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Real Analysis, Harmonic Analysis and Applications

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Abstract. In the last few years there have been important developments in the point of view and methods of harmonic analysis, and at the same time significant concurrent progress in the application of these to various other fields. The workshop has focused on harmonic analysis and its applications, such as dispersive PDE's, with particular emphasis to the interactions with ergodic theory.

Mathematics Subject Classification (2000): 42xx, 43xx, 44xx, 22xx, 35xx.

Introduction by the Organisers

This workshop, which continued the triennial series at Oberwolfach on Real and Harmonic Analysis that started in 1986, has brought together experts and young scientists working in harmonic analysis and its applications (such as to dispersive PDE's and ergodic theory) with the objective of furthering the important interactions between these fields.

Three prominent experts, Elon Lindenstrauss (Princeton), Amos Nevo (Technion, Haifa), and Terence Tao (UCLA), gave survey respectively introductory lectures. Their topics included "Effective equidistribution on the torus", "Non-Euclidean lattice point counting problems, and the ergodic theory of lattice subgroups," and "The van der Corput lemma, equidistribution in nilmanifolds, and the primes."

Major further areas and results represented at the workshop are:

- Application of Time Frequency analysis: this is an outgrowth of the method of "tile decomposition" which has been so successful in solving the problems of the bilinear Hilbert transform. Recent progress includes applications of these techniques and the theory of multilinear singular integral operators to ergodic theory and an extension of the celebrated Carleson-Hunt theorem to the "polynomial Carleson operator."

- Estimates for maximal functions: this includes recent progress on best weak $(1,1)$ constants for the Hardy-Littlewood maximal function on metric measure spaces, estimates for maximal functions associated to monomial polyhedra, with applications to sharp estimates for the Bergman kernel on a general class of weakly pseudoconvex domains of finite type in $\mathbb{C}^n$, as well as estimates for maximal functions for the Schrödinger and the wave equation.
- Fourier and spectral multipliers: a breakthrough has been obtained on the characterization of radial Fourier multipliers. Contrary to a general belief that for $p \neq 1, 2$ or ∞, no "concrete" characterization of Fourier multipliers for $L^p(\mathbb{R}^d)$ would be possible, radial Fourier multipliers have been characterized for the range $1 < p < 2d/(d+1)$, at least when acting on radial functions, and in sufficiently high dimension even when acting on arbitrary L^p- functions, in terms of Fourier localized pieces of the convolution kernels. Moreover, improvements on Wolff's inequality for the cone multiplier have been achieved.

  Also, a theory of Hardy spaces on metric measure spaces with exponential growth has been developed, which allows for instance to significantly improve on a spectral multiplier theorem for Riemannian manifolds with bounded geometry by M.Taylor. For instance, the new results apply to complex powers of the Laplacian, which could not be handled before.
- Oscillatory and Fourierintegral operators: this includes endpoint $L^p - L^q$ and Sobolev inequalities for certain broad classes of highly degenerate Radon-like averaging operators.
- Applications to PDE's: this includes optimal global existence theorems by means of abstract Strichartz estimates for small amplitude nonlinear wave equations associated to certain linear wave equations involving compact perturbations of the standard Laplacian, global well-posedness and scattering in H^1 for defocusing nonlinear Schrödinger equations on hyperbolic space, and a smoothing property for the L^2-critical nonlinear Schrödinger equation.

The meeting took place in a lively and active atmosphere, and greatly benefited from the ideal environment at Oberwolfach. It was attended by 43 participants. The program consisted of 3 survey lecture series and 25 lectures. The organisers made an effort to include young mathematicians, and greatly appreciate the support through the joint Oberwolfach/NSF program "US Junior Oberwolfach Fellows," which allowed to invite several outstanding young scientists from the United States.

Workshop: Real Analysis, Harmonic Analysis and Applications

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Abstracts

Effective equidistribution on the torus

Elon Lindenstrauss
(joint work with J. Bourgain, A. Furman, P. Michel, S. Mozes, A. Venkatesh)

In my talks I have presented two effective results related to distribution of orbits of groups (or semigroups) of endomorphisms on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^d$.

First consider the action of the semigroup generated by two multiplicatively independent integers[1] on $\mathbb{T}$. In [3], Furstenberg showed that the only closed, infinite subset of $\mathbb{R}/\mathbb{Z}$ invariant under the maps $t_a : x \mapsto a.x$ and $t_b : x \mapsto b.x$ is $\mathbb{R}/\mathbb{Z}$ (with $a.x = ax \mod 1$). This implies that for any irrational x,

$$\overline{\{a^k b^\ell.x : k, \ell \geq 0\}} = \mathbb{R}/\mathbb{Z}. \tag{1}$$

Furstenberg raised the question of what are the t_a, t_b invariant measures on $\mathbb{R}/\mathbb{Z}$, conjecturing that the only nonatomic such measure[2] is the Lebesgue measure λ. A theorem of Rudolph for a, b relatively prime [6], generalized by Johnson to the case of a, b multiplicatively independent, asserts that a probability measure on the circle $\mathbb{R}/\mathbb{Z}$ that is invariant and ergodic with respect to the semigroup generated by the maps $t_a : x \mapsto ax$ and $t_b : x \mapsto bx$, and has positive entropy with respect to t_a, is equal to λ.

In joint work with J. Bourgain, P. Michel and A. Venkatesh [2], we give an effective versions of the Rudolph-Johnson theorem, and use it (among other things) to obtain effective versions of Furstenberg's theorem, in particular giving an estimate on the rate in (1) in terms of the Diophantine properties of x.

We now consider the case of $\mathbb{T}^d$, and consider the action on $\mathbb{T}^d$ generated by two (or more) automorphisms, say A and B. The case of actions generated by commuting automorphisms is quite similar to the action of the semigroup generated by multiplication by a and b on the $\mathbb{T}$. In this generality we do not know yet how to extend the effective density results of BLMV to $\mathbb{T}^d$ but this has been carried out by Zhiren Wang [7] for several interesting families.

In joint work with J. Bourgain, A. Furman and S. Mozes [1] we consider the case where A and B generate a big group (e.g. a Zariski dense subgroup of $\mathrm{SL}_d(\mathbb{R})$ (so in particular A and B do not commute). In this case a much more satisfactory picture can be given:

Theorem 1 (Invariant Measures)**.** *Let $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ be a subgroup Zariski dense in $\mathrm{SL}_d(\mathbb{R})$ or, more generally, a group whose Zariski closure $\overline{\Gamma}^Z < \mathrm{SL}_d(\mathbb{R})$*

(2) *acts strongly irreducibly on $\mathbb{R}^d$ and contains a proximal element.*

[1] I.e. not powers of the same integer, or equivalently so that $\log a/\log b \notin \mathbb{Q}$ — for example, a, b relatively prime.

[2] I.e. a measure which gives measure zero to any single point.

If μ is a probability measure on $\mathbb{T}^d$ invariant under the Γ-action, then it is a convex combination of the Haar measure m on $\mathbb{T}^d$ and an atomic measure supported by rational points.

Recall that the linear action of a group $G < \mathrm{GL}_d(\mathbb{R}^d)$ on $\mathbb{R}^d$ is *strongly irreducible* if no finite union of proper subspaces is G-invariant; equivalently, if every finite index subgroup of G acts irreducibly. An element $g \in \mathrm{GL}_d(\mathbb{R})$ is called *proximal* if g has a dominant eigenvalue: $|\lambda_1| > \max_{2\le i\le d} |\lambda_i|$. In [5] it is proved that if the Zariski closure $\overline{\Gamma}^Z < \mathrm{GL}_d(\mathbb{R})$ is strongly irreducible and contains a proximal element then so does Γ itself.

Given a probability measure ν on $\mathrm{SL}_d(\mathbb{Z})$ and a probability measure μ on $\mathbb{T}^d$ the convolution $\nu * \mu \in \mathbb{T}^d$ is

$$\nu * \mu = \sum_{g\in\Gamma} \nu(g)\, g_*\mu.$$

If $\nu * \mu = \mu$ we say that μ is ν-stationary. Any Γ-invariant probability measure is ν-stationary for any distribution ν on Γ, but the converse (even for a fixed ν) is not true in general. Actions of the group Γ on a space X for which all ν-stationary measures are Γ-invariant, are called *ν-stiff* (Furstenberg [4]).

Theorem 2 (Stiffness)**.** *Let ν be a probability measure on $\mathrm{SL}_d(\mathbb{Z})$ so that the group $\Gamma = \langle \mathrm{supp}(\nu)\rangle$ is as in Theorem 1, and for some $\epsilon > 0$*

$$\sum_{g\in\Gamma} \|g\|^{\epsilon}\nu(g) < +\infty. \tag{3}$$

Then any ν-stationary measure μ on $\mathbb{T}^d$ is Γ-invariant, and is, therefore, a convex combination of the Haar measure on $\mathbb{T}^d$ and an atomic measure supported by rational points.

Theorem 3 (Quantitative)**.** *Let ν and $\Gamma = \langle \mathrm{supp}(\nu)\rangle < \mathrm{SL}_d(\mathbb{Z})$ be as in Theorem 2. Then there exist $c > 0$ and $k < \infty$ so that: if for a point $x \in \mathbb{T}^d$ the measure $\mu_n = \nu^{*n} * \delta_x$ has Fourier coefficient $|\hat{\mu}_n(a)| > \delta$ for some $a \in \mathbb{Z}^d\backslash\{0\}$, then x admits a rational approximation*

$$\left\| x - \frac{p}{q} \right\| < e^{-cn} \quad \textit{for some} \quad p \in \mathbb{Z}^d, \quad q < \left(\frac{\|a\|}{\delta}\right)^k. \tag{4}$$

Theorem 3 answers the question of equidistribution, posed by Y. Guivarc'h .

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A sharp multiplier theorem for the Kohn sublaplacian on the sphere in $\mathbb{C}^n$

Michael G. Cowling

(joint work with Oldrich Klima, Adam Sikora)

The Kohn sublaplacian $\mathcal{L}$ on the unit sphere S in $\mathbb{C}^n$ is of interest in complex analysis, and as a model subelliptic operator. We take $\mathcal{L}$ to be $\Delta + T^2$, where $-\Delta$ and T denote the Laplace–Beltrami operator on S and the unit vector field in the iz direction at the point z (we take Laplacians to be positive operators).

Given a self-adjoint positive operator $\mathcal{L}$ on $L^2(X)$, where X is a measure space, one can define the bounded operator $F(\mathcal{L})$ on $L^2(X)$ for any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ by spectral theory. If we specialise to radial functions in Hörmander's Fourier multiplier theorem [7, Theorem 2.5],then it becomes the following prototypical spectral multiplier theorem for the Laplacian Δ, in which we denote by $H^s(\mathbb{R})$ the Sobolev space of functions on $\mathbb{R}$ with s derivatives in $L^2(\mathbb{R})$ and by $\delta_t G$ the function $G(t\cdot)$.

Theorem. *Suppose that $G\colon [0,\infty) \to \mathbb{C}$ is continuous. Suppose also that $0 \neq \eta \in C_c^\infty(0,\infty)$, that $s > n/2$, and that*

$$\|\eta\delta_t G\|_{H^s} \leq C \qquad \forall t > 0.$$

Then $G(\Delta)$, initially defined on $L^2(\mathbb{R}^n)$, extends continuously to a bounded operator on the space $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$.

In this result, the "index" $n/2$ cannot be improved, and is therefore called critical. M. Christ [2] and G. Mauceri and S. Meda [9] proved that the index for a homogeneous sublaplacian on a homogeneous Lie group is at most half the homogeneous dimension of the group. D. Müller and E.M. Stein [11], W. Hebisch [6] and some of the present authors [5] showed that the critical index is half the topological dimension for Heisenberg-like groups and for SU(2). Our main result here is the following.

Theorem. *Suppose that $G\colon [0,\infty) \to \mathbb{C}$ is continuous. Suppose also that $0 \neq \eta \in C_c^\infty(0,\infty)$, that $s > n - 1/2$, and that*

$$\|\eta\delta_t G\|_{H^s} \leq C \qquad \forall t > 0. \tag{1}$$

Then $G(\mathcal{L})$, initially defined on $L^2(S)$, extends continuously to a bounded operator on the space $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$.

If $F(\lambda) = G(\lambda^2)$, then F satisfies (1) if and only if G does. Therefore one can consider $F(\sqrt{\mathcal{L}})$ instead of $G(\mathcal{L})$. This simplifies some homogeneity arguments.

There is a subriemannian or control distance dist_0 associated to the Kohn Laplacian. R.B. Melrose [10] showed that the distribution $\cos(t\sqrt{\mathcal{L}})\delta_w$ is supported in the dist_0 ball with centre w and radius t. Since dist_0 is hard to work with, we use a simpler equivalent distance, dist:

$$\mathrm{dist}(w,z) = |1 - \langle w, z\rangle|^{1/2} \qquad \forall w, z \in \mathbb{C}^n.$$

Equipped with the distance dist and the standard surface measure σ, S is a space of homogeneous type in the sense of R.R. Coifman and G.L. Weiss [3], of homogeneous dimension $2n$. Using this, Coifman and Weiss [3, 4] proved a Hörmander multiplier theorem with index n.

The key step in proving the multiplier theorem is to associate to the operator $F(\sqrt{\mathcal{L}})$ a kernel $k_F\colon S \times S \to \mathbb{C}$ and to establish that

$$\int_{\mathrm{dist}(w,z)>2\,\mathrm{dist}(z,z')} |k_F(w,z) - k_F(w,z')|\,dw \le C \tag{2}$$

for all y and y' in S.

Take smooth functions $\varphi_n\colon [0,\infty] \to [0,1]$ such that $\varphi_{n+1} = \varphi_1(2^{-n}\cdot)$ when $n \ge 1$, $\mathrm{supp}(\varphi_0) \subseteq [0,1]$, $\mathrm{supp}(\varphi_1) \subseteq [1/2,2]$, and $\sum_{n=0}^{\infty} \varphi_n = 1$. Write F_n for $\varphi_n F$, and k_{F_n} for the corresponding kernels. The kernel k_{F_0} is smooth and poses no problems. We estimate the integral (2) by

$$\sum_{n=0}^{\infty} \int_{\mathrm{dist}(w,z)>2\,\mathrm{dist}(w,z')} |k_{F_n}(w,z) - k_{F_n}(w,z')|\,dw.$$

The worst terms are where $\mathrm{dist}(y,y')$ is close to $2^{-n/2}$. For smaller n, $\mathrm{dist}(y,y')$ is small compared to the scale on which k_{F_n} oscillates, and the two kernels tend to cancel. For larger n, $k_{F_n}(\cdot,y)$ and $k_{F_n}(\cdot,y')$ are nearly disjoint, and we control the integral by

$$2\int_{\mathrm{dist}(w,z)>\mathrm{dist}(z,z')} |k_{F_n}(w,z)|\,dw;$$

here we integrate where the kernel is small. Then the real task is to control

$$\int_{\mathrm{dist}(w,z)>\epsilon} |k(w,z)|\,dw.$$

Hörmander used Fourier analysis to prove his multiplier result. On $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x)|\,dx \le \Big(\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{2s}}\,dx\Big)^{1/2} \Big(\int_{\mathbb{R}^n} (1+|x|)^{2s}|f(x)|^2\,dx\Big)^{1/2};$$

the first factor on the right hand side converges provided that $s > n/2$, and the second is essentially the $H^s(\mathbb{R}^n)$ norm of $\hat{f}$, the Fourier transform of f; this is why $n/2$ is the index for the classical Hörmander theorem. We decompose the integral into integrals over annuli:

$$\int_{\mathrm{dist}(w,z)>\epsilon} |k(w,z)|\, dw = \sum_{n=0}^{\infty} \int_{2^{n+1}\epsilon \geq \mathrm{dist}(w,z) > 2^n\epsilon} |k(w,z)|\, dw,$$

and in each annulus we use the trivial estimate

$$\int_{B(z,2\delta)\setminus B(z,\delta)} |k(w,z)|\, dw \leq \left|B(z,2\delta) \setminus B(z,\delta)\right|^{1/2} \Big(\int_S |k(w,z)|^2\, dw \Big)^{1/2}. \tag{3}$$

We use Melrose's finite propagation speed result in an argument due to J. Cheeger, M. Gromov and M. Taylor [1] to control the decay of $k_{F_n}(w,z)$ as $\mathrm{dist}(w,z)$ grows. However, this still only yields a multiplier theorem when $s > n$, because $|B(y,t)|$ behaves like a multiple of t^{2n} for small t.

The trick needed is to use a weight: we replace the right hand side of (3) by

$$\Big(\int_{B(z,2\delta)\setminus B(z,\delta)} \mathrm{dist}(w,z)^{-\alpha}\, dx \Big)^{1/2} \Big(\int_S |k(w,z)|^2\ \mathrm{dist}(w,z)^{\alpha}\, dw \Big)^{1/2}.$$

Then the first integral on the right hand side behaves as $\delta^{n-\alpha/2}$. The price that we pay is that we need weighted L^2 estimates for the second integral.

In [5], we used weighted L^2 estimates for the sphere in $\mathbb{C}^2$ using harmonic analysis on SU(2). We can now prove the general theorem for the sphere in $\mathbb{C}^n$ using the weighted L^2 estimates in the M.Sc. thesis of Klima [8] — the key to these is a careful study of complex spherical harmonics.

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Applications of time-frequency analysis in ergodic theory

Christoph Thiele

The classical paradigm of Calderón and Zygmund provides a set of techniques invariant under translations and dilations that allow to prove estimates for operators acting on function spaces on $\mathbb{R}^n$. We focus on the real line as underlying space and further concentrate on estimates for operators that are themselves invariant under translation and dilation. The only such operators bounded on $L^p(R)$ are the linear combinations of the identity operator and the Hilbert transform

$$p.v. \int f(x-t)\frac{dt}{t} \quad .$$

Passing to invariant sub-linear operators, we have the well known examples of the maximal operators and maximal truncated Hilbert transform

$$\sup_\epsilon \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} f(x-t)\,dt, \quad \sup_\epsilon \int_{R\setminus[-\epsilon,\epsilon]} f(x-t)\,\frac{dt}{t} \quad .$$

To obtain further invariant operators we may replace the supremum norm in the parameter ϵ by other norms such as the V^r variation norm

$$\sup_{N,\epsilon_0,\epsilon_1,\dots,\epsilon_N} (\sum_{j=1}^{N} |f(\epsilon_j)-f(\epsilon_{j-1})|^r)^{1/r}$$

or variants thereof with names such as jump counting norms and oscillation norms that we shall not elaborate on in detail. For $r>2$ the operators

$$\|\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} f(x-t)\,dt\|_{V^r(\epsilon)}, \quad \|\int_{R\setminus[-\epsilon,\epsilon]} f(x-t)\,\frac{dt}{t}\|_{V^r(\epsilon)}$$

are bounded in L^p for $1<p<\infty$.

By a classical transfer principle bounds on the former can be used to prove Birkhoff's ergodic theorem: For a probability space X and a measure preserving transformation T of X, and for $f\in L^p(X)$, the ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$$

converge for almost every x in X.

Time-frequency analysis provides an additional paradigm to Calderón-Zygmund analysis that allows to estimate operators that are invariant under translation, dilation, and modulation. Modulation is defined as

$$M_\eta f(x) = f(x)e^{i\eta x} \quad .$$

There are no bounded linear operators on $L^p(\mathbb{R})$ with non-trivial translation, dilation and modulation symmetry other than the identity operator, so we immediately turn to sublinear operators. The classical operator in this class is Carleson's operator. It is the maximal operator over the family of operators formed by pre-composing the Hilbert transform with a modulation:

$$Cf(x) = \sup_\eta p.v. \int f(x-t)e^{i\eta(x-t)} \frac{dt}{t} \quad .$$

Boundedness of this operator in $L^p(R)$ is the celebrated Carleson-Hunt theorem [2],[8] with alternative approaches by Fefferman [5] and by Lacey and the author [11].

Taking a supremum or variation norm over the dilation parameter one obtains the operators

$$C^* f(x) = \sup_\epsilon \sup_\eta p.v. \int_{R\setminus[-\epsilon,\epsilon]} f(x-t)e^{i\eta(x-t)} \frac{dt}{t}$$

$$C^V f(x) = \sup_\eta \|p.v. \int_{R\setminus[-\epsilon,\epsilon]} f(x-t)e^{i\eta(x-t)} \frac{dt}{t} \|_{V^r(\epsilon)}$$

While the first operator can somewhat surprisingly be bounded by a superposition of the classical Carleson operator, it is not known whether the second operator is bounded. Lacey and Terwilleger have estimated a variant of the second operator with variation norm replaced by some oscillation norm, motivated by a family of Wiener-Wintner theorems in ergodic theory [10].

For another strengthening of Carleson's operator we recall that the L^∞ norm of a function f is the same as the M_2 Fourier multiplier norm, i.e. the norm of the operator

$$g \to F^{-1}(fF(g))$$

where F denotes the Fourier transform. Given a family f_ϵ of function, one can form the maximal operator associated to the family of multipliers. We denote by $\|f\|_{M_2^*}$ the norm of this maximal operator on $L^2(R)$:

$$g \to \sup_\epsilon F^{-1}(f_\epsilon F(g)) \quad .$$

We then have the sublinear operator

$$C^M f(x) = \|p.v \int f(x-t)e^{i\eta(x-t)} \frac{dt}{t} \|_{M_2^*(\eta)}$$

This operator has been estimated in L^p for $1 < p < 2$ in [4]. By transfer to ergodic theory this result implies Bourgain's Return Times theorem [1] as well as an extension of the range of exponents p and q in that theorem compared to previously known results. The Return Times Theorem states: If Y is a probability space and S a measure preserving transformation on Y and g in $L^p(Y)$, then for almost every $y \in Y$ the sequence $a_n = g(S^n y)$ is a good sequence for a weighted Birkhoff theorem: Given a probability space X, a measure preserving transformation T on X and $f \in L^q(X)$, then the averages

$$\frac{1}{N} \sum_{n=1}^{N} a_n f(T^n x)$$

converge for almost every $x \in X$.

Another class of objects with translation, dilation, and modulation invariance can be found among multilinear operators and forms. Fix a vector $\beta = (\beta_1, \beta_2, \beta_3)$ of unit length in R^3 that is perpendicular to $(1, 1, 1)$. Then the trilinear form

$$\Lambda(f_1, f_2, f_3) = \int p.v. \int f_1(x - \beta_1 t) f_2(x - \beta_2 t) f_3(x - \beta_3 t) \frac{dt}{t} \, dx$$

is invariant under simultaneous translations and dilations of the three functions, as well as under the modulations

$$\Lambda(f_1, f_2, f_3) = \Lambda(M_{\alpha_1} f_1, M_{\alpha_2} f_2, M_{\alpha_3} f_3)$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is perpendicular to $(1, 1, 1)$ and β.

One has to distinguish two cases: If two of the components β_j coincide, then the trilinear form is called degenerate and can be reduced to the combination of a pointwise product and a bilinear form. In this degenerate case $L^p(R)$ bounds follow trivially by Hölder's inequality and bounds for the linear Hilbert transform. In the non-degenerate case, the trilinear form has been proved to be bounded in [12] and [13] for $\sum_j 1/p_j = 1$ and $1 < p_1, p_2, p_3 \leq \infty$.

Interesting questions concern uniform bounds in the vicinity of the degenerate cases. For any triple of exponents p_1, p_2, p_3 for which one has bounds both in the non-degenerate case and at some degenerate point β, one may expect that one has uniform bounds in the vicinity of this degenerate point. This has been established only for a certain sub-optimal range of triples (p_1, p_2, p_3) in a series of papers [15],[7],[14].

Maximal and oscillation norm operators derived from the dual bilinear operator (the bilinear Hilbert transform) have been considered in [9] and [3] in connection with ergodic averages of the form

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) g(T^{2n} x) \quad .$$

A longstanding open problem in ergodic theory concerns the almost everywhere convergence of ergodic averrages formed using two commuting but otherwise arbitrary measure preserving transformations S and T on a probability space X:

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) g(S^n x)$$

for two bounded measurable functions f and g. An approach to this problem using time frequency analysis suggests the study of the two dimensional analogue of the above trilinear Hilbert form with $x \in \mathbb{R}^2$ and $\beta_1, \beta_2, \beta_3$ vectors in $\mathbb{R}^2$. No L^p estimates are known for this form. An apparently easier question arises when one replaces t by a variable in $\mathbb{R}^2$. Thus we consider the trilinear form

$$\int_{\mathbb{R}^2} p.v. \int_{\mathbb{R}^2} f_1(x - B_1 t) f_2(x - B_2 t) f_3(x - B_3 t) \, K(t) dt \, dx$$

where K is some Calderón Zygmund kernel on $\mathbb{R}^2$ and B_1, B_2, B_3 are 2×2 real matrices.

In all interesting cases one may by a change of variables and possible permutation of indices assume that $B_1 = 0$, $B_2 = \mathrm{id}$ and B_3 is in Jordan canonical form. As in the one dimensional case one encounters certain degenerate behaviour when eigenvalues of B_3 coincide with the eigenvalues of B_1 and B_2. We have five essentially different cases:

(1) Neither of 0 and 1 is an eigenvalues of B_3.
(2) B_3 is equal to B_1 or B_2.
(3) B_3 has two different eigenvalues of which exactly one is equal to 0 or 1.
(4) B_3 is a Jordan block of size 2 with eigenvalue 0 or 1.
(5) Both 0 and 1 are eigenvalues of B_3.

Cases 1 and 2 are relatively straight-forward generalizations of the non-degenerate and the degenerate cases in one dimension. Cases 3 and 4 are discussed in [6] and bounds are proved in the range $\sum_{i=1}^{3} 1/p_i = 1$ and $2 < p_1, p_2, p_3 < \infty$. Interestingly, while evidently Case 3 implies bounds for the one dimensional Hilbert form, the bound in Case 4 can be used as a an alternative approach to the Carleson-Hunt theorem. This provides a somewhat unified approach to these two objects.

It is not known whether the trilinear form satisfies any L^p bounds in Case 5. This is a basic challenge in harmonic analysis. In the following paragraphs we rephrase it in more concrete terms.

Let ϕ be a smooth approximation to the characteristic function of $[-1, 1]$ in the Schwartz class $S(\mathbb{R})$ with integral 1. Define

$$\phi_k(x) = 2^k \phi(2^k x) \quad ,$$

$$\psi_k = \phi_k - \phi_{k-1} \quad ,$$

and define the "sum" and "difference" operators

$$P_k f(x) = f * \phi_k \quad ,$$

$$Q_k f(x) = f * \psi_k \quad .$$

Let $P_{k,1}$ and $Q_{k,1}$ be the corresponding operators on functions in $\mathbb{R}^2$ that act only in the first variable, e.g.

$$P_{k,1} f(x, y) = \int f(x - t, y) \phi_k(t) \, dt$$

and define similarly $P_{k,2}$ and $Q_{k,2}$. The bounds for the trilinear form in Case 5 essentially reduce to finding any L^p bounds for the bilinear paraproduct type operator

$$B(f, g) = \sum_{k=-\infty}^{\infty} (P_{k,1} f)(Q_{k,2} g) \quad .$$

No such bound is known. Any bound together with analoguous bounds for the dual forms and interpolation would imply the bound

$$\|B(f, g)\|_{3/2} \leq C \|f\|_3 \|g\|_3$$

which may therefore be the main estimate in question.

There is an analoguous question when P_k and Q_k are replaced by the natural dyadic martingale sum and difference operators on the real line. No L^p bounds are known for this dyadic analogue, which therefore presents another closely related and possibly simpler problem.

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Heat-flow monotonicity related to some inequalities in euclidean analysis

Jonathan Bennett

The purpose of this talk is to draw attention to some heat-flow monotonicity phenomena underlying certain integral inequalities in euclidean analysis.

We begin by clarifying what we mean by this on an informal level. Suppose that $X_1, \ldots X_n$ are function spaces and that the quantities $Q_L, Q_R : X_1 \times \cdots \times X_n \to \mathbb{R}$ satisfy the inequality $Q_L(f_1, \ldots, f_n) \leq Q_R(f_1, \ldots, f_n)$ for all $f_j \in X_j$, $1 \leq j \leq n$. In such a situation it is conceivable that one may define "flows" $t \mapsto (f_j)_t$, $t > 0$

under which the quantity $t \mapsto Q_L((f_1)_t, \dots, (f_n)_t)$ is nondecreasing and

$$Q_L((f_1)_t, \dots, (f_n)_t) \longrightarrow \begin{cases} Q_L(f_1, \dots, f_n) & \text{as } t \to 0 \\ Q_R(f_1, \dots, f_n) & \text{as } t \to \infty. \end{cases}$$

This type of flow approach to proving inequalities, by its nature, tends to generate sharp constants and identify extremisers. Such information will be a byproduct of all of the examples we discuss here.

In certain situations where the quantities Q_L and Q_R are appropriately "geometric", and the flows are variants of euclidean heat flow, this sort of phenomenon does indeed exist. This was observed explicitly by Carlen, Lieb and Loss in [7], and later by Bennett, Carbery, Christ and Tao in [5]. Such heat-flow monotonicity methods also appeared in the work of Bennett, Carbery and Tao [6] on certain multilinear analogues of the longstanding Kakeya conjecture (see also [6] for applications to the closely related joints problem from incidence geometry). The reader is referred to [1] for a discussion of the historical context of such ideas within geometric analysis and for some further references.

Here we shall indicate how this phenomenon manifests itself in various contexts. We shall largely restrict our attention to settings in which the function spaces X_j are Lebesgue L^p spaces and the inputs $f_1, \dots, f_n$ flow by classical heat-flows conjugated by powers; that is, flows of the form $t \mapsto (H_t * f^p)^{1/p}$ for some p, where H_t denotes a euclidean heat kernel.

1. Heat-flow monotonicity underlying geometric inequalities

As is implicit in several works, beginning with Carlen, Lieb and Loss [7], monotone quantities underlying many geometric inequalities turn out to be generated by algebraic closure properties of solutions to heat inequalities.

We illustrate this with the classical Hölder inequality, which states that for $1 \le p_1, p_2 \le \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = 1$,

$$\int f_1 f_2 \le \|f_1\|_{p_1} \|f_2\|_{p_2}$$

for all nonnegative functions $f_1 \in L^{p_1}(\mathbb{R}^d)$ and $f_2 \in L^{p_2}(\mathbb{R}^d)$. One way of observing a heat-flow monotonicity phenomenon underlying this inequality is to appeal to a certain algebraic closure property of solutions to the heat inequality

$$\partial_t u \ge \frac{1}{4\pi} \Delta u. \tag{1}$$

Namely, if $u_1, u_2 : (0, \infty) \times \mathbb{R}^d \to (0, \infty)$ satisfy (1) then their geometric mean $u := u_1^{1/p_1} u_2^{1/p_2}$ also satisfies (1). As a corollary to this, provided that u_1 and u_2 are sufficiently well-behaved, it follows from differentiating under the integral and the divergence theorem that the quantity

$$Q(t) := \int_{\mathbb{R}^d} u_1(t, x)^{1/p_1} u_2(t, x)^{1/p_2} \, dx$$

is nondecreasing for all $t > 0$. Now, on insisting that, for $j = 1, 2$, u_j satisfies (1) with equality and nonnegative initial data $f_j^{p_j}$, it follows that

$$\int_{\mathbb{R}^d} f_1(x) f_2(x) dx = \lim_{t \to 0} Q(t) \leq \lim_{t \to \infty} Q(t) = \|f_1\|_{p_1} \|f_2\|_{p_2};$$

that is, we recover the classical Hölder inequality in the manner in which we sought. The nonnegative inputs f_1 and f_2 have associated flows $t \mapsto (H_t * f_1^{p_1})^{1/p_1}$ and $t \mapsto (H_t * f_2^{p_2})^{1/p_2}$ respectively, where $H_t(x)$ is the appropriate heat kernel.

These considerations may be generalised considerably to the so-called geometric Brascamp–Lieb inequalities, where the associated closure property of solutions to (1) involves "nonisotropic geometric means". This was first observed by Carlen, Lieb and Loss [7], and later by Carbery, Christ, Tao and the author in some generality.

In addition to such geometric means and the trivial operation of ordinary addition, solutions to certain heat inequalities are closed under convolution-based operations of the form $(u_1, u_2) \mapsto (u_1^{1/p_1} * u_2^{1/p_2})^p$, yielding various sharp Young's convolution inequalities of Beckner and Brascamp–Lieb. It is perhaps also interesting to note that the solutions of (1) are also closed under harmonic addition $(u_1, u_2) \mapsto (1/u_1 + 1/u_2)^{-1}$. See [2].

2. More "exotic" inequalities

As we have seen, a byproduct of such heat-flow monotonicity phenomena is the existence of centred gaussian extremisers to the inequalities under consideration. It is natural at this stage to turn our attention to other types of inequalities (which are not manifestly geometric) where we know, or at least suspect that gaussians are extremisers.

The classical mixed-norm Strichartz inequalities for the free Schrödinger equation take the form

$$\|e^{i\pi s\Delta} f\|_{L^p_s(L^q_x(\mathbb{R}^d))} \leq c\|f\|_{L^2(\mathbb{R}^d)}, \tag{2}$$

for certain indices p and q. Here c denotes a constant depending on at most d, p and q. A necessary condition for this inequality to hold is that

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{3}$$

See [10] for historical references and a full treatment of (2) for suboptimal c.

Recently Foschi [8] (see also Hundermark and Zharnitsky [9]) showed that in the cases where one can "multiply out" the Strichartz norm (i.e. when $q \in 2\mathbb{N}$ and $q \mid p$) the sharp constants c in (2) are obtained by testing on centred gaussians.

Due to the presence of an L^2 norm on the right hand side of Inequality (2), it is natural to flow the input $f \in L^2(\mathbb{R}^d)$ under a quadratic heat flow. For such a function f and such p, q let

$$Q_{p,q}(t) := \|e^{i\pi s\Delta}(e^{\pi t\Delta}|f|^2)^{1/2}\|_{L^p_s(L^q_x(\mathbb{R}^d))}.$$

Theorem 3 (B., Bez, Carbery, Hundertmark [4]). *If in addition to* (3) *we have that* $q \in 2\mathbb{N}$ *and* $q \mid p$, *then* $Q_{p,q}(t)$ *is nondecreasing for all* $t > 0$*; i.e.* $Q_{p,q}$ *is nondecreasing in the cases* $(d, p, q) = (1, 8, 4), (1, 6, 6), (2, 4, 4)$.

There is strong anecdotal evidence to suggest that the monotonicity of $Q_{p,q}$ will in general fail when $q \notin 2\mathbb{N}$ or q does not divide p. We refer the reader to [3] where we provide counterexamples to an analogous question in the setting of the Hausdorff–Young inequality.

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Ergodic Theory, lattice subgroups and lattice points

Amos Nevo

1. General set-up.

Let :

– G be a locally compact second countable group,

– $\Gamma \subset G$ a discrete lattice subgroup, namely G/Γ carries a G-invariant probability measure, $m = m_{G/\Gamma}$,

– $B_t \subset G$ a growing family of sets, for example $B_t = \{g \in G\,;\, N(g) \le t\}$ for some gauge function N.

Basic problem : Let (X, μ) be an arbitrary measure preserving action of Γ. Consider the lattice points in B_t, and the uniform averages supported on them. Do these averages

$$\lambda_t f(x) = \frac{1}{|\Gamma \cap B_t|} \sum_{\gamma \in \Gamma \cap B_t} f(\gamma^{-1} x)$$

converge, for a given function f on X ??

2. Ergodic theorems for Lattices in semisimple Lie groups

Let now

– G = connected semisimple Lie group with finite center and no compact factors.

– $S = G/K$ the symmetric space (K = maximal compact subgroup), d the distance function.

– $B_t = \{g \in G : d(gK, K) < t\}$.

– Γ = any lattice subgroup, λ_t = uniform averages on $\Gamma \cap B_t = \Gamma_t$.

Theorem 1. (Gorodnik-Nevo, 2005).

If the Γ-action is ergodic then the pointwise ergodic theorem holds: for every $f \in L^p$, $p > 1$, for almost every x,

$$\lim_{t\to\infty} \lambda_t f(x) = \int_X f d\mu \,.$$

In particular, the mean ergodic theorem holds: for every $f \in L^p$, $1 \leq p < \infty$

$$\lim_{t\to\infty} \left\| \lambda_t f - \int_X f d\mu \right\|_p = 0 \,.$$

An important phenomenon that arises for semisimple groups (but not for amenable groups) is the appearance of a spectral gap.

Definition : The Γ-action has a *spectral gap* in $L^2(X)$ if there exists a probability measure ν on Γ with full support, satisfying

$$\left\| \pi(\nu) f - \int_X f d\mu \right\| < (1-\eta) \left\| f \right\|$$

for $f \in L^2(X)$ and a fixed $\eta > 0$.

Theorem 2. (Gorodnik-Nevo, 2005).

If the Γ-action has a spectral gap then the fast pointwise ergodic theorem holds: for every $f \in L^p$, $p > 1$, for almost every x,

$$\left| \lambda_t f(x) - \int_X f d\mu \right| \leq C_p(x,f) \text{vol } (B_t)^{-\theta_p} \,.$$

In particular, the fast mean ergodic theorem holds : for every $f \in L^p$, $1 \leq p < \infty$

$$\lim_{t\to\infty} \left\| \lambda_t f - \int_X f d\mu \right\|_p \leq C_p \text{vol } (B_t)^{-\theta_p} \,,$$

where $\theta_p > 0$.

3. Equidistribution is isometric actions

In certain situations, it is possible to assert that convergence holds *everywhere*, not just almost everywhere.

Theorem 3. (Gorodnik-Nevo, 2006).

Let (X, d) be a compact metric space, and Γ act by isometries, ergodically with respect to an invariant probability measure μ of full support on X. Then for every continuous function f on X, uniformly

$$\lim_{t\to\infty} \max_{x\in X} \left| \lambda_t f(x) - \int_X f d\mu \right| = 0 .$$

If the Γ-action has a spectral gap (and the measure satisfies $\mu(B_\epsilon) \geq C\epsilon^d$), then for every continuous function f, and for **every** point x

$$\left| \frac{1}{|\Gamma_t|} \sum_{\gamma\in\Gamma_t} f(\gamma^{-1}x) - \int_X f d\mu \right| \leq C(x, f)e^{-\kappa t} ,$$

for an explicit rate $\kappa > 0$ (independent of f).

4. Solution of the lattice point counting problem

Let us demonstrate the basic idea in the simplest possible case. Consider the action induced to G by the trivial action of Γ on one point. Consider the simplest ergodic theorem for G, namely the mean ergodic theorem for the Haar uniform averages β_t supported on a family of sets B_t. Let us assume

A) The validity of the (stable) mean ergodic theorem in the G-action on $L^2(G/\Gamma)$, in the form

$$\left\| \beta_t f - \int_{G/\Gamma} f dm \right\|_2 \leq C\text{vol } (B_t)^{-\theta}$$

B) The regularity condition for the sets B_t given by ($t > t_0$, $0 < \epsilon < \epsilon_0$)

$$\text{vol } (\mathcal{O}_\epsilon B_t \mathcal{O}_\epsilon) \leq (1 + c\epsilon)\text{vol } (\cap_{u,v\in\mathcal{O}_\epsilon} uB_t v)$$

where $\mathcal{O}_\epsilon$ is a family of decreasing symmetric neighborhoods of $e \in G$ satisfying : vol $(\mathcal{O}_\epsilon) \geq C\epsilon^d$.

Theorem 4. (Gorodnik-Nevo, 2006).

For any lcsc group G and any lattice Γ, under conditions A and B, the lattice point counting problem in the domains B_t has the solution

$$\frac{|\Gamma \cap B_t|}{\text{vol } (B_t)} = 1 + O\left(\text{vol } (B_t)^{-\theta/(d+1)}\right)$$

Theorem 4 gives rise to a an explicit quantitative solution of the following lattice points counting problems, among others :

(1) Number of integral points in a norm ball on any semisimple Lie group defined over $\mathbb{Q}$.
(2) Number of S-integral points in a ball defined by a height function on any semisimple S-algebraic group.
(3) The number of rational points in a ball defined by a height function on any semisimple algebraic group.
(4) The number of lattice points in a norm ball on a homogeneous affine symmetric variety.

(5) Number of lattice points in an angular sector in the preceding situations.

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Estimates for Poisson kernels and stationary measures

Ewa Damek
(joint work with S.Broferio, D.Buraczewski, Y.Guivarc'h, A.Hulanicki, R.Urban)

[1]

Consider a very simple example: the upper half plane

$$\{(x,a) : x \in \mathbf{R}, a > 0\}$$

and the hyperbolic Laplacian there.

$$\mathcal{L} = a^2\partial_a^2 + a^2\partial_x^2 = a^2(\partial_a^2 + \partial_x^2)$$
$$= (a\partial_a)^2 - a\partial_a + a^2\partial_x^2$$

It is well known that bounded $\mathcal{L}$ harmonic functions are Poisson integrals

$$F(xa) = \frac{1}{\pi}\int_{\mathbf{R}} f(y)\frac{a}{(x-y)^2 + a^2}\ dy$$
$$= \frac{1}{\pi}\int_{\mathbf{R}} f(x+ay)\frac{1}{y^2+1}\ dy, \quad f \in L^\infty(\mathbf{R})$$

against the kernel

$$\frac{1}{\pi}\frac{1}{y^2+1}, \quad y \in \mathbf{R}.$$

The operator $\mathcal{L}$ is left invariant on the affine group $S = \mathbf{R}\times\mathbf{R}^+$ with multiplication $(x,a)(y,b) = (x+ay, ab)$.

Consider some more examples of invariant operators:

$$\mathcal{L}_1 = (a\partial_a)^2 - \alpha a\partial_a + a^2\partial_x^2 + \beta a\partial_x, \quad \alpha \geq 0$$
$$\mathcal{L}_2 = -a\partial_a + a^2\partial_x^2 + \beta a\partial_x$$
$$\mathcal{L}_3 = (a\partial_a)^2 - \alpha a\partial_a - a\partial_x, \quad \alpha \geq 0$$

After a change of variables all left-invariant Hörmander type operators on S with negative drift are of this form. Let ν_j be the Poisson kernel corresponding to the operator $\mathcal{L}_j$. Functions

$$F(xa) = \int_{\mathbf{R}} f(xa \circ y)d\nu_j(y) = \int_{\mathbf{R}} f(x+ay)d\nu_j(y), f \in L^\infty$$

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are harmonic with respect to $\mathcal{L}_j$ (for $\alpha = 0$ we have to assume $f \in L_c^\infty$). Moreover, if $\alpha > 0$ all bounded harmonic functions are of this form. The corresponding Poisson kernels are given by the formulas:

$$
\begin{aligned}
&\mathcal{L}_1 \qquad \nu_1(x) = c(1+x^2)^{-\frac{1+\alpha}{2}} e^{\beta \arctan x} \\
&\mathcal{L}_2 \qquad \nu_2(x) = ce^{-\frac{(x-\beta)^2}{2}} \\
&\mathcal{L}_3 \qquad \nu_3(x) = c|x|^{-1-\alpha} e^{-\frac{1}{|x|}}, \quad x < 0 \\
&\qquad\quad\;\; \nu_3(x) = 0, \quad x \geq 0
\end{aligned}
$$

As far as $(a\partial_a)^2$ is present in the operator, decay at infinity as $|x|^{-1-\alpha}$ is observed provided a give half line is in the support of the kernel. If $\alpha = 0$, the kernel is not integrable and there are no bounded harmonic functions.

Let $\check{\mu}_t$ be the semigroup with the infinitesimal generator $\mathcal{L}_j$. Then $\check{\mu}_t * \nu_j = \nu_j$. It is natural to ask what happens if instead of the semigroup we consider a single measure μ? Is there a stationary measure ν i.e a measure such that $\mu * \nu = \nu$? Is it unique? If so what is it's behavior at infinity?

We make the following assumptions:

$$
\begin{aligned}
& S = \mathbf{R}^n \times \mathbf{R}^+ \\
& axa^{-1} = (a^{d_1}x_1, ..., a^{d_n}x_n), \ d_j > 0 \\
\text{contraction in mean} \ & \int_S \log a \ d\mu(x,a) < 0 \\
\text{there is } \alpha > 0 \ & \int_S a^\alpha \ d\mu(x,a) = 1 \\
& \int_S (a^\alpha |\log a| + |x|^\alpha) \ d\mu(x,a) < \infty,
\end{aligned}
$$

where $| \ |$ is a homogeneous norm corresponding to dilations axa^{-1}. Moreover, to simplify the presentation we assume that suppμ generates S.

Theorem 1. (Buraczewski, Guivarc'h, Hulanicki, Urban, Damek, [2]**)** *Let μ be a probability measure on S satisfying the above assumptions. Then there is a Radon measure Λ on $\mathbf{R}^n \setminus \{0\}$ such that for any $f \in C_c(\mathbf{R}^n \setminus \{0\})$ we have*

$$\lim_{a\to\infty} a^\alpha \int_{\mathbf{R}^n} f(\delta_{a^{-1}}x) \ d\nu(x) = \int_{\mathbf{R}^n} f(x) \ d\Lambda(x).$$

Λ is homogeneous i.e

$$
\begin{aligned}
\delta_a * \Lambda =& a^\alpha \Lambda \\
\int_{\mathbf{R}^n} f(ax) \ \Lambda(x) =& a^\alpha \int_{\mathbf{R}^n} f(x) \ \Lambda(x).
\end{aligned}
$$

In radial coordinates

$$\int_{\mathbf{R}^n} f(x) d\Lambda(x) = \int_{S^1 \times \mathbf{R}^+} f(r\omega) d\sigma(\omega) \frac{dr}{r^{1+\alpha}}$$

for a finite not zero measure σ.

In particular

$$\lim_{a\to\infty} a^{\alpha} \nu(\{x : |x| > a,\ \delta_{|x|^{-1}}(x) \in W\}) = \frac{1}{\alpha}\sigma(W)$$

for $W \subset S^1, \sigma(\partial W) = 0$.

Theorem (1) holds in a much more general setting: $\mathbf{R}^n$ may be replaced by a nilpotent group and dilations by a direct product of dilations and the norm preserving transformations.

The above estimate should be compared with the one obtained for Poisson kernels corresponding to differential operators.

Theorem 2. (D.Buraczewski, A.Hulanicki, E.Damek, [3]**)** *Let* $S = NA$, *be a semi-direct product of a nilpotent group* N *and* $A = \mathbf{R}^+$ *acting on* N *by contracting automorphisms* δ_a *(i.e. real parts of eigenvalues of the adjoint action on the Lie algebra of* N *are positive). Let*

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y$$

be a left-invariant operator satisfying the Hörmander condition i.e.

$$\mathcal{S} = Lie(Y_0, ..., Y_m, Y)$$

Assume that via the homomorphism $\pi : S \mapsto A$, $\pi(xa) = a$, $\mathcal{L}$ *is mapped onto*

$$\pi(\mathcal{L}) = (a\partial_a)^2 - \alpha a \partial_a, \alpha \geq 0.$$

Let μ_t *be the semigroup of measures with the infinitesimal generator* $\mathcal{L}$. *Then, for the unique positive measure* ν *on* N *such that*

$$\check{\mu}_t * \nu = \nu,$$

we have

$$\lim_{a\to\infty} a^{\alpha+Q} \nu(\delta_a(x)) = c(x),$$

and $c(x)$ *is a continuous function on* $\Sigma = \{x : |x| = 1\}$. *If* $Lie(Y_0, ..., Y_m) = \mathcal{S}$ *then* $c(x) > 0$.

Notice that $\alpha \geq 0$ in Theorem (2) i.e. we reach the bottom of the spectrum, while in Theorem (1), $\alpha > 0$. The case $\int_S \log a \ d\mu(x,a) = 0$ has been treated separately and differently:

Theorem 3. (S.Broferio, Buraczewski, Damek, [1]) *Assume that* $S = \mathbf{R}^n \times \mathbf{R}^+$ *with* $axa^{-1} = (ax_1, ..., ax_n)$, $x \in \mathbf{R}^n$, $a \in \mathbf{R}^+$. *Let* μ *be a probability measure on* S *such that*

$$\int_S \log a \; d\mu(x,a) = 0,$$

suppμ *generates* S,

$$\text{there is a positive } \delta \int_S \int_S (a^\delta + a^{-\delta} + |x|^\delta) \; d\mu(x,a) < \infty$$

Then

$$\lim_{a\to\infty} \int_{\mathbf{R}^n} f(a^{-1}x) \; \nu(x)$$
$$= \int_{\mathbf{R}^n} f(r\omega) \; \frac{dr}{r} \; \sigma(\omega)$$

for $f \in C_c(\mathbf{R}^d \setminus \{0\})$ *and* $\sigma \neq 0$ *provided some additional assumptions (supp*μ *compact is sufficient).*

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Abstract Strichartz estimates and existence theorems for nonlinear wave equations

Christopher D. Sogge

(joint work with K. Hidano, J. Metcalfe, H. Smith, and Y. Zhou)

Our purpose is to show how local energy decay estimates for certain linear wave equations involving compact perturbations of the standard Laplacian lead to optimal global existence theorems for the corresponding small amplitude nonlinear wave equations with power nonlinearities. To achieve this goal, at least for spatial dimensions $n = 3$ and 4, we shall show how the aforementioned linear decay estimates can be combined with "abstract Strichartz" estimates for the free wave equation to prove corresponding estimates for the perturbed wave equation.

Let us start by describing the local energy decay assumption that we shall make throughout. We shall consider wave equations on the exterior domain $\Omega \subset \mathbb{R}^n$ of a compact obstacle:

$$
(1)\qquad \begin{cases} (\partial_t^2 - \Delta_{\mathrm{g}})u(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u(0,\,\cdot\,) = f \\ \partial_t u(0,\,\cdot\,) = g \\ (Bu)(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases}
$$

where for simplicity we take B to either be the identity operator (Dirichlet-wave equation) or the inward pointing normal derivative ∂_ν (Neumann-wave equation).

The operator Δ_{g} is the Laplace-Beltrami operator associated with a smooth, time independent Riemannian metric $\mathbf{g}_{jk}(x)$ which we assume equals the Euclidean metric δ_{jk} for $|x| \geq R$, some R. The set Ω is assumed to be either all of $\mathbb{R}^n$, or else $\Omega = \mathbb{R}^n \backslash \mathcal{K}$ where $\mathcal{K}$ is a compact subset of $|x| < R$ with smooth boundary.

We can now state the main assumption that we shall make.

Hypothesis B. *Fix the boundary operator B and the exterior domain $\Omega \subset \mathbb{R}^n$ as above. We then assume that given $R_0 > 0$*

$$
(2)\qquad \int_0^\infty \Big(\|u(t,\,\cdot\,)\|^2_{H^1(|x|<R_0)} + \|\partial_t u(t,\,\cdot\,)\|^2_{L^2(|x|<R_0)} \Big)\, dt \lesssim \|f\|^2_{H^1} + \|g\|^2_{L^2} + \int_0^\infty \|F(s,\,\cdot\,)\|^2_{L^2}\, ds,
$$

whenever u is a solution of (1) *with data $(f(x), g(x))$ and forcing term $F(t,x)$ that both vanish for $|x| > R_0$.*

Having described the main assumption about the linear problem, let us now describe the nonlinear equations that we shall consider. They are of the form

$$
(3)\qquad \begin{cases} (\partial_t^2 - \Delta_{\mathrm{g}})u(t,x) = F_p\big(u(t,x)\big), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ Bu = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x), \quad x \in \Omega, \end{cases}
$$

with B as above. We shall assume that the nonlinear term behaves like $|u|^p$ when u is small, and so we assume that $\sum_{0\le j\le 2} |u|^j \left| \partial_u^j F_p(u) \right| \lesssim |u|^p$, when u is small.

If we let $\{Z\} = \{\partial_l,\, x_j\partial_k - x_k\partial_j : 1 \le l \le n,\, 1 \le j < k \le n\}$ then we can now state our existence theorem for (3).

Theorem 1. *Let $n = 3$ or 4, and fix $\Omega \subset \mathbb{R}^n$ and boundary operator B as above. Assume further that Hypothesis B is valid.*

Let $p = p_c$ be the positive root of $(n-1)p^2 - (n+1)p - 2 = 0$, and fix $p_c < p < (n+3)/(n-1)$. Then if $\gamma = \frac{n}{2} - \frac{2}{p-1}$, there is an $\varepsilon_0 > 0$ depending on Ω, B and p so that (3) *has a global solution satisfying $(Z^\alpha u(t,\,\cdot\,), \partial_t Z^\alpha u(t,\,\cdot\,)) \in \dot{H}^\gamma_B \times \dot{H}^{\gamma-1}_B$, $|\alpha| \le 2$, $t \in \mathbb{R}_+$, whenever the initial data satisfies the boundary conditions of order 2, and*

$$
(4)\qquad \sum_{|\alpha|\le 2} \Big(\|Z^\alpha f\|_{\dot{H}^\gamma_B(\Omega)} + \|Z^\alpha g\|_{\dot{H}^{\gamma-1}_B(\Omega)} \Big) < \varepsilon, \quad 0 < \varepsilon < \varepsilon_0.
$$

To prove Theorem 1, we shall use certain "abstract Strichartz estimates" which we now describe. Earlier works ([1], [5], [6]) have focused on establishing certain mixed norm, $L^q_t L^r_x$ estimates on $\mathbb{R}_+ \times \Omega$ for solutions of (1). For certain applications, such as obtaining the Strauss conjecture in various settings, it is convenient to replace the L^r_x norm with a more general one. To this end, we consider pairs of normed function spaces $X(\mathbb{R}^n)$ and $X(\Omega)$. The spaces are localizable, in that $\|f\|_X \approx \|\beta f\|_X + \|(1-\beta)f\|_X$ for smooth, compactly supported β, with $\beta = 1$ on a neighborhood of $\mathbb{R}^n \backslash \Omega$ in case $X = X(\Omega)$. Finally, we assume that $\|(1-\beta)f\|_{X(\Omega)} \approx \|(1-\beta)f\|_{X(\mathbb{R}^n)}$ for such β. Weighted mixed L^p spaces, as well as $\big(\dot{H}^\gamma(\mathbb{R}^n), \dot{H}^\gamma_B(\Omega)\big)$, are the examples used in the proof of Theorem 1.

We shall let $\|\cdot\|_{X'}$ denote the dual norm (respectively over $\mathbb{R}^n$ and Ω) so that $\|u\|_X = \sup_{\|v\|_{X'}=1}\left|\int u\,\overline{v}\,dx\right|$. An important example for us is when $\|u\|_X = \|\,|x|^\alpha u\|_{L^p}$, for a given $1 \le p \le \infty$ and $|\alpha| < n/p$, in which case the dual norm is $\|v\|_{X'} = \|\,|x|^{-\alpha} v\|_{L^{p'}}$, with p' denoting the conjugate exponent.

We shall consider time Lebesgue exponents $q \ge 2$ and assume that we have the global Minkowski abstract Strichartz estimates

$$\|v\|_{L^q_t X(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|v(0,\,\cdot\,)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t v(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\,, \tag{5}$$

assuming that $(\partial_t^2 - \Delta)v = 0$ Here $\|v\|_{L^q_t X(I\times\mathbb{R}^n)} = \Big(\int_I \|v(t,\,\cdot\,)\|^q_X\,dt\Big)^{1/q}$. We shall also consider analogous norms on $I \times \Omega$ $\|u\|_{L^q_t X(I\times\Omega)} = \Big(\int_I \|u(t,\,\cdot\,)\|^q_{X(\Omega)}\,dt\Big)^{1/q}$.

In addition to Hypothesis B and (5), we shall assume that we have the local abstract Strichartz estimates for Ω:

$$\|u\|_{L^q_t X([0,1]\times\Omega)} \lesssim \|f\|_{\dot{H}^\gamma_B(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_B(\Omega)}, \tag{6}$$

assuming that u solves (1) with vanishing forcing term.

Definition 2. When (5) and (6) hold we say that (X, γ, q) is an admissible triple.

We can now state our main estimate.

Theorem 3. *Let $n \ge 2$ and assume that (X, γ, q) is an admissible triple with $q > 2$ and $\gamma \in [-\frac{n-3}{2}, \frac{n-1}{2}]$. Then if Hypothesis B is valid and if u solves* (1) *and $(\partial_t^2 - \Delta_g)u \equiv 0$, we have the global abstract Strichartz estimates*

$$\|u\|_{L^q_t X(\mathbb{R}\times\Omega)} \lesssim \|f\|_{\dot{H}^\gamma_B(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_B(\Omega)}. \tag{7}$$

The condition on γ ensures that γ and $1-\gamma$ are both $\le (n-1)/2$, which is what the proof seems to require. Unfortunately, for $n = 2$, this forces γ to be equal to $1/2$, while a larger range of $\gamma \in (0,1)$ is what certain applications require. For this reason, we are unable at present to show that the Strauss conjecture for obstacles holds when $n = 2$.

Corollary 4. *Assume that (X, γ, q) and $(Y, 1-\gamma, r)$ are admissible triples and that Hypothesis B is valid. Also assume that* (7) *holds for (X, γ, q) and $(Y, 1-\gamma, r)$, and* *that* $0 \leq \gamma \leq 1$. *Then we have the following global abstract Strichartz estimates for the solution of* (1)

$$\|u\|_{L^q_t X(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\dot H^\gamma_B(\Omega)} + \|g\|_{\dot H^{\gamma-1}_B(\Omega)} + \|F\|_{L^{r'}_t Y'(\mathbb{R}_+\times\Omega)}, \tag{8}$$

where r' *denotes the conjugate exponent to* r *and* $\|\cdot\|_{Y'}$ *is the dual norm to* $\|\cdot\|_Y$.

One can adapt arguments from Hidano [2], [3] to show that

$$\begin{aligned} \Big\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \Big\|_{L^p_t L^p_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^n)} &\lesssim \|u(0,\cdot)\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\partial_t u(0,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)} \\ &+ \Big\| |x|^{-\frac{n}{2}+1-\gamma}(\partial_t^2-\Delta)u \Big\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\mathbb{R}^n)}, \end{aligned} \tag{9}$$

provided that $\frac{1}{2}-\frac{1}{p} < \gamma < \frac{n}{2}-\frac{1}{p}$ and $\frac{1}{2} < 1-\gamma < \frac{n}{2}$

This inequality is strong enough to prove the variant of Theorem 1 for the nonobstacle case when $2 \leq n \leq 4$. Using Corollary 4, one can show that a variant of (9) is also valid for the obstacle case under the above assumptions when $n = 3$ and $n = 4$, which is strong enough to prove Theorem 1.

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The Hardy-Littlewood maximal inequality in metric measure spaces

Terence Tao

(joint work with Assaf Naor)

We study the best weak $(1,1)$ constant $\|M\|_{L^1\to L^{1,\infty}}$ of the Hardy-Littlewood maximal function $Mf := \sup_{r>0} \frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f|\, d\mu$ in a metric measure space (X,μ,d). A classical Vitali-type covering argument of Wiener gives the bound $\|M\|_{L^1\to L^\infty} \leq K$ whenever the metric measure space has doubling constant at most K, thus $\mu(B(x,2r)) \leq K\mu(B(x,r))$ for all x, r. We give various examples that show that Wiener's bound is essentially sharp, even when assuming the metric measure space is an abelian group with translation-invariant measure and metric, assuming L^p operator bounds on M, and also restricting attention to dyadic radii r.

On the other hand, if we instead assume a *micro-doubling* condition $\mu(B(y,(1+\frac{1}{d})r)) \leq K\mu(B(x,r))$ whenever $r > 0$ and $y \in B(x,r)$, then one has the improved bound $\|M\|_{L^1 \to L^{1,\infty}} \leq CK^2 d \log d$, recovering a result of Stein and Strömberg (and also closely related to a maximal inequality of Lindenstrauss); if one restricts to dyadic radii then one can improve this to $CK \log d$. More generally, if M_I denotes the maximal function with radii restricted to some set $I \subset \mathbb{R}^+$, then we have the localisation result $\|M_I\|_{L^1 \to L^{1,\infty}} \leq CK \sup_r \|M_{I \cap [r,dr]}\|_{L^1 \to L^{1,\infty}}$. Thus, in order to analyse the weak $(1,1)$ constant in the Hardy-Littlewood maximal inequality, it suffices to restrict attention to a range of scales such as $[1,d]$.

Finally, we show that the Hardy-Littlewood inequality can hold in some contexts with no doubling properties whatsoever, such as the free group or hyperbolic space.

An affine invariant inequality and applications in harmonic analysis

James Wright

(joint work with Spyridon Dendrinos, Magali Folch-Gabayet, Norberto Laghi)

Recently there has been considerable attention given to certain euclidean harmonic analysis problems associated to a surface or curve (for example, the problems of Fourier restriction and the smoothing effects of generalised Radon transforms) where the underlying surface measure is replaced by the so-called affine arclength or surface measure. See [1], [2], [3], [5], [8], [11], [12], [13], [14], [15], [16], [17] and [18]. This has the effect of making the problem affine invariant as well as invariant under reparametrisations of the underlying curve or surface. For this reason there have been many attempts to obtain universal results, establishing uniform bounds over a large class of surfaces or curves. The affine arclength or surface measure also has the mitigating effect of dampening any curvature degeneracies of the variety and therefore one expects that the universal bounds one seeks will be the same as those arising from the most non-degenerate situation.

In this talk we will be concerned with problems associated to curves in $\mathbb{R}^d$ defined with respect to the affine arclength measure. If $\Gamma : I \to \mathbb{R}^d$ parametrises a smooth curve in $\mathbb{R}^d$ on an interval I, set

$$L_\Gamma(t) = \det(\Gamma'(t) \cdots \Gamma^{(d)}(t));$$

this is the determinant of a $d \times d$ matrix whose jth column is given by the jth derivative of Γ, $\Gamma^{(j)}(t)$. The affine arclength measure $\nu = \nu_\Gamma$ on Γ is defined on a test function ϕ by

$$\nu(\phi) = \int_I \phi(\Gamma(t)) |L_\Gamma(t)|^{\frac{2}{d(d+1)}} \, dt;$$

one easily checks that this measure is invariant under reparametrisations of Γ.

We will be mainly interested here in obtaining universal bounds for two problems in euclidean harmonic analysis. One such problem lies in the theory of Fourier restriction where one would like to determine the exponents p and q so that the apriori estimate

$$\|\widehat{f}|_\Gamma\|_{L^q(\Gamma, d\nu)} \leq C\|f\|_{L^p(\mathbb{R}^d)} \qquad (1)$$

holds uniformly for a large class of curves Γ. This problem was first considered by Sjölin in [19] where he showed that (1) holds uniformly over all smooth convex curves in the plane if and only if $p' = 3q$ (here $p' = p/(p-1)$ denotes the conjugate exponent to p) and $1 \leq p < 4/3$. See also [15]. The convexity assumption implies that $L_\Gamma(t)$ remains single-signed and Sjölin produced a plane curve Γ where L_Γ rapidly changes sign and (1) fails for any $p' = 3q$ and $1 < p < 4/3$ (Sjölin's argument establishing (1) for convex curves works for any smooth plane curve as long as the number of sign changes of L_Γ remains bounded).

Another basic problem arises in the theory of averaging operators along curves where one wants to determine the optimal exponents p and q so that the apriori estimate

$$\|Tf\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)} \tag{2}$$

holds uniformly for a large class of curves Γ where

$$Tf(x) = f * \nu(x) = \int_I f(x - \Gamma(t))|L_\Gamma(t)|^{\frac{2}{d(d+1)}}\, dt.$$

Again simple examples where L_Γ changes sign too often show that (2) can fail in such situations.

Therefore in both problems it is natural to restrict to families of curves Γ where one has control over the number of sign changes of L_Γ. Of course convex curves in the plane is one such natural family. We will be interested in families of curves $\Gamma(t) = (Q_1(t), \ldots, Q_d(t))$ where each Q_j is a real polynomial (more generally we also consider the class where each Q_j is a rational function). Both cases are natural families as L_Γ is either a polynomial or rational function and the number of sign changes is controlled by the degrees of the polynomials defining Γ. One method for establishing (1) or (2) is the so-called T^*T method where one examines the regularity properties of the measure ν by considering its d-fold convolution $\nu * \cdots * \nu$. The map which carries this d-fold convolution is $\Phi_\Gamma(t_1, \ldots, t_d) = \Gamma(t_1) + \cdots + \Gamma(t_d)$ and the determinant of the Jacobian matrix for this map, $J_\Gamma(t_1, \ldots, t_d) = \det(\Gamma'(t_1) \cdots \Gamma'(t_d))$, then determines the density of $\nu * \cdots * \nu$.

M. Christ introduced two methods to establish (1) and (2), see [6] and [7] respectively, reducing matters (in part) to proving the following affine invariant inequality relating the jacobi-determinant J_Γ and L_Γ:

$$|J_\Gamma(t_1, \ldots, t_d)| \geq \epsilon \prod_{j=1}^{d} |L_\Gamma(t_j)|^{\frac{1}{d}} \prod_{j<k} |t_j - t_k|. \tag{3}$$

In [12] Drury and Marshall proved (3) for monomial curves $\Gamma(t) = (t^{k_1}, \ldots, t^{k_d})$ where $k_1, \ldots, k_d$ are positive integers and thereby establishing (1) for this family of curves. In joint work with S. Dendrinos, we extend the work of Drury and Marshall from monomial curves to general polynomial curves $\Gamma(t) = (P_1(t), \ldots, P_d(t))$. For such Γ one easily sees that (3) cannot possibly hold for all $t_j \in \mathbb{R}$. Nevertheless we have

Theorem 1. *For $\Gamma(t) = (P_1(t), \ldots, P_d(t))$ where each P_j is a real polynomial, there is a decomposition of the reals $\mathbb{R} = \cup I$ into M disjoint intervals so that on each I^d, (3) holds. Moreover M and ϵ can be taken to depend only on d and the degrees of the polynomials P_j.*

As a consequence we establish (1) for general polynomial curves Γ on the sharp line $p' = \frac{d(d+1)}{2}q$ but only in the range $1 \le p < \frac{d^2+2d}{d^2+2d-d}$. We expect that this should be extended to the full range $1 \le p < \frac{d^2+d+2}{d^2+d}$; see [2] for this extension in particular cases. As another consequence of Theorem 1, in joint work with S. Dendrinos and N. Laghi, we extend some work of Oberlin [16] and obtain sharp universal bounds for (2) over the class of polynomial curves in dimensions two and three. Finally in joint work with S. Dendrinos and M. Folch-Gabayet, we have extended Theorem 1 and the corresponding consequences for (1) and (2) to the family of curves $\Gamma(t) = (R_1(t), \ldots, R_d(t))$ where each R_j is a general rational function.

2. Outline of proof of Theorem 1

The decomposition is produced in two stages. The first stage produces an elementary decomposition of $\mathbb{R} = \cup J$ so that on each interval J, various polynomial quantities (more precisely, certain determinants of minors of the $d \times d$ matrix $(\Gamma'(t) \cdots \Gamma^{(d)})$, including L_Γ) are single-signed. This allows us to write down a formula relating J_{Φ_Γ} and L_Γ. When $d = 2$ this formula is particularly simple; namely,

$$J_{\Phi_\Gamma}(s,t) = P_1'(s)P_1'(t) \int_s^t \frac{L_\Gamma(w)}{P_1'(w)^2} dw$$

for any $s, t \in J$ (here $\Gamma = (P_1, P_2)$). Simple examples show that (3) can fail on some J and therefore we need to decompose each $J = \cup I$ further so that on each I^d, (3) holds.

This second stage decomposition $J = \cup I$ is much more technical and derived from a certain algorithm which uses two further decomposition procedures generated by individual polynomials; one of these decomposition procedures has been used in other problems and first appeared in [4]. The algorithm exploits in a crucial way the affine invariance of the inequality (3); that is, the inequality is left invariant when Γ is replaced by $A\Gamma$ for any invertible $d \times d$ matrix A.

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On Stein's conjecture on the Polynomial Carleson Operator

Victor Lie

The topic that we address in our talk follows from the work in [5], [6] and has its origin in the celebrated Carleson-Hunt Theorem ([1],[3]):

Theorem (Carleson-Hunt). *If for $f \in C^1(\mathbb{T})$ we define the expression*

$$Cf(x) := \sup_{\substack{a\in\mathbb{Z}\\(a\in\mathbb{R})}} \left| \int_{\mathbb{T}} \frac{1}{y} e^{iay} f(x-y)\,dy \right|,$$

then
i) (Carleson) C is of weak type $(2,2)$.
ii) (Hunt) C is of strong type (p,p) for $1 < p < \infty$.

The proof of the above theorem brought to light a series of new techniques, which developed further, set the foundation of a new field: the time-frequency analysis. Following this, many analysts focused on further finding different approaches (proofs) and/or extending this result. In terms of different proofs two more are known: one due to Fefferman ([2]) and the other due to Lacey and Thiele ([4]). With respect to the second direction (extensions), Stein proposed the following

Conjecture (Stein). *Define*

$$C_d f(x) := \sup_{Q\in\mathbb{Q}_d}\left|p.v.\int_{\mathbb{T}}\frac{1}{y}e^{iQ(y)}f(x-y)dy\right|$$

where here $d\in\mathbb{N}$, $\mathbb{Q}_d$ is the class of all integer (real) coefficient polynomials Q with $deg(Q)\le d$, and $f\in C^1(\mathbb{T})$; then
i) C_d is of weak type $(2,2)$.
ii) C_d is of strong type (p,p) for $1<p<\infty$.

If the supremum in the above expression is restricted only to polynomials without linear term (call the corresponding operator $\tilde{C}_d$), then Stein (for $d=2$, [8]) and respectively Stein and Wainger (for general d, [9]) proved that ii) holds with C_d replaced by $\tilde{C}_d$.

Our aim in this talk is to provide a (partial) positive answer to this conjecture. Indeed the main results presented can be summarized as follows:

Theorem A. *i) C_d is of weak type $(2,2)$.*
ii) C_d is of strong type (p,p) for $1<p<2$.

More generally:

Theorem B. *Let $1<r<p<\infty$; then*

$$\|C_d f\|_{L^r(\mathbb{T})}\lesssim_{p,r,d}\|f\|_{L^p(\mathbb{T})}.$$

It is worth noting that Theorem A is a consequence of Theorem B via the results appearing in [7].

A key ingredient in the proof of Theorem B is a new perspective on determining the time-frequency localization of an object, which we call the *relational* time-frequency perspective; this perspective is based on the following loose assertion: "Interactions among objects (scalar products) precede the objects themselves in terms of importance." An important consequence of this viewpoint is a method of representing our objects (by drawing pictures) in which we encode their oscillation as well as their magnitude.

Further, we will develop analytic and combinatorial techniques (many of them inspired by [2]), adapting them to this relational perspective.

The actual proof of Theorem B will follow two basic steps:

Step 1 - a discretization procedure, in which we split our operator into "small pieces" that are "well-localized" in both time and frequency.

Step 2 - a selection algorithm, which relies on finding qualitative and quantitative criteria depending on which we decide how to glue the above-mentioned pieces together to obtain a global estimate on our operator.

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Geometric configurations in Euclidean, integer, and finite field geometries

Alex Iosevich

A theorem due to Furstenberg, Katznelson, and Weiss says that any δ-neighborhood of a subset of $\mathbb{R}^2$ of positive density contains a copy of every sufficiently large triangle. Bourgain removed the δ-neighborhood assumption for k-simplexes in $\mathbb{R}^d$, $d \geq k$.

In this talk we study finite field models and prove the following results:

Theorem 1. *Let $E \subseteq \mathbb{F}_q^d$, $d \geq 2$ with $|E| \gg q^{d\left(\frac{k-1}{k}\right)+\frac{k-1}{k}}$. Then E contains a copy of every k-point configuration.*

Theorem 2. *Let $E \subseteq \mathbb{F}_q^2$ with $|E| = pq$. Then E determines at least pq^3 non-congruent triangles.*

The methods are Fourier analytic and bootstrapping plays a major role.

Semilinear Schrödinger flows on hyperbolic spaces

Alexandru Ionescu
(joint work with Gigliola Staffilani)

We consider the defocusing semilinear Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{\mathbf{g}})u = u|u|^{2\sigma}; \\ u(0) = \phi, \end{cases} \tag{1}$$

on Riemannian manifolds $(M, \mathbf{g})$ of dimensions $d \geq 2$, for exponents $\sigma \in (0, \infty)$. The initial-value problem (1) has been studied extensively in the Euclidean geometry, for more general classes of nonlinearities, see the recent books [2] and [7] and the references therein. For example, on Euclidean spaces, it is known that the defocusing H^1 subcritical initial-value problem (1), which corresponds to exponents $\sigma \in (0, 2/(d-2))$, is globally well-posed in the energy space H^1; moreover, scattering to linear solutions is known in the restricted range $\sigma \in (2/d, 2/(d-2))$. In recent years, the more delicate problems that correspond to critical power nonlinearities, both in $\dot{H}^1$ (with $\sigma = 2/(d-2)$, $d \geq 3$) and L^2 (with $\sigma = 2/d$, $d \geq 2$), have also been considered.

Suitable solutions on the time interval $(-T, T)$ of (1) satisfy (at least formally) mass and energy conservation,

$$\begin{aligned} &E^0(u)(t) := ||u(t)||_{L^2(M)} = E^0(u)(0); \\ &E^1(u)(t) := \frac{1}{2}\int_M |\nabla u(t)|^2 \, d\mu + \frac{1}{2\sigma+2}\int_M |u(t)|^{2\sigma+2} \, d\mu = E^1(u)(0), \end{aligned} \tag{2}$$

for any $t \in (-T, T)$. In the case $\sigma \in (0, 2/(d-2)]$ these conservation laws suggest that H^1 is a suitable space to study the global behaviour of solutions of (1). Let $M = \mathbb{H}^d$, the hyperbolic space of dimension d. Our main theorem concerns global well-posedness and scattering in $H^1(\mathbb{H}^d)$ of the initial-value problem (1).

Theorem 1. *Assume $\sigma \in (0, 2/(d-2))$ is fixed.*

(a) (Global well-posedness) If $\phi \in H^1(\mathbb{H}^d)$ then there exists a unique global solution $u \in C(\mathbb{R} : H^1(\mathbb{H}^d))$ of the initial-value problem (1). *In addition, for any $T \in [0, \infty)$, the mapping*

$$\phi \to U_T(\phi) = \mathbf{1}_{(-T,T)}(t) \cdot u$$

is a continuous mapping from $H^1(\mathbb{H}^d)$ to $C((-T, T) : H^1(\mathbb{H}^d)$, and the conservation laws (2) *are satisfied.*

(b) (Scattering) For any $\phi \in H^1(\mathbb{H}^d)$ there exist unique $u_\pm \in H^1(\mathbb{H}^d)$ such that

$$||u(t) - e^{it\Delta_{\mathbf{g}}} u_\pm||_{H^1(\mathbb{H}^d)} = 0 \quad \text{as } t \to \pm\infty. \tag{3}$$

The main conclusion of the theorem is the H^1 scattering, particularly in the case of small exponents $\sigma \in (0, 2/d]$. We also emphasize that this theorem does not require radial symmetry. The conclusion of Theorem 1 (b) is in sharp contrast with its Euclidean analogue: on Euclidean spaces scattering to linear solutions is only known for exponents $\sigma \in (2/d, 2/(d-2))$, see [4, 6], and also for $\sigma =$

$2/(d-2)$, see [4, 3]. Moreover, on Euclidean spaces, scattering in H^1 is known to fail for exponents $\sigma \in (0, 1/d]$. Even the easier question of existence of the wave operator in the range $(1/d, 2/d]$ is not settled yet, and in particular no (unweighted) scattering results are known for the range $\sigma \in (1/d, 2/d]$.

Our proof of Theorem 1 depends on two main noneuclidean ingredients. The first ingredient is the inequality

$$\| f * |K| \|_{L^2(\mathbb{H}^d)} \leq C \|f\|_{L^2(\mathbb{H}^d)} \cdot \int_0^\infty |K(r)| e^{-\rho r} (r+1) (\operatorname{sh} r)^{2\rho} \, dr, \tag{4}$$

for any $f, K \in C_0^\infty(\mathbb{H}^d)$, provided that K is a radial kernel. For comparison, $\|K\|_{L^1(\mathbb{H}^d)} = C \int_0^\infty |K(r)| (\operatorname{sh} r)^{2\rho} \, dr$, thus the factor $e^{-\rho r}(r+1)$ in (4) represents a nontrivial gain[1] over the (best possible) Euclidean inequality $\| f * |K| \|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)} \|K\|_{L^1(\mathbb{R}^d)}$. We exploit this gain to prove the noneuclidean Strichartz estimates (as well as suitable inhomogeneous estimates)

$$\|e^{it\Delta}\phi\|_{L^q(\mathbb{R}\times\mathbb{H}^d)} \leq C_q \|\phi\|_{L^2} \text{ for any } q \in (2, (2d+4)/d]. \tag{5}$$

On Euclidean spaces, this global inequality holds only for $q = (2d+4)/d$.

The second noneuclidean ingredient we need is the existence of a smooth radial function $a : \mathbb{H}^d \to [0, \infty)$ with the properties

$$\Delta a = 1, \quad |\nabla a| \leq C, \quad \mathbf{D}^2 a \geq 0 \quad \text{on } \mathbb{H}^d.$$

We use this function and standard arguments to prove the Morawetz inequality

$$\|u\|^{2\sigma+2}_{L^{2\sigma+2}((-T,T)\times\mathbb{H}^d)} \leq C_\sigma \sup_{t\in(-T,T)} \|u(t)\|_{L^2(\mathbb{H}^d)} \|u(t)\|_{H^1(\mathbb{H}^d)}, \tag{6}$$

for any solution u of the nonlinear Schrödinger equation (1), with a constant C_σ that does not depend on T. Theorem 1 follows, using mostly standard arguments, from this Morawetz inequality and the noneuclidean Strichartz estimates (5).

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[1] This gain is related to the Kunze–Stein phenomenon $L^2(\mathbb{G}) * L^p(\mathbb{G}) \subseteq L^2(\mathbb{G})$ for any $p \in [1, 2)$, where $\mathbb{G}$ is the Lorentz group $SO(d, 1)$, see [5].

Holomorphic Sobolev spaces associated to compact symmetric spaces

Sundaram Thangavelu

Given a compact Lie group G and $f \in L^2(G)$ consider the convolution $f * h_t$ of f with the heat kernel h_t associated to the Laplacian on G. The Segal-Bargmann transform of f, also known as the heat kernel transform, is just the holomorphic extension of $f * h_t$ to the complexification $G_{\mathbb{C}}$ of G. In 1994 Hall [2] characterised the image of $L^2(G)$ under this transform as a weighted Bergman space. This extended the classical results of Segal and Bargmann, where the same problem was considered on $\mathbb{R}^n$. Later, Stenzel [5] treated the case of compact symmetric spaces obtaining a similar characterisation.

In 2004, Hall and Lewkeeratiyutkul [3] studied the Segal-Bargmann transform on Sobolev spaces $\mathbb{H}^{2m}(G)$ associated to compact Lie groups. They have shown that the images can be characterised as certain holomorphic Sobolev spaces. The problem of treating the Segal-Bargmann transform on Sobolev spaces defined over compact symmetric spaces remained open until recently. Our aim in this talk is to describe our results in [6] where we have characterised the images of the Sobolev spaces $\mathbb{H}^m(X)$ under the Segal-Bargmann transform as holomorphic Sobolev spaces when X is a compact symmetric space.

Using an interesting formula due to Lassalle [4], called the Gutzmer's formula, Faraut [1] gave a different proof of Stenzel's result. The same arguments can be extended to treat Sobolev spaces as well. For the proof of our main theorem we need some estimates on derivatives of the heat kernel on a noncompact Riemannian symmetric space. This is achieved by using a result of Flensted-Jensen. We also remark that the images of the Sobolev spaces turn out to be Bergman spaces defined in terms of certain weight functions which are not necessarily nonnegative. Nevertheless, they can be used to define weighted Bergman spaces. This is reminiscent of the case of the heat kernel transform on the Heisenberg group.

We consider a compact Riemannian symmetric space $X = U/K$. Let $\mathbf{u}$ and $\mathbf{k}$ be the Lie algebras of U and K and let θ be an involutive automorphism so that $\mathbf{u} = \mathbf{k} \oplus \mathbf{p}$ where $\mathbf{p} = \{Y \in \mathbf{u} : d\theta(Y) = -Y\}$. Let $\mathbf{a}$ be a Cartan subspace of $\mathbf{p}$. Let Δ be the Laplace-Beltrami operator on X with heat kernel h_t. Let $U_{\mathbb{C}}$ (resp.$K_{\mathbb{C}}$) be the universal complexification of U (resp.K). The complex homogeneous space $X_{\mathbb{C}} = U_{\mathbb{C}}/K_{\mathbb{C}}$, is a complex variety and gives the complexification of the symmetric space $X = U/K$. For every $f \in L^2(X)$ the convolution $f * h_t$ extends to $X_{\mathbb{C}}$ as a holomorphic function. We let $Y = G/K$ where G is a closed subgroup of $U_{\mathbb{C}}$, be the noncompact dual of X. Let γ_t^1 be the heat kernel associated to the Laplace-Beltrami operator on Y.

Recall that for each real umber s, the Sobolev space $\mathbb{H}^s(X)$ of order s can be defined as the completion of $C^\infty(X)$ under the norm $\|f\|_{(s)} = \|(1-\Delta)^{\frac{s}{2}} f\|_2$. We define the holomorphic Sobolev space $\mathbb{H}_t^s(X_{\mathbb{C}})$ to be the image of $\mathbb{H}^s(X)$ under the heat kernel transform. This can be made into a Hilbert space, simply by transfering the Hilbert space structure of $\mathbb{H}^s(X)$ to $\mathbb{H}_t^s(X_{\mathbb{C}})$. This means that if $F = f * \gamma_t, G = g * \gamma_t$, where $f, g \in \mathbb{H}^s(X)$ then $(F, G)_{\mathbb{H}_t^s(X_{\mathbb{C}})} = (f, g)_{\mathbb{H}_t^s(X)}$. Then, it is clear that the heat kernel transform is an isometric isomorphism from $\mathbb{H}^s(X)$ onto $\mathbb{H}^s_t(X_{\mathbb{C}})$. We are interested in realising $\mathbb{H}^s_t(X_{\mathbb{C}})$ as weighted Bergman spaces.

Theorem 1. *Let m be any integer. Then there exists a non-negative weight function w_t^m such that $F \in \mathbb{H}^m_t(X_{\mathbb{C}})$ if and only if*

$$\int_{X_{\mathbb{C}}} |F(z)|^2 w_t^m(z) dm(z) < \infty.$$

Moreover, the norm on $\mathbb{H}^m_t(X_{\mathbb{C}})$ is equivalent to the above weighted L^2 norm.

The weight function w_t^m involves derivatives of the heat kernel γ_t^1 associated to the noncompact dual of the symmetric space X. And hence, proving the non-negativity of w_t^m is not easy since we do not have explicit formulas for γ_t^1. However, the case of $\mathbb{H}^{-s}_t(X_{\mathbb{C}}), s > 0$ is much simpler, as the weight function is given by the Riemann-Liouville fractional integral

$$w_t^{-s}(\exp H) = \frac{1}{\Gamma(s)} \int_0^{2t} (2t-r)^{s-1} e^r \gamma_r^1(\exp 2H) dr.$$

Holomorphic Sobolev spaces can also be characterised in terms of pointwise estimates. We make use of the duality between $\mathbb{H}^s_t(X_{\mathbb{C}})$ and $\mathbb{H}^{-s}_t(X_{\mathbb{C}})$ in obtaining pointwise estimates. Once we have such a characterisation the following result for the image of $C^\infty(X)$ under the Segal-Bargmann transform can be proved, as the intersection of all Sobolev spaces is just $C^\infty(X)$.

Theorem 2. *A holomorphic function F on $X_{\mathbb{C}}$ is of the form $F = f * \gamma_t$ with $f \in C^\infty(X)$ if and only if it satisfies*

$$|F(u \exp(H))| \leq C_m (1 + |H|^2)^{-m/2} (\Phi(H))^{\frac{1}{2}} e^{\frac{1}{4t}|H|^2}$$

for all $u \in U, H \in i\mathbf{a}$ and for all positive integers m.

In the above theorem Φ is a certain function defined in terms of restricted roots. We also have a characterisation of the image of distributions on X under the heat kernel transform. If f is a distribution, $f * \gamma_t$ still makes sense and extends to $X_{\mathbb{C}}$ as a holomorphic function.

Theorem 3. *A holomorphic function F on $X_{\mathbb{C}}$ is of the form $F = f * \gamma_t$ for a distribution f on X if and only if it satisfies the estimate*

$$|F(u \exp(H))| \leq C (1 + |H|^2)^{m/2} (\Phi(H))^{\frac{1}{2}} e^{\frac{1}{4t}|H|^2}$$

for some positive integer m for all $u \in U$ and $H \in i\mathbf{a}$.

This result settles a conjecture stated in Hall-Lewkeeratiyutkul [3]. The proof is based on results for holomorphic Sobolev spaces as the union of all Sobolev spaces is precisely the space of distributions.

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Improvements in Wolff's inequality for the cone multiplier

Gustavo Garrigós
(joint work with W. Schlag and A. Seeger)

In this work we pursue a better understanding of an inequality proposed by T. Wolff in relation with two outstanding problems in Harmonic Analysis: the so-called *cone multiplier problem* and the local smoothing inequality for the wave equation (*Sogge's conjecture*). Wolff's deep methods imply the first optimal results for these two problems, which are valid in L^p only for large values of the exponent p (see [9], and [4] for the higher dimensional case). In our work we organize more efficiently the original proof to obtain improvements in the range of p, as well as stronger versions of Wolff's inequality with applications to other problems.
To be more precise, for small $\delta > 0$, consider δ-neighborhoods of the truncated light-cone in $\mathbb{R}^{d+1}$

$$\Gamma_\delta = \{(\tau,\xi) \in \mathbb{R}^{d+1} \ : \ 1 \le \tau \le 2 \quad \text{and} \quad \big|\tau - |\xi|\big| \le \delta\},$$

and the usual plate decomposition of Γ_δ subordinated to a covering of the sphere by $\sqrt{\delta}$-caps. Namely, given a maximal $\sqrt{\delta}$-separated sequence $\{\omega_k\} \subset S^{d-1}$ and a constant $c \approx 1$, we let

$$\Pi_k^{(\delta)} = \Big\{(\tau,\xi) \in \Gamma_\delta \ : \ \big|\xi/|\xi| - \omega_k\big| \le c\sqrt{\delta}\,\Big\}.$$

Wolff's inequality can then be written as follows (see [9]): *for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ (independent of δ and $\{\omega_k\}$) so that*

$$(1) \qquad \Big\|\sum_k f_k\Big\|_p \le C_\varepsilon\, \delta^{-\alpha(p)-\varepsilon}\Big(\sum_k \|f_k\|_p^p\Big)^{1/p}, \quad \forall\, f_k \ : \ \operatorname{supp}\widehat{f_k} \subset \Pi_k^{(\delta)},$$

where $\alpha(p) := d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$ is the standard Bochner-Riesz critical index in d dimensions. The inequality is conjectured to hold for all $p > 2(d+1)/(d-1)$, and for each such p, the power $\alpha(p)$ is optimal (except perhaps for $\varepsilon > 0$).

The methods developed by T. Wolff in his fundamental paper [9] give a positive answer to (1) for large p; namely $p > 74$ when $d = 2$, and $p > 2 + \min\{\frac{32}{3d-7}, \frac{8}{d-3}\}$

when $d \geq 3$ (the latter contained in the paper by Łaba and Wolff [4]). In both papers the authors announce that improvements over these indices should certainly be possible, although perhaps still far from the conjectured exponents. In fact, a slight improvement was already presented by Garrigós and Seeger in [2].

The most recent results, which appear in [3], give the following improved range.

Theorem 1. *Let $d \geq 2$. Then, for all $\varepsilon > 0$ the inequality (1) holds when $p > p_d$, where*

$$p_d = 2 + \tfrac{8}{d-2}\big(1 - \tfrac{1/2}{d+1}\big) \quad \textit{for} \quad d \geq 3, \quad \textit{and} \quad p_2 = 20 \quad \textit{for} \quad d = 2. \tag{2}$$

In [3] we also consider a stronger inequality than (1), which is motivated by questions on the Bergman projection for tube domains over light cones (see [1]). Namely, we consider a mixed norm $\ell^2(L^p)$ version of (1):

$$\Big\| \sum_k f_k \Big\|_p \leq C_\varepsilon \, \delta^{-\beta(p)-\varepsilon} \Big(\sum_k \|f_k\|_p^2 \Big)^{1/2}, \quad \forall \, f_k \ : \ \operatorname{supp} \widehat{f_k} \subset \Pi_k^{(\delta)}, \tag{3}$$

where now

$$\beta(p) = \frac{d-1}{4} - \frac{d+1}{2p}. \tag{4}$$

Note that the Wolff inequality (1) is implied by (3) and Hölder's inequality, since

$$\# \, k'\text{s} \; = \; \text{Card}\,(\Omega) \; \leq \; C \, \delta^{-\frac{d-1}{2}},$$

and $\beta(p) + \frac{d-1}{2}(\frac{1}{2} - \frac{1}{p}) = \alpha(p)$. As before, (3) is conjectured to hold for all $p > 2 + \frac{4}{d-1}$, in which case the power $\beta(p)$ in (4) is also best possible. It may also be conjectured the validity of (3) for all $2 < p \leq 2 + \frac{4}{d-1}$, in which case one must let $\beta(p) = 0$. The hardest case should be $p = 2 + \frac{4}{d-1}$, which by interpolation will imply all the other cases $p > 2$. However, none of these inequalities with the optimal exponent $\beta(p)$ seem to have been known before, even in the simpler setting when the cone is replaced by a sphere. Our contribution in [3] can be stated as follows.

Theorem 2. *Let $d \geq 2$. Then, the inequality (3) holds for all $\varepsilon > 0$ when*

$$p > p_d \quad \textit{for} \quad d \geq 3, \quad \textit{and} \quad p > 23 + \tfrac{1}{3} \quad \textit{for} \quad d = 2. \tag{5}$$

Moreover, when $d = 2$, an optimal $\ell^4(L^p)$ inequality (and in particular an $\ell^p(L^p)$ as in (1)) holds for all $p > 20$.

Among the consequences of these theorems we can list the following:

(i) *Sogge's conjecture:* For all $p > p_d$ and $\alpha > \frac{d-1}{2} - \frac{d}{p}$, we have

$$\Big(\int_1^2 \big\| e^{it\sqrt{-\Delta}} f \big\|_{L^p(\mathbb{R}^d)}^p dt \Big)^{1/p} \leq C \, \|f\|_{L^p_\alpha(\mathbb{R}^d)}.$$

(ii) *Cone multiplier*: For all $p \in (p_d, \infty)$ and $\alpha > \frac{d-1}{2} - \frac{d}{p}$, the Fourier multiplier

$$m_\alpha(\tau, \xi) = (1 - |\xi|^2/\tau^2)_+^\alpha$$

defines a bounded operator in $L^p(\mathbb{R}^{d+1})$.

(iii) *Averages over curves in* $\mathbb{R}^3$ *(see* [7]*):* Let $s \mapsto \gamma(s) \in \mathbb{R}^3$ be a smooth curve satisfying $\sum_{j=1}^n |\langle\theta, \gamma^{(j)}(s)\rangle| \neq 0$ for every unit vector θ, and let $\chi \in C_0^\infty(\mathbb{R})$. Then the convolution operator

$$A_t f(x) = \int f(x - t\gamma(s))\chi(s)ds$$

maps $L^p(\mathbb{R}^3)$ into $L^p_{1/p}(\mathbb{R}^3)$ for every $t > 0$, provided $\max\{n, (p_2+2)/2\} < p < \infty$.

(iv) *Radial multipliers (see* [5]*):* Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be radial, and $K_t = \mathcal{F}^{-1}[\varphi\widehat{K}(t\cdot)]$, for a fixed radial $\varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ (not identically zero). Then when $1 < q < p_d'$, K defines a bounded convolution operator in $L^q(\mathbb{R}^d)$ provided that

$$\sup_{t>0} \big\| K_t \|_{L^q(\mathbb{R}^d;(1+|x|)^\varepsilon\, dx)} < \infty, \quad \text{for some } \varepsilon > 0. \tag{6}$$

(v) *Bergman projections in tubes over light-cones (see* [1, 3]*).* Let $\mathcal{T} = \mathbb{R}^{d+1} + i\Omega$ denote the tube domain in $\mathbb{C}^{d+1}$ over the open light-cone $\Omega \subset \mathbb{R}^{d+1}$, and $Q(y) = y_0^2 - |y'|^2$ the associated Lorentz form. The (weighted) Bergman projections $\mathcal{P}_\gamma$ are bounded in $L^p(\mathcal{T}^{d+1}, Q(y)_+^\gamma\, dx\, dy)$ in the optimal range

$$1 + \frac{d-1}{2(\gamma+d+1)} < p < 1 + \frac{2(\gamma+d+1)}{d-1},$$

provided $\gamma \geq \max\big\{-1 + \frac{d-1}{4}(p_d - \frac{2(d+1)}{d-1}), \frac{d-1}{2}(p_d - \frac{2(d+1)}{d-1} - 1)\big\}$. (The conjectured range of weights is $\gamma > -1$.)

(vi) *Square function estimates (see* [8, 2]*):* For all $\alpha > \frac{1}{9}$ and all f_k with $\operatorname{supp} \widehat{f}_k \subset \Pi_k^{(\delta)}$, we have

$$\Big\| \sum_k f_k \Big\|_{L^4(\mathbb{R}^3)} \leq C_\alpha\, \delta^{-\alpha} \Big\| \big(\sum_k |f_k|^2\big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}.$$

This is a slight improvement over the previously known $\alpha > 5/44$, due to Tao-Vargas and Wolff [8, 10].

As a last comment, we observe that concerning the application to radial multipliers (iv), Nazarov and Seeger have recently shown a characterization theorem (with $\varepsilon = 0$ in (6)) for dimensions $d \geq 5$ in the range $1 < q < 2(d^2-2d-3)/(d^2-5)$, which in addition to the endpoint $\varepsilon = 0$, gives also a better range of p for large dimensions ($d \geq 6$). The proof involves different and very interesting techniques (see [NS], or the survey by A. Seeger in this issue).

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The Vitali covering lemma, equidistribution in nilmanifolds, and the primes

Terence Tao

In this talk we survey (in broad terms, without details) the basic strategy in additive prime number theory in being able to count the solutions to additive patterns in the primes, such as arithmetic progressions, originating from the classical arguments of Hardy-Littlewood and Vinogradov. The starting point is van der Corput's lemma, which gives a criterion for a sequence x_n in a torus $(\mathbb{R}/\mathbb{Z})^d$ to be equidistributed; a generalisation of this lemma also works for other spaces, such as nilmanifolds. This lemma gives quantitative bounds on exponential sums such as $\sum_n e(\phi(n))$ for polynomials ϕ, or $\sum_n F(g^n x)$ for some *nilsequence* $F(g^n x)$, in the case when the polynomial or nilsequence is suitably "irrational", "minor arc", or "ergodic". (We deliberately suppress the range of summation for the index n.) Using the TT^* method (or the large sieve inequality), this allows us to control bilinear summations such as $\sum_n \sum_m a_n b_m e(\phi(nm))$ or $\sum_n \sum_m a_n b_m F(g^{nm} x)$, where the sequences a_n, b_m obey size bounds but are otherwise allowed to be rough. Using divisor sum identities such as Vaughan's identity (which are based on truncating standard identities such as $\mu(n) = \sum_{abc=n} \mu(a)\mu(b)$), one can obtain non-trivial control of sums such as $\sum_n \mu(n) F(g^n x)$ in the minor arc case. Meanwhile, classical analytic number theory results (such as the prime number theorem in arithmetic progressions) control the major arc case, in which the nilsequence is approximately periodic.

The classical circle method of Hardy and Littlewood, based on the linear Fourier transform, lets one convert bounds on linear-phase exponential sums such as $\sum_n \mu(n)e^{2\pi i\alpha n}$, or $\sum_n \mu(n)F(g^n x)$ where $g^n x$ is an orbit in a 1-step nilmanifold (i.e. a torus), into an accurate count on solutions to some additive problems, such as the count of arithmetic progressions of primes of length 3. More recently, the nascent theory of *quadratic Fourier analysis* has allowed one to use bounds on quadratic-phase exponential sums such as $\sum_n e^{2\pi i\alpha^2 n}$, or $\sum_n \mu(n)F(g^n x)$ where $g^n x$ is an orbit in a 2-step nilmanifold, such as a quotient of the Heisenberg group, to control more complex additive patterns, such as arithmetic progressions of length 4. Some partial results are now also available for even more complex patterns.

The structure of monomial polyhedra and applications

Malabika Pramanik
(joint work with Alexander Nagel)

Let $\mathcal{P} = \{P_1, \cdots, P_d\}$ be a finite set of polynomials on $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For every $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{F}^n$ and every multi-radius $\vec{\delta} = (\delta_1, \cdots, \delta_d) \in (0, \infty)^d$, we define a *polynomial ball* $B_{\mathcal{P}}(\mathbf{x}; \vec{\delta})$ as the connected component containing $\mathbf{x}$ of the set

$$\widetilde{B}_{\mathcal{P}}(\mathbf{x}; \vec{\delta}) := \Big\{\mathbf{y} \in \mathbb{R}^n \Big| |P_j(\mathbf{y}) - P_j(\mathbf{x})| < \delta_j, \quad 1 \leq j \leq d\Big\}.$$

The structure and size of balls of the form $B_{\mathcal{P}}$ are of interest in various problems in analysis. Two examples are:

- **Study of maximal operators associated to polynomial polyhedra in $\mathbb{R}^n$.** Let us set for instance
$$\mathcal{M}_{\mathcal{P}}[f](\mathbf{x}) := \sup_{\vec{\delta}} |B(\mathbf{x}; \vec{\delta})|^{-1} \int_{B(\mathbf{x};\vec{\delta})} |f(\mathbf{y})|\, d\mathbf{y}.$$
The operator $\mathcal{M}_{\mathcal{P}}$ may be thought of as a natural generalization of the classical strong maximal function, and one would like to determine for which collections $\mathcal{P}$ of polynomials and which exponents $p \in [1, \infty]$ the operator $\mathcal{M}_{\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^n)$.
- **Estimation of the Bergman kernel on certain domains in $\mathbb{C}^n$,** such as Reinhardt or weakly pseudoconvex domains with polynomial defining inequalities.

In joint work with Alexander Nagel, we address these problems in the special case where $\mathcal{P}$ consists of monomials. Our structure theorem for monomial balls leads to L^p bounds for the maximal operator $\mathcal{M}_{\mathcal{P}}$ and sharp Bergman kernel estimates for a general class of weakly pseudoconvex domains of finite type, in particular the "cross of iron" domain.

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Radon-like operators and rank conditions

Philip T. Gressman

The purpose of this talk is to disucss new endpoint L^p-L^q and Sobolev inequalities for certain broad classes of highly degenerate Radon-like averaging operators. The study of such questions was begun by Phong and Stein [11], [12]. Since that time, the literature relating to this problem has grown both broad and deep, including but not limited to the works of Bak, Oberlin, and Seeger [1]; Cuccagna [2]; Greenblatt [3]; Greenleaf and Seeger [5], [6]; Lee [7], [8]; Phong and Stein [13]; Phong, Stein, and Sturm [14]; Pramanik and Yang [15]; Rychkov [16]; Seeger [17]; and Tao and Wright [18]. This literature provides a comprehensive theory of Radon transforms in the plane (optimal L^p-L^q and Sobolev bounds were established by Seeger [17] and others). Tao and Wright [18] have also established sharp (up to ϵ-loss) L^p-L^q inequalities for completely general averaging operators over curves in any dimension.

In the remaining cases, though, little has been proved regarding optimal inequalities for Radon-like operators. Among the reasons for this is that the rotational curvature (in the sense of Phong and Stein [9], [10]) is essentially controlled by a scalar quantity for averaging operators in the plane, but is governed in higer dimensions (and higher codimension) by a matrix condition which is increasingly difficult to deal with using standard tools. While it is generally impossible for rotational curvature to be nonvanishing in this case, the corresponding matrix can be expected to have nontrivial rank. Under this assumption, works along the lines of Cuccagna [2] and Greenleaf, Pramanik, and Tang [4] have been able to use this weaker information as a replacement for nonvanishing rotational curvature. In particular, Greenleaf, Pramanik, and Tang showed that optimal L^2-decay inequalities for "generic" oscillatory integral operators can be established in the highly degenerate case with only the knowledge that the corresponding matrix quantity has rank one or higher at every point away from the origin.

Fix positive integers n' and n'', and let S be a smooth mapping into $\mathbb{R}^{n''}$ which is defined on a neighborhood of the origin in $\mathbb{R}^{n'} \times \mathbb{R}^{n''} \times \mathbb{R}^{n'}$. We consider operators of the form

$$Tf(x', x'') := \int f(y', x'' + S(x', x'', y'))\psi(x', x'', y')dy', \tag{1}$$

where $x', y' \in \mathbb{R}^{n'}$ and $x'' \in \mathbb{R}^{n''}$ (n' represents the dimension of the manifolds over which f is averaged, and n'' represents the codimension). When no confusion arises, the variable x will stand for the pair (x', x''), and n will refer to the sum $n' + n''$.

The assumption to be made on S is that it exhibits a sort of approximate homogeneity (aka semiquasihomogeneity). The notation to be used to describe this scaling will be as follows: given any multiindex $\gamma := (\gamma_1, \ldots, \gamma_m)$ of length m, any $z := (z_1, \ldots, z_m) \in \mathbb{R}^m$, and any integer j, let $2^{j\gamma} z := (2^{j\gamma_1} z_1, \ldots 2^{j\gamma_m} z_m)$. The order of the multiindex γ will be denoted $|\gamma|$, is the sum of the entries, i.e., $\gamma_1 + \cdots + \gamma_m$, and may be negative in some cases. With this notation, it will be assumed that there exist multiindices α' and β' of length n' and α'' and β'' of length n'' such that the limit of

$$\lim_{j \to \infty} 2^{j\beta''} S(2^{-j\alpha'} x', 2^{-j\alpha''} x'', 2^{-j\beta'} y') =: S^P(x', x'', y') \tag{2}$$

as $j \to \infty$ exists and is a smooth function of x', x'', and y' which does not vanish identically (note that, given a smooth mapping S, there is always at least one choice of multiindices so that this condition holds). Furthermore, it will be assumed that $\beta''_i > \alpha''_i$ for $i = 1, \ldots, n''$. The assumption on α'' and β'' will together with (2) be referred to as the homogeneity conditions.

Additionally, for each pair (x, y') in the support of the cutoff ψ in (1) and each $\eta'' \in \mathbb{R}^{n''} \setminus \{0\}$, consider the $n' \times n'$ mixed Hessian matrix H^P whose (i, j)-entry is given by

$$H^P_{ij}(x', x'', y', \eta'') := \frac{\partial^2}{\partial x'_i y'_j} \left(\eta'' \cdot S^P(x', x'', y') \right). \tag{3}$$

The main theorems can be stated as follows:

Theorem 1. *Suppose that the operator* (1) *satisfies the homogeneity conditions and that the mixed Hessian* (3) *has rank at least* r *whenever* $(x', x'', y') \neq (0, 0, 0)$

and $\eta'' \neq 0$*. If the support of* ψ *is sufficiently near the origin and* $\frac{r}{n''} > \frac{|\alpha'|+|\beta'|}{|\beta''|}$ *then* T *maps* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$ *provided that the following inequalities are satisfied:*

$$\frac{|\beta'|+|\beta''|}{p} - \frac{|\alpha'|+|\beta''|}{q} < |\beta'|, \tag{4}$$

$$\left|\frac{1}{p} + \frac{1}{q} - 1\right| < 1 - \frac{2n''+r}{r}\left(\frac{1}{p} - \frac{1}{q}\right). \tag{5}$$

Additionally, T *maps* L^p *to* L^q *if either one of the inequalities* (4) *or* (5) *are replaced with equality. If both inequalities are replaced with equality, then* T *is of restricted weak-type* (p,q)*.*

Theorem 2. *Suppose that* T *satisfies the rank and homogeneity conditions and* $\frac{r}{n''} > \frac{|\alpha'|+|\beta'|}{|\beta''|}$ *(and the support of* ψ *is sufficiently near the origin). Then the operator* T *maps the space* $L^p(\mathbb{R}^n)$ *to the Sobolev space* $L^p_s(\mathbb{R}^n)$ $(s \geq 0)$ *provided that the following two conditions are satisfied:*

$$s \max\{\beta''_1, \ldots, \beta''_{n''}\} \leq \frac{|\alpha'|}{p} + |\beta'|\left(1 - \frac{1}{p}\right), \tag{6}$$

$$\frac{s}{r} < \frac{1}{2} - \left|\frac{1}{2} - \frac{1}{p}\right|. \tag{7}$$

In both theorems, the shape of the regions defined by the given inequalities are quadrilaterals; this is also the same shape that was first identified by Phong and Stein [11].

In addition to these theorems, quantitative estimates of the largest possible value of r are obtained in many cases. When the multiindices α', β' and α'' are considered fixed, a "positive fraction" of the choices of β'' will be examined and shown to generically admit large values of r. There are a variety of ways to formulate this concept; here a set of multiindices E of length n'' will be said to have lower density ϵ provided that

$$\liminf_{N\to\infty} \frac{\#\{\beta'' \in E \mid \beta''_i \leq N \ \forall i = 1, \ldots, n''\}}{N^{n''}} \geq \epsilon.$$

Let $\Lambda_{\alpha,\beta}$ be the space of all n''-tuples of real polynomials $(p_1, \ldots, p_{n''})$ in the variables x', y', and x'' (for $x', y' \in \mathbb{R}^{n'}$ and $x'' \in R^{n''}$) such that

$$p_l(2^{j\alpha'}x', 2^{j\alpha''}x'', 2^{j\beta'}y') = 2^{\beta''_l} p_l(x', x'', y')$$

for each integer j and $l = 1, \ldots, n''$; suppose further that $\Lambda^{\alpha,\beta}$ is given the topology of a real, finite-dimensional vector space. Each element $(p_1, \ldots, p_{n'})$ naturally induces an operator of the form (1) which satisfies the homogeneity condition. The strength of the condition (3) can now be quantified as follows:

Theorem 3. *Fix* α', α''*, and* β'*. Let* K_1 *be the least common multiple of the entries of* α', α'' *and* β'*; let* K_2 *be the number of distinct values (modulo* K_1*) taken by the sum* $\alpha'_i + \beta'_j$ *for* $i, j = 1, \ldots, n'$*. Then for any* β'' *in some set of lower* *density $K_1^{-n''}$, the operators* (1) *corresponding to the polynomials $\Lambda_{\alpha,\beta}$ generically satisfy the rank condition provided*

$$r < n' - \sqrt{(1-K_2^{-1})(n')^2 + 2n}.$$

In the context of averages over hypersurfaces with isotropic homogeneity (taking the entries of α', α'', and β' to equal one, corresponding to the case $n_X = n_Z$ in the work of Greenleaf, Pramanik, and Tang [4]), a generic mixed Hessian (3) has everywhere (exept the origin) rank at least $n' - 1 - \sqrt{2n'+2}$, and the hypotheses of the theorems 1 and 2 are satisfied for *any* choice of $\beta'' \geq 3$ when $n' > 25$. On the opposite extreme, a rank one condition holds provided that $n'' < \frac{n'(n'-4)}{2}$ (an extremely large codimension, similar to those encountered by Cuccagna [2] and well beyond the range of nonvanishing rotational curvature) and theorems 1 and 2 hold with $r = 1$ for all multiindices β'' satisfying $|\beta''| > (n')^2(n'-4)$.

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On matrices with (almost) prime entries, lattice points and ergodic theorems

Amos Nevo

1. Matrices with prime entries

Let P = primes in $\mathbb{Z}$.

Consider 2×2 matrices with determinant 2 and prime entries.

Question 1 : Is this set infinite ?

A positive answer will be quite close to the best result ever proved towards the twin primes problem (Chen, 1967), namely : There exist infinitely many pairs p, q with $q - p = 2$, such that p is prime and q has at most two prime factors.

Question 1 is still open. Let us formulate an even stronger quantitative version of it.

Fix any norm on $M_2(\mathbb{R})$, for example $\|A\|^2 = \mathrm{tr}\ (A^t A)$. Restrict the resulting metric to the algebraic variety

$$V = \{A \in M_2(\mathbb{R})\,;\, \det A = 2\}\,,$$

and consider the balls

$$B_T = \{A \in V\,;\, \|A\| \leq T\}\,.$$

The number of integral matrices in B_T, namely $|B_T \cap M_2(\mathbb{Z})|$, is well-known to be asymptotic to cT^2, where $0 < c < \infty$.

Question 2 : How many of these integral matrices have all their entries prime ?

2. Counting heuristics

The prime number theorem asserts that

$$\frac{1}{x}\,|\{p \in P\,;\, p \leq x\}| \sim \frac{1}{\log x}$$

Therefore, the expected number of matrices with prime entries in B_T is the total number of integral matrices, multiplied by the "probability" that the four entries of size $\leq T$ are all prime, namely $(1/\log T)^4$.

We are therefore led to formulating the following

Conjecture (Prime matrix theorem) (Nevo-Sarnak, 2006).

$$|\{A \in B_T\,;\, a, b, c, d \in P\}| \sim \frac{CT^2}{(\log T)^4}\,.$$

3. Progress towards the conjecture

Let P_r=integers with at most r prime factors (r-almost primes).
Theorem 1. (Nevo-Sarnak, 2006).
1) There exists a constant C such that the following upper bound holds

$$|\{A \in B_T\,;\, a, b, c, d \in P\}| \leq \frac{CT^2}{(\log T)^4}\,,$$

2) There exist constants r and D such that the following lower bound holds

$$|\{A \in B_T\,;\, a, b, c, d \in P_r\}| \geq \frac{DT^2}{(\log T)^4}\,.$$

The method of proof is based on the following ingredients

(1) Sieve theory,
(2) Congruence groups,
(3) Counting lattice points,
(4) Ergodic theory and spectral estimates.

Parts (3) and 4 consist of an ergodic-theoretic estimate of the error term in the lattice point counting problem, uniformly over all congruence subgroups and their cosets. Namely, $G/\Gamma = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is a probability space with an ergodic action of $SL_2(\mathbb{R})$. The same holds for G/Δ for all finite-index subgroups $\Delta \subset \Gamma$. Consider the averaging operators on $L^2(G/\Delta)$, given by

$$\beta_T f(x) = \frac{1}{\text{vol }(B_T)} \int_{g \in B_T} f(g^{-1}x)dg$$

Theorem 2. (Nevo-Sarnak, 2006, see also Gorodnik-Nevo, 2005). Let Γ be any lattice in $G = SL_2(\mathbb{R})$, $\Delta \subset \Gamma$ a finite index subgroup, and fix any norm on $M_2(\mathbb{R})$. If the corresponding averaging operators β_T on $L^2(G/\Delta)$ satisfy the norm bound

$$\left\| \beta_T f - \int_{G/\Delta} f(x)d\lambda(x) \right\| \leq C(\text{vol } B_T)^{-\theta}\,\|f\|$$

then the lattice point counting problem in the norm ball B_T has the solution

$$|B_T \cap \gamma\Delta| = \frac{\text{vol } B_T}{[\Gamma : \Delta]} + O\left(\text{vol } B_T)^{1-\theta/4}\right)$$

for every coset $\gamma\Delta$.

4. General results

Theorems 1 and 2 are instances of much more general results, as follows.

(1) **(almost) prime points on principal homogeneous spaces**. Theorem 1 applies to almost prime points in every principal homogeneous space of a semisimple algebraic group defined over $\mathbb{Q}$, with certain necessary caveats. Property τ for congruence subgroups of the group of integer points gives uniformity of the spectral gap for β_T (property τ was recently established in full generality by Clozel).

(2) **Counting lattice points.** Theorem 2 is valid for all semisimple S-algebraic groups and in fact has a version in considerably greater generality, valid also for semisimple groups over the ring of adeles. In addition, it is possible to apply similar ideas to solve the uniform lattice points counting problem on homogeneous affine symmetric varieties defined over $\mathbb{Q}$, We refer to [GN2] for these generalizations.
(3) **(almost) prime points on homogeneous symmetric varieties**. Using the previous two items, the methods presented give a lower bound on the number of almost prime points on the an affine homogeneous symmetric variety. A quantitative version of this result is currently being developed.

5. Matrices with genuine prime entries

The case of 2×2 matrices is actually the hardest. In the case of $n \times n$ matrices where $n \geq 3$ one can establish :

Theorem 3. (Nevo-Sarnak, 2006).

The set of $n \times n$ matrices with prime entries and determinant 2^{n-1} is Zariski dense in the set of $n \times n$ matrices with determinant 2^{n-1}.

In fact, one can establish that there are at least $T^{a(n)}$ prime matrices in a ball of radius T, with some $a(n) > 0$. The conjecture calls for $C_n T^{n^2-n}/(\log T)^{n^2}$ such matrices.

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Estimates for oscillatory integrals related to Fourier restriction to curves

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Let $\gamma \subset \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$ defined by a differentiable map $\varphi : [-1, 1] \mapsto \mathbb{R}^n$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ be a fixed smooth function with compact support. Consider the following measure (charge) concentrated on that curve: $d\mu_\alpha := (\varphi', \varphi')^\alpha \psi(\varphi(t))dt$, where α is a real number satisfying the condition: $\int_\gamma (\varphi'(t), \varphi'(t))^\alpha |\psi(\varphi(t))| dt < \infty$.

Then the Fourier transform of that charge is defined by: $\widehat{d\mu_\alpha}(\xi) = \int_\gamma e^{2\pi i(\xi,x)} d\mu_\alpha$.

Proposition of the main problem: Find $q(\gamma, \alpha) := \inf\{q \in [1, +\infty] : \widehat{d\mu_\alpha} \in L^q(\mathbb{R}^n)\}$, where $L^q(\mathbb{R}^n)$ space of summation functions with degree q $(1 \leq q < \infty)$, in the case $q = \infty$ we have the space of essentially bounded functions.

Remark 1. Note that in general may be $q(\gamma,\alpha)=\infty$. For example if $\gamma\in\mathbb{R}^3$ is the unite circle and ψ is a nontrivial function then we have $\widehat{d\mu_\alpha}(\xi)\notin L^q(\mathbb{R}^n)$ for any $q<\infty$.

1. Brief outline of history and motivations of the problem.

The analogical problem for the following trigonometric integrals:

$$J(\xi):=\int_0^1 e^{2\pi i P(t,\xi)}dt, \quad \text{where} \quad P(t,\xi):=\xi_1 t+\xi_2 t^2+\cdots+\xi_n t^n$$

has been considered by I.M. Vinogradov [13]. He obtain the following estimate for the trigonometric integral $J(\xi)$:

$$|J(\xi)|\le \min\{1,\, 32|\xi|_\infty^{-1/n}\}, \quad \text{where} \quad |\xi|_\infty:=\max_{1\le j\le n}|\xi_j|.$$

From the last estimate it follows that if $q>n^2$ then $J(\xi)\in L^q(\mathbb{R}^n)$.

Further, Hua-Lo-Geng [7] proved that if $q>0.5n^2+n$ then $J(\xi)\in L^q(\mathbb{R}^n)$.

Finally, by Arkhipov G.I., Karatsuba A.A. and Chubarikov V.N. proved that $J(\xi)\in L^q(\mathbb{R}^n)$, whenever $q>n(n+1)/2+1$ and $J(\xi)\notin L^q(\mathbb{R}^n)$ for $q\le n(n+1)/2+1$. Moreover, they considered trigonometric integrals with phase function $P(t,\xi):=\xi_1 t^{k_1}+\xi_2 t^{k_2}+\cdots+\xi_n t^{k_n}$, where $k_1<k_2<\cdots<k_n$ are positive integer numbers satisfying the condition $q_0:=k_1+k_2+\cdots+k_n>n(n+1)/2$. It is proved that if $q>q_0$ then $J(\xi)\in L^q(\mathbb{R}^n)$ and if $q\le q_0$ then $J(\xi)\notin L^q(\mathbb{R}^n)$.

The problem on summation of trigonometric integrals for multidimensional cases remains open until now. Although, there are some partial results (see [1], [10] and also [9]).

It is well-known that the restriction problem of Fourier transform to smooth surfaces is connected to the summation property of trigonometric integrals [10].

The restriction problem for smooth curves with torsion has been considered by M. Christ [4], S. Drury [5]. Moreover, estimates for oscillatory integral operators associated to the restriction problem are connected to the summation problem (see [6], [2]).

Note that from the results of the paper [1] it follows sharpness of estimates obtained by S. Drury [5] and also J.-G.-Bak and S. Lee [2] for model curves with torsion.

Therefore, it is important to obtain analogical estimates for curves with torsion and for more general finite type curves.

2. Discussion of results.

Let $\gamma\subset\mathbb{R}^n$ be a differentiable curve. We assume $\varphi:[-1,1]\mapsto\mathbb{R}^n$ is a fixed parameter.

The group C^∞ of diffeomorphisms of the interval $[-1,1]$ is denoted by H and $G:=\mathbb{R}^n\otimes O(n,\mathbb{R})$ is the group of motion of Euclidian space.

Denote by M the set of differentiable curves in $\mathbb{R}^n$. The group (G,H) acts on that set by [8]: $\rho(g,h)(\varphi)(t):=g\varphi(h^{-1}(t))$.

Definition 2. The map $s : M \mapsto \mathbb{R}$ is called to be an invariant with weight r of the group (G, H) if the following relation $s(\rho(g,h)(\varphi))(t) = ((h^{-1})'(t))^r s(\varphi)(h^{-1})(t)$ holds for any $(g,h) \in (G,H)$.

We define the following invariants with weights 2 and $n(n+1)/2$ respectively:

$$s_d(t) := (\varphi'(t), \varphi'(t)), \quad s_\kappa(t) = det[\varphi'(t), \, \varphi''(t), \, \dots, \varphi^{(n)}(t)].$$

The curve γ is called to be a curve with torsion if $s_\kappa(t) \neq 0$ for any $t \in [-1,1]$. It is called a finite type curve if $s_\kappa(t)$ has no roots of infinite order. Note that in the book by E.M. Stein [5] it is defined a notion of finite type surfaces. For curves the both definitions are equivalent. It is easy to see that the definitions are well-defined i.e. there are do not depend on parametrization.

Theorem 3. *Let $\gamma \in \mathbb{R}^n$ be a curve with torsion and $d\mu_\alpha := s_d^\alpha(t)\psi(\varphi(t))dt$ be a smooth charge on the curve γ. Then we have $\widehat{d\mu_\alpha(\xi)} \in L^q(\mathbb{R}^n)$ for any $q > n(n+1)/2+1$. Moreover, if $\psi(0) \neq 0$ and ψ is a smooth function concentrated in a sufficiently small neighborhood of the origin, then $\widehat{d\mu_\alpha(\xi)} \notin L^q(\mathbb{R}^n)$ for $q \leq n(n+1)/2+1$. In particular, for any real number α we have: $q(\gamma,\alpha) = n(n+1)/2+1$.*

Note that the sharp exponent of summation $n(n+1)/2+1$ of such oscillatory integrals is connected to weight of the invariant $s_\kappa(t)$. Since $s_d(t) \neq 0$ the result does not on a real number α.

Let $s_\kappa(t)$ be a function which has no roots of infinite order. We introduce some notation

$$m_d := \max\{k_d(t) : s_d(t) = 0\}, \quad m_\kappa := \max\{k_\kappa(t) : s_\kappa(t) = 0\},$$

where $k(t)$ is a multiplicity of root t of the function $s(t)$.

For any $\alpha > -\frac{1}{m_d}$ we define:

$$q(\gamma,\alpha) = \max\left\{\frac{n(n+1)}{2} + 1, \, \frac{2k_\kappa(t) + n(n+1)}{2(1+k_d(t)\alpha)} \, : \, s_\kappa(t) = 0\right\}.$$

Theorem 4. *Let $\gamma \in \mathbb{R}^n$ be a finite type curve and $d\mu_\alpha = \psi(\varphi(t))s_d^\alpha(t)dt$ be an associated charge. Then we have $\widehat{d\mu_\alpha} \in L^q(\mathbb{R}^n)$ for any $q > q(\alpha,\gamma)$. Moreover, if $t_0 \in [-1,1]$ is a point with properties*

$$q(\gamma,\alpha) = \frac{2(k_\kappa(t_0) + n(n+1))}{2(1+k_d(t_0)\alpha)}$$

for $q(\gamma,\alpha) > n(n+1)/2+1$ and t_0 is a point satisfying the condition $k_\kappa(t_0) \neq 0$ in the case $q(\gamma,\alpha) \leq n(n+1)/2+1$ and also ψ is a smooth function concentrated in a sufficiently small neighborhood of the point $\varphi(t_0)$ with $\psi(\varphi(t_0)) \neq 0$, then $\widehat{d\mu_\alpha} \notin L^{q(\alpha,\gamma)}(\mathbb{R}^n)$.

Corollary 5. *Let $\gamma \in \mathbb{R}^n$ be a finite type curve and $d\mu_\alpha = \psi(\varphi(t))s_d^\alpha(t)dt$. Then the following operator $R^*f(t) := \int_\gamma f(t)e^{2\pi i(\xi,x)}d\mu_\alpha$ has no type (p,q) for any $q \leq q(\gamma,\alpha)$ and $p \geq 1$.*

Let's consider the oscillatory integral operator:

$$T_\lambda f(x) := \int_{\mathbb{R}} e^{i\lambda\Phi(x,t)} a(x,t) f(t) dt, \tag{2.1}$$

where $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ and $\Phi(x,t)$ is a real-valued smooth phase function.

From the Theorem 3 and from results proved by J. Mokenhaupt [10] and J. C. Bak and S. Lee [2] it follows the following Theorem.

Theorem 6. *Let T_λ be a family of oscillatory integral operators defined by the relation (2.1) and the phase function $\Phi(x,t)$ satisfies the following Carleson–Sjölin* [3] *condition:*

$$det(\partial_t \nabla\Phi(x,t), \partial_t^2 \nabla\Phi(x,t), \dots, \partial_t^n \nabla\Phi(x,t)) \neq 0$$

on the support of the amplitude function $a(x,t)$ and $a(0,0) \neq 0$. Then the following conditions are equivalent:
1) $\frac{1}{p} + \frac{n(n+1)}{2q} \leq 1, \quad q > \frac{n^2+n+2}{2}$;
2) there exists a positive real number C such that, for any $f \in L^p(\mathbb{R})$ and $\lambda > 1$ it holds the following estimate:

$$\|T_\lambda f\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{-\frac{n}{q}} \|f\|_{L^p(\mathbb{R})}. \tag{2.2}$$

Now, we consider estimates for oscillatory integral operators associated to finite type curves. Suppose that the phase function has the form:

$$\Phi(x,t) = x_1 t^{k_1} b_1(t) + x_2 t^{k_2} b_2(t) + \cdots + x_n t^{k_n} b_n(t), \tag{2.3}$$

where $b_1, b_2, \dots, b_n$ are smooth functions satisfying the condition: $b_1(0) b_2(0) \dots b_n(0) \neq 0$.

Theorem 7. *Let T_λ be a family of oscillatory integral operators defined by the relation (2.1) and the phase function $\Phi(x,t)$ is defined by (2.3) and $a(x,t) \equiv a(t)$ is a smooth function concentrated in a sufficiently small neighborhood of the origin of $\mathbb{R}$ and $a(0) \neq 0$. Assume $q_0 := k_1 + k_2 + \cdots + k_n > n(n+1)/2$. Then the following conditions are equivalent:*
1) $\frac{1}{p} + \frac{q_0}{q} \leq 1, \quad q > q_0$;
2) there exists a positive real number C such, that the estimate (2.2) holds for any $f \in L^p(\mathbb{R})$ and $\lambda > 1$.

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Cauchy Integrals for non-smooth domains in $\mathbb{C}^n$ vs. $\mathbb{C}$: the effect of dimension

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(joint work with E. M. Stein)

1. Introduction

This is an announcement of work that will appear in the forthcoming paper [LS]. Depending on context, here and below σ will denote arc-length, or surface measure. Our main goal is to extend to several complex variables the celebrated result by A. Calderon [Ca] and Coifman-McIntosh-Meyer [CMM] concerning the regularity on $L^p(bD, \mathrm{d}\sigma)$ of the singular integral operator associated with the classical Cauchy integral:

$$C\varphi(z) \;=\; \frac{1}{2\pi i}\int\limits_{w\in bD}\frac{\varphi(w)}{w-z}\,\mathrm{d}w, \quad z\in D \tag{1}$$

where $D\subset\mathbb{C}$ is a domain with Lipschitz boundary. This result is most effectively proved by means of a $T(1)$ theorem applied to the closely related operator

$$T_C\varphi(z) \;=\; \frac{1}{2\pi i}\int\limits_{t\in\mathbb{R}}\frac{\varphi(t+if(t))}{x-t+i(y-f(t))}\,\mathrm{d}t, \quad z=x+iy\in D, \tag{2}$$

where $f:\mathbb{R}\to\mathbb{R}$ satisfies a Lipschitz condition:

$$|f(t)-f(s)|\le C|t-s|, \quad s,t\in\mathbb{R}.$$

By Cauchy formula we have

$$T_C(1) = -i\,T_C(f')$$

where $f'=\mathrm{d}f/\mathrm{d}t$. If f is merely Lipschitz then $T_C(f')$ will, in general, be of bounded mean oscillation and no better. Thus, although far from being optimal, in complex dimension 1 (the planar domain setting) the Lipschitz condition already captures the main difficulty of the general case[1]: this is our motivation for the investigation of an analogue condition in higher dimension.

The kernel that is canonically associated with the Lipschitz condition in $\mathbb{C}^n$ is the Martinelli-Bochner kernel:

$$K(z,w) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\bar{w}_j - \bar{z}_j}{|w-z|^{2n}}\, \mathrm{d}w_j \wedge \left(\bigwedge_{\ell \neq j} \mathrm{d}\bar{w}_\ell \wedge \mathrm{d}w_\ell \right), \tag{3}$$

see e.g. [Ra]. This kernel is intimately related to the double layer potential operator (see e.g. [K] and [CMM]) in the sense that the latter is the analogue, for the Martinelli-Bochner integral, of the operator (2). In particular, it follows that the singular integral operator associated with the Martinelli-Bochner integral for a Lipschitz domain $D \subset \mathbb{C}^n$ is bounded: $L^p(bD, \mathrm{d}\sigma) \to L^p(bD, \mathrm{d}\sigma)$, $1 < p < +\infty$. Note, however, that the one-dimensional Cauchy kernel:

$$C(z,w) = \frac{1}{2\pi i} \frac{\mathrm{d}w}{w-z} \tag{4}$$

is holomorphic in the parameter $z \in D$, see (1), whereas $K(z,w)$ is not (unless $n = 1$). This lack of holomorphicity drastically reduces the applications of the Martinelli-Bochner integral to complex function theory[2], so we refine our goal and investigate higher dimensional versions of Cauchy-type integrals with kernels that are *holomorphic* in the parameter $z \in D$. The simplest such example is given by the Leray kernel:

$$L(z,w) = \frac{1}{(2\pi i)^n} \frac{\partial\rho(w) \wedge (\overline{\partial}\partial\rho)^{n-1}(w)}{\langle \partial\rho(w), w-z \rangle^n}, \tag{5}$$

see Leray [Le] and Norguet [No], where $\rho : \mathbb{C}^n \to \mathbb{R}$ is a defining function for D and we have set

$$\langle \partial\rho(w), w-z \rangle = \sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)(z_j - w_j). \tag{6}$$

Note that when $n = 1$ both of (3) and (5) reduce to (4). Also, note that the Leray kernel is not canonical in the sense that it is defined in terms of a defining function of the domain (this is not the case for the one-dimensional Cauchy kernel, see (4), nor for the Martinelli-Bochner kernel); this fact has a deep impact on both of our two main concerns:

• *Holomorphicity of the kernel.* The Leray kernel is of course holomorphic in $z \in D$ provided the condition

$$\langle \partial\rho(w), w-z \rangle \neq 0 \quad z \in D, w \in bD \tag{7}$$

[1] that is the case when the domain has Ahlfors-regular boundary

[2] to name two: the $\overline{\partial}$-problem; representation of Hardy spaces of holomorphic functions, see [Ra]. For additional applications see Kerzman-Stein [KS].

is satisfied. On account of the identity

$$|\,\langle\partial\rho(w), w-z\rangle\,| = \frac{\mathrm{d}^E(z, T_w^{\mathbb{C}}(bD))}{|\nabla\rho(w)|},$$

where $T_w^{\mathbb{C}}(bD)$ denotes the complex tangent space to D at $w\in bD$ (see e.g. [Ra]) and d^E denotes the Euclidean distance in $\mathbb{C}^n$, we see that naturally associated with (7) is the notion of *strong* $\mathbb{C}$*-linear convexity*, see [APS]:

Definition 1. *A bounded open set $D\subset\mathbb{C}^n$ is said to be strongly $\mathbb{C}$-linearly convex if the boundary of D is of class C^1 and, moreover, there is $0<C<+\infty$ such that the following inequality:*

$$\mathrm{d}^E(z, T_w^{\mathbb{C}}(bD)) \geq C|z-w|^2 \tag{8}$$

holds for all $w\in bD$ and $z\in\bar{D}$.

• *Identification of the correct analogue of the Lipschitz boundary condition.* Strong $\mathbb{C}$-linear convexity puts a constraint on the minimal amount of boundary regularity that is required to define the Leray kernel, namely: $D\in C^1$. Looking at the numerator of $L(z,w)$ we see that, in fact, an additional amount of regularity is needed; an obvious condition is: $D\in C^2$ but we claim it suffices to require: $D\in C^{1,1}$, that is $\rho\in C^{1,1}(\mathbb{C}^n)$. Passing from C^2 to $C^{1,1}$ raises an interesting question: on the one hand, by the Rademecher theorem, we have that $\overline{\partial}\partial\rho(w)$ exists in the sense of distributions for a.e. $w\in\mathbb{C}^n$. On the other hand, in dealing with the Leray integral one only uses $w\in bD$, and the latter has measure zero in $\mathbb{C}^n$. Whereas $\overline{\partial}\partial\rho$ may indeed not exist on bD if $D\in C^{1,1}$, it turns out that its *tangential component*: $j^*(\overline{\partial}\partial\rho)(w)$ exists as a distribution for σ-a.e. $w\in bD$, and this amount of control is enough to establish the existence of the Leray integral oprator.

2. Statement of the main results

In studying the one-dimensional Cauchy integral (1) the auxiliary operator (2) is introduced in order to deal with the fact that the measure occurring in the Cauchy kernel (4) is non positive:

$$\int_{bD} \mathrm{d}w = 0.$$

In higher dimension this is no longer an issue: the measure induced on the boundary by the numerator of the Leray kernel is

$$\mathrm{d}\mu_\rho(w) = \frac{1}{(2\pi i)^n} j^*(\partial\rho\wedge(\overline{\partial}\partial\rho)^{n-1})(w)\,, \qquad w\in bD,$$

where $j^*:\Lambda_r(\mathbb{C}^n)\to\Lambda_r(bD)$ denotes the pull back of the inclusion: $j: bD\hookrightarrow\mathbb{C}^n$ acting on r-forms ($0\leq r\leq 2n-1$). We will henceforth refer to μ_ρ as the *Leray ρ-measure for bD.*

Theorem 2 ([LS]). *Suppose $D = \{\rho < 0\} \subset \mathbb{C}^n$ is a strongly $\mathbb{C}$-linearly convex, bounded domain of class $C^{1,1}$. Then we have that the density $\mathrm{d}\mu_\rho(w)$ is defined for σ-a.e. $w \in bD$ and, moreover*

$$\mathrm{d}\mu_\rho \approx \mathrm{d}\sigma.$$

In particular, it follows that the Leray-ρ measure is positive.

Proposition 3 ([LS]). *With same setting as above, we have*

$$\varphi(z) = \int\limits_{w\in bD} \frac{\varphi(w)}{\langle \partial\rho(w), w - z\rangle^n} \mathrm{d}\mu_\rho(w), \quad z \in D, \quad \varphi \in \vartheta(D) \cap C(\bar{D}). \tag{9}$$

Here, $\vartheta(D)$ denotes the space of holomorphic functions.

Theorem 4 ([LS]). *With same setting as above, we have that (bD, μ_ρ, d) is a space of homogeneous type, where we have set*

$$d(z, w) = |\langle \partial\rho(w), w - z\rangle|^{\frac{1}{2}}, \quad z, w \in bD. \tag{10}$$

Moreover, we have

$$d(z, w) \approx |z - w| + |\mathrm{Im}\langle \partial\rho(w), z - w\rangle|^{\frac{1}{2}}, \quad z, w \in bD; \tag{11}$$

$$\int\limits_{d(z,w)<R} d(z, w)^{-2n+\delta} \mathrm{d}\mu_\rho(w) \le C\, R^{\delta} \quad \textit{for all } \delta > 0. \tag{12}$$

We may now define the Leray integral as an operator acting on the boundary:

$$\mathbb{L}\varphi(z) = \int\limits_{w\in bD} \frac{\varphi(w) - \Phi(z)}{\langle \partial\rho(w), z - w\rangle^n}\, \mathrm{d}\mu_\rho(w) + \Phi(z), \; z \in \overline{D}, \tag{13}$$

where $\Phi \in C^1(\bar{D})$ is an extension[3] of φ to $\overline{D}$. It follows from (11) that the absolute value of the integrand in (13) is bounded by $d(z, w)^{-2n+1}$, and the latter (as a function of $w \in bD$) belongs to $L^1(bD, \mu_\rho)$, see (12). Thus, $\mathbb{L}\varphi(z)$ is an absolutely convergent integral for all $\varphi \in C^1(bD)$ and $z \in \overline{D}$. On account of the reproducing formula for holomorphic functions, see Proposition 3, for $z \in D$ we have that $\mathbb{L}\varphi(z)$ agrees with the value of the boundary integral with kernel (5). When $z \in bD$ the definition of $\mathbb{L}\varphi(z)$ does not depend on the choice of the C^1 extension of φ.

Theorem 5 ([LS]). *With same setting as above, we have*

$$\mathbb{L}^*(1) \in Lip^{\frac{1}{2}}(bD, d), \tag{14}$$

that is

$$|\mathbb{L}^*(1)(z) - \mathbb{L}^*(1)(w)| \le C d^{\frac{1}{2}}(z, w), \quad z, w \in bD.$$

Here, $\mathbb{L}^*$ denotes the formal $L^2(bD, \mu_\rho)$-adjoint of $\mathbb{L}$.

We may now state our main result.

[3] the extension function is obtained by applying a linear operator to φ, so $\mathbb{L}$ is linear in φ.

Theorem 6 ([LS]). *Suppose $D = \{\rho < 0\} \subset \mathbb{C}^n$ is a strongly $\mathbb{C}$-linearly convex, bounded domain of class $C^{1,1}$.*

Then, we have that $\mathbb{L}$ extends to a bounded operator: $L^p(bD, \mu_\rho) \to L^p(bD, \mu_\rho)$, $1 < p < +\infty$.

Corollary 7 ([LS]). *With same setting as above, we have that $\mathbb{L}$ extends to a bounded operator: $L^p(bD, \sigma) \to L^p(bD, \sigma)$, $1 < p < +\infty$.*

3. CONCLUSION

On the one hand, in light of the previous remarks we see that, for the Leray kernel, the correct higher dimensional analog of the Lipschitz condition for planar domains appears to be the notion of strong $\mathbb{C}$-linearly convexity paired up with $C^{1,1}$-boundary regularity.

On the other hand, as recalled earlier, the proof of the one-dimensional result requires the application of the $T(1)$ theorem in its sharpest form: $T(1), T^*(1) \in BMO$; whereas on account of Theorem 5, the proof of Theorem 6 is obtained by applying the $T(1)$ theorem, for spaces of homogeneous type, in its simplest form: $T(1) = T^*(1) = 0$, to the auxiliary operator

$$T_{\mathbb{L}}\varphi(z) = \mathbb{L}\varphi(z) - \frac{1}{\mu_\rho(bD)} \int\limits_{w \in bD} \mathbb{L}^*(\bar{1})(w)\varphi(w) \mathrm{d}\mu_\rho(w),$$

see e.g. [Ch]. The reason for this phenomenon is purely dimensional and is due to the fact that when $n = 1$, passing from C to T_C brings up a change of measure (from $\mathrm{d}w$ to $\mathrm{d}\sigma$), whereas for $n \geq 2$ passing from $\mathbb{L}$ to $T_{\mathbb{L}}$ does not.

One may ask whether either of the assumptions: strong $\mathbb{C}$-linear convexity (resp. $C^{1,1}$-boundary regularity) may be weakened, to $\mathbb{C}$-linear convexity (resp. $C^{1,\alpha}$-boundary regularity). The answer is no: in Barrett-Lanzani [BL], with different methods, the following examples are obtained:

- **Example 1**. $D \Subset \mathbb{C}^2$ complete Reinhardt: $bD \in C^\infty$; D weakly $\mathbb{C}$-linearly convex.
- **Example 2.** $D \Subset \mathbb{C}^2$ complete Reinhardt: $bD \in C^{1,\alpha}$ (for any given $0 < \alpha < 1$); D strongly $\mathbb{C}$-linearly convex.

In either case we have that $\mathbb{L}$ is unbounded: $L^2(bD, \mu) \to L^2(bD, \mu)$, where $\mu \in \{\sigma;\ \mu_\rho\}$.

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Sharp weighted bound for Calderón-Zygmund Singular Integral Operators and Sobolev inequalities

Carlos Pérez

(joint work with A. Lerner, S. Ombrosi, K. Moen and R. Torres)

In the first part of this talk we will present recent results about sharp weighted estimates for Calderón-Zygmund singular integral operator assuming that the weight satisfy the A_1 condition. To be more precise we presented the following result for Calderón-Zygmund Singular Integral Operators.

Theorem A (The linear growth theorem)
Let T be a Calderón-Zygmund operator and let $w \in A_1$. Then for $1 < p < \infty$

$$\|Tf\|_{L^p(w)} \leq c\, pp'\, [w]_{A_1} \|f\|_{L^p(w)} \quad (1 < p < \infty) \tag{1}$$

where $c = c(n, T)$.

Recall that w is an A_1 weight if there is a finite constant c such that $Mw \leq c\, w$, a.e.. $[w]_{A_1}$ denotes the smallest of these c. This result can be found in [3] (see also [2]) improving a corresponding result from [1] for just $p = 2$ and the Hilbert transform. An application of this result is the following weak type result.

Theorem B (The logarithmic growth theorem)
Let T as above and let $w \in A_p$ with $1 \leq p < \infty$. Then

$$\|Tf\|_{L^{p,\infty}(w)} \leq c\, \varphi\big([w]_{A_p}\big)\, \|f\|_{L^p(w)}, \tag{2}$$

where $c = c(n, p, T)$ and $\varphi(t) = t(1 + \log^+ t)$.

We explained in the lecture that the right result should be with $\varphi(t) = t$ instead, namely the linear growth. In the case $1 < p < \infty$ we used the notation

$$[w]_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty.$$

This result is related to a problem of Muckenhoupt-Wheeden that we discussed, namely

$$\|Hf\|_{L^{1,\infty}(w)} \leq c\, \varphi\big([w]_{A_1}\big)\, \|f\|_{L^1(Mw)} \tag{3}$$

where H is the Hilbert transform. Observe that no condition is assumed on w.

We also discussed that all these results are related to the A_2 theorems obtained in [5] for the Ahlfors-Beurling transform [6] the Hilbert transform and [7] for the Riesz transforms which we believe is true for any Calderón-Zygmund Singular Integral Operator T.

We will show that the endpoint result follows by proving first a corresponding sharp weighted L^p estimate both sharp on p and the A_1 constant of the weight. The connection of this result with the A_2 conjecture for Singular Integrals Operators will be discussed as well. Finally we will show some sharp string type weighted estimates for Sobolev type inequalities. They will be derived from corresponding weak type results for fractional integrals.

In the second part of the talk we consider very recent results for the classical fractional integral operator I_α. The classical result is due to Muckenhoupt-Wheeden which establishes that if $1 < p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$

$$\|w\, I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq C\, \|w\, f\|_{L^p(\mathbb{R}^n)}$$

holds if and only if w satisfies the $A_{p,q}$ condition:

$$[w]_{A_{p,q}} \equiv \sup_Q \left(\frac{1}{|Q|}\int_Q w^q\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{q/p'} < \infty$$

Theorem C (The strong fractional integral case)
Let p, q and α as above and let $w \in A_{p,q}$, then

$$\|w\, I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq C\, [w]_{A_{p,q}}^{\max\{1,(1-\frac{\alpha}{n})\frac{p'}{q}\}}\, \|w\, f\|_{L^p(\mathbb{R}^n)}$$

We discussed that the right result is with the exponent $(1-\frac{\alpha}{n})\max\{1, \frac{p'}{q}\}$ instead.

The result is proved by means of an "off-diagonal" extrapolation theorem with sharp bounds. We also discussed the following weak version.

Theorem D (The weak fractional integral case)
Let p, q, α and w as above with $p \geq 1$. Then

$$\|I_\alpha f\|_{L^{q,\infty}(w^q)} \leq C[w]_{A_{p,q}}^{1-\frac{\alpha}{n}}\, \|fw\|_{L^p}$$

These type of estimates are of interest because results of Sobolev type for the gradient which are important in applications.

Theorem E (Sobolev estimate)
Let $p \geq 1$ and $w \in A_{p,p^*}$, where p^* is the Sobolev exponent given by $p^* = \frac{np}{n-p}$. Then

$$\left(\int_{\mathbb{R}^n} |w(x)f(x)|^{p^*}\, dx\right)^{1/p^*} \leq C\, [w]_{A_{p,p^*}}^{\frac{1}{n'}} \left(\int_{\mathbb{R}^n} |w(x)\nabla f(x)|^p\, dx\right)^{1/p}$$

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A smoothing property for the L^2-critical NLS equation

Ana Vargas
(joint work with Sahbi Keraani)

We consider the initial value problem for the L^2–critical nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + \kappa |u|^{\frac{4}{d}} u = 0 \ ; \quad u_{|t=0} = u_0 \in H^s(\mathbb{R}^d). \tag{1}$$

It is well known that for every $u_0 \in L^2(\mathbb{R}^n)$, there is a unique maximal solution

$$u \in C((-T_*, T^*); L^2(\mathbb{R}^n)) \cap L_{loc}^{\frac{2(n+2)}{n}}((-T_*, T^*); L^{\frac{2(n+2)}{n}}(\mathbb{R}^n)).$$

The solution is global for data with small L^2 norm. In the focusing case, (+), there are solutions that blow up in finite time. It has been conjectured that in the defocusing case, (-) conjecture the solution is global.

For data in the Sobolev space H^1, the conjecture is known to be true. Moreover, it is known that in the focusing case and for H^1–data, with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution is global. Here Q is the ground state, i.e., the unique positive solution of $\Delta Q - Q + |Q|^{4/n} Q = 0$.

Bourgain [4] proved that in the defocusing case, in dimension 2, there is some $s < 1$, so that for $u_0 \in H^s(\mathbb{R}^2)$, the solution is global. This result was improved by Colliander-Keel-Staffilani-Takaoka-Tao [3], Fang-Grillakis [6] and Colliander-Grillakis-Tzirakis [2]. It has been extended to other dimensions by De Silva-Pavlovic-Staffilani-Tzirakis [4] [5]. In this situation, the energy method can not be used, and more refined techniques are needed. Very recently, global existence has been proved for any radial datum in the defocusing case, and for datum with

$\|u_0\|_{L^2} < \|Q\|_{L^2}$ in the focusing case (Tao-Visan-Zhang [12], Killip-Tao-Visan [7], Killip-Visan-Zhang [9]).

In this talk we show the following theorem

Theorem 1. *For $d = 1$, set $s_1 = 3/4$. For $2 \leq d \leq 4$, set $s_d = \frac{d}{d+2}$. Finally, for $d \geq 5$, set $s_d = \frac{d^2+2d-8}{d(d+2)}$. The solution of (1) with initial data $u_0 \in H^s(\mathbb{R}^d)$, $s > s_d$, can be written*

$$u(t) = e^{it\Delta}u_0 + w(t), \qquad t \in [0, T^*[\,,$$

with $w \in C([0, T^*[, H^1(\mathbb{R}^d))$.

This type of result was firstly discovered by Bourgain [4].

The theorem says that the blowup phenomenon has an H^1 mechanism. In fact, any singular solution can be split en two parts: an H^s part which is global (since it is linear) and an H^1 part, which blows up.

Combining the Theorem with the local smoothing result for the linear Schrödinger equation (Sjölin [10], Vega [13]), we obtain the following

Corollary 2. *Under the assumptions of the Theorem, $u(t) \in H^1_{\rm loc}(\mathbb{R}^d)$ for almost every $t \in [0, T^*[$.*

For the proof of the theorem, we use the norms defined by Bourgain

$$\|\phi\|_{\mathbf{X}^{s,b}} = \left[\iint (1 + |\xi|^{2s})(1 + |\lambda - |\xi|^2|)^{2b}|\tilde{\phi}(\xi,\lambda)|^2 d\xi dx\right]^{\frac{1}{2}}.$$

The main proposition is

Proposition 3. *For $b > \frac{1}{2}$, $\|v_1 \nabla v_2\|_{L^{\frac{n+2}{n}}_{t,x}} \leq C_{s,b}\|v_1\|_{\mathbf{X}^{s+,b}}\|v_2\|_{\mathbf{X}^{1-s,b}}$, for every $s < \frac{2}{n+2}$ when $n \geq 2$, and for for every $s < \frac{1}{4}$, when $n = 1$.*

This is a consequence of the following

Proposition 4. *For all $s < \frac{2}{n+2}$, $n \geq 2$, and for all $s < \frac{1}{4}$ if $n = 1$,*

$$\|e^{it\Delta}\psi_1 \nabla e^{it\Delta}\psi_2\|_{L^{\frac{n+2}{n}}(\mathbb{R}^{n+1})} \leq C_b\|\psi_1\|_{W^s(\mathbb{R}^n)}\|\psi_2\|_{W^{1-s,2}(\mathbb{R}^n)}.$$

Using a Littlewood–Paley decomposition, it is enough to prove an estimate for functions having a Fourier transform supported in dyadic annuli.

Proposition 5. *For $s = \frac{1}{4}$ if $n = 1$, for $s = \frac{1}{2}$ if $n = 2$, and for every $s < \frac{2}{n+2}$ for $n \geq 3$, there exists $C = C(s)$, such that the following estimate*

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L^{\frac{n+2}{n}}(\mathbf{R}^{n+1})} \leq C\left(\frac{M}{L}\right)^s \|f\|_{L^2}\|g\|_{L^2}$$

holds for all functions f *and* g *with* $\operatorname{supp}\widehat{f} \subset \{\xi \in \mathbb{R}^n : M \leq |\xi| \leq 2M\}$ *and* $\operatorname{supp}\widehat{g} \subset \{\xi \in \mathbb{R}^n : L \leq |\xi| \leq 2L\}$ *for all* $0 < M \leq L$.

The two dimensional case of this proposition was already stated in [4].

The ingredients to prove this estimate are the bilinear restriction theorem by Tao [5] and a bilinear L^2–estimate that is proven using geometric methods.

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Space time estimates for Schrödinger and wave equations

Andreas Seeger

(joint work with Fedor Nazarov, Keith Rogers)

The talk was based on work with Fedor Nazarov [10] on the wave equation and related multiplier results and work with Keith Rogers on Schrödinger type equations [13].

Consider the operators T_t^a given by

$$\widehat{T_t^a f}(\xi) = e^{i|\xi|^a}\widehat{f}(\xi)$$

By results of Stein, Miyachi and Peral ([15], [2], [8], [9], [11]) there are the fixed time regularity results

$$\|T_t^a f\|_q^q \le C(t)^q \|f\|_{L^q_\beta}^q \tag{1}$$

where

$$\beta \ge \beta_{\rm cr}(a,q) := \begin{cases} ad|\frac{1}{2}-\frac{1}{q}| & \text{if } a\neq 1 \\ (d-1)|\frac{1}{q}-\frac{1}{2}| & \text{if } a=1 \end{cases};$$

here L^q_β denotes the standard Sobolev space.

It is also conjectured that for suitable $q \gg 2$ the estimate improves with a gain of $1/q$ derivatives after integrating over a compact time interval, i.e. one expects

$$\int_I \|T_t^a f\|_q^q dt \le C_I^q \|f\|_{L^q_{\beta-1/q}}^q, \qquad \beta \ge \beta_{\rm cr}(a,q) \tag{2}$$

to hold. The conjectured range for this inequality is $q \in (2(d+1)/d, \infty)$ if $a \neq 1$ and $q \in (2d/(d-1), \infty)$ if $a = 1$.

In joint work with Rogers [13] it is shown that (2) holds for $a \neq 1$, in the range $q > 2(d+3)/(d+1)$ (see [12] for a prior result for $a = 2$ and $\beta > \beta_{\rm cr}(a,q)$). The proof uses the (bilinear version of) Fourier extension theorems in [1], [18] and [16], and an argument from [7]. A more recent variant, namely control of $\|\|\chi T_*^a f\|_{B^q_{1/q,1}(dt)}\|_{L^q(dx)}$ for $f \in L^q_\beta$, with χ smooth and compactly supported in $(-2,-2)$, also leads to new bounds for the maximal function $\sup_{|t|\le 1} |T_t^a f|$.

The above mentioned conjecture on (2) for $a = 1$ had been formulated by Sogge [14]; this case corresponds to the wave equation. With an ϵ-loss this conjecture was proven by T. Wolff [17], for $q > 74$ and $d = 2$ (for extensions and improvements see [6] and [3], [5]).

In joint work with Nazarov [10] a proof of the sharp from of Sogge's conjecture for a suitable (non-optimal) range of q, is obtained in dimension $d \ge 5$. In high dimensions our range is larger than the previously known on Wolff's inequality for plate decompositions; we show that (2) holds for $a = 1$, $d \ge 5$ and $q > \frac{2(d^2-2d-3)}{d^2-4d-1}$.

The smoothing estimate in the case $a = 1$ is also closely related to another result with Nazarov [10] which gives characterizations of L^p boundedness for convolution operators with radial kernels. To formulate this let ϕ be a nontrivial C^∞ function supported in $(1,2)$ and $m \in L^\infty[0,\infty)$. Set $K_t[m] = \mathcal{F}^{-1}[\phi(|\cdot|)m(t|\cdot|)]$. Then for $d \ge 5$ and $1 < p < \frac{2(d^2-2d-3)}{d^2-5}$ there is the equivalence

$$\|m(\sqrt{-\Delta})\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \approx \sup_{t>0} \|K_t[m]\|_{L^p(\mathbb{R}^d)}. \tag{3}$$

We remark that the corresponding result on radial functions, i.e.

$$\|m(\sqrt{-\Delta})\|_{L^p_{\rm rad}(\mathbb{R}^d)\to L^p_{\rm rad}(\mathbb{R}^d)} \approx \sup_{t>0} \|K_t[m]\|_{L^p(\mathbb{R}^d)}$$

had been obtained in work with Garrigós [4], in the optimal range $1 < p < \frac{2d}{d+1}$, $d > 1$. One may conjecture that (3) holds in the same range.

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Hardy spaces on metric measure spaces of exponential growth

Giancarlo Mauceri
(joint work with Andrea Carbonaro, Stefano Meda, Maria Vallarino)

There are interesting operators on metric measure spaces of exponential growth which are bounded on L^p for every $p \in (1,\infty)$, but are neither bounded on L^1, nor of weak type $(1,1)$. Examples include higher order complex powers of the Laplacian and higher order Riesz transforms on non-compact Riemannian symmetric spaces and on homogeneous trees [1] [3]. The purpose of our work is to introduce subspaces of L^1 on certain measure metric spaces of exponential growth that allow us to obtain endpoint estimates for such operators.

We assume that (M, ρ, μ) is a metric measure space which is unbounded and satisfies the following properties

(LD) *locally doubling property*: for every $s > 0$ there exists a constant D_s such that for every ball B of radius $\leq s$

$$\mu(2B) \geq D_s\,\mu(B),$$

where $2B$ denotes the ball with the same centre and twice the radius of B.

(AM) *approximate midpoint property*: there exist two constants $R_0 > 0$ and $\beta < 1$ such that for all x, y in M there exists z in M such that $\max\big(\rho(z,x), \rho(z,y)\big) \leq \beta\rho(x,y)$;

(I) *isoperimetric property*: There exist positive constants C and κ_0 such thar for every bounded open set A in M

$$\mu\{x \in A : \rho(x, A^c) \leq t\} > C\,t\mu(A) \qquad \forall t \in (0, \kappa_0).$$

These properties are satisfied, for instance, by Riemannian manifolds with a lower bound on Ricci curvature, positive injectivity radius and positive spectral gap.

First, in a joint work with A. Carbonaro and S. Meda [2] we define an atomic Hardy space $H^1(\mu)$, where atoms are supported only on "small balls", and a corresponding space $BMO(\mu)$ of functions of "bounded mean oscillation", where the control is only on the oscillation over small balls. We prove that $BMO(\mu)$ is the dual of $H^1(\mu)$ and that an inequality of John–Nirenberg type on small balls holds for functions in $BMO(\mu)$. Furthermore, we show that, even though $H^1(\mu)$ and $BMO(\mu)$ are much smaller and much larger, respectively, than the classical spaces defined in terms of all balls, the $L^p(\mu)$ spaces are intermediate spaces between $H^1(\mu)$ and $BMO(\mu)$. Next we develop a theory of singular integral operators acting on function spaces on M. Namely, we show that singular integral operators which satisfy the Hörmander integral condition

$$\sup_B \sup_{x,x' \in B} \int_{(2B)^c}^{\infty} |k(x,y) - k(x',y)| \,\mathrm{d}\mu(y) < \infty,$$

where B varies over all the "small balls", are bounded from $H^1(\mu)$ to $L^1(\mu)$. There is a corresponding $L^\infty(\mu)$-$BMO(\mu)$ result for operators satisfying the dual condition. This result allows us to prove an endpoint $H^1(\mu)$-$H^1(\mu)$ estimate for the functions of the Laplacian on Riemannian manifolds considered by M. Taylor in [5], under weaker bounded geometry assumptions. Namely,

Theorem 1. *Let M be a Riemannian manifold of dimension d, with positive injectivity radius and Ricci curvature bounded from below. Denote by κ the growth exponent of M. Let $b > 0$ be the bottom of the spectrum of the Laplacian $\mathcal{L}$ on $L^2(M)$ and denote by H the operator $\sqrt{\mathcal{L} - b}$. If m is an even bounded holomorphic function on the strip $\{z \in \mathbb{C} : |\Im z| < \kappa/2\}$ whose boundary values satisfy a Mihlin condition at infinity of order $N > d/2 + 2$, i.e.*

$$\left|D_s^j m(s \pm i\kappa/2)\right| \leq C(1 + |s|)^{-j} \qquad \forall s \in \mathbb{R} \quad \forall j = 0, 1, \ldots, N$$

then the operator $m(H)$ is bounded on H^1 and from $L^\infty(M)$ to BMO.

The space $H^1(\mu)$ turns out to be too large to obtain endpoint estimates for singular integrals which are "singular at infinity" such as certain complex powers of the Laplacian or Riesz transforms on symmetric spaces. Thus, in a second paper with M. Vallarino and S. Meda [4], we introduce a sequence $X_k,\ k \in \mathbb{N}_+$ of subspaces of $L^1(\mu)$ associated to the generator $\mathcal{L}$ of a semigroup of contractions on $L^1(\mu)+L^2(\mu)$, which is ultracontractive and has a positive spectral gap on $L^2(\mu)$. For each positive integer k the space X_k is defined as the range of the restriction to $H^1(\mu)$ of the operator $\mathcal{J}^k = \mathcal{L}^k(\mathcal{I}+\mathcal{L})^{-k}$, with norm $\|f\|_{X_k} = \|\mathcal{J}^{-k}f\|_{H^1}$. We prove that, on Riemannian manifolds, (X_k) is a strictly decreasing sequence of subspaces of $H^1(\mu)$. Moreover we show that the complex interpolation space $[X_k, L^2(\mu)]_\theta$ is $L^p(\mu)$, $1/p = 1-\theta/2$, for all $\theta \in (0,1)$. Finally, by using these spaces we are able to obtain endpoint estimates for the complex powers of the Laplacian and all Riesz transforms on symmetric spaces.

Theorem 2. *Let M be a noncompact Riemannian symmetric space of dimension d and let $\mathcal{L}$ be the Laplacian on M. Then*

(i) *if $\Re z > d/2 - k$ the operator $\mathcal{L}^z$ is bounded from X_k to $L^1(\mu)$*
(ii) *for all positive integers m the Riesz transform $\nabla^m \mathcal{L}^{-m/2}$ is bounded from $X_{[m/2]+1}$ to $L^1(\mu)$ Here $[m/2]$ denotes the greatest integer $\leq m$.*

We also obtain endpoint estimates for a class of spectral multipliers on Riemannian manifolds, more general than that considered by M. Taylor in [5].

Theorem 3. *Let M and H be as in Theorem 1. If m is an even holomorphic function on the strip $\{z \in \mathbb{C} : |\Im z| < \kappa/2\}$ whose boundary values satisfy*

$$\left|D_s^j m(s \pm i\kappa/2)\right| \leq C\ |s|^{-\gamma-j} \qquad \forall s \in \mathbb{R} \quad \forall j = 0, 1, \ldots, N$$

for some $\gamma \geq 0$ and for some $N > d/2+2$. Then the operator $m(H)$ is bounded from X_k to $H^1(\mu)$ for all $k > \gamma + d/2 + 2$.

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Large sets with limited tube occupancy

Anthony Carbery

A subset E of $\mathbb{R}^d$ is a Kakeya-type set, (or a Besicovitch–Kakeya–Furstenberg-type set) if each of a large set of tubes (say one in each direction, or one passing through each point of a hyperplane) contains a relatively large amount of E. The natural question for such sets is how small they can be, and this question has received a great deal of attention over the last 40 years.

In this talk we are concerned, in contrast, with "Anti-Kakeya"-type sets, that is, subsets E of $\mathbb{R}^d$ such that for *every* tube, the amount of mass of E contained in the tube is small. The question now is how large such sets may be.

This question naturally arises in X-ray tomography, but we are interested in its connections with harmonic analysis and PDEs.

In the late 1970's Stein proposed that the disc multiplier operator should be controlled by a maximal function involving averages over eccentric rectangles via an L^2-weighted inequality. Parallel to this, it is natural to ask the same question (and indeed in some model cases the questions are equivalent) for the extension operator for the Fourier transform associated to a hypersurface of non-vanishing gaussian curvature such as the unit sphere.

In the mid 1980's, Mizohata and Takeuchi, in connection with estimates for solutions to the Helmholtz equation, and apparently unaware of the connection with Stein's proposal, suggested that the following inequality should hold:

$$\int_{\mathbb{R}^d} |\widehat{gd\sigma}(x)|^2 w(x)dx \leq C \sup_T w(T) \int_{S^{d-1}} |g|^2 \tag{1}$$

where the sup is taken over all 1-tubes T, i.e. 1-neighbourhoods of lines in $\mathbb{R}^d$.

This question is still open, but was resolved in the affirmative for radial weights w independently by Barceló, Ruiz and Vega and by Carbery and Soria about 10 years ago.

The work of Carbery and Soria concerned analogues of Riemann's Localisation Theorem for Fourier transforms in higher dimensions. For $f \in L^2(\mathbb{R}^d)$ let

$$S_R f(x) = \int_{|\xi| \leq R} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

If f is identically zero on the unit ball $\mathbb{B}$ of $\mathbb{R}^d$, in what senses can we expect pointwise convergence of $S_R f(x)$ to zero on $\mathbb{B}$? Three results were given (some in later work of Carbery, Soria and Vargas):

- If $E \subseteq \mathbb{B}$ supports a positive measure μ with
$$\sup_{r\text{-tubes } T} \frac{\mu(T)}{r^{d-1}} \leq C$$
uniformly in r then, conditional on (1) holding, $S_R f \to 0$ a.e $d\mu$.

- If $d - 1/2 < \beta \le d$, if $0 < \mathcal{H}_\beta(E) < \infty$, and if E is radial, then
$$\sup_{r\text{-tubes } T} \frac{\mathcal{H}_\beta(T \cap E)}{r^{d-1}} \le C$$
uniformly in r (and so $S_R f(x)$ converges to 0 a.e. with respect to $\mathcal{H}_\beta|_E$ if $f \equiv 0$ on $\mathbb{B}$).
- There is no $E \subseteq \mathbb{B}$ with $0 < \mathcal{H}_{d-1}(E) < \infty$ such that
$$\sup_{r\text{-tubes } T} \frac{\mathcal{H}_{d-1}(T \cap E)}{r^{d-1}} \le C$$
uniformly in r (and moreover if $\mathcal{H}_{d-1}(E)$ is σ-finite, there is an $f \in L^2(\mathbb{R}^d)$, identically zero on $\mathbb{B}$ such that $S_R f(x)$ diverges on E.)

(Here, r-tubes are r-neighbourhoods of lines in $\mathbb{R}^d$, and $\mathcal{H}_\beta$ denotes β-dimensional Hausdorff measure.)

Thus the following problem naturally arises: determine the set of pairs (β, γ) such that there is an $E \subseteq \mathbb{B}$ with $0 < \mathcal{H}_\beta(E) < \infty$ such that

$$\sup_r \sup_{r\text{-tubes } T} \frac{\mathcal{H}_\beta(T \cap E)}{r^\gamma} < \infty. \tag{2}$$

It is easy to see that if either $\gamma > d-1$ or $\beta < \gamma$, then (2) implies that $\mathcal{H}_\beta(E) = 0$.

Theorem 1. *If $\gamma < d - 1$ and $\beta > \gamma$, then there exists a set $E \subseteq \mathbb{B}$ with $0 < \mathcal{H}_\beta(E) < \infty$ and such that* (2) *holds.*

Returning to (1), recall the Stein–Tomas restriction theorem which says, in its weighted formulation, that

$$\int_{\mathbb{R}^d} |\widehat{gd\sigma}(x)|^2 w(x)dx \le C\|w\|_{(d+1)/2} \int_{S^{d-1}} |g|^2.$$

So, in considering testing (1), it only makes sense to do so on weights w for which

$$\sup_T w(T) << \|w\|_{(d+1)/2}.$$

Because of the uncertainty principle we may assume that the finest scale appearing is 1, and to suitably quantify the previous inequality, we introduce a scale $N >> 1$ and consider weights w supported in $B(0, N)$ which are essentially constant on unit scale. For such weights it is easy to see that for $p \ge 1$,

$$\|w\|_p \le C_{d,p} N^{(d-1)/p} \sup_{1\text{-tubes } T} w(T).$$

Theorem 2. *There exists a w taking integer values such that*

$$\sup_{1\text{-tubes } T} w(T) \le C_d \log N$$

while

$$\|w\|_1 \ge C_d \log N \, N^{d-1}.$$

Corollary 3. *With the same w, if $p > 1$ we have*

$$\|w\|_p \geq C_{d,p} \frac{N^{(d-1)/p}}{(\log N)^{1/p'}} \sup_{1\text{-tubes } T} w(T).$$

Such weights should in principle be good candidates on which to test (1).

In fact, there is a finer version of Theorem 2:

Theorem 4. *For $2 \leq k \leq N^{1/2}$ there is a collection of at least $C_d k N^{d-1} N^{-(d-1)/k}$ lattice points in $\{1, 2, \ldots, N\}^d$ so that no 1-tube contains more than k of them.*

We caution that the collection may not consist of distinct points. Notice that $C_d k N^{d-1}$ would be best possible for no 1-tube to contain more than k points, and that when $k \geq \log N$ the term $N^{-(d-1)/k}$ essentially disappears. On the other hand, when $k = 2$, the result is easy as can be seen by placing $\sim N^{(d-1)/2}$ points at roughly equal spacings on a sphere of radius $\sim N$.

The proof of Theorem 4 is probabalistic and is closely related to work of Komlós, Pintz and Szemerédi on the Heilbronn triangle problem. In fact there is a logarithmic improvement of the case $k = 2,\ d = 2$ of Theorem 4 implied by the work of those authors and our argument is based on a simplified version of that analysis. Theorem 2 can be obtained by a simpler large deviation/Bernoulli trials analysis; Michael Christ has also made a similar observation. Theorem 1 is obtained by building Cantor sets based upon the examples furnished by Theorem 2 or 4. Since the examples are generated probabalistically rather than deterministically their potential as counterexamples to (1) is perhaps limited: for example it is not hard to show that if we write $g \in L^2(S^{d-1})$ in its wave packet representation and then put random ± 1's on the coefficients, then (1) holds for all weights w almost surely.

Details of the proofs will appear elsewhere.

Bochner-Riesz analysis on on asymptotically conic manifolds

Adam Sikora
(joint work with Colin Guillarmou and Andrew Hassell)

Suppose that X is a measure space, equipped with a measure μ, and that L is a self-adjoint positive definite operator on $L^2(X)$. Then L has a spectral resolution:

$$L = \int_0^\infty \lambda dE_L(\lambda),$$

where the $E_L(\lambda)$ are spectral projectors. For any bounded Borel function $F\colon [0, \infty) \to \mathbb{C}$, we define the operator $F(L)$ by the formula

$$F(L) = \int_0^\infty F(\lambda) dE_L(\lambda). \tag{1}$$

By the spectral theorem, $F(L)$ is well defined and bounded on $L^2(X)$. To define the Bochner-Riesz means of the operator L we put

$$\sigma^\alpha(\lambda) = \begin{cases} (1-\lambda)^\alpha & \text{for } \lambda \le 1 \\ 0 & \text{for } \lambda > 1. \end{cases}$$

We then define the operator $\sigma_R^\alpha(tL)$ using (1). We call $\sigma^\alpha(tL)$ the Riesz or the Bochner-Riesz means of order α. The basic question in Bochner-Riesz analysis is to establish the critical exponent for the continuity and convergence of the Riesz means. More precisely we want to study the optimal range of α for which

$$\sup_{t>0} \|\sigma^\alpha(tL)\|_{p\to p} < \infty.$$

for a given $p \neq 2$.

We study Bochner-Riesz analysis in the setting of asymptotically conic manifold. We say that complete noncompact manifold M° with a Riemannian metric g is an asymptotically conic manifold on M° if it compactifies to a manifold with boundary M in such a way that g becomes a scattering metric on M. A good illustrative example is an exact metric cone (that is manifold $\mathbb{R}_+ \times Y$ with Riemannian metric $dr^2 + r^2 h$ where (Y, h) is a compact Riemannian manifold) with a smoothed out vertex. Let Δ be the positive Laplacian associated to g, and $L = \Delta + V$, where V is a potential function obeying certain conditions. We analyze the asymptotics of the spectral measure $dE(\lambda) = \frac{1}{2\pi i} R(\lambda + i0) - R(\lambda - i0)$, where $R(\lambda) = (L - \lambda^2)^{-1}$, as $\lambda \to 0$, in a similar way as in [2] and [1]. Using this analysis, we obtain $L^1 \to L^\infty$ estimates on derivatives (in λ) of the spectral measure under the assumption that (M, g) has no conjugate points. Then we show that such $L^1 \to L^\infty$ estimates imply restriction theorems, i.e. $L^p \to L^{p'}$ continuity of the spectral projection $dE(\lambda)$, which are as good as those currently known for flat Euclidean space.

As an immediate application, spectral multiplier and Bochner Riesz summability results for H, similar to those described in [3, 4, 5], hold under the same assumption. Moreover, when there are conjugate points, then we show that the restriction estimates definitely fail to hold.

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An Algebra Containing the Two-Sided-Convolution Operators

Brian Street

Let G be a stratified Lie group. Given a Calderón-Zygmund distribution kernel $K \in C_0^\infty(G)'$, one obtains two "Calderón-Zygmund singular integral operators":

$$\mathrm{Op}_{\mathrm{L}}(K) f := f * K$$

$$\mathrm{Op}_{\mathrm{R}}(K) f := K * f$$

The operators of the form $\mathrm{Op}_{\mathrm{L}}(K)$ form an algebra:

$$\mathrm{Op}_{\mathrm{L}}(K_1)\,\mathrm{Op}_{\mathrm{L}}(K_2) = \mathrm{Op}_{\mathrm{L}}(K_1 * K_2)$$

In addition, they are bounded on L^p $(1 < p < \infty)$, and are pseudolocal. The same is true for operators of the form $\mathrm{Op}_{\mathrm{R}}(K)$. Also, if we consider:

$$\mathrm{Op}_{\mathrm{L}}(K_1)\,\mathrm{Op}_{\mathrm{R}}(K_2) f = (K_2 * f) * K_1 = K_2 * (f * K_1) = \mathrm{Op}_{\mathrm{R}}(K_2)\,\mathrm{Op}_{\mathrm{L}}(K_1) f$$

we see that $\mathrm{Op}_{\mathrm{L}}(K_1)$ and $\mathrm{Op}_{\mathrm{R}}(K_2)$ commute.

Hence, it follows that:

$$\mathrm{Op}_{\mathrm{L}}(K_1)\,\mathrm{Op}_{\mathrm{R}}(K_2)\,\mathrm{Op}_{\mathrm{L}}(K_3)\,\mathrm{Op}_{\mathrm{R}}(K_4) = \mathrm{Op}_{\mathrm{L}}(K_1 * K_3)\,\mathrm{Op}_{\mathrm{R}}(K_4 * K_2)$$

and so operators of the form $\mathrm{Op}_{\mathrm{L}}(K_1)\,\mathrm{Op}_{\mathrm{R}}(K_2)$ are closed under composition. It is also evident that they are bounded on L^p $(1 < p < \infty)$ and are pseudolocal. This raises the question as to what the algebra is that is "generated" by operators of the form $\mathrm{Op}_{\mathrm{L}}(K_1)\,\mathrm{Op}_{\mathrm{R}}(K_2)$. An extrinsic answer to the question is quite easy. Indeed, operators of the form:

$$Tf(x) = \int K(y,z)\, f\left(z^{-1}xy^{-1}\right)\, dydz \tag{1}$$

where K is a suitable "product kernel," form a natural algebra containing operators of the form $\mathrm{Op}_{\mathrm{L}}(K_1)\,\mathrm{Op}_{\mathrm{R}}(K_2)$.

In this talk, we present an intrinsic approach to an appropriate algebra containing the above operators. This turns out to be somewhat more difficult, and requires singular integrals that have a truly two-parameter nature. Indeed, the cancellation conditions that these operators satisfy are defined in terms of a one-parameter family of Carnot-Carathéadory metrics. These results can be found in [1].

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Structure of sets with small additive doubling

Nets Hawk Katz
(joint work with Paul Koester)

We give a new basic result in the study of sets with small additive doubling. For simplicity, in this talk we restrict to the setting of characteristic 2 so that the set A in question is a subset of F_2^N for some N, presumably very large. We assume a bound on the size of the sumset $|A + A| \leq K|A|$, with K a relatively small constant. By the Cauchy-Schwarz inequality, this implies a lower bound on the additive energy $E(A) \geq \frac{1}{K}$. Here we define additive energy by

$$E(A) = \frac{1}{|A|^3}|\{(a, b, c, d) \in A^4 : a + b = c + d\}|.$$

Our theorem is

Theorem 1. *There is an $\epsilon > 0$ and $C > 0$ so that if A is as above, there is a subset $B \subset A$ so that $|B| > K^{-C}|A|$ and so that $E(B) > K^{\epsilon-1}$.*

Part of the motivation of this theorem was an attempt to settle the celebrated Polynomial Freiman-Ruzsa conjecture which says

Conjecture 2. *Let A be as above, then there is a subspace $H \subset F_2^N$ so that $|H| \leq K^C|A|$, and $|H \cap A| \geq K^{-C}|A|$.*

The work was inspired by the incrementation argument of Green and Tao, and would have yielded a proof of the conjecture if the result obtained on the set B estimated its additive doubling. As it is, our result shows that the enemy of the conjecture is a set whose energy is substantially higher than what is predicted by additive doubling. In characteristic zero, such a set is a generalized arithmetic progression of high dimension which of course must be encoded into a characteristic zero Polynomial Freiman-Ruzsa conjecture.

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The complex Monge-Ampère equation and the Szegö kernel

Duong H. Phong and Jacob Sturm

We describe recent progress on the following Dirichlet problem: let M be a complex manifold with smooth boundary ∂M, Ω_0 be a non-negative $(1,1)$-form on M, and Φ_b a smooth function on ∂M. Find Φ satisfying $\Omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi \geq 0$ and

$$(\Omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi)^{\dim M} = 0 \text{ on } M, \qquad \Phi_{|\partial M} = \Phi_b. \tag{1}$$

A case of particular interest is $M = X \times A$, where (X, ω_0) is a polarized compact Kähler manifold, and $A = \{e^{-T} < |w| < 1\} \subset \mathbf{C}$ is an annulus. The parameter T can be infinite, in which case A is a punctured disk, and by convention, we define ∂M to be then $X \times \{|w| = 1\}$. Polarized means that ω_0 arises as the curvature $-\frac{i}{2}\partial\bar{\partial}\log h_0$ of a metric h_0 on a holomorphic line bundle $L \to X$.

The equation (1) is motivated by the still unsolved problem of determining when the Kähler class $[\omega_0]$ of a given polarized compact Kähler manifold X contains a metric of constant scalar curvature. A well-known conjecture of Yau [28] is that the existence of such a metric should be equivalent to the stability of the polarization $L \to X$ in the sense of geometric invariant theory. There are several candidate notions of stability: classical notions of Chow-Mumford and Hilbert-Mumford stability [17] (now known to be asymptotically equivalent [16]), notions of K-stability due to Tian [26] and Donaldson [13]; and infinite-dimensional notions due to Donaldson [12] and the authors [18]. The notion of stability in [12] is based on the geometry of the space

$$\mathbf{K} = \{\phi \in C^\infty(X); \omega_\phi \equiv \omega_0 + \frac{i}{2}\partial\bar{\partial}\phi > 0\}, \tag{2}$$

of Kähler potentials, which is a symmetric space of non-positive curvature with respect to the metric $\|\delta\phi\|^2 = \int_X |\delta\phi|^2 \omega_\phi^n$ [12, 15, 22]. The polarization $L \to X$ is then defined not to be stable in the sense of [12] if there exists an infinite geodesic ray $(-\infty, 0] \ni t \to \phi(\cdot, t) \in \mathbf{K}$ with $\int_X \dot{\phi}(R - \langle R \rangle)\omega_\phi^n \le 0$ for all $t \in (-\infty, 0]$. Here R is the scalar curvature of the metric ω_ϕ and $\langle R \rangle$ its average. Now the geodesic equation for a path $\phi(\cdot, t)$ in $\mathbf{K}$ is $\ddot{\phi} - \omega_\phi^{j\bar{k}}\partial_j\dot{\phi}\partial_{\bar{k}}\dot{\phi} = 0$. If we set $\Phi(z, w) \equiv \phi(z, \log|w|)$, $\Omega_0 = \omega_0$ viewed as a form on M, then this equation is actually equivalent to (1), with $M \equiv X \times A$, and $\Phi_b = 0$ on $|w| = 1$. Thus the existence and regularity of geodesics in $\mathbf{K}$ reduces to the existence and regularity of solutions of (1).

Our results are of two types. In the first, generalized solutions of (1) are constructed explicitly as limits of geodesics in $SL(N+1)/U(N+1)$ as $N \to \infty$. More precisely, for each $k \in \mathbf{N}^*$, let $\underline{s} = \{s_\alpha\}_0^{N_k}$ be an orthonormal basis of the space $H^0(X, L^k)$ of holomorphic sections of L^k, with respect to the L^2 metric defined by h_0 and the volume form ω_0^n. Given weights $\lambda_\alpha^{(k)}$, set

$$\Phi_k(z, w) \quad \equiv \quad \frac{1}{k}\log\sum_{\alpha=0}^{N_k}|w|^{2\lambda_\alpha^{(k)}}|s_\alpha(z)|^2 h_0^k - n\frac{\log k}{k} \tag{3}$$

and, with $*$ denoting the upper semi-continuous envelope,

$$\Phi(z, w) \quad \equiv \quad \lim_{\ell\to\infty}\Big[\sup_{k\ge\ell}\Phi_k(z, w)\Big]^*. \tag{4}$$

Theorem 1. [19] *Let $\phi_1 \in \mathbf{K}$. Let $A = \{e^{-1} < |w| < 1\}$, $M = X \times A$, and define the boundary value Φ_b in the Dirichlet problem (1) by $\Phi_b = 0$ when $|w| = 1$, $\Phi_b = -\phi_1$ when $|w| = e^{-1}$. Then a generalized solution of (1) can be obtained in the form (4), by choosing the weights $\lambda_\alpha^{(k)}$ to be the coefficients of proportionality*

$s_\alpha^{(1)} = e^{\lambda_\alpha^{(k)}} s_\alpha$ *between two bases* $\underline{s} = \{s_\alpha\}_0^{N_k}$, $\underline{s}^{(1)} = \{s_\alpha^{(1)}\}_0^{N_k}$ *of* $H^0(X, L^k)$, *orthonormal with respect to* h_0, ω_0^n *and* $h_1 \equiv e^{-\phi_1} h_0, \omega_{\phi_1}^n$ *respectively.*

Theorem 2. [20] *Let* $\mathbf{L} \to \mathbf{X} \to \mathbf{C}$ *be a test configuration for* $L \to X$ *in the sense of Donaldson. Let* $\lambda_\alpha^{(k)}$ *be the weights of the corresponding* $\mathbf{C}^\times$ *action on* $H^0(X_0, L_0^k)$ *where* X_0 *is the central fiber of* $\mathbf{X}$ *and* L_0 *the corresponding fiber of* $\mathbf{L}$. *Then the formula (4) defines a generalized solution of the Dirichlet problem (1), with* $M = X \times \{0 < |w| < 1\}$, $\Phi_b = 0$ *when* $|w| = 1$. *The solution is not identically* 0 *when the test configuration is non-trivial.*

The proof of the preceding theorems makes essential use of two fundamental ingredients, namely the Tian-Yau-Zelditch theorem [27, 25, 29] and the Bedford-Taylor pluripotential theory. Geometrically, the TYZ theorem asserts that an arbitrary metric on a positive line bundle L can be approximated by Fubini-Study metrics. It insures that the ansatz (4) satisfies the desired boundary values. A proof can be found in [29] and Catlin [7], using an asymptotic expansion for the Szegö kernel due to Boutet de Monvel-Sjöstrand [5] and Fefferman [11]. The Bedford-Taylor theory [2] provides conditions for the uniqueness and weak convergence of the Monge-Ampère determinants. Precise estimates on the rate of convergence in (4) when X is a toric variety have recently been obtained by Song and Zelditch [23, 24]. A different approximation scheme with a twist of L^k by the canonical bundle K_X, in the spirit of the L^2 estimates of Hörmander and Akizuki-Kodaira-Nakano, has been proposed by Berndtsson [3].

Theorems 1 and 2 provide constructions in terms of Fubini-Study metrics on $\mathbf{CP}^N$, which are expected to provide the vital link between infinite-dimensional and finite-dimensional notions of stability. However, they give only generalized solutions. A priori estimates for the Monge-Ampere equation [6, 14, 8, 4] can also be applied to give $C^{1,1}$ solutions for the completely degenerate case. In the case of $A = \{e^{-T} < |w| < 1\}$ with T finite, this was done by X.X. Chen [8]. Applying a priori estimates as well as a resolution of singularities, we have

Theorem 3. [21] *Let* $\mathbf{L} \to \mathbf{X} \to \mathbf{C}$ *be a test configuration for* $L \to X$ *in the sense of Donaldson. Let* $p : \tilde{\mathbf{X}} \to \mathbf{X} \to \mathbf{C}$ *be any smooth,* S^1 *equivariant resolution of* $\mathbf{X}$. *Then the Dirichlet problem (1) admits a solution* Φ, *with* $\Omega_0 + \frac{i}{2}\partial\bar\partial\Phi$ *the restriction to* $p^{-1}(\mathbf{X}_{|_{\{0<|w|<1\}}})$ *of a non-negative current* $\Omega + \frac{i}{2}\partial\bar\partial\Psi$ *on* $p^{-1}(\mathbf{X}_{|_{\{|w|<1\}}})$ *satisfying* $(\Omega + \frac{i}{2}\partial\bar\partial\Psi)^{n+1} = 0$, *with* Ω *a smooth Kähler form.*

Other constructions of geodesic rays have also been given in [1] and [9, 10].

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