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Calculus of Variations

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Abstract. The Calculus of Variations is at the same time a classical subject, with long-standing open questions which have generated exciting discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and by emergent applications in other fields of science such as physics, economics, and engineering. The July 2008 Oberwolfach workshop devoted to the Calculus of Variations showcased a blend of continued progress in traditional areas with surprising developments which emerged from the exploration of new lines of research.

Mathematics Subject Classification (2000): 49-xx, 35Jxx, 53Cxx, 58Exx.

Introduction by the Organisers

This workshop attracted 44 participants, some 30% of them recent PhDs and 10% of them women. Its main themes could be divided into three large groups (i) differential geometry; (ii) physics and materials; (iii) optimal transportation and its applications.

The first general area encompassed the role of calculus of variations in differential geometry, including minimal surface theory and general relativity. One of the highlights was a substantial simplification of Almgren's formidable regularity theory for Q-valued functions, presented by Camillo DeLellis. This theory restricts the singularities which can occur on area minimizing currents in codimension higher than one. Another highlight was the confirmation by Del Pino (with Kowalczyk and Wei) of the counterexample long expected in dimensions $N \geq 9$ to DeGiorgi's conjecture about the symmetry of transition profiles in the standard model of phase segregation. Further progress showcased at the 2008 meeting included a new approach to regularity theory for harmonic maps (Rivière, Struwe), and new existence and regularity results of Willmore surfaces from a PDE point of view (Kuwert-Schätzle, Rivière).

Exciting developments were also discussed concerning the structure of spacelike surfaces in general relativity. Foliations by constant mean curvature surfaces and by Willmore surfaces are being used to describe the center of mass and quasilocal energy distribution of isolated gravitating systems: Tobias Lamm and Jan Metzger described their recent work with Schulze showing that Willmore surfaces behave in a similar way and can serve as an asymptotic center of mass.

Turning to physics and materials science, the highlights included a review presented by Elliott Lieb of variational approaches to calculating approximate quantum ground states for atoms and molecules. A related challenge in applied mathematics is to find a satisfactory explanation for the emergence of periodic patterns in physical models, a problem which has received important recent contributions described by Andrea Malchiodi, Sylvia Serfaty, and Gero Friesecke. These involved blending techniques from the calculus of variations with other kinds of mathematics — in one case as far off as number theory. Another highlight, falling somewhere between physics and the optimization problems described below, was the idiosyncratic lecture by Eitan Bachmat explaining how supermarket queueing, airline boarding, and optimal controlling of disk drive read/write operations can be boiled down to variational problems concerning the longest future directed geodesic in a Lorentzian spacetime.

Turning to the theory of optimal transportation and its connections to geometry, we recall the conference's opening lecture by Cedric Villani, addressing local to global curvature principles. In it he described his own work with John Lott (and independently Sturm), using the displacement convexity from optimal transportation to develop a theory of Ricci bounds in metric measure spaces. At the same occasion, he discussed the stability of the curvature conditions used by Ma-Trudinger-Wang to prove the regularity of optimal mappings. His first theme was reprised in the talk of Peter Topping, which used displacement convexity to give a new understanding of Perelman's monotonicity of entropy along the Ricci flow. Further talks by Figalli and Kim described new regularity results concerning free boundary problems in optimal transportation, and continuity of optimal maps for non-negatively cross-curved costs. The coordinate independence of these problems leads to phenomena which are governed by invariant structures such as curvatures, thus providing a strong point of thematic and methodological contact between this topic and the more geometric questions described in the first paragraphs above.

Only a small sampling from the July 2008 Oberwolfach Workshop has been mentioned above. Lectures which fell outside these three themes included those by Gianni Dal Maso, Yury Grabovsky, Manuel Ritoré, Tatiana Toro, and Tobias Weth, and there were lively informal discussions in which many young researchers participated. Still, we hope this brief summary gives some flavor for the exciting developments presented in the workshop, and which continue to take place in the Calculus of Variations.

Workshop: Calculus of Variations

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Abstracts

Sufficient conditions for smooth strong local minimizers in classical Calculus of Variations.

Yury Grabovsky
(joint work with Tadele Mengesha)

1. A classical variational problem

A classical variational problem is to find necessary conditions and sufficient conditions for $\boldsymbol{y}_0(\boldsymbol{x}) \in \mathcal{A}$ to be a strong local minimizer of

$$E(\boldsymbol{y}) = \int_\Omega W(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}), \nabla \boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x},$$

where $\mathcal{A} = \{\boldsymbol{y} \in C^1(\overline{\Omega};\mathbb{R}^m) : \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}),\ \boldsymbol{x} \in \partial\Omega_1\}$, $\boldsymbol{g} \in C^1(\partial\Omega;\mathbb{R}^m)$, $\Omega \subset \mathbb{R}^d$—open, bounded and C^1, $\partial\Omega_1 \subset \partial\Omega$—relatively open, W—continuous, and C^2 on $(\boldsymbol{x}, \boldsymbol{y}_0(\boldsymbol{x}), \nabla \boldsymbol{y}_0(\boldsymbol{x}))$. The following necessary conditions are well-known.

Euler-Lagrange equation:

$$\int_\Omega \{(W_{\boldsymbol{y}}(\boldsymbol{x}), \boldsymbol{\phi}(\boldsymbol{x})) + (W_{\boldsymbol{F}}(\boldsymbol{x}), \nabla\boldsymbol{\phi}(\boldsymbol{x}))\}d\boldsymbol{x} = 0 \tag{1}$$

for all $\boldsymbol{\phi} \in \mathrm{Var}(\mathcal{A}) = \{\boldsymbol{\phi} \in C^1(\overline{\Omega};\mathbb{R}^m) \colon \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0},\ \boldsymbol{x} \in \partial\Omega_1\}$, where $W(\boldsymbol{x}) = W(\boldsymbol{x}, \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{y}_0(\boldsymbol{x}))$.

Non-negativity of second variation:

$$\int_\Omega \delta^2 W(\boldsymbol{x}, \boldsymbol{\phi}, \nabla\boldsymbol{\phi})d\boldsymbol{x} \geq 0, \qquad \boldsymbol{\phi} \in \mathrm{Var}(\mathcal{A}), \tag{2}$$

$$\delta^2 W = \frac{1}{2}((W_{\boldsymbol{yy}}(\boldsymbol{x})\boldsymbol{\phi}, \boldsymbol{\phi}) + 2(W_{\boldsymbol{Fy}}(\boldsymbol{x})\boldsymbol{\phi}, \boldsymbol{H}) + (W_{\boldsymbol{FF}}(\boldsymbol{x})\boldsymbol{H}, \boldsymbol{H})).$$

These conditions have been obtained by testing the minimality by weak variations.

Definition: *A weak variation is a sequence $\{\boldsymbol{\phi}_n\} \subset Var(\mathcal{A})$ such that*

$$\lim_{n\to\infty} \|\boldsymbol{\phi}_n\|_\infty = \lim_{n\to\infty} \|\nabla\boldsymbol{\phi}_n\|_\infty = 0.$$

A typical weak variation is

$$\boldsymbol{y}_0(\boldsymbol{x}) \to \boldsymbol{y}_0(\boldsymbol{x}) + \epsilon\boldsymbol{\phi}(\boldsymbol{x}). \tag{3}$$

In fact, the variation (3) has been used to obtain the necessary conditions (1) and (2). In addition to these necessary conditions we also have

Quasiconvexity in the interior:

$$⨍_B W(\nabla\boldsymbol{y}_0(\boldsymbol{x}_0) + \nabla\boldsymbol{\phi}(\boldsymbol{z}))d\boldsymbol{z} \geq W(\nabla\boldsymbol{y}_0(\boldsymbol{x}_0)), \quad \boldsymbol{\phi} \in C_0^\infty(B;\mathbb{R}^m) \tag{4}$$

for every $\boldsymbol{x}_0 \in \Omega$, where B—unit ball in $\mathbb{R}^d$. This condition is the analog of the classical Weierstrass positivity condition. In sharp contrast to the one-dimentional case $d = 1$, this condition has no local formulation and is usually very difficult to verify.

Quasiconvexity on the free boundary:

$$\fint_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}} W(\nabla \boldsymbol{y}_0(\boldsymbol{x}_0) + \nabla \boldsymbol{\phi}(\boldsymbol{z}))d\boldsymbol{z} \geq W(\nabla \boldsymbol{y}_0(\boldsymbol{x}_0)), \quad \boldsymbol{\phi} \in V_{\boldsymbol{n}(\boldsymbol{x}_0)}, \tag{5}$$

where $\boldsymbol{x}_0 \in \partial\Omega \setminus \overline{\partial\Omega_1}$ and $V_{\boldsymbol{n}} = \{\boldsymbol{\phi} \in C^\infty(\overline{B^-_{\boldsymbol{n}}}; \mathbb{R}^m) \colon \boldsymbol{\phi}(\boldsymbol{z}) = \boldsymbol{0} \text{ on } \partial B \cap \overline{B^-_{\boldsymbol{n}}}\}$, $B^-_{\boldsymbol{n}} = \{\boldsymbol{z} \in B : (\boldsymbol{z}, \boldsymbol{n}) < 0\}$. This condition was discovered relatively recently by Ball and Marsden [2]. It is even more difficult to check than quasiconvexity in the interior. These necessary conditions were derived by testing the minimality of extremal by strong variations.

Definition: A strong variation is a sequence $\{\boldsymbol{\phi}_n\} \subset \mathrm{Var}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \|\boldsymbol{\phi}_n\|_\infty = 0$$

A typical strong variation is the “Weierstrass needle”

$$\boldsymbol{y}_0(\boldsymbol{x}) \to \boldsymbol{y}_0(\boldsymbol{x}) + \epsilon\boldsymbol{\phi}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_0}{\epsilon}\right), \quad \boldsymbol{\phi} \in C_0^\infty(B; \mathbb{R}^m), \quad \boldsymbol{x}_0 \in \overline{\Omega}.$$

Conditions (4) and (5) have been derived using this variation.

Stability with respect to the two types of variation determine the two types of local minima

Definition: $\boldsymbol{y}_0(\boldsymbol{x})$ *is called a weak local minimizer if* $E(\boldsymbol{y}_0) \leq E(\boldsymbol{y}_0 + \boldsymbol{\phi}_n)$ *for every weak variation and* n *large enough.*

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Now we want to strengthen the necessary condition in a “small way” but such as to obtain sufficient conditions for local minima. In the case of weak variations Taylor expansion can be used to express

$$\Delta E(\boldsymbol{\phi}_n) = \int_\Omega \{W(\boldsymbol{x}, \boldsymbol{y}_0 + \boldsymbol{\phi}_n, \nabla \boldsymbol{y}_0 + \nabla \boldsymbol{\phi}_n) - W(\boldsymbol{x})\}d\boldsymbol{x}$$

in terms of second variation, the uniform positivity of which is sufficient for $\boldsymbol{y}_0$ to be a weak local minimizer.

If $\|\nabla \boldsymbol{\phi}_n\|_\infty \not\to 0$, as $n \to \infty$, then neither Taylor expansion nor Weierstrass field theory methods work ($d > 1$, $m > 1$). In fact field theory has been applied to vectorial variational problems (see e.g. [3, 5, 12, 15]), but the sufficient conditions were never close to the necessary ones. The reason was pointed out by Ball in [1]. The field theory uses translations by null-Lagrangians (see e.g. [4, 7, 16]), and is thus associated with polyconvexity, while the necessary conditions involve quasiconvexity. Ball conjectured in [1, Section 6.2] that uniform quasiconvexity together with sufficient conditions for weak local minima should be sufficient for strong local minima. In [8, 9] we have proved Ball’s conjecture.

In addition to field theory we are aware of only two other approaches to the problem of sufficient conditions. Levi’s “expansion method” [13] (see also [14]) and Hestenes’s normalized or directional convergence method [10]. Levi’s method was specific to $d = 1$, $m = 1$ case. Hestenes's method, however, could be modified to suite our purposes. Following Hestenes we consider the normalized functional increment

$$\delta E(\{\boldsymbol{\phi}_n\}) = \varliminf_{n\to\infty} \frac{\Delta E(\boldsymbol{\phi}_n)}{\|\boldsymbol{\phi}_n\|_{1,2}^2}$$

and prove that it is positive. Following Hestenes, we start with an arbitrary strong variation $\{\boldsymbol{\phi}_n\}$. If $\|\nabla\boldsymbol{\phi}_n\|_p \to \infty$, then $\delta E(\{\boldsymbol{\phi}_n\}) = \infty$, provided $W(\boldsymbol{F}) \geq C_1|\boldsymbol{F}|^p - C_2$. If $\|\nabla\boldsymbol{\phi}_n\|_2$ is bounded, but not small, then we can use uniform quasiconvexity to prove that $\delta E(\{\boldsymbol{\phi}_n\}) > 0$. If $\|\nabla\boldsymbol{\phi}_n\|_2 \to 0$, as $n \to \infty$, we use Decomposition Lemma of Kristensen [11] and Fonseca, Müller and Pedregal [6] to split $\boldsymbol{\phi}_n$ into pure strong and weak parts. To be more precise, we represent the variation $\boldsymbol{\phi}_n$ as $\boldsymbol{\phi}_n = \alpha_n\boldsymbol{z}_n + \alpha_n\boldsymbol{v}_n$, where $\alpha_n = \|\nabla\boldsymbol{\phi}_n\|_2 \to 0$. The weak part $\alpha_n\boldsymbol{z}_n$ is characterized by the property that $|\nabla\boldsymbol{z}_n|^2$ is equiintegrable and the strong part by $|\{\boldsymbol{x} \in \Omega : \nabla\boldsymbol{v}_n(\boldsymbol{x}) \neq \boldsymbol{0}\}| \to 0$, as $n \to 0$. We can also arrange for the boundary conditions to be statisfied $\boldsymbol{z}_n = \boldsymbol{v}_n = \boldsymbol{\phi}_n = \boldsymbol{0}$ on $\partial\Omega_1$.

2. Sufficiency theorem

Assume that

(H1) the partial derivatives of $W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})$ of first and second order in $(\boldsymbol{y}, \boldsymbol{F})$ exist and are continuous on $\overline{\Omega} \times \mathcal{O}$, where $\mathcal{O}$ is an open and bounded neighborhood of $(\boldsymbol{x}, \boldsymbol{y}_0(\boldsymbol{x}), \nabla\boldsymbol{y}_0(\boldsymbol{x}))$;

(H2) $c_1(\boldsymbol{y})|\boldsymbol{F}|^p - c_2(\boldsymbol{y}) \leq W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F}) \leq C(\boldsymbol{y})(1 + |\boldsymbol{F}|^p)$;

(H3) For every $r > 0$ and $\epsilon > 0$ there exists $\delta > 0$ so that for every $\boldsymbol{x} \in \Omega$, $\{\boldsymbol{y}, \boldsymbol{y}'\} \subset \mathbb{R}^m$, and $\{\boldsymbol{F}, \boldsymbol{F}'\} \subset \mathbb{M}$, such that $|\boldsymbol{y}| < r$, $|\boldsymbol{y}'| < r$, $|\boldsymbol{y} - \boldsymbol{y}'| < \delta$ and

$$\frac{|\boldsymbol{F} - \boldsymbol{F}'|}{1 + |\boldsymbol{F}| + |\boldsymbol{F}'|} < \delta,$$

we have

$$\left| \frac{W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{F})}{1 + |\boldsymbol{F}|^p} - \frac{W(\boldsymbol{x}, \boldsymbol{y}', \boldsymbol{F}')}{1 + |\boldsymbol{F}'|^p} \right| < \epsilon$$

(H4) For every $r > 0$ and $\epsilon > 0$ there exists $\delta > 0$, so that for every $\boldsymbol{F} \in \mathbb{M}$, $|\boldsymbol{y}| < r$ and $\{\boldsymbol{x}', \boldsymbol{x}''\} \subset \overline{\Omega}$, such that $|\boldsymbol{x}' - \boldsymbol{x}''| < \delta$, we have

$$\frac{|W(\boldsymbol{x}', \boldsymbol{y}, \boldsymbol{F}) - W(\boldsymbol{x}'', \boldsymbol{y}, \boldsymbol{F})|}{1 + |\boldsymbol{F}|^p} < \epsilon.$$

Suppose $\boldsymbol{y}_0 \in \mathcal{A}$ solves the Euler-Lagrange equation (1). Suppose, in addition, the following conditions are satisfied

- $\delta^2 E(\boldsymbol{\phi}) \geq \beta\|\boldsymbol{\phi}\|_{1,2}^2, \quad \boldsymbol{\phi} \in \mathrm{Var}(\mathcal{A})$
- for all $\boldsymbol{x}_0 \in \Omega$ and all $\boldsymbol{\phi} \in C_0^\infty(B; \mathbb{R}^m)$

$$\int_B \{W(\nabla\boldsymbol{y}_0(\boldsymbol{x}_0) + \nabla\boldsymbol{\phi}(\boldsymbol{z})) - W(\nabla\boldsymbol{y}_0(\boldsymbol{x}_0))\} d\boldsymbol{z} \geq \beta\|\nabla\boldsymbol{\phi}\|_2^2$$

- for all $\boldsymbol{x}_0 \in \partial\Omega_2 = \partial\Omega \setminus \overline{\partial\Omega_1}$ and all

$$\boldsymbol{\phi} \in V_{\boldsymbol{n}(\boldsymbol{x}_0)} = \{\boldsymbol{\phi} \in C^\infty\left(\overline{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}};\mathbb{R}^m\right) : \boldsymbol{\phi}(\boldsymbol{z}) = \boldsymbol{0} \text{ on } \partial B \cap \overline{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}}\}$$

$$\int_{B^-_{\boldsymbol{n}(\boldsymbol{x}_0)}} \{W(\nabla \boldsymbol{y}_0(\boldsymbol{x}_0) + \nabla\boldsymbol{\phi}(\boldsymbol{z})) - W(\nabla \boldsymbol{y}_0(\boldsymbol{x}_0)\} d\boldsymbol{z} \geq \beta \|\nabla\boldsymbol{\phi}\|_2^2.$$

Then $\boldsymbol{y}_0$ is a strong local minimizer of the functional $E(\boldsymbol{y})$. Moreover, $\delta E(\{\boldsymbol{\phi}_n\}) \geq \beta\|\nabla\boldsymbol{\phi}_n\|_2$.

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On a conjecture by De Giorgi in large dimensions

Manuel del Pino
(joint work with Michal Kowalczyk, Juncheng Wei)

We consider the Allen-Cahn equation

$$\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N. \tag{1}$$

E. De Giorgi [3] formulated in 1978 the following celebrated conjecture:

(DG) *Let u be a bounded solution of equation* (1) *such that* $\partial_{x_N} u > 0$. *Then the level sets* $[u = \lambda]$ *are hyperplanes, at least for dimension* $N \le 8$.

De Giorgi conjecture has been fully established in dimensions $N = 2$ by Ghoussoub and Gui [4] and for $N = 3$ by Ambrosio and Cabré [1]. Savin [5] proved its validity for $4 \le N \le 8$ under a mild additional assumption. A counterexample to (DG) in dimension $N \ge 9$ has long been believed to exist, but the question has remained open. In this paper we disprove (DG) for $N \ge 9$. (DG) is a statement parallel to *Bernstein's theorem* for minimal graphs which in its most general form, due to Simons [7], states that any minimal hypersurface in $\mathbb{R}^N$, which is also a graph of a function of $N-1$ variables, must be a hyperplane if $N \le 8$. Bombieri, De Giorgi and Giusti [2] proved that this fact is false in dimension $N \ge 9$, by constructing a nontrivial solution to the problem

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8. \tag{2}$$

by means of the super-subsolution method. Let us write

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \cdots + x_4^2}, \quad v = \sqrt{x_5^2 + \cdots + x_8^2}.$$

The BDG solution has the form $F(x') = F(u,v)$ with the symmetry property $F(u,v) = -F(v,u)$ if $u \ge v$. In addition we can show that F becomes asymptotic to a function homogeneous of degree 3 that vanishes on the cone $u = v$. Let $\Gamma = \{x_9 = F(x')\}$ be the minimal BDG graph so predicted, and let us consider for $\alpha > 0$ its dilation $\Gamma_\alpha = \alpha^{-1}\Gamma$, which is also a minimal graph. Our result, which disproves statement (DG) in dimensions 9 or higher is the following:

Theorem. Let $N \ge 9$. For all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(x)$ of equation (1) such that $u_\alpha(0) = 0$,

$\partial_{x_N} u_\alpha(x) > 0$ for all $x \in \mathbb{R}^N$, and $|u_\alpha(x)| \to 1$ as $\operatorname{dist}(x, \Gamma_\alpha) \to +\infty$, uniformly in all small $\alpha > 0$.

The proof provides accurate information on u_α. If $t = t(y)$ denotes a choice of signed distance to the graph Γ_α then, for a small fixed number $\delta > 0$, the solution looks like $u_\alpha(x) \sim w(t)$, if $|t| < \frac{\delta}{\alpha}$. with $w(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$, the one-dimensional heteroclinic solution to (1). Let us us assume $N = 9$ (which is sufficient), and consider Fermi coordinates to describe points in $\mathbb{R}^9$ near Γ_α, $x = y + z\nu_\alpha(y)$, $y \in \Gamma_\alpha$, $|z| < \frac{\delta}{\alpha}$ where ν_α is the unit normal to Γ_α for which

$\nu_{\alpha 9} > 0$. Then we choose as a first approximation $\mathtt{w}(x) := w(z + h(\alpha y))$ where h is a smooth, small function on $\Gamma = \Gamma_1$, to be determined. Looking for a solution of the form $\mathtt{w} + \phi$, it turns out that the problem becomes essentially reduced to

$$\Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi + E + N(\phi) = 0 \quad \text{in } \Gamma_\alpha \times \mathbb{R}$$

where $S(\mathtt{w}) = \Delta \mathtt{w} + f(\mathtt{w})$, $E = \chi_{|z|<\alpha^{-1}\delta}\, S(\mathtt{w})$, $N(\phi) = f(\mathtt{w}+\phi) - f(\mathtt{w}) - f'(\mathtt{w})\phi + B(\phi)$, $f(\mathtt{w}) = \mathtt{w}(1-\mathtt{w}^2)$, and $B(\phi)$ is a second order linear operator with small coefficients, also cut-off for $|z| > \delta\alpha^{-1}$. Rather than solving the above problem directly we consider a projected version of it:

$$\mathcal{L}(\phi) := \Delta_{\Gamma_\alpha}\phi + \partial_{zz}\phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z) \quad \text{in } \Gamma_\alpha \times \mathbb{R} \tag{3}$$

$$\int \phi(y,z)w'(z)\,dz \;=\; 0 \quad \text{for all } y \in \Gamma_\alpha \tag{4}$$

A solution to this problem can be found in such a way that it respects the size and decay rate of the error E, which is roughly of the order $\sim r(\alpha y)^{-3}e^{-|z|}$, this is made precise with the use of a linear theory for the projected problem in weighted Sobolev norms and an application of contraction mapping principle. Finally h s found so that $c(y) \equiv 0$. We have $c(y)\int {w'}^2 dz = \int (E + N(\phi))w'\,dz$ and thus we get reduced to a (nonlocal) nonlinear PDE in Γ of the form

$$\mathcal{J}(h) := \Delta_\Gamma h + |A|^2 h = O(\alpha)r(y)^{-3} + M_\alpha(h) \quad \text{in } \Gamma, \quad h = 0 \quad \text{on } \Gamma \cap [u = v], \tag{5}$$

where $M(h)$ is a small operator which includes nonlocal terms. A solvability theory for the Jacobi operator in weighted Sobolev norms is then devised, with the crucial ingredient of the presence of explicit barriers for inequalities involving the linear operator above, and asymptotic curvature estimates by Simon [6]. Using this theory, problem (5) is finally solved by means of contraction mapping principle. The monotonicity property follows from maximum principle applied to the linear equation satisfied by $\partial_{x_9}u$.

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Boundary regularity for solutions of divergence form operators on non-smooth domains

Tatiana Toro
(joint work with E. Milakis)

A few years ago Kenig and Toro (see [12]) introduced two types of non-smooth domains in $\mathbb{R}^n$. Reifenberg flat domains with small constant are domains whose boundary can be well approximated by affine $(n-1)$-dimensional spaces in the Hausdorff distance sense. Chord arc domains are sets of locally finite perimeter which are NTA domains and whose surface measure at the boundary σ is Ahlfors regular (i.e. $\sigma(B(Q,r)) \sim r^{n-1}$). A chord arc domain with small constant is a chord arc domain for which the unit normal vector to the boundary has small BMO norm. In joint work with Kenig we studied the boundary regularity of the solutions to the Laplacian in these domains by analyzing the doubling properties of the harmonic measure and the existence and regularity of the Poisson kernel. Since then, several authors have studied PDE problems, free boundary regularity problems and potential theory questions on these domains (see [1], [2], [3], [4], [5], [9] [10], [11], [12], [13], [14]). In particular, chord arc domains with vanishing constant (i.e those for which the unit normal vector to the boundary in in $VMO(\sigma)$) are good substitutes for C^1 domains in this context.

In current joint work work with Milakis ([15]) we consider two types of operators.

We say that elliptic operator $L \in \mathcal{L}(\lambda, \Lambda, \alpha)$ if

$$Lu = \operatorname{div}(A(X)\nabla u) \text{ in } \Omega \tag{1}$$

with symmetric coefficient matrix $A(X) = (a_{ij}(X))$, such that there are $\lambda, \Lambda > 0$ satisfying

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{2}$$

for all $X \in \Omega$ and $\xi \in \mathbb{R}^n$. Furthermore we require that $A \in C^\alpha(\overline{\Omega})$.

An elliptic operator $Lu = \operatorname{div}(A(X)\nabla u)$ defined on a chord arc domain $\Omega \subset \mathbb{R}^n$ is a perturbation of the Laplacian if the deviation function

$$a(X) = \sup\{|\mathrm{Id} - \mathrm{A(Y)}| : \mathrm{Y} \in \mathrm{B(X}, \delta(\mathrm{X})/2)\} \tag{3}$$

where $\delta(X)$ is the distance of X to $\partial\Omega$, satisfies the following Carleson measure property: i.e. there exists $C > 0$ such that

$$\sup_{0<r<\operatorname{diam}\Omega} \sup_{Q\in\partial\Omega} \left\{ \frac{1}{\sigma(B(Q,r))} \int_{B(Q,r)\cap\Omega} \frac{a^2(X)}{\delta(X)} dX \right\} \le C, \tag{4}$$

where $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$. Note that in this case $L = \Delta$ on $\partial\Omega$.

We now state our results.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a Reifenberg flat domain with vanishing constant, let $L \in \mathcal{L}(\lambda, \Lambda, \alpha)$ and let ω be its elliptic measure. Then for all $\tau \in (0, 1)$,*

$$\lim_{\rho \to 0} \inf_{Q \in \partial\Omega} \frac{\omega(B(Q, \tau\rho))}{\omega(B(Q, \rho))} = \lim_{\rho \to 0} \sup_{Q \in \partial\Omega} \frac{\omega(B(Q, \tau\rho))}{\omega(B(Q, \rho))} = \tau^n.$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a chord arc domain with vanishing constant. Assume that $L \in \mathcal{L}(\lambda, \Lambda, \alpha)$. Then $\log k \in VMO(\partial\Omega)$.*

Theorems 1 and 2 are analogue to the results obtained for the Laplacian in [12]. The structure of the proofs is similar to those in [12]. We require the Hopf maximum principle (see [8] or [6]), the regularity theory for this type of operator (see [6]), the maximum principle and several comparison theorems which are valid on NTA domains.

Theorem 3. *Let Ω be a chord arc domain. Let L be such that (4) holds. There exists $\delta(n) > 0$ such that if $\Omega \subset \mathbb{R}^n$ is a chord arc domain with constant $\delta \in (0, \delta(n)$ then $\omega \in A_\infty(d\sigma)$.*

The proof of Theorem 3 requires an argument along the lines of the one in [7], as well as the use of Semmes decomposition theorem for chord arc domains with small constant.

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The optimal partial transport problem

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The aim of the talk is to describe some recent results on a variant of the classical optimal transport problem. The problem considered is the following: given two densities f and g, we want to transport a fraction $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ of the mass of f onto g minimizing the transportation cost $c(x,y) = |x-y|^2$. More precisely, let $f, g \in L^1(\mathbb{R}^n)$ be two nonnegative functions, and denote by $\Gamma_{\leq}(f,g)$ the set of nonnegative Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginals are dominated by f and g respectively. Fix a certain amount $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ which represents the mass one wants to transport, and consider the following partial transport problem:

$$\text{minimize } C(\gamma) := \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 \, d\gamma(x,y)$$

among all $\gamma \in \Gamma_{\leq}(f,g)$ with $\int d\gamma = m$.

Using weak topologies, it is simple to prove existence of minimizers for any fixed amount of mass $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$. We remark however that in general one cannot expect uniqueness of minimizers: if $m \leq \int_{\mathbb{R}^n} f \wedge g$, any γ supported on the diagonal $\{x = y\}$ with marginals dominated by $f \wedge g$ is a minimizer with zero cost. To ensure uniqueness, in [1] the authors assume f and g to have disjoint supports. Under this assumption they are able to prove (as in the classical Monge-Kantorovich problem) that there exists a (unique) convex function ψ such that the unique minimizer is concentrated on the graph of $\nabla\psi$. This ψ is also shown to solve in a weak sense a Monge-Ampère double obstacle problem. Then, strengthening the disjointness assumption into the hypothesis on the existence of a hyperplane separating the supports of the two measures, the authors prove a semiconvexity result on the free boundaries. Furthermore, under some classical regularity assumptions on the measures and on their supports, local $C^{1,\alpha}$ regularity of ψ and on the free boundaries of the active regions is shown.

In [2], we study what happens if one removes the disjointness assumption. Although minimizers are non-unique for $m < \int_{\mathbb{R}^n} f \wedge g$ (but in this case the set of minimizers can be trivially described), uniqueness holds for any $m \geq \int_{\mathbb{R}^n} f \wedge g$. Moreover, exactly as in [1], the unique minimizer is concentrated on the graph of the gradient of a convex function.

We can also prove that the marginals of the minimizers always dominate the common mass $f \wedge g$ (that is all the common mass is both source and target). This property, which has an interest on its own, plays also a crucial role in the regularity of the free boundaries. Indeed, one can show that the free boundary has zero Lebesgue measure (under some mild assumptions on the supports of the two densities), and as a consequence of this fact one can apply Caffarelli's regularity theory for the Monge-Ampère equation whenever the support of g is assumed to be convex, and f and g are bounded away from zero and infinity on their respective support. One can therefore deduce local $C^{0,\alpha}$ regularity of the transport map, and that it extends to an homeomorphism up to the boundary if both supports are assumed to be strictly convex.

On the other hand, in this situation where the supports of f and g can intersect, something new happens: usually, assuming C^∞ regularity on the density of f and g (together with some convexity assumption on their supports), one can show that the transport map is C^∞ too. In our case we will show that $C^{0,\alpha}_{\mathrm{loc}}$ regularity is in some sense optimal: we can find two C^∞ densities on $\mathbb{R}$, supported on two bounded intervals and bounded away from zero on their supports, such that the transport map is not C^1.

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New entire solutions to semilinear elliptic equations in $\mathbb{R}^n$

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We construct new entire solutions of the equation

$$(E_p) \qquad -\Delta u + u = u^p \qquad \text{in } \mathbb{R}^n,$$

where $p \in \left(1, \frac{n+2}{n-2}\right)$. These solutions decay exponentially away from three half lines and are asymptotically periodic in their three directions.

The study of (E_p) has several motivations: as basic examples we mention nonlinear scalar field equations like the *Nonlinear Klein-Gordon* or the *Nonlinear Schrödinger*. More precisely a special class of solutions of the latter, $i\hbar\frac{\partial\tilde{\psi}}{\partial t} = -\hbar^2\Delta\tilde{\psi} + V(x)\tilde{\psi} - |\tilde{\psi}|^{p-1}\tilde{\psi}$, called *standing waves*, are complex-valued functions $\psi(x,t)$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, of the form $\psi(x,t) = e^{-i\omega t}u(x)$, where ω is a real constant and $u : \mathbb{R}^n \to \mathbb{R}$ a real-valued function which satisfies the equation (adding ω to V)

$$(NLS) \qquad -\hbar^2\Delta u + V(x)u = u^p \qquad \text{in } \mathbb{R}^n$$

($V : \mathbb{R}^n \to \mathbb{R}$ is the potential and, still, $p > 1$). Problem (E_p) plays a role in the understanding of both the loss of compactness in (NLS) or the profile of solutions in the semiclassical limit, when $\hbar$ tends to zero.

Other motivations for considering (E_p) arise in the study of models from biology: for example, the *Gierer-Meinhardt* system, see [8], can be approached by studying first the equation $-\varepsilon^2\Delta u + u = u^p$ in a domain $\Omega \subseteq \mathbb{R}^n$, with Neumann boundary conditions. There is a broad literature on this problem, concerning existence and multiplicity results on *spike layers*, namely solutions u_ε which concentrate at a finite number of points of $\overline{\Omega}$, with the profile $u_\varepsilon(x) \simeq U\left(\frac{x-x_0}{\varepsilon}\right)$, $x_0 \in \overline{\Omega}$, where U solves (E_p). For the above issues we refer the interested reader to [1], where a rather complete list of references is given.

Recently, a different kind of solutions (whose existence has been conjectured for some time, see [8]) has been shown to exist, for both the above mentioned singularly perturbed problems. These have a different profile and scale only in one direction (or, more generally, in k directions, with $k \in \{1, \dots, n-1\}$), corresponding to solutions of (E_p) which are independent of some of the variables, see [5], and references therein.

Except when some symmetry is present, this kind of results asserts that concentration occurs provided we restrict ourselves to suitable sequences $\varepsilon_j \to 0$: the reason is that these solutions have a larger and larger Morse index, and therefore resonance occurs. As a consequence, if one wishes to employ local inversion arguments, it is necessary to avoid some values of the parameter ε, so that the linearized equation is invertible. Under symmetry assumptions one can work in spaces of invariant functions and obtain existence for all epsilon's: however the resonance phenomenon is still underlying, and this generates bifurcation phenomena, see [2].

This bifurcation is indeed also present for a class of solutions of (E_p). For example, one can start from entire (decaying) solutions of the equation in lower dimension, say in $\mathbb{R}^{n-1}$, and extend them (with obvious notation) to the whole $\mathbb{R}^n$ by setting $\tilde{U}(x_1, x') = U_{n-1}(x')$. In [4] N.Dancer proved bifurcation of non-cylindrical solutions from $\tilde{U}$ which are periodic in x_1, considering the Morse index of $\tilde{U}$ restricted to the strip $D_L := \left\{-\frac{L}{2} \le x_1 \le \frac{L}{2}\right\}$, and showing that this diverges when $L \to +\infty$. A similar strategy was previously used by R.Schoen to prove multiplicity of solutions for the Yamabe problem, see [9], and in fact other geometric problems exhibit this kind of phenomenon, like that of finding surfaces in $\mathbb{R}^3$ which have constant mean curvature, giving rise to *Delaunay unduloids*. These are used as *building blocks* to produce complete surfaces in $\mathbb{R}^3$ with constant mean curvature which are union of a compact set and a finite number of *ends*, namely subsets with the topology of the cylinder which are asymptotically close to Delaunay surfaces. Analogous constructions can be done with Yamabe metrics which are defined on domains of $\mathbb{R}^n$ with a finite number of points removed, and which are singular at these points. We refer for example to [7], and its references for details.

Denoting points of $\mathbb{R}^n$ by couples $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, we consider first a family of solutions u_L to (E_p) which are periodic in the x_1 variable and which decay to zero at an exponential rate away from $x' = 0$, counterparts of the Delaunay surfaces. We focus on the case of large period L, which allows us to construct the solutions of [4] using perturbative methods. In fact, setting $\mathbf{z}_i = (iL, 0, \dots, 0)$ and considering the function $u_{0,L} = \sum_{i \in \mathbb{N}} U(\cdot - \mathbf{z}_i)$, this satisfies the Neumann boundary conditions on ∂D_L and is an approximate solution of (E_p) for L large.

Using the implicit function theorem, one can add a correction $\overline{w}_L$ to $u_{0,L}$ so that $u_L = u_{0,L} + \overline{w}_L$ solves (E_p) exactly.

To state our result, we introduce some extra notation: set

$$\Pi = \left\{ (z_1, z_2, 0, \dots, 0) \ : \ (z_1, z_2) \in \mathbb{R}^2 \right\} \subseteq \mathbb{R}^n$$

and also, given $\theta \in S^{n-1}(\subseteq \mathbb{R}^n) \cap \Pi$, we define the ray $l_\theta = \{t\theta \ : \ t \geq 0\}$. We also let R_θ denote the rotation in the plane Π (extended naturally to all of $\mathbb{R}^n$) of the angle θ. The distance function between two points (or between two sets) of $\mathbb{R}^n$ is denoted by $dist(\cdot,\cdot)$. In the statement of Theorem 1 u_L stands for the solution of (E_p) periodic in x_1 just described.

Theorem 1. *([6]) Problem (E_p) admits a three-dimensional (up to rotations and translations) family of solutions which decay exponentially away from three rays originating from a common point, and which have an asymptotic periodic profile along the rays. More precisely, there exist a positive constant C, a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^3$, smooth functions $\theta_1, \theta_2, \theta_3 : \mathcal{U} \to S^{n-1} \cap \Pi$, $L_1, L_2, L_3 : \mathcal{U} \to \mathbb{R}$, $y_1, y_2, y_3 : \mathcal{U} \to \Pi$ and a map from $\mathcal{U}$ into $L^\infty(\mathbb{R}^n)$, $\zeta \in \mathcal{U} \mapsto u_\zeta$, such that the following properties hold*

- **(i):** *u_ζ is a positive solution of (E_p);*
- **(ii):** *if $l_{\theta_1}, l_{\theta_2}, l_{\theta_3}$ are the rays corresponding to the directions θ_1, θ_2 and θ_3 respectively, then*

$$u_\zeta(x) \leq C e^{-\frac{1}{C} dist(x, (l_{\theta_1} \cup l_{\theta_2} \cup l_{\theta_3}))} \qquad \text{for every } x \in \mathbb{R}^n;$$

- **(iii):** *for any $t_i \to +\infty$, given any compact set K of $\mathbb{R}^n$ there holds*

$$\| u\left(\cdot - t_i \theta_a\right) - u_{L_a}\left(R_{\theta_a}\left(\cdot - y_a\right)\right) \|_{C^2(K)} \leq C_K e^{\frac{1}{C}|t_i|}, \qquad \text{for } a = 1,2,3.$$

We can indeed characterize more precisely these solutions in terms of their asymptotic behavior at infinity. In our construction the values of the numbers L_a, $a = 1,2,3$, can be chosen arbitrarily large, but the differences $|L_a - L_b|$, with $a \neq b$, stay uniformly bounded. Also, we have $\theta_a \angle \theta_b > \frac{\pi}{3}$ for every $a \neq b$, where $\theta_a \angle \theta_b$ stands for the angle between the two versors θ_a and θ_b. It is also possible to prove that the following function is positive and monotone in L ($L \gg 1$)

$$G(L) := \frac{1}{4} \int_{\partial D_L} \left(|\nabla u_L|^2 + u_L^2 \right) d\mathcal{S} - \frac{1}{2(p+1)} \int_{\partial D_L} |u_L|^{p+1} d\mathcal{S},$$

and that it determines uniquely the asymptotic period and profile of the functions u_L. In analogy with a *balance condition* for the CMC surfaces or the singular Yamabe metrics we have the following result.

Theorem 2. *([6]) Let u be a function satisfying the properties* **(i)**-**(iii)** *in Theorem 1, and let θ_a, L_a, $a = 1,2,3$, be the corresponding quantities. Assume that the angles $\theta_a \angle \theta_b$ between any two different θ's are greater than $\frac{\pi}{3}$. Then $\sum_{a=1,2,3} \theta_a G(L_a) = 0$.*

Remark 3. *(a) Existence of solutions of semilinear elliptic equations with infinitely-many bumps has been considered in other works, but from different points of view from ours. For example, in* [3]*, similar equations in the presence of a slowly-oscillating potential have been considered. While in that paper is the potential which mainly determines the locations of the* bumps*, here are precisely their mutual interactions which allow us to perform the construction of Theorem 1.*

(b) Concerning the Neumann problem mentioned above, we believe that the functions constructed in Theorem 1, scaled in ε, might lead to the existence of solutions concentrating at a singular set in Ω, with a triple point. This would be a new type of phenomenon, since so far concentration at sets of dimension greater than zero has been proved for smooth curves or manifolds only.

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Almgren's Q–valued functions revisited

Camillo De Lellis

(joint work with Emanuele Spadaro)

A central problem in geometric measure theory is to understand the regularity of integer rectifiable currents in the euclidean space, which arise as solutions of the classical Plateau's problem. There is a wide literature for the case of currents of codimension 1. Instead much less is available in codimension higher than 1. The big difference between these two cases is that in higher codimension a new phenomenon can occur: that of branching. Building on an old observation of Wirtinger, Federer showed that any holomorphic variety $\mathcal{V}$ in $\mathbf{C}^n = \mathbf{R}^{2n}$ is an area minimizing current. That is, for any smooth bounded open set Ω, $\mathcal{V} \cap \Omega$ is the unique area–minimizing current which bounds $\mathcal{V} \cap \partial\Omega$. Since holomorphic varieties can branch on a set of real codimension 2, clearly the singular set of an area–minimizing current has codimension at most 2.

Remark 1. *In particular, the following example is a good model for the type of singularities that might occur. Consider in* $\mathbf{C}^2$ *the algebraic variety* $\mathcal{V} := \{(z,w) \in \mathbf{C}^2 : z^2 = w^3\}$*. This defines an integer rectifiable current of dimension 2 in* $\mathbf{R}^4$*. By the discussion above, this is an area–minimizing current. However, in a neighborhood of the origin there is no system of coordinates in which* $\mathcal{V}$ *can be represented as the graph of a function.*

In his fundamental work [1], Almgren proved the following

Theorem 1. *Let* Γ *be a* k*–dimensional area–minimizing integer rectifiable current in* $\mathbf{R}^n$*. Denote by* $supp\,(\partial\Gamma)$ *the support of the current* $\partial\Gamma$*. Then* Γ *is an embedded analytic submanifold outside on* $\mathbf{R^n} \setminus (supp\,(\Gamma) \cup \Sigma)$*, where* Σ *is a subset of Hausdorff dimension at most* $k-2$ *(which is relatively closed in* $\mathbf{R^n} \setminus supp\,(\Gamma)$*).*

Relying on [1] this theorem was later improved by Chang in [2] in the case $k = 2$:

Theorem 2. *If* $k = 2$*, then* Σ *consists of isolated points.*

The book [1] contains almost 1000 pages and it would be desirable to have a more manageable proof. One reason for the complexity of this work is due to branching. Indeed, one of the key ideas of the regularity theory for minimal surfaces, dating back to the pioneering works of De Giorgi, is that, when the current is approximately flat in an average sense, then it can be well approximated by the graph of an harmonic functions. Such a statement which holds under general assumptions in the codimension 1 case, fails in the higher codimension when branching occurs. For instance, consider the holomorphic varieties $\mathcal{V}_\varepsilon = \{z^2 = \varepsilon^2 w^3\} \subset \mathbf{C}^2 = \mathbf{R}^4$. As ε approaches 0, $\mathcal{V}_\varepsilon$ converge to a double copy of the plane $\{z = 0\}$. However, none of these currents can be approximated by the graph of a function in a neighborhood of the origin.

One of the main ideas of Almgren is to bypass this difficulty by considering "multiple–valued" functions. After defining these objects and introducing a suitable concept of Dirichlet energy for them, he dedicates a first part of his book to the existence and regularity of minimizers. We note in passing that there is no freedom in the choice of the energy: since the goal is to devise an appropriate first order approximation to the area of an integer rectifiable currents, when the multiple valued functions consists of separate sheets, its energy must be the sum of the Dirichlet energies of the single sheets.

In [3] we revisit Almgren's theory of multiple valued functions by giving two different simplified approaches to it. The first is based on Almgren's original proofs, suitably shortened. The other substitutes part of his theory with a more intrinsic one, which draws heavily on existing results for Sobolev spaces of functions with metric targets. As an example of how this theory works, we will give here a quick definition of Q–valued functions and of their energy.

Definition: [Unordered Q-tuples] $[[P_i]]$ denotes the Dirac mass in $P_i \in \mathbf{R}^n$ and

$$\mathcal{A}_Q(\mathbf{R}^n) := \left\{ \sum_{i=1}^{Q} [[P_i]] \,:\, P_i \in \mathbf{R}^n \text{ for every } i = 1, \ldots, Q \right\}.$$

Remark 2. *$(\mathcal{A}_Q(\mathbf{R}^n), \mathcal{G})$ is a closed subset of a "convex" complete metric space. Indeed, $\mathcal{G}$ coincides with the well–known L^2-Wasserstein distance, W_2, on the space $\mathcal{M}_2(\mathbf{R}^n)$ of positive measures with finite second moment.*

In order to simplify the notation, we use $\mathcal{A}_Q$ in place of $\mathcal{A}_Q(\mathbf{R}^n)$ and we write $\sum_i [[P_i]]$ when n and Q are clear from the context. Clearly, the points P_i do not have to be distinct: for instance $Q[[P]]$ is an element of $\mathcal{A}_Q(\mathbf{R}^n)$. We endow $\mathcal{A}_Q(\mathbf{R}^n)$ with a metric which makes it a complete metric space (the completeness is an elementary exercise left to the reader).

Definition: For every $T_1, T_2 \in \mathcal{A}_Q(\mathbf{R}^n)$, with $T_1 = \sum_i [[P_i]]$ and $T_2 = \sum_i [[S_i]]$, we set

$$\mathcal{G}(T_1, T_2) \;:=\; \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_i \left|P_i - S_{\sigma(i)}\right|^2}\,, \tag{1}$$

where $\mathcal{P}_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

From now on, we will denote by Ω a bounded open subset of the euclidean space $\mathbf{R}^m$ with sufficiently regular boundary (in fact, the Lipschitz regularity will be enough for our purposes).

Definition: [Sobolev Q-valued functions] A measurable $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,p}$ ($1 \le p \le \infty$) if there exist m functions $\varphi_j \in L^p(\Omega; \mathbf{R}^+)$ such that

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,p}(\Omega)$ for all $T \in \mathcal{A}_Q$;
(ii) $|\partial_j\, \mathcal{G}(f, T)| \le \varphi_j$ a.e. in Ω for all $T \in \mathcal{A}_Q$ and for all $j \in \{1, \ldots, m\}$.

It is not difficult to show the existence of minimal functions $\tilde{\varphi}_j$ fulfilling (ii), i.e. such that, for any other φ_j satisfying (ii), $\tilde{\varphi}_j \le \varphi_j$ a.e.. We denote them by $|\partial_j f|$. We then set

$$|Df|^2 \;:=\; \sum_{j=1}^{m} |\partial_j f|^2 \tag{2}$$

and the Dirichlet energy of $f \in W^{1,2}(U; \mathcal{A}_Q)$ is given by

$$\mathrm{Dir}\,(f, U) := \int_U |Df|^2\,.$$

Finally, the usual notion of trace at the boundary can be easily generalized to this setting.

Definition: [Trace of Sobolev Q-functions] Let $\Omega \subset \mathbf{R}^m$ be a Lipschitz bounded open set and $f \in W^{1,p}(\Omega; \mathcal{A}_Q)$. A function $g \in L^p(\partial\Omega; \mathcal{A}_Q)$ is said to be the trace of f at $\partial\Omega$ (and we denote it by $f|_{\partial\Omega}$) if, for every $T \in \mathcal{A}_Q$, the trace of the real-valued Sobolev function $\mathcal{G}(f, T)$ coincides with $\mathcal{G}(g, T)$.

We are now ready to state the main results of Almgren's theory.

Theorem 3 (Existence for the Dirichlet Problem)**.** *Let* $g \in W^{1,2}(\Omega; \mathcal{A}_Q)$*. Then, there exists a* Dir *-minimizing* $f \in W^{1,2}(\Omega; \mathcal{A}_Q)$ *such that* $f|_{\partial\Omega} = g|_{\partial\Omega}$*.*

Theorem 4 (Hölder regularity)**.** *There is a constant* $\alpha = \alpha(m, Q) > 0$ *with the following property. If* $f \in W^{1,2}(\Omega; \mathcal{A}_Q)$ *is* Dir *-minimizing, then* $f \in C^{0,\alpha}(\Omega')$ *for every* $\Omega' \subset\subset \Omega \subset \mathbf{R}^m$*. For two-dimensional domains, we have the explicit constant* $\alpha(2, Q) = 1/Q$*.*

For the second regularity theorem we need the definition of singular set of f.
Definition: [Regular and singular points] A Dir -minimizing f is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i : B \to \mathbf{R}^n$ such that

$$f(y) = \sum_i [[f_i(y)]] \qquad \text{for almost every } y \in B \tag{3}$$

and either $f_i(x) \neq f_j(x)$ for every $x \in B$, or $f_i \equiv f_j$. The singular set Σ_f of f is the complement of the set of regular points.

Theorem 5 (Estimate of the singular set)**.** *Let* f *be* Dir *-minimizing. Then, the singular set* Σ_f *of* f *is relatively closed in* Ω*. Moreover, if* $m = 2$*, then* Σ_f *is at most countable, and if* $m \geq 3$*, then the Hausdorff dimension of* Σ_f *is at most* $m - 2$*.*

In the paper [3], following in part ideas of [2], we improve this last theorem in the following way.

Theorem 6 (Improved estimate of the singular set)**.** *Let* f *be* Dir *-minimizing and* $m = 2$*. Then, the singular set* Σ *of* f *consists of isolated points.*

Our paper [1] is considerably simpler than the first part of Almgren's book. However, the complexity of Almgren's proofs is due in part to applications in later chapters of his book, whereas we have completely ignored any issue which is not directly relevant to the theory of Dir–minimizing Q–valued functions. Finally, we wish to point out that parts of our "intrinsic theory" intersects with previous works of Jordan Goblet (see for instance [4] and [5]).

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Foliations of asymptotically flat manifolds by surfaces of Willmore type

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(joint work with Tobias Lamm and Felix Schulze)

1. Introduction

In this talk we consider a problem motivated partly by General Relativity. There 3-dimensional Riemannian manifolds (M, g) appear as initial data sets for the Einstein equations. Therefore it is of interest to assign physical quantities such as mass to such manifolds. An extremely useful concept for measuring the mass contained in a region bounded by an embedded surface Σ in M is the *Hawking mass*

$$m_H(\Sigma) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_\Sigma H^2 \, d\mu\right).$$

Here $|\Sigma|$ denotes the area and H the mean curvature of Σ. This quantity has interesting properties which were for example exploited in [HI01] to show the Riemannian Penrose inequality. Its key properties needed are non-negativity and monotonicity on a class of particular surfaces, in the given reference for example along inverse mean curvature flow. However, as one can see from the inequality

$$16\pi \leq \int_{\Sigma'} H^2 \, d\mu$$

for all surfaces Σ' in Euclidean space, non-negativity can only be expected to hold on a subset of all surfaces. In Euclidean space the surfaces with non-negative Hawking mass are exactly the round spheres with mass zero.

Here we want to introduce a variational principle to produce a canonical selection of good surfaces. The most natural way is to maximize the Hawking mass among all surfaces. If (M, g) is an asymptotically flat manifold representing an isolated system with non-negative matter density, then one expects that increasing the size of a surface also increases the contained mass. Therefore a maximum might not be attained. To circumvent this issue, we consider the constrained problem

$$\begin{aligned} \text{Maximize} \quad & m_H(\Sigma) \\ \text{subject to} \quad & |\Sigma| = a_0. \end{aligned}$$

Note that the area constraint renders this problem equivalent to minimizing the Willmore energy

$$\begin{aligned} \text{Minimize} \quad & \mathcal{W}(\Sigma) = \int_\Sigma H^2 \, d\mu \\ \text{subject to} \quad & |\Sigma| = a_0. \end{aligned}$$

The Euler-Lagrange equation of this problem is

$$\Delta H + H|\mathring{A}|^2 + H \operatorname{Ric}(\nu, \nu) + \lambda H = 0. \tag{1}$$

Here $\mathring{A} = A - \frac{1}{2}H\gamma$ is the trace free part of the second fundamental form of Σ, γ denotes the induced metric on Σ, Ric the Ricci-tensor on M, and ν is the normal to Σ. The constant λ is the Lagrange parameter.

The goal of the work described here will be to foliate certain asymptotically flat manifolds by surfaces satisfying (1) for varying λ. The detailed assumptions will be presented below.

Previous work in this direction was done by Huisken and Yau [HY96] who showed the existence of foliations by surfaces of constant mean curvature (CMC) and used these foliations to define a center of mass in such manifolds. Further directions include CMC foliations in other settings, such as asymptotically hyperbolic manifolds.

2. Model spaces

If one considers equation (1) in Euclidean space, we find that all round spheres satisfy the equation with $\lambda = 0$. This is a manifestation of the invariance of the functional $\mathcal{W}$ under translations and dilations. In particular the linearization of (1) has an (at least) four dimensional kernel.

Now consider the spatial Schwarzschild manifold of mass $m > 0$,

$$(\mathbf{R}^3 \setminus \{0\}, g^S = \phi^4 g^{\mathrm{eucl}}) \quad \text{with} \quad \phi = 1 + \frac{m}{2r}.$$

Here r denotes the Euclidean radius. This manifold describes the exterior of a single isolated body of mass m. It will be the background on which we model the asymptotically flat manifolds we consider.

Centered spheres $S_r(0)$ in this metric have

$$H = \text{const}, \quad \mathring{A} = 0, \qquad \text{and} \quad \mathrm{Ric}(\nu,\nu) = -2mR^{-3} = \text{const}$$

with $R = \phi^2 r$. Hence they satisfy (1) with $\lambda = -\mathrm{Ric}(\nu,\nu) = 2mR^{-3}$. Unlike in flat space λ now depends on r and thus (1) distinguishes spheres of different radius. It is convenient to express the linearization of (1) in terms of the Jacobi-operator of minimal surfaces

$$Lu = -\Delta u - u\big(|A|^2 + \mathrm{Ric}(\nu,\nu)\big).$$

Then the linearization M of the right hand side of (1) on centered spheres in Schwarzschild becomes

$$\mathcal{M}u = L^2 u + \tfrac{1}{2}H^2(Lu - 3\lambda u) + \lambda Lu.$$

In Schwarzschild, the spectrum of L is

$$\sigma(L) = \big\{ -\tfrac{1}{2}H^2 + \lambda, 3\lambda, c_k R^{-2} + O(R^{-3})\big\},$$

where the c_k are positive numbers. The eigenvalue 3λ has a three dimensional eigenspace corresponding to the variations induced by translations. This implies that the spectrum of $\mathcal{M}$ is

$$\begin{aligned}\sigma(\mathcal{M}) &= \big\{\mu^2 + \tfrac{1}{2}H^2(\mu - 3\lambda) + \lambda\mu : \mu \in \sigma(L)\big\} \\ &= \big\{ -\tfrac{1}{2}H^2\lambda = -\tfrac{12m}{R^5} + O(R^{-6}),\ 6\lambda^2 = \tfrac{24m}{R^6},\ \tfrac{\tilde{c}_k}{R^4} + O(R^{-5})\big\}\end{aligned}$$

where $\tilde{c}_k$ are also positive numbers. In particular $\mathcal{M}$ is invertible and the implicit function theorem can be used to deform individual surfaces under slight perturbations of the metric. However, the main difficulty is the construction of a whole foliation.

3. Results and sketch of proof

Before we formulate the main results, we make the following definition

Definition 1. *A three dimensional Riemannian manifold (M, g) is called (m, η, σ)-asymptotically Schwarzschild if there is a compact subset $B \subset M$ and coordinates $x : M \setminus B \to \mathbf{R}^3 \setminus B_\sigma(0)$ such that in these coordinates*

$$r^2|g - g^S| + r^3|\nabla - \nabla^S| + r^4|\operatorname{Ric} - \operatorname{Ric}^S| + r^5|\nabla \operatorname{Ric} - \nabla^S \operatorname{Ric}^S| \leq \eta \tag{2}$$

where g^S is the Schwarzschild metric of mass m. Furthermore ∇ is the Levi-Civita connection of g and the quantities with the superscript S are taken with respect to g^S.

Theorem 1. *For all C' and $m > 0$ there exist $\eta_0 = \eta_0(m)$ such that for all $\sigma > 0$ there is λ_0 and C depending only on C, m, η and σ with the following properties.*

If (M, g) is (m, η, σ)-asymptotically Schwarzschild and satisfies

1. *$r^{-5}|\operatorname{Scal}| \leq C'$, where Scal is the scalar curvature of (M, g), and*
2. *$\eta \leq \eta_0$*

then for all $\lambda \in (0, \lambda_0)$ there exists a surface Σ_λ which solves (1) for the given λ. The collection of the Σ_λ forms a foliation near the asymptotic end of M.

The surface is well approximated in C^2-norm by a coordinate sphere $S_{r_\lambda}(a_\lambda)$ with $|a_\lambda| \leq C$.

Before we turn to the question of uniqueness, we have to describe some features of the existence proof. A key observation is that the assumptions $H > 0$ and $\lambda > 0$ allow to derive the estimate

$$\int_\Sigma \lambda + |\overset{\circ}{A}|^2 + |\nabla \log H|^2 + \tfrac{1}{4} H^2 \, d\mu \leq 4\pi + C r_{\min}^{-1}$$

for a connected surface Σ satisfying (1) with $r \geq r_{\min}$. This follows by dividing (1) by H and the Gauss equation together with Gauss-Bonnet. Together with the estimate

$$\int_\Sigma H^2 \, d\mu \geq 16\pi - C r_{\min}^{-1}$$

this implies that

$$\int_\Sigma H^2 + |\nabla \log H|^2 \, d\mu \leq C r_{\min}^{-1}.$$

Using methods from [KS01] these estimates can be improved to

$$\int_\Sigma H^2 + |\nabla \log H|^2 \, d\mu \leq |\Sigma| \int_\Sigma |\omega|^2 + \big(\operatorname{Ric}(\nu, \nu) + \lambda\big)^2 \, d\mu$$

where $\omega = \operatorname{Ric}(\nu, \cdot)^T$. Since in Schwarzschild both of these terms vanish on centered spheres, this estimate can be eventually improved such that the right hand side is dominated by the error from (2), provided we can argue that the surfaces in question are close to centered spheres. Taking advantage of the conformal invariance of the L^2-norm of $\overset{\circ}{A}$, we get that the surfaces are well approximated by spheres from [DLM05, DLM06].

To get position estimates we take advantage of the Gauss equation to rewrite the functional as

$$\mathcal{W}(\Sigma) = 8\pi(1 - \mathrm{genus}(\Sigma)) + 2\mathcal{V}(\Sigma) + \mathcal{U}(\Sigma) \tag{3}$$

where

$$\mathcal{V}(\Sigma) = \int_\Sigma \mathrm{Ric}(\nu,\nu) - \tfrac{1}{2}\,\mathrm{Scal}\, d\mu \quad \text{and} \quad \mathcal{U}(\Sigma) = \int_\Sigma |\overset{\circ}{A}|^2\, d\mu.$$

The main idea is to identify portions of this decomposition which are sensitive to translations. The functional $\mathcal{U}$ is translation invariant in Schwarzschild and hence it is clear that it does not contribute a centering force. We therefore concentrate on the other two terms in (3).

The variation of $\mathcal{W}$ can be computed using equation (1), whereas the variation of $\mathcal{V}$ requires some care. The main issue is that to first order $\mathcal{V}$ is translation invariant on coordinate spheres in Schwarzschild. However, taking advantage of the conformally flat structure and the geometric Pohozaev identity, we can explicitly calculate the variation of $\mathcal{V}$ and get a definite term, which combines with the variation of $\mathcal{W}$ to the contribution expected by the size of the $6\lambda^2$ eigenvalue of $\mathcal{M}$.

Recall that the estimates imply that solutions Σ_λ to (1) are approximated by spheres $S_{r_\lambda}(a_\lambda)$. Then $\tau_\lambda = |a_\lambda|/r_\lambda$ is a scale invariant measure for how off-center Σ_λ is. Now we formulate the uniqueness theorem.

Theorem 2. *Assume that (M, g) is as in the existence theorem. Then for all $\varepsilon > 0$ there exists $r_0 = r_0(m, \eta, \sigma, C, \varepsilon)$ with the following properties.*

Let Σ'_λ be a family of surfaces satisfying (1) with $\lambda > 0$ and $H > 0$. Assume that the approximating spheres $S_{r_\lambda}(a_\lambda)$ for Σ'_λ satisfy,

$$r_0 < \min_{\Sigma_\lambda} r, \quad r_\lambda < (\min_{\Sigma_\lambda} r)^{2-\varepsilon}, \quad \textit{and} \quad \tau_\lambda < (\min_{\Sigma_\lambda} r)^{-\varepsilon}.$$

Then all Σ'_λ coincide with the surfaces Σ_λ constructed in the existence theorem.

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Partial regularity for biharmonic maps, revisited

Michael Struwe

In [10], jointly with Tristan Rivière we presented a new approach to the partial regularity for stationary weakly harmonic maps in dimension $m \geq 2$ as a special case of a regularity result for elliptic systems

$$-\Delta u^i = \Omega^{ij} \cdot \nabla u^j \text{ in } B \tag{1}$$

on a ball $B = B^m \subset \mathbb{R}^m$ with $\Omega = (\Omega^{ij}) \in L^2(B, \wedge^1\mathbb{R}^m \otimes so(n))$ and with $u = (u^1, \dots, u^n) \in H^1(B, \mathbb{R}^n)$ satisfying the Morrey growth assumption

$$\sup_{x\in B,\, r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x)\cap B} (|\nabla u|^2 + |\Omega|^2)\, dx \right)^{1/2} < \varepsilon(m). \tag{2}$$

A key ingredient in this new approach is the natural use of gauge theory, which is motivated by the anti-symmetry of the 1-form $\Omega = \Omega^{ij}$. Previously, Rivière [9] already had recognized this structure as the essential structure of the harmonic map system in $m = 2$ space dimensions, allowing him to obtain an equivalent formulation of this equation in divergence form. His results generalize to a large number of conformally invariant equations of second order. Subsequently, Lamm and Rivière [7] obtained a similar equivalent formulation of the biharmonic map system as a conservation law in the *"conformal"* case of $m = 4$ space dimensions. However, just as the methods of [9] no longer seem applicable when $m > 2$, also the approach in [7] seems to fail in dimensions $m > 4$.

The paper [11] extends the approach in [10] to fourth order equations, allowing to recover the known partial regularity results for stationary (extrinsic) biharmonic maps into a closed target manifold $N \subset \mathbb{R}^n$ by a simpler method and under less stringent, possibly optimal regularity assumptions. In particular, we obtain the following result which improves the pioneering work of Chang-Wang-Yang [4] and the later results by Changyou Wang [16] and Strzelecki [12] in this regard.

Theorem 1. *Let $N^k \subset \mathbb{R}^n$ be a closed submanifold of class C^3. Let $m \geq 4$ and suppose $u \in H^2(B; N)$ is a stationary biharmonic map on a ball $B = B^m \subset \mathbb{R}^m$. There exists a constant $\varepsilon_0 > 0$ depending only on N and m with the following property. Whenever on some ball $B_R(x_0) \subset B$ there holds*

$$R^{4-m} \int_{B_R(x_0)} (|\nabla^2 u|^2 + |\nabla u|^4) dx < \varepsilon_0, \tag{3}$$

then u is Hölder continuous (and hence as smooth as the target permits) on $B_{R/3}(x_0)$. In particular, u is smooth off a set $S \subset B$ of vanishing $(m-4)$-dimensional Hausdorff measure.

Recall that for any $1 \le p < \infty$ and any $s < m$ a function $f \in L^p(B)$ belongs to the homogeneous Morrey space $L^{p,s}(B)$ on a ball $B \subset \mathbb{R}^m$, provided that

$$\|f\|^p_{L^{p,s}(B)} = \sup_{x_0 \in B\,,\, r>0} \left(\frac{1}{r^s} \int_{B_r(x_0)\cap B} |f|^p dx \right) < \infty\,, \tag{4}$$

and $f \in L^{p,m-kp}_k(B)$, provided that $f \in W^{k,p}(B)$ with $\nabla^l f \in L^{p,m-lp}(B)$ for $0 < l \le k$.

Assume that condition (3) is satisfied on $B_3(0)$ for some sufficiently small number $\varepsilon_0 = \varepsilon_0(m,N) > 0$. As in [4], Lemma 4.8, or [16], Lemma 5.3, a first important step in the proof of Theorem 1 then is the derivation of the Morrey estimate

$$\varepsilon_1^4 := ||\nabla^2 u||^2_{L^{2,m-4}(B_2(0))} + ||\nabla u||^4_{L^{4,m-4}(B_2(0))} < C\varepsilon_0 \tag{5}$$

for a stationary biharmonic map u, with a constant $C = C(N,m)$. In [11] we observe that this bound directly follows from the monotonicity formula for stationary biharmonic maps due to Chang-Wang-Yang [4] and Angelsberg [2], whereas [4], [16] in addition again use the biharmonic map system for the derivation of (5).

Next we cast the equation for a biharmonic map $u \in H^2(B, \mathbb{R}^n)$ into the form

$$\Delta^2 u = \Delta(D \cdot \nabla u) + div(E \cdot \nabla u) + F \cdot \nabla u \text{ in } B \tag{6}$$

previously considered in [7] in dimension $m = 4$. Unlike [7], we decompose the function F as $F = G + \Delta\Omega$ with $\Omega = (\Omega^{ij}) \in H^1(B, \wedge^1\mathbb{R}^n \otimes so(n))$. The coefficient functions D, E, G, and Ω naturally depend on u and satisfy the growth conditions

$$|D| + |\Omega| \le C|\nabla u|,\ |E| + |\nabla D| + |\nabla\Omega| \le C|\nabla^2 u| + C|\nabla u|^2,\ etc. \tag{7}$$

Thus they also naturally inherit bounds in appropriate Morrey norms.

As in [10] we interpret the 1-form $\Omega \in H^1(B; \wedge^1\mathbb{R}^n \otimes so(n))$ arising in equation (6) as a connection in the $SO(n)$-bundle $u^*T\mathbb{R}^n$. An extension of the classical result of Uhlenbeck [15] on the existence of Coulomb gauges for connections in Morrey spaces, due to Meyer-Rivière [8], Theorem I.3, and Tao-Tian [14], Theorem 4.6, then permits to find a gauge transformation P and an $(m-2)$-form ξ satisfying

$$dPP^{-1} + P\Omega P^{-1} = *d\xi \text{ on } B \tag{8}$$

with natural Morrey bounds, thereby transforming Ω into Coulomb gauge. Applying the gauge transformation P to Δu, we obtain the gauge-equivalent form

$$\Delta(P\Delta u) = div^2(D_P \otimes \nabla u) + div(E_P \cdot \nabla u) + G_P \cdot \nabla u + *d\Delta\xi \cdot P\nabla u \tag{9}$$

of equation (6), where now $|D_P| \le C(|\nabla u| + |\nabla P|)$, etc.

We regard (9) and (8) as a coupled system of equations for u and P.

On any ball $B_R(x_1) \subset B_{R_0}(x_0) \subset B_2(0)$ we split

$$P\Delta u = f + h, \tag{10}$$

where $\Delta h = 0$ in $B_R(x_1)$ and where $f\big|_{\partial B_R(x_1)} = 0$ in the weak sense. Similar to [3] and following also [10] in this regard, for numbers $1 < p < m/2 < q < m$ with

$1/p + 1/q = 1$ we proceed to estimate f by duality, and we obtain

(11)
$$R^{2p-m} \int_{B_R(x_1)} |f|^p \, dx \le C\varepsilon_1 \big(||\nabla u||^p_{L^{2p,m-2p}(B_R(x_1))} + ||\nabla P||^p_{L^{2p,m-2p}(B_R(x_1))} + ||\nabla u||^{2p}_{L^{4,m-4}(B_R(x_1))} + ||\nabla P||^{2p}_{L^{4,m-4}(B_R(x_1))} \big).$$

We can close the estimates by means of the following local version of a Gagliardo-Nirenberg type interpolation result, due to Adams-Frazier [1] with later refinements by Meyer-Rivière [8] and Strzelecki [13]. A similar result is stated in [16], Proposition 4.3.

Proposition 2. *For any $1 < s \le m/2$ there exists a constant C such that for any ball $B \subset \mathbb{R}^m$ and any $u \in L_2^{s,m-2s}(B)$ there holds*

$$||\nabla u||^2_{L^{2s,m-2s}(B)} \le C ||\nabla u||_{L^{1,m-1}(B)} \big(||\nabla^2 u||_{L^{s,m-2s}(B)} + ||\nabla u||_{L^{s,m-s}(B)} \big).$$

Finally, with the help of the Campanato estimates for harmonic functions, as in Giaquinta [5], proof of Theorem III.2.2, p.84 f., we derive a Morrey-type decay estimate

$$\int_{B_r(x_0)} |\nabla u|^p \, dx \le C r^{m-p+\alpha p} \tag{12}$$

for all $x_0 \in B_1(0)$ and all $0 < r < 1$ with uniform constants C and $\alpha > 0$. By Morrey's Dirichlet growth theorem then $u \in C^{0,\alpha}(B_1(0))$, as claimed.

Note that in view of (11) we need to improve the estimates for P along with the estimates for u; thus it is essential for our approach that the coefficient functions Ω smoothly depend on u. An interesting question, that we discussed with Tristan Rivière, is whether for a given $\Omega \in L_1^{2,m-4} \cap L^{4,m-4}(B, \wedge^1 \mathbb{R}^m \otimes so(n))$ in analogy with our results for (1) weak solutions $u \in L_2^{2,m-4}(B; \mathbb{R}^n)$ to the linear equation

$$\Delta^2 u^i = \Delta \Omega^{ij} \nabla u^j, \tag{13}$$

are Hölder continuous, provided u and Ω satisfy the analogue of (2).

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Minimizers of the Willmore functional under fixed conformal class

Reiner Schätzle
(joint work with E. Kuwert)

We prove the existence of a smooth minimizer of the Willmore energy in the class of conformal immersions of a given closed Riemann surface into $\mathbb{R}^n, n = 3, 4$, if there is one conformal immersion with Willmore energy smaller than a certain bound $\mathcal{W}_{n,p}$ depending on codimension and genus p of the Riemann surface. For tori in codimension 1, we know $\mathcal{W}_{3,1} = 8\pi$.

Some minimization problems in quantum mechanics

Elliott H. Lieb
(joint work with Rupert Frank, Robert Seiringer, Heinz Siedentop and Barry Simon)

ABSTRACT: Three examples are given, in order of historical development, of minimization problems in quantum mechanics arising from attempts to model the N–body Schrödinger equation by simpler energy functionals involving only densities. These simpler models are Thomas-Fermi theory, Hartree-Fock theory and the Müller density matrix functional theory.

1. Introduction

The triumph of quantum mechanics was the explanation of the fact that a negatively charged electron (with charge -1) does not fall into a positively charged nucleus (of charge $+Z$). If the nucleus is located at $\boldsymbol{R} \in \mathbb{R}^3$ the Hamiltonian (energy function) is $H = p^2 + V(x)$ with $V(x) = -Z/|x - \boldsymbol{R}|$. In classical mechanics the minimum of H is $-\infty$, obtained with $p = 0$, $x = \boldsymbol{R}$. Schrödinger's

Hamiltonian, instead, is an *operator* $H = -\Delta + V(x)$ acting on $L^2(\mathbb{R}^3)$. Thanks to Sobolev's inequality, $||\nabla\psi||_2 > C||\psi||_6$, one easily shows that $E := \inf\{(\psi, H\psi) : ||\psi||_2 = 1\} = -C'Z^2$. The minimizing $\psi(x) \sim \exp(-Z|x|/2)$.

The next problem to consider is a molecule comprised of many, say M, nuclei located at $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, ..., $\boldsymbol{R}_M$ in $\mathbb{R}^3$. For simplicity we shall assume here that they all have the same charge $+Z$. There are N electrons now and the Hilbert space is the tensor product $\mathcal{H} = \otimes^N L^2(\mathbb{R}^3)$. (To be precise it is $L^2(\mathbb{R}^3; \mathbb{C}^2)$, but this does not change the qualitative picture.) The Hamiltonian operator is now

$$H = -\sum_{j=1}^{N} \Delta_j + W(X)$$

where X denotes the electron coordinates $x_1, \ldots, x_N$ in $\mathbb{R}^3$ and

$$W(X) = \sum_{j=1}^{N} V(x_j) + \sum_{1\le i<j\le N} |x_i - x_j|^{-1} + U$$

with U being a constant and with V being the potential given by

$$U = Z^2 \sum_{1\le i<j\le M} |\boldsymbol{R}_i - \boldsymbol{R}_j|^{-1} \qquad V(x) = -Z\sum_{j=1}^{M} |x - \boldsymbol{R}_j|^{-1}.$$

These terms constitute the total electrostatic potential given by Coulomb's law. Again $E := \inf\{(\psi, H\psi) : ||\psi||_2 = 1\}$

The proof of the finiteness of E is the same as for an atom, but we would like to know that when N and M are large, E is bounded below by $E > -A(Z)(N + M)$ for some Z–dependent constant A. This fact, known as *Stability of Matter*, was proved in 1967 by Dyson and Lenard [1] and is absolutely essential if we are to explain the real world. A much simpler proof was given in 1975 in [2]. It is *not* true, however, unless we impose an additional condition on $\mathcal{H}$, namely $\mathcal{H} = \wedge^N L^2(\mathbb{R}^3) \subset \otimes^N L^2(\mathbb{R}^3)$. That is, ψ must be an antisymmetric function of its N arguments The key analytic fact at play in this (physical) $\mathcal{H}$ is the Lieb-Thirring inequality for any normalized, antisymmetric ψ

$$\sum_{j=1}^{N} ||\nabla\phi_j||_2^2 \ge C\int \rho_\psi(x)^{5/3} dx\ ,$$

where $\rho_\psi(x) = \gamma_\psi(x, x)$ is the electron *density* and

$$\gamma_\psi(x, y) := N\int_{\mathbb{R}^{3(N-1)}} \psi(x, x_2, \ldots, x_N)\psi^*(y, x_2, \ldots, x_N)dx_2 \ldots x_N\ .$$

is the electron *density matrix* (which depends on ψ). This inequality fails miserably without antisymmetry.

What we want is an accurate evaluation of E as a function of $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_M$ and then minimize E over all configurations of the $\boldsymbol{R}_j$. This project is of vast commercial importance (apart from its purely mathematical interest) because success means the ability to compute the shape of molecules for pharmaceuticals, etc. without trial and error in the laboratory. Large sums of money are obviously involved here.

Most attempts proceed by focusing on ρ or γ and forgetting about ψ. One posits the existence of some functional $\mathcal{E}$ of ρ or γ (depending parametrically on the $\boldsymbol{R}_j$) whose minimum is supposedly E. The condition imposed on ρ and γ is $\int \rho = \operatorname{trace}\gamma = N$ and $0 \leq \gamma \leq 1$ as an operator. The last condition comes from the antisymmetry of the underlying ψ.

History is replete with attempts to find accurate functionals. Some of them work well in some circumstances, but there is no universal functional yet.

2. Thomas–Fermi Theory

The first attempt to find E was invented independently by Thomas and Fermi one year after Schrödinger's quantum mechanics. In a modern formulation [3, 4]

$$\mathcal{E}^{TF}(\rho) = K \int_{\mathbb{R}^3} \rho(x)^{5/3} dx + \int_{\mathbb{R}^3} V(x)\rho(x)dx + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(x)\rho(y)|x-y|^{-1} dxdy + U\ .$$

$\mathcal{E}^{TF}$ leads, of course, to a convex minimization problem. It was thoroughly studied in[3, 4] and all essential properties are known. The most important conclusions are:

(1) There is a minimizing ρ if and only if $N \leq MZ$. This coincides well with what is expected for the Schrödinger equation.
(2) There is no binding, i.e., the configuration of the nuclei that minimizes the energy is one of infinite separation.
(3) Nevertheless, E^{TF} is asymptotically exact in Z, i.e., $E^{TF}/E \to 1$ as $Z \to \infty$.

We can conclude that TF theory is accurate on some points, but does not tell us anything about molecules.

3. Hartree–Fock Theory

Another approximation scheme that came shortly after Schrödinger's equation was developed by Hartree and Fock. One way to construct an antisymmetric ψ is to take N orthonormal functions of one variable, $\Phi = \phi_1, \ldots, \phi_N$ and then form a determinant:

$$\psi(x_1, \ldots, x_N) = (N!)^{-1/2} \det \phi_i(x_j)\big|_{i,j=1}^{N}\ .$$

For such a ψ we can compute $\gamma(x,y) = \sum_{i=1}^N \phi_i(x)\phi_i^*(y)$. We can also evaluate

$$\begin{aligned}(\psi,\, H\,\psi) &= \mathcal{E}^{HF}(\gamma)\\ &:= \operatorname{tr}(-\Delta + V(x))\gamma\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left[\gamma(x,x)\gamma(y,y) - |\gamma(x,y)|^2\right] |x-y|^{-1} dxdy + U.\end{aligned}$$

This defines a minimization problem (for the N O.N. ϕ_i). Since it is *not* convex there can be more than one minimizer. There is one if $N < MZ+1$ (see [5]), and it is known that there is no minimizer if $N > MZ + \mathit{constant}$ for large Z [6]. It is also known that this HF minimization problem is more accurate than TF, but it is difficult to get an accurate evaluation of the minimum since N functions are involved.

It was finally realized years later [7] that $\mathcal{E}^{HF}(\gamma)$ defines a variational problem in its own right. That is, if we ignore the N functions and simply try to minimize $\mathcal{E}^{HF}(\gamma)$ with respect to γ (but taking into account that $0 \le \gamma \le 1$ and $\operatorname{tr}\gamma = N$) then this infimum is precisely what one obtains by assuming that γ has N eigenvalues 1, and the rest zero — which is the original HF problem. The determinantal function ψ thus gives the minimizing γ for $\mathcal{E}^{HF}(\gamma)$. We shall call this version the HF–γ theory.

We have a minimization problem that is a little unusual in the calculus of variations. We minimize a functional of a function, γ of two variables, but the condition on this function is an eigenvalue condition on the function considered as an integral kernel.

4. Müller–Functional Theory

To remedy a defect of HF-γ theory M'üller proposed the following change. Replace the term $\Xi^{HF}(\gamma) = -\int\int |\gamma(x,y)|^2 |x-y|^{-1} dxdy$, the so called *exchange* energy, by the term $\Xi^{M}(\gamma) = -\int\int |\gamma^{1/2}(x,y)|^2 |x-y|^{-1} dxdy$; everything else remains the same. Here $\gamma^{1/2}$ means the square root in the operator sense, i.e., $\int \gamma^{1/2}(x,z)\gamma^{1/2}(z,y)dz = \gamma(x,y)$. This new minimization problem is obviously technically more complicated but it has one big advantage over the HF problem. It is convex!

The new exchange term $\Xi^{M}(\gamma)$ is convex because of the following operator inequality for operators $A \ge 0$. $A \to -\operatorname{tr} A^{1/2} L A^{1/2} L^{\dagger}$ is convex for any L. This was proved in [8] and generalized in [9] to the convexity of $A \to -\operatorname{tr} A^{p} L A^{1-p} L^{\dagger}$ for all $0 \le p \le 1$. It is the basis for the strong subadditivity of quantum entropy and other basic facts in quantum information theory.

This Müller minimization problem was studied in [10] where it was shown that a minimizer exists up to $N = MZ$; it is not known if there is a maximum N or not. Convexity helps, for it implies that the minimizer has a unique density $\rho(x) = \gamma(x,x)$. Nevertheless there are several interesting open problems in [10] and everyone is welcome to try to solve them.

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Some applications of variational problems related to Lorentzian geometry

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(joint work with M. Elkin, V. Khachaturov, D. Berend, S. Skiena, L. Sapir, H. Sarfati)

We consider several seemingly unrelated problems which arise in the analysis of various systems. The first problem concerns the use of express lines in a supermarket. The express line is defined by a bound on gthe number of items, this being related to the service time of the customer. We may extend this to all the lines in the supermarket, by setting bounds on the number of items which are allowed in each line. For example, the first line will handle up to 10 items, the next 11-20 items, the third, 21-27 items and so on. Such a line management policy was suggested in the context of computer servers by Mor Harchol Balter and her collaborators. The question is how to set the bounds so that the resulting system will be as efficient as possible, for example, will minimize the average waiting time of customers.

A second problem concerns airplane boarding. Given an airplane, with known leg room and number of passengers per row, how should the airline control the queueing of passengers. Should it allow first the passengers from the back of the airplane to board, following the passengerfs from the front, or perhaps it should allow windows passengers first, followed by middle and then aisle passengers? or perhaps some combination of these policies?

A third problem is related to the identification of horizontal cracks in a 2D or 3D picture. Mathematically, the question is the following, given a distribution of points in the plane or in space, how many ofthese points can reside on the graph of a Lipschitz function with constant 1. This question was originally considered by E. Arias-Castro, D. Donoho, X. Hou and C. Tovey.

A fourth problem concerns scheduling of I/O to a disk drive. Disk drives, such as the ones in a PC or laptop can handle many concurrent requests to read and write data. Due to mechanical motion, the order in which the requests are served has a considerable effect on the total service time (performance) of the disk drive. the question is to find the optimal order and to estimate the resulting service time. This problem was considered by M. Andrews, M. Bender and L. Zhang.

As it turns out all these problems are related to Lorentzian geometry and the answers are given in terms of variational problems. The process of passengers boarding an airplane and the process of optimally scheduling I/O to a disk are both described as a wave front propagating in a 2 dimensional Lorentzian domain. Optimizing the number of items allowed in each line of the supermarket leads to a variational problem which is very similar to the geodesic equation in a Lorentzian domain, while the Lipschitz graph problem is a discrete variant of a maximal hypersurface, the Lorentzian analogue of a minimal surface in Riemannian geometry.

In the talk we discussed all of the above examples and showed how the discrete setting leads to a nice interpretation of Einstein's law of geodesic motion as being probabilistically inevitable, assuming only causality.

Variational problems related to the area functional in the sub-Riemannian Heisenberg groups

Manuel Ritoré

(joint work with César Rosales)

The Heisenberg group $\mathbb{H}^m$ is the product $\mathbb{C}^m \times \mathbb{R}$ with the Lie group structure

$$(z,t) * (z',t') = (z+z', t+t' + \sum_{i=1}^{m} \operatorname{Im}(z_i \bar{z}_i')).$$

The Lie algebra of left-invariant vector fields is given by

$$X_j := \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \qquad Y_j := \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial t}, \qquad T := \frac{\partial}{\partial t}, \qquad j = 1, \ldots, m.$$

The *horizontal distribution* is given, at $p \in \mathbb{H}^m$, by $\mathcal{H}_p := \operatorname{span}\{(X_j)_p, (Y_j)_p : j = 1, ..., m\}$. It is not integrable since $[X_j, Y_j] = -2T$ for any j. We shall consider on $\mathbb{H}^m$ the left-invariant Riemannian metric g so that $\{X_j, Y_j, T : j = 1, ..., m\}$ is an orthonormal basis of tangent vectors at every point. Since the horizontal distribution is bracket-generating, Chow's Theorem [13] implies that any pair of points in $\mathbb{H}^m$ can be joined by a piecewise smooth horizontal curve. The *Carnot-Carathéodory* distance between p, $q \in \mathbb{H}^m$ is given by the infimum of the Riemannian length in $(\mathbb{H}^m, g)$ in the class of piecewise smooth horizontal curves joining p and q.

The Riemannian volume element dv_g associated to g coincides with the Haar measure of $\mathbb{H}^m$ and with the Lebesgue measure of $\mathbb{C}^m \times \mathbb{R}$, and it is taken as the standard volume in $\mathbb{H}^m$. Several notions of hypersurface area can be defined on $\mathbb{H}^m$, such as the Minkowski content associated to the Carnot-Carathéodory distance, the horizontal perimeter [12], [11], or the spherical Hausdorff measure. For C^2 hypersurfaces $\Sigma \subset \mathbb{H}^m$ they are all equivalent and coincide with the sub-Riemannian area given by

$$A(\Sigma) := \int_\Sigma |N_H| \, d\Sigma,$$

where N is a local unit normal to Σ in $(\mathbb{H}^m, g)$, N_H is the orthogonal projection over the horizontal distribution, and $d\Sigma$ is the Riemannian hypersurface area element in Σ.

Variational problems related to this sub-Riemannian area functional have been the object of recent intensive research. These include the *isoperimetric problem*, consisting on finding the minimizers of perimeter under a volume constraint, and problems of *Bernstein type*, concerning the classification of entire graphs over given planes which are critical points of area. The reader is referred to the monograph [6] for an exhaustive list of references. Interesting results on the *Plateau problem* for t-graphs in $\mathbb{H}^m$ can be found in [8].

The first step toward a complete comprehension of critical points of area is the computation of the first variation of area

Lemma ([19],[20]). *Let $\Sigma \subset \mathbb{H}^m$ be an immersed oriented C^2 hypersurface. Let U be a smooth vector field with compact support, and $\{\varphi_s\}_{s\in\mathbb{R}}$ the associated flow. Assume that* $\operatorname{div}_\Sigma \nu_H \in L^1_{loc}(\Sigma)$. *Then*

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(\Sigma)) = -\int_\Sigma \operatorname{div}_\Sigma \left(u\,(\nu_H)^\top\right) d\Sigma + \int_\Sigma u \, \operatorname{div}_\Sigma \nu_H \, d\Sigma. \tag{1}$$

Here ν_H is the horizontal unit normal to Σ, defined out of the singular set $\Sigma_0 := \{p \in \Sigma : T_p\Sigma = \mathcal{H}_p\}$, and $u := \langle U, N \rangle$. The function $\operatorname{div}_\Sigma \nu_H$, defined on $\Sigma - \Sigma_0$, is the *mean curvature* of Σ. We must remark that a related first variation formula for t-graphs was obtained in [8].

To characterize the critical points of area the Hausdorff dimension of the singular set Σ_0 must be estimated [2], [8]. It turns out that Σ_0 is of sufficiently low Hausdorff dimension in $\mathbb{H}^m$, $m > 1$, so that the first integral in (1) does not contribute to the first derivative of area. For $m = 1$ the structure of the singular set has been described by Cheng, Hwang, Malchiodi and Yang [7]: Σ_0 is composed of isolated points and singular C^1 curves, and the first integral in (1) has in fact a non trivial contribution to the first derivative of area. It follows that critical points of area in $\mathbb{H}^1$ have zero (or constant if there is a volume constraint) mean curvature, and the *characteristic curves must meet orthogonally the singular curves* [20].

Using this characterization, the ruling property of constant mean curvature surfaces in $\mathbb{H}^1$, and properties of Jacobi fields in $\mathbb{H}^1$, it has been recently proven

Alexandrov Theorem in $\mathbb{H}^1$ ([20]). *Let $\Sigma \subset \mathbb{H}^1$ be a compact connected embedded C^2 surface which is a critical point of area for any variation keeping constant the volume enclosed by Σ. Then Σ is congruent to a Pansu sphere.*

A Pansu sphere is the union of all minimizing segments of horizontal geodesics with the same non-vanishing curvature starting from a given point. A complete description of geodesics and Pansu's spheres can be found in [20, § 3]. This result gives a positive answer to Pansu's conjecture [15] assuming C^2 regularity of solutions. Some other partial results can be found in [10], [14], [18].

Improving the characterization of Cheng, Hwang, Malchiodi and Yang [7] of entire minimal xy-graphs, one gets the following Bernstein type result

Theorem ([20]). *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 entire t-graph which is a critical point of area for any variation with compact support. Then Σ is congruent either to a plane or to the hyperbolic paraboloid $t = xy$.*

We should mention that the regularity of critical points of area in the Heisenberg group is still an open question. Examples of area-minimizing t-graphs in $\mathbb{H}^1$ with Euclidean Lipschitz regularity have been given in [8], [17]. Existence and regularity results have been obtained in [8], [9], see also [16]. Regularity properties of $\mathbb{H}$-regular surfaces [1] in terms of regularity properties of the horizontal unit normal have been obtained by Bigolin and Serra-Cassano [3]. Strong regularity results for Euclidean Lipschitz viscosity solutions of the minimal surface equation have been obtained by Capogna, Citti and Manfredini [4], [5].

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Superquadratic curvature functionals and approximation of the Willmore energy

Tobias Lamm

(joint work with Ernst Kuwert and Yuxiang Li)

For an immersed, closed surface $f : \Sigma \to \mathbb{R}^n$ the Willmore functional is defined by

$$W(f) = \frac{1}{4} \int_\Sigma |H|^2 d\mu_g$$

where H denotes the mean curvature vector of f, g is the pull-back metric and μ_g is the induced area measure on Σ. By the Gauss equations and the Gauss-Bonnet theorem we have the equivalent expression

$$W(f) = \frac{1}{4} \int_\Sigma |A|^2 d\mu_g + 2\pi(1-q),$$

where A is the second fundamental form of the immersion and q denotes the genus of the surface Σ. We also define the functional F by

$$F(f) = \frac{1}{4} \int_\Sigma |A|^2 d\mu_g.$$

¿From the above we see that the critical points of F and W coincide and they are called Willmore surfaces. In the case $n = 3$ the Euler-Lagrange equation of both functionals is given by

$$\Delta_g H_S + |\overset{\circ}{A}|^2 H_S = 0,$$

where $\overset{\circ}{A} = A - \frac{1}{2}Hg$ is the tracefree second fundamental form and $H_S = H \cdot \nu$ is the scalar mean curvature.

One of the most ineresting properties of the Willmore functional is its conformal invariance. This means that for every Möbius transformation $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ and every immersion $f : \Sigma \to \mathbb{R}^n$ one has

$$W(\Psi \circ f) = W(f).$$

Another interesting fact is that for all closed immersions one has the estimate

$$W(f) \geq 4\pi \tag{1}$$

and equality is attained only by round spheres. This shows that the round spheres are the absolute minimizers of the Willmore functional among all closed immersions. A natural question which then arises is whether there exist minimizers of W with different genus?

To simplify the notation we define the numbers

$$\beta_q = \inf\{W(f) | f : \Sigma \to \mathbb{R}^n \text{ smooth, closed immersion, } \text{genus}(\Sigma) = q\}.$$

¿From (1) it follows that $\beta_0 = 4\pi$ and the famous *Willmore conjecture* states that $\beta_1 = 2\pi^2$, which is the energy of the Clifford torus. In 1993, Leon Simon [8] showed that β_1 is attained and that moreover the same is true for β_q $(q \geq 2)$ if a Douglas-type condition is satisfied. The Douglas-type condition was subsequently shown to be true by Bauer & Kuwert [1].

In this talk we want to outline a different proof of the existence of a minimizer for the Willmore functional for any genus. In order to do so we study regularizations F_p and W_p $(p > 2)$ of the Willmore energy. These functionals are defined by

$$F_p(f) = \frac{1}{4}\int_\Sigma (1 + |A|^2)^{\frac{p}{2}} d\mu_g \quad \text{respectively}$$
$$W_p(f) = \frac{1}{4}\int_\Sigma (1 + |H|^2)^{\frac{p}{2}} d\mu_g.$$

A similar approximation has previously been studied by Sacks & Uhlenbeck [7] in the case of the Dirichlet energy for maps from a Riemannian surface into a Riemannian manifold.

It is interesting to note that the functionals F_p and W_p arise naturally in the research area of image reconstruction and therefore their study is of independent interest.

The critical points of F_p and W_p are weak solutions of a fourth order nonlinear system of partial differential equations. Locally one can write these immersions as the graph of a function $u : D_1 \to \mathbb{R}^{n-2}$ with $u \in W^{2,p}$. The Euler-Lagrange equations, written in terms of the function u, become strictly elliptic and one can apply the standard difference quotient method to conclude that the graph functions are actually in $W^{3,2}$. Moreover, by using a higher integrability result of Bildhauer, Fuchs & Zhong [2], we show that $u \in C^2$. By reinserting this improved regularity into the difference quotient procedure we prove the Hölder continuity of the second derivatives of u and by standard Schauder theory we finally obtain the smoothness of critical points of F_p and W_p.

Next we show the existence of minimizers of the functionals F_p and W_p with prescribed genus. This result is obtained by using standard variational techniques once a compactness result for minimizing sequences has been obtained. In the case of F_p the desired compactness result has been proved by Langer [5]. He showed that for every $p > 2$ and every sequence of immersions $f_k : \Sigma \to \mathbb{R}^n$ with uniformly bounded energy $F_p(f_k) \leq c$ there exist diffeomorphims $\phi_k : \Sigma \to \Sigma$ such that $f_k \circ \Phi_k$ converges (modulo the choice of a subsequence) in C^1 to a limiting immersion $f_\infty : \Sigma \to \mathbb{R}^n$ with $F_p(f_\infty) \leq c$.

We prove the corresponding result for a sequence of immersions $f_k : \Sigma \to \mathbb{R}^n$ with uniformly bounded energy W_p and with Willmore energy $W(f_k) \leq 8\pi - \delta$ for some $\delta > 0$.

By connecting two-spheres with a catenoidal neck whose diameter tends to zero along the sequence one sees that one can not drop the bound on the Willmore energy in order to get the desired compactness property in the case of the functional W_p.

After having obtained the existence of critical points f_p of F_p and W_p with genus q we study their behavior as $p \to 2$. In a first step we prove the so called small energy estimates. These estimates show that if the second fundamental form of f_p is locally small in L^2 then we get a uniform local control on the $W^{3,2}$-norm of f_p. For the Willmore flow corresponding estimates have been obtained by Kuwert & Schätzle [3].

By a Pohozaev type argument it is easy to see that the area of the critical immersions is converging to zero as $p \to 2$. This means that in order to obtain a non-zero limit we have to rescale f_p such that the rescaled immersions f_p' have strictly positive and finite area. After having done this we conclude, by using the small energy estimates, that f_p' converges to a limiting Willmore immersion f^0 weakly in $W^{3,2}$ away from at most finitely many points. Around the singular points we perform a blow-up and show that it converges to a smooth, non-trivial and non-compact Willmore immersion f^1. By repeating this procedure finitely many times we can find all possible blow-up's f^i, $1 \leq i \leq m$.

In the case that the sequence of critical points f_p of F_p or W_p satisfies additionally $\liminf_{p\to 2} W(f_p) < 8\pi$ we show that the point removability results of Kuwert & Schätzle [4] and Rivière [6] apply and therefore the weak limit f is an analytic Willmore embedding. Moreover, after suitably inverting the blow-up's

f^i, $1 \le i \le m$, we get smooth, closed and non-trivial Willmore immersions $I(f^i)$, $1 \le i \le m$.

In summary we can state one of our main results as follows

Theorem 1. *Let Σ be a surface with genus q and let $f_p : \Sigma \to \mathbb{R}^n$ be a sequence of critical points of F_p respectively W_p with uniformly bounded energy. We assume moreover that $\liminf_{p\to 2} W(f_p) < 8\pi$. Then there exists a subsequence $p_k \to 2$ and an analytic Willmore embedding $f^0 : \Sigma_0 \to \mathbb{R}^n$ of a surface Σ_0 with genus q_0 such that $f_{p_k} \to f^0$ weakly in $W^{3,2}$ away from at most finitely many points. Moreover there are at most finitely many smooth, closed and non-trivial Willmore immersions $I(f^1) : \Sigma_1 \to \mathbb{R}^n$, ..., $I(f^m) : \Sigma_m \to \mathbb{R}^n$ with genus$(\Sigma_i) = q_i$ such that*

$$q = \sum_{i=0}^{m} q_i, \tag{2}$$

and

$$\liminf_{k\to\infty} F_{p_k}(f_{p_k}) \ge W(f^0) + \sum_{i=1}^{m} (W(I(f_i)) - 4\pi), \tag{3}$$

respectively

$$\liminf_{k\to\infty} W_{p_k}(f_{p_k}) \ge W(f^0) + \sum_{i=1}^{m} (W(I(f_i)) - 4\pi) \tag{4}$$

When restricting the attention to a sequence of minimizers of F_p respectively W_p we show that

$$\lim_{p\to 2} \beta_q^p = \beta_q < 8\pi, \tag{5}$$

where

$$\beta_q^p = \inf\{F_p(f) \text{ resp. } W_p(f) \,|\, f : \Sigma \to \mathbb{R}^n \text{ smooth, closed immersion, } \text{genus}(\Sigma) = q\}.$$

(5) with Theorem 1 and the Douglas condition of Bauer & Kuwert [1] we are then able to give a new proof of the existence of a minimizing Willmore surface with given genus.

Theorem 2. *Let $q \in \mathbb{N}_0$. Then there exists a smooth Willmore embedding $f : \Sigma \to \mathbb{R}^n$ with genus$(\Sigma) = q$ and $W(f) = \beta_q$.*

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From the Ginzburg-Landau Model to Vortex Lattice Problems

Sylvia Serfaty
(joint work with Etienne Sandier)

$$G_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + |h-h_{\text{ex}}|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2}, \tag{1}$$

in the asymptotic regime of $\varepsilon \to 0$. The main objects of interest are the vortices, i.e. zeroes of u with nonzero topological degree, studied via the vorticity

$$\mu(u,A) = \operatorname{curl}(iu, \nabla_A u) + \operatorname{curl} A. \tag{2}$$

In a first stage, we derived (see [2]) a mean-field model describing the optimal vorticity at leading order: Considering $h_{\text{ex}} = \lambda|\log\varepsilon|$ we established by a Γ-convergence approach that minimizers $(u_\varepsilon, A_\varepsilon)$ of G_ε satisfy

$$\mu(u_\varepsilon, A_\varepsilon)/h_{\text{ex}} \text{ converges to } \mu_\lambda, \text{ as } \varepsilon \to 0,$$

where $\mu_\lambda = -\Delta h_\lambda + h_\lambda$, and h_λ is the solution of the following minimization problem

$$\min_{h-1\in H_0^1(\Omega)} \frac{1}{2\lambda}\int_\Omega |-\Delta h + h| + \frac{1}{2}\int_\Omega |\nabla h|^2 + |h-1|^2. \tag{3}$$

This problem is in turn equivalent to an obstacle problem, and as a consequence, if h_{ex} is large enough (larger than a value $H_{c_1} = \lambda_0|\log\varepsilon|$) there exists a subdomain ω_λ of Ω such that

$$\mu_\lambda = m_\lambda \mathbf{1}_{\omega_\lambda}, \text{where } m_\lambda = 1 - \frac{1}{2\lambda}.$$

This mean field description tells us that the vortices tend to arrange uniformly in ω_λ but is insensitive to the pattern formed by vortices. This pattern is in fact, as we shall see, selected by the minimization of the next term in the asymptotic expansion of the energy as $\varepsilon \to 0$. The proof of this is achieved in this paper by a splitting of the energy that separates the leading order term from a remainder term, and then studying via Γ-convergence techniques the remainder term after blow up at the scale of the expected intervortex distance $1/\sqrt{h_{\text{ex}}}$.

Our main result can be stated in a rough way as

Theorem 1 ([3]). *Assume that*

$$\log|\log \varepsilon| \ll h_{\text{ex}} - H_{c_1}, \quad h_{\text{ex}} \ll 1/\varepsilon^2 \tag{4}$$

and that $(u_\varepsilon, A_\varepsilon)$ *is a minimizer of* G_ε. *Then, choosing* x_ε *at random in* ω_λ, *and letting* $\tilde{\mu}_\varepsilon$ *be the push-forward of* $\mu(u_\varepsilon, A_\varepsilon)$ *under the blow-up map* $x \mapsto (x - x_\varepsilon)\sqrt{m_\lambda h_{\text{ex}}}$, *we have that almost surely* $\tilde{\mu}_\varepsilon$ *converges weakly in the sense measures to a measure* $\tilde{\mu}_*$ *of the form*

$$\tilde{\mu}_* = \sum_{p\in\Lambda} 2\pi\delta_p,$$

where Λ *is a discrete subset of* $\mathbb{R}^2$, *and* $\tilde{\mu}_*$ *minimizes a certain function* W, *moreover,* $\min G_\varepsilon$ *may be computed up to* $o(h_{\text{ex}})$.

The interaction function W is a logarithmic type interaction between points in the plane. It is a sort of analogue of the renormalized energy W of [1] but for an infinite number of points in the whole plane. Let us give its precise definition. Given a function H in $\mathbb{R}^2$ such that

$$-\Delta H = 2\pi \sum_i \delta_{a_i} - 1. \tag{5}$$

Consider a family of cutoff functions $\{\chi_R\}_{R>0}$ such that there exists $C > 0$ such that for every $R > 0$,

(6) i) $\chi_R = 0$ outside B_R, ii) $\chi_R = 1$ in B_{R-C}, iii) $|\nabla \chi_R| \le C$ in $\mathbb{R}^2$.

We then define

$$W(H) = \liminf_{R\to\infty} \frac{W(H,\chi_R)}{|B_R|}, \tag{7}$$

where

$$W(H,\chi_R) = \lim_{\alpha\to 0}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_i B(a_i,\alpha)} \chi_R|\nabla H|^2 + \sum_i \chi_R(a_i)(\gamma + \pi\log\alpha)\right). \tag{8}$$

Here γ is a universal constant introduced in [1].

The renormalized energy W allows to distinguish among vortex patterns: indeed we show that if we consider perfect lattice configurations (with a fixed density) then W is uniquely minimized by the triangular lattice. This is, to our knowledge, the first rigorous justification of the Abrikosov triangular lattice in this regime. At least the triangular lattice is the best among perfect lattice configurations. The proof of this fact involves ingredients from number theory.

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Partial symmetry of solutions to semilinear elliptic equations via Morse index estimates

Tobias Weth
(joint work with Francesca Gladiali and Filomena Pacella)

In the talk I discussed a new type of symmetry results for classical solutions of the semilinear elliptic equation

$$-\Delta u = f(|x|, u) \quad \text{in } \Sigma, \tag{1}$$

see [1, 2]. Here Σ is a radially symmetric (bounded or unbounded) domain in $\mathbb{R}^N$, $N \geq 2$, so we consider the cases

$$\Sigma = B, \qquad \Sigma = A, \qquad \Sigma = \mathbb{R}^N, \qquad \Sigma = \mathbb{R}^N \setminus B,$$

where B denotes a ball and $A = \{x \in \mathbb{R}^N \;:\; r_1 < |x| < r_2\}$ denotes an annulus centered at the origin. We also require Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Sigma, \tag{2}$$

which are empty if $\Sigma = \mathbb{R}^N$. The nonlinearity $f : \overline{\Sigma} \times \mathbb{R} \to \mathbb{R}$ is (locally) a $C^{1,\alpha}$-function depending only on the radial space variable. Our aim is to derive symmetry properties from Morse index information. Recently there has been a growing interest in understanding the structure of the set of stable and finite Morse index solutions of semilinear elliptic equations. One reason for this is the fact that existence and multiplicity results based on variational methods often include Morse index information. To state our results I recall the following definitions.

Definition 2. *A function $u \in C(\overline{\Sigma})$ is said to be foliated Schwarz symmetric if there is a unit vector $p \in \mathbb{R}^N$, $|p| = 1$ such that $u(x)$ only depends on $r = |x|$ and $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$ and u is nonincreasing in θ.*

Hence a foliated Schwarz symmetric function is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis. Now let Q_u denote the quadratic form corresponding to a solution of (1) and (2), i.e.

$$Q_u(\psi) = \int_\Sigma \left[|\nabla\psi|^2 - V_u(x)\psi^2(x)\right] dx, \quad \psi \in C_0^1(\Sigma), \tag{3}$$

where $V_u(x) := f'(|x|, u(x))$. Here and in the following, f' stands for the derivative of f with respect to u.

Definition 3. *We say that a C^2-solution of (1) and (2)*

- *is stable if $Q_u(\psi) \geq 0$ for all $\psi \in C_0^1(\Sigma)$;*
- *has Morse index equal to $j \geq 1$ if j is the maximal dimension of a subspace X of $C_0^1(\Sigma)$ such that $Q_u(\psi) < 0$ for all $\psi \in X$.*

If Σ is bounded, the Morse index of u is precisely the number of negative eigenvalues of the operator $-\Delta + V_u \ : \ H^2(\Sigma) \cap H_0^1(\Sigma) \subset L^2(\Sigma) \ \to \ L^2(\Sigma)$.

Our first result is concerned with bounded radial domains.

Theorem 1. *Suppose that Σ is either a ball or an annulus in $\mathbb{R}^N$, and that $f(|x|, \cdot)$ or $f'(|x|, \cdot)$ is convex in u for every $x \in \Sigma$. Then every solution of* (1), (2) *with Morse index $j \leq N$ is foliated Schwarz symmetric.*

For bounded radial domains Σ, it is well known that – without any convexity assumption on f or f' – any *stable* solution of (1), (2) is radial. Easy examples show that this is not true any more in the case of unbounded domains unless additional assumptions on the asymptotic behavior of u are imposed. We have the following result:

Theorem 2. *Every bounded stable solution u of (1),(2) with $V_u \in L^\infty(\Sigma)$ and $|\nabla u| \in L^2(\Sigma)$ is radial.*

For higher Morse index solutions, we obtain foliated Schwarz symmetry under stronger decay assumptions.

Theorem 3. *Suppose that Σ is an unbounded radial domain and that $f(|x|, \cdot)$ or $f'(|x|, \cdot)$ is convex for every $x \in \Sigma$. Suppose furthermore that u is a bounded solution of (1),(2) with Morse index $j \leq N$, $V_u \in L^\infty(\Sigma)$ and such that*

$$u \in H_0^1(\Sigma) \qquad \text{or} \qquad u \in D_0^{1,2}(\Sigma) \text{ and } V_u \in L^{N/2}(\Sigma).$$

Then u is foliated Schwarz symmetric.

In the case where the nonlinearity f does not depend on x, the following nonexistence result can be deduced.

Theorem 4. *Suppose that Σ is an unbounded radial domain, $f = f(u)$ does not depend on x and f or f' is convex. Suppose furthermore that u is a bounded solution of (1),(2) with Morse index $j \leq N$, $V_u \in L^\infty(\Sigma)$ and such that*

$$u \in H_0^1(\Sigma) \qquad \text{or} \qquad u \in D_0^{1,2}(\Sigma) \text{ and } V_u \in L^{N/2}(\Sigma).$$

Then we have:

- *if $\Sigma = \mathbb{R}^N$, then u does not change sign.*
- *if $\Sigma = \mathbb{R}^N \setminus B$, then $u \equiv 0$.*

The assumption that f does not depend on x is crucial here. Indeed, for a large class of nonlinearities depending on $|x|$, sign changing solutions having Morse index two can be constructed in both cases $\Sigma = \mathbb{R}^N$ and $\Sigma = \mathbb{R}^N \setminus B$.
We conjecture that the decay assumptions on u in Theorems 3 and 4 can be replaced by the weaker assumption $|\nabla u| \in L^2(\Sigma)$. This is part of current work.

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The canonical shrinking soliton associated to a Ricci flow

Peter M. Topping

(joint work with Esther Cabezas-Rivas)

In 1982, Hamilton [2] introduced the study of Ricci flow, which evolves a Riemannian metric g on a manifold $\mathcal{M}$ under the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g(t)), \tag{1}$$

for t in some time interval $I \subset \mathbb{R}$.

In this talk, given a Ricci flow on a manifold $\mathcal{M}$ over a time interval $I \subset \mathbb{R}$, we introduce a second time parameter, and define a natural shrinking Ricci soliton on the space-time $\mathcal{M} \times I$.

Theorem 1. *Suppose $g(\tau)$ is a (reverse) Ricci flow – i.e. a solution of $\frac{\partial g}{\partial \tau} = 2\,\mathrm{Ric}(g(\tau))$ – defined for τ within a time interval $(a,b) \subset (0,\infty)$, on a manifold $\mathcal{M}$ of dimension $n \in \mathbb{N}$. Suppose $\Omega \subset\subset \mathcal{M}$, $I \subset\subset (a,b)$, and $N \in \mathbb{N}$ is sufficiently large to give a positive definite metric $\hat{g}$ on $\Omega \times I \subset \hat{\mathcal{M}} := \mathcal{M} \times (a,b)$ defined by*

$$\hat{g}_{ij} = \frac{g_{ij}}{\tau}; \qquad \hat{g}_{00} = \frac{N}{2\tau^3} + \frac{R}{\tau} - \frac{n}{2\tau^2}; \qquad \hat{g}_{0i} = 0,$$

where i, j are coordinate indices on the $\mathcal{M}$ factor, 0 represents the index of the time coordinate $\tau \in (a,b)$, and the scalar curvature of g is written as R.

Then up to errors of order $\frac{1}{N}$, the metric $\hat{g}$ is a gradient shrinking Ricci soliton on the higher dimensional space $\hat{\mathcal{M}}$:

$$\mathrm{Ric}(\hat{g}) + \mathrm{Hess}_{\hat{g}}\left(\frac{N}{2\tau}\right) \simeq \frac{1}{2}\hat{g}, \tag{2}$$

by which we mean that on $\Omega \times I \subset \hat{\mathcal{M}}$, the quantity

$$N\left[\mathrm{Ric}(\hat{g}) + \mathrm{Hess}_{\hat{g}}\left(\frac{N}{2\tau}\right) - \frac{1}{2}\hat{g}\right]$$

is bounded independently of N, with respect to any fixed metric on $\hat{\mathcal{M}}$.

We show how part of the existing theory of Ricci flow is encoded in our soliton.

This geometric construction was discovered by consideration of the theory of optimal transportation, and in particular by reconciling the results of [5] and [3], and we discuss how this occured.

For more information, see [1].

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Continuity of optimal transport maps under a degenerate MTW conditioin

Young-Heon Kim

(joint work with Alessio Figalli and Robert McCann)

This abstract reports a work in progress regarding continuity of optimal transport maps for general cost functions. To focus on the presentation, our assumptions in the following are not necessarily the most optimal ones and we refer the reader to [3] for details.

Let Ω, $\bar\Omega$ be two smooth bounded domains in $\mathbb{R}^n$. Let $c \in C^4(\mathrm{cl}(\Omega \times \bar\Omega))$ be a cost function that governs the transportation between mass distributions $\rho, \bar\rho > 0$, $\rho \in L^\infty(\Omega)$, $\bar\rho \in L^\infty(\bar\Omega)$. That is, we seek for an optimal map $F : \Omega \to \bar\Omega$, which minimizes the total cost functional

$$\mathcal{C}(T) := \int_\Omega c(x, T(x))\rho(x)dx$$

among all Borel measurable maps $T : \Omega \to \bar\Omega$ with the highly nonlinear constraint

$$\int_\Omega f(T(x))\rho(x)dx = \int_{\bar\Omega} f(\bar x)\bar\rho(\bar x)d\bar x\,, \quad \forall f \in C_c^\infty(\bar\Omega).$$

We further assume that for all $(x, \bar x) \in \Omega \times \bar\Omega$, the maps

$$\bar x \in \bar\Omega \mapsto D_x c(x, \bar x) \in T_x^*\Omega, \quad x \in \Omega \mapsto D_{\bar x} c(x, \bar x) \in T_{\bar x}^*\bar\Omega$$

are one-to-one, and the mixed second-order derivative $D_x D_{\bar x} c$ is nondegenerate. By the results of pioneers including Brenier, Caffarelli, Carlier, Gangbo, Ma, McCann, Trudinger and Wang, there exists a unique optimal map F. Moreover, F satisfies in a weak sense

$$-D_x c(x, F(x)) = D_x u(x)$$

where the function u, called the c-potential, is given in pairs as

$$u(x) = \sup_{\bar x \in \bar\Omega} -c(x, \bar x) + \bar u(\bar x)$$
$$\bar u(\bar x) = \sup_{x \in \Omega} -c(x, \bar x) + u(x),$$

and u is Lipschitz continuous.

For the case $c(x,\bar{x}) = |x-\bar{x}|^2$, a successful regularity theory for F is known due to Caffarelli, Delanoë and Urbas. For general cost functions satisfying the above assumptions, a significant regularity theory for F is initiated by Ma, Trudinger and Wang. Their theory further requires on c that

$$MTW(p,\bar{p}) := (-c_{ij\bar{k}\bar{l}} + c_{ij\bar{a}}c^{\bar{a}b}c_{\bar{k}\bar{l}b})p^i p^j \bar{p}^{\bar{k}}\bar{p}^{\bar{l}} \geq 0$$

for all $p \in T_x\Omega$, $\bar{p} \in T_{\bar{x}}\bar{\Omega}$ with $p^i c_{i\bar{l}}\bar{p}^{\bar{l}} = 0$. Let $MTW^{\perp} \geq 0$ denote this condition, and further use $MTW^{\perp} > 0$ to denote the same condition but with strict inequality. Due to Loeper, $MTW^{\perp} \geq 0$ is known to be a necessary condition for the continuity of F. Namely, without this condition, there are smooth ρ, $\bar{\rho}$ on nice domains Ω, $\bar{\Omega}$, whose optimal transport map F is discontinuous. Moreover, it is shown [4] that $MTW(p,\bar{p})$ is the sectional curvature of the pseudo-Riemannian metric h on the product $\Omega \times \bar{\Omega}$ defined as

$$h := \frac{1}{2}\begin{pmatrix} 0 & -c_{i\bar{l}} \\ -c_{\bar{k}j} & 0 \end{pmatrix}.$$

That is, $MTW(p,\bar{p}) = R_h(p \oplus 0 \wedge 0 \oplus \bar{p}, p \oplus 0 \wedge 0 \oplus \bar{p})$ where R_h denotes the Riemann curvature tensor of the pseudo-metric h, and $p \oplus \bar{p}$ denotes the canonical decomposition of tangent vectors in the product $\Omega \times \bar{\Omega}$.

For the domains Ω, $\bar{\Omega}$, we assume that they are uniformly c-convex with respect to each other. That is, under the maps $x \mapsto D_{\bar{x}}c(x,\bar{x}) \in T^*_{\bar{x}}\bar{\Omega}$, $\bar{x} \mapsto D_x c(x,\bar{x}) \in T^*_x\Omega$ for all $x \in \Omega$, $\bar{x} \in \bar{\Omega}$, the image of Ω, $\bar{\Omega}$, respectively, is uniformly convex. Ma, Trudinger and Wang show that under $MTW^{\perp} \geq 0$, smooth mass distributions $\rho, \bar{\rho}$ induce a smooth optimal map F. With a stronger condition $MTW^{\perp} > 0$, Loeper obtain Hölder continuity of F for $\rho, \bar{\rho} \in L^\infty$ (in fact, even for ρ not necessarily absolutely continuous with respect to the Lebesgue measure). His theory is further extended by Delanoë, Figalli, Ge, Kim, Loeper, McCann, Rifford, Trudinger, Villani and Wang. Such a lower regularity of F with $\rho, \bar{\rho}$ merely in L^∞ is missing for the degenerate condition $MTW^{\perp} \geq 0$, except in two dimensions, where Figalli and Loeper verify continuity of F. In [3], we show interior Hölder continuity of F for $\rho, \bar{\rho} \in L^\infty$, under a slightly stronger but still degenerate condition that $MTW(p,\bar{p}) \geq 0$ for all $(p,\bar{p}) \in T_x\Omega \times T_{\bar{x}}\bar{\Omega}$, not necessarily satisfying $p^i c_{i\bar{l}}\bar{p}^{\bar{l}} = 0$. Let $MTW \geq 0$ denote this condition. Examples of cost functions satisfying this degenerate MTW condition (but not $MTW^{\perp} > 0$) include Riemannian distance squared on the product of standard round spheres and Euclidean space $S^{n_1} \times \cdots \times S^{n_k} \times \mathbb{R}^n$, and their Riemannian submersion quotients [5], for example, products of complex projective spaces $\mathbf{CP}^n \times \mathbf{CP}^m \times \cdots$. Note that there are cost functions that satisfy $MTW^{\perp} \geq 0$ but not $MTW \geq 0$; e.g. $c(x,\bar{x}) = |x-\bar{x}|^{-2}$ due to Trudinger. It is an interesting open question whether $MTW^{\perp} \geq 0$ implies $MTW \geq 0$ for the Riemannian distance squared cost.

One of our key observations in [3] is to use the map $x \mapsto q := D_{\bar{x}}c(x,\bar{x}_0)$ for fixed $\bar{x}_0 \in \bar{\Omega}$, to have a change of variables $c(x,\bar{x}) = \tilde{c}(q,\bar{x})$. For each $\bar{x} \in \bar{\Omega}$, the function $m_{\bar{x}}(q) := -\tilde{c}(q,\bar{x}) + \tilde{c}(q,\bar{x}_0)$ is convex under the assumption $MTW \geq 0$. (Under $MTW^{\perp} \geq 0$, $m_{\bar{x}}$ is level-set convex due to Loeper, Trudinger, Wang, Kim, McCann, Figalli and Villani. This level-set convexity is a key necessary property for the regularity theory of optimal transport maps.) Thus under the same change of variables, the modified potential function

$$\tilde{u}(q) := \sup_{\bar{x}\in\bar{\Omega}} -\tilde{c}(q,\bar{x}) + \tilde{c}(q,\bar{x}_0) + \bar{u}(\bar{x})$$

becomes a convex function. This then enables us to extend Caffarelli's theory [1, 2] for convex potentials of optimal maps for $c(x,\bar{x}) = |x-\bar{x}|^2$. Two of the most important difficulties we overcome, are (1) to get Alexandrov-type estimates for the convex $\tilde{c}$-potential $\tilde{u}(q)$ whose supporting functions $-\tilde{c}(q,\bar{x}) + \tilde{c}(q,\bar{x}_0) + \bar{u}(\bar{x})$ are not affine, and (2) to deal with the fact that the cost function c may not be defined and smooth on the whole $\mathbb{R}^n \times \mathbb{R}^n$. Our result is used by Liu, Trudinger and Wang [6] to show the optimal transport map F is $C^{1,\alpha}$ for $\rho, \bar{\rho} \in C^\alpha$.

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From interatomic potentials to Wulff shapes, via Gamma convergence.

Gero Friesecke

(joint work with Yuen Au Yeung and Bernd Schmidt)

1. Introduction

In this note we describe recent progress in the understanding of exact or approximate minimizers of N particles interacting via interatomic potentials as N tends to infinity.

In case of zero temperature, two dimensions, and short range pair potentials, we establish

1. formation of a cluster of constant density and finite perimeter
2. formation of a local crystal lattice structure.

Moreover in the special case of the Heitmann-Radin potential, we show

3. emergence of a well defined anisotropic "surface energy", in the sense of Gamma convergence
4. emergence of a unique overall geometric (Wulff) shape.

A key new idea compared to previous studies of discrete-to-continuum limits for crystals is not to try to parameterize particle positions x_j by displacements from some reference configuration (Lagrangian viewpoint), but to study the (appropriately re-scaled) empirical measure $\mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j/N^{1/d}}$ (Eulerian viewpoint). Point 1 can then be implemented as the mathematical statement that the weak* limit measure as the number of particles gets large has the structure of Lebesgue measure restricted to a set whose characteristic function belongs to BV. This forms the starting point for connecting atomistic energy minimization to a continuum surface energy problem.

2. Set-up

Our precise results are obtained in the following setting. We consider the following energy on $(\mathbb{R}^2)^N$:

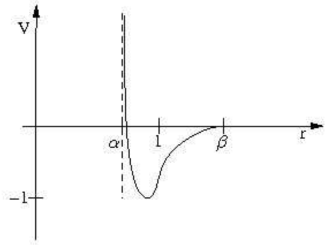
$$E(x_1, .., x_N) = \sum_{i\neq j} V(|x_i - x_j|),$$

where the pair potential V satisfies

(H1) (minimum at $r = 1$) $V(1) = -1$, $V(r) > -1$ for all $r \neq 1$

(H2) (behaviour at short and long range) There exist constants $\alpha \in (0, 1]$, $\beta \in [1, \infty)$ such that $V(r) = +\infty$ for $r < \alpha$, $V(r) = 0$ for $r > \beta$, V continuous on (α, β)

(H3) (narrow potential well) α, β are close to 1.



A finite set $S \subset \mathbb{R}^2$ of particle positions is called *connected* if for any two x, $y \in S$ there exist $x_0, .., x_N \in S$ such that $x_0 = x$, $x_N = y$, and $|x_j - x_{j-1}| < \beta$ (i.e., the nearest neighbours along the connecting chain always lie within the interaction range of the potential) for all $j = 1, .., N$. It is easy to show that minimizers are always connected. In case of disconnected configurations, our analysis can be applied separately to the connected components.

3. Clusters of finite perimeter

Our first result makes precise part 1 of the above program. Implicitly, its proof requires to also implement part 2.

Theorem 1. *Let $E : (\mathbb{R}^2)^N \to \mathbb{R}$ be a short range pair potential energy. Let $\{x_1^{(N)}, .., x_N^{(N)}\}$ be any sequence of connected N-particle configurations satisfying the energy bound*

$$E(\{x_1^{(N)} .., x_N^{(N)}\}) \le -6N + const\, N^{1/2}.$$

Let $\{\mu_N\}$ be the sequence of re-scaled empirical measures

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^{(N)}/\sqrt{N}}.$$

As $N \to \infty$, and up to subsequences,

$$\lim \mu_N = \rho\chi_{E},$$

where $\rho = \frac{2}{\sqrt{3}}$ and E is some set of finite perimeter of volume $\frac{\sqrt{3}}{2}$.

Here the limit is taken in the sense of weak* convergence of Radon measures. Note that he assumptions of the theorem allow atoms to "do what they like" in regions of size $N^{1/4}$.

4. Rigorous interfacial energy result

Parts 3 and 4 of the above program require us to restrict to the case of the Heitmann-Radin potential

$$V(r) = \begin{cases} +\infty, & 0 \le r < 1 \\ -1, & r = 1 \\ 0 & r > 1. \end{cases}$$

We start by re-writing the atomistic energy of a particle configuration in terms of the associated empirical measure,

$$I_N(\mu) = \begin{cases} +\infty, & \mu \ne \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i/\sqrt{N}} \text{ with } x_i \in \mathcal{L} \\ E(x_1, .., x_N), & \text{otherwise,} \end{cases}$$

We then subtract the leading order (bulk) term $e \cdot N$, where $e := \lim_{N\to\infty} \frac{\inf I_N}{N}$, and extract a rigorous surface energy functional, via Gamma convergence:

Theorem 2. *$N^{-1/2}(I_N - e \cdot N)$ Gamma-converges to the Wulff/Herring type functional*

$$I(\mu) := \begin{cases} +\infty, \mu \ne \frac{2}{\sqrt{3}}\chi_E \text{ for some set } E \text{ of finite per. and mass } \frac{\sqrt{3}}{2} \\ \int_{\partial^* E} \Gamma(\nu_E) dH^1(x), \text{otherwise,} \end{cases}$$

where Γ is the $2\pi/6$-periodic function

$$\Gamma(\nu) = 2\left(\nu_2 - \frac{\nu_1}{\sqrt{3}}\right) \text{ for } \nu = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix},\ \varphi \in [0, \tfrac{2\pi}{6}].$$

As a corollary, we obtain that the rescaled empirical measures of atomistic minimizers must converge to Lebesgue measure restricted to a regular hexagon. This follows from the uniqueness theorem of minimizers of Wulff/Herring type energies, due to Taylor (in a geometric measure theory setting) and to Fonseca and Müller (in a, for our purposes more convenient, BV setting).

Finally, we remark that atomistic minimizers are trivially non-unique, because one can re-arrange surface atoms. However, the natural conjecture that non-uniqueness is only due to surface atoms (of which there are $O(N^{1/2})$) is false. The true amount of non-uniqueness in much bigger. We prove that for infinitely many choices of the particle number N, one can re-arrange $O(N^{3/4})$ atoms. This scaling is sharp, that is to say we establish a matching upper bound for the number of atoms one can re-arrange.

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Quasistatic crack growth in elasto-plastic materials

Gianni Dal Maso

(joint work with Rodica Toader)

We present here a variational model for the quasistatic evolution of an elasto-plastic body with cracks. The variational approach to fracture mechanics [1] is based on Griffith's idea [9] that the crack growth is determined by the competition between the elastic energy released when the crack grows and the energy dissipated to produce new crack. Except for the models where the crack path is prescribed, all mathematical results obtained so far in this framework deal with the case of brittle fracture (see [8], [4], [2], [5], [7], [3]).

Our model is developed in *dimension two*, assuming an *a priori bound on the number of connected components of the cracks*, which are represented by one dimensional closed sets. The plastic behaviour is described within the framework of the *linearized theory of small strain elasto-plasticity*.

The *reference configuration* is a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Given a positive integer m, the set of *admissible cracks* $\mathcal{C}_m$ is defined as

$$\mathcal{C}_m := \{C : C \text{ is closed, } C \subset \overline{\Omega},\ \mathcal{H}^1(\Gamma) < +\infty\},$$

where $\mathcal{H}^1$ is the *one dimensional Hausdorff measure*. For every $\Gamma \in \mathcal{C}_m$ the *displacement* $u\colon \Omega\backslash\Gamma \to \mathbb{R}^2$ belongs to $L^1_{loc}(\Omega\backslash\Gamma;\mathbb{R}^2)$ and its symmetrized gradient Eu is defined by $Eu := \frac{1}{2}(\nabla u + \nabla u^t)$.

The linearized strain Eu is *additively decomposed* as $Eu = e + p$, where e is the *elastic part* and p is the *plastic part* of the strain. The *stress* σ depends only on the elastic part of the strain and is given by $\sigma := \mathbb{C}e$, where $\mathbb{C}$ is the *elasticity tensor*. The stress must satisfy the constraint $\sigma(x) \in \mathbb{K}$ for a.e. $x \in \Omega \setminus \Gamma$, where $\mathbb{K}$ is a prescribed bounded closed convex subset of the space $\mathbb{M}^{2\times 2}_{sym}$ of 2×2 symmetric matrices, whose boundary $\partial\mathbb{K}$ plays the role of the *yield surface*.

The *boundary condition* on $\partial\Omega \setminus \Gamma$ is prescribed by assigning a function $w \in H^1(\Omega \setminus \Gamma;\mathbb{R}^2)$. The weak formulation of the problem requires the use of the space $M_b(\overline{\Omega} \setminus \Gamma;\mathbb{M}^{2\times 2}_{sym})$ of *bounded Radon measures* on $\overline{\Omega} \setminus \Gamma$ with values in $\mathbb{M}^{2\times 2}_{sym}$. Given

w and Γ, the set $A(w,\Gamma)$ of *admissible strains* with prescribed boundary condition w and crack Γ is defined as the set of all pairs (e,p), with

$$e \in L^2(\Omega \setminus \Gamma; \mathbb{M}^{2\times 2}_{sym}) \quad \text{and} \quad p \in M_b(\overline{\Omega} \setminus \Gamma; \mathbb{M}^{2\times 2}_{sym}),$$

for which there exists $u \in L^1_{loc}(\Omega \setminus \Gamma; \mathbb{R}^2)$, with $Eu \in M_b(\overline{\Omega} \setminus \Gamma; \mathbb{M}^{2\times 2}_{sym})$ and trace $u \in L^1(\partial\Omega \setminus \Gamma; \mathbb{R}^2)$, which satisfies the *kinematic admissibility conditions*

$$\begin{cases} Eu = e + p \quad \text{in } \Omega \setminus \Gamma & \textit{(additive decomposition)}, \\ p = (w-u) \odot \nu\, \mathcal{H}^1 \quad \text{in } \partial\Omega \setminus \Gamma & \textit{(relaxed boundary condition)}, \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$ and $\odot$ is the *symmetrized tensor product*. The prescribed boundary condition $u = w$ is relaxed in the usual way: it is satisfied only on $\partial\Omega \setminus (\Gamma \cup \mathrm{supp}(p))$.

The *stored elastic energy* $\mathcal{Q}(e)$ is the quadratic form defined by

$$\mathcal{Q}(e) := \int_{\Omega\setminus\Gamma} \sigma(x) : e(x)\, dx = \int_{\Omega\setminus\Gamma} \mathbb{C}e(x) : e(x)\, dx,$$

where the colon denotes the Euclidean scalar product in $\mathbb{M}^{2\times 2}_{sym}$. To define the dissipative terms, we introduce the *support function* H of $\mathbb{K}$, defined by

$$H(\xi) := \sup_{\sigma\in\mathbb{K}} \sigma : \xi \qquad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}\,.$$

For every $p \in M_b(\overline{\Omega} \setminus \Gamma; \mathbb{M}^{2\times 2}_{sym})$ we consider the measure $H(p)$ on $\overline{\Omega} \setminus \Gamma$ defined by

$$H(p)(B) := \int_B H(\frac{dp}{d|p|}(x))\, d|p|(x),$$

for every Borel set $B \subset \overline{\Omega} \setminus \Gamma$, where $\frac{dp}{d|p|}$ is the Radon-Nikodym derivative of p with respect to its variation $|p|$.

Given two cracks $\Gamma_1 \in \mathcal{C}_m$ and $\Gamma_2 \in \mathcal{C}_m$, with $\Gamma_1 \subset \Gamma_2$, and two plastic strains $p_1 \in M_b(\overline{\Omega} \setminus \Gamma_1; \mathbb{M}^{2\times 2}_{sym})$ and $p_2 \in M_b(\overline{\Omega} \setminus \Gamma_2; \mathbb{M}^{2\times 2}_{sym})$, the *dissipation distance* $\mathcal{D}((p_2,\Gamma_2),(p_1,\Gamma_1))$ represents the energy dissipated during the transition from (p_1,Γ_1) to (p_2,Γ_2). It is given by

$$\mathcal{D}((p_2,\Gamma_2),(p_1,\Gamma_1)) := H(p_2 - p_1)(\overline{\Omega} \setminus \Gamma_2) + \mathcal{H}^1(\Gamma_2 \setminus \Gamma_1).$$

The former term is the usual *plastic dissipation* in the region $\overline{\Omega} \setminus \Gamma_2$, while the latter is the *energy spent to produce the new crack* $\Gamma_2 \setminus \Gamma_1$. If the inclusion $\Gamma_1 \subset \Gamma_2$ does not hold, we set $\mathcal{D}((p_2,\Gamma_2),(p_1,\Gamma_1)) = +\infty$. This reflects the *irreversibility* of crack growth. The total *dissipation* in the time interval $[0,T]$ is defined by

$$\mathrm{Diss}_{\mathcal{D}}(p,\Gamma;0,T) := \sup \sum_{i=1}^{k} \mathcal{D}((p(t_i),\Gamma(t_i)),(p(t_{i-1}),\Gamma(t_{i-1}))),$$

where the supremum is over all partitions $0 = t_0 < t_1 < \cdots < t_k = T$. Of course, if $\mathrm{Diss}_{\mathcal{D}}(p,\Gamma;0,T) < +\infty$, then $t \mapsto \Gamma(t)$ is increasing with respect to inclusion.

Given a crack $\Gamma_0 \in \mathcal{C}_m$ and a function $t \mapsto w(t)$ from $[0,T]$ into $H^1(\Omega \setminus \Gamma_0; \mathbb{R}^2)$, a *quasistatic evolution* with initial crack Γ_0 and boundary condition $w(t)$ is a function $t \mapsto (e(t), p(t), \Gamma(t))$, with $\Gamma(0) = \Gamma_0$ and $t \mapsto e(t)$ in $L^\infty([0,T]; L^2(\Omega \setminus \Gamma; \mathbb{M}^{2\times2}_{sym}))$, which satisfies the following conditions for every $t \in [0,T]$:

- *Global stability*: $(e(t), p(t)) \in A(w(t), \Gamma(t))$ and
$$\mathcal{Q}(e(t)) \le \mathcal{Q}(\hat{e}) + \mathcal{D}((\hat{p}, \hat{\Gamma}), (p(t), \Gamma(t)))$$
for every $\hat{\Gamma} \in \mathcal{C}_m$ and every $(\hat{e}, \hat{p}) \in A(w(t), \Gamma(t))$;
- *Energy balance*: setting $\sigma(t) := \mathbb{C}e(t)$, we have
$$\mathcal{Q}(e(t)) + \mathrm{Diss}_{\mathcal{D}}(p, \Gamma; 0, t) = \mathcal{Q}(e(0)) + \int_0^t \langle \sigma(s), E\dot{w}(s) \rangle \, ds,$$
where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega \setminus \Gamma(s); \mathbb{M}^{2\times2}_{sym})$.

In [6] we proved the following theorem.

Theorem 1 (Dal Maso-Toader 2008)**.** *Assume that the prescribed boundary condition $t \mapsto w(t)$ belongs to $AC([0,T]; H^1(\Omega \setminus \Gamma_0; \mathbb{R}^2))$ and that $(e_0, p_0) \in A(w(0), \Gamma_0)$. Then there exists a quasistatic evolution $t \mapsto (e(t), p(t), \Gamma(t))$ with initial crack Γ_0 and boundary condition $w(t)$, such that $e(0) = e_0$ and $p(0) = p_0$.*

Variations of e in the global stability condition lead to the *equilibrium conditions*
$$\mathrm{div}_x \sigma(t,x) = 0 \quad \text{in } \Omega \setminus \Gamma(t) \quad \text{and} \quad \sigma(t,x)\nu(x) = 0 \quad \text{on } \Gamma(t)$$
in the usual weak sense. Variations with $p = -e$ lead to the *stress constraint*
$$\sigma(t,x) \in \mathbb{K} \quad \text{for a.e. } x \in \Omega \setminus \Gamma(t).$$

The energy balance, together with the differential conditions obtained from the global stability, allows to prove the next theorem (see [6]) about the *flow rule*, that for $p(t,x)$ smooth is expressed by the classical inclusion
$$\dot{p}(t,x) \in N_{\mathbb{K}}(\sigma(t,x)) \quad \text{for } x \in \Omega \setminus \Gamma(t),$$
where $N_{\mathbb{K}}(\sigma)$ is the normal cone to $\mathbb{K}$ at σ.

Theorem 2 (Dal Maso-Toader 2008)**.** *For a.e. $t \in [0,T]$ the limits*
$$\frac{p(t) - p(t-h)}{h} \rightharpoonup \dot{p}(t) \quad \textit{weakly}^* \textit{ in } M_b(\overline{\Omega} \setminus \Gamma(t); \mathbb{M}^{2\times2}_{sym})$$
$$\frac{\mathcal{H}^1(\Gamma(t)) - \mathcal{H}^1(\Gamma(t-h))}{h} \to \dot{s}(t)$$
exist as $h \to 0+$. Let $t \in [0,T]$ be a Lebesgue point of $t \mapsto \|E\dot{w}(t)\|_{L^2}$ such that the above limits exist and $\sigma(t) \in C(\overline{\Omega} \setminus \Gamma(t); \mathbb{M}^{2\times2}_{sym})$. Then
$$\frac{d\dot{p}(t)}{d|\dot{p}(t)|}(x) \in N_{\mathbb{K}}(\sigma(t,x)) \quad \textit{for } |\dot{p}(t)|\textit{-a.e. } x \in \overline{\Omega} \setminus \Gamma(t).$$

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Local-to-global principles in Riemannian geometry and optimal transport

Cédric Villani

1. An analogy

Convexity in $\mathbb{R}^n$ has a local side and a global side. An example of local property is, $\nabla^2\Phi \geq \kappa\, I_n$ ($\kappa \in \mathbb{R}$); the corresponding global property is $\Phi((1-t)x+ty) \leq (1-t)\,\Phi(x)+t\,\Phi(y)-\kappa\, t(1-t)|x-y|^2/2$. Both formulations have their advantages: the local formulation is effective (can be checked in practice); the global formulation on the other hand is often useful, more general and more stable. Observe that it is not a priori obvious that the global property has a local reformulation.

2. Local-to-global principles for curvature

The main local-to-global principles associated with curvature bounds in Riemannian geometry are those associated with lower sectional curvature bounds, upper sectional curvature bounds, and lower Ricci curvature bounds. As pointed out by G. Huisken at the end of my talk, there is also an important local-to-global principle associated with scalar curvature: this is the content of the mass gap theorem.

3. Sectional curvature bounds

Let (M,g) be a Riemannian manifold with its geodesic distance d. The inequality $\mathrm{Sect} \geq \kappa$ (sectional curvatures bounded by $\kappa \in \mathbb{R}$) can be seen as a local inequality on the geometry of M: if u, v are two unit tangent vectors at x, and $\ell(t) = d(\exp_x(tu), \exp_x(su))$, then the inequality amounts to

$$\ell(t) \leq \sqrt{2(1-\cos\theta)}\, t \left(1 - \frac{\kappa \cos^2(\theta/2)}{6}\, t^2 + O(t^4)\right).$$

There is also an equivalent inequality in the large: for $t \in [0,1]$, $\delta > 0$, $L > 0$, define

$$\mathcal{D}(t,\delta,L) = \inf\Big\{ d(\exp_x(tu), \exp_x(tv));\ |v| = |w| = \delta;\ d(\exp_x u, \exp_x v) = L \Big\}$$

then $\mathcal{D}(t,\delta,L) \geq \mathcal{D}^{(\kappa)}(t,\delta,L)$, where $\mathcal{D}^{(\kappa)}$ is the $\mathcal{D}$ function for the reference space with constant sectional curvature κ. This reformulation is at the basis of the Toponogov theorem.

For upper sectional curvature bounds, the story is the same, reversing all inequalities, except that now one needs an additional assumption of simple connectedness.

4. Ricci curvature bounds

The Ricci curvature bound $\mathrm{Ric} \geq K\, g$ of a Riemannian manifold, or more generally the curvature-dimension condition $\mathrm{CD}(K,N)$ for a measured Riemannian manifold, also has an interpretation in terms of local geometry, expressed either via the Bochner formula or equivalently via estimates on the Jacobian of the Riemannian exponential of a symmetric vector field. Let me recall that $\mathrm{CD}(K,N)$ means $\mathrm{Ric}_{N,\nu} \geq K\, g$, where $\mathrm{Ric}_{N,\nu} = \mathrm{Ric} - (\nabla V \otimes \nabla V)/(N-n) + \nabla^2 V$, where ν is the reference measure (absolutely continuous with respect to the volume measure) and $V = -\log(d\nu/d\mathrm{vol})$.

The discussion is similar to the one for sectional curvatures, with now the function

$$\mathcal{J}_{M,\nu}(t,\delta,J) = \inf\Big\{ \mathrm{Jac}_x(\exp t\nabla\psi);\ |\nabla\psi(x)| = \delta,\ \mathrm{Jac}_x(\exp \nabla\psi) = J \Big\},$$

where ν is the reference measure on M, the (geometric) Jacobian is defined with respect to this reference measure, and ψ is a C^2 function defined in the neighborhood of x, such that there $\mathrm{Jac}_x(\exp(t\nabla\psi))$ stays positive for all $t \in [0,1]$ (no focalization).

However, this is not really global, since ψ is defined only locally!

5. Globalization via optimal transport

Optimal transport canonically produces gradients of globally defined (alas, not smooth) gradients, such that there is no focalization. This can be seen from the following theorem of McCann: *On (M,g) compact let $\mu_0(dx) = f(x)\,\mathrm{vol}(dx)$ and $\mu_1(dy)$ be two probability measures, then the solution of the variational problem*

$$C(\mu_0,\mu_1) = \inf_{T_\#\mu_0 = \mu_1} \int d(x, T(x))^2\, \mu_0(dx)$$

is given by $T = \exp(\nabla\psi)$, where ψ is a semiconvex function $M \to \mathbb{R}$.

Then the formula $T_{\#}\mu_0 = \mu_1$ gives a way to express the Jacobian in terms of μ_0 and μ_1 via the change of variables formula.

This leads to the possibility to define global reformulations of Ricci bounds, either via the convexity of certain functionals along minimizing curves for optimal transport, or via estimates of contraction rates along the heat equation, in the distance of optimal transport. A very incomplete list of people who have worked in this direction over the past years are Cordero-Erausquin, Lott, McCann, Otto, Schmuckenschläger, Sturm, and myself.

The simplest example is the following: $\mathrm{Ric} \geq 0$ if and only if for all $t \in [0,1]$ and for any curve $(\mu_t)_{0\leq t\leq 1}$ defined from McCann's theorem by $\mu_t = (\exp(t\nabla\psi))_{\#}\mu_0$, one has

$$H(\mu_t) \leq (1-t)\, H(\mu_0) + t\, H(\mu_1),$$

where $H(\mu) = \int \rho \log \rho \, d\mathrm{vol}$, $\rho = d\mu/d\mathrm{vol}$. There is another statement according to which $\mathrm{Ric} \geq 0$ if and only if the heat equation is a (possibly nonstrict) contraction in the space of probability measures, equipped with the distance $W_2(\mu_0, \mu_1) = \sqrt{C(\mu_0, \mu_1)}$.

6. Applications

The global reformulation of Ricci curvature bounds makes sense in terms of metric and measure only. This leads to

• more generality. For instance, for any norm on $\mathbb{R}^N$, the metric-measure space $(\mathbb{R}^N, \|\cdot\|, \lambda_N)$ satisfies $\mathrm{CD}(0,N)$ ($\mathrm{Ric} \geq 0$, dimension $\leq N$), although this is not a Riemannian manifold (neither an Alexandrov space) if the norm is non-Hilbert. This might seem shocking, but after all these spaces have common geometric features, for instance they share the same Sobolev inequalities.

• stability. For instance, the geodesic reformulation is very well adapted to the Gromov–Hausdorff convergence. Using these techniques, Lott, Sturm and myself established the following theorem (as a particular case of more general results): If a sequence of compact Riemannian manifolds (M_k, g_k, e^{-V_k}) converges in measured Gromov–Hausdorff topology to a Riemannian manifold (M, g, e^{-V}), and any M_k satisfies $\mathrm{CD}(K,N)$, then also M satisfies the same bound.

• robustness under discretization or other perturbation. Using these formulations we can easily cook up definitions of Ricci curvature bounds for discrete spaces, using approximate geodesics in replacement for geodesics, or Markov chains instead of heat flows. In this way Ollivier could prove that the Ricci curvature of the hypercube $\{0,1\}^N$, at scale $O(1)$, is bounded below by const.$/N$ as $N \to \infty$. Motivations for such a study come from statistical mechanics.

• a remarkable algebra with links to the Ricci flow. For instance, part of Perelman's work in this topic(such as the monotonicity formula for the reduced volume) can be reinterpreted in terms of optimal transport, as was done by Lott and Topping.

7. Yet another local-to-global principle

Recent research by Delanoë, Figalli, Ge, Kim, Loeper, McCann, Rifford, Trudinger and myself has focused on the Ma–Trudinger–Wang tensor: for $(x, y) \in M \setminus \mathrm{cut}(M)$, $\xi \in T_xM$, $\eta \in T_yM$ define in coordinates

$$\mathfrak{S}_{(x,y)}(\xi, \eta) = \sum_{ijk\ell rs} \Big(c_{ij,r} c^{r,s} c_{s,k\ell} - c_{ij,k\ell} \Big) \, \xi^i \, \xi^j \, \eta^k \, \eta^\ell,$$

where $c(x, y) = d(x, y)^2/2$, and $c_{ij,r} = \partial^3 c / \partial x^i \partial x^j \partial y^r$, etc.

Assuming that all tangent injectivity loci on M are convex, there is equivalence between the local condition ($\mathfrak{S} \geq 0$ if $c_{i,j} \xi^i \eta^j = 0$), and the global condition

$$\forall x, \overline{x}, y_0, y_1 \;:\; d(x, y_t)^2 - d(\overline{x}, y_t)^2 \geq \min\big[d(x, y_0)^2 - d(\overline{x}, y_0)^2, \; d(x, y_1)^2 - d(\overline{x}, y_1)^2 \big],$$

where $y_t = \exp_{\overline{x}}((1 - t)(\exp_{\overline{x}})^{-1}(y_0) + t(\exp_{\overline{x}})^{-1}(y_1))$.

This story is not unrelated to the previous one. First, bounds on the Ma–Trudinger–Wang tensor also enjoy stability under Gromov–Hausdorff convergence (although this tensor is of fourth order in the Riemannian metric!). Secondly, it is intimately related to optimal transport theory; in fact, adequate positivity conditions on this tensor come close to be equivalent to the smoothness of the function ψ appearing in McCann's theorem, for arbitrary measures μ_0 and μ_1 with smooth positive densities.

A last word on the assumption of convexity of the tangent injectivity loci: I conjecture that it follows automatically from the Ma–Trudinger–Wang condition. Together with Loeper, I have proved some partial results in this direction, but the general case remains to be done. This in my opinion is remarkable, since the cut locus is a quite mysterious object. Research in this subject is currently advancing with works by Figalli, Rifford, myself and others.

8. References

The main reference for this talk is my book *Optimal transport, old and new* (which grew up of notes for the 2005 Saint-Flour summer school), to appear in *Grundlehren der mathematischen Wissenschaften*; precise statements and references can be found there. One may also consult the proceedings *Optimal transport and curvature*, from the 2008 CIME Course in Cetraro, *Nonlinear partial differential equations and applications.*

Measuring the Geodesic Radon Transform with Mass Transport

Benjamin K. Stephens

In tomography one extracts averages of an input density field along slices. If an input density with sharp features is disturbed by local and global displacements, how does this affect the slice averages? We use as model the geodesic Radon transform R, acting on spherical measures as the dual of the map

$$R_0[f](u) = \frac{1}{|E(u)|} \int_{E(u)} f, \qquad f \in C^0(S^{n-1}).$$

Here $E(u)$ is the equator of u, composed of all points geodesic distance $\pi/2$ from u. Restricting to probability measures $\mathcal{P}(S^{n-1})$, we have available the Wasserstein-p distance

$$d_{W_p}(\mu,\nu) = \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{S^{n-1} \times S^{n-1}} d_{S^{n-1}}(x,y)^p \, d\gamma(x,y) \right)^{1/p}$$

for $p \in [1,\infty)$. Here $\Gamma(\mu,\nu)$ is the set of all transference plans $\gamma \in \mathcal{P}(S^{n-1} \times S^{n-1})$ which push forward to $\mu = \pi^1_{\#}\gamma$ and $\nu = \pi^2_{\#}\gamma$ via the projections π^1, π^2 to the respective sphere factors. This is a natural way to measure displacement type errors; note that a translating delta measure $\delta_{P(t)}$ moves continuously in Wasserstein-p distance, but immediately jumps distance 2 away from $\delta_{P(0)}$ in total-variation. Our conclusion is that R is a contraction, with Lipschitz constant that is sharp, less than one, and given by a simple trigonometric formula:

$$C(p,n) = \left(\frac{\int_0^{\frac{\pi}{2}} (\sin t)^{n-3} (\cos t)^p \, dt}{\int_0^{\frac{\pi}{2}} (\sin t)^{n-3} \, dt} \right)^{1/p}.$$

One consequence is a new range criterion for R: If $\mu \in \mathcal{P}(S^{n-1})$ satisfies $\mu = R[\nu]$ for $\nu \in \mathcal{P}(S^{n-1})$ then for all $p \geq 1$,

$$d_{W_p}(\mu, Z) \leq C(p,n) D(p,n)$$

where Z is the uniform probability measure and

$$D(p,n) = \sup_{\mu \in \mathcal{P}(S^{n-1})} d_{W_p}(\mu, Z) \leq \pi.$$

The contraction constant $C(p,n)$ appears as the rate at which the equator measure $R[\delta_{p(t)}]$ of a delta measure $\delta_{p(t)}$ moves away from the equator measure $R[\delta_{p(0)}]$ when $\delta_{p(t)}$ moves at speed 1 and all distances are measured in Wasserstein-p. The speaker would like to thank Elliott Lieb for pointing out that this result may be seen as a manifestation of the principle that convolution by a fixed measure causes Wasserstein distances to contract. (See for example Villani, [4, Chapter 7], for the case of normed vector spaces.) Note that continuity questions for R (and inverse R) have been extensively studied with respect to L^p type norms; we cite for the Euclidean Radon transform [3] and [1], also the work [2] with respect to weak-topology-metrizing distances that are not Wasserstein. A question for further study is whether there is a uniform inequality in the other direction — a factor $C'(p,n)$ which is the most that Wasserstein-p distance may contract under R.

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Conservation laws for 2-dimensional conformally invariant variational problems

Tristan Rivière

The presentation which took place on july 11th 2008 at 10:30 am was divided in three parts.

In the first one we proved a compensation compactness result satisfied by sequences of solutions to elliptic linear systems of the form

$$-\Delta u = \Omega \cdot \nabla u \quad , \tag{1}$$

where u is a $W^{1,2}-$map from a 2-dimensional domain ω into $\mathbb{R}^n$ (n is an arbitrary integer) and Ω is an L^2 vector-field taking values into antisymmetric matrices and where we have used matrix-vector multiplication. Precisely (1) reads : $\forall i \in \{1 \cdots n\}$ $-\Delta u^i = \sum_{j=1}^n \Omega^i_j \cdot \nabla u^j$ where $\cdot$ is the scalar product between vectorfields in the 2-dimensional domain. The compensation compactness result says then the following :

Theorem 1. [Ri1] *Let u_k and Ω_k be two uniformly bounded sequences in respectively $W^{1,2}(\omega, \mathbb{R}^n)$ and $L^2(\wedge^1\omega \otimes so(n))$. Assume that there exists f_k converging strongly to zero in $H^{-1}(\omega, \mathbb{R}^n)$ such that*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k + f_k \quad .$$

Then, modulo extraction of a subsequence, u_k and Ω_k converge weakly, respectively in $W^{1,2}$ and L^2, to a pair (u, Ω) which solves (1). □

This compensation compactness result is based on the discovery of conservation laws equivalent to (1) which can be written locally in divergence form. These conservation laws imply moreover the Hölder continuity $C^{0,\alpha}$ (for some $\alpha > 0$) of the solution u (see [Ri2]).

In the second part of the talk we show that the critical points $u \in W^{1,2}(\omega, N^k)$, to continuously differentiable conformally invariant lagrangians, with quadratic growth in the gradient of the map u - where ω is a 2-dimensional domain and where N^k is a closed submanifold of an euclidian space - satisfy elliptic systems of the form (1) for antisymmetric potentials Ω in L^2. We then deduce from the first part of the presentation the Hölder continuity of these critical points. This solves a conjecture posed by Stefan Hildebrandt in the seventies.

In the last part of the talk we present corresponding conservation laws for critical points to the Willmore Functional (see [Ri3]). From these conservation laws we deduce that Palais-Smale sequences to the *Conformal Willmore Equation* converge to solutions to the *Conformal Willmore Equation* see [BR].

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