marginals. Note that the PASEP has a parameter $p \in (1/2, 1)$ and the measure depends on p deeply.

Some additional results that extend conditionally to the PASEP: On the event that 0 and 1 never swap the joint density of U_0, U_1 is $c(U_1 - U_0)$. On the event that $U_0 = U_1$ particles 0 and 1 a.s. swap infinitely often.

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A quenched large deviation principle for words in a letter sequence

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(joint work with M. Birkner, A. Greven)

When we cut an i.i.d. sequence of letters into words according to an independent renewal process, we obtain an i.i.d. sequence of words. In the *annealed* large deviation principle (LDP) for the *empirical process* of words, the rate function is the specific relative entropy of the observed law of words w.r.t. the reference law of words. We consider the *quenched* LDP, i.e., we condition on a typical letter sequence.

The rate function of the quenched LDP turns out to be a sum of two terms, one being the annealed rate function, the other being proportional to the specific relative entropy of the observed law of letters w.r.t. the reference law of letters, with the former obtained after a "randomised concatenation" of words. The proportionality constant equals the tail exponent of the renewal process.

The annealed and the quenched LDP can be combined to prove that the radius of convergence of the moment generating function of the collision local time of two transient random walks on $\mathbb{Z}^d$, $d \geq 1$, strictly increases when we condition on one of the random walks. The presence of this gap implies the existence of an *intermediate phase* for the long-time behaviour of a class of coupled branching processes, interacting diffusions, and directed polymers in random environments.

Further applications of the quenched LDP include random copolymers near selective interfaces (work in progress with E. Bolthausen), and polymers pinned at an interface by random charges (work in progress with D. Cheliotis).

Ageing in spin glass models at intermediate time scales: Universality of the trap model

Anton Bovier
(joint work with G. Ben Arous, J. Černý)

Aging has become one of the main paradigms to describe the long-time behavior of complex and/or disordered systems. Systems that have strongly motivated this research are *spin glasses*. The theoretical modeling of aging phenomena took a major leap with the introduction of so-called *trap models* by Bouchaud and Dean in the early 1990'ies [BD95] (see [BCKM98] for a review). These models reproduce the characteristic power law behavior seen experimentally while being sufficiently simple to allow for detailed analytical treatment. While trap models are heuristically motivated to capture the behavior of the dynamics of spin glass models, there is no clear theoretical, let alone mathematical derivation of these from an underlying spin-glass dynamics. The first attempt to establish such a connection was made in [BBG03a, BBG03b] where it was shown that starting from a particular Glauber dynamics of the Random Energy Model (REM), at low temperatures and at the time scale slightly shorter than the equilibration time of the dynamics, the aging of the time-time correlation function of the dynamics converged to that given by Bouchaud's REM-like trap model.

On the other hand, in a series of papers [BČ05, BČM06, BČ07a, BČ07b] a systematic investigation of a variety of trap models was initiated. In this process, it emerged that there appears to be an almost universal aging mechanism based on α-stable subordinators that governs aging in most of the trap models. It was also shown that the same feature holds for the dynamics of the REM at shorter time scales. For a general review on trap models see [BČ06].

In all models considered so far, however, the random variables describing the quenched disorder were considered to be independent, be it in the REM or in the trap models. Here we report first results obtained in [BBČ07] that show that the same mechanisms are at work in correlated spin glasses.

Let us describe the class of models we are considering. Our state spaces will be the N-dimensional hypercube, $\mathcal{S}_N \equiv \{-1,1\}^N$. $R_N : \mathcal{S}_N \times \mathcal{S}_N \to [-1,1]$ denotes as usual the normalized overlap, $R_N(\sigma,\tau) \equiv N^{-1}\sum_{i=1}^N \sigma_i\tau_i$. The Hamiltonian of the p-spin SK-model is defined as $\sqrt{N}H_N$, where $H_N : \mathcal{S}_N \to \mathbb{R}$ is the centered normal process indexed by $\mathcal{S}_N$ with covariance

$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = R_N(\sigma,\tau)^p, \tag{1}$$

and $p \in \mathbb{N}$, $p > 2$.

We define the classical trap-model dynamics as a nearest neighbor continuous time Markov chain $\sigma_N(\cdot)$ on $\mathcal{S}_N$ with transition rates

$$w_N(\sigma,\tau) = \begin{cases} N^{-1}e^{-\beta\sqrt{N}H_N(\sigma)}, & \text{if } \operatorname{dist}(\sigma,\tau)=1, \\ 0, & \text{otherwise;} \end{cases} \tag{2}$$

here $\operatorname{dist}(\cdot,\cdot)$ is the Hamming distance.

This dynamics is a time change of a simple random walk on $\mathcal{S}_N$: Let $Y_N(k) \in \mathcal{S}_N$ be the simple unbiased random walk (SRW) on $\mathcal{S}_N$. Define the *clock-process* by

$$S_N(k) = \sum_{i=0}^{k-1} e_i \exp\left\{\beta\sqrt{N}H_N\big(Y_N(i)\big)\right\}, \tag{3}$$

where $\{e_i, i \in \mathbb{N}\}$ are i.i.d. standard exponential random variables. Then the process $\sigma_N(\cdot)$ can be written as

$$\sigma_N(t) \equiv Y_N(S_N^{-1}(t)). \tag{4}$$

The main result on the dynamics will be the following theorem that provides the asymptotic behavior of the clock process.

Theorem 1. There exists a function $\zeta(p)$ such that for all $p \geq 3$ and γ satisfying

$$0 < \gamma < \min\left(\beta^2, \zeta(p)\beta\right), \tag{5}$$

under the conditional distribution $\mathbb{P}[\cdot|\mathcal{Y}]$ the law of the stochastic process

$$\bar{S}_N(t) = e^{-\gamma N} S_N\big(\lfloor tN^{1/2}e^{N\gamma^2/2\beta^2}\rfloor\big), \qquad t \geq 0, \tag{6}$$

defined on the the space of càdlàg functions equipped with the Skorokhod M_1-topology, converges, $\mathcal{Y}$-a.s., to the law of γ/β^2-stable subordinator $V_{\gamma/\beta^2}(Kt), t \geq 0$, where K is a positive constant depending on γ, β and p.

Moreover, the function $\zeta(p)$ is increasing and it satisfies

$$\zeta(3) \simeq 1.0291 \qquad \text{and} \qquad \lim_{p\to\infty} \zeta(p) = \sqrt{2\log 2}. \tag{7}$$

To control the behaviour of spin-spin correlation functions that are commonly used to characterize aging, we need to know more on how these jumps occur at finite N. What we will show, is that if we the slightly coarse-grain the process $\bar{S}_N$ over blocks of size $o(N)$, the rescaled process does converge in the J_1-topology. What this says, is that the jumps of the limiting process are compounded by smaller jumps that are made over $\leq o(N)$ steps of the SRW. In other words, the jumps of the limiting process come from waiting times accumulated in one slightly extended trap, and during this entire time only a negligible fraction of the spins are flipped. That will imply the following aging result.

Theorem 2. Let $A_N^\varepsilon(t,s)$ be the event defined by

$$A_N^\varepsilon(t,s) = \{R_N\big(\sigma_N\big(te^{\gamma N}\big), \sigma_N\big((t+s)e^{\gamma N}\big)\big) \geq 1-\varepsilon\}. \tag{8}$$

Then, under the hypothesis of Theorem 1, for all $\varepsilon \in (0,1)$, $t > 0$ and $s > 0$,

$$\lim_{N\to\infty} \mathbb{P}[A_N^\varepsilon(t,s)] = \frac{\sin\alpha\pi}{\pi} \int_0^{t/(t+s)} u^{\alpha-1}(1-u)^{-\alpha}\, \mathrm{d}u. \tag{9}$$

Let us discuss the meaning of these results. $e^{\gamma N}$ is the time-scale at which we want to observe the process. According to Theorem 1, at this time the random walk will make of the order of $N^{1/2}e^{N\gamma^2/2\beta^2} \ll e^{\gamma N}$ steps. Since this number is also much smaller than 2^N (as follows from (7)), the random walk will essentially visit that number of sites.

If the random process H_N was i.i.d., then the maximum of H_N along the trajectory would be $\big(2\ln(N^{1/2}e^{N\gamma^2/2\beta^2})\big)^{1/2} \sim N^{1/2}\gamma/\beta$, and the time spent in that site would be of order $e^{\gamma N}$. Since Theorem 1 holds also in the i.i.d. case, that is in the REM (see [BČ07a]), the time spent in the maximum is comparable to the total time and the convergence to the α-stable subordinator implies that the total accumulated time is composed of pieces of order $e^{\gamma N}$ that are collected along the trajectory. In fact, each jump of the subordinator corresponds to one visit to a site that has waiting times of that order. In a common metaphor, the sites are referred to as traps and the mean waiting times as their depths.

The theorem in the general case states that in the p-spin model, the same is essentially true. The difference will be that the traps here will not consist of a single site, but consist of a deep valley (along the trajectory) whose bottom that has approximately the same energy as in the i.i.d. case and whose shape and width we will be able to describe quite precisely. Remarkably, the number of sites contributing significantly to the residence time in the valley is essentially finite, and different valleys are statistically independent.

The proof of Theorem 1 relies on the combination of detailed information on the properties of simple random walk on the hypercube and comparison of the process H_N on the trajectory of the SRW to a simpler Gaussian process using interpolation techniques.

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The Signature of a Path

Terry Lyons

This work is joint, primarily with Ben Hambly, but also with Daniel Levin, Nadia Sidorova, Phillip Yam, Christian Litterer, Greg Gyurko, Tom Fawcett.

Paths, that is to say maps from an interval into a space, are a critical tool for representing evolving systems. Many of the most interesting probabilistic models, such as Bownian motion, self-avoiding walks, ..., describe ways to choose randomly from certain classes of paths. Within this general framework of stochastic processes, there are many that do not fit within the classical Markovian framework.

There are a number of contexts where one would like to be able to model interactions between stochastic processes. This represents a mathematical challenge because of the raher arbitrary nature of the sample paths which arise in interesting examples. The theory of differential equations has been extended to the setting where the stochastic processes are semimartingales (Ito calculus), but there are many important examples where the interactions are not of this kind. The theory of rough paths provides a broader framework and a wider range of new models.

A key technical perspective in the development of the theory of rough paths concerns the way that one describes a smooth path. One is able to describe a path segment in a graded, top-down, way, so as to give progressively more information about the effects that this evolving path or process will have on other systems. The simplest and most frequently used description, or approximation, or a segment is the chord from its starting to finishing point. This chordal approximation to a path is the foundation of classical Newtonian calculus.

The increment, or chord, describing the difference between the beginning and any point on the path segment is the first non-trivial term in an algebraic transformation of the path, known as its signature.

If $\gamma : J \to R^d$ is a path, then we can introduce its signature

$$\begin{aligned} S(\gamma) \quad : \quad &= 1 + \int_{u \in j} d\gamma_u + \ldots + \int \ldots \int_{u_1 < \ldots < u_k \in j} d\gamma_{u_1} \otimes \ldots \otimes d\gamma_{u \cdot k} + \ldots \\ &\in \quad R \oplus R^d \oplus \ldots \oplus \left(R^d\right)^{\otimes k} \oplus \ldots \end{aligned}$$

This transformation maps the path γ into an algebraic sequence, taking its values in the tensor series. This signature, which has its origins in seminal work of K.T. Chen, is a fundamental and natural description of the path γ. It is a homomorphism from the space of paths in the vector space R^d with the operation of concatenation, into the associative algebra of the tensor series.The range of the map is a group. Hambly and Lyons had proved that the map is essentially faithful for paths of finite variation, and in particular two paths have the same signature, if and only if the concatenation of the first with the second run backwards is tree-like. In other words, it characterises the paths up to reparameterisation. In particular, it contains sufficient information about a loop to determine its winding, knotting etc, and setermines a self-avoiding lattice path completely.

One critical feature of the signature for probabilists is the way that the expected signature determines the lawe of the signature, at least in the case where the law of the signature is compactly supported. This is an immediate consequence of the well nown fact that the tensor algebra is the enveloping algebra of the free Lie algebra, and so the shuffle product on the dual tensor algebra allows one to provide Stone-Wierstrass and identify the dual space as a dense subspace of the continuous functions on any compact subset of the range of the signature. The expected signature had been computed in a number of interesting examples, but much more work is needed to be done.

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