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# C*-Algebras

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Abstract. The theory of C*-algebras plays a major rôle in many areas of modern mathematics, like Non-commutative Geometry, Dynamical Systems, Harmonic Analysis, and Topology, to name a few. The aim of the conference "C*-algebras" is to bring together experts from all those areas to provide a present day picture and to initiate new cooperations in of this fast growing mathematical field.

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## Introduction by the Organisers

A C*-algebra is an involutive Banach algebra $A$ which satisfies the C*-condition $\|a^*a\| = \|a\|^2$ for all $a \in A$. The theory of C*-algebras goes back to work of Murray and von Neumann, who first studied a special variant now known as von Neumann algebras. The theory developed rapidly after some ground breaking work of Gelfand and Naimark in 1943 in which they showed that

- every commutative C*-algebra can be realized up to isomorphism as an algebra of continuous functions which vanish at infinity on a locally compact Hausdorff space,
- every C*-algebra can be realized as a closed *-subalgebra of the algebra of bounded operators $B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$.

In the 70's and 80's of the last century, the first of the above items lead to the point of view that non-commutative C*-algebras should be regarded as function spaces of "non-commutative" topological spaces. As a consequence, completely new areas in mathematics, like Non-commutative Geometry or Free Probability evolved and we now see that the theory of C*-algebras became a very active field with applications in and interactions with almost all areas of modern mathematics.

The aim of the workshop *C*-algebras*, organized by Claire Anantharaman-Delaroche, Siegfried Echterhoff, Uffe Haagerup, and Dan Voiculescu, is to bring together leading researchers from basically all areas related to the field. This gives a unique opportunity to maintain a broad view on the subject and to create new cooperations between researchers with different background. Among the 42 participants was a good number of young researchers, some of them already on the top of the field. There have been 27 lectures presented at the workshop with topics ranging from classification of C*-algebras, group actions on C*-algebras, orbit equivalence of dynamical systems, Jones' theory of subfactors, fundamental groups of $\mathrm{II}_1$-factors, $L^2$-invariants, duality for quantum groups and quantum groupoids, Index theorems, C*-algebras related to number theory, free probability, the relation between C*-algebras and Harmonic Analysis, and others.

Among the most exciting recent developments in the field we mention the breathtaking advances of Popa and Vaes on the fundamental groups of $\mathrm{II}_1$-factors in the theory of von Neumann algebras, with beautiful applications to the classification of equivalence relations and dynamical systems. Other milestones are the progress of Kirchberg and coworkers on the classification of actions on $\mathcal{O}_2$ and the beautiful results of Dadarlat on continuous bundles of C*-algebras. Important progress has been achieved also in the classification theory of simple and non-simple C*-algebras by Elliot, Rørdam, Toms, and Winter–with some further progress being made during the workshop. There were many other exciting contributions to this very successful workshop, which can be seen from the reports presented in this issue.

It is a pleasure for the organizers to thank all participants of the workshop for their beautiful lectures and fruitful discussions. We also want to use this opportunity to thank the Mathematisches Forschungsinstitut Oberwolfach for providing a very stimulating environment and strong support for organizing this conference. Special thanks also go to the very competent and helpful staff of the institute.

## Workshop: C*-Algebras

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# Abstracts

## Compact $C^*$-quantum groupoids

Thomas Timmermann

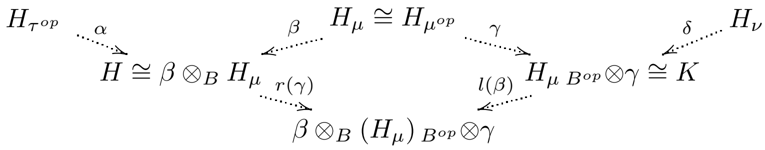
Quantum groupoids have successfully been axiomatized and studied in the finite case by Böhm, Szlachányi, Nikshych, Vainerman and others [3], and in the measurable case by Enock, Lesieur and Vallin [2] who were motivated by depth 2 inclusions of factors. In this talk, we turn to the setting of $C^*$-algebras and introduce compact $C^*$-quantum groupoids and an analogue of the fundamental multiplicative unitaries of Baaj and Skandalis [1] that is adapted to $C^*$-quantum groupoids. These unitaries form the basis for further developments like the construction of Pontrjagin duals and of reduced crossed products for coactions.

### 1. A fiber product of $C^*$-algebras and Hopf $C^*$-bimodules

1.1. **Introduction.** Recall that a groupoid consists of a space of units $G^0$, a total space $G$, range and source maps $r, s\colon G \to G^0$, and a multiplication map $G \times G \supseteq G_s{\times}_r G \to G$, subject to several axioms. Replacing spaces by algebras and reversing the maps as in the transition from groups to Hopf algebras, we arrive at the notion of a Hopf bimodule: two algebras $B$, $A$ with range and source maps $\rho, \sigma\colon B \to A$ and a comultiplication $\Delta\colon A \to A_\sigma *_\rho A$. In the purely algebraic setting, $A_\sigma *_\rho A$ is defined using Takeuchi's $\times_R$-product; in the setting of von Neumann algebras, $A_\sigma *_\rho A$ is the fiber product defined by Sauvageot and Vallin. For the setting of $C^*$-algebras, we propose a definition of $A_\sigma *_\rho A$ in this section.

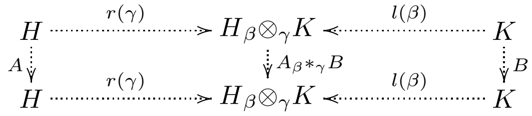
1.2. **Hilbert $C^*$-modules and $C^*$-algebras over KMS-weights.** Let $B, C$ be $C^*$-algebras with faithful proper KMS-weights $\mu, \nu$ and associated GNS-spaces $H_\mu, H_\nu$. We identify $B$ with a $C^*$-subalgebra of $\mathcal{L}(H_\mu)$, and $H_\mu$ with the GNS-space $H_{\mu^{op}}$ for the opposite weight $\mu^{op}$ on $B^{op}$; then $B^{op} \subseteq B' \subseteq \mathcal{L}(H_\mu)$. A *Hilbert $C^*$-$\mu$-module* $H_\alpha$ consists of a Hilbert space $H$ and a space $\alpha \subseteq \mathcal{L}(H_\mu, H)$ satisfying $[\alpha H_\mu] = H$, $[\alpha^*\alpha] = B$, $[\alpha B] = \alpha$, where, $[\,\cdot\,]$ denotes the closed linear span. Given such a Hilbert $C^*$-$\mu$-module, we can view $\alpha$ as a Hilbert $C^*$-$B$-module, identify the internal tensor product $\alpha \otimes_B H_\mu$ with $H$ via $\xi \otimes_B \zeta \mapsto \xi\zeta$, and construct a representation $\rho_\alpha\colon B' \to \mathcal{L}(H)$, $x \mapsto \mathrm{id}_\alpha \otimes_B x$. A *$C^*$-$\mu$-algebra* $A^\alpha_H$ consists of a Hilbert $C^*$-$\mu$-module $H_\alpha$ and a nondegenerate $C^*$-algebra $A \subseteq \mathcal{L}(H)$ such that $[\rho_\alpha(B^{op})A] = A$. A *Hilbert $C^*$-$(\mu,\nu)$-module* ${}_\alpha H_\beta$ is formed by a Hilbert $C^*$-$\mu$-module $H_\alpha$ and a Hilbert $C^*$-$\nu$-module $H_\beta$ such that $[\rho_\alpha(B^{op})\beta] = \beta$ and $[\rho_\beta(C^{op})\alpha] = \alpha$. Finally, there exist natural notions of a $C^*$-$(\mu,\nu)$-algebra $A^{\alpha,\beta}_H$ and of morphisms of Hilbert $C^*$-$\mu/(\mu,\nu)$-modules and $C^*$-$\mu/(\mu,\nu)$-algebras [4].

1.3. **The relative tensor product.** Let $\mu, \nu, \tau$ be faithful proper KMS-weights, ${}_\alpha H_\beta$ a Hilbert $C^*$-$(\tau^{op},\mu)$-module, and ${}_\gamma K_\delta$ a Hilbert $C^*$-$(\mu^{op},\nu)$-module. Then we can form a Hilbert space $H_\beta \otimes_\gamma K := \beta \otimes_B (H_\mu)_{B^{op}} \otimes \gamma$ and associate to each $\xi \in \beta$ and $\eta \in \gamma$ operators $l(\xi)\colon K \to H_\beta {\otimes}_\gamma K$ and $r(\eta)\colon H \to H_\beta {\otimes}_\gamma K$ that are given by tensor multiplication on the left and on the right, respectively. The *relative tensor product* ${}_\alpha H_\beta \otimes_\gamma K_\delta := {}_{[r(\gamma)\alpha]}(H_\beta \otimes_\gamma K)_{[l(\beta)\delta]}$ is a Hilbert-$C^*$-$(\tau^{op}, \nu)$-module. The situation is depicted in the diagram below:



The assignment $({}_\alpha H_\beta, {}_\gamma K_\delta) \mapsto {}_\alpha H_\beta \otimes {}_\gamma K_\delta$ is functorial, associative, and unital [4].

1.4. **The fiber product and concrete Hopf $C^*$-bimodules.** Let $A_H^{\alpha,\beta}$ be a $C^*$-$(\tau^{op}, \mu)$-algebra and $D_K^{\gamma,\delta}$ a $C^*$-$(\mu^{op}, \nu)$-algebra. Then the space $A_\beta *_\gamma D := \{T \in \mathcal{L}(H_\beta \otimes_\gamma K) \mid T^{(*)} l(\beta) \subseteq [l(\beta)D],\, T^{(*)} r(\gamma) \subseteq [r(\gamma)A]\}$ is a $C^*$-algebra. If it is nondegenerate, then it is a $C^*$-$(\tau^{op}, \nu)$-algebra on ${}_\alpha H_\beta \otimes {}_\gamma K_\delta$ and we call it the *fiber product* $A_H^{\alpha,\beta} * D_K^{\gamma,\delta}$. The situation can be depicted as follows:



This fiber product construction is functorial, but neither associative nor unital. A *concrete Hopf $C^*$-bimodule over* $\mu$ is a $C^*$-$(\mu, \mu^{op})$-algebra $A_H^{\alpha,\beta}$ with a morphism $\Delta$ from $A_H^{\alpha,\beta}$ to $A_H^{\alpha,\beta} * A_H^{\alpha,\beta}$ such that $(\Delta * \mathrm{id}) \circ \Delta = (\mathrm{id} * \Delta) \circ \Delta$, where the image of this map is contained in $(A_H^{\alpha,\beta} * A_H^{\alpha,\beta}) * A_H^{\alpha,\beta} \cap A_H^{\alpha,\beta} * (A_H^{\alpha,\beta} * A_H^{\alpha,\beta}) \subseteq \mathcal{L}(H_\alpha \otimes_\beta H_\alpha \otimes_\beta H)$.

## 2. A definition of compact $C^*$-quantum groupoids

A compact $C^*$-quantum groupoid consists — roughly — of the following ingredients: a unital $C^*$-algebra $B$ with faithful KMS-state $\mu$, a unital $C^*$-algebra $A$, unital embeddings $\rho\colon B \to A$ and $\sigma\colon B^{op} \to A$, faithful conditional expectations $\phi\colon A \to \rho(B) \cong B$ and $\psi\colon A \to \sigma(B^{op}) \cong B^{op}$, and a Radon-Nikodym-derivative $\delta = d\nu/d\nu^{-1}$, where $\nu = \mu \circ \phi$ and $\nu^{-1} = \mu^{op} \circ \psi$. Using (partial) GNS-constructions for $\mu, \nu$ and $\phi, \psi, \phi^{op}, \psi^{op}$, one can construct a Hilbert $C^*$-$(\mu, \mu^{op}, \mu, \mu^{op})$-module $(H, \hat\alpha, \hat\beta, \alpha, \beta)$ such that $A_H^{\alpha,\beta}$ is a $C^*$-$(\mu, \mu^{op})$-algebra. Further ingredients are a morphism $\Delta$ such that $(A_H^{\alpha,\beta}, \Delta)$ is a Hopf $C^*$-bimodule and $\phi, \psi$ are left- and right-invariant with respect to $\Delta$, and a unitary antipode $R$ which is an anti-automorphism of $A$ and satisfies a strong invariance condition involving $\phi$ and $\Delta$. The precise definition can be found in [4].

## 3. $C^*$-pseudo-multiplicative unitaries

3.1. **Definition and associated Hopf $C^*$-bimodules.** In the theory of quantum groups, a fundamental role is played by the multiplicative unitaries of Baaj, Skandalis and Woronowicz. For the study of quantum groupoids in the setting of $C^*$-algebras, we propose the following generalization. A *$C^*$-pseudo-multiplicative* *unitary* over a proper, faithful KMS-weight $\mu$ consists of a Hilbert $C^*$-$(\mu^{op}, \mu, \mu^{op})$-module $(H, \hat{\beta}, \alpha, \beta)$ and a unitary $V\colon H_{\hat{\beta}}\otimes_{\alpha} H \to H_{\alpha}\otimes_{\beta} H$ that satisfies certain compatibility conditions with respect to $\hat{\beta}, \alpha, \beta$ and the *pentagon equation* $V_{12}V_{13}V_{23} = V_{12}V_{23}$. Here, the $V_{ij}$ are operators that act on relative tensor products of three copies of $H$ like $V$ at the positions $i, j$ and like the identity at the remaining position. If $V$ satisfies a certain regularity condition, we can associate to $V$ two concrete Hopf $C^*$-bimodules $(\hat{A}(V)^{\hat{\beta},\alpha}_{H}, \hat{\Delta})$ and $(A(V)^{\alpha,\beta}_{H}, \Delta)$. Using representations and corepresentations of the unitary $V$, we can also form two universal $C^*$-algebras $\hat{A}_u(V)$ and $A_u(V)$ with surjections onto $\hat{A}(V)$ and $A(V)$, respectively, but we have no universal fiber product construction to define the target of the comultiplication on these $C^*$-algebras.

3.2. **Examples.** To every compact $C^*$-quantum groupoid as above, we can associated a regular $C^*$-pseudo-multiplicative unitary using similar formulas as in the setting of quantum groups. One of the associated Hopf $C^*$-bimodules coincides with the initial compact $C^*$-quantum groupoid, and the other is the Hopf $C^*$-bimodule of the generalized Pontrjagin dual. We hope to axiomatize these duals as *étale $C^*$-quantum groupoids* and to establish a duality between étale and compact $C^*$-quantum groupoids.

As another example, every locally compact Hausdorff groupoid $G$ yields a regular $C^*$-pseudo-multiplicative unitary $V$ such that $\hat{A}(V) \cong C_0(G)$ and $A(V) \cong C^*_r(G)$. Further examples of $C^*$-pseudo-multiplicative unitaries can be obtained from continuous bundles of multiplicative unitaries, tracial conditional expectations, and via natural constructions like tensor products and direct sums.

3.3. **Duality for coactions.** There exists a natural notion of a coaction of a concrete Hopf $C^*$-bimodule over $\mu$ on $C^*$-$\mu$-algebras. Given a regular $C^*$-pseudo-multiplicative unitary $V$ with a suitable additional symmetry, we can construct for each coaction of one of the associated Hopf $C^*$-bimodules a reduced crossed product with a dual coaction of the other Hopf $C^*$-bimodule, and identify the iterated reduced crossed product with a stabilization of the initial coaction. In particular, this duality result covers actions of and Fell bundles on locally compact groupoids.

## 4. Transition to the setting of von Neumann algebras

All concepts introduced above — the relative tensor product, the fiber product, Hopf $C^*$-bimodules, and $C^*$-pseudo-multiplicative unitaries — can be mapped functorially to their respective counterparts in the setting of von Neumann algebras, which were introduced by Connes, Sauvageot, Vallin, Enock, Lesieur and others.

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## Liberation of orthogonal Lie groups

Teodor Banica
(joint work with Roland Speicher)

The notion of free quantum group appeared in Wang's papers. The idea is as follows: let $G \subset U_n$ be a compact group. The $n^2$ matrix coordinates $u_{ij}$ satisfy certain relations $R$, and generate the algebra $C(G)$. One can define then the universal algebra $A$ generated by $n^2$ noncommuting variables $u_{ij}$, satisfying the relations $R$. For a suitable choice of $R$ we get a Hopf algebra in the sense of Woronowicz, and we have the heuristic formula $A = C(G^+)$, where $G^+$ is a compact quantum group, called free version of $G$. (Clearly, if $A$ is not commutative then $G^+$ is a fictional object and any statement about $G^+$ has to be interpreted in terms of $A$ to make rigorous sense.)

This construction is not axiomatized, in the sense that $G^+$ depends on the relations $R$, and it is not known in general what the good choice of $R$ is. For instance any choice with $R$ including the commutativity relations $u_{ij}u_{kl} = u_{kl}u_{ij}$ would be definitely a bad one, because in this case we would get $G^+ = G$. Moreover, any choice with $R$ including certain relations which imply these commutativity relations would be a bad one as well.

The purpose of this talk is to bring some advances on the axiomatization and general study of free quantum groups.

The starting object is a compact group satisfying $S_n \subset G \subset U_n$. The main problem with the construction of the liberation $A = C^*(u_{ij}|R)$ is whether the normality of the generators should be included or not into the relations $R$.

For instance in the case $G = U_n$ the normality of generators has definitely to be avoided, simply in order to get a Hopf algebra, while in the case of the complex reflection groups $G = H_n^s$, the normality of generators has to be included into the relations $R$, for the "liberation" to be compatible in some natural sense with Voiculescu's free probability theory.

Summarizing, in the unitary case the main problem, recently discovered and not solved yet, concerns the normality of the generators.

In this talk we investigate the orthogonal case, $G \subset O_n$. The matrix coordinates $u_{ij}$, being in this case real functions, are self-adjoint in the $C^*$-algebra sense. So, it is natural to assume that the relations $R$ contain the self-adjointness conditions $u_{ij} = u_{ij}^*$, and the above-mentioned normality issue dissapears.

With this observation at hand, the main problem if to find the relevant "extra relations" between the generators $u_{ij}$. Inspired by Tannakian philosophy as developed by Woronowicz, we propose here the following answer: the relevant "extra relations" should be those corresponding to the "noncrossing partitions appearing as intertwiners between the tensor powers of $u$".

Our main results can be described as follows. First, we classify on one hand the orthogonal groups having "noncrossing presentations", and on the other hand, the orthogonal free quantum groups. These groups and quantum groups are in a natural correspondence, as follows:

$$\begin{matrix} B_n & \subset & B'_n & \subset & O_n \\ \cup & & \cup & & \cup \\ S_n & \subset & S'_n & \subset & H_n \end{matrix} \qquad \Longleftrightarrow \qquad \begin{matrix} B_n^+ & \subset & B'^+_n & \subset & O_n^+ \\ \cup & & \cup & & \cup \\ S_n^+ & \subset & S'^+_n & \subset & H_n^+ \end{matrix}$$

Here $S_n, O_n$ are the symmetric and orthogonal groups, $B_n, H_n$ are the bistochastic and hyperoctahedral groups, and we use the notation $G' = \mathbb{Z}_2 \times G$.

The classification is done by computing all possible categories of partitions, respectively of noncrossing partitions. We show that each of these two categorical problems has exactly 6 solutions, given by:

$$\begin{matrix} \left\{\begin{matrix}\text{singletons and}\\ \text{pairings}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{singletons and}\\ \text{pairings (even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all}\\ \text{pairings}\end{matrix}\right\} \\ \cap & & \cap & & \cap \\ \left\{\begin{matrix}\text{all}\\ \text{partitions}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{all partitions}\\ \text{(even part)}\end{matrix}\right\} & \supset & \left\{\begin{matrix}\text{with blocks of}\\ \text{even size}\end{matrix}\right\} \end{matrix}$$

We discuss now a basic problem, belonging at the same time to representation theory and to probability, namely the computation of the asymptotic laws of truncated characters for the above 6 groups and 6 quantum groups. These laws, depending on a truncation parameter $t \in (0, 1]$, are as follows:

$$\begin{matrix} s_t & - & s'_t & - & g_t \\ | & & | & & | \\ p_t & - & p'_t & - & b_t \end{matrix} \qquad \Longleftrightarrow \qquad \begin{matrix} \sigma_t & - & \sigma'_t & - & \gamma_t \\ | & & | & & | \\ \pi_t & - & \pi'_t & - & \beta_t \end{matrix}$$

Here $p_t, g_t, s_t, b_t$ are the Poisson, Gaussian, shifted Gaussian and Bessel laws, $\pi_t, \gamma_t, \sigma_t, \beta_t$ are the free Poisson, semicircular, shifted semicircular and free Bessel laws, and the prime signs denote the symmetric versions.

The laws at the corners of the above two rectangles are known to form semigroups with respect to convolution and free convolution, respectively, and correspond to each other via the Bercovici-Pata bijection. We have a simple proof for this fact, by using cumulants and free cumulants.

Finally, we have some more general classification problems. The idea is that the above 6-classification results concern the following two situations:

$$\begin{array}{ccccc} S_n^+ & \subset & G_{free} & \subset & O_n^+ \\ \cup & & & & \cup \\ S_n & \subset & G_{class} & \subset & O_n \end{array}$$

The unifying problem concerns the classification of the quantum groups satisfying $S_n \subset G \subset O_n^+$. We don't have an answer here, but have several advances on the problem, notably the construction of 3 more examples.

These new quantum groups, that we denote $O_n^*, B_n^*, B_n'^*$, are constructed via Tannakian duality, by using certain categories of partitions. They sit as follows with respect to the previously known examples of orthogonal quantum groups:

$$\begin{array}{ccccc} B_n^+ & \subset & B_n'^+ & \subset & O_n^+ \\ \cup & & \cup & & \cup \\ B_n^* & \subset & B_n'^* & \subset & O_n^* \\ \cup & & \cup & & \cup \\ B_n & \subset & B_n' & \subset & O_n \end{array}$$

As a conclusion, the present results bring us one step further into the clarification of the relationship between free quantum groups and free probability, from the representation theory point of view.

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## The Free Product of Planar Algebras

Dietmar Bisch
(joint work with Vaughan Jones)

Let $N \subset M$ be an (extremal) inclusion of $\mathrm{II}_1$ factor with finite Jones index ([7]). The *standard invariant* of $N \subset M$, given by the system of higher relative commutants

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} = M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \cdots \end{array}$$

can be axiomatized as a (subfactor) *planar algebra* ([8], [10]). It is a very rich group-like object that captures the "quantum symmetries" encoded in the subfactor. The planar algebra formalism allows one to carry out sophisticated computations with the standard invariant in an extremely efficient way, and hence one can analyze the structure of subfactors via Popa's reconstruction theorem [10]. Note that the standard invariant is a complete invariant for *amenable* subfactors of the hyperfinite $\mathrm{II}_1$ factor ([9]). In the remainder of this report, we will mean "subfactor planar algebra" whenever we say "planar algebra". See [8] for the notation used here.

We will proceed with the definition of what we call the *free product* of two planar algebras $\mathfrak{A}$ and $\mathfrak{B}$.

**Definition 1.** Given labelled $n$-tangles $T_A$ of $\mathfrak{A}$ and $T_B$ of $\mathfrak{B}$ we say that $T_A$ is *free with respect to* $T_B$ if there is a Temperley-Lieb $n$-tangle $T_0$ (called the *separating tangle*) such that:

(i) The interior of each shaded boundary interval of $T_0$ contains exactly two marked boundary points of $T_B$ and the interior of every unshaded boundary interval of $T_0$ contains exactly two marked boundary points of $T_A$.

(ii) The starred boundary interval of $T_B$ contains that of $T_0$ which contains that of $T_A$.

(iii) $T_B$ is contained entirely within the shaded regions of $T_0$ and $T_A$ is contained entirely within the unshaded regions of $T_0$.

**Definition 2.** A *free product $n$-tangle* $T$ is a $2n$-tangle labelled with $\mathfrak{A} \sqcup \mathfrak{B}$ which is the union of a $T_A$ and a $T_B$ with $T_B$ free with respect to $T_A$.

Isotopy classes of free product $n$-tangles span a graded vector space denoted by $\mathfrak{A}\mathfrak{B} = \{(\mathfrak{A}\mathfrak{B})_n\}$. We define an action of the planar operad on $\mathfrak{A}\mathfrak{B}$ as follows. Given a planar (unlabelled) $n$-tangle $\mathfrak{T}$ with $k$ internal disks, add close parallels to each of its strings on the side of the shaded regions to form the planar $2n$-tangle $\hat{\mathfrak{T}}$. Then given appropriate free product tangles $T_1, T_2, ..., T_k$, $Z_{\hat{\mathfrak{T}}}(T_1, T_2, ..., T_k)$ is a well defined labelled $2n$-tangle which is obviously a free product tangle whose separating tangle is $Z_{\hat{\mathfrak{T}}}$ applied to the separating tangles of $T_1$, $T_2$,..., $T_k$. Isotopies of $\mathfrak{T}$ and the $T_i$ only affect $Z_{\hat{\mathfrak{T}}}(T_1, T_2, ..., T_k)$ by isotopies. Thus $\mathfrak{A}\mathfrak{B}$ becomes a planar algebra.

**Lemma 3.** *Let $T$ be a free product tangle as above. The map $T \mapsto Z(T_{\mathfrak{A}}) \otimes Z(T_{\mathfrak{B}})$ defines a planar algebra homomorphism $\Psi$ from $\mathfrak{A}\mathfrak{B}$ to the tensor product planar algebra $\mathfrak{A} \otimes \mathfrak{B}$.*

We can therefore define the *free product planar algebra* $\mathfrak{A} * \mathfrak{B}$ as the image $\Psi(\mathfrak{A}\mathfrak{B})$ in $\mathfrak{A} \otimes \mathfrak{B}$ ([6]). It is obvious from the definition in [3] that the Fuss-Catalan planar algebras are the free product of two Temperley-Lieb planar algebras. Note that our notion of free product planar algebra corresponds to what is called *free composition* of subfactors in [3] (see also [2]).

It turns out that the structure of the free product planar algebra $\mathfrak{A} * \mathfrak{B}$ can be determined explicitly. For instance, the dimensions of the vector spaces $(\mathfrak{A} * \mathfrak{B})_n$ can be computed from the dimensions of $\mathfrak{A}_k$ and $\mathfrak{B}_l$ using Voiculescu's free multiplicative convolution ([11]). Principal graphs and fusion algebras can be determined as well using similar methods as in [3] and [4].

Another interesting feature of the free product planar algebra is the fact that it has the following universal property. If $\mathfrak{P}$ is a planar algebra which contains a biprojection $p \in P_2$ ([1], see also [5]), one can define two natural projections in the annular algebra associated to $\mathfrak{P}$ using this biprojection (see [6] for details). One can then show that the images of $\mathfrak{P}$ under these projections become planar algebras $\mathfrak{A}$ and $\mathfrak{B}$, and under suitable irreducibility assumptions one can prove that $\mathfrak{A} * \mathfrak{B}$ is a sub-planar algebra of $\mathfrak{P}$. This is a subtle and nontrivial result from which the structure theorem below follows. If $N \subset P \subset M$ is a composition of subfactors $N \subset P$ and $P \subset M$, then the Jones projection onto the intermediate subfactor $P$ is a biprojection in the planar algebra $\mathfrak{P}_{N \subset M}$ associated to $N \subset M$ by [1]. Thus we obtain:

**Theorem 4.** *Let $N \subset P \subset M$ be an inclusion of (extremal) subfactors with finite index. Assume $N' \cap P = \mathbb{C}$ and $P' \cap M = \mathbb{C}$. Then $\mathfrak{P}_{N \subset P} * \mathfrak{P}_{P \subset M} \subset \mathfrak{P}_{N \subset M}$.*

Thus, in general, the planar algebra of $N \subset M$ will contain more structure then just the Fuss-Catalan planar algebra whenever an intermediate subfactor is present.

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## Some applications of free stochastic calculus

Dimitri Shlyakhtenko

**Hypotesis A.** We say that a $\mathrm{II}_1$ factor $M$ satisfies this hypothesis, if there exists a one-parameter family of embeddings $\alpha_t : M \to M * L(\mathbb{F}_\infty)$, having the property that for some generators $X_1, \ldots, X_n$ of $M$,

$$\alpha_t(X_j) = X_j + tQ_j + O(t^2)$$

with $Q_j \in \overline{\mathrm{span}(MS_1M + \cdots + MS_nM)}^{L^2}$, where $S_1, S_2, \ldots, S_n$ are a free semicircular system in $L(\mathbb{F}_\infty)$.

**Theorem** [5] Assume that a $\mathrm{II}_1$ factor $M$ satisfies Hypothesis A for some generators $X_1, \ldots, X_n$, and that $M$ is embeddable into the ultrapower of the hyperfinite $\mathrm{II}_1$ factor. Then Voiculescu's free entropy dimension $\delta_0$ satisfies

$$\delta_0(X_1, \ldots, X_n) \geq \dim_{M \otimes M^o} \overline{\mathrm{span}(MQ_1M + \cdots + MQ_nM)}^{L^2}$$

(here $MS_1M + MS_2M + \cdots$ is isometrically identified, as an $M, M$-bimodule, with $L^2(M)\bar{\otimes}L^2(M)^{\oplus n}$).

Combined with previous results of Voiculescu, Ge and others (cf. Voiculescu's survey [6]), this theorem implies that any algebra satisfying Hypothesis A is non-$\Gamma$, has no Cartan subalgebras and is prime.

We show that several classes of von Neumann algebras satisfy Hypothesis A. These include von Neumann algebras of groups in the smallest class of groups closed under passage to finite index subgroups/extensions, amalgamated free products over finite groups, and containing all groups with vanishing first $L^2$-Betti number which are $R^\omega$-embeddable. Combined with earlier estimates on free entropy dimension [3], this implies that for such groups $\delta_0$ is equal to the following combination of $L^2$-Betti numbers: $\beta_1^{(2)} - \beta_0^{(2)} + 1$.

Another class of von Neumann algebras satisfying Hypothesis A are von Neumann algebras coming from random matrix models with polynomial potentials satisfying a convexity property [4].

We also show that von Neumann algebras generated by $q$-semicircular systems [2] satisfy Hypothesis A for small values of $|q|$.

To show that Hypothesis A holds for these von Neumann algebras, at the suggestion of A. Guionnet, we utilize free stochastic calculus [1]. We prove a technical result, showing existence of stationary solutions to certain free stochastic differential equations with coefficients given by analytic power series. If such a stationary solution has the same law as a generating set of $M$, then $M$ satisfies Hypothesis A.

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## Towards a Universal Coefficient Theorem for Kirchberg's bivariant K-theory

Ralf Meyer
(joint work with Ryszard Nest)

I present here some recent joint work with Ryszard Nest that aims at a Universal Coefficient Theorem that would help to extend classification results for simple purely infinite C*-algebras to the non-simple case.

First I discuss the results for simple C*-algebras that we would like to generalise. These build upon two rather deep classification results by Eberhard Kirchberg:

**Theorem 1.** *Let A and B be simple, purely infinite, stable, separable, nuclear* C*-*algebras. Then any* KK-*equivalence between them lifts to a* *-*isomorphism.*

**Theorem 2.** *Any separable, nuclear* C*-*algebra is* KK-*equivalent to a simple, purely infinite, stable, separable, nuclear* C*-*algebra.*

Roughly speaking, these two results mean that classifying simple, purely infinite, stable, separable, nuclear C*-algebras up to isomorphism is equivalent to classifying separable, nuclear C*-algebras up to KK-equivalence.

The main point of the second theorem is to replace a nuclear C*-algebra by a KK-equivalent *simple* nuclear C*-algebra. It is useful to allow non-simple C*-algebras because the category of all separable, nuclear C*-algebras with $\mathrm{KK}(A, B)$ as morphisms from $A$ to $B$ has good formal properties: it is a triangulated category. This provides some tools to study separable, nuclear C*-algebras up to KK-equivalence.

The main tool here is the Universal Coefficient Theorem. It holds for a certain class of separable, nuclear C*-algebras called the *bootstrap class*, and it computes Kasparov's bivariant K-theory in terms of ordinary K-theory. As a consequence, two C*-algebras in the bootstrap class are KK-equivalent if and only if they have isomorphic K-theory; more precisely, any K-theory isomorphism lifts to a KK-equivalence and, in the purely infinite, stable, simple case, even to a *-isomorphism.

We want to develop an analogous theory for non-simple algebras. In this case, the primitive ideal space is another important invariant, which we have to take into account. To extend Kirchberg's Theorem 1, we need a version of Kasparov theory that takes into account the primitive ideal spaces in question. This theory was developed by Eberhard Kirchberg as well.

Let $X$ be a sober topological space, possibly non-Hausdorff. A C*-*algebra over* $X$ is a C*-algebra $A$ together with a continuous map from its primitive ideal space to $X$. Since the lattice of open subsets in the primitive ideal space of $A$ agrees with the ideal lattice of $A$, such maps correspond to maps from the lattice of open subsets in $X$ to the ideal lattice in $A$ that preserve arbitrary suprema and finite infima. Roughly speaking, a C*-algebra over $X$ is a C*-algebra $A$ with distinguished ideals $A(U)$ for open subsets $U$ of $X$ satisfying natural assumptions. We call a C*-algebra over $X$ *tight* if the map from its primitive ideal space to $X$ is a homeomorphism.

Let $A$ and $B$ be two C*-algebras over $X$. We may consider Kasparov cycles for $A$ and $B$ that are compatible with the distinguished ideals in a suitable sense which ensures that our cycle restricts to Kasparov cycles for $A(U)$ and $B(U)$ for all open subsets $U$ of $X$. Homotopy classes of such cycles define Kirchberg's bivariant K-theory for C*-algebras over $X$. It has the same formal properties as Kasparov's bivariant K-theory. In particular, it is the universal split-exact C*-stable functor on the category of separable C*-algebras over $X$.

Kirchberg already extended Theorem 1 to the non-simple case:

**Theorem 3.** *Let $A$ and $B$ be tight, strongly purely infinite, stable, separable, nuclear* C* *-algebras over the same topological space $X$. Then any invertible element in* $\mathrm{KK}_0^X(A,B)$ *lifts to a* *-*isomorphism from $A$ to $B$.*

The tightness assumption replaces simplicity, otherwise the assumptions are the same. A C*-algebra is strongly purely infinite if and only if it absorbs the Cuntz algebra $\mathcal{O}_\infty$.

Theorem 2 above is harder to extend because not every topological space is the primitive ideal space of a C*-algebra. So far, we can only prove an analogue of Theorem 2 if the underlying space $X$ is finite:

**Theorem 4.** *Let $X$ be a finite $T_0$-space. Any separable nuclear* C* *-algebra over $X$ is* $\mathrm{KK}^X$*-equivalent to a tight, strongly purely infinite, stable, separable, nuclear* C* *-algebra over $X$.*

In particular, this contains the well-known fact that any finite $T_0$-space is the primitive ideal space of some separable nuclear C*-algebra. We can make it strongly purely infinite and stable by tensoring with the Cuntz algebra $\mathcal{O}_\infty$ and with the C*-algebra of compact operators.

As above, this theorem means that classifying strongly purely infinite, stable, separable, nuclear C*-algebras with finite primitive ideal space $X$ up to isomorphism over $X$ is equivalent to classifying separable, nuclear C*-algebras over $X$ up to $\mathrm{KK}^X$-equivalence. Again, the latter problem is simpler because $\mathrm{KK}^X$ is a triangulated category, so that we may study it using homology theories. It is not clear to what extent Theorem 4 extends to infinite $X$.

The next issue is the right analogue of the bootstrap class for C*-algebras over $X$. Several of the equivalent characterisations make sense in this generality, but it is somewhat unclear which of them remain equivalent. We only define the bootstrap class for finite spaces $X$ here. We prefer the characterisation by generators.

For each $x \in X$, we let $(\mathbb{C}, x)$ be the elementary C*-algebra $\mathbb{C}$ viewed as a C*-algebra over $X$ via the constant map from its primitive ideal space – a single point – to $X$ with value $x$. We let $\mathcal{B}(X)$ be the localising subcategory of $\mathrm{KK}^X$ generated by $(\mathbb{C}, x)$; that is, we take the smallest class of C*-algebras over $X$ that contains $(\mathbb{C}, x)$ for all $x \in X$ and is closed under suspensions, extensions, countable direct sums, and $\mathrm{KK}^X$-equivalence.

**Proposition 5.** *Let $X$ be a finite $T_0$-space. A* $\mathrm{C}^*$*-algebra $A$ over $X$ belongs to $\mathcal{B}(X)$ if and only if $\mathrm{KK}_*^X(A,D) = 0$ whenever $\mathrm{K}_*\bigl(D(U)\bigr) = 0$ for all open subsets $U$ of $X$.*

Roughly speaking, this means that $\mathcal{B}(X)$ is exactly the right category on which a Universal Coefficient Theorem should be formulated. If $\mathrm{K}_*\bigl(D(U)\bigr) = 0$ for all open subsets $U$ of $X$, then all K-theoretic data we can attach to $D$ vanishes; hence any Universal Coefficient Theorem will predict $\mathrm{KK}_*^X(A,D) = 0$ for such $D$. That is, $\mathcal{B}(X)$ is the largest possible category on which a Universal Coefficient Theorem can hold.

The main point in the proof of the proposition is the natural isomorphism

$$\mathrm{KK}_*^X\bigl((\mathbb{C},x),D\bigr) \cong \mathrm{K}_*\bigl(D(U_x)\bigr),$$

where $U_x$ is the minimal open neighbourhood of $x$, which exists because $x$ is finite.

It turns out that a separable, nuclear $\mathrm{C}^*$-algebra $A$ over $X$ belongs to $\mathcal{B}(X)$ if and only if $A(U)$ belongs to the usual bootstrap category for all open subsets $U$ of $X$ – it even suffices to assume this for the minimal open subsets $U_x$ for $x \in X$ mentioned above. As a consequence, any $\mathrm{C}^*$-algebra over $X$ of type 1 belongs to $\mathcal{B}(X)$. Conversely, a $\mathrm{C}^*$-algebra over $X$ that belongs to $\mathcal{B}(X)$ is $\mathrm{KK}^X$-equivalent to a type 1 $\mathrm{C}^*$-algebra over $X$ by adapting the proof of Theorem 4. Even more, we can achieve that the subquotients $A_x = A(U_x) \bigm/ A(U_x \setminus \{x\})$ are all commutative. It seems likely that we can even achieve that $A$ itself is commutative – this description of the bootstrap category was suggested by Kirchberg – but it is unclear how to prove this.

Now we know for which class of $\mathrm{C}^*$-algebras over $X$ a Universal Coefficient Theorem is expected to hold. It remains to find a Universal Coefficient Theorem suitable for classification purposes. There is some general machinery available for this purpose, but it only produces spectral sequences in general. A spectral sequence is merely a bookkeeping tool that guides computations, but it is usually not enough to actually carry them out to the finish. Roughly speaking, it involves a recursive procedure that converges towards the group we want to compute and whose starting point has a reasonably simple description. But in each step, we need certain boundary maps that provide obstructions to lifting the results of our computation so far to the group we wish to compute. In particular, these may obstruct the lifting of an isomorphism on the invariant to a $\mathrm{KK}^X$-equivalence. Therefore, an invariant is only suitable for classification purposes if the spectral sequence that it produces degenerates at the very first step.

Here an invariant is a homological functor $F$ from $\mathrm{KK}^X$ to some Abelian category $\mathcal{C}$ that is universal in the sense that any other homological functor with a larger kernel on morphisms factors through it; this ensures that the category $\mathcal{C}$ sees all the internal symmetries of $F$. We must check that $\mathrm{Ext}^p_{\mathcal{C}}\bigl(F(A),F(B)\bigr)$ vanishes for $p \geq 2$: then we get a Universal Coefficient Theorem of the usual form:

$$\mathrm{Ext}^1_{\mathcal{C}}\bigl(F(A),F(B)\bigr) \rightarrowtail \mathrm{KK}_*^X(A,B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}}\bigl(F(A),F(B)\bigr).$$

This ensures that an isomorphism $F(A) \cong F(B)$ in $\mathcal{C}$ lifts to a $\mathrm{KK}^X$-equivalence.

For some particularly simple spaces $X$, for instance, in the linearly ordered case, filtrated K-theory is an invariant with this property. But there are also simple spaces with four points for which filtrated K-theory still has non-vanishing $\mathrm{Ext}^2$. It seems likely that there are spaces $X$ for which it is impossible to find an invariant with a UCT as above. But in the simple counterexamples studied so far, filtrated K-theory may be enriched to a complete invariant.

Finally, I briefly discuss filtrated K-theory. The main issue is to describe the target Abelian category $\mathcal{C}$. For any subset $Y$ of $X$ of the form $Y = U \setminus V$ with open subsets $V \subseteq U \subseteq X$, we want to consider the K-theory of the subquotient $A(Y) = A(U) \;/\; A(V)$. These K-theory groups admit certain natural transformations. The target category $\mathcal{C}$ is the category of all countable graded modules over the ring of natural transformations between $\mathrm{K}_*\big(A(Y)\big)$ for subsets $Y$ as above. More precisely, we should also restrict to subsets $Y$ that are non-empty and connected here.

To describe the ring of natural transformations, we construct C*-algebras $\mathcal{R}_Y$ over $X$ with $\mathrm{KK}^X(\mathcal{R}_Y, D) \cong \mathrm{K}_*\big(D(Y)\big)$. Then the natural transformations from $\mathrm{K}_*\big(D(Y)\big)$ to $\mathrm{K}_*\big(D(Z)\big)$ are given by $\mathrm{KK}^X(\mathcal{R}_Z, \mathcal{R}_Y) \cong \mathrm{K}_*\big(\mathcal{R}_Y(Z)\big)$. The construction of $\mathcal{R}_Y$ is quite explicit and uses certain natural subcomplexes of the order complex of the partial order on $X$ induced by the topology. Hence the ring of natural transformations may be computed explicitly. In the examples studied so far, this ring may also be described by obvious generators and relations coming from K-theory six-term exact sequences.

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## The automorphism groups for non-simple Cuntz-Krieger algebras


Søren Eilers

(joint work with Gunnar Restorff and Efren Ruiz)


The profound isomorphism theorem of Kirchberg ([5]) allows a reduction of the study of $*$-automorphisms of certain non-simple $C^*$-algebras to the study of their ideal-related $KK$-theory. However, the ideal-related $KK$-groups have proved to be quite difficult to compute in general, as explained below and indicated also by the work of Meyer and Nest (ibid.), and thus at this stage we do not even in the finite ideal case have a canonical candidate for a univariant $K$-functor to classify such objects or indeed their automorphism groups. We report on a complete solution of the latter type of problem in a special case leading to computations of automorphism groups of certain non-simple Cuntz-Krieger algebras (cf. [1]) the novel feature of which is a new kind of $K$-groups which could lead to a deeper understanding of the general problem.

We focus on the fundamental case of $C^*$-algebras with precisely one non-trivial ideal, as it is in this case quite clear from the seminal work of Rørdam ([8]) that any useful invariant must be based on the six-term exact sequence in $K$-theory. In fact, we have managed to generalize this as follows:

**Theorem 1.** [3] *Let two unital extensions*

$$\mathfrak{e}_i : \qquad 0 \longrightarrow B_i \otimes \mathbb{K} \longrightarrow E_i \longrightarrow A_i \longrightarrow 0$$

*be given. If all algebras $A_i$ and $B_i$ are unital and fall in either of the classes of simple $C^*$-algebras classified by Lin ([7]) or by Kirchberg-Phillips ([6]) then $E_1 \otimes \mathbb{K} \simeq E_2 \otimes \mathbb{K}$ precisely when the six term sequences associated to $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are isomorphic with order isomorphisms at $K_0(A_i)$ and $K_0(B_i)$.*

**Remark 2.** The case when all $A_i$ and $B_i$ are Kirchberg algebras is [8]; the mixed cases are relevant for applications such as Matsumoto algebras and graph algebras.

To classify automorphisms on non-simple $C^*$-algebras up to approximate unitary equivalence, one naturally follows the lead of the work of Dadarlat and Loring ([2]) which gave such a characterization of the automorphism groups of certain stably finite $C^*$-algebras of real rank zero as a corollary to their Universal Multi-Coefficient Theorem (UMCT). Combining this result with the classification of isomorphisms on simple Kirchberg algebras it is clear that one needs to consider $K$-theory with coefficients in a similar manner.

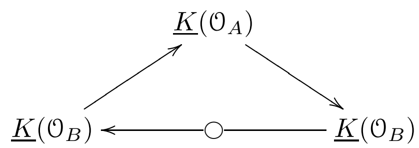
To our initial surprise, we could prove ([4]) that this collection is not sufficient to support a model for the automorphisms even for Cuntz-Krieger algebras with one ideal. For instance, $\mathcal{O}_A$ with

$$A = \begin{bmatrix} B & 0 \\ C & B \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

we get that

$$\overline{\mathrm{Inn}}(\mathcal{O}_A) \to \mathrm{Aut}(\mathcal{O}_A) \to \mathrm{Aut}(\underline{K}_\Delta(\mathcal{O}_A))$$

is not injective, where $\underline{K}_\Delta(\mathcal{O}_A)$ is the six term exact sequence of total $K$-theory which in triangular form looks like




Investigating cases of this type has lead us to a new form of $K$-groups which can be used to compute $\mathrm{Aut}(A)/\overline{\mathrm{Inn}}(A)$ for $A$ a Cuntz-Krieger algebra of the type mentioned above. They are given by using certain extensions $\mathfrak{e}_n$ as "test objects" in Kirchberg's ideal-related $KK$-theory, where

$$\mathfrak{e}_0 : \qquad 0 \longrightarrow M_n(C_0(0,1)) \longrightarrow \mathbb{I}_n \longrightarrow \mathbb{C} \longrightarrow 0$$

is the canonical extension associated to the nonunital dimension drop algebra $\mathbb{I}_n$, and the subsequent extensions are given by mapping cones of their predecessors. The new groups $KK^i_{\mathcal{E}}(\mathfrak{e}_n, A)$ are 6-periodic in the sense that $KK^i_{\mathcal{E}}(\mathfrak{e}_3, A) = KK^{i+1}_{\mathcal{E}}(\mathfrak{e}_0, A)$, and must be considered along with certain natural transformations similar to the Bockstein operations. Thus, with $\underline{\mathbf{K}}_{\blacktriangle}(-)$ defined taking all this structure into account (notation may be subject to change), we get

**Theorem 3.** *For a Cuntz-Krieger algebra $\mathcal{O}_A$ with property (II) and precisely one ideal, we have short exactness of*

$$\{1\} \to \overline{\mathrm{Inn}}(\mathcal{O}_A) \to \mathrm{Aut}(\mathcal{O}_A) \to \mathrm{Aut}(\underline{\mathbf{K}}_{\blacktriangle}(\mathcal{O}_A)) \to \{1\}$$

We have not been able to determine whether or not this invariant allows a UMCT in general, but by using the special propeties of the six-term exact sequence for Cuntz-Krieger algebras we have established a UMCT in this particular case. In fact, as a part of the proof we show that only a small subset of the invariants mentioned above (and in particular not all of the new groups) are necessary when all $K$-groups are finitely generated, all $K_1$-groups are free, and the exponential map vanishes.

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**Fundamental groups of $\mathrm{II}_1$ factors and equivalence relations**

Stefaan Vaes
(joint work with Sorin Popa)

Murray and von Neumann [6] introduced in 1943 the *fundamental group* of a $\mathrm{II}_1$ factor and of a $\mathrm{II}_1$ equivalence relation and showed that it equals $\mathbb{R}_+$ in the hyperfinite case. Since then, the question which subgroups of $\mathbb{R}_+$ can arise as a fundamental group, attracted a lot of attention. The first breakthrough was obtained by Connes, who proved [2] that the group von Neumann algebra $\mathcal{L}(\Gamma)$ has countable fundamental group whenever $\Gamma$ is a group with property (T) and infinite conjugacy classes (ICC).

Although Popa proved in [8] that every countable subgroup of $\mathbb{R}_+$ is the fundamental group of a $\mathrm{II}_1$ factor $M$, as well as of a $\mathrm{II}_1$ equivalence relation $\mathcal{R}$, the question whether uncountable subgroups different from $\mathbb{R}_+$ can occur, remained open. If you moreover require that $M = \mathrm{L}^\infty(X) \rtimes \Gamma$, resp. $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$, is given by a free ergodic probability measure preserving (p.m.p.) action $\Gamma \curvearrowright (X,\mu)$ of a countable group $\Gamma$ on a standard probability space $(X,\mu)$, only the groups $\{1\}$ and $\mathbb{R}_+$ had been realized as $\mathcal{F}(M), \mathcal{F}(\mathcal{R})$, see [3, 4, 7].

In [9], we exhibit a large family $\mathcal{S}$ of subgroups of $\mathbb{R}$ and prove that for every $H \in \mathcal{S}$, there exist uncountably many free ergodic p.m.p. actions of the free group $\mathbb{F}_\infty \curvearrowright (X,\mu)$ such that the associated $\mathrm{II}_1$ factors and equivalence relations have fundamental group $\exp(H)$ and are non-isomorphic. The family $\mathcal{S}$ includes $\mathbb{R}$, all its countable subgroups and subgroups of arbitrary Hausdorff dimension in the interval $(0,1)$. In [10], it is shown that the same statement holds if we replace $\mathbb{F}_\infty$ by an arbitrary group of the form $\Gamma^{*\infty} * \Sigma$, where $\Gamma$ and $\Sigma$ are infinite groups and $\Sigma$ is amenable. On the other hand, let $\Gamma$, $\Sigma$ be infinite, finitely generated groups such that $\Gamma$ is ICC and either has property (T) or can be written as a direct product $\Gamma = \Gamma_1 \times \Gamma_2$ of infinite groups with $\Gamma_1$ non-amenable. We prove in [10] that the fundamental group of $M = \mathrm{L}^\infty(X) \rtimes (\Gamma * \Sigma)$ is trivial for any free ergodic p.m.p. action of $\Gamma * \Sigma$. Note that the equivalence relation $\mathcal{R}(\Gamma * \Sigma \curvearrowright X)$ has trivial fundamental group by [3], since $\Gamma * \Sigma$ has first $\mathrm{L}^2$-Betti number strictly between 0 and $\infty$.

Given a free ergodic p.m.p. action $\Gamma \curvearrowright (X,\mu)$, in order to prove that $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ and $\mathcal{R}(\Gamma \curvearrowright X)$ have the same fundamental group, one has to show that whenever $\theta : M \to pMp$ is a $^*$-isomorphism with $p$ a projection in $\mathrm{L}^\infty(X)$, the Cartan subalgebras $\theta(\mathrm{L}^\infty(X))$ and $\mathrm{L}^\infty(X)p$ are unitarily conjugate. In the setup of [9], involving rigid actions of the free group $\mathbb{F}_\infty$, this follows directly from [7], while in [10], we rely on several results of [5].

Finally, we provide in [10] examples of $\mathrm{II}_1$ equivalence relations $\mathcal{R}$ having property (T) in the sense of Zimmer (see [1, 11]) and yet having fundamental group $\mathbb{R}_+$. In particular, neither $\mathcal{R}$ nor any of its amplifications $\mathcal{R}^t \cong \mathcal{R}$, $t > 0$, can be implemented by a free action of a group. We also give examples of $\mathrm{II}_1$ factors $M$ with $\mathcal{F}(M) = \mathbb{R}_+$, but such that nevertheless $M \overline{\otimes} \mathrm{B}(\ell^2)$ admits no trace scaling action of $\mathbb{R}_+$.

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## $\mathbb{R}$-Actions on the Cuntz algebra $\mathcal{O}_2$ and Reconstruction


Eberhard Kirchberg
(joint work with N.C. Phillips)


In the following all algebras $A, \ldots$ are separable, all spaces are second countable and "$G$-actions" are injective group-morphisms $G \hookrightarrow \mathrm{Aut}(A)$.

It is folklore that a $C^*$-algebra $A$ is *simple* if it is prime and is $G$-simple for a *compact* group $G$. In joint work with Chris Phillips [4] we show that, for each *non-compact* second countable amenable l.c. group $G$, there exists separable amenable(=nuclear) $A$ with a $G$-action such that $A \rtimes G \cong \mathcal{O}_2 \otimes \mathbb{K}$, $A \cong A \otimes \mathcal{O}_2$, and $A$ is prime and is *not simple*, but is $G$-simple (One finds in the case $G = \mathbb{Z}$ also a $\mathbb{Z}$-simple, prime, non-simple and unital AF-algebra, — but by different methods —). Our proof uses the following characterization of primitive ideal spaces $\mathrm{Prim}(A)$ of amenable separable $C^*$-algebras and the "reconstruction" theorem, respectively its $G$-equivariant corollary (their proofs are scattered over [2],[3],[5] and use [6]):

**Theorem [H.Harnisch, E.K., M.Rørdam]**
*Let $X$ a* point-complete *(or* spectral, sober*) $T_0$-space. TFAE:*

(i) $X \cong \mathrm{Prim}(E)$ *for some separable exact $C^*$-algebra $E$.*
(ii) *The lattice of open sets $\mathbb{O}(X)$ is isomorphic to an* sup–inf *invariant sublattice of $\mathbb{O}(Y)$ for some l.c. Polish space $Y$.*

*If $X$ satisfies (ii), then there is a* nuclear *$C^*$-algebra $A$ with $A \cong A \otimes \mathcal{O}_2 \otimes \mathbb{K}$, and a homeomorphism $\psi \colon X \to \mathrm{Prim}(A)$, that is unique in the following sense: For every nuclear $B$ with $B \cong B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and every homeomorphism $\phi \colon X \to \mathrm{Prim}(B)$, there is an isomorphism $\alpha \colon A \to B$ with $\alpha(\psi(x)) = \phi(x)$ for $x \in X$.*
*The isomorphism $\alpha$ is uniquely determined up to unitary homotopies (in $\mathcal{U}(\mathcal{M}(B))$ — in particular up to approximately inner automorphisms.*
(Notice here that $\psi(x)$ and $\phi(x)$ are — in particular — ideals of $A$.)

**Reconstruction-Theorem [H.Harnisch, E.K.]**
*Suppose that $A$ is a nuclear and stable, that $\Omega$ is a* sup–inf *closed sub-lattice of $\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A))$ with $\mathrm{Prim}(A), \emptyset \in \Omega$. Then there is a* non-degenerate *-mono*morphism $H_0 \colon A \to \mathcal{M}(A)$ with following properties:*

(i) *The infinite repeat $\delta_\infty \circ H_0$ is unitarily equivalent to $H_0$.*
(ii) *For every $U \in \mathbb{O}(\mathrm{Prim}(A))$ holds $H_0(J(V)) = H_0(A) \cap \mathcal{M}(A, J(U))$, where $V \in \Omega$ is given by $V = \bigcup\{W \in \Omega ;\ W \subset U\}$.*

*The $H_0$ is uniquely determined up to unitary homotopy.*
*The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ of the Hilbert $A$-$A$-module $\mathcal{H} := (A, H_0)$ is stable and strongly purely infinite; and it is the same as the $C^*$-Fock algebra $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$.*
*The natural embedding of $A$ into $\mathcal{O}_{\mathcal{H}}$ defines a lattice isomorphism from $\Omega$ onto $\mathbb{O}(\mathrm{Prim}(\mathcal{O}_{\mathcal{H}}))$ and is a $\mathrm{KK}(\Omega; \cdot, \cdot)$-equivalence.*

One can use the pentagon-rule unitary on $L_2(G \times G)$ to see that the uniqueness (in conjunction with the classification of non-simple amenable s.p.i. algebras) implies the following:

**Corollary [$G$-equivariant Reconstruction]**
*Suppose that $A$ and $\Omega$ are as in the Reconstruction Theorem. If a locally compact group $G$ acts on $A$ by $\alpha \colon G \to \mathrm{Aut}(A)$ with $\alpha(g)(J) \in \Omega$ for all $J \in \Omega$, then $H_0 \colon A \to \mathcal{M}(A)$ can be found such that — in addition —, $H_0$ can be taken $G$-equivariant.*
*More precisely: there is an action $\gamma \colon G \to \mathrm{Aut}(A)$ of $G$ on $A$ that is outer conjugate to $\alpha$ such that that $H_0$ can be found with $\gamma(g)\,(H_0(a)b) = H_0\,(\gamma(g)(a))\,\gamma(g)(b)$ for $g \in G$, $a \in A$.*
*In particular, $G$ acts on $\mathcal{O}_{\mathcal{H}}$ in a way that the inclusion map $A \hookrightarrow \mathcal{O}_{\mathcal{H}}$ is $G$-equivariant.*
(Notice: If above $A$ is of type I, then $\mathcal{O}_{\mathcal{H}}$ is a $\mathbb{Z}$-crossed product of an inductive limit of type I algebras.
The Corollary remains true for some Quantum groups including all Kac algebras (Ring groups) — with obvious modifications of the formulation —.)

The **proof of the existence** of prime, non-simple and $G$-minimal $A$ with the quoted properties proceeds as follows:

(1) We define a $T_0$ space $X(G)$ by $X(G) = G \cup \{\infty\}$ with topology given by $\{\emptyset, (G \setminus C) \cup \{\infty\}\,;\ G \supset C \text{ compact}\}$. Then we show that $X(G)$ satisfies (ii) of the first of the above theorems. It gives amenable $B$ with $B \cong B \otimes \mathcal{O}_2 \otimes \mathbb{K}$ and $\mathrm{Prim}(B) \cong X(G)$. Clearly, $G$ acts on $X(G)$ naturally, the action is topologically minimal, and $B$ is *prime*, because $\overline{\{\infty\}} = X(G)$.
(2) The action of $G$ on $X(G)$ defines a homeomorphism $\ell\colon (t,s) \to (s+t, s)$ of $X(G) \times G \cong \mathrm{Prim}(B \otimes \mathrm{C}_0(G))$. It lifts to an automorphism $\kappa$ of $C := B \otimes \mathrm{C}_0(G)$ — by the uniqueness part of the characterization of $\mathrm{Prim}(C)$ —, because $C \cong C \otimes \mathcal{O}_2 \otimes \mathbb{K}$.

(3) We apply the $G$-equivariant Reconstruction to $C$, $\Omega := \{U \times G\,;\ U \in \mathbb{O}(X(G))\,\}$ and the action $\alpha\colon g \in G \mapsto \kappa^{-1} \circ (\mathrm{id} \otimes \rho_g) \circ \kappa$. Since $\Omega \cong \mathbb{O}(X(G))$ by a $G$-equivariant isomorphism, the resulting amenable stable separable $C$*-algebra $A := \mathcal{O}_{\mathcal{H}}$ (for $G$-equivariant $H_0$ with respect to $\rho\colon G \to \mathrm{Aut}(C)$ — outer conjugate to $\alpha$ —), carries a $G$-action such that the natural monomorphism $C \hookrightarrow A$ is $G$-equivariant and is a $\mathrm{KK}(\Omega; \cdot, \cdot)$-equivalence. Since $C \cong C \otimes \mathcal{O}_2 \otimes \mathbb{K}$, we get that the stable separable amenable $A$ must satisfy $A \cong A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ (by classification of amenable s.p.i. algebras) and the $G$-action on $A$ . The pure infiniteness and primeness of $A$ implies that $A$ contains a full projection.

(4) Suppose that $G$ is amenable. We replace the action of $G$ on $A$ by the diagonal action of $G$ on $A \otimes \mathcal{O}_\infty \cong A$, where we consider $\mathcal{O}_\infty$ as the $C$*-Fock-algebra of $L_2(G)$. Then we obtain that $A \rtimes G$ is a simple nuclear algebra. We can again tensor by $\mathcal{O}_2 \otimes \mathbb{K}$ (with trivial $G$-action) and get $A \rtimes G \cong \mathcal{O}_2 \otimes \mathbb{K}$.

(5) If $G = \mathbb{R}$ then one finds actions $\beta\colon \mathbb{R} \to \mathrm{Aut}(A)$ and $\gamma\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_2)$, such that $\beta$ fixes a full projection of $A$, $\mathcal{O}_2 \rtimes_\gamma \mathbb{R} \cong A$ and $\beta$ and the dual action $\widehat{\gamma}$ on $A$ are both outer conjugate to the action on $A$ that was defined finally in part (4). (See a footnote [1]). □

In our talk, we have also discussed some applications of the Reconstruction theorem to questions concerning the range of possible "Universal Coefficient Theorems", e.g. as follows:
We say that a $C$*-subalgebra $A \subset B$ is *regular* (in $B$) if

$$(A \cap I) + (A \cap J) = A \cap (I + J) \quad \forall \text{ closed ideals } I, J \subset B\,.$$

Suppose that a regular Abelian $A \subset B$ exists such that — in addition — the inclusion defines in $\mathrm{KK}(X; A, B)$ a $\mathrm{KK}(X; \cdot, \cdot)$–equivalence of $A$ and $B$ for $X := \mathrm{Prim}(B)$. Such $A$ is *not* uniquely determined by $B$, but it has the property that $A$ and the action of $X$ on $A$ determine $B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ by the Reconstruction Theorem (and by classification of non-simple amenable stable s.p.i. algebras) up to $X$-equivariant isomorphisms *if $B$ is nuclear*, i.e., there is a *canonical reconstruction* of $B$ from $(A, X)$ if $B$ is strongly purely infinite, separable, stable and nuclear.

If such $A \subset B$ exists, we say that $B$ is in the *strong UCT-class* for $\mathrm{KK}(X; \cdot, \cdot)$. (Notice: The action of the *infinite* $\mathrm{T}_0$ space $X := \mathrm{Prim}(B)$ on $A$ satisfies in particular the required $\cap$-$\cup$-compatibility by R. Meyer and R. Nest, see their abstract.)

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# Singular foliations: C*-algebra and pseudodifferential calculus


Georges Skandalis

(joint work with Iakovos Androulidakis)


We construct the holonomy groupoid of *any* singular foliation. Our construction generalizes that of Ehresmann, Winkelnkemper, Connes [5, 10, 15] in the regular case and of Pradines, Bigonnet, Debord [3, 4, 8, 9, 13, 14] in an"almost regular" case.

Our holonomy groupoid is a quite ill behaved geometric object (but often has a nice longitudinal smooth structure). Nevertheless, we are able to use it in order to generalize Connes' construction of:

- the $C^*$-algebra $C^*(M, \mathcal{F})$ of the foliation;
- the longitudinal pseudo-differential calculus;
- the analytic index of a longitudinally elliptic operator which is an element of $K(C^*(M, \mathcal{F}))$.

*This talk is an account of results published in* [1] *and work in progress.*

## 1. Singular foliations

The kind of foliations that we deal with are the *Stefan-Sussman foliations:* such a foliation is a partition into connected submanifolds (of *non constant dimension*) given by vector fields more precisely with a (locally) finitely generated submodule of $C_c^\infty(M; TM)$ stable under Lie brackets.

We actually wish to keep track which vector fields we used to define our foliation. So, our definition is:

**Definition.** A *foliation* on $M$ is a locally finitely generated submodule $\mathcal{F} \subset C_c^\infty(M; TM)$ stable under Lie brackets.

It is immediate that Lie algebroids and in particular Lie groupoids (hence Lie group actions) give rise to foliations.

The foliation $\mathcal{F}$ is *regular* if it is a direct summand of $C_c^\infty(M; TM)$, *i.e.* it is the set of (smooth compactly supported) sections of an integrable subbundle of the tangent bundle $TM$.

The foliation $\mathcal{F}$ is *almost regular* if the module $\mathcal{F}$ is projective.

**Examples**

(1) Consider $\mathbb{R}$ foliated by three leaves: $(-\infty, 0)$, $\{0\}$, $(0, +\infty)$. One can take $\mathcal{F}$ to be generated by $x^n \partial/\partial x$ $(n > 0)$. From our definition, we consider these as *different foliations* for each $n$.

(2) Consider $\mathbb{R}^2$ foliated by two leaves $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. There is no obvious best choice for $\mathcal{F}$. The foliation can be given by an action of a Lie group $G$, where $G$ can be $GL(2, \mathbb{R})$, $SL(2, \mathbb{R})$, or $\mathbb{C}^*$. Moreover, there are many more choices: one could take vector fields that are $o(\|x\|^k)$...

## 2. The holonomy groupoid

In the singular case, we do not define directly the holonomy groupoid. We define *submersions* into the holonomy groupoid. These are given by a manifold $U$ with two submersions $s, t : U \to M$ such that each fiber of $s$ maps through $t$ into the same leaf in a "submersive" way. More precisely:

**Definition.** Let $(M, \mathcal{F})$ be a foliation. A *bi-submersion* of $(M, \mathcal{F})$ is a manifold $U$ with two submersions $s, t : U \to M$ such that $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = \ker dt + \ker ds$.

There is an obvious meaning of the *inverse* of a bi-submersion, the *composition* of two bi-submersions. Moreover, one may define a notion of a bi-submersion "near the identity".

There is an easy definition of equivalence of two germs of bi-submersions.

We then define an *atlas* to be a family of bi-submersions stable - up to equivalence by composition and inverse and which contains (up to equivalence) the bi-submersions near the identity. There is a minimal atlas $(U_i, t_i, s_i)_{i \in I}$ corresponding somewhat to paths.

**Definition.** The *holonomy groupoid of the foliation* is the quotient of $\bigcup_{i \in I} U_i$ by the above mentioned equivalence relation.

## 3. The C*-algebra, pseudodifferential calculus

In order to construct the analogue of smooth functions on our groupoid, we push a little further the construction of Alain Connes in the case of non Hausdorff holonomy groupoids [6]:

Our algebra is $\mathcal{A} = \left(\bigoplus_i C_c^\infty(U_i)\right)/\sim$ where $\sim$ is a equivalence relation. Its $*$-algebra is quite easily defined. Note that in our construction we *need* to consider *half densities* (on $\ker ds \oplus \ker dt$).

We then define unitary representations of the groupoid, and associate to them $*$-representations of our algebra (thanks to a - not so obvious - $L^1$ estimate). We then define $C^*(M, \mathcal{F})$ to be $C^*$-algebra obtained by completing $\mathcal{A}$ with respect to these representations.

NB. If the groupoid is smooth along the leaves we can further define regular represenations (on $L^2(G_x)$) and form the reduced $C^*$-algebra $C_r^*(M, \mathcal{F})$.

In order to construct the pseudodifferential calculus, we discuss the cotangent space to the foliation and construct pseudodifferential operators using the usual oscillatory integrals.

**Theorem.** These operators satisfy:

- A negative order pseudodifferential operator belongs to $C^*(M, \mathcal{F})$;
- A zero order pseudodifferential operator is a multiplier of $C^*(M, \mathcal{F})$;
- A positive order *elliptic* pseudodifferential operator is a *regular* (*cf.* [2]) unbounded multiplier of $C^*(M, \mathcal{F})$.

We discuss a few applications of these results. In particular, they tell us that we actually constructed the right object.

## 4. Further developments

We presented several developments and generalizations of our constructions. In particular, we discussed:

- Generalization to our case of Connes' tangent groupoid ([7]) and its relation to analytic index (see also [11]).
- Smooth singular foliations on locally compact spaces in the spirit of *continuous family groupoids* of [12].

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## KMS-states on Bost-Connes type systems

Nadia S. Larsen
(joint work with M. Laca and S. Neshveyev)

In a remarkable paper, Bost and Connes [1] introduced and studied a quantum statistical dynamical system which has deep connections with number theory. The underlying $C^*$-algebra is associated to a Hecke pair $(P^+_{\mathbb{Q}}, P^+_{\mathbb{Z}})$ coming from the rational $ax+b$-group. From its presentation in terms of generators and relations Laca and Raeburn [8] formalized an alternative description of this $C^*$-algebra as the semigroup crossed product $C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^*$, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the compact group of integral adeles and the action $\alpha$ by endomorphisms of $C(\hat{\mathbb{Z}})$ satisfies

$$\alpha_n(f)(x) = \begin{cases} f(n^{-1}x) & \text{if } x \in n\hat{\mathbb{Z}} \\ 0 & \text{otherwise.} \end{cases}$$

There is a natural one-parameter group of automorphisms (a dynamics) $\sigma_t \in \mathrm{Aut}(C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^*)$ satisfying $\sigma_t(\mu_m^* f \mu_n) = (m^{-1}n)^{it} \mu_m^* f \mu_n$, for $t \in \mathbb{R}$, $f \in C(\hat{\mathbb{Z}})$ and $n \in \mathbb{N}^*$ (here $\mu_m^* f \mu_n$ are spanning monomials of the crossed product, with $\mu_n$ isometries). The compact group $\hat{\mathbb{Z}}^* = \prod_p \mathbb{Z}_p^*$ acts as symmetry group, i.e. it implements automorphisms commuting with $\sigma$. Later Laca [5] also obtained $C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^*$ as the full corner in $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}_+^*$ determined by the projection corresponding to the characteristic function of $\hat{\mathbb{Z}}$.

We recall that a $\mathrm{KMS}_\beta$-state of a $C^*$-algebra $B$ with a dynamics $\sigma$ is a $\sigma$-invariant state $\varphi$ of $B$ such that $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for $a$, $b$ in a set of $\sigma$-analytic elements with dense linear span in $B$. A ground state is a $\sigma$-invariant state $\varphi$ such that the holomorphic function $z \mapsto \varphi(a\sigma_z(b))$ is bounded on the upper half-plane for $a$, $b$ analytic. A *$KMS_\infty$-state* is a weak* limit point of a sequence $\{\varphi_n\}_n$ with $\varphi_n$ a $\mathrm{KMS}_{\beta_n}$-state and $\beta_n \to +\infty$ as $n \to \infty$ [2].

The main theorem of [1] establishes several outstanding properties of the Bost-Connes system: a phase transition occurs at the pole of the Riemann zeta function, namely in the range $0 < \beta \le 1$ there is a unique $\mathrm{KMS}_\beta$-state which is a type $\mathrm{III}_1$ factor state, while in the range $\beta > 1$ the symmetry group of the system acts freely and transitively on the extremal $\mathrm{KMS}_\beta$-states. Moreover, each extremal $\mathrm{KMS}_\beta$-state for $\beta > 1$ is a type $\mathrm{I}_\infty$ state, and the partition function of the system is the Riemann zeta function. Further, the system encodes explicit class field theory for the rationals in the existence of a $\mathbb{Q}$-subalgebra such that values of the extremal KMS$_\infty$-states at elements of this algebra give algebraic numbers on which the Galois action of $Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ is intertwined via the class-field theory isomorphism with the action of the symmetry group $\hat{\mathbb{Z}}^*$.

The generalization of the Bost-Connes theorem to arbitrary number fields has been the focus of much research, and was achieved in the case of $K = \mathbb{Q}(\sqrt{-d})$ by Connes, Marcolli and Ramachandran [3]. For more general fields $K$ we refer to [2, Section 1.4] for a discussion of several constructions of systems exhibiting some, but not all, of the properties of the Bost-Connes system. A proposal of Bost-Connes systems for arbitrary number fields due to Ha and Paugam [4] materializes as a special case of their more general construction of systems associated to Shimura data.

In our approach the $C^*$-algebra $\mathcal{A}$ arises from a restricted groupoid associated to a transformation groupoid. However, our systems are isomorphic to the systems for number fields from [4], and for totally imaginary fields of class number one are also isomorphic to the systems constructed in [6] using Hecke algebras and crossed products by semigroups of endomorphisms.

We fix some notation. We let $\mathcal{O}$ denote the ring of integers in an algebraic number field $K$ and $V_K$ the set of places (or equivalence classes of valuations) of $K$. The finite places corresponding to non-archimedean valuations is $V_{K,f}$. For $v \in V_K$ denote by $K_v$ the corresponding completion of $K$ and, when $v$ is finite, let $\mathcal{O}_v$ be the closure of $\mathcal{O}$ in $K_v$. The ring of finite integral adeles is $\hat{\mathcal{O}} = \prod_{v\in V_{K,f}} \mathcal{O}_v$, and $A_{K,f} = K \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ is the ring of finite adeles. Denoting by $K_\infty = \prod_{v|\infty} K_v$ the completion of $K$ at all infinite places, we get the ring $A_K = K_\infty \times A_{K,f}$ of adeles. The idele group is $I_K = A_K^*$. We let $J_K$ (and $J_K^+$) be the free abelian group (semigroup) on the non-zero prime ideals $\mathfrak{p}$ of $\mathcal{O}$.

The multiplicative group $A_{K,f}^*$ acts on $Gal(K^{ab}/K) \times A_{K,f}$ by

$$j \cdot (\alpha, a) = (\alpha s(j)^{-1}, ja),$$

where $s : I_K \to Gal(K^{ab}/K)$ is the Artin map. Upon quotienting out the action of the (compact open) subgroup $\hat{\mathcal{O}}^*$ of $A_{K,f}^*$ (similarly to [3]) we get the space

$$X := Gal(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} A_{K,f}$$

with a quotient action of $A_{K,f}^*/\hat{\mathcal{O}}^*$ (note that this group is known to be isomorphic to $J_K$). Finally, note that $Y := Gal(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ is clopen in $X$, let $1_Y$ be the characteristic function of $Y$, and set $(\mathcal{A}, \sigma) = (1_Y(C_0(X) \rtimes J_K)1_Y, \sigma)$ with $\sigma_t$ implemented by the absolute norm of an ideal, so $N(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$ for $\mathfrak{a} \in J_K^+$. Alternatively, $\mathcal{A}$ is the semigroup crossed product $C(Y) \rtimes J_K^+$.

The zeta function of the BC-system for $K$ is $\zeta_K(\beta) = \sum_{\mathfrak{a}\in J_K^+} N(\mathfrak{a})^{-\beta}$, with the series converging for $\beta > 1$ and diverging for $\beta \in (0, 1]$. Our main result is the following theorem:

**Theorem 1.** *([7]) For the system $(\mathcal{A}, \sigma)$ we have:*
*(i) for $\beta < 0$ there are no KMS$_\beta$-states;*

(*ii*) *for each* $0 < \beta \leq 1$ *there is a unique* $KMS_\beta$*-state;*

(*iii*) *for each* $1 < \beta < \infty$ *the extremal* $KMS_\beta$*-states are indexed by* $Y_0 := Gal(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}^*$*, with the state corresponding to* $w \in Y_0$ *given by*

$$\varphi_{\beta,w}(f) = \frac{1}{\zeta_K(\beta)} \sum_{\mathfrak{a} \in J_K^+} N(\mathfrak{a})^{-\beta} f(\mathfrak{a}w) \quad \text{for} \quad f \in C(Gal(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}});$$

*moreover,* $Gal(K^{ab}/K)$ *acts freely and transitively on these states;*

(*iv*) *the extremal ground states are indexed by* $Y_0$*, with the state corresponding to* $w \in Y_0$ *given by* $\varphi_{\infty,w}(f) = f(w)$*, and all ground states are* $KMS_\infty$*-states.*

Connes and Marcolli [2] defined $n$-dimensional $\mathbb{Q}$-lattices as pairs $(L, \varphi)$ with $L \subset \mathbb{R}^n$ a lattice and $\varphi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}L/L$ a homomorphism. Connes, Marcolli and Ramachandran [3] defined 1-dimensional $K$-lattices for $K$ an imaginary quadratic field. Similarly we define $n$-dimensional $K$-lattices for an arbitrary number field $K$ and $n \geq 1$ a positive integer. An *$n$-dimensional $\mathcal{O}$-lattice* is a lattice $L$ in $K_\infty^n$ such that $\mathcal{O}L = L$. An *$n$-dimensional $K$-lattice* is a pair $(L, \varphi)$ where $L \subset K_\infty^n$ is an $n$-dimensional $\mathcal{O}$-lattice and $\varphi : K^n/\mathcal{O}^n \to KL/L$ is an $\mathcal{O}$-module map. As in [2] and [3] we say that two $n$-dimensional $K$-lattices $(L_1, \varphi_1)$ and $(L_2, \varphi_2)$ are commensurable if $KL_1 = KL_2$ and $\varphi_1 = \varphi_2$ modulo $L_1 + L_2$. We let $\mathcal{R}_{K,n}$ denote the equivalence relation of commensurability of $n$-dimensional $K$-lattices, and identify the resulting groupoid with a noncommutative space associated to a restricted transformation groupoid. Next, we consider a scaling action of $K_\infty^*$ on 1-dimensional $K$-lattices which is similar to the scaling action of $\mathbb{C}^*$ on the 1-dimensional $K$-lattices for imaginary quadratic fields from [3]; namely we let $k(L, \varphi) = (kL, k\varphi)$ for $(L, \varphi)$ a $K$-lattice and $k \in K_\infty^*$. The quotient of $\mathcal{R}_{K,1}$ by the scaling action of $(K_\infty^*)^\circ$ is a groupoid isomorphic to the noncommutative space

$$(A_{K,f}^*/\hat{\mathcal{O}}^*) \boxtimes ((A_K^*/K^*(K_\infty^*)^\circ) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}) \tag{1}$$

arising from the action of $A_{K,f}^*$ on $K^* \backslash A_K^* \times A_{K,f}$ given by right-multiplication in the first component and left multiplication in the second, followed by restriction to $\hat{\mathcal{O}}$ in the last component. By taking the closure of $K^*(K_\infty^*)^\circ$ in (1) we recover $\mathcal{A}$ (after invoking the class-field theory isomorphism), see [7] for details. Thus $\mathcal{A}$ can be introduced solely in terms of $K$-lattices.

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## Permanence Properties of the Bost Conjecture

Walther Paravicini

Let $G$ be a locally compact second countable Hausdorff group. If $A$ is a $G$-$\mathrm{C}^*$-algebra, then the Bost assembly map is a homomorphism

$$\mu_{\mathrm{L}^1}^{G,A} \colon \mathrm{K}_*^{G,\mathrm{top}}(\underline{\mathrm{E}}G; A) \to \mathrm{K}_*(\mathrm{L}^1(G, A));$$

the Bost conjecture with $\mathrm{C}^*$-algebra coefficients for $G$ asserts that $\mu_{\mathrm{L}^1}^{G,A}$ is an isomorphism for all $A$. This conjecture is known to be true for a large class of groups by the work of Lafforgue [Laf02], including closed subgroups of reductive Lie groups, all groups having the Haagerup property and all hyperbolic groups. The homomorphism $\mu_{\mathrm{L}^1}^{G,A}$ can be constructed in analogy to the Baum-Connes assembly map by the use of Vincent Lafforgue's bivariant K-theory $\mathrm{KK}^{\mathrm{ban}}$, which is defined for Banach algebras. It should be mentioned that most of the proofs of the following results can be carried out in a purely Banach algebraic framework, we need to have $\mathrm{C}^*$-coefficients only to ensure good properties of the left-hand side of the Bost conjecture.

It is known that the Bost conjecture for discrete groups is stable under colimits of groups with injective structure maps [BEL07]; it is also true for non-injective structure maps if we combine the first part of Theorem 0.7 of [BEL07] with the following permanence result:

**Theorem 1.** *If the Bost conjecture is true for $G$ and arbitrary separable $C^*$-coefficients, then the same is true for all open subgroups of $G$.*

This theorem is known in the Baum-Connes context even for closed subgroups [CE01], and the only obstruction to proving it also for closed subgroups in our context is a technical subtlety concerning $\mathrm{L}^1$-completions which can hopefully be dealt with at a later date.

Using very similar techniques, we also extend a theorem of Oyono-Oyono [OO98] about the Baum-Connes conjecture to the Bost conjecture, obtaining the following result:

**Theorem 2.** *Let $G$ act on an oriented tree $X$. Then the group $G$ satisfies the Bost conjecture with arbitrary separable $C^*$-coefficients if and only if the stabilisers of all vertices of $X$ satisfy it.*

The machinery that we deploy to prove both theorems uses a version of $\mathrm{KK}^{\mathrm{ban}}$ which is equivariant with respect to an action of a locally compact Hausdorff groupoid $\mathcal{G}$, see [Laf06]: A $\mathcal{G}$-Banach algebra is an upper semi-continuous field of

Banach algebras over the unit space of $\mathcal{G}$ carrying a continuous action by isometric Banach algebra isomorphisms. The definition of the group $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$ for $\mathcal{G}$-Banach algebras $A$ and $B$ is analogous to the group case. If $\mathcal{G}$ carries a Haar system, then we obtain a descent homomorphism

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B) \to \mathrm{KK}^{\mathrm{ban}}(\mathrm{L}^1(\mathcal{G},A),\mathrm{L}^1(\mathcal{G},B))$$

which can be used to construct an assembly map

$$\mu^{\mathcal{G},A}_{\mathrm{L}^1}\colon \mathrm{K}^{\mathcal{G},\mathrm{top}}_*(\underline{\mathrm{E}}\mathcal{G};A) \to \mathrm{K}_*(\mathrm{L}^1(\mathcal{G},A)),$$

where $A$ is a $\mathcal{G}$-C*-algebra. The Bost conjecture for the groupoid $\mathcal{G}$ and coefficients in $A$ asserts that this assembly map is an isomorphism.

There is a standard notion of equivalence for locally compact Hausdorff groupoids, see for example [LG94]. If we are given an equivalence between such groupoids $\mathcal{G}$ and $\mathcal{H}$, then we can use it to implement an induction construction which assigns to every $\mathcal{H}$-Banach algebra $A$ a $\mathcal{G}$-Banach algebra $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}A$. It turns out that equivalent groupoids have Morita equivalent $\mathrm{L}^1$-algebras (in the sense of Lafforgue, see [Par07c]):

$$\mathrm{L}^1(\mathcal{G},\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}A) \sim \mathrm{L}^1(\mathcal{H},A).$$

This implies that their K-theory is the same. Even more is true:

**Theorem 3.** *Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids admitting Haar systems and having $\sigma$-compact unit spaces. If $\mathcal{G}$ and $\mathcal{H}$ are equivalent, then the Bost conjecture with arbitrary C*-coefficients is true for $\mathcal{G}$ if and only it is true for $\mathcal{H}$.*

The theorem can be read off the following commutative square which has to be established with some technical ado.

$$\begin{array}{ccc}
\mathrm{K}^{\mathcal{H},\mathrm{top}}_0(\underline{\mathrm{E}}\mathcal{H};A) & \longrightarrow & \mathrm{K}_0(\mathrm{L}^1(\mathcal{H},A)) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{K}^{\mathcal{G},\mathrm{top}}_0(\underline{\mathrm{E}}\mathcal{G};\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}A) & \longrightarrow & \mathrm{K}_0(\mathrm{L}^1(\mathcal{G},\mathrm{Ind}^{\mathcal{G}}_{\mathcal{H}}A))
\end{array}$$

The key point in the construction of this diagram is that $\mathrm{KK}^{\mathrm{ban}}$ is functorial with respect to generalised morphisms of groupoids in the sense of Le Gall ([LG94], [LG99]).

In the special case that $\mathcal{H}$ is a closed subgroup $H$ of a locally compact Hausdorff group $G$ and that $\mathcal{G}$ is the transformation groupoid $G \ltimes G/H$, which is equivalent to $H$, the above diagram can be extended as follows:

$$\begin{array}{ccc}
\mathrm{K}_0^{H,\mathrm{top}}(\underline{\mathrm{E}}H;A) & \longrightarrow & \mathrm{K}_0(\mathrm{L}^1(H,A)) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{K}_0^{\mathcal{G},\mathrm{top}}(\underline{\mathrm{E}}\mathcal{G};\mathrm{Ind}_H^{\mathcal{G}}A) & \longrightarrow & \mathrm{K}_0(\mathrm{L}^1(\mathcal{G},\mathrm{Ind}_H^{\mathcal{G}}A)) \\
\cong\downarrow & & \uparrow ? \\
\mathrm{K}_0^{G,\mathrm{top}}(\underline{\mathrm{E}}G;\mathrm{Ind}_H^{G}A) & \longrightarrow & \mathrm{K}_0(\mathrm{L}^1(G,\mathrm{Ind}_H^{G}A))
\end{array}$$

Note that $\mathrm{Ind}_H^{\mathcal{G}}A$ is by definition a $G \ltimes G/H$-C*-algebra. In particular, it is a field of C*-algebras over $G/H$. The $G$-C*-algebra $\mathrm{Ind}_H^G A$ is the standard induced algebra that can be found in the literature, see for example [CE01]. There is a close link between the two algebras: the algebra $\mathrm{Ind}_H^G A$ is $\mathrm{Ind}_H^{G\ltimes G/H}A$ after forgetting the additional fibration over $G/H$.

The "forgetful map" can also be defined on the level of KK-theory, yielding a forgetful homomorphism from $\mathrm{K}_0^{\mathcal{G},\mathrm{top}}(\underline{\mathrm{E}}\mathcal{G};\mathrm{Ind}_H^{\mathcal{G}}A)$ to $\mathrm{K}_0^{G,\mathrm{top}}(\underline{\mathrm{E}}G;\mathrm{Ind}_H^{G}A)$. It is a theorem of Chabert, Echterhoff and Oyono-Oyono [CEOO03] that this forgetful homomorphism is actually an isomorphism.

Note that there is a $*$-isomorphism between $\left(\mathrm{Ind}_H^{\mathcal{G}}A\right) \rtimes_r \mathcal{G}$ and $\left(\mathrm{Ind}_H^{G}A\right) \rtimes_r G$. So for the Baum-Connes conjecture, there is no problem finding the analogue of the dashed arrow in the above diagram and it is an isomorphism. For the Bost conjecture, we have to compare $\mathrm{L}^1(\mathcal{G},\mathrm{Ind}_H^{\mathcal{G}}A)$ and $\mathrm{L}^1(G,\mathrm{Ind}_H^{G}A)$.

It turns out that there is always a homomorphism with dense image between these Banach algebras, but it is certainly not an isomorphism of Banach algebras in most cases, and it is not clear whether it still is an isomorphism in K-theory in general. In case that $H$ is an open subgroup of $G$, there is a dense hereditary subalgebra of both algebras, $\mathrm{L}^1(\mathcal{G},\mathrm{Ind}_H^{\mathcal{G}}A)$ and $\mathrm{L}^1(G,\mathrm{Ind}_H^{G}A)$, so we obtain an isomorphism in K-theory at least for open $H$. This proves Theorem 1 by the above extended diagram.

Because the stabilisers of vertices are open subgroups, Theorem 2 can be proved with the same techniques as Theorem 1, combined with the ideas of [OO98].

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## Topological dimension and $\mathcal{Z}$-stability

Wilhelm Winter

Consider the following regularity conditions on a $C^*$-algebra $A$ which, at first glance, do not seem to have much in common, but occur simultaneously in many of our stock-in-trade examples; even more surprisingly, there are large natural classes of $C^*$-algebras for which they are equivalent.

(A) $A$ is topologically finite-dimensional.
(B) $A$ absorbs a suitable strongly self-absorbing $C^*$-algebra tensorially.
(Γ) $A$ allows comparison of its positive elements in the sense of Murray and von Neumann.
(Δ) The natural order structures on suitable homological invariants of $A$ are complete in the sense that they are sufficiently unperforated.

Obviously, these conditions are of a somewhat philosophical nature, and require interpretation. All of them may be viewed as regularity properties, with (A) of a topological nature and (B) and (Γ) of a ($C^*$-)algebraic type, thus approaching the homological condition (Δ) from quite different directions. The following conjecture, suggested in this form by A. Toms, gives a concrete interpretation of the properties listed above.

Conjecture: For a separable, finite, nonelementary, simple, unital and nuclear $C^*$-algebra $A$, the following are equivalent:

($\alpha$) $A$ has finite decomposition rank.
($\beta$) $A$ absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially.
($\gamma$) $A$ has strict comparison of positive elements.
($\delta$) $A$ has almost unperforated Cuntz semigroup.

Some of these implications are known, but none is trivial. The conjecture has been verified by Toms and Winter for the class of Villadsen algebras of the first type. Rørdam has shown that ($\gamma$) is equivalent to ($\delta$), and that ($\beta$) implies ($\delta$).

Little is known about implication $(\gamma) \Longrightarrow (\beta)$; it has been confirmed for $A$ strongly self-absorbing by Rørdam and Winter. There are partial verifications of $(\beta) \Longrightarrow (\alpha)$, but only under additional hypotheses on $A$ (e.g., asking for locally finite decomposition rank and for many projections or few traces); these results factorize through classification theorems, and hence additionally require the Universal Coefficient Theorem (UCT) to hold for $A$. There are no direct proofs known. The situation for $(\alpha) \Longrightarrow (\beta)$ up to this point was similar: it had been known under additional structural conditions (e.g., for $A$ approximately homogeneous, or when $A$ has many projections and few traces, and only in the presence of the UCT).

Our main result is a direct proof of $(\alpha) \Longrightarrow (\beta)$. Using earlier work of the author and of H. Lin, Q. Lin and N. C. Phillips, this has a number of consequences for the classification program for nuclear $C^*$-algebras. In particular, we can now classify simple, unital $C^*$-algebras with locally finite decomposition rank which satisfy the UCT and for which projections separate tracial states. This includes the real rank zero case as well as the monotracial case; it completes the classification of $C^*$-algebras associated to smooth, minimal, uniquely ergodic dynamical systems. We can also characterize the Jiang–Su algebra as the uniquely determined monotracial $C^*$-algebra with finite decomposition rank which is $KK$-equivalent to the complex numbers.

In a joint project with J. Zacharias, we introduce another interpretation of property (A) above, the so-called weak decomposition rank. In a sense, this has even better permanence properties than the original decomposition rank. Moreover, it turns out that a simple infinite $C^*$-algebra with finite weak decomposition rank absorbs the Cuntz algebra $\mathcal{O}_\infty$ tensorially. Conversely, every Kirchberg algebra with UCT has finite weak decomposition rank. Together with earlier results of Kirchberg, this verifies the infinite version of the conjecture above, at least in the UCT case. Moreover, weak decomposition rank allows an interesting connection to coarse geometry: If a discrete metric space with bounded geometry has finite asymptotic dimension $n$, then its associated uniform Roe algebra has weak decomposition rank at most $n$.

## Traces on $C^*$-algebras of arithmetic groups

Bachir Bekka

Let $\Gamma$ be a countable group. Two distinguished $C^*$-algebras are associated with $\Gamma$: the reduced $C^*$-algebra $C^*_r(\Gamma)$ and the maximal (or full) $C^*$-algebra $C^*(\Gamma)$ of $\Gamma$. One is interested in a description of the convex set of all tracial states on $C^*_r(\Gamma)$ and on $C^*(\Gamma)$.

The reduced $C^*$-algebra $C^*_r(\Gamma)$ has a canonical tracial state $\tau_{\text{reg}}$ given by $\tau_{\text{reg}}(T) = \langle T\delta_e|\delta_e\rangle$. The maximal $C^*$-algebra $C^*(\Gamma)$ has, besides the lifting of $\tau_{\text{reg}}$ to $C^*(\Gamma)$, other tracial states: for every finite dimensional unitary representation $\pi$ of $\Gamma$, the normalized character of $\pi$ induces such a state on $C^*(\Gamma)$.

The main theme of the talk is that, for several arithmetic groups $\Gamma$, these are the only tracial states on $C_r^*(\Gamma)$ respectively on $C^*(\Gamma)$.

For a discrete group $\Gamma$, an obvious necessary condition for $\tau_{\text{reg}}$ to be the unique tracial state on $C_r^*(\Gamma)$ is that the amenable radical $\mathrm{R_a}(\Gamma)$ (i.e. the largest amenable normal subgroup) of $\Gamma$ is trivial. Let $\Gamma$ be a linear group (that is, $\Gamma$ is a subgroup of $GL_n(\mathbf{C})$ for some $n$) with trivial $\mathrm{R_a}(\Gamma)$. Let $G$ be the connected component of the Zariski closure of $\Gamma$ in $GL_n(\mathbf{C})$. If $G$ has no compact factor, then, by results of [2] and [3], $\tau_{\text{reg}}$ is the unique tracial state on $C_r^*(\Gamma)$ and $C_r^*(\Gamma)$ is simple. The assumption that $G$ has a compact factor seems to have been removed in recent work by T. Poznansky, yielding the result that the reduced $C^*$-algebra of a linear group $\Gamma$ with $\mathrm{R_a}(\Gamma) = \{e\}$ is simple and has a unique tracial state. Here are two open problems:

- Does there exist a group $\Gamma$ with $\mathrm{R_a}(\Gamma) = \{e\}$ for which $C_r^*(\Gamma)$ is not simple or does not have a unique tracial state?
- Does there exist a group $\Gamma$ for which $C_r^*(\Gamma)$ is simple but does not have a unique tracial state? Vice versa, does there exist a group $\Gamma$ for which $C_r^*(\Gamma)$ is not simple but has a unique tracial state?

As to the maximal $C^*$-algebra $C^*(\Gamma)$, it turns out that, if $\Gamma$ is higher rank arithmetic lattice like $PSL_n(\mathbf{Z})$ for $n \geq 3$, the only tracial states on $C^*(\Gamma)$ are the obvious ones: $\tau_{\text{reg}}$ and the normalized characters of finite dimensional unitary representations of $\Gamma$ (see [1]). As a consequence, this answers to the negative a question of E. Kirchberg ([4]): there is no faithful trace on $C^*(SL_4(\mathbf{Z}))$.

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## Super moonshine and operator algebras

Yasuyuki Kawahigashi

The Moonshine conjecture is on mysterious relations between the Monster group, the largest among 26 sporadic finite simple groups, and modular invariant $j$-functions. A mathematical object to study is the Moonshine vertex operator algebra, where a vertex operator algebra gives an algebraic axiomatization of a family of operator-valued distributions on the circle appearing in a conformal field theory. We constructed the operator algebraic counterpart of the Moonshine vertex operator algebra as a family of von Neumann algebras parameterized by intervals on the circle and showed that its automorphism group in the operator algebraic sense is indeed the Monster group with Longo in [3]. It has all expected nice properties.

Duncan constructed an "super" analogue of the Moonshine vertex operator algebra and showed that its automorphism group fixing a "super conformal element" is Conway's sporadic finite simple group $Co_1$ in [2]. Here "super" refers to a $\mathbf{Z}_2$-grading. We now present its operator algebraic counterpart. We study this within a general operator algebraic framework for super conformal field theory [1], where the $N = 1$ super Virasoro algebras play an essential role.

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## Separable states and positive maps

Erling Størmer

Let $A$ be a $C^*-$algebra, $H$ a Hilbert space and $B(A, H)$,( resp.$B(A, H)^+$) the bounded (resp. positive) linear maps of $A$ into $B(H)$. Let $t$ denote the transpose map of $B(H)$ with respect to a fixed orthonormal basis. For simplicity of this abstract we assume $A$ and $H$ are finite dimensional. The work is based on [3] and the following fact: There is a duality between maps $\phi \in B(A, H)$ and linear functionals $\tilde{\phi}$ on $A \otimes B(H)$ given by

$$\tilde{\phi}(a \otimes b) = Tr(\phi(a)b^t), a \in A, b \in B(H),$$

where $Tr$ is the usual trace on $B(H)$. Furthermore, $\phi$ is positive if and only if $\tilde{\phi}$ is positive on the cone $A^+ \otimes B(H)^+$ generated by tensors $a \otimes b$ with $a$ and $b$ positive.

We say a nonzero cone $K$ in $B(B(H), H)^+$ is a *mapping cone* if $\alpha \in K$ implies $\beta \circ \alpha \circ \gamma \in K$ for all $\beta, \gamma \in CP(H)$ - the completely positive maps of $B(H)$ into itself. Well known examples are $B(B(H), H)^+, CP(H)$, the copositive maps, i.e. maps $\alpha \circ t$ with $\alpha \in CP(H)$, and $S(H)$ consisting of maps of the form $\sum b_i \omega_i$ where $\omega_i$ is a state and $b_i$ a positive operator. The latter maps are often called *entanglement breaking* or *superpositive*. We denote by

$$P(A, K) = \{x \in A \otimes B(H)_{sa} : \iota \otimes \alpha(x) \geq 0, \forall \alpha \in K\}$$

where $\iota$ denotes the identity map. $P(A, K)$ is a proper closed cone in $A \otimes B(H)$ containing the cone $A^+ \otimes B(H)^+$, which can be shown to be equal to $P(A, B(B(H), H)^+)$.

We say $\phi$ is *K-positive* if $\tilde{\phi}$ is positive on $P(A, K)$. Then, see [3], $\phi$ is completely positive if and only if $\tilde{\phi}$ is a positive functional if and only if $\phi$ is $CP(H)-$positive. Let $K^d$ denote the cone consisting of maps $t \circ \alpha^* \circ t$ with $\alpha \in K$, where $\alpha^*$ is the adjoint of $\alpha$ considered as an operator on the Hilbert-Schmidt operators. Then we have the following characterization of K-positivity.

**Theorem 1.** *Let $C_K$ denote the closed cone generated by maps of the form $\alpha \circ \psi$ with $\alpha \in K^d$ and $\psi$ completely positve in $B(A, H)$. Then $\phi$ is K-positive if and only if $\phi \in C_K$.*

A state $\rho$ on $A \otimes B(H)$ is called *separable* if it is a convex sum of product states. Otherwise $\rho$ is called *entangled*. The following result is essentially proved in [2], but appears in the present form in [4].

**Theorem 2.** *Let $\phi \in B(A, H)^+$. Then the following are equivalent.*
*(i) $\tilde{\phi}$ is a separable state.*
*(ii) $\phi$ is $S(H)-$positive.*
*(iii) $\phi$ is of the form $\phi(a) = \sum \omega_i(a) b_i, \omega_i$ state of $A$, $b_i \in B(H)^+$.*

Assume $A$ is the complex $n \times n$ matrices with matrix units $(e_{ij})$. Then the *Choi matrix* for $\phi$ is $C_\phi = \sum_{ij} e_{ij} \otimes \phi(e_{ij})$. If $\phi^t = t \circ \phi \circ t$, then $C_{\phi^t}$ is the density matrix for $\tilde{\phi}$. Furthermore we have

$$P(A, S(H)) = \{C_\phi : \phi \geq 0\}.$$

Using this identity it is easy to prove the Horodecki Theorem [1] which characterizes separable states in terms of the action of positive maps on their density matrices. One version of the Horodecki Theorem is as follows, see [5]

**Theorem 3.** *If $A$ is a matrix algebra contained in $B(L)$ with $dimL \leq dimH < \infty$, and $\rho$ is a state on $A \otimes B(H)$, then $\rho$ is separable if and only if $\rho \circ (\iota \otimes \psi) \geq 0$ for all $\psi \in B(B(H), H)^+$.*

This theorem has a natural extension to $C^*-$algebras, [5].

**Theorem 4.** *Let $A$ be a nuclear $C^*-$algebra, $B$ a UHF-algebra, and $\rho$ a state on $A \otimes B$. then $\rho$ is a $w^*-$limit of separable states if and only if $\rho \circ (\iota \otimes \psi) \geq 0$ for all positve maps $\psi \colon B \to B$.*

A state $\rho$ satisfies the *Peres (or PPT) condition* if $\rho \circ (\iota \otimes t) \geq 0$. If $\rho = \tilde{\phi}$ then $\rho$ satisfies the Peres condition if and only if $\phi$ is both completely positive and copositive. The *definite set* $D_\phi$ of $\phi$ is the set of operators $a \in A_{sa}$ - the self-adjoint operators in $A$- such that $\phi(a^2) = \phi(a)^2$. Then $D_\phi$ is a Jordan algebra. Furthermore if $\phi$ satisfies the Peres condition, then $D_\phi$ is the self-adjoint part of an abelian $C^*-$algebra. The next theorem connects this to separability, [5].

**Theorem 5.** *Assume $\phi(1) = 1$ and that $\phi(A_{sa}) = \phi(D_\phi)$. Then $\tilde{\phi}$ is separable if and only if $\phi(A)$ is an abelian $C^*-$algebra.*

If $\phi \colon A \to A$ is positive and unital let $P_\phi$ denote the positive projection of $A$ onto the fixed point set $A_\phi$ of $\phi$. If $P_\phi$ is faithful, e.g. when there exists a faithful $\phi-$invariant state, then $A_\phi$ is a Jordan subalgebra of $D_\phi$. Then we have that $P_\phi$ is *decomposable* i.e. the sum of a completely positive map and a copositive map, if and only if $A_\phi$ is a reversible Jordan algebra, i.e. if $a_1, ..., a_k \in A_\phi$, then the symmetric product $a_1...a_k + a_k...a_1 \in A_\phi$. It follows that if $A_\phi$ is a nonreversible Jordan algebra, for example that its self-adjoint part is a spin factor, the $\phi$ is nondecomposable.

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## Turbulence, representations, and trace-preserving actions

David Kerr
(joint work with Hanfeng Li and Mikaël Pichot)

The last couple of decades have witnessed great progress in the study of classification problems from the descriptive set theory viewpoint, as illustrated for example in [5, 6]. The descriptive theory of classification complexity revolves around the concept of Borel reducibility. Suppose we have a collection of objects that can be viewed as elements in a Polish space $X$ and an equivalence relation $E$ on $X$ encoding the isomorphism relation between the objects. Given another equivalence relation $F$ on a standard Borel space $Y$, one says that $E$ is *Borel reducible* to $F$ if there is a Borel map $\theta : X \to Y$ such that, for all $x_1, x_2 \in X$, $x_1 E x_2$ if and only if $\theta(x_1) F \theta(x_2)$. We can thus measure the complexity of $E$ by the way it relates to other equivalence relations in the hierarchy of Borel reducibility. For example, the classical notion of smoothness can be described as Borel reducibility to the relation of equality on $\mathbb{R}$. At a higher level of complexity one has the notion of classification by countable structures, as exemplified by the Halmos-von Neumann classification of measure-preserving transformations with discrete spectrum.

In the 1990s Hjorth introduced the topological-dynamical notion of turbulence as an obstruction to classifiability by countable structures in the case that $E$ arises as the orbit equivalence relation of the continuous action of a Polish group [5]. A point is said to be turbulent if its orbit satisfies a certain local density condition, and an action is said to be turbulent if every point is turbulent and every orbit is meager and dense. In fact to obtain the nonclassifiability conclusion it is enough that some orbit be dense, some point be turbulent, and every orbit be meager, in which case one speaks of generic turbulence.

Generic turbulence has been shown to occur in various examples such as the space of unitary operators on a separable infinite-dimensional Hilbert space [7], certain subspaces of irreducible representations of non-type I groups [4], the space of ergodic measure-preserving actions of a countably infinite amenable group on a standard atomless probability space [2], and certain subspaces of free ergodic measure-preserving actions of arbitrary countably infinite groups (see Section 13 of [6]). Following the line of this work, Hanfeng Li, Mikaël Pichot, and I have developed a general spectral approach to the identification of turbulent behaviour in spaces of representations and actions [8].

We show that the nondegenerate representations of a separable $C^*$-algebra $A$ on a fixed separable infinite-dimensional Hilbert space do not admit classification by countable structures as soon as the spectrum of $A$ is uncountable. This specializes for example to the unitary representations of any separable noncompact Lie group. The nonclassifiability conclusion is based on spectral criteria for generic turbulence in spaces of representations under the action of the unitary group on the given Hilbert space. By a standard second quantization procedure using the the canonical commutation or anticommutation relations, this also produces nonclassifiability by countable structures for actions of a second countable locally compact group with uncountable dual on a standard atomless probability space and on the hyperfinite $\mathrm{II}_1$ factor $R$.

In the case that the acting group $G$ is countably infinite and amenable, we establish turbulence in the space of free $G$-actions on $R$, yielding a noncommutative analogue of the result of Foreman and Weiss from [2]. The meagerness of orbits is obtained by developing a noncommutative version of Foreman and Weiss's entropy and disjointness argument. This requires the use of Connes-Narnhofer-Thirring entropy [1] and of von Neumann algebra correspondences viewed in terms of completely positive maps. For general second countable locally compact $G$ we show that there exists a turbulent point with dense orbit. When $G$ is countably infinite and amenable we can then apply Ocneanu's result that any two free actions are cocycle conjugate [9, Thm. 1.4] to conclude that every free action is a turbulent point. Our method for demonstrating the existence of a turbulent point with dense orbit also works the commutative situation, in which case we use it to show generic turbulence in the space of ergodic measure-preserving flows on a standard atomless probability space.

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## Towards a Generalized Dixmier-Douady Theory

Marius Dadarlat

The importance of continuous fields of C*-algebras (continuous $C(X)$-C*-algebras) is evident from their role as topological bundles in the C*-algebra theory. Continuous fields appear naturaly as C*-algebras with Hausdorff primitive spectrum but also as versatile tools in a large array of contexts.

Let $X$ be a metrizable locally compact space of finite dimension and let $A$ be a separable continuous field over $X$ with fibers isomorphic to the compact operators $\mathcal{K}$ on an infinite dimensional separable Hilbert space. Dixmier and Douady proved that the field $A$ is locally trivial if and only it satisfies the *Fell's Condition*: for any $x_0 \in X$, there is a closed neighborhood $V$ of $x_0$ and there is a projection $p \in A(V)$ such that $\mathrm{rank}(p(x)) = 1$ for all $x \in V$. Moreover they showed that these locally trivial fields are classified up to isomorphism by a characteristic class $\delta(A) \in H^3(X;\mathbb{Z})$ (Cech cohomology) with the propery that $\delta(A \otimes A') = \delta(A) + \delta(A')$.

It is natural to investigate generalizations of the results of Dixmier and Douady to continuous with fibers nuclear C*-algebras. In what follows we report on some results obtained in this direction.

A separable simple nuclear purely infinite algebra is called a Kirchberg algebra. Kirchberg proved that the Cuntz algebra $\mathcal{O}_2$ is the unique unital Kirchberg algebra which KK-equivalent to zero. His remarkable work on the classification of non-simple purely infinite C*-algebras [5] led to the question of automatic triviality of separable unital continuous fields with all fibers isomorphic to $\mathcal{O}_2$. We have answered this question in [1]:

*Theorem. All separable unital continuous fields with fiber $\mathcal{O}_2$ over a finite dimensional compact metrizable $X$ space are trivial, i.e. they are isomorphic to $C(X) \otimes \mathcal{O}_2$.*

The result was extended to strongly selfabsorbing algebras in our joint work with Winter. The finite dimensionality of the spectrum is essential. For a nonempty set $P$ of prime numbers, let $\mathbb{Z}(P)$ be the subgroup of $\mathbb{T}$ consisting of all elements whose orders have all prime factors in $P$. Let $G_P$ be the group of all continuous functions from the Cantor set to $\mathbb{Z}(P)$. We exhibit in [2]:

*Examples. For each set of primes $P$, there is a separable unital continuous field with fiber $\mathcal{O}_2$ over the Hilbert cube whose $K_0$-group is isomorphic to $G_P$.*

Since $P \neq P' \Rightarrow G_P \ncong G_{P'}$, we obtain a continuum of separable, unital, non-isomorphic $\mathcal{O}_2$-continuous fields over a contractible compact space. The input in the construction of our examples is a locally nontrivial continuous field with fibers the universal UHF algebra over an infinite product of two spheres, due to Hirshberg, Rørdam and Winter, [4].

In his work on the Novikov conjecture, Kasparov has introduced parametrized KK-theory groups $\mathrm{KK}_X(A,B)$ for $C(X)$-algebras $A$ and $B$. These groups, admit a Kasparov product $KK_X(A,B) \times KK_X(B,C) \to KK_X(A,C)$. The invertible elements in $KK_X(A,B)$ are denoted by $KK_X(A,B)^{-1}$. Recently we have obtained the following comparison result which shows that fibrewise KK-equivalence does imply $\mathrm{KK}_X(A,B)$-equivalence under suitable assumptions [2].

*Theorem. Let $A$ and $B$ be separable nuclear continuous $C(X)$-algebras over a finite dimensional compact metrizable space $X$. If $\sigma \in \mathrm{KK}_X(A,B)$, then $\sigma \in \mathrm{KK}_X(A,B)^{-1}$ if and only if $\sigma_x \in KK(A(x),B(x))^{-1}$ for all $x \in X$.*

Let us also note that the theorem is easily applicable if the fibers of $A$ and $B$ satisfy the UCT. The case $A = C(X) \otimes D$ together with the identification $\mathrm{KK}_X(C(X) \otimes D, B) \cong \mathrm{KK}(D,B)$ gives a very simple Fell-type condition for the $\mathrm{KK}_X$-triviality of $B$:

*Corollary. Let $A$ be a separable nuclear continuous $C(X)$-algebra over a finite dimensional compact metrizable space $X$ with fibers stable Kirchberg C*-algebras. Then $A$ is locally trivial if and only if for any $x_0 \in X$, there is a closed neighborhood $V$ of $x_0$ and there is $\sigma \in \mathrm{KK}(A(x_0), A(V))$ such that $\sigma_x \in \mathrm{KK}(A(x_0), A(x))^{-1}$ for all $x \in V$.*

The finite dimensionality assumption is again essential as one sees by considering a unital embedding of one of the nontrivial $\mathcal{O}_2$-fields over the Hilbert cube $X$ into $C(X) \otimes \mathcal{O}_2$ given by E. Blanchard's embedding theorem.

Once we have an effective criterium for local triviality is natural to investigate invariants for locally trivial $C(X)$-algebras with a fixed fiber $D$. This leads to the question of computing homotopy invariants of the automorphism group $\mathrm{Aut}(D)$ and of the classifying space $B\mathrm{Aut}(D)$ for principal $\mathrm{Aut}(D)$-bundles.

Let $D$ be a unital Kirchberg algebra. The unital inclusion $\nu : \mathbb{C} \to D$ induces a morphism $KK(D,SD) \xrightarrow{\nu^*} KK(\mathbb{C},SD) \xrightarrow{\sim} K_1(D)$ whose cokernel is denoted by $K_1(D)/\nu$. Let $C_\nu D$ denote the mapping cone of $\nu$ and let $KK(D,D)_u^{-1}$ denote the invertible elements of the ring $KK(D,D)$ which map $[1_D]$ to $[1_D]$. We prove in [3]

*Theorem. $\pi_n \mathrm{Aut}(D) \cong KK(C_\nu D, S^{n+1}D)$ for $n \geq 1$. There is an exact sequence $1 \to K_1(D)/\nu \to \pi_0\mathrm{Aut}(D) \to KK(D,D)_u^{-1} \to 1$.*

More generally, we compute the homotopy classes $[X, \mathrm{Aut}(D)]$ for finite dimensional metric spaces $X$. In the case of $\mathcal{O}_n$ we have that for any compact metrizable space $X$ there is a bijection $[X, \mathrm{Aut}(\mathcal{O}_n)] \to K_1(C(X) \otimes \mathcal{O}_n)$. The $k^{th}$-homotopy group $\pi_k(\mathrm{Aut}(\mathcal{O}_n))$ is isomorphic to $\mathbb{Z}/(n-1)$ if $k$ is odd and it vanishes if $k$ is even.

Let $E$ be a complex vector bundle of rank $n$ over a compact Hausdorff space $X$, i.e. $E \in \mathrm{Vect}_n(X)$. After putting a hermitian metric on $E$, the continuous sections in $E$ becomes a $C(X)$-Hilbert module. denoted again by $E$. The Cuntz-Pimsner algebra $\mathcal{O}_E$ is a locally trivial continuous field with fiber $\mathcal{O}_n$ over $X$, [6]. Cuntz suggested the following question. What invariants of the vector bundle $E$ are captured by the isomorphism class of the continuous field $\mathcal{O}_E$?

The geometric motivation behind this question is that given a smooth compact manifold $M$ of dimension $n$, with tangent space $TM$, one can associate to it the continuous field of C*-algebras $\mathcal{O}_{TM\otimes\mathbb{C}}$ with fibers $\mathcal{O}_n$. It will interesting to see what invariants for $M$ will emerge from the invariants of $\mathcal{O}_{TM\otimes\mathbb{C}}$. Assuming that $X$ is finite dimensional, Vasselli showed that $\mathcal{O}_E$ depends only on the K-theory class of $E$. In other words, if $E, F \in \mathrm{Vect}_n(X)$ and $[E] = [F]$ in $K^0(X)$, then $\mathcal{O}_E \cong \mathcal{O}_F$ as continuous fields.

*Theorem. If* $E \in \mathrm{Vect}_n(X)$*, then the continuous field* $\mathcal{O}_E$ *is trivial if and only if* $[E] - n$ *is divisible by* $(n-1)$ *in the group* $K^0(X)$*.*

Let $\mathcal{C}_n(X)$ denote the isomorphism classes of unital locally trivial $C(X)$-algebras with fiber $\mathcal{O}_n$. Using our computation of the homotopy groups of $\mathrm{Aut}(\mathcal{O}_n)$ [3], we introduced a characteristic class $\delta : \mathcal{C}_n(X) \to H^2(X; \mathbb{Z}/n-1)$ corresponding to a generator of $H^2(B\,\mathrm{Aut}(\mathcal{O}_n); \mathbb{Z}/n-1) \cong \mathbb{Z}/n-1$. We showed that $\delta(\mathcal{O}_E)$ can be identified with the reduction mod $(n-1)$ of the first Chern class of $E$. Thus $\delta(\mathcal{O}_E) = \rho c_1(E)$, where $\rho : H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}/n-1)$ is the natural coefficient map. Consider the Bockstein exact sequence

$$H^2(X;\mathbb{Z}) \xrightarrow{n-1} H^2(X;\mathbb{Z}) \xrightarrow{\rho} H^2(X;\mathbb{Z}/n-1) \xrightarrow{\beta} H^3(X;\mathbb{Z}) \xrightarrow{n-1} H^3(X;\mathbb{Z}).$$

It turns out that the map $\Delta = \beta\rho$ gives an obstruction for a locally trivial $\mathcal{O}_n$-field to come from a vector bundle of rank $n$. For spaces of low dimension the picture is complete:

*Theorem. If* $\dim(X) \leq 3$ *and* $n \geq 2$*, then:*
*(a)* $\delta : \mathcal{C}_n(X) \to H^2(X; \mathbb{Z}/n-1)$ *is a bijection.*
*(b) If* $E, F \in Vect_n(X)$*, then* $\mathcal{O}_E \cong \mathcal{O}_F$ *if and only if* $c_1(E) - c_1(F) \in (n-1)H^2(X; \mathbb{Z})$*.*
*(c) A continuous field* $A \in \mathcal{C}_n(X)$ *is isomorphic to* $\mathcal{O}_E$ *for some* $E \in Vect_n(X)$ *if and only if* $\Delta(A) = 0$ *in* $H^3(X; \mathbb{Z})$*.*
*(d) If* $\tau \in H^3(X; \mathbb{Z})$ *and* $(n-1)\tau = 0$ *then* $\Delta(A) = \tau$ *for some* $A \in \mathcal{C}_n(X)$*.*

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## The Cuntz semigroup and the structure of C*-algebras

Andrew Toms

In his 1978 paper [2], Cuntz generalised Murray-von Neumann comparison for projections to the realm of positive elements. If $a, b$ are positive elements of a C*-algebra $A$, then we write $a \precsim b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ in $A$ such that

$$\|v_n b v_n^* - a\| \to 0.$$

We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. It is straightforward to show that $\sim$ is an equivalence relation on positive elements in $A$. We can construct an Abelian monoid from the set of $\sim$ equivalence classes in arbitrarily large matrices over $A$, in perfect analogy with the construction of the Murray-von Neumann semigroup. This new monoid is called the *Cuntz semigroup*, and we denote it by $W(A)$. We use $\langle a \rangle$ to denote the Cuntz equivalence class of a positive operator $a$.

Interest in the Cuntz semigroup has been high lately, due in part to the discovery that it is a very sensitive invariant in the matter of distinguishing simple separable amenable C*-algebras (see [4]). But this invariant does not appear in most classification theorems for simple separable amenable C*-algebras via topological K-theory and traces. This invites the question of how one might recover the Cuntz semigroup from K-theory and traces in the case of well behaved C*-algebras.

Let $A$ be a simple unital C*-algebra with a trace. It is shown in [3] that

$$W(A) = V(A) \sqcup W(A)_+,$$

where $V(A)$ denotes the semigroup of Cuntz equivalence classes whose representatives are Cuntz equivalent to a projection, and $W(A)_+$ is the semigroup of Cuntz equivalence classes whose representatives have zero as an accumulation point of the spectrum. Let $\mathrm{T}(A)$ denote the tracial state space of $A$, and let $L(\mathrm{T}(A))$ denote the space of strictly positive, lower semicontinuous, bounded, affine functions on $\mathrm{T}(A)$. Define a map $\iota : W(A)_+ \to L(\mathrm{T}(A))$ by

$$\iota(\langle a \rangle)(\tau) = d_\tau(a) := \lim_{n\to\infty} \tau(a^{1/n}),\ \forall \tau \in \mathrm{T}(A).$$

Now define a map $\phi : W(A) \to V(A) \sqcup L(\mathrm{T}(A))$ by $\phi(x) = x$ if $x \in V(A)$, and $\phi(x) = \iota(x)$ otherwise. It turns out that $\phi$ is an order-preserving semigroup map.

**Theorem 1** (Brown-Perera-T, [1])**.** *Let $A$ be a unital simple exact finite C*-algebra which absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially. It follows that*

$$\phi : W(A) \to V(A) \sqcup L(\mathrm{T}(A))$$

*is an isomorphism.*

All classes of simple separable amenable C*-algebras known to be classified by K-theory and traces are also known to consist of algebras which absorb $\mathcal{Z}$ tensorially, and so our theorem applies to them. Note that absorbing $\mathcal{Z}$ yields stable rank one, whence the monoid $V(A)$ carries the same information as $\mathrm{K}_0(A)$.

The spaces $\mathrm{T}(A)$ and $L(\mathrm{T}(A))$ are in natural duality. It follows that $W(A)$ is recovered from K-theory and traces for algebras $A$ as in the theorem.

A close analysis of the order structure on the Cuntz semigroup (inherited from the relation $\precsim$ above) yields an invariant which may be thought of as a $\mathbb{R}$-valued topological dimension. Assume that $A$ is unital, exact (so that we can dispense with the consideration of quasi-traces), and stably finite, and let $r > 0$. Say that $A$ has $r$-strict comparison if $a \precsim b$ in (matrices over) $A$ whenever

$$d_\tau(a) + r < d_\tau(b), \ \forall \tau \in \mathrm{T}(A).$$

We define the *radius of comparison* of $A$ to be the infimum of the set of $r \in \mathbb{R}^+$ such that $A$ has $r$-strict comparison. We denote this quantity by $\mathrm{rc}(A)$.

**Theorem 2** (T, [5])**.** *Let $X$ be a CW-complex of dimension $d < \infty$, and let $k$ be the largest integer such that $2k < n$. It follows that*

$$k - 1 \leq \mathrm{rc}(\mathrm{C}(X)) \leq \frac{d-1}{2}.$$

*In particular,* $\mathrm{rc}(\mathrm{C}(X)) \approx d/2$.

This result shows that the radius of comparison recovers topological dimension in the commutative case.

**Theorem 3** (T, [5])**.** *For each $r \in \mathbb{R}^+ \cup \{\infty\}$ there is a unital simple $C^*$-algebra $A_r$ such that* $\mathrm{rc}(A_r) = r$.

In other words, C*-algebras admit a continuous *topological* dimension theory. The minimal instance of this dimension can be viewed as the condition that the C*-algebra exhibits no unstable homotopy phenomena.

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## On sums of Hermitian operators in finite von Neumann algebras

Ken Dykema

(joint work with Hari Bercovici, Benoît Collins, Wing Suet Li and Dan Timotin)

Suppose $A, B \in M_n(\mathbf{C})_{s.a.}$ have eigenvalues

$$\lambda_A(1) \geq \lambda_A(2) \geq \cdots \geq \lambda_A(n) \text{ of } A$$
$$\lambda_B(1) \geq \lambda_B(2) \geq \cdots \geq \lambda_B(n) \text{ of } B.$$

What can the eigenvalues of $A+B$ be? This is an old question. Equivalently (in a symmetric reformulation), which triples

$$(\lambda_A(i))_{i=1}^n, \qquad (\lambda_B(j))_{j=1}^n, \qquad (\lambda_C(k))_{k=1}^n,$$

of sequences in $\mathbf{R}^n_{\geq}$ arise as eigenvalues of $A, B, C \in M_n(\mathbf{C})_{s.a.}$ with $A+B+C=0$? An obvious necessary condition is

$$\sum_{i=1}^n \lambda_A(i) + \sum_{j=1}^n \lambda_B(j) + \sum_{k=1}^n \lambda_C(k) = 0. \tag{1}$$

A. Horn [8] considered inequalities of the form

$$\sum_{i\in I} \lambda_A(i) + \sum_{j\in J} \lambda_B(j) + \sum_{k\in K} \lambda_C(k) \leq 0, \tag{2}$$

where $I$, $J$, and $K$ are subsets of $\{1,\ldots,n\}$ of the same cardinality. He recursively defined sets $T_r^n$ of such triples $(I,J,K)$ with $|I|=|J|=|K|=r$. and conjectured that $\lambda_A$, $\lambda_B$, $\lambda_C$ arise as eigenvalue sequences of $A$, $B$ and $C$ with $A+B+C=0$ if and only if

- the trace equality (1) holds
- the inequality (2) holds for all $(I,J,K) \in \bigcup_{r=1}^{n-1} T_r^n$.

Here is Horn's recursive definition: Consider triples $(I,J,K)$ of subsets of $\{1,\ldots,n\}$ with $|I|=|J|=|K|=r$. Let

$$U_r^n = \left\{ (I,J,K) \,\middle|\, \sum_{i\in I} i + \sum_{j\in J} j + \sum_{k\in K} k = \frac{r(4n-r+3)}{2} \right\}.$$

When $r=1$, set $T_1^n = U_1^n$. Otherwise, writing $I = \{i_1 < i_2 < \cdots < i_r\}$, etc., let

$$T_r^n = \left\{ (I,J,K) \in U_r^n \,\middle|\, \sum_{f\in F} i_f + \sum_{g\in G} j_g + \sum_{h\in H} k_h \geq \frac{p(4n-p+3)}{2}, \right.$$
$$\left. \text{for all } p<r \text{ and } (F,G,H) \in T_p^r \right\}.$$

One way of proving that a Horn inequality (2) for a given $(I,J,K)$ always holds is to show the existence of a projection $P \in M_n(\mathbf{C})$ such that

$$\sum_{i\in I} \lambda_A(i) \leq \mathrm{Tr}(PAP), \qquad \sum_{j\in J} \lambda_B(j) \leq \mathrm{Tr}(PBP),$$
$$\sum_{k\in K} \lambda_C(k) \leq \mathrm{Tr}(PCP)$$

and then to use that $\mathrm{Tr}(PAP+PBP+PCP)=0$. To find $P$ so that

$$\sum_{i\in I} \lambda_A(i) \leq \mathrm{Tr}(PAP), \tag{3}$$

holds, write $I = \{i_1 < i_2 < \cdots < i_r\}$ and let $v_1,\ldots,v_n$ be orthonormal eigenvectors for $A$ with $Av_i = \lambda_A(i)v_i$. Consider the flag $\mathcal{E} = (E_m)_{m=0}^n$ where $E_m$ is the rank $m$ projection onto $\mathrm{span}\{v_1,\ldots,v_m\}$. If

- $\operatorname{rank}(P) = r$
- $\operatorname{rank}(P \wedge E_{i_\ell}) \geq \ell$ for all $1 \leq \ell \leq r$,

then (3) holds. The set of projections $P$ satisfying the two conditions above is the *Schubert variety* $S(\mathcal{E}, I)$. Thus, to prove that (2) holds for a particular $(I, J, K)$, it suffices to show that for all flags $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, we have

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K) \neq \emptyset.$$

Horn's conjecture was proved approximately ten years ago, due to work of Klyachko, Knutson, Tao, and others. We will now describe some elements of proof. To each $(I, J, K) \in U_r^n$, one associates a nonnegative integer $c_{IJK}^{(n)}$, which is the *Littlewood–Richardson coefficient* $c_{\alpha,\beta}^{\gamma}$ where $\alpha$, $\beta$ and $\gamma$ are certain partitions of integers obtained from $I$, $J$ and $K$, respectively. These coefficients appear in the representation theory of permutation groups and general linear groups, the have a combinatorial description in terms of integer fillings of Young diagrams and they appear in the cohomology ring of the Grassmanian $G(r, \mathbf{C}^n)$ (when multiplying Schubert cycles). Moreover, from Horn's recursive definition one can show

$$T_r^n = \{(I, J, K) \in U_r^n \mid c_{IJK}^{(n)} > 0\},$$

- Many authors, ([9], [13], [7], [10]) showed that for every $(I, J, K) \in T_r^n$, the Horn inequality (2) holds (whenever $A + B + C = 0$). This proves half of Horn's conjecture.
- Klyachko [10] showed that the reverse direction would follow, i.e. the Horn inequalities together with the trace equality would determine the set of possible eigenvalues of $A, B, C$ such that $A + B + C = 0$, if the *saturation conjecture* were known to hold, i.e.,

$$K \in \mathbf{N} \text{ and } c_{K\alpha,K\beta}^{K\gamma} > 0 \quad \implies \quad c_{\alpha,\beta}^{\gamma} > 0.$$

- Knutson and Tao [11] proved the saturation conjecture.
- Belkale [1] showed that the $(I, J, K)$ such that $c_{IJK}^{(n)} > 1$ are redundant.
- Knutson, Tao, and Woodward [12] give a direct proof of Horn's conjecture, and show that the set of Horn inequalties (2) over all $(I, J, K)$ with $c_{IJK}^{(n)} = 1$ is minimal.

The question in a $\mathrm{II}_1$–factor $\mathcal{M}$ with trace $\tau$ that is analogous to Horn's question is: if "spectral data" of $a, b \in \mathcal{M}_{s.a.}$ are specified, what can be the "spectral data" of $a + b$? Or, in symmetric form: what are the possible spectral data of $a$, $b$ and $c$ when $a + b + c = 0$? For "spectral data" we have the distribution $\mu_a$ of $a$, a Borel probability measure on $\mathbf{R}$, such that

$$\frac{\tau(a^k)}{\tau(1)} = \int_{\sigma(a)} t^k \, d\mu_a(t), \qquad (k \in \mathbf{N}).$$

Moreover, there is the *eigenvalue function* $\lambda_a : [0, \tau(1)) \to \mathbf{R}$, given by

$$\lambda_a(t) = \sup \big\{ s \in \mathbf{R} \mid \mu_a((s, \infty)) > \frac{t}{\tau(1)} \big\},$$

which is bounded, right–continuous, nonincreasing. This function $\lambda_a : [0, \tau(1)) \to \mathbf{R}$ has properties expected of eigenvalues. For example,

$$\tau(a^k) = \int_0^{\tau(1)} \lambda_a(t)^k \, dt.$$

If $(I, J, K) \in T_r^n$, we say that *the corresponding Horn inequality holds in* $\mathcal{M}$ if, taking the trace normalization $\tau(1) = n$, we have

$$\sum_{i\in I} \int_{i-1}^{i} \lambda_a + \sum_{j\in J} \int_{j-1}^{j} \lambda_b + \sum_{k\in K} \int_{k-1}^{k} \lambda_c \leq 0$$

whenever $a, b, c \in \mathcal{M}_{s.a.}$ and $a + b + c = 0$.

- Bercovici and Li observed in [4] that if a $\mathrm{II}_1$–factor $\mathcal{M}$ is embeddable in $R^\omega$, then all Horn inequalities hold in $\mathcal{M}$.
- Bercovici and Li showed in [3] that the analogues of the Freede–Thompson inequalities hold in all $\mathrm{II}_1$–factors.
- With Benoît Collins in [6] we showed that the Horn inequalities for all $(I, J, K) \in T_3^n$ with $c_{IJK}^{(n)} = 1$ hold in all $\mathrm{II}_1$–factors.

Here, now, is our main result:

Theorem [2]: All Horn inequalities hold in all $\mathrm{II}_1$–factors.

The proof can be described as "practical Schubert calculus," because we solve the intersection problem for Schubert varieties in $\mathrm{II}_1$–factors. The same methods work in $n \times n$ matrices, and give a new constructive solution in this case.

Here is an outline of the proof. We need only consider those $(I, J, K) \in T_r^n$ with $c_{IJK}^{(n)} = 1$. Let $\mathcal{M}$ be a $\mathrm{II}_1$–factor (or $M_n(\mathbf{C})$) with trace $\tau$, normalized so that $\tau(1) = n$. We consider flags $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{M}$, of the form

$$\mathcal{E} : \quad 0 = E_0 \leq E_1 \leq E_2 \leq \cdots \leq E_{n-1} \leq E_n = 1$$

with $\tau(E_j) = j$. We will show

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K) \neq \emptyset.$$

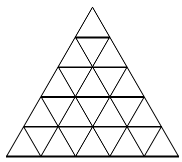
For this, we must prove there is a projection $P \in \mathcal{M}$ such that $\tau(P) = r$ and

$$\tau(P \wedge E_{i_\ell}) \geq \ell, \qquad \tau(P \wedge F_{j_\ell}) \geq \ell, \qquad \tau(P \wedge G_{k_\ell}) \geq \ell$$

for all $1 \leq \ell \leq r$.

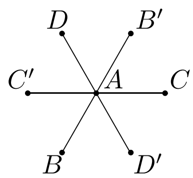
We describe an algorithm for constructing $P$ from the projections in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, using lattice operations $\wedge$ and $\vee$, (but only in the case $c_{IJK}^{(n)} = 1$.)

Following [11], triples $(I, J, K) \in T_r^n$ with $c_{IJK}^{(n)} = 1$ correspond to certain positive measures $m$ supported on the edges of a triangular grid $\Delta_r$ (shown here in the case $r = 5$)



such that the small edges have integer masses.

The measures $m$ must satisfy a balance condition: around every vertex




we must have

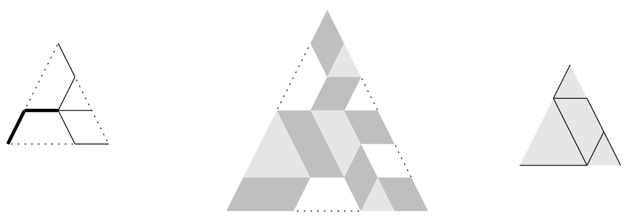
$$m(AB) - m(AB') = m(AC) - m(AC') = m(AD) - m(AD').$$

Then $I$, $J$, and $K$ are read off from values of $m$ on edges at the boundary of the triangle $\Delta_r$. We consider the cone of all positive measures on the grid that satisfy the balance condition (+ some further orientation conditions).

Step I: Decompose. Write the measure $m$ as a sum of extremal ones, which are characterized (because of the balance condition and a characterization from [12] for $c^{(n)}_{IJK} = 1$) by their supports.

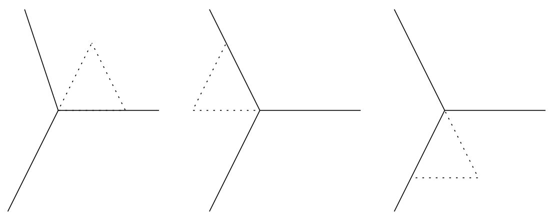
Step II: Reduce. It is possible to select one of these extremal measures $\mu_1$ and to show that the intersection problem for $m$ reduces to solving the intersection problems for $\mu_1$ and for $m - \mu_1$ (which has fewer extremal measures).

Step III: Dualize. The intersection problem for a measure $\mu$ is equivalent to the intersection problem for a dual measure $\mu^*$. This duality (following [12]) is realized by *inflation* and *deflation*. For example:



If $\mu$ is an extremal measure supported, then $\mu^*$ is a sum of extremal measures whose supports have lower complexity.

Step IV: The most reduced case.



corresponds to an intersection problem that has a trivial solution.

QED

Connes' embedding problem asks whether every $II_1$–factor having separable predual embeds in $R^\omega$. We were actually inspired by the earlier result [4] of Bercovici and Li, that all Horn inequalities hold in all $II_1$–factors that embed in $R^\omega$, to try to construct a $II_1$–factor in which some Horn inequality fails to hold. Our main result, above, shows that this is impossible. However, with Benoît Collins in [5], have shown that a certain generalization of the Horn question, "with matrix coefficients" is equivalent to Connes' embedding problem.

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**Applications of classification results to perturbation questions.**

Erik Christensen
(joint work with Allan M. Sinclair, Roger R. Smith, Stuart White)

Given a C*-algebra $\mathcal{C}$, two sub C*-algebras $\mathcal{A}$, $\mathcal{B}$ and $\gamma > 0$ then we define

$$\mathcal{A} \overset{\gamma}{\subseteq} \mathcal{B} \text{ if } \forall a \in \mathcal{A}\, \exists b \in \mathcal{B} : \|a - b\| \leq \gamma \|a\|$$

$$\|\mathcal{A} - \mathcal{B}\| := \inf\{\gamma \,|\, \mathcal{A} \overset{\gamma}{\subseteq} \mathcal{B} \text{ and } \mathcal{B} \overset{\gamma}{\subseteq} \mathcal{A}\,\}$$

In 1973 Kadison & Kastler [8] raised the *Question:* If $\mathcal{A}$ and $\mathcal{B}$ are sufficiently close, are they then isomorphic ? Some answers have been obtained, and we know that the general answer is no [1], [6], but in some special situations the answer is yes. Especially so for *injective von Neumann algebras and AF-C*-algebras,* [3], [10].

## Strategy

**The classification program** has obtained a great number of results of the type: *If two C*-algebras inside a certain class have isomorphic invariants then they are isomorphic,* [5], [7], [Elliott, Kirchberg, Phillips + many more]
**The class** is defined by certain properties among which one finds algebras with some of the following properties
*unital, separable, simple, nuclear, purely infinite, inductive limits of certain types, UCT, real rank zero*
**The invariants** were in the first place $K$-groups, traces and the pairing of traces and $K_0$, but much more is involved now [Lin, Winter, ... ]
**This work does not discuss perturbations** of the UCT-property, $KK-$groups, topological stable rank and the many properties introduced in recent results.
**This article will show** that many of the named properties and the classical invariants are stable under small perturbations **if the algebras are of finite lengths.**

## Near inclusions, finite length and $K-$theory

We will not discuss the concept of length of a C*-algebra here. It was first introduced by Gilles Pisier [11] under the name *similarity degree,* because Pisier showed that a C*-algebra will have finite similarity degree if and only if any bounded homomorphism of this C*-algebra into some $B(H)$ is similar to a *-representation. The importance of finite length for perturbation questions comes from the following theorem.

**Theorem 1.** *Let $\mathcal{A} \overset{\gamma}{\subseteq} \mathcal{B}$. If $\mathcal{A}$ has finite length then there exists $L > 0$ s. t.*

$$\forall n : M_n(\mathcal{A}) \overset{L\gamma}{\subseteq} M_n(\mathcal{B}.)$$

Given a situation such that

$$\forall n : M_n(\mathcal{A}) \overset{L\gamma}{\subseteq} M_n(\mathcal{B})$$

then it is quite easy to obtain the following result:

**Theorem 2.** *If $L\gamma < 1$ then there exist homomorphisms*

$$\Phi_0 : K_0(\mathcal{A}) \to K_0(\mathcal{B}),\ \Phi_1 : K_1(\mathcal{A}) \to K_1(\mathcal{B}).$$

*For $p, u \in M_n(\mathcal{A})$ a projection and a unitary, let $q, v \in M_n(\mathcal{B})$ be a projection and a unitary such that $\|p - q\| < 1/2$ and $\|u - v\| < 1$ then $\Phi_0([p]_{K_0}) = [q]_{K_0}$ and $\Phi_1([u]_{K_1}) = [v]_{K_1}$.*
*There exists $\alpha > 0$ s. t. if $\|\mathcal{A} - \mathcal{B}\| < \alpha$ then $\Phi_0$ and $\Phi_1$ are isomorphisms.*

**Which algebras have finite lengths ?** Let $\mathcal{A}$ be a C*-algebra and let $\ell(\mathcal{A})$ denote its length.

(i) $\ell(\mathcal{A}) \leq 2$ if and only if $\mathcal{A}$ is nuclear. **Pisier**
(ii) $\mathcal{A}$ has no bounded traces $\Rightarrow \ell(\mathcal{A}) \leq 3$.

## Stability of properties when $\|\mathcal{A} - \mathcal{B}\| < \gamma$

**Length.** There exists an $\alpha > 0$ such that if $\gamma < \alpha$ then $\mathcal{A}$ has finite length if and only if $\mathcal{B}$ has finite length.

**Unitality.** If $\gamma < 1$ then $\mathcal{A}$ is unital if and only if $\mathcal{B}$ is unital.

**Separability.** If $\gamma < 1$ then $\mathcal{A}$ is separable if and only if $\mathcal{B}$ is separable.

**Simplicity.** If $\gamma < 0.4$ then $\mathcal{A}$ is simple if and only if $\mathcal{B}$ is simple.

**Nuclearity.**

$$\|\mathcal{A} - \mathcal{B}\| < 1/100 \Rightarrow \{\mathcal{A} \text{ is nuclear if and only if } \mathcal{B} \text{ is nuclear }\}$$

The proof is by [C ] from 1980 and is based on Choi-Effros and Connes' works on nuclear C*-algebras and injective von Neumann algebras.

**Real rank zero.** There exists an $\alpha > 0$ such that if $\gamma < \alpha$ then $\mathcal{A}$ has real rank zero if and only if $\mathcal{B}$ has real rank zero.

**Finiteness**

If $\gamma < 0.5$. and any isometry in $\mathcal{A}$ is unitary, then any isometry in $\mathcal{B}$ is unitary.

**Purely infiniteness** If $\gamma < 1/100$ and the algebras are simple then $\mathcal{A}$ is purely infinite if and only if $\mathcal{B}$ is purely infinite.

## Stability of tracial state spaces under perturbations

**Theorem 3.** *Let $0 < \gamma < 1/102$. If $\|\mathcal{A} - \mathcal{B}\| < \gamma$ then there exists an affine isomorphism $\Psi$ of the w*-compact convex set of tracial states on $\mathcal{B}$ onto that of $\mathcal{A}$ s. t.*

(i) *If $\gamma < 2\,560\,000^{-1} = 1\,600^{-2}$ and the C*-algebra $\mathcal{A}$ has real rank zero then $\Psi$ is an affine w*-homeomorphism of $TS(\mathcal{A})$ onto $TS(\mathcal{B})$.*

(ii) *If $\mathcal{A}$ has finite length then there exists an $\alpha > 0$ s. t. if $\|\mathcal{A} - \mathcal{B}\| < \alpha$, then for any tracial state $\tau$ on $\mathcal{B}$, the induced homomorphisms $\tau_0$ on $K_0(B)$ and $\Psi(\tau)_0$ on $K_0(\mathcal{A})$, are connected via the isomorphism $\Phi_0 : K_0(\mathcal{A}) \to K_0(\mathcal{B})$, s. t.*

$$\forall g \in K_0(\mathcal{A}) : \quad \Psi(\tau)_0(g) \;=\; \tau_0(\Phi_0(g)).$$

## Combined results

**Theorem 4.** *There exists an $\alpha > 0$ s. t. if $\|\mathcal{A} - \mathcal{B}\| < \alpha$, and $\mathcal{A}$ is unital separable, nuclear, simple and of real rank zero, then:*

(i) *the algebra $\mathcal{B}$ is unital separable, nuclear, simple and of real rank zero.*

(ii) *There exists an isomorphism $\Phi_0 : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ such that for any pair of projections*

$$p \in M_n(\mathcal{A}), q \in M_n(\mathcal{B}) : \; \|p - q\| < 1/2 \Rightarrow \Phi_0([p]) = [q].$$

(iii) *There exists an isomorphism $\Phi_1 : K_1(\mathcal{A}) \to K_1(\mathcal{B})$ such that for any pair of unitaries*

$$u \in M_n(\mathcal{A}), v \in M_n(\mathcal{B}) : \; \|u - v\| < 1 \Rightarrow \Phi_1([u]) = [v].$$

(iv) *There exists an affine w*-homeomorphism $\Psi$ of $TS(\mathcal{B})$ onto $TS(\mathcal{A})$ s. t. for $\tau$ in $TS(\mathcal{B})$ the homomorphisms $\tau_0$, and $\Psi(\tau)_0$ on $K_0(\mathcal{B})$ and $K_0(\mathcal{A})$ satisfy $\forall g \in K_0(\mathcal{A}) : \Psi(\tau)_0(g) \;=\; \tau_0(\Phi_0(g))$.*

You may add *finite or purely infinite* to the list of properties.

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## $\ell^2$-invariants and simplicity of the reduced group $C^*$-algebra

Andreas Thom
(joint work with Jesse Peterson)

### 1. Introduction

The computations of $\ell^2$-homology have been algebraized through the seminal work of W. Lück, which is summarized and explained in detail in his nice compendium [2]. This extended abstract is a report about results obtained in [3].

Our first theorem gives an identification of dimensions of cohomology groups, where the coefficients vary among the canonical choices $LG, \ell^2 G$ and $\mathcal{U}G$.

**Theorem 1.1.** *Let $G$ be a countable discrete group.*

$$\beta_k^{(2)}(G) = \dim_{LG} H^k(G, \mathcal{U}G) = \dim_{LG} H^k(G, \ell^2 G) = \dim_{LG} H^k(G, LG).$$

*Moreover, if $\beta_k^{(2)}(G) = 0$ for some $k$, then $H^k(G, \mathcal{U}G) = 0$.*

### 2. Free subgroups

2.1. **Restriction maps and free subgroups.** Throughout this section, we are assuming that $G$ is a torsionfree discrete countable group and most of the time also that it satisfies the following condition:

($\star$) Every non-trivial element of $\mathbb{Z}G$ acts without kernel on $\ell^2 G$.

Condition ($\star$) is known to hold for all right orderable groups and all residually torsionfree elementary amenable groups. No counterexample is known.

Let $G$ be a discrete group, we use the notation $\dot{G}$ to denote the set $G \setminus \{e\}$. The main result here is the following theorem.

**Theorem 2.1.** *Let $G$ be a torsionfree discrete countable group. There exists a family of subgroups $\{G_i \mid i \in I\}$, such that*

(i) *We can write $G$ as the disjoint union:*

$$G = \{e\} \cup \bigcup_{i \in I} \dot{G}_i.$$

(ii) *The groups $G_i$ are mal-normal in $G$, for $i \in I$.*
(iii) *If $G$ satisfies condition ($\star$), then $G_i$ is free from $G_j$, for $i \neq j$.*
(iv) *$\beta_1^{(2)}(G_i) = 0$, for all $i \in I$.*

*Remark* 2.2. It follows from Theorem 7.1 in [3], that the set $I$ is infinite if the first $\ell^2$-Betti number of $G$ does not vanish.

**Corollary 2.3.** *Let $G$ be a discrete countable group satisfying condition ($\star$). Assume that the first $\ell^2$-Betti number does not vanish. Let $F$ be a finite subset of $G$. There exists $g \in G$, such that $g$ is free from each element in $F$. In particular, $G$ contains a copy of $F_2$.*

*Remark* 2.4. Corollary 2.3 confirms the feeling that a sufficiently non-amenable group contains a free subgroup. Note, that various weaker conditions like *non-amenability* itself or *uniform non-amenability* have been proved to be insufficient to ensure the existence of free subgroups, at least in the presence of torsion.

Using results from [1] we obtain the following result.

**Corollary 2.5.** *Let $G$ be a torsionfree discrete countable group satisfying condition ($\star$). If the first $\ell^2$-Betti number does not vanish, then the reduced group $C^*$-algebra $C^*_{red}(G)$ is simple and carries a unique trace.*

The following result is a generalization of the main result of J. Wilson in [4] for torsionfree groups which satisfy ($\star$). For this, note that a group $G$ with $n$ generators and $m$ relations satisfies $\beta_1^{(2)}(G) \geq n - m - 1$.

**Corollary 2.6** (Freiheitssatz)**.** *Let $G$ be a torsionfree discrete countable group which satisfies ($\star$). Assume that $a_1, \ldots, a_n \in G$ generate $G$ and $\lceil \beta_1^{(2)}(G) \rceil \geq k$. There exist $k+1$ elements $a_{i_0}, \ldots, a_{i_k}$ among the generators such that the natural map*

$$\pi \colon F_{k+1} \to \langle a_{i_0}, \ldots, a_{i_k} \rangle \subset G$$

*is an isomorphism.*

**Corollary 2.7.** *Let $G$ be a finitely generated torsionfree discrete countable group which satisfies ($\star$). Then*

$$e_S(G) \geq 2\lceil \beta_1^{(2)}(G) \rceil + 1,$$

*for any generating set $S$. Here, $e_S(G)$ denotes the exponential growth rate w.r.t. the generating set $S$.*

In particular, a torsionfree group satisfying condition ($\star$) has uniform exponential growth if its first $\ell^2$-Betti number is positive.

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## MASAs in C*-algebras

Simon Wassermann
(joint work with N. Christopher Phillips)

### I. Tensor products of MASAs.

For C*-algebras $A$ and $B$, $A \otimes B$ will denote the minimal C*-tensor product $A \otimes_{min} B$. By a **MASA** of $A$ we mean a maximal abelian C*-subalgebra of $A$. The following result was obtained in 1973.

**Theorem 1** [3] *Let $A_1$ and $A_2$ be C*-algebras. If $C$ is a MASA of $A_1$, then $(C \otimes 1)^c = C \otimes A_2$. If $C_1$ and $C_2$ are MASAs in $A_1$ and $A_2$, respectively, then $C_1 \otimes C_2$ is a MASA in $A_1 \otimes A_2$.*

This suggested the more general question: if $C_1 \subset A_1$ and $C_2 \subset A_2$ are MASAs, is $C_1 \otimes C_2$ a MASA in $A_1 \otimes_\beta A_2$ for any C*-norm $\| \ \|_\beta$ on $A_1 \odot A_2$, in particular if $\beta = max \neq min$? I have recently obtained a fairly complete answer to this question [4], though further interesting questions are raised. Provided one of the MASAs has the following extension property and the other contains an approximate unit for the containing algebra, Theorem 1 extends to arbitrary tensor products.

**Definition.** A MASA $C$ in a C*-algebra $A$ has the **extension property** if

(i) Every character of $C$ has a unique (pure) state extension to $A$.

(ii) If $A$ is non-unital then no pure state of $A$ restricts to the zero functional on $C$.

**Theorem 2** *Let $A_1$ and $A_2$ be C*-algebras with $A_2$ unital and let $C$ be a MASA of $A_1$ with the extension property. Let $(C \otimes 1)^c$ denote the algebra*

$$\{x \in A_1 \otimes_\beta A_2 : (c \otimes 1)x = x(c \otimes 1) \text{ for } c \in C\},$$

*for a given C*-norm $\| \ \|_\beta$ on $A_1 \odot A_2$. Then*

$$(C \otimes 1)^c = C \otimes A_2$$

*in $A_1 \otimes_\beta A_2$ .*

**Corollary 3** *Let $A_1$ and $A_2$ arbitrary C*-algebras and let $C_1$ and $C_2$ be MASAs of C*-algebras $A_1$ and $A_2$, respectively. If $C_1$ has the extension property and $C_2$ contains an approximate unit $\{e_i\}_{i \in I}$ for $A_2$, then $C_1 \otimes C_2$ is a MASA in $A_1 \otimes_\beta A_2$ for any C*-norm $\| \ \|_\beta$ on $A_1 \odot A_2$.*

The conditions on the MASAs in this corollary cannot be relaxed. In the unital case $C_1 \otimes C_2$ can fail to be a MASA in $A_1 \otimes_\beta A_2$ if neither MASA has the extension property, as the following example shows.

Let

$$A = C_r^*(\mathbb{F}_2) + K(\ell^2(\mathbb{F}_2))$$

in $B(\ell^2(\mathbb{F}_2))$, where $\mathbb{F}_2$ is the free group on two generators. Let $\{\xi_g : g \in \mathbb{F}_2\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{F}_2)$, let $e_g$ be the projection onto $\mathbb{C}\xi_g$, and let $C$ be the abelian C*-algebra generated by $\{e_g : g \in \mathbb{F}_2\} \cup \{1\}$. Then $C$ is a MASA in $A$.

**Proposition 4** *There is a C*-norm $\alpha$ on the algebraic tensor product $A \odot A$ such that $C \otimes C$ is not maximal abelian in $A \otimes_\beta A$ for any C*-norm $\| \ \|_\beta$ satisfying $\|x\|_\beta \geq \|x\|_\alpha$ on $x \in A \odot A$, in particular if $\| \ \|_\beta = \| \ \|_{max}$.*

When $A_2$ is non-unital, $C_1 \otimes C_2$ can fail to be a MASA in $A_1 \otimes_\beta A_2$ if $C_2$ does not contain an approximate unit for $A_2$, even if $C_1$ has the extension property in $A_1$. In fact a modification of the definition of $A$ and $C$ in the unital case gives a separable non-unital C*-algebra $A_0$ with a MASA $C_0$ such that, for any MASA $C_1$ in $C_r^*(\mathbb{F}_2)$, $C_1 \otimes C_0$ is not a MASA in $C_r^*(\mathbb{F}_2) \otimes_{max} A_0$. It is known that the abelian C*-subalgebras of $C_r^*(\mathbb{F}_2)$ generated by each of the canonical unitary generators are MASAs with the extension property.

**Open Questions.**

1. By [5], $\| \ \|_{max} \neq \| \ \|_{min}$ on $B(H) \odot C_r^*(\mathbb{F}_2)$ if $H \cong \ell^2(\mathbb{N})$. If $C$ is a MASA of $B(H)$ isomorphic to $L^\infty(0,1)$, is it true that
$$(C \otimes 1)^c = C \otimes C_r^*(\mathbb{F}_2)$$
in $B(H) \otimes_{max} C_r^*(\mathbb{F}_2)$?

2. By [1], $\| \ \|_{max} \neq \| \ \|_{min}$ on $B(H) \odot B(H)$. Is it true that
$$(C \otimes 1)^c = C \otimes B(H)$$
in $B(H) \otimes_{max} B(H)$ for any MASA $C$ of $B(H)$? Is $C_1 \otimes C_2$ maximal abelian in $B(H) \otimes_{max} B(H)$ for any (all) MASAs $C_1$ and $C_2$?

3. If $C \cong \ell^\infty(\mathbb{N})$ is a MASA of $B(H)$, does $C$ have the extension property? (This is the celebrated Kadison-Singer question [2].) Is $C \otimes C$ a MASA in $B(H) \otimes_{max} B(H)$? A negative answer to this last question would imply a negative answer to the Kadison-Singer question.

4. Can any compact Hausdorff space occur as the spectrum of a MASA $C$ in a C*-algebras $A$ exhibiting the pathology in Proposition 4? By an extension of the above construction, it can be shown that $[0,1]$ occurs as such a spectrum.

**II. MASAs of the CAR algebra (jointly with N.C. Phillips).**

This is a brief report on a project which is at a relatively early stage. The CAR algebra can be defined as the infinite tensor product $\otimes M_2(\mathbb{C})$ of a countable set of copies of $M_2(\mathbb{C})$. A MASA is obtained by taking the infinite tensor product of 2-dimensional diagonal algebras from each $2 \times 2$ matrix algebra. MASAs constructed in this way have the extension property and their spectra are totally disconnected. A question which we are currently investigating is which compact metric spaces can occur as the spectra of MASAs of the CAR algebra. Using an alternative characterisations of the CAR algebra as an inductive limit of building blocks of the form $C([0,1], M_n)$ with appropriately defined connecting maps, we are able to show that a wide variety of compact metric spaces do occur as the spectra of MASAs, though MASAs without the extension property. The set of possible spectra includes for example the unit interval $[0,1]$ and, more generally, any finite graph. For a given spectrum, the construction gives a continuum of mutually non-unitarily conjugate MASAs.

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## $K_1$-injectivity and proper infiniteness


Mikael Rørdam

(joint work with Etienne Blanchard, Randi Rohde)


We discuss how certain permanence problems of proper infiniteness is related to $K_1$-injectivity. We begin by defining and motivating the basic concepts.

A unital $C^*$-algebra $A$ is said to be *properly infinite* if the unit $1_A$ for $A$ is a properly infinite projection, i.e., if $1_A \oplus 1_A \precsim 1_A$. Equivalently, $A$ is properly infinite if one can embed any of the Cuntz-Toeplitz algebras $\mathcal{T}_n$ ($n \geq 2$) unitally into $A$, or if one can embed the Cuntz algebra $\mathcal{O}_\infty$ unitally into $A$.

The significance of being properly infinite lies perhaps primarily in its tight connection with existence of traces (or 2-quasi-traces). Blackadar and Handelman proved in [1] that a unital $C^*$-algebra admits a 2-quasitrace if and only if $M_n(A)$ fails to be properly infinite for all $n$; and Haagerup proved in [5] that all 2-quasitraces on a unital $C^*$-algebra are equal to (or lift to) a tracial state.

These results raise the question if proper infiniteness is a stable property, i.e., if $M_n(A)$ is properly infinite for some $n$ implies that $A$ itself is properly infinite. This question turns out to have negative answer (see [8]), even in the simple nuclear case (see [9]).

A unital $C^*$-algebra $A$ is said to be $K_1$*-injective* if the canonical mapping $\mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$ is injective. One can usually calculate the $K_1$-class of a unitary in a $C^*$-algebra and decide if it is zero. In presence of $K_1$-injectivity, this will imply that such a unitary is homotopic to 1; a useful fact which often is hard to obtain with bare hands.

It is know that a $C^*$-algebra is $K_1$-injective if it is simple and purely infinite (Cuntz, [4]); of stable rank one (Rieffel, [7]); of real rank zero (Lin, [6]); or extremally rich (Brown-Pedersen, [3]).

Our main result, from [2], relating permanence properties of proper infiniteness and $K_1$-injectivity can be stated as follows.

**Theorem 1.** The following statements are equivalent:

(1) All unital properly infinite $C^*$-algebras are $K_1$-injective.
(2) Proper infiniteness is closed under forming pull-backs.
(3) All unital $C(X)$-algebras with properly infinite fibres are properly infinite.
(4) All sub-trivial $C(\mathbb{T})$-algebras with properly infinite fibres are properly infinite.
(5) For any unital (properly infinite) $C^*$-algebra $A$ and for any full, properly infinite projections $p, q$ in $A$ there exist full, properly infinite projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim_h q_0$.
(6) There exists a projection $p \in \mathcal{O}_\infty$ with $p \neq 0, 1$ such that $\iota_1(p) \sim_h \iota_2(p)$ in the set of projections in $\mathcal{O}_\infty * \mathcal{O}_\infty$.
(7) The (sub-trival) $C([0,1])$-algebra

$$\{f \in C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty) \mid f(0) \in \mathcal{O}_\infty * \mathbb{C}, \quad f(1) \in \mathbb{C} * \mathcal{O}_\infty\}$$

contains a non-trivial projection (or is properly infinite).
(8) $\mathcal{O}_\infty * \mathcal{O}_\infty$ is $K_1$-injective.

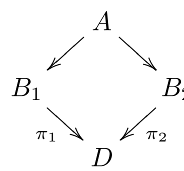
It is quite likely that all these equivalent properties are false!

We indicate below why the first four conditions are equivalent. (1) $\Rightarrow$ (2) follows from Proposition 3, (2) $\Rightarrow$ (3) follows from Proposition 4 (and its proof), (3) $\Rightarrow$ (4) is trivial, and the proof of (4) $\Rightarrow$ (1) is given at the end.

The lemma below is well-known (and can be easily proved using the Whitehead lemma):

**Lemma 2.** Let $A$ be a unital $C^*$-algebra, let $u$ be a unitary element in $A$, and let $p$ be a projection in $A$ such that $[u] = 0$ in $K_1(A)$, $\|up - pu\| < 1$, and $p$ and $1-p$ are properly infinite and full. Then $u \sim_h 1$.

**Proposition 3.** Given a pull-back diagram of unital $C^*$-algebras:




If $B_1$ and $B_2$ are properly infinite, then $M_2(A)$ is properly infinite; and if $D$ is also assumed to be $K_1$-injective, then $A$ itself is properly infinite.

**Proof:** Take unital $^*$-homomorphisms $\rho_j \colon \mathcal{T}_3 \to B_j$, $j = 1, 2$. With $s_1, s_2, s_3$ the canonical generators of $\mathcal{T}_3$, a standard argument gives a unitary $u$ in $D$ such that

$[u] = 0$ in $K_1(D)$ and $(\pi_1 \circ \rho_1)(s_j) = u(\pi_2 \circ \rho_2)(s_j)$ for $j = 1, 2$ (we use the "free" space from $s_3 s_3^*$ to arrange that $u$ is unitary with trivial $K_1$-class). If $D$ were $K_1$-injective, then $u$ would lift to a unitary $v$ in $B_1$. The $^*$-homomorphisms $\pi_j' \colon \mathcal{T}_2 \to B_j$ given by $\pi_1'(s_i) = v\pi_1(s_i)$, $i = 1, 2$, and $\pi_2' = \pi_{2|\mathcal{T}_2}$, satisfy $\pi_1' \circ \rho_1 = \pi_2' \circ \rho_2$, and hence give rise to a unital $^*$-homomorphism $\mathcal{T}_2 \to A$, whence $A$ is properly infinite.

In general, if $D$ is not injective, then by tensoring the pull-back diagram by $M_2$, the unitary $u$ above is being replaced with $\mathrm{diag}(u, u)$, and this unitary is homotopic to 1 by Lemma 2, and hence lifts to a unitary in $M_2(B_1)$. One can now argue as above to get that $M_2(A)$ is properly infinite. □

**Proposition 4.** Let $A$ be a $C(X)$-algebra and assume that all fibres $A_x$, $x \in X$, are properly infinite. Then $M_n(A)$ is properly infinite for some $n$.

**Proof:** Each $x \in X$ has a closed neighborhood $F_x$ for which $A_{|F_x}$ is properly infinite. Hence one can cover the compact space $X$ with finitely many closed subsets $F_1, F_2, \dots, F_m$ such that $A_{|F_j}$ is properly infinite for each $j$. Apply Proposition 3 to conclude that $M_{2^{m-1}}(A)$ is properly infinite.

(4) ⇒ (1). Assume that (4) holds, let $A$ be a unital properly infinite $C^*$-algebra, and let $u$ be a unitary in $A$ such that $[u] = 0$ in $K_1(A)$. Then $\mathrm{diag}(u, 1) \sim_h 1$ in $M_2(A)$ (by Lemma 2), and so there is a continuous path $t \mapsto v_t$ of unitaries in $M_2(A)$ such that $v_0 = 1$ and $v_1 = \mathrm{diag}(u, 1)$. Put $p(t) = v_t \mathrm{diag}(1, 0) v_t^*$. Then $p(0) = p(1) = \mathrm{diag}(1, 0)$, and so $p$ is an element in $C(\mathbb{T}, M_2(A))$.

The (sub-trivial) $C(\mathbb{T})$-algebra $B = pC(\mathbb{T}, M_2(A))p$ has properly infinite fibres $B_t = p(t)M_2(A)p(t) \cong A$, $t \in \mathbb{T}$. If (4) holds, then $B$ is properly infinite, whence $p$ is a full properly infinite projection in $C(\mathbb{T}, M_2(A))$.

The partial isometry $t \mapsto v_t \mathrm{diag}(0, 1)$ belongs to $C(\mathbb{T}, M_2(A))$ and induces an equivalence between $1 - p$ and $\mathrm{diag}(0, 1)$. As $p$ is properly infinite and full, we conclude that $p$ is unitarily equivalent to $\mathrm{diag}(1, 0)$. We can therefore find a continous path of unitaries $t \mapsto w_t \in M_2(A)$ such that $w_0 = w_1 = 1$ and $[w_t v_t, \mathrm{diag}(1, 0)] = 0$. It follows that $w_t v_t = \mathrm{diag}(u_t, z_t)$, $u_0 = 1$ and $u_1 = u$, whence $u \sim_h 1$ in $\mathcal{U}(A)$ as desired. □.

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## On non-existense of certain finite depth subfactors

Marta Asaeda
(joint work with Seidai Yasuda)

Since Jones introduced the index theory of subfactors in [6], the theory of operator algebras have been achieving a remarkable development, having relations with various other areas of mathematics. One of the important objective is to find *exotic* subfactors: subfactors that are not constructed from other known mathematical objects, such as finite groups and quantum groups. One of the most important invariants of subfactors are (dual) principal graphs. Haagerup started a systematic search for exotic subfactors in 1991, and gave the list of graphs in [5, §7] as candidates which might be realized as (dual) principal graphs of subfactors. Haagerup and I proved that two pairs of graphs: the case $n = 3$ of (2) (see Figure 1) as well as the case (3) in [5, §7], are realized as (dual) principal graphs of subfactors, and that such subfactors are unique respectively ([2]).

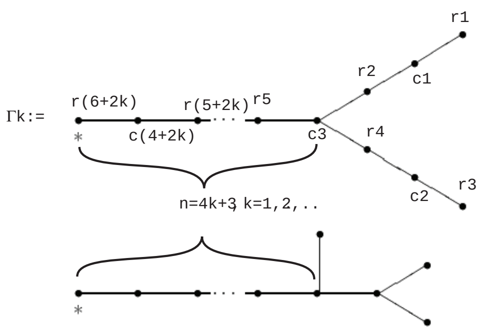


Figure 1. The pairs of graphs (2) in the list of Haagerup

In 2005, Etingof, Nikshych, and Ostrik showed in [4, Theorem 8.51], that the index of a subfactor has to be a cyclotomic integer, namely an algebraic integer that lies in a cyclotomic field. This implies that if the square of the Perron-Frobenius eigenvalue (PFEV) of a graph is not a cyclotomic integer, the graph cannot be the (dual) principal graph of a subfactor. In this talk we prove that none of the graphs in Figure 1 can be realized a (dual) principal graph for $k \geq 2$.

## 1. Minimal polynomials

Let $d_k$ be the square of PFEV of the graph $\Gamma_k$ in Fig. 1. In [1] the adjacency matrix $A_k$ of $\Gamma_k$ was given. The characteristic polynomial of the matrix $N_k := {A_k}^t A_k$ divided by $(x-2)^2$, which is denoted by $q_k(x)$, satisfies the following recursive formula

$$\begin{aligned} q_k(x) &= (x^2-4x+2)q_{k-1}(x) - q_{k-2}, \\ q_0(x) &= x^2-5x+3, \\ q_1(x) &= (x^3-8x^2+17x-5)(x-1). \end{aligned}$$

**Theorem 1.** *([3]) Let*

$$r_k(x) = \begin{cases} q_k(x)/(x-1), & \text{if } k \equiv 1 \bmod 3, \\ q_k(x), & \text{else.} \end{cases}$$

*Then $r_k(x)$ is irreducible for any $k$, thus it is the minimal polynomial of $d_k$.*

I do not get into the proof of this theorem here. However note that $q_k$'s are generally ugly. By the change of variable one obtains better polynomials. Let

$$P_k(q) := q_k(x)|_{x=q+q^{-1}+2} q^{2k+2}.$$

Then

$$P_{k-1}(q) = q^{4k} - q^{4k-1} - q^{4k-2} - q^{4k-3} + q^{4k-4} - \cdots - q^5 + q^4 - q^3 - q^2 - q + 1.$$

for any $k \geq 1$. The polynomials $P_k(q)$'s are one of the key players in proving Theorem 1, as well as further argument.

## 2. $d_k$'s are not cyclotomic integers for $k \geq 2$

Let me briefly describe Hilbert's theory on ramification of ideals.

Let $K$ be a finite extension of $\mathbb{Q}$, We denote by $O_K$ *the ring of integers of* $K$, namely the set of algebraic integers contained in $K$. For example, $O_{\mathbb{Q}} = \mathbb{Z}$. Let $p$ be a prime number. It generates a prime ideal $(p)$ in $\mathbb{Z}$.Consider the ideal $pO_K$, generated by $p$ in $O_K$. This is not generally a prime ideal, however, it factorizes into a product of prime ideals uniquely:

$$pO_K = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g},$$

where $\mathfrak{P}_i$'s are distinct prime ideals of $O_K$. We call $e_i$ *the ramification index of* $\mathfrak{P}_i$. For a prime ideal $\mathfrak{P}$ of $O_K$, $O_K/\mathfrak{P}$ is a field extension over $k := \mathbb{Z}/p\mathbb{Z}$. We call $[O_K/\mathfrak{P}, k] =: h(\mathfrak{P})$ *the degree of* $\mathfrak{P}$ *over* $k$.

The ramification theory concerns the factorization described above, for a given prime $p$ and a field extension $K$. When $K$ is a Galois extension of $\mathbb{Q}$, it is known that $e_i$'s and $h(\mathfrak{P}_i)$ does not depend on $i$. We modify a well-known classical theorem by Dedekind ([7], Theorem 4.33) and obtained the following theorem that we use for our purpose:

**Theorem 2.** *Let $d$ be an algebraic integer, $K = \mathbb{Q}(d)$, and $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of $d$ with degree $n$. Suppose that $K/\mathbb{Q}$ is Galois. Let $p$ be a prime number, and $k := \mathbb{Z}/pZ$. Let $e$, $f$, and $g$ be integers such that*

$$pO_K = (\mathfrak{P}_1 \cdots \mathfrak{P}_g)^e,$$

*where $\mathfrak{P}_i$'s are distinct prime ideals of $O_K$, and $h(\mathfrak{P}_i) = h$ for all $i = 1, \ldots, g$. Then $f(x)$ factorizes mod $p$ as follows:*

$$\bar{f}(x) = (f_1 \cdots f_g)^e \mod p,$$

*where $f_i \in k[x]$ with $\deg f_i = h$ for all $i$ and each $f_i$ is of the form $f_i = g_i^{e'_i}$, where $g_i \in k[x]$ is irreducible.*

Note that if $d_k$ is a cyclotomic integer, then $K = \mathbb{Q}(d_k)$ is automatically a Galois extension of $\mathbb{Q}$. Thus, in order to prove that $d_k$ is not a cyclotomic integer, it suffices to show that $K$ is not Galois over $\mathbb{Q}$. We study the factorization of the minimal polynomial modulo suitable prime $p$ and derive a contradiction by the above theorem. For simplicity, we prove the equivalent statement that $e_k = d_k - 2$ is not cyclotomic integers for $k \geq 2$. We shift the variable of all the polynomials accordingly: The minimal polynomial for $e_k$ is $m_k(x) := r_k(x+2)$, and $p_k(x) := q_k(x+2)$. Here we list the key observations.

(1) $p_{k-1}(0) = (-1)^k(2k+1)$, $p'_{k-1}(0) = (-1)^k k$
(2) If $p > 3$, $p_{k-1}(x)$ mod $p$ has multiplicities at most 4.

Note that for a prime $p$ so that $p|2k+1$, $p \not| k$. Thus, (1) and Theorem 2 imply that, if $\mathbb{Q}(e_{k-1})/\mathbb{Q}$ is Galois, we have

$$m_{k-1}(x) = x \prod_{0 \neq a \in \mathbb{Z}/p\mathbb{Z}} (x-a)^{n_a} \mod p.$$

We demonstrate the proof for the simplest case here:
**Case 1**: $2k+1$ is not a prime, nor a power of 3, and $k \neq 2$ mod 3.
By the assumption, there is a prime number $p \neq 2k+1, 3$ that divides $2k+1$. Since $2 \not| 2k+1$, $2k+1$ is divisible by some number larger or equal to 5, thus $p \leq \frac{2k+1}{5}$. Since $\deg m_{k-1} = \deg p_{k-1} = 2k$ and $n_a \leq 4$,

$$2k \leq 1 + 4(p-1) \leq 1 + 4(\frac{2k-4}{5}) = \frac{8k}{5} - \frac{11}{5} < 2k,$$

thus contradiction.

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## The Effros-Ruan conjecture for bilinear forms on C*-algebras

Magdalena Musat
(joint work with Uffe Haagerup)

In 1956 Grothendieck published the celebrated "Résumé de la théorie métrique des produits tensoriels topologiques", containing a general theory of tensor norms on tensor products of Banach spaces, describing several operations to generate new norms from known ones, and studying the duality theory between these norms. Since 1968 it has had considerable influence on the development of Banach space theory. The highlight of the paper [2], now referred to as the "Résumé" is a result that Grothendieck called "The fundamental theorem on the metric theory of tensor products". Grothendieck's theorem asserts that given compact spaces $K_1$ and $K_2$ and a bounded bilinear form $u : C(K_1) \times C(K_2) \to \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) , then there exist probability measures $\mu_1$ and $\mu_2$ on $K_1$ and $K_2$ , respectively, such that

$$|u(f,g)| \leq K_G^{\mathbb{K}} \|u\| \left( \int_{K_1} |f(t)|^2 \, d\mu_1(t) \right)^{1/2} \left( \int_{K_2} |g(t)|^2 \, d\mu_2(t) \right)^{1/2} ,$$

for all $f \in C(K_1)$ and $g \in C(K_2)$ , where $K_G^{\mathbb{K}}$ is a universal constant.

The non-commutative version of Grothendieck's inequality (conjectured in the "Résumé") was first proved by Pisier under some approximability assumption (cf. [6]) , and obtained in full generality in [3]. The theorem asserts that given C*-algebras $A$ and $B$ and a bounded bilinear form $u : A \times B \to \mathbb{C}$ , then there exist states $f_1 , f_2$ on $A$ and states $g_1 , g_2$ on $B$ such that for all $a \in A$ and $b \in B$ ,

$$|u(a,b)| \leq \|u\| (f_1(a^*a) + f_2(aa^*))^{1/2} (g_1(b^*b) + g_2(bb^*))^{1/2} .$$

As a corollary, it was shown in [3] that given C*-algebras $A$ and $B$ , then any bounded linear operator $T : A \to B^*$ admits a factorization $T = SR$ through a Hilbert space $H$ , where $A \xrightarrow{R} H \xrightarrow{S} B^*$ , and

$$\|R\| \|S\| \leq 2\|T\| .$$

Let $E \subseteq A$ and $F \subseteq B$ be operator spaces sitting in C*-algebras $A$ and $B$ , and let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form. Then, there exists a unique bounded linear operator $\widetilde{u} : E \to F^*$ such that

$$u(a\,, b) := \langle \widetilde{u}(a)\,, b \rangle \,, \quad a \in E\,, b \in F\,, \tag{1}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the duality bracket between $F$ and $F^*$ . The map $u$ is called *jointly completely bounded* (for short, j.c.b.) if the associated map $\widetilde{u} : E \to F^*$ is completely bounded, in which case we set

$$\|u\|_{\mathrm{jcb}} := \|\widetilde{u}\|_{\mathrm{cb}}\,. \tag{2}$$

(Otherwise, we set $\|u\|_{\mathrm{jcb}} = \infty$ .) It is easily checked that

$$\|u\|_{\mathrm{jcb}} = \sup_{n\in\mathbb{N}} \|u_n\|\,, \tag{3}$$

where for every $n \geq 1$ , the map $u_n : M_n(E) \otimes M_n(F) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is given by

$$u_n\left(\sum_{i=1}^{k} a_i \otimes c_i\,, \sum_{j=1}^{l} b_j \otimes d_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i, b_j) c_i \otimes d_j\,,$$

for all finite sequences $\{a_i\}_{1\leq i\leq k}$ in $E$ , $\{b_j\}_{1\leq j\leq l}$ in $F$ , $\{c_i\}_{1\leq i\leq k}$ and $\{d_j\}_{1\leq j\leq l}$ in $M_n(\mathbb{C})$ , $k\,, l \in \mathbb{N}$ . Moreover, $\|u\|_{\mathrm{jcb}}$ is the smallest constant $\kappa_1$ for which, given arbitrary C*-algebras $C$ and $D$ and finite sequences $\{a_i\}_{1\leq i\leq k}$ in $E$ , $\{b_j\}_{1\leq j\leq l}$ in $F$ , $\{c_i\}_{1\leq i\leq k}$ in $C$ and $\{d_j\}_{1\leq j\leq l}$ in $D$ , where $k\,, l \in \mathbb{N}$ , the following inequality holds

$$\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j) c_i \otimes d_j\right\|_{C\otimes_{\min} D} \leq \kappa_1 \left\|\sum_{i=1}^{k} a_i \otimes c_i\right\|_{E\otimes_{\min} C} \left\|\sum_{j=1}^{l} b_j \otimes d_j\right\|_{F\otimes_{\min} D}.$$

It was conjectured by Effros and Ruan in 1991 (cf. [1] and [7], Conjecture 0.1) that if $A$ and $B$ are C*-algebras and $u : A\times B \to \mathbb{C}$ is a jointly completely bounded bilinear form, then there exist states $f_1$ , $f_2$ on $A$ and states $g_1$ , $g_2$ on $B$ such that for all $a \in A$ and $b \in B$ ,

$$|u(a, b)| \leq K\|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2})\,, \tag{4}$$

where $K$ is a universal constant.

In [7] Pisier and Shlyakhtenko proved an operator space version of (4), namely, if $E \subseteq A$ and $F \subseteq B$ are exact operator spaces with exactness constants $\mathrm{ex}(E)$ and $\mathrm{ex}(F)$ , respectively, and $u : E \times F \to \mathbb{C}$ is a j.c.b. bilinear form, then there exist states $f_1$ , $f_2$ on $A$ and states $g_1$ , $g_2$ on $B$ such that for all $a \in E$ and $b \in F$ ,

$$|u(a, b)| \leq 2^{3/2}\mathrm{ex}(E)\mathrm{ex}(F)\|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2})\,.$$

Moreover, by the same methods they were able to prove the Effros-Ruan conjecture for C*-algebras with constant $K = 2^{3/2}$ , provided that at least one of the C*-algebras $A$ , $B$ is exact (cf. [7], Theorem 0.5).

Our main result from [5] is that the Effros-Ruan conjecture is true. Moreover, it holds with constant $K = 1$ , that is,

**Theorem 1.** Let $A$ and $B$ be C*-algebras and $u : A \times B \to \mathbb{C}$ a jointly completely bounded bilinear form. Then there exist states $f_1$ , $f_2$ on $A$ and states $g_1$ , $g_2$ on $B$ such that for all $a \in A$ and $b \in B$ ,

$$|u(a, b)| \leq \|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2})\,.$$

We also prove that $K = 1$ is the best constant in the inequality (4) .

It follows from Theorem 1 that every completely bounded linear map $T : A \to B^*$ from a C*-algebra $A$ to the dual $B^*$ of a C*-algebra $B$ has a factorization $T = vw$ through $H_r \oplus K_c$ (the direct sum of a row Hilbert space and a column Hilbert space), such that

$$\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq 2\|T\|_{\mathrm{cb}}.$$

Furthermore, thanks to Theorem 1 we can strengthen a number of results from [7]. For instance, it follows that if an operator space $E$ and its dual $E^*$ both embed in noncommutative $L_1$-spaces, then $E$ is completely isomorphic to a quotient of a subspace of $H_r \oplus K_c$, for some Hilbert spaces $H$ and $K$.

While the approach by Pisier and Shlyakhtenko relies on free probability techniques, our proof uses more classical operator algebra theory, namely, Tomita-Takesaki theory and special properties of the Powers factors of type $\mathrm{III}_\lambda$, $0 < \lambda < 1$ (cf. [4]).

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