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Arithmetic Algebraic Geometry

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Abstract. Arithmetic geometry lies between number theory and algebraic geometry. It deals with schemes over the rings of integers of a numberfield or also over a p-adic completion. For them one investigates geometric porperties, integral points, or the cohomology. The present workshop had a heavy emphasis on p-adic cohomologies.

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Introduction by the Organisers

The workshop covered developments in the field in the last four years. Roughly speaking arithmetic geometry consider algebraic schemes over rings of integers of numberfields. However an important tool is to first extend the base to a p-adic completion. Although both global and local problems matter this time there was a heavy emphasis on p-adic topics.

One of them is the deformation-theory of Galois-representations, leading to a proof of Serre's conjecture. here one starts with a global Galois-representation modulo p, then lifts modulo p^2, etc. For the lifts one requires certain local conditions (like being unramified outside a given set of places), and the most important and difficult such conditions arise at primes dividing p. Here the most important tool is J.M.Fontaine's theory which relates Galois-representations to filtered Frobenius-crystals.

Another spectacular progess is the proof (by Ngo) of the fundamental lemma in the theory of automorphic representations. It postulates identities of p-adicorbital integrals and is reduced to a geometric statement about perverse sheaves on Hitchin-fibrations in positive characteristic.

Concerning p-adic cohomology theories we are getting closer to a p-adic theory of D-modules, and of overconvergent crystals, over singular schemes. Also the long awaited etale coverings of p-adic period domains have finally been constructed, after it has been understood that they have "holes" which are visible in the Berkovich-space but not in the conventional rigid space. That they exist is suggested by Fontaine's theory. These period domains classify p-divisible groups. Some of them (Drinfeld, Lubin-Tate) can be covered by explicit affinoid domains, thus giving some type of reduction theory for p-divisible groups. There are attempts to extend this to finite flat group-schemes.

Concerning K-theory the classical Borel-regulator from K-theory to Deligne-cohomology has been extended to syntomic cohomology, as well as the computation of its values on Eisenstein-symbols. For the l-adic etale theory general finiteness theorems can now be shown for quasi-excellent schemes. A further topic was the theory of p-adic Banach-representations of p-adic Lie-groups.

On a more global level we had talks about (p-adic!) constructions of rational points on elliptic curves, the association of K-classes to abelian varieties, and the theory of tame fundamental groups. Finally the theory of small points has been extended to function fields (over numberfields it leads to equidistribution) using tropical geometry.

Workshop: Arithmetic Algebraic Geometry

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Abstracts

Applications of $\mathcal{D}$-module theory to finiteness conditions in crystalline cohomology

Pierre Berthelot

Let k be a perfect field of characteristic $p > 0$, $S_0 = \mathrm{Spec}(k)$, $S = \mathrm{Spec}(W_n(k))$ for some fixed integer $n \geq 1$, and let X_0 be a proper and smooth S_0-scheme. We assume that $n = 1$ if $p = 2$, and we denote here by $\mathrm{Crys}(X_0/S)$ the nilpotent crystalline site [1, III, 1.3] and by $\mathcal{O}_{X_0/S}$ its structural sheaf. Using $\mathcal{D}$-module theory, we define two triangulated subcategories of the derived category $DF(\mathcal{O}_{X_0/S})$ of complexes of filtered $\mathcal{O}_{X_0/S}$-modules, whose objects are called respectively $\mathcal{D}$-perfect and $\mathcal{D}^\vee$-perfect complexes. For these complexes, the classical finiteness and duality theorems can be generalized.

$\mathcal{D}$-perfect complexes. For any smooth S-scheme X, we denote by $\mathcal{D}_X$ the sheaf of PD-differential operators on X (see [1, II, 2.1] or [2, Ch. 4]).

Let X_0 be a smooth S_0-scheme which can be lifted as a smooth S-scheme X, and let

$$C_{X_0} : \{\text{left } \mathcal{D}_X\text{-modules}\} \xrightarrow{\approx} \left\{ \begin{matrix} \text{crystals of} \\ \mathcal{O}_{X_0/S}\text{-modules} \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{sheaves of} \\ \mathcal{O}_{X_0/S}\text{-modules} \end{matrix} \right\}$$

be the classical functor which associates to a left $\mathcal{D}_X$-module a sheaf of $\mathcal{O}_{X_0/S}$-modules (which is a crystal). This functor has a right adjoint M_X. We denote respectively by $\mathbb{L}C_{X_0}$ and $\mathbb{R}M_X$ the left and right derived functors of C_{X_0} and M_X. A complex $E \in D^-(\mathcal{O}_{X_0/S})$ is called a *crystalline complex* if, for any morphism $v : (U', T') \to (U, T)$ of $\mathrm{Crys}(X_0/S)$, the canonical morphism $\mathbb{L}v^*E_T \to E_{T'}$ is an isomorphism of $D^-(\mathcal{O}_{T'})$.

Theorem 1. *The functors $\mathbb{L}C_{X_0}$ and $\mathbb{R}M_X$ induce quasi-inverse equivalences between the full subcategory of $D^{\mathrm{b}}(\mathcal{D}_X)$ whose objects have finite Tor-dimension and quasi-coherent cohomology, and the full subcategory of $D^{\mathrm{b}}(\mathcal{O}_{X_0/S})$ whose objects are crystalline complexes which have finite Tor-dimension, and quasi-coherent cohomology on every thickening in* $\mathrm{Crys}(X_0/S)$.

Without liftability assumption, we define $\mathcal{D}$-*perfect complexes* on $\mathrm{Crys}(X_0/S)$ to be complexes $E \in D^{\mathrm{b}}(\mathcal{O}_{X_0/S})$ such that there exists an open covering $(U_{0,\alpha})$ of X_0, smooth liftings (U_α) of the subschemes $U_{0,\alpha}$, perfect complexes $\mathcal{E}_\alpha \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_{U_\alpha})$ and isomorphisms $E_{|U_{0,\alpha}} \simeq \mathbb{L}C_{U_{0,\alpha}}(\mathcal{E}_\alpha)$. The previous theorem implies that, if X_0 has a smooth lifting X on S, this local condition is equivalent to the similar global condition on X.

We assume that X_0 has constant relative dimension d over S_0. For $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_X)$, we set

$$CR_{X_0}(\mathcal{E}) \;=\; \mathbb{L}C_{X_0}(\mathcal{E})[d].$$

Theorem 2. *Let $f_0 : X_0 \to Y_0$ be a smooth morphism of smooth S_0-schemes.*

(i) *If* f_0 *can be lifted as a morphism* $f : X \to Y$ *of smooth* S*-schemes, and if* $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{perf}}(\mathcal{D}_X)$*, there exists in* $D^{\mathrm{b}}(\mathcal{O}_{Y_0/S})$ *a canonical isomorphism*

$$CR_{Y_0}(f_+(\mathcal{E})) \xrightarrow{\sim} \mathbb{R}f_{0\,\mathrm{crys}\,*}(CR_{X_0}(\mathcal{E})).$$

(ii) *Without liftability assumption, but if in addition* f_0 *is proper,* $\mathbb{R}f_{0\,\mathrm{crys}\,*}$ *preserves* $\mathcal{D}$*-perfection.*

Local duality and $\mathcal{D}^\vee$-perfection. The previous theorem does not hold when f_0 is a closed immersion. To obtain a finiteness condition which is stable under direct images by arbitrary proper maps, we introduce another notion of perfection, dual to $\mathcal{D}$-perfection. We will now use filtered versions of the previous results: this will be indicated by the addition of the symbol "F" to the notations. The sheaf $\mathcal{O}_X$ is filtered by the divided powers of the ideal $p\mathcal{O}_X$, the sheaf $\mathcal{D}_X$ is endowed with the tensor product filtration of the filtration by the order of differential operators with the filtration of $\mathcal{O}_X$, and the sheaf $\mathcal{O}_{X_0/S}$ is filtered by the divided powers of its canonical PD-ideal. Our treatment of derived categories and derived functors for filtered modules is based on Laumon's constructions in [4].

If A is a filtered ring, and E a complex of filtered A-modules, we denote by E^{f} the subcomplex defined by $E^{\mathrm{f}} = \cup_{i\in\mathbb{Z}} E^i$, and we say that E is *exhaustive* if the morphism $E^{\mathrm{f}} \to E$ is a quasi-isomorphism. We endow $\mathrm{Crys}(X_0/S)$ with the dualizing complex

$$\mathcal{K}_{X_0/S} = \mathcal{O}_{X_0/S}(d)[2d],$$

and, for any exhaustive complex $E \in D^-F(\mathcal{O}_{X_0/S})$, we define

$$E^\vee = \mathbb{R}\mathcal{H}om^{\mathrm{f}}_{\mathcal{O}_{X_0/S}}(E, \mathcal{K}_{X_0/S}).$$

Theorem 3. *Let* $E \in D^{\mathrm{b}}F(\mathcal{O}_{X_0/S})$ *be a* $\mathcal{D}$*-perfect complex.*

(i) E *is exhaustive, and* $E^\vee$ *is exhaustive and bounded.*

(ii) *There exists a canonical isomorphism* $E \xrightarrow{\sim} (E^\vee)^\vee$ *in* $D^{\mathrm{b}}F(\mathcal{O}_{X_0/S})$*.*

Without liftability assumption, we can use this biduality theorem to define $\mathcal{D}^\vee$-*perfect complexes* as being exhaustive complexes such that $E^\vee$ is $\mathcal{D}$-perfect and the biduality morphism $E \to (E^\vee)^\vee$ is an isomorphism. The functor $E \mapsto E^\vee$ induces an anti-equivalence between the categories of $\mathcal{D}$-perfect and $\mathcal{D}^\vee$-perfect complexes.

Stability and duality. If X is a smooth S-scheme lifting X_0, we will define a functor $CRF^\vee_{X_0}$ on $D^{\mathrm{b}}F_{\mathrm{perf}}(\mathcal{D}_X)$ by setting $CRF^\vee_{X_0}(\mathcal{E}) = CRF_{X_0}(\mathcal{E})^\vee$.

Theorem 4. *Let* $f_0 : X_0 \to Y_0$ *be a* S_0*-morphism between proper and smooth* S_0*-schemes.*

(i) *If* f_0 *can be lifted as a morphism* $f : X \to Y$ *of proper and smooth* S*-schemes, and if* $\mathcal{E} \in D^{\mathrm{b}}F_{\mathrm{perf}}(\mathcal{D}_X)$*, there exists a canonical isomorphism*

$$CRF^\vee_{Y_0}(f_+(\mathcal{E})) \xrightarrow{\sim} \mathbb{R}f_{0\,\mathrm{crys}\,*}(CRF^\vee_{X_0}(\mathcal{E})).$$

(ii) *Without liftability assumption,* $\mathbb{R}f_{0\,\mathrm{crys}\,*}$ *preserves* $\mathcal{D}^\vee$*-perfection.*

The proof of assertion (ii) uses Theorem 2 and the following relative duality theorem, where r is the relative dimension of X_0 over Y_0.

Theorem 5. *Let $f_0 : X_0 \to Y_0$ be a proper and smooth morphism.*

(i) *There exists in $D^{\mathrm{b}}F(\mathcal{O}_{Y_0/S})$ a trace morphism*

$$\mathrm{Tr}_{f_0} : \mathbb{R}f_{0\,\mathrm{crys}\,*}(\mathcal{K}_{X_0/S}) \to \mathcal{K}_{Y_0/S},$$

compatible with the usual trace morphism $R^r f_{0*}(\Omega^r_{X_0/Y_0}) \to \mathcal{O}_{Y_0}$.

(ii) *If E is a $\mathcal{D}$-perfect complex on* $\mathrm{Crys}(\mathcal{O}_{X_0/S})$*, the natural pairing and the previous trace morphism induce a perfect duality pairing in $D^{\mathrm{b}}F(\mathcal{O}_{Y_0/S})$*

$$\mathbb{R}f_{0\,\mathrm{crys}\,*}(E) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{Y_0/S}} \mathbb{R}f_{0\,\mathrm{crys}\,*}(E^{\vee}) \to \mathcal{K}_{Y_0/S}.$$

Remarks on the logarithmic case. One can endow S and S_0 with the log structures associated to the pre-log structures defined by the morphism of monoids $\mathbb{N} \to \mathcal{O}_S$ sending 1 to 0, and work within the category of fine log schemes over S, using the PD nilpotent variant of the log crystalline cohomology defined by Kato [3]. Then, under various additional hypotheses (see in particular [5], [6]), similar results hold for smooth logarithmic schemes over S_0. In particular, one can use Theorem 4 to show that, when X_0 is the special fiber of a proper semi-stable scheme over a totally ramified extension of W, and $X_0 \hookrightarrow Y$ is a closed immersion of X_0 into a smooth usual scheme over S, the $\mathcal{D}_Y$-module constructed by Tsuji in [6] computes the Hyodo-Kato cohomology of X_0.

References

[1] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math. **407** (1974), Springer-Verlag.

[2] P. Berthelot, A. Ogus, *Notes on crystalline cohomology*, Math. Notes **21**, Princeton University Press (1978).

[3] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in *Algebraic analysis, geometry, and number theory* (edited by J.-I. Igusa), Johns Hopkins Univ. Press, Baltimore (1989), 191–224.

[4] G. Laumon, *Sur la catégorie dérivée des $\mathcal{D}$-modules filtrés*, in *Algebraic Geometry* (edited by M. Raynaud and T. Shioda), Lecture Notes in Math. **1016** (1983), 151-237, Springer-Verlag.

[5] T. Tsuji, *Poincaré Duality for Logarithmic Crystalline Cohomology*, Compositio Mathematica **118** (1999), 11-41.

[6] T. Tsuji, *On nearby cycles and $\mathcal{D}$-modules of log schemes in characteristic* $p > 0$, preprint (2008).

p-adic Rankin *L*-series

Massimo Bertolini

(joint work with Henri Darmon, Kartik Prasanna)

The general theme of this talk is the p-adic construction of rational points on elliptic curves, by means of values (or derivatives) of p-adic L-functions. The following two basic examples should be kept in mind.

(1) Rubin's formula [1], which expresses the square of the p-adic formal group logarithm of a rational point on a CM elliptic curve as a value of Katz's 2-variable p-adic L-function outside the range of p-adic interpolation. (We mention, in this connection, also the conjectural generalization of Rubin's formula obtained by Perrin-Riou [2].)
(2) The main result of [3], which identifies the p-adic logarithm of a Heegner point on an elliptic curve with multiplicative reduction at p with the first derivative of a (square-root) anticyclotomic p-adic L-function.

In this talk we present new results of a similar nature, resulting from collaborations with H. Darmon, and with H. Darmon and K. Prasanna. We also attempt to fit these results and examples in a unified setting.

Let A be a elliptic curve over $\mathbf{Q}$ of squarefree conductor N, and let p be an ordinary prime for E. Fix an imaginary quadratic field K of discriminant $-D$. Assume for simplicity that K has class number one, and that D is greater than 3 and does not divide pN.

We also make the basic assumption that the complex L-series $L(A/K, s)$ vanishes to *odd* order at $s = 1$. In this case, the Birch and Swinnerton-Dyer conjecture predicts that the Mordell-Weil group $A(K)$ contains a point of infinite order. In many instances, the theory of complex multiplication provides a construction of such a point, via a Shimura curve parametrization of A.

It will be crucial to distinguish the following two cases:

(1) Case I (the "exceptional case"): p divides exactly N (so that A has multiplicative reduction at p).
    In this case, we assume that p is inert in K.
(2) Case II (the "generic case"): p does not divide N (so that A has good ordinary reduction at p).
    In this case, we assume that p is split in K, and that all the primes dividing N are split in K.

The work of Hida [4] attaches to the triple (A, K, p) a p-adic L-function $L(A, K; k)$ in a weight variable k.

In the exceptional case, this p-adic L-function satisfies the interpolation property

$$L_p(A, K; k)^2 \doteq L(f_k \times \theta_1, k/2) \text{ for } k \in \mathbf{Z}_{\geq 2} \text{ even,}$$

where the symbol $\doteq$ denotes equality up to an explicit constant. Here, f_k is the weight k specialization of the Hida family of normalized eigenforms attached to A (thus $f_2 = f_A$ is the normalised eigenform attached to A). Furthermore, θ_1 is the canonical theta series of weight one attached to K, such that the Rankin L-series $L(f_k \times \theta_1, s)$ is equal to the L-series of the base-change of f_k to K.

In the generic case, the interpolation formula has the shape

$$L_p(A, K; k)^2 \doteq L(f_A \times \theta_k, (k+1)/2) \text{ for } k \in \mathbf{Z}_{\geq 3} \text{ odd.}$$

Here θ_k is the theta series of weight k attached to ψ^{k-1}, where ψ is the canonical Hecke character of infinity type $(1, 0)$ and minimal conductor attached to K, and $L(f_A \times \theta_k, s)$ is the Rankin convolution L-series.

Note that in both the exceptional and the generic case $L_p(A,K;k)$ should be regarded as a "square-root" p-adic L-function, since it interpolates square-roots of special values of complex L-functions.

We now state our results alluded to above.

Theorem 0.1. *Assume we are in the exceptional case. Then* $L'_p(A,K;2) = \log_A(P)$*, where* P *is a global point in* $A(K)\otimes \mathbf{Q}$ *and* $\log_A$ *denotes the formal group logarithm on* A*.*

The proof of this result in based on the Cerednik-Drinfeld theory of p-adic uniformization of Shimura curves, on the theory of complex multiplication and on Coleman's theory of p-adic integration. See [5] for details.

Theorem 0.1 is related the the second example mentioned above in the following way. The work of Hida [4] yields a 2-variable p-adic L-function $L_p(A,K;k,\ell)$ that interpolates the central critical values at $s=(k+\ell-1)/2$ attached to the products $f_k\times\theta_\ell$, for $1\le\ell\le k-1$. By setting $k=2$, one obtains the anticyclotomic p-adic L-function of example (2), whose derivative $L'_p(A,K;2,1)$ at $\ell=1$ gives rise to the logarithm of a global point.

Theorem 0.2. *Assume we are in the generic case. Then* $L_p(A,K;1)=\log_A(P)$*, where* P *is a global point in* $A(K)\otimes\mathbf{Q}$*.*

Note that the value $L_p(A,K;1)$ is outside the range of p-adic interpolation. The proof of this result, based on the theory of p-adic modular forms, can be found in [6].

What is the relation (if any) between theorem 0.2 and Rubin's formula (example (1))? It should be noted that Rubin's formula holds for an elliptic curve A with good ordinary reduction at p, having CM by the ring of integers of K. This implies that the L-series $L(A/K,s)=L(A,s)^2$ vanishes to *even* order at $s=1$, so that our basic assumption fails in this new setting. Rubin's formula can be understood in the framework of theorem 0.2 by replacing the modular form f_A (equal to θ_2 in the new setting) by a theta series θ_{2+r} of *odd* weight $2+r\ge 3$. This leads to a higher dimensional generalization of theorem 0.2, which relates the value at $r+1$ of the Hida p-adic L-function interpolating the central critical values $L(\theta_{2+r}\times\theta_{1+r+2j}, r+1+j)$, for $j\ge 1$, to the image by the p-adic Abel-Jacobi map of a $r+1$-codimensional algebraic cycle Δ. More specifically, let X_r be the $2r+1$-dimensional product variety $A^r\times W_r$, where W_r denotes the r-fold fiber product of the universal generalised elliptic curve over the modular curve on which θ_{2+r} arises. Then Δ defines an element of the Chow group $\mathrm{CH}^{r+1}(X_r)_0$ of $r+1$-codimensional cycles on X_r modulo rational equivalence, which are homologically trivial. Rubin's theorem implies that the Abel-Jacobi image of Δ in the continuous Galois cohomology group $H^1(K,V_p(A))$, $V_p(A)$ being the Tate module with $\mathbf{Q}_p$ coefficients of A, arises from a global point. The reader is referred to [6] and [7] for more details and explanations.

References

[1] K. Rubin, *p-adic L-functions and rational points on elliptic curves with complex multiplication*, Inventiones Math. **107** (1992), 323–350.
[2] B. Perrin-Riou, *Fonctions L p-adiques d'une courbe elliptique et points rationnels*, Ann. Inst. Fourier **43** (1993), 945–995.
[3] M. Bertolini, H. Darmon, *Heegner points, p-adic L-functions, and the Cerednik-Drinfeld uniformisation*, Inventiones Math. **131** (1998), 453–491.
[4] H. Hida, *Elementary theory of L-functions and Eisenstein series*, London Math. Society Student Texts **26** (1993), Cambridge University Press.
[5] M. Bertolini, H. Darmon, *Hida families and rational points on elliptic curves*, Inventiones Math. **168** (2007), 371–431.
[6] M. Bertolini, H. Darmon, K. Prasanna, *Generalised Heegner cycles and p-adic Rankin L-series*, in preparation.
[7] M. Bertolini, H. Darmon, K. Prasanna, *Exotic Heegner points on CM elliptic curves and periods of binary theta series*, in preparation.

Pseudo-reductive groups and finiteness theorems

Brian Conrad

1. Motivation

Let G be an affine algebraic group scheme over a global field k. Let $\mathbf{A}_k$ denote the locally compact adele ring of k, and consider the double coset space

$$\Sigma_{G,S,K} = G(k)\backslash G(\mathbf{A}_k)/G(k_S)K = G(k)\backslash G(\mathbf{A}_k^S)/K \tag{1.1}$$

for a finite non-empty set S of places of k that contains the archimedean places, $k_S = \prod_{v\in S} k_v$ and $\mathbf{A}_k^S$ the factor ring of adeles with vanishing component along S (so $\mathbf{A}_k = k_S \times \mathbf{A}_k^S$ as topological rings), and K a compact open subgroup of $G(\mathbf{A}_k^S)$. Such double cosets arise in many contexts in number theory, and it is an important result of Borel from the 1960's that double coset spaces as in (1.1) are always finite when k is a number field. We say that G *has finite class numbers* if $\Sigma_{G,S,K}$ is finite for all S and for all (equivalently, one) K.

Over global function fields the finiteness of class numbers for solvable G was proved by Oesterlé, and the connected reductive case can be deduced from results of Borel, Prasad, and Harder. However, the general case of function fields does not follow from such methods. For various reasons (some indicated below) it is desired to handle all affine algebraic groups, so we need a new ingredient to get beyond the usual framework of reductive groups.

2. Main finiteness results

By using the new results described in §3, we developed an entirely different approach to the problem of finiteness of class numbers, and in the number field case we get a proof which is far from the original one of Borel (i.e., no use of reduction theory). Our first main finiteness result is a generalization to all global fields (in odd characteristic) of Borel's theorem over number fields.

Theorem 2.1. *Let k be a global function field with* $\mathrm{char}(k) \neq 2$*. Every affine algebraic k-group scheme has finite class numbers.*

As an application of theorem 2.1 and the main result in §3 below, we obtain the following theorem that is an analogue of a result of Borel and Serre over number fields.

Theorem 2.2. *Let k be a global function field with* $\mathrm{char}(k) \neq 2$*, and let S be a finite (possibly empty) set of places of k. Let G be an affine k-group scheme of finite type.*

(1) *The natural localization map* $\theta_{S,G} : \mathrm{H}^1(k, G) \to \prod_{v \notin S} \mathrm{H}^1(k_v, G)$ *has finite fibers.*
(2) *Let X be a k-scheme equipped with a right action by G. For $x \in X(k)$, the set of points $x' \in X(k)$ in the same $G(k_v)$-orbit as x in $X(k_v)$ for all $v \notin S$ consists of finitely many $G(k)$-orbits.*

Our proof of Theorem 2.2 uses the known finiteness of $Ш^1_S(k, G) := \ker \theta_{S,G}$ when G is a smooth connected commutative affine k-group, which was proved by Oesterlé over all global fields by a uniform method.

Remark 2.3. The proof of Theorem 2.2(2) is an easy consequence of part (1) applied to the stabilizer group G_x that is generally not smooth. The proof of both parts of Theorem 2.2 involves reduction to cases in which all relevant group schemes are smooth. The main ingredients we use are Harder's vanishing theorem for $\mathrm{H}^1(k, G)$ for any global function field k and any (connected and) simply connected semisimple k-group G, as well as the main result in §3 below. We expect Theorems 2.1 and 2.2 to be true in characteristic 2. Ongoing work with O. Gabber and G. Prasad should settle this point.

Our ability to get *beyond* the reductive case is the key innovation in this work, and such generality is needed to even prove Theorem 2.2(2) for connected semisimple G. The work in §3 below was largely motivated by its role in the proofs of the above results.

3. Pseudo-reductive groups

The technical heart of the proofs of the preceding finiteness theorems is a structure theory for a class of groups that was first studied by Borel and Tits. What follows is joint work with Gabber and Prasad.

For an arbitrary field k, a smooth affine k-group G is *pseudo-reductive* over k if it is connected and the maximal smooth connected unipotent normal k-subgroup $\mathrm{R}_{u,k}(G)$ is trivial. In general if G is connected then $G/\mathrm{R}_{u,k}(G)$ is pseudo-reductive and the canonical exact sequence

$$1 \to \mathrm{R}_{u,k}(G) \to G \to G/\mathrm{R}_{u,k}(G) \to 1 \tag{3.1}$$

over k expresses G as an extension of a pseudo-reductive k-group by a smooth connected unipotent k-group.

By (3.1), pseudo-reductive groups naturally arise in solving rationality questions for general smooth affine groups over general fields. The main reason for interest in the structure of pseudo-reductive groups is as a very useful device for proving theorems about general affine algebraic groups (over imperfect fields) that were previously only known in the reductive case.

Example 3.1. Let k'/k be a finite extension of fields with arbitrary characteristic and let G' be a connected reductive k'-group. The affine Weil restriction $\mathrm{Res}_{k'/k}(G')$ is pseudo-reductive over k, but if k'/k is not separable and $G' \neq 1$ then it is *not* reductive.

Pseudo-reductive groups were the topic of courses by Tits and the Collège de France in 1991–92 and 1992–93. Tits attempted to classify pseudo-reductive groups over separably closed fields of positive characteristic, but he ran into many complications and so did not write up all of his results. Away from characteristic 2 we give a canonical description of pseudo-reductive groups that isolates the mysteries of the commutative case in a useful manner.

Example 3.2. Let k' be a nonzero finite reduced k-algebra and let G' be a k'-group whose fiber over each factor field of k' is an absolutely simple and connected semisimple group that is simply connected. Let $j : T' \hookrightarrow G'$ be a maximal k'-torus, $Z_{G'}$ the (scheme-theoretic) center of G', and $\overline{T}' = T'/Z_{G'}$. Suppose that there is given a factorization

$$\mathrm{Res}_{k'/k}(T') \xrightarrow{\varphi} C \to \mathrm{Res}_{k'/k}(\overline{T}') \tag{3.2}$$

of the canonical map of k-groups $\mathrm{Res}_{k'/k}(T') \to \mathrm{Res}_{k'/k}(\overline{T}')$ with C a commutative (connected) pseudo-reductive k-group; it is *not* assumed that φ is surjective (e.g., for $C = \mathrm{Res}_{k'/k}(\overline{T}')$ the map φ can fail to be surjective).

Let C act on $\mathrm{Res}_{k'/k}(G')$ on the left through conjugation by its image in $\mathrm{Res}_{k'/k}(\overline{T}')$, so there arises a semidirect product group $\mathrm{Res}_{k'/k}(G') \rtimes C$. Using the pair of homomorphisms

$$\varphi : \mathrm{Res}_{k'/k}(T') \to C, \ \ \mathrm{Res}_{k'/k}(j) : \mathrm{Res}_{k'/k}(T') \to \mathrm{Res}_{k'/k}(G'),$$

the twisted product map $\alpha : \mathrm{Res}_{k'/k}(T') \to \mathrm{Res}_{k'/k}(G') \rtimes C$ defined by $t' \mapsto (\mathrm{Res}_{k'/k}(j)(t')^{-1}, \varphi(t'))$ is readily checked to be an isomorphism onto a central subgroup. The k-group $G := \mathrm{coker}\,\alpha$ turns out to be non-commutative and pseudo-reductive over k.

The non-commutative pseudo-reductive groups arising as in Example 3.2 are called *standard*. For such G the pair $(G', k'/k)$ is uniquely determined by G up to a non-canonical k-isomorphism, and the choice of T' corresponds to a choice of maximal k-torus T in G (in which case C is the associated Cartan k-subgroup $Z_G(T)$). Our main result is:

Theorem 3.3. *Assume* $\mathrm{char}(k) \neq 2$ *and let G be a non-commutative pseudo-reductive k-group. If k is imperfect with* $\mathrm{char}(k) = 3$ *then assume* $G^{\mathrm{ss}}_{\overline{k}}$ *has no simple factor of type* G_2. *The k-group G is standard.*

For imperfect k with $\mathrm{char}(k) = 3$ we can also classify the structure of all pseudo-reductive groups G for which $G^{ss}_{\overline{k}}$ has G_2-factors. Our classification provides some new insight into Tits' results on root systems and pseudo-parabolic subgroups of pseudo-reductive groups: the root groups and minimal pseudo-parabolic subgroups in Tits' theory are obtained from (possibly non-étale) Weil restrictions of root groups and parabolic subgroups in semisimple groups over certain intrinsically associated finite extensions of k, at least away from characteristic 2.

A key difference between pseudo-reductive k-groups and connected reductive k-groups is that the Cartan k-subgroups in pseudo-reductive groups may not be tori, although they are always commutative. If $\mathrm{char}(k) \neq 2$ then a pseudo-reductive k-group is reductive if and only if its Cartan k-subgroups are tori; this is false in characteristic 2. In this sense, a basic reason that pseudo-reductive groups are more difficult to understand than connected reductive groups is that we do not understand the general structure of the commutative objects.

Coverings of p-adic period domains

Gerd Faltings

The theory of Fontaine relates crystalline p-adic Galois-representations to certain objects which typically arise as crystalline cohomology. Namely a Frobenius-crystal over an unramified field K_0 and a Hodge filtration on it defined over a finite extension K of K_0. The correspondence is fully faithful and its essential image consists of weakly admissible objects (some type of stability condition). For infinite extensions K one can still associate to certain crystals with Hodge-filtration a $\mathbb{Q}_p$-vectorspace, but it has been shown by U.Hartl that "weakly admissible" is not sufficient for that. If we fix the Frobenius crystal the possible Hodge-filtrations are parametrised by a flag-variety, the weakly admissible points form a rigid analytic subspace, and the admissible points in it an open Berkovich-subspace which has the same conventional rigid points but fewer Berkovich-points.

If the Hodge-filtrations has length one (corresponding to representations which should be associated to p-divisible groups) we show that over this subspace there exist etale local systems naturally associated to the tautological Hodge-filtration. Here "associated" is defined by generalising Fontaine's theory from discrete valuation-rings to higher dimensional p-adic domains. The proof uses a simple deformation-argument:

One has to solve certain equations. A solution at one point extends to an approximate solution in an open neighbourhood, which then can be corrected to yield an exact solution.

Finally one can show that the resulting Galois-representations arise from p-divisible groups, by reducing to the known (Breuil, Kisin) result over conventional rigid points.

For Hodge filtrations of higher length the method should work in the same way. However one needs at some stage Griffiths' transversality so there exist (like in classical Hodge theory) no universal families, and the optimal generality in which a theorem should be formulated remains unclear.

References

[1] C. Breuil *Groupes p-divisibles, groupes finies et modules filtrés*, Ann. of Math.**152** (2000), 489-549.

[2] P. Colmez, J.M. Fontaine *Construction des représentations semi-stables*, Invent. math. **140** (2000), 1-43.

[3] J.M. Fontaine *Modules galoisiennes, modules filtrés et anneaux de Barsotti-Tate*, Astérisque**223** (1994), 3-80.

[4] U. Hartl *On a conjecture of Rapoport and Zink*, preprint**152** (2006).

Reduction theory for p-divisible groups

Laurent Fargues

In [1], Gerd Faltings has shown the existence of a link between Lubin-Tate and Drinfeld towers (see [3] too). Using the link between Drinfeld's Ω space and the Bruhat-Tits building of $\mathrm{PGL}_{n/\mathbb{Q}_p}$ one can use this to define an Hecke equivariant "parametrization" by the geometric realization of this Bruhat-Tits building of the Berkovich space associated to the Lubin-Tate tower with infinite level. This parametrization has been studied in details in [4].

The Lubin-Tate tower can be seen as tubes (or p-adic Milnor fibers) over some "supersingular" points in the reduction mod p of some particular type of Shimura varieties (unitary type with signature $(1, n-1) \times (0, n) \times \cdots \times (0, n)$ at a split prime p). In fact one can extend the results of [4] to define and study an Hecke equivariant parametrization of the p-adic Berkovich analytic space associated to this Shimura variety with infinite level at p by compactifications of the preceding buildings. In this parametrization the boundary stratification of the compactified building corresponds to the Newton stratification of the Shimura variety. For $n = 2$ one finds back Lubin's theory of canonical subgroup. For general n this should be helpful to construct a theory of p-adic automorphic forms on those Shimura varieties generalizing Katz theory for modular curves (for $n = 2$ the structure of the Bruhat-Tits tree being "simple" one can compute everything, but in general the structure of the building is more complicated).

We propose a new way to stratify and define fundamental domains for the action of p-adic Hecke correspondences on more general moduli spaces (Rapoport-Zink spaces or general PEL type Shimura varieties). In [2] the author has defined Harder-Narasimhan type filtrations for finite flat group schemes over unequal characteristic complete valuation rings. Stuhler and Grayson have developed reduction theory for the action of arithmetic groups on archimedean symmetric spaces using Harder-Narasimhan filtrations for hermitian vector bundles in the sense of Arakelov geometry. We use our theory for finite flat group schemes to define a reduction theory for p-divisible groups (like reduction theory for quadratic forms) and define fundamental domains for the action of Hecke correspondences on some

Rapoport-Zink spaces and Shimura varieties. Those fundamental domains are interesting from the point of view of the associated period mapping. When one starts from a $\overline{\mathbb{Q}}$-point in our Shimura variety the associated point in the fundamental domain is a point in the Hecke orbit where the Faltings height of the associated abelian variety is minimized in its p-isogeny class.

References

[1] G. Faltings. A relation between two moduli spaces studied by V. G. Drinfeld. In *Algebraic number theory and algebraic geometry*, volume 300 of *Contemp. Math.*, pages 115–129, 2002.

[2] L. Fargues. La filtration de Harder-Narasimhan des schmas en groupes finis et plats. *Prpublication, http://www.math.u-psud.fr/˜fargues/Prepublications*, 2008.

[3] L. Fargues. L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques. In *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in math., 262, pages 1–325. Birkhuser, 2008.

[4] Laurent Fargues. Application de Hodge-Tate duale d'un groupe de Lubin-Tate, immeuble de Bruhat-Tits du groupe linéaire et filtrations de ramification. *Duke Math. J.*, 140(3):499–590, 2007.

Finiteness theorems in étale cohomology

Ofer Gabber

Theorem 1 [1]. *Let ℓ be a prime and $f : X \to Y$ a morphism of finite type between quasi-excellent noetherian $\mathbb{Z}[1/\ell]$-schemes and F a constructible sheaf of $\mathbb{Z}/\ell^r$-modules on X_{et}. Then the sheaves $R^i f_* F$ on Y_{et} are constructible and vanish except for finitely many i's.*

There are analogues of theorem 1 for non-abelian coefficients. For direct images of sheaves of sets the quasi-excellence condition is not needed. The conclusion of theorem 1 also holds for morphisms between S-schemes of finite type where S is a noetherian $\mathbb{Z}[1/\ell]$-scheme of dimension ≤ 1 (this is reduced to results of [3]). If X is normal excellent and $j : U \to X$ an open immersion with $\mathrm{cod}(X - U, X) \geq 2$ one can show by another method (which uses ultraproducts) that the sheaves $R^1 j_* G$ are constructible for every finite group G.

Let S be a noetherian scheme and $\mathcal{C}$ the category of reduced S-schemes of finite type $X \xrightarrow{f} S$ such that f sends maximal points to maximal points and is generically finite. The alteration topology is the Grothendieck topology on $\mathcal{C}$ defined by the pretopology generated by Zariski covers and proper surjective maps. A family $\{X_i \to X\}$ in $\mathcal{C}$ is covering for the alteration topology iff it is covering for the Voevodsky h-topology. For a prime ℓ, the ℓ'-alteration topology on $\mathcal{C}$ is the Grothendieck topology generated by

(i) families of étale morphisms $\{X_i \to X\}$ such that every $x \in X$ is the image of some point $y \in X_i$ with $[\kappa(y) : \kappa(x)]$ prime to ℓ,

(ii) proper surjective maps $X' \to X$ satisfying the same condition for maximal points of X.

Theorem 2. *Let X be a reduced quasi-excellent noetherian scheme and $Z \subset X$ a nowhere dense closed subset. Then there is a covering family $\{X_i \to X\}$ for the alteration topology such that each X_i is regular and the inverse image of Z in X_i is the support of a normal crossings divisor.*

The conclusion of theorem 2 can be reformulated in terms of points of the alteration topology, which correspond to valuations. There is a refined form with the ℓ'-alteration topology when a prime ℓ is invertible. With further work one can add the condition that the $X_i \to X$ are generically étale.

One proof of a part of theorem 1 (constructibility for each i) uses theorem 2 and cohomological descent for oriented product toposes, and the proof of the full result uses the refined form of theorem 2. We discussed the use of reduction to the complete case (Artin-Popescu approximation) and Epp's theorem in the proof of theorem 2, and a modification theorem (theorem 3) which is used in the proof of the refined form.

Let X be a locally noetherian scheme and $Z \subset X$ a closed subset. We say that (X, Z) is log-regular if X is log-regular when equipped with some fs (étale) log-structure with locus of triviality $X - Z$. The log-structure is then given by the subsheaf of $\mathcal{O}_X$ of functions invertible outside Z.

Theorem 3. *Let (X, Z) be a noetherian log-regular scheme equipped with an action of a finite group G which is generically free, admissible (so that $p : X \to X/G$ exists) and such that for every $x \in X$ the order of the inertia group G_x is invertible in $\kappa(x)$. Then there is a projective birational map $q : Y' \to X/G$ such that (Y', Z') is log-regular where $Z' = q^{-1}p$ ($Z \cup$ (locus of non-trivial inertia)).*

This requires some form of canonical desingularization in characteristic zero.

As a consequence of theorem 3 (version with log-smoothness over a base) and de Jong's results one has

Theorem 4. *Let ℓ be a prime and R an excellent Dedekind ring over $\mathbb{Z}[1/\ell]$ and X a separated integral flat R-scheme of finite type. Then there is an alteration $q : X' \to X$ of generic degree prime to ℓ (generically étale if R is a perfect field), with X' regular integral quasi-projective over R, such that X' has an fs log-structure such that there is a log-smooth R-morphism $X' \to \mathrm{Spec}(R')$ where R' is the normalization of R in a finite extension of* $\mathrm{Frac}(R)$ *and the log-structure of* $\mathrm{Spec}(R')$ *is defined by a finite set of codimension one points.*

Theorem 4 is used in a new proof of absolute cohomological purity in the mixed characteristic case.

We mentioned cohomological dimension results

(i) Affine Lefschetz: If R is an excellent strictly henselian noetherian local ring with $1/\ell \in R$, the ℓ-cohomological dimension of affine open subschemes of $\mathrm{Spec}(R)$ is $\leq \dim(R)$.

(ii) For R as above a domain, $cd_\ell(\mathrm{Frac}(R)) = \dim(R)$ holds without the excellence assumption.
(iii) Kato's conjecture on p-cohomological dimension [2].

One can prove the existence of dualizing complexes adapted to dimension functions [4].

References

[1] O. Gabber, *Finiteness theorems for étale cohomology of excellent scheme*, lecture in Deligne's conference, Princeton (2005).
[2] O. Gabber and F. Orgogozo, *Sur la p-dimension des corps*, Invent. math. **174** (2008), 47–80.
[3] P. Deligne, *Théorèmes de finitude en cohomologie ℓ-adique*, in SGA $4\frac{1}{2}$, LNM **569** (1977).
[4] J. Riou, *Dualité (d'après Ofer Gabber)*, preprint, 2007.

Equidistribution of small points

Walter Gubler

Let K be a number field or a function field (of any dimension) over the constant field k (of any characteristic). We consider the Néron–Tate height $\widehat{h}$ on an abelian variety A over K. For a closed subvariety X of A, the Bogomolov conjecture claims that there is a well-understood exceptional set E in X such that $X(\overline{K}) - E$ is discrete with respect to the semidistance given by the positive semidefinite quadratic form $\widehat{h}$. In case of number fields, this was proved by Ullmo [8] for curves inside the Jacobian and by Zhang [10] in general.

Theorem 1. *Gu2 The Bogomolov conjecture holds for abelian varieties over function fields which are totally degenerate with respect to some place.*

The proofs of the Bogomolov conjecture rely on the equidistribution of small points. The original equidistribution theorem of Szpiro–Ullmo–Zhang was recently generalized by Yuan in the following way: Let (P_m) be a generic net in the in the d-dimensional projective variety X such that $\lim_m h(P_m) = \frac{1}{(d+1)\deg_L(X)} h(X)$, where the heights are with respect to an ample line bundle L endowed with semipositive admissible metrics $\| \ \|_v$. The absolute Galois group G of the algebraic closure $\overline{K}$ over K acts on $X(\overline{K})$. We fix a place v of K to get a discrete probability measure μ_m on X_v^{an} which is supported and equidistributed on the Galois orbit GP_m.

Theorem 2. *The measures μ_m converge weakly to the regular probability measure $\frac{1}{\deg_L(X)} c_1(L, \| \ \|_v)^{\wedge d}$ on X_v^{an}.*

Yuan [9] proved the number field case. For a non-archimedean place v, X_v^{an} is the Berkovich analytic space associated to X and the limit is given by Chambert-Loir's measures. In [6], the function field case of Theorem 2 is proved. The special case $h(X) = 0$ was shown independently by Faber [3].

This theorem is only useful for applications if we have a good understanding of the limit measure. This poses no special problem in the archimedean case as we get volume forms on the complex space $X_v^{\rm an}$. We attack now the non-archimedean case in the following special case relevant for the Bogomolov conjecture:

Let v be a discrete valuation on the field K and let X be a closed d-dimensional subvariety of the abelian variety A over K. We assume for simplicity that X has a strictly semistable model $\mathfrak{X}$. Then Berkovich has defined the skeleton $S(\mathfrak{X})$ as a compact subset of $X_v^{\rm an}$ (see [1], [2]). It is a proper deformation retraction of $X_v^{\rm an}$. Moreover, it is a union of canonical simplices Δ_S corresponding to the strata S of the special fibre of $\mathfrak{X}$. This is a higher dimensional generalization of the dual graph. Let b be the dimension of the abelian variety of good reduction in the Raynaud extension of A.

Theorem 3 ([6])**.** *There is $e \in \{0, \min(b,d)\}$ and a list $(\Delta_S)_{S\in I}$ of canonical simplices such that*

a) *all the simplices of the list have dimension $\geq d-b$ and the maximal ones have dimension $d-e$;*
b) *for any ample line bundle L on A endowed with the canonical metric $\| \ \|_{\rm can}, v$, Chambert-Loir's measure $\mu := c_1(L|_X, \| \ \|_{\rm can}, v)^{\wedge d}$ is supported in $\bigcup_{S\in I}\Delta_S$.*
c) *For any $S \in I$, the restriction of μ to the relative interior of Δ_S is a positive multiple r_S of the relative Lebesgue measure μ_S and we have*

$$\mu = \sum_{S\in I} r_S \mu_S.$$

Using the Raynaud extension of A, one can define the tropical variety associated to X which turns out to be a polytopal subset of pure dimension $d-e$ (see [6]). This is based on an analytic generalization of tropical algebraic geometry (see [4]).

If A is totally degenerate at the place v, then we have $b = 0$ by definition and hence $e = 0$. Then the supporting simplices of μ in Theorem 3 are all of dimension d. Using a tropical adaption of Zhang's original proof, we can deduce Theorem 1 (see [5], Section 6).

For an application, we consider a discrete valuation v on the number field or function field K. Let K^{nr} be a maximal algebraic extension of K which is unramified over v.

Theorem 4 ([5])**.** *Let A be an abelian variety over K which is totally degenerate at v. Then the following two statements hold:*

a) *The set of torsion points in $A(K^{nr})$ is finite.*
b) *The Néron–Tate height $\widehat{h}$ has a positive lower bound on $A(K^{nr}) - A_{\rm tors}$.*

The proof uses Theorem 1 and a tropical version of the equidistribution theorem.

References

[1] V.G. Berkovich, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), 1–84.
[2] V.G. Berkovich, *Smooth p-adic analytic spaces are locally contractible. II*, Adolphson, Alan (ed.) et al., Geometric aspects of Dwork theory. Vol. I. Berlin: de Gruyter (2004). 293–370.

[3] X.W.C. Faber: *Equidistribution of dynamically small subvarieties over the function field of a curve*, Preprint available at arXiv:math.NT:0801.4811v2.
[4] W. Gubler, *Tropical varieties for non-archimedean analytic spaces*, Invent. Math. **169** (2007), 321–376.
[5] W. Gubler, *The Bogomolov conjecture for totally degenerate abelian varieties*, Invent. Math. **169** (2007), 377–400.
[6] W. Gubler: *Equidistribution over function fields*, Preprint available at arXiv:math.NT:0801.4508v3.
[7] W. Gubler, *Non-archimedean canonical measures on abelian varieties*, Preprint available at arXiv:math.NT:0801.4503v2.
[8] E. Ullmo, *Positivité et discrétion des points algébriques des courbes*, Ann. Math. (2) **147** (1998), 167–179.
[9] X. Yuan, *Positive line bundles over arithmetic varieties*, Preprint available at arXiv:math.NT:0612424v1.
[10] S. Zhang, *Equidistribution of small points on abelian varieties*, Ann. of Math. **147** (1998), 159-165.

The image of the Rapoport-Zink period morphism

Urs Hartl

Fix a Barsotti-Tate group $\mathbb{X}$ over $\mathbb{F}_p^{\rm alg}$ of height h and dimension d. Let $W := W(\mathbb{F}_p^{\rm alg})$ be the ring of Witt vectors and let $K_0 := W[\frac{1}{p}]$. Let (D,φ_D) be the covariant F-isocrystal of $\mathbb{X}$, let $\mathcal{F} := \mathrm{Grass}(h-d;D)$ be the Grassmannian of $(h-d)$-dimensional subspaces of D, and let $\mathcal{F}^{\rm an}$ be the K_0-analytic space associated with $\mathcal{F}$ in the sense of Berkovich [2, 3]. A point $L_K \in \mathcal{F}^{\rm an}$ with values in a complete, rank one valued field extension K of K_0 is viewed as a K-subspace $L_K \subset D_K := D\otimes_{K_0} K$. One defines the *Newton slope* $t_N(D,\varphi_D,L_K) := \mathrm{ord}_p(\det\varphi_D)$ and the *Hodge slope* $t_H(D,\varphi_D,L_K) := \dim_K L_K - \dim_K D_K$. Following Fontaine [4] and Rapoport-Zink [10, 1.18], the point $L_K \in \mathcal{F}$ is called *weakly admissible* if

$$t_N(D,\varphi_D,L_K) \;=\; t_H(D,\varphi_D,L_K) \;=\; -d \quad \text{and}$$

$$t_N(D',\varphi_D|_{D'},L_K\cap D'_K) \;\geq\; t_H(D',\varphi_D|_{D'},L_K\cap D'_K)$$

for all φ_D-stable K_0-subspaces $D'\subset D$. There is the following

Theorem 1. *(Rapoport-Zink* [10, Proposition 1.36]*, see also* [6, Proposition 1.3]*) The set $\mathcal{F}^{\rm an}_{wa} := \{\, L_K \in \mathcal{F}^{\rm an} : L_K$ is weakly admissible$\}$ is an open K_0-analytic subspace of $\mathcal{F}^{\rm an}$.*

The space $\mathcal{F}^{\rm an}_{wa}$ is an example for the *p-adic period domains* constructed more generally in [10, Proposition 1.36] for arbitrary filtered isocrystals. Likewise Rapoport and Zink have constructed moduli spaces for Barsotti-Tate groups isogenous to $\mathbb{X}$ in the following way. Let $\mathcal{N}ilp_W$ be the category of W-schemes on which p is locally nilpotent. For an $S\in\mathcal{N}ilp_W$ we set $\bar S := \mathrm{V}(p)\subset S$.

Theorem 2. *(Rapoport-Zink* [10, Theorem 2.16]*) The functor $\mathcal{G} : \mathcal{N}ilp_W \to \mathcal{S}ets$*

$$S \;\longmapsto\; \big\{\textit{isomorphism classes of pairs } (X,\rho) \textit{ where } X \textit{ is a Barsotti-Tate group over } S \textit{ and } \rho : \mathbb{X}_{\bar S}\to X_{\bar S} \textit{ is a quasi-isogeny}\big\}$$

is pro-representable by a formal scheme locally formally of finite type over W.

With $\mathcal{G}$ one can associate a K_0-analytic space $\mathcal{G}^{\mathrm{an}}$. Rapoport and Zink [10, 5.16] construct a period morphism $\breve{\pi}_1^{\mathrm{an}} : \mathcal{G}^{\mathrm{an}} \to \mathcal{F}_{wa}^{\mathrm{an}}$ as follows. By the theory of Grothendieck-Messing [9], the universal Barsotti-Tate group X over $\mathcal{G}$ gives rise to an extension

$$0 \longrightarrow (\mathrm{Lie}\, X^\vee)^\vee_{\mathcal{G}^{\mathrm{an}}} \longrightarrow \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\mathrm{an}}} \longrightarrow \mathrm{Lie}\, X_{\mathcal{G}^{\mathrm{an}}} \longrightarrow 0$$

of locally free sheaves on $\mathcal{G}^{\mathrm{an}}$, where $\mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\mathrm{an}}}$ is the crystal of Grothendieck-Messing evaluated on $\mathcal{G}^{\mathrm{an}}$. The quasi-isogeny $\rho : \mathbb{X}_{\bar{\mathcal{G}}} \to X_{\bar{\mathcal{G}}}$ induces by the crystalline nature of $\mathbb{D}(\,.\,)$ an isomorphism $\mathbb{D}(\rho)_{\mathcal{G}^{\mathrm{an}}} : \mathbb{D}(\mathbb{X})_{\mathcal{G}^{\mathrm{an}}} \xrightarrow{\sim} \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\mathrm{an}}}$ and the preimage $\mathbb{D}(\rho^{-1})_{\mathcal{G}^{\mathrm{an}}}(\mathrm{Lie}\, X^\vee)^\vee_{\mathcal{G}^{\mathrm{an}}}$ defines a $\mathcal{G}^{\mathrm{an}}$-valued point of $\mathcal{F}^{\mathrm{an}}$. By [10, 5.27] the induced morphism $\mathcal{G}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}$ factors through $\mathcal{F}_{wa}^{\mathrm{an}}$. This is the period morphism $\breve{\pi}_1^{\mathrm{an}}$.

In our talk we determined the image of $\breve{\pi}_1^{\mathrm{an}}$ as follows. In a construction similar to Berger's [1, §II] we associated with each K-valued point $L = L_K \in \mathcal{F}^{\mathrm{an}}$ a φ-module $\mathcal{M}_L$ over the ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ ("the algebraic closure of the Robba ring") associated with K (see also [6, Proposition 4.1]).

Theorem 3. *(Hartl* [6, Theorem 5.2]*) The subset*

$$\mathcal{F}_a^{\mathrm{an}} \;:=\; \{\, L \in \mathcal{F}^{\mathrm{an}} : \mathcal{M}_L \textit{ is isoclinic of slope zero}\,\}$$

is an open K_0-analytic subspace of $\mathcal{F}_{wa}^{\mathrm{an}}$.

In general the inclusion $\mathcal{F}_a^{\mathrm{an}} \subset \mathcal{F}_{wa}^{\mathrm{an}}$ is strict. We gave an example with $h = 5, d = 3$ for this fact; see [6, 5.4].

Theorem 4. *(Hartl* [5, Theorem 3.5]*)*
$\mathcal{F}_a^{\mathrm{an}}$ is the image of the period morphism $\breve{\pi}_1^{\mathrm{an}} : \mathcal{G}^{\mathrm{an}} \to \mathcal{F}_{wa}^{\mathrm{an}}$.

Finally we would like to mention that the ideas for the above results were inspired by our analogous theory in equal characteristic [7], see also our dictionary [8].

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References

[1] L. Berger: *Équations différentielles p-adiques et (φ, N)-modules filtrés*, Preprint on `arXiv:math.NT/0406601`, version from June 29th, 2004.

[2] V.G. Berkovich: *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs **33**, Amer. Math. Soc., Providence 1990.

[3] V.G. Berkovich: Étale cohomology for non-Archimedean analytic spaces, *Publ. Math. I.H.E.S.* **78** (1993), 5–161.

[4] J.-M. Fontaine: Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in *Journées de géometrie algebrique de Rennes (III), Astérisque* **65** (1979), 3–80.

[5] U. Hartl: On period spaces for p-divisible groups, to appear in *Comptes Rendus Math.*, Acad. Sci. Paris, Ser. I (2008), also available at `arXiv:0709.3444`.
[6] U. Hartl: *On a Conjecture of Rapoport and Zink*, Preprint on `arXiv:math.NT/0605254`, version from May 10, 2006.
[7] U. Hartl: *Period Spaces for Hodge Structures in Equal Characteristic*, Preprint on `arXiv:math.NT/0511686`.
[8] U. Hartl: *A Dictionary between Fontaine-Theory and its Analogue in Equal Characteristic*, Preprint on `arXiv:math.NT/0607182`.
[9] W. Messing: *The Crystals Associated to Barsotti-Tate Groups*, LNM **264**, Springer-Verlag, Berlin etc. 1972.
[10] M. Rapoport, T. Zink: *Period Spaces for p-divisible Groups*, Ann. Math. Stud. **141**, Princeton University Press, Princeton 1996.

A p-adic Borel regulator

Annette Huber
(joint work with Guido Kings)

1. Introduction

The aim of the project is to develop a p-adic version of the theory that led to Borel's computation of ranks of K-groups of number fields and the regulator formula for Dedekind-Zeta-functions up to rational factors. This follows the philosophy of Bloch and Kato, who conjecture such formulae for special values of L-functions of motives at integral points up to sign.

2. Definition

Let p be a fixed prime, $K/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_K$.

Definition 2.1. Let $\mathcal{G}$ be a K-Lie group with K-Lie algebra $\mathfrak{g}$. The morphism

$$\mathcal{L} : H^i_{\mathrm{an}}(\mathcal{G}, K) \to H^i(\mathfrak{g}, K)$$
$$f_1 \otimes \cdots \otimes f_i \mapsto (df_1)_e \wedge \cdots \wedge (df_i)_e$$

is called *Lazard morphism*. Here H^i_{an} denotes cohomology with locally analytic cochains.

For $K = \mathbb{Q}_p$ and suitable $\mathcal{G}$ this can be proved to be the same morphism as considered by Lazard, [2] chapter V.

Proposition 2.2. *Let $G/\mathcal{O}_K$ be a smooth group scheme with G_K connected. Let $\mathcal{G} \subset G(\mathcal{O}_K)$ a sub-Lie group. Then the Lazard morphism is an isomorphism.*

Proof. We reduce to Lazard's case. □

We now consider the composition

$$H^{2n-1}(\mathfrak{gl}_N, K) \cong H^{2n-1}_{\mathrm{an}}(\mathrm{GL}_N(\mathcal{O}_K), K) \to H^{2n-1}(\mathrm{GL}_N(\mathcal{O}_K), K) .$$

For $N \geq n$, there is a distinguished element p_n on the left-hand side, the suitably normalized primitive element. These elements are compatible for varying N and the usual $\mathrm{GL}_N \to \mathrm{GL}_{N+1}$.

Definition 2.3. The image $r_p \in H^{2n-1}(\mathrm{GL}_N(\mathcal{O}_K), K)$ of p_n under the above composition is called *p-adic Borel regulator.*

Elements in cohomology can be viewed as maps on homology. For N big enough, algebraic K-theory embeds into group homology; hence this is indeed a regulator.

3. Relation to Tamagawa numbers

In the spirit of Borel's regulator computation, we want to evaluate top degree Lie algebra cohomology classes on fundamental classes in group homology.

By the work of Lazard ([2] V.2.5), torsion free p-adic Lie groups are Poincaré groups with trivial dualizing module. This implies an isomorphism

$$\mathbb{Z}_p \cong \mathrm{Hom}(H^0_{\mathrm{an}}(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p) \cong H^d_{\mathrm{an}}(\mathcal{G}, \mathbb{Z}_p)$$

where $\mathcal{G}$ has dimension d. We call the image $\eta_{\mathcal{G}} \in H^d_{\mathrm{an}}(\mathcal{G}, \mathbb{Z}_p)$ of 1 the *fundamental class* of $\mathcal{G}$.

In particular, let $G/\mathbb{Z}_p$ be a smooth group scheme with $G_{\mathbb{Q}_p}$ connected reductive and $\mathcal{G} \subset G(\mathbb{Z}_p)$ torsion free. Let ω be a non-vanishing invariant d-form on G. It defines a cohomology class

$$[\omega] \in H^d_{\mathrm{dR}}(G_{\mathbb{Q}_p}) \cong H^d(\mathfrak{g}, \mathbb{Q}_p) \cong H^d_{\mathrm{an}}(\mathcal{G}, \mathbb{Q}_p)$$

Its image can be written as $\mathrm{per} \cdot \eta_{\mathbb{G}}$ with a period number $\mathrm{per} \in \mathbb{Q}_p$.

Definition 3.1.

$$\int_{G(\mathbb{Z}_p)} [\omega] = [G(\mathbb{Z}_p) : \mathcal{G}]\, |\mathrm{per}|_p$$

Proposition 3.2. *If G is abelian (i.e. a torus), then*

$$\int_{G(\mathbb{Z}_p)} [\omega] := \tau_\omega\, (G(\mathbb{Z}_p))$$

where τ_ω is the local Tamagawa measure on $G(\mathbb{Q}_p)$ attached to ω.

4. Relation to the Bloch-Kato conjecture

We return to $K/\mathbb{Q}_p$ a finite extension, $\mathcal{O}_K$ its ring of integers. In the formulation of the Bloch-Kato conjecture, the Soulé regulator is used, i.e. the Chern class in étale cohomology. By definition

$$r_p^{\mathrm{Sou}} : K_{2n-1}(K) \to H^1_{\mathrm{Gal}}(K, \mathbb{Q}_p(n))$$

is given by a compatible system of elements in $H^{2n-1}(\mathrm{GL}_N(\mathcal{O}_K), H^1_{\mathrm{Gal}}(K, \mathbb{Q}_p(n))$. Recall the Bloch-Kato exponential $\exp : K \to H^1_{\mathrm{Gal}}(K, \mathbb{Q}_p(n))$, which is also used in the formulation of the Bloch-Kato conjecture. It is an isomorphism for $n \geq 2$.

Theorem 4.1 ([1] Theorem 1.3.2)**.** *Let $N \geq n \geq 2$. Then the Soul'e regulator is given by the p-adic Borel regulator, i.e., under* exp

$$H^{2n-1}(\mathrm{GL}_N(\mathcal{O}_K), K) \cong H^{2n-1}(\mathrm{GL}_N(\mathcal{O}_K), H^1_{\mathrm{Gal}}(K, \mathbb{Q}_p(n))$$
$$r_p \mapsto r_p^{\mathrm{Sou}}$$

In particular, this means that the Soulé regulator is continuous (even analytic) in the sense that it is given by continuous cochains in group cohomology.

Idea of proof: We replace étale cohomology by a version of rigid syntomic cohomology. This is viewed as a p-adic analogue of absolute Hodge cohomology. Hence the theorem is a p-adic analogue of Beilinson's comparison of the Beilinson regulator (chern class in absolute Hodge cohomology) and the Borel regulator. The same arguments go through. The above mentioned analyticity is a non-formal ingredient that has to be checked directly. □

References

[1] A. Huber, G. Kings, *A p-adic analogue of the Borel regulator and the Bloch-Kato exponential*, Preprint 2006, revised 2007.

[2] M. Lazard, *Groupes analytiques p-adiques*, Publ. IHES **26** (1965).

Semistable reduction for overconvergent F-isocrystals

Kiran S. Kedlaya

We report on results from our recently completed series of papers [3, 4, 5, 6]. Let k be a field of characteristic $p > 0$. Let K be a complete discretely valued field of characteristic 0 with residue field k, equipped with a continuous endomorphism $\sigma_K : K \to K$. Using this data, for any open immersion $X \hookrightarrow Y$, we have a category F-$\mathrm{Isoc}^\dagger(X, Y)$ of *F-isocrystals on X overconvergent within Y*. For $X = Y$, we abbreviate to F-$\mathrm{Isoc}(X)$ and call these *convergent F-isocrystals*; for Y proper, we abbreviate to F-$\mathrm{Isoc}^\dagger(X)$ and call these *overconvergent F-isocrystals* (the choice of Y not mattering in this case). If X carries a log-structure, we also have a category of *convergent log-F-isocrystals*; if X is smooth and the log-structure is the canonical one associated to a normal crossings divisor Z, we denote this category by F-$\mathrm{Isoc}((X, Z))$. In this case, we can restrict to objects with nilpotent residues; we denote the resulting subcategory by F-$\mathrm{Isoc}^{\mathrm{nil}}((X, Z))$.

Our main theorem is the following [6, Theorem 2.4.4]; this answers a conjecture of Shiho [8, Conjecture 3.1.8]. (In fact, [6, Theorem 2.4.4] also includes an analogous assertion for partially overconvergent F-isocrystals, which we omit.)

Theorem 1 (Global semistable reduction)**.** *Let X be a smooth k-variety, and take $\mathcal{E} \in F$-$\mathrm{Isoc}^\dagger(X)$. Then there exists an alteration $f : X' \to X$ (in the sense of de Jong* [2]*) and an open immersion $X' \hookrightarrow \overline{X}'$ with $\overline{X}'$ proper over k and $Z' = \overline{X}' - X'$ a normal crossings divisor, with the following property: the pullback $f^*\mathcal{E}$ is the restriction of an object in F-$\mathrm{Isoc}^{\mathrm{nil}}((\overline{X}', Z'))$.*

The resulting object of $F\text{-Isoc}^{\text{nil}}((\overline{X}', Z'))$ is unique if it exists, because the restriction functor to $F\text{-Isoc}^{\dagger}(X')$ is fully faithful [3, Theorem 6.4.5]. Moreover, the existence of the extending object in $F\text{-Isoc}^{\text{nil}}((\overline{X}', Z')$ can be checked in codimension 1 on $\overline{X}'$ [3, Theorem 6.4.5].

Using the compactness of Zariski-Riemann spaces [4, Proposition 3.3.4], Theorem 1 can be reduced to the following.

Theorem 2 (Local semistable reduction)**.** *Let X be a smooth irreducible k-variety, let v be a valuation on $k(X)$ trivial on k, and take $\mathcal{E} \in F\text{-Isoc}^{\dagger}(X)$. Then there exists an irreducible alteration $f : X' \to X$ and an open immersion $X' \hookrightarrow \overline{X}'$ with $\overline{X}'$ proper over k and $Z' = \overline{X}' - X'$ a normal crossings divisor, with the following property: for some open $U \subseteq \overline{X}'$ on which some extension of v to $k(X')$ is centered, the restriction to $F\text{-Isoc}^{\dagger}(U \cap X', U)$ of $f^*\mathcal{E}$ is the restriction of an object in $F\text{-Isoc}^{\text{nil}}((U, U \cap Z'))$.*

Furthermore, it suffices to check Theorem 2 in case k is algebraically closed, v has real rank 1, and the residue field of the valuation ring of v equals k (see [4, §4]).

We prove Theorem 2 by induction on the *corank* of v, which under our hypotheses is just $\dim(X)$ minus the rational rank of v. Abhyankar's inequality [4, Theorem 2.5.2] implies that the corank is nonnegative, and that the case of corank 0 is particularly simple: such valuations (*Abhyankar valuations*) can be described in suitable local coordinates $x_1, \ldots, x_n$ in which $v(x_1), \ldots, v(x_n)$ are linearly independent over $\mathbb{Q}$ and generate the value group. For such valuations, we may obtain Theorem 2 using a generalization of the theory of (F, ∇)-modules over the Robba ring, together with a numerical argument involving an analogue of the Swan conductor for overconvergent F-isocrystals. See [5].

To complete the proof of Theorem 2, we must use a different argument because no local uniformization exists which is simple enough to allow for suitable p-adic analytic arguments. Instead, we induct on corank. Given v of positive corank, we construct a fibration $X \to X_1$ in curves, such that the restriction v_1 of v to $k(X_1)$ has lower corank. We then view v as a real valuation on $\ell[x]$, for ℓ the completion of $k(X_1)$ with respect to v_1 and x restricting to a local parameter of the center of v within its fibre. We identify v with a point in the Berkovich affine line over ℓ; under the tree structure on this Berkovich space, v corresponds to an end, while all other points on a path leading to v correspond to valuations of lower corank. We show (using new results on p-adic differential equations from [7]) that local semistable reduction at v can be reduced to another point on the path, to which the induction hypothesis applies. See [6].

We note that Theorem 1 has strong consequences in the theory of coefficients in p-adic cohomology. For instance, Caro and Tsuzuki [1] have shown that overconvergent F-isocrystals belong to the category of overholonomic arithmetic $\mathcal{D}$-modules; this implication is a key step in Caro's proof that the latter category is stable under the expected cohomological operations.

References

[1] D. Caro and N. Tsuzuki, Overholonomicity of overconvergent F-isocrystals over smooth varieties, arXiv preprint `0803.2105v1` (2008).
[2] A.J. de Jong, Smoothness, semi-stability and alterations, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93.
[3] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, I: Unipotence and logarithmic extensions, *Compos. Math.* **143** (2007), 1164–1212.
[4] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, II: A valuation-theoretic approach, *Compos. Math.* **144** (2008), 657–672.
[5] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, III: Local semistable reduction at monomial valuations, arXiv preprint `math/0609645v3` (2008); to appear in *Compos. Math.*
[6] K.S. Kedlaya, Semistable reduction for overconvergent F-isocrystals, IV: Local semistable reduction at nonmonomial valuations, arXiv preprint `0712.3400v2` (2008).
[7] K.S. Kedlaya, p-adic differential equations (version of 29 Jul 08), course notes available at `http://math.mit.edu/~kedlaya/papers/`.
[8] A. Shiho, Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, *J. Math. Sci. Univ. Tokyo* **9** (2002), 1–163.

p-adic elliptic polylog and Katz Eisenstein measure

Guido Kings
(joint work with Kenichi Bannai)

The specializations of the motivic elliptic polylogarithm on the universal elliptic curve to the modular curve are referred to as Eisenstein classes. In this talk, we explain that the syntomic realization of the Eisenstein classes restricted to the ordinary locus of the modular curve may be expressed using p-adic Eisenstein series of negative weight, which are p-adic modular forms defined using the two-variable p-adic measure with values in p-adic modular forms constructed by Katz. This answers a question raised by Beilinson and Levin and is a key step towards Perrin-Riou's p-adic Beilinson conjecture for Hecke characters over imaginary quadratic fields for ordinary p. The reference for this result is [1].

Consider the universal elliptic curve $\pi : E \to M$ over the moduli scheme M with full level-N-structure, $N \geq 3$ and let p be a prime number, which does not divide N.

Let $H^{k+1}_{mot}(E^k, \mathbb{Q}(k+1))(\varepsilon)$ be the ε-eigen part of the motivic cohomology group of $H^{k+1}_{mot}(E^k, \mathbb{Q}(k+1))$.

Recall from [2] 6.4.3. that for each non-zero torsion point $t \in E(M)$ the motivic elliptic polylog gives a class

$$t^* pol^{k+1}_{mot} \in H^{k+1}_{mot}(E^k, \mathbb{Q}(k+1))(\varepsilon).$$

Let $\varphi = \sum a_t t$ be a formal linear combination of non zero torsion sections $t \in E_{tors}(M)$ with coefficients in $a_t \in \mathbb{Q}$, then we define the *motivic Eisenstein class* to be

$$Eis^{k+2}_{mot}(\varphi) := \sum_{t \in E[N]-\{0\}} a_t t^* pol^{k+1}_{mot} \in H^{k+1}_{mot}(E^k, \mathbb{Q}(k+1))(\varepsilon).$$

This Eisenstein class was used to prove the Beilinson conjecture for CM elliptic curves, modular forms and Dirichlet L-functions by Bloch, Beilinson, Deninger, Scholl and Huber/Kings. For this one needs an explicit knowledge of the image of the Eisenstein class under the regulator to Deligne cohomology. To obtain similar results for the p-adic Beilinson conjecture of Perrin-Riou, one needs an explicit formula for the syntomic realization.

Our main theorem concerns the description (on the ordinary locus) of the image of this Eisenstein class under the syntomic regulator

$$r_{syn} : H^{k+1}_{mot}(E^k, \mathbb{Q}(k+1))(\varepsilon) \to H^1_{syn}(M, Sym^k H(1)),$$

here H is the filtered overconvergent F-isocrystal $R^1\pi_*\mathbb{Q}_p(1)$. This syntomic cohomology group has the following description: Let $[\alpha] \in H^1_{syn}(M, Sym^k H(1))$. Then $[\alpha]$ is given uniquely by pairs of sections (α, ξ) for $\alpha \in \Gamma(\overline{M}_{\mathbb{Q}_p}, Sym^k H(1)_{rig})$ and $\xi \in \Gamma(\overline{M}_{\mathbb{Q}_p}, F^{-1} Sym^k H \otimes \Omega^1(log))$ satisfying the conditions $\nabla(\alpha) = (1-\Phi)\xi$ and $\nabla(\xi) = 0$. Define

$$E_{k+2,0,\varphi}(\tau, g) := \frac{(-N)^{k+2}}{2(2\pi i)^{k+2}}(k+1)! \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{\widehat{g\varphi}(m,n)}{(m+n\tau)^{k+2}},$$

where τ is the coordinate on the upper half plane and $g \in GL_2(\mathbb{Z}/N)$. Let $N\Theta$ be the operator on p-adic modular forms defined by Katz, which is the operator $q\frac{d}{dq}$ on the q-expansion. Define

$$E_{k,r,\varphi} := (N\Theta)^r E_{k-r,0,\varphi}$$

if $k \geq r$ and $E_{k,r,\varphi} := (N\Theta)^k E_{r-k,0,\varphi}$ if $r \geq k$. The main theorem in Katz [3] implies the following result:

Let $k > 0$, $r > 0$, then there is a p-adic measure μ^k on $\mathbb{Z}_p^* \times (\mathbb{Z}/N)^2$ such that

$$\int_{\mathbb{Z}_p^* \times (\mathbb{Z}/N)^2} y^r \varphi d\mu^k = (1 - p^r Frob) E_{k,r,\varphi}.$$

With the help of this theorem we can now define for $k > 0$ and $r \in \mathbb{Z}$ the *p-adic Eisenstein series of negative weight*:

$$E^{(p)}_{k,r,\varphi} := \int_{\mathbb{Z}_p^* \times (\mathbb{Z}/N)^2} y^r \varphi d\mu^k$$

Now let $\widetilde{M}^{ord}$ be the moduli space of elliptic curves with full level N structure and a trivialization $\widehat{\mathbf{G}_m} \cong \widehat{E}$. On $\widetilde{M}^{ord}$ the isocrystal H has a trivialization by global sections $\widetilde{\omega}^\vee, \widetilde{u}^\vee$ given by the unit root subspace. Define

$$\widetilde{\alpha}^{k+2}_\varphi := \sum_{n=0}^{k} \frac{(-1)^n k!}{(k-n)!} E^{(p)}_{k+1-n,-1-n,\varphi} (\widetilde{\omega}^\vee)^n (\widetilde{u}^\vee)^{k-n}.$$

This descends to a section $\alpha_\varphi^{k+2} \in \Gamma(\overline{M}^{ord}_{\mathbb{Q}_p}, Sym^k H(1)_{rig})$ and we if we let

$$E_{DR}^{k+2}(\varphi) := \frac{2}{N^{k+1}k!} E_{k+2,0,\varphi} \frac{dq}{q} \wedge dz_1 \wedge \ldots \wedge dz_k,$$

then the pair

$$(\alpha_\varphi^{k+2}, E_{DR}^{k+2}(\varphi)) \in H^1_{syn}(M^{ord}, Sym^k H(1))$$

describes the realization of the syntomic Eisenstein class.

References

[1] K. Bannai and G. Kings, p-adic elliptic polylogarithm, p-adic Eisenstein series and Katz measure, arXiv:0707.3747

[2] A. Beilinson and A. Levin, The Elliptic Polylogarithm, *Motives*, (Seattle, WA, 1991), 123-192. Proc. Sympos. Pure Math., **55**, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[3] N. Katz, p-adic interpolation of real analytic Eisenstein series, Ann. of Math. (2) **104** (1976), no. 3, 459–571.

Truncated Hitchin Fibration

Gérard Laumon

(joint work with Pierre-Henri Chaudouard)

Our main result is the weighted Fundamental Lemma formulated by James Arthur. This is a combinatorial statement which extends the Langlands-Shelstad Fundamental Lemma and which is needed for stabilizing the Arthur-Selberg trace formula.

The Langlands-Shelstad Fundamental Lemma has been proved in general by Ngô Bao Châu as a consequence of his cohomological study of the elliptic part of the Hitchin fibration. In the same way, we obtain the weighted Fundamental Lemma by extending Ngô's cohomological study to the hyperbolic part of the Hitchin fibration.

Here I consider the GL(n) case. It is trivial from the point of view of the Fundamental Lemma, but Ngô's main cohomological result and our extension of it are not.

1. Hitchin Fibration

Let k be an algebraically closed field and let C be a smooth, connected, projective curve over k of genus g.

We fix an even effective divisor $D = 2D' \subset C$ of degree $> 2g$ and a closed point ∞ of C which is not in the support of D.

We also fix an integer $n \geq 1$ and we assume that either k is of characteristic 0 either it is of characteristic $p > n$. We denote by G the full general linear group GL(n) and by $T \subset G$ its diagonal maximal torus.

Let $\mathcal{M} = \mathcal{M}_G$ be the algebraic stack of quadruples $(\mathcal{E}, \theta, t_\infty, e_\infty)$ where:

- $\mathcal{E}$ is a rank n vector bundle over C of degree 0,

- $\theta : \mathcal{E} \to \mathcal{E}(D)$ is a twisted endomorphism such that its fiber at ∞, $\theta_\infty \in \mathrm{End}(\mathcal{E}_\infty)$, is regular semisimple,
- $t_\infty = (t_1, \ldots t_n)$ is an ordering on the set of eigenvalues of θ_∞,
- e_∞ is a basis of the 1-dimensional eigenspace of θ_∞ for the eigenvalue t_1.

Let $\mathbb{A} = \mathbb{A}_G$ be the affine scheme of pairs $a = (P(U), t_\infty)$ where:

- $P(U) := U^n + a_1 U^{n-1} + \cdots + a_n$ with $a_i \in H^0(C, \mathcal{O}_C(iD))$,
- $t_\infty = (t_1, \ldots t_n)$ is an ordering on the set of roots of the polynomial

  $$P(\infty)(U) = U^n + a_1(\infty)U^{n-1} + \cdots + a_n(\infty) \in k[U]$$

  that we assume two by two distinct.

For us the Hitchin fibration is the morphism

$$f = f_G : \mathcal{M} \to \mathbb{A}$$

which takes $(\mathcal{E}, \theta, t_\infty, e_\infty)$ to $a = (P(U), t_\infty)$ where $P(U)$ is the characteristic polynomial of θ.

2. The elliptic part

Let $\mathcal{L}(T)$ the set of the proper Levi subgroups M of G containing T.

For each $M \cong \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_s) \in \mathcal{L}(T)$ we have the base $\mathbb{A}_M = \prod_{j=1}^s \mathbb{A}_{\mathrm{GL}(n_j)}$ of the Hitchin fibration of M, the open subset

$$\mathbb{A}_M^{G-\mathrm{reg}} \subset \mathbb{A}_M$$

of s-uples $(P_j(U), t_{j,\infty} = (t_{j,1}, \ldots, t_{j,n_i}))_{j=1,\ldots,s}$ such that the $t_{j,i}$'s are two by two distinct, and a closed embedding

$$\mathbb{A}_M^{G-\mathrm{reg}} \hookrightarrow \mathbb{A}$$

which maps $(P_j(U), t_{j,\infty})_{j=1,\ldots,s}$ to $(P(U), t_\infty)$ where

$$P(U) = \prod_{j=1}^{s} P_j(U)$$

and t_∞ is $(t_{1,1}, \ldots, t_{1,n_1}, \ldots, t_{s,1}, \ldots, t_{s,n_s})$ up to some reordering of the entries.

The elliptic part $\mathbb{A}^{\mathrm{ell}}$ of $\mathbb{A}$ is the complementary open subset of the union of the $\mathbb{A}_M^{G-\mathrm{reg}}$ for $M \in \mathcal{L}(T)$.

Let $f^{\mathrm{ell}} = \mathcal{M}^{\mathrm{ell}} \to \mathbb{A}^{\mathrm{ell}}$ be the restriction of the Hitchin fibration to the elliptic open subset.

Proposition 1 (Altman-Kleiman, Mumford-Langton, Faltings)**.** *The algebraic stack $\mathcal{M}^{\mathrm{ell}}$ is a smooth scheme over k and the morphism f^{ell} is proper.*

3. Ngô's main cohomological result

Let us fix a prime number ℓ invertible in k. We may consider the complex of ℓ-adic sheaves

$$Rf^{\mathrm{ell}}_{*}\mathbb{Q}_{\ell}[\dim_{\mathcal{M}}]$$

over $\mathbb{A}$ and the direct sum of its perverse cohomology sheaves

$${}^{\mathrm{p}}\mathcal{H}^{\bullet}(Rf^{\mathrm{ell}}_{*}\mathbb{Q}_{\ell}[\dim_{\mathcal{M}}])=\bigoplus_{i}{}^{\mathrm{p}}\mathcal{H}^{i}(Rf^{\mathrm{ell}}_{*}\mathbb{Q}_{\ell}[\dim_{\mathcal{M}}])$$

By Deligne's theorem, these cohomology sheaves are all pure and their direct sum is thus semisimple. We thus have a canonical decomposition

$${}^{\mathrm{p}}\mathcal{H}^{\bullet}(Rf^{\mathrm{ell}}_{*}\mathbb{Q}_{\ell}[\dim_{\mathcal{M}}])=\bigoplus_{a\in\mathbb{A}^{\mathrm{ell}}} i_{a,*}j_{a,!*}\mathcal{F}^{\bullet}_{a}[\dim_{a}]$$

where $i_a:\overline{\{a\}}\hookrightarrow\mathbb{A}^{\mathrm{ell}}$ and $j_a:\{a\}\hookrightarrow\overline{\{a\}}$ are the inclusion, $\dim_a$ is the dimension of $\overline{\{a\}}$ and $\mathcal{F}^{\bullet}_{a}$ is a graded local system over a which extends to a local system over a dense open subset of $\overline{\{a\}}$. Almost all $\mathcal{F}^{\bullet}_{a}$ are zero. The socle of ${}^{\mathrm{p}}\mathcal{H}^{\bullet}(Rf^{\mathrm{ell}}_{*}\mathbb{Q}_{\ell}[\dim_{\mathcal{M}}])$ is the finite set of a's such that $\mathcal{F}^{\bullet}_{a}\neq(0)$.

The main result of Ngô in the $\mathrm{GL}(n)$ case is:

Theorem 1 (Ngô)**.** The socle of ${}^{\mathrm{p}}\mathcal{H}^{\bullet}(Rf^{\mathrm{ell}}_{*}\mathbb{Q}_{\ell}[\dim_{\mathcal{M}}])$ is reduced to the generic point of $\mathbb{A}^{\mathrm{ell}}$.

Remark 1. As it is stated, the above theorem is completely proved only if k is of characteristic 0. If k is of characteristic $p>0$, there is a slightly weaker more technical statement which is sufficient for the Fundamental Lemma.

4. ξ-stability

Let us fix $\xi=(\xi_1,\ldots,\xi_n)\in\mathbb{Q}^n$ such that $\xi_1+\cdots+\xi_n=0$.

Let $(\mathcal{E},\theta,t_\infty,e_\infty)$ be a Hitchin quadruple. Let us recall that $\deg(\mathcal{E})=0$ and that $t_\infty=(t_1,\ldots,t_n)$ is an ordering on the set of eigenvalues of the regular semisimple endomorphism θ_∞ of $\mathcal{E}_\infty$.

Then, for any θ-stable subbundle $\mathcal{F}\subset\mathcal{E}$ the set of eigenvalues of θ_∞ on $\mathcal{F}_\infty$ is of the form $\{t_i\mid i\in I_{\mathcal{F}}\}$ for some uniquely determined subset $I_{\mathcal{F}}\subset\{1,\ldots,n\}$ with $\mathrm{rank}(\mathcal{F})$ elements.

For any non empty proper subset $I\subset\{1,\ldots,n\}$ let us set $\xi_I:=\sum_{i\in I}\xi_i$.

Definition 1. We say that $(\mathcal{E},\theta,t_\infty,e_\infty)$ is ξ-stable if for every non trivial proper θ-stable subbundle $\mathcal{F}$ of $\mathcal{E}$ we have the inequality

$$\deg(\mathcal{F})<\xi_{I_{\mathcal{F}}}.$$

If ξ_I is not an integer for any non empty proper subset I of $\{1,\ldots,n\}$, we can replace in the above definition the strict inequality by less or equal. For such a ξ we have:

Theorem 2. The ξ-stable Hitchin quadruples form an open substack $\mathcal{M}^{\xi-\mathrm{st}}$ of $\mathcal{M}$ which contains the elliptic locus. It a smooth algebraic space over k and the restriction $f^{\xi-\mathrm{st}} : \mathcal{M}^{\xi-\mathrm{st}} \to \mathbb{A}$ of f to $\mathcal{M}^{\xi-\mathrm{st}}$ is proper.

The proof is a variant of Mumford-Langton and Faltings' one. We also prove an analogous result for a general semisimple group by using the uniformization method of Heinloth.

5. Our main result

Theorem 3. The socle of the graded perverse sheaf ${}^{\mathrm{p}}\mathcal{H}^{\bullet}(Rf_*^{\xi-\mathrm{st}}\mathbb{Q}_\ell[\dim_{\mathcal{M}}])$ is reduced to the generic point of $\mathbb{A}$.

The proof is a variant of Ngô's one.

Remark 2. Same as Remark 1.

Fujiwara's theorem for stacks

Martin Olsson

We discuss a generalization of Fujiwara's theorem (formerly known as Deligne's conjecture) on traces of correspondences [2]. Throughout we use Grothendieck's six operations for sheaves on stacks as developed in [3, 4]. The preprint containing this work is [5].

Let $\mathbb{F}_q$ be a finite field, and let $\mathrm{X}_0/\mathbb{F}_q$ be an algebraic stack of finite type. A *correspondence* on X_0 is a morphism of algebraic stacks over $\mathbb{F}_q$

$$c = (c_1, c_2) : \mathrm{C}_0 \to \mathrm{X}_0 \times \mathrm{X}_0.$$

Fix an algebraic closure k of $\mathbb{F}_q$, and let X (resp. C) denote the base change of X_0 (resp. C_0) to k.

Let $F_{\mathrm{X}_0} : \mathrm{X}_0 \to \mathrm{X}_0$ denote the q-power Frobenius morphism. For any integer $n \geq 0$, we get a correspondence

$$c^{(n)} = (F^n_{\mathrm{X}_0} \circ c_1, c_2) : \mathrm{C}_0 \to \mathrm{X}_0 \times \mathrm{X}_0.$$

For a correspondence c, let $\mathrm{Fix}(c)$ denote the fiber product of the diagram of stacks (over k)

$$\begin{array}{ccc} & & \mathrm{C} \\ & & \downarrow{\scriptstyle c} \\ \mathrm{X} & \xrightarrow{\Delta} & \mathrm{X} \times \mathrm{X}. \end{array}$$

Theorem 1. *Let $\mathrm{X}_0/\mathbb{F}_q$ be an algebraic stack of finite type, and let $c : \mathrm{C}_0 \to \mathrm{X}_0 \times \mathrm{X}_0$ be a correspondence with c_2 representable and quasi-finite. Then there exists an integer n_0 such that for every $n \geq n_0$ the maximal reduced substack of $\mathrm{Fix}(c^{(n)})$ is isomorphic to a finite disjoint union of stacks of the form BH, with H a finite group.*

For the rest of this report, we assume furthermore that c_1 is proper with finite diagonal, and that c_2 is representable and quasi-finite.

A *Weil complex* on X is an object $\mathcal{F} \in D^b_c(\mathrm{X}, \mathbb{Q}_\ell)$ (where as usual ℓ is a prime not dividing q) and $\varphi : F^*_{\mathrm{X}}\mathcal{F} \to \mathcal{F}$ is an isomorphism (here F_{X} is the base change to k of the q-power Frobenius morphism on X_0).

A C-*structure* on $\mathcal{F} \in D^b_c(\mathrm{X}, \mathbb{Q}_\ell)$ is a morphism

$$u : c_{2!}c_1^*\mathcal{F} \to \mathcal{F} \quad \text{(equivalently a map } c_1^*\mathcal{F} \to c_2^!\mathcal{F}).$$

A c-structure on $\mathcal{F}$ induces an endomorphism of $R\Gamma_c(\mathrm{X}, \mathcal{F})$ as the composite

$$\begin{array}{ccc} R\Gamma_c(\mathrm{X}, F) \longrightarrow R\Gamma_c(\mathrm{X}, c_{1*}c_1^*\mathcal{F}) \xrightarrow{\simeq} R\Gamma_c(\mathrm{C}, c_1^*\mathcal{F}) \\ \swarrow \simeq \\ R\Gamma_c(\mathrm{X}, c_{2!}c_1^*\mathcal{F}) \xrightarrow{u} R\Gamma_c(\mathrm{X}, F). \end{array}$$

Here the second isomorphism results from a natural isomorphism $c_{1*} \simeq c_{1!}$ constructed in [5].

For a fixed point $(x, \lambda) \in \mathrm{Fix}(\mathrm{C})(k)$, with $x \in \mathrm{C}(k)$ and $\lambda : c_2(x) \to c_1(x)$ an isomorphism in $\mathrm{X}(k)$, we get for any $\mathcal{F} \in D^b_c(\mathrm{X}, \mathbb{Q}_\ell)$ with a c-structure $u : c_{2!}c_1^*\mathcal{F} \to \mathcal{F}$ an endomorphism

$$u_{(x,\lambda)} : \mathcal{F}_{c_2(x)} \to \mathcal{F}_{c_2(x)}$$

defined as follows.

Since $c_2 : \mathrm{C} \to \mathrm{X}$ is representable and quasi-finite, we have

$$(c_{2!}c_1^*\mathcal{F})_{c_2(x)} = \oplus_{(y,\tau)}\mathcal{F}_{c_1(y)},$$

where the sum is taken over isomorphism classes of pairs (y, τ) with $y \in \mathrm{C}(k)$ and $\tau : c_2(y) \simeq c_2(x)$ an isomorphism in $\mathrm{X}(k)$. The map $u_{(x,\lambda)}$ is defined to be the composite

$$\mathcal{F}_{c_2(x)} \xrightarrow{\lambda} \mathcal{F}_{c_1(x)} \xhookrightarrow{x} \oplus_{(y,\tau)}\mathcal{F}_{c_1(y)} = (c_{2!}c_1^*\mathcal{F})_{c_2(x)} \xrightarrow{u} \mathcal{F}_{c_2(x)}.$$

For a Weil complex with c-structure $(\mathcal{F}, \varphi, u)$, we obtain a $c^{(n)}$-structure $u^{(n)}$ from the composite

$$c_1^*F_{\mathrm{X}}^{n*}\mathcal{F} \xrightarrow{\varphi^n} c_1^*\mathcal{F} \xrightarrow{u} c_2^!\mathcal{F}.$$

Let $(\mathcal{F}, \varphi, u)$ be a Weil complex with c-structure. If n_0 is as in theorem 1 and $n \geq n_0$, then for $(x, \lambda) \in \mathrm{Fix}(c^{(n)})(k)$ define the *local term*

$$\mathrm{LT}((x, \lambda), (\mathcal{F}, \varphi, u))$$

to be the trace of $u^{(n)}_{(x,\lambda)}$ divided by the order of the stabilizer group of (x, λ). This depends only on the connected component β of (x, λ) in $\mathrm{Fix}(c^{(n)})$.

Based on the case of separated schemes in [2], one would expect that for n sufficiently big that there is an equality

$$\mathrm{tr}(u^{(n)}|R\Gamma_c(\mathrm{X}, \mathcal{F})) = \sum_\beta \mathrm{LT}(\beta, (\mathcal{F}, \varphi, u)).$$

This does not make sense as written as $R\Gamma_c(\mathcal{X}, \mathcal{F})$ is usually an unbounded complex so the left side is not in general defined. However, using a suitable notion of convergence one can make sense of both sides. The main result is then the following:

Theorem 2. *Suppose either of the following conditions hold:*
(i) $\mathcal{X}_0$ *is an Artin stack with finite diagonal;*
(ii) $\mathcal{X}_0 = [X_0/G_0]$ *is a global quotient of a separated algebraic space by a finite type group scheme* G_0*,* $\mathcal{C}_0 = [C_0/G_0]$ *where* C_0 *is a separated algebraic space,* c_1 *is induced by a morphism* $C_0 \to X_0$ *which is* α*-equivariant for some finite homomorphism* $\alpha : G_0 \to G_0$*, and* c_2 *is induced by a* G_0*-equivariant morphism* $C_0 \to X_0$*.*
Then there exists an integer n_0 *such that for* $n \geq n_0$ *the trace*

$$\mathrm{tr}(u^{(n)}|R\Gamma_c(\mathcal{X}, \mathcal{F}))$$

converges and is equal to

$$\sum_\beta \mathrm{LT}(\beta, (\mathcal{F}, \varphi, u)).$$

Remark 3. *In the case of Frobenius, theorem 2 holds in complete generality and the result is due to Behrend* [1]*.*

References

[1] K. A. Behrend. Derived l-adic categories for algebraic stacks. *Mem. Amer. Math. Soc.*, 163(774), 2003.
[2] K. Fujiwara, *Rigid geometry, Lefschetz-Verdier trace formula and Deligne's conjecture*, Invent. math. **127** (1997), 489–533.
[3] Y. Laszlo and M. Olsson, *Grothendieck's six operations for sheaves on stacks I: finite coefficients*, to appear in Publ. Math. IHES.
[4] Y. Laszlo and M. Olsson, *Grothendieck's six operations for sheaves on stacks II: adic coefficients*, to appear in Publ. Math. IHES.
[5] M. Olsson, *Fujiwara's theorem for equivariant correspondences*, preprint (2008).

Tamely branched covers and degenerations

Brian Osserman

The classical Riemann existence theorem describes complex branched covers of the projective line in terms of the monodromy over the branch points. In characteristic p, if we restrict to the case of tame branching, Grothendieck shows that every cover is still characterized by its monodromy, and that when the monodromy group has order prime to p, every possibility for the monodromy group in the complex case also occurs in characteristic p. However, when p divides the order of the monodromy group, very little is known about which possibilities for monodromy are realized by covers.

We present a sharp Riemann existence theorem for two families of tamely branched covers of the projective line. Both families consist of genus-0 "pure-cycle" covers, meaning that there is a single ramified point over each branch point.

In [6], we define a complicated but purely elementary notion of *p-admissibility* for prospective tuples of monodromy. Our main theorem is then the following.

Theorem 1. Suppose we have $(e_1,\dots,e_r)$ and d with $2d-2=\sum_i(e_i-1)$, and $p \nmid e_i$ for $i=1,\dots,r$. Suppose further that either $r=3$, or $e_i<p$ for $i=1,\dots,r$. Then a tuple of cycles $(\sigma_1,\dots,\sigma_r)$ having lengths $(e_1,\dots,e_r)$ occurs as a the monodromy of a tamely branched cover of the projective line in characteristic p if and only if $(\sigma_1,\dots,\sigma_r)$ is p-admissible.

The proofs of both cases of the theorem make use of a shift in point of view from branched covers to linear series: that is, instead of controlling the branch points on the target, we control the ramification points on the source, and work up to automorphism of the target.

The case of three branch points is relatively straightforward: we show that a related intersection number is 1, and conclude that a map of the specified type exists if and only if an inseparable linear series of degree d with at least the required ramification does not exist. Studying the possibilities for inseparable linear series then reduces the problem to a purely combinatorial condition.

The proof of the second case is more involved. Here, the idea is to use degenerations from the linear series point of view, using the Eisenbud-Harris theory of limit linear series. Existence statements are relatively straightforward, but non-existence statements are much more difficult. The first step is to understand the linear series situation from a numerical standpoint: that is, when we have a map of degree d with ramification indices e_i at r general ramification points. This is carried out in [4], with the crucial point being to prove that in the given situation, separable maps will always degenerate to separable maps. The second step is to translate from linear series to branched covers, which amounts to seeing that if any map exists with the specified ramification, then one exists which also has general ramification points. This is a consequence of the following theorem, proved in [5].

Theorem 2. Given $e_1,\dots,e_r$ and d with $2d-2=\sum_i(e_i-1)$ and $e_i<p$ for $i=1,\dots,r$, assume also that each e_i is odd. Then for any distinct points $P_1,\dots,P_r$ on the projective line, the number of rational functions of degree d from the projective line to itself, ramified exactly to order e_i at each P_i, and counted up to linear fractional transformation, is finite.

The theorem is false if we allow some e_i to be greater than p, highlighting the contrast with the case of branched covers, where the equivalent statement would be true much more generally for arbitrary tame ramification indices and covers of any genus. Despite the elementary nature of the statement, the proof involves exploiting a relationship between such rational functions and Mochizuki's dormant torally indigenous bundles, which are certain bundles of rank 2 on the projective line with logarithmic connection and vanishing p-curvature.

Finally, to go from numerical statements involving ramification indices to group-theoretic statements involving monodromy, we use the following theorem on complex Hurwitz spaces, proved with Fu Liu in [3].

Theorem 3. Complex genus-0 pure-cycle Hurwitz spaces are always irreducible.

Putting everything together gives the proof of Theorem 1.

There at two natural approaches to attempting to generalize these results to higher-genus covers. The first approach is to imitate the structure of the above argument, using a degeneration to a curve with one component of genus 0, with g elliptic tails. In this case, the natural context would be to consider maps with at least $3g$ simply ramified points, which specialize to the tails, and r additional points with arbitrary ramification indices $e_i < p$, which specialize to the genus-0 component. In principle, this would allow us to reduce to the genus-0 case. The two main obstacles are controlling degeneration to inseparable maps, and understanding the relationship between linear series and branched covers.

To study degenerations, one promising approach is to generalize the relationship between rational functions and dormant torally indigenous bundles. Mochizuki proves that the latter are finite and flat in families, so a family of maps specializing to an inseparable one would correspond to a family of indigenous bundles specializing to an indigenous bundle with certain pathological properties, and studying possibilities for indigenous bundles on the degenerate curve, one might be able to rule this out. To make the transition from linear series to branched covers, and then to translate from numerical to group-theoretic results, a positive answer to the following question (in characteristic p when $e_i < p$ and in characteristic 0 respectively) would be sufficient:

Question 4. Fix $r, g \geq 0$, $d \geq 1$ and $(e_1, \ldots, e_r)$ with $2d-2-g = \sum_i (e_i - 1)$. Let $\mathcal{H}$ be the Hurwitz space of genus-g pure-cycle covers branched to orders e_i over r points and simply branched over $3g$ additional points. Is it the case that every component of $\mathcal{H}$ maps dominantly to $\mathcal{M}_{g,r}$ under the map induced by forgetting the covering map and the $3g$ simple ramification points?

Finally, a rather different approach to understanding higher-genus covers would be to study degenerations entirely within the point of view of branched covers. Here the main obstacle is that we have very few tools for controlling specialization to inseparable maps, although another problem is that the geometry does not simplify as much under degeneration as it does in the case of linear series. Examples of Irene Bouw [1] show that we do not always obtain good degenerations for p-rank reasons, but degenerations in the branched cover setting nonetheless seem to be surprisingly well-behaved. For instance, in [2] (with Bouw) we carry out the following calculation (with possibly finitely many exceptions):

Theorem 5. Suppose we have $e_1 \leq e_2 \leq e_3 \leq e_4 < p$ all odd, with $2p - 2 = \sum_i (e_i - 1)$. Then every degree p pure-cycle genus-0 cover with ramification indices e_1, e_2, e_3, e_4 has good degeneration under the degeneration sending the first two branch points to one component and the last two to the other.

References

[1] I. Bouw, *Construction of covers in positive characteristic via degeneration*, preprint.
[2] I. Bouw and B. Osserman, *Some 4-point Hurwitz numbers in positive characteristic*, in preparation.

[3] F. Liu and B. Osserman, *Irreducibility of certain pure-cycle Hurwitz spaces*, American Journal of Mathematics, to appear.
[4] B. Osserman, *Rational functions with given ramification in characteristic p*, Compositio Mathematica **142** (2006), 433–450.
[5] B. Osserman, *Mochizuki's crys-stable bundles: a lexicon and applications*, Publications of RIMS **43** (2007), 95–119.
[6] B. Osserman, *Linear series and existence of branched covers*, Compositio Mathematica **144** (2008), 89–106.

Elements of the group K_1 associated to abelian schemes

Damian Rössler
(joint work with Vincent Maillot)

In this short description of results, we shall use the general terminology of higher-dimensional Arakelov theory (cf. [7]).

Let B be a regular arithmetic variety over $\mathbb{Z}$ and let $\pi : A \to B$ be an abelian scheme over B. We choose a Kähler fibration structure ω on $A(\mathbb{C})$, such that the metrics induced on the fibers are translation invariant. We choose a line bundle L on A, which is rigidified along the 0-section and such that there exists $k \in \mathbb{N}^*$, such that there exists an isomorphism $L^{\otimes k} \simeq \mathcal{O}_B$ respecting the rigidification. We equip this line bundle with the unique hermitian metric $h_{L(\mathbb{C})}$, whose curvature form vanishes and such that the rigidification is an isometry. We shall write $T(A(\mathbb{C}), \omega, h_{L(\mathbb{C})}) \in \widetilde{A}(B)$ for the higher analytic torsion form of $L(\mathbb{C})$ over $B(\mathbb{C})$ (see [1]). We shall denote by reg the regulator map

$$\mathrm{reg} : K_1(B) \to \oplus_{p\geqslant 0} H^{2p-1}_{D,\mathrm{an}}(B(\mathbb{C}), \mathbb{R}(p)).$$

Here $H^{2p-1}_{D,\mathrm{an}}(B(\mathbb{C}), \mathbb{R}(p))$ is the p-th analytic Deligne cohomology of $B(\mathbb{C})$. There is a natural inclusion of groups $\oplus_{p\geqslant 0} H^{2p-1}_{D,\mathrm{an}}(B(\mathbb{C}), \mathbb{R}(p)) \subseteq \widetilde{A}(B)$ (see [2]).

The object of the talk was to present the following

Proposition 0.1. *Suppose that $L|_{A_b} \not\simeq \mathcal{O}_{A_b}$ for all fibers A_b of π. Then*

(1) *The element $T(A(\mathbb{C}), \omega, h_{L(\mathbb{C})})$ does not depend on the choice of ω. We shall thus henceforth write $T(A(\mathbb{C}), L)$ for $T(A(\mathbb{C}), \omega, h_{L(\mathbb{C})})$.*

(2) *We have*

$$T(A, L) \in \mathrm{image}(\mathrm{reg} \otimes \mathbb{Q}).$$

(3) *Let $n \in \mathbb{N}$ be such that $(n, k) = 1$. Suppose that the dual abelian scheme $A^\vee \to B$ has n^{2g} disjoint n-torsion sections. Let $M_1, \dots, M_{n^{2g}}$ be the corresponding rigidified line bundles on A. Then*

$$T(A, L^{\otimes n}) = \sum_{j=1}^{n^{2g}} T(A, L \otimes M_j).$$

In the case where $\dim(A/B)$ (elliptic fibrations), it is shown in [6] that the function part of $T(A, L)$ is a certain elliptic unit. The No. 2 in the Proposition 0.1 contains in particular the reciprocity law for this elliptic unit. The No. 3 is a generalisation of part of the distributivity law for (certain) elliptic units.

Sketch of proof of Proposition 0.1. No.1 is a consequence of the anomaly formula [1, Th. 3.10]. No. 2 is a direct consequence of the arithmetic Riemann-Roch theorem in all degrees proven in [3]. No. 3 results from a computation with the Fourier-Mukai transform and from the main result of [5].■

During the talk, G. Kings made the very interesting suggestion that the elements $T(A, L)$ coincide with the Hodge realisations of certain elements of $K_1(B) \otimes \mathbb{Q}$ constructed using the motivic polylogarithmic sheaf on abelian scheme (see [4]). Since these elements are constructed using an (apparently) completely different method, the proof of such an identity would be of great interest.

References

[1] Jean-Michel Bismut and Kai Köhler, *Higher analytic torsion forms for direct images and anomaly formulas*, J. Algebraic Geom. 1 (1992), 647-684.

[2] Jose Ignacio Burgos and Steve Wang, *Higher Bott-Chern forms and Beilinson's regulator*, Invent. Math. 132 (1998), 261-305.

[3] Henri Gillet, Damian Rössler, and Christophe Soulé, *An arithmetic Riemann-Roch theorem in higher degrees*, (To appear in Annales de l'Institut Fourier).

[4] Guido Kings, *K-theory elements for the polylogarithm of abelian schemes*, J. Reine Angew. Math. 517 (1999), 103-116.

[5] Xiaonan Ma, *Formes de torsion analytique et familles de submersions. I*, Bull. Soc. Math. France 127 (1999), 541-621. (French, with English and French summaries)

[6] D. B. Ray and I. M. Singer, *Analytic torsion for complex manifolds*, Ann. of Math. (2) 98 (1973), 154-177.

[7] C. Soulé, *Lectures on Arakelov geometry*, Cambridge Studies in Advanced Mathematics, vol. 33, Cambridge University Press, Cambridge, 1992, ISBN 0-521-41669-8, With the collaboration of D. Abramovich, J. F. Burnol, and J. Kramer.

Smooth representations and (φ, Γ)-modules

Peter Schneider

(joint work with Marie-France Vigneras)

The classical local Langlands correspondence (proved by Harris/Taylor and Henniart) establishes a distinguished bijection between n-dimensional discrete semisimple representations of the Weil-Deligne group of the nonarchimedean local field $\mathbb{Q}_p$ on the one hand and irreducible smooth representations of the group $GL_n(\mathbb{Q}_p)$ on the other hand. The Weil-Deligne group is a modification of the absolute Galois group of the field $\mathbb{Q}_p$ and its discrete representations are closely related to the ℓ-adic Galois representations where ℓ is any prime number different from p. If we consider p-adic Galois representations instead then the picture becomes much more complicated. On the other hand one can reduce it modulo p. By a theorem of Fontaine the category of p-adic Galois representations is equivalent to the category of etale (φ, Γ)-modules. So it seems a natural attempt to relate smooth representations of $GL_n(\mathbb{Q}_p)$ with torsion coefficients to etale (φ, Γ)-modules. In spectacular recent work Colmez has managed to do exactly this, and surprisingly even in a functorial way, in the special case of the group $GL_2(\mathbb{Q}_p)$.

In this talk I describe the general construction of a δ-functor from the category of smooth representations of $G(\mathbb{Q}_p)$ in $\mathbb{Z}_p$-torsion modules where G is any split reductive group over $\mathbb{Q}_p$ to the category of etale (φ, Γ)-modules but which are not required to be finitely generated. The crucial technique consists in introducing a much more general noncommutative analog of (φ, Γ)-modules which can be related to the action of the dominant submonoid in a Borel subgroup. The passage to commutative (φ, Γ)-modules uses a nondegenerate character of the unipotent radical of Borel. In the case of the group $GL_2(\mathbb{Q}_p)$ and under the finiteness assumptions on the representations imposed by Colmez we show that our zeroth functor coincides with Colmez' functor and that our higher functors vanish.

References

[1] P. Colmez: (φ, Γ)-*modules et représentations du mirabolique de* $GL_2(\mathbb{Q}_p)$, Preprint 2004.
[2] P. Colmez: *La série principale unitaire de* $GL_2(\mathbb{Q}_p)$, Preprint 2007.
[3] J.-M. Fontaine: *Représentations p-adiques des corps locaux*, Grothendieck Festschrift, vol. 2, pp. 249 - 309. Progress in Math. 87, Birkhäuser 1990.

On the Tate and Langlands–Rapoport Conjectures for Shimura Varieties

Adrian Vasiu

Let $p \in \mathbb{N}$ be a prime. Let $\mathbb{F}$ be an algebraic closure of the field $\mathbb{F}_p$ with p elements. By an *algebraic cycle* on an abelian variety $\triangle$ we mean a $\mathbb{Q}$–linear combination of irreducible subvarieties of products of $\triangle$. We report on work whose main goal is:

• to show that many crystalline cycles on abelian varieties over $\mathbb{F}$ are in fact algebraic (i.e., are crystalline realizations of algebraic cycles) and to get combinatorial descriptions of certain isogeny classes of principally polarized abelian varieties over $\mathbb{F}$ endowed with families of crystalline (presumed algebraic) cycles.

1. Preliminaries. Let (W, ψ) be a symplectic space over $\mathbb{Q}$. Let $d \in \mathbb{N}$ be such that $\dim(W) = 2d$. Let $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ be an injective map of Shimura pairs. Let $(v_\alpha)_{\alpha \in \mathcal{J}}$ be the family of all tensors of the tensor algebra $\mathcal{T}(\mathrm{End}(W))$ of $\mathrm{End}(W) = W \otimes_{\mathbb{Q}} W^*$ that are fixed by G. Let L be a $\mathbb{Z}$-lattice of W such that we have a perfect alternating form $\psi : L \times L \to \mathbb{Z}$. We will assume that:

(*) The schematic closure $G_{\mathbb{Z}_{(p)}}$ of G in $\boldsymbol{GL}_{L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}}$ is a reductive group scheme.

Let $E(G, X)$ be the reflex field of (G, X). Let $\mathrm{Sh}(G, X)$ be the canonical model over $E(G, X)$ of the complex Shimura variety defined by (G, X), cf. [1]. Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of $\mathbb{Q}$ with the p-component omitted; we have $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}_p \times \mathbb{A}_f^{(p)}$. The group $G(\mathbb{A}_f)$ acts naturally on $\mathrm{Sh}(G, X)$. For a compact subgroup $\dagger$ of $G(\mathbb{A}_f)$, let $\mathrm{Sh}_\dagger(G, X)$ be the quotient of $\mathrm{Sh}(G, X)$ by $\dagger$. Let $K_p := \boldsymbol{GSp}(L, \psi)(\mathbb{Z}_p)$ and $H_p := K_p \cap G(\mathbb{Q}_p) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. We have $\mathrm{Sh}(G, X)(\mathbb{C}) =$

$G(\mathbb{Q})\backslash[X\times G(\mathbb{A}_f)]$ and $\mathrm{Sh}(\boldsymbol{GSp}(W,\psi),S)(\mathbb{C}) = \boldsymbol{GSp}(W,\psi)(\mathbb{Q})\backslash[S\times\boldsymbol{GSp}(W,\psi)(\mathbb{A}_f)]$ (cf. [2]) and a functorial closed embedding $\mathrm{Sh}(G,X) \to \mathrm{Sh}(\boldsymbol{GSp}(W,\psi),S)_{E(G,X)}$ (cf. [1], Cor. 5.4). The last two sentences imply that we also have a functorial closed embedding $\mathrm{Sh}_{H_p}(G,X) \hookrightarrow \mathrm{Sh}_{K_p}(\boldsymbol{GSp}(W,\psi),S)_{E(G,X)}$.

We say (G,X) has *compact factors* if for each simple factor H of G^{ad}, there exists a simple, compact factor of $H_{\mathbb{R}}$. For *Hodge cycles* on abelian schemes over reduced $\mathbb{Q}$–schemes we refer to [3].

Let v be a prime of $E(G,X)$ that divides p. Let O be the localization of the ring of integers of $E(G,X)$ at v. Property (*) implies that O is an étale $\mathbb{Z}_{(p)}$-algebra. Let $\mathcal{M}$ be Mumford's moduli scheme over $\mathbb{Z}_{(p)}$ that parametrizes isomorphism classes of principally polarized abelian schemes of relative dimension d over $\mathbb{Z}_{(p)}$-schemes that have compatibly level-N symplectic similitude structures for all $N \in \mathbb{N}$ prime to p. We can identify $\mathrm{Sh}_{K_p}(\boldsymbol{GSp}(W,\psi),S) = \mathcal{M}_{\mathbb{Q}}$ (cf. [1], Prop. 4.17) and thus one can speak about the schematic closure $\mathcal{N}^{\mathrm{cl}}$ of $\mathrm{Sh}_{H_p}(G,X)$ in $\mathcal{M}_O$. Let $\mathcal{N}$ be the normalization of $\mathcal{N}^{\mathrm{cl}}$. The morphism $\mathcal{N} \to \mathcal{M}_O$ is finite.

1.2. On $\mathbb{F}$-valued points. To each point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ one associates naturally a principally polarized abelian variety (A,λ_A) over $\mathbb{F}$ and an isomorphism $\eta_N : (L/NL)_{\mathbb{F}} \tilde{\to} A[N]$ of constant étale group schemes over $\mathbb{F}$ that defines a level-N symplectic similitude structure on (A,λ_A). Let (M,φ,ψ_M) be the principally quasi-polarized Dieudonné module of the principally quasi-polarized p-divisible group of (A,λ_A). Let t_α be the tensor of the tensor algebra $\mathcal{T}(\mathrm{End}(M[\frac{1}{p}]))$ of $\mathrm{End}(M[\frac{1}{p}])$ which is the crystalline realization of the Hodge cycle on the lift of A to V defined by any lift $\tilde{z} : \mathrm{Spec}(V) \to \mathcal{N}$ of y that corresponds to v_α, with V as a finite, discrete valuation ring extension of $W(\mathbb{F})$.

1.3. Basic Theorem. **(a)** *The O-scheme $\mathcal{N}$ is regular and formally smooth. Thus $\mathcal{N}$ is the integral canonical model of* $\mathrm{Sh}_{H_p}(G,X)$ *over O in the strongest sense of* [6]*, Def. Def. 3.2.3 6).*

(b) *There exist isomorphisms* $(M,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M)\tilde{\to}(L^*\otimes_{\mathbb{Z}} W(\mathbb{F}),(v_\alpha)_{\alpha\in\mathcal{J}},\psi)$.

(c) *If (G,X) has compact factors, then $\mathcal{N}$ is a pro-étale cover of a projective O-scheme.*

Part (a) is proved in [6] and [8] for $p \geq 5$ and for the unitary case; see [11] for the general case. Part (b) is proved in [9], Thm. 1.2 and Rm. 4.4 (a) for $p \geq 3$; the case $p = 2$ follows from [11]. See [7], Cor. 4.3 for (c). Due to (b), we can speak about the reductive subgroup $\mathcal{G}$ of $\boldsymbol{GL}_M$ whose generic fibre is the subgroup of $\boldsymbol{GL}_{M[\frac{1}{p}]}$ that fixes each t_α with $\alpha \in \mathcal{J}$.

2. Conjecture (adèlic version of a conjecture of Tate). *Let y : Spec$(\mathbb{F}) \to \mathcal{N}$. Then each t_α is the crystalline realization of an algebraic cycle on A.*

3. Pointwise properties. Let $\mathfrak{E}(y)$ be the set of those $h \in \mathcal{G}(B(\mathbb{F}))$ such that the pair $(h(M),\varphi)$ is a Dieudonné module over k and there exists $h_1 \in \mathcal{G}(W(\mathbb{F}))$ for which we have $\varphi^{-1}(ph(M)) = hh_1(\varphi^{-1}(pM))$. Let $\mathfrak{E}_0(y) := \{h \in \mathfrak{E}(y)|h$ fixes $\psi_M\}$. For each $h \in \mathfrak{E}_0(y)$, let $(A(h),\lambda_{A(h)})$ be the principally quasi-polarized abelian variety over $\mathbb{F}$ which is $\mathbb{Z}[\frac{1}{p}]$-isogenous to (A,λ_A) and whose principally quasi-polarized Dieudonné module is canonically identified with $(h(M), \varphi, \psi_M)$. Let $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ be the $\mathbb{F}$-valued point defined by $(A(h), \lambda_{A(h)})$ and its level-N symplectic similitude structures defined by η_N's.

3.1. Definitions. **(a)** We say the *isogeny property* holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, if for each $h \in \mathfrak{E}_0(y)$ the point $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ factors through a morphism $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ in such a way that for a (any) lift $z(h) : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ of it, every tensor $t_\alpha \in \mathcal{T}(\mathrm{End}(M[\frac{1}{p}]))$ is the crystalline realization of the Hodge cycle on the abelian scheme over $W(\mathbb{F})$ defined by $z(h)$ that corresponds to v_α.

(b) We say the *endomorphism property* holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, if there exists a reductive subgroup $\mathcal{E}$ of $\boldsymbol{Aut}(A)_{\mathbb{Q}}$ that has the following two properties: (i) its extension to $W(\mathbb{F})[\frac{1}{p}]$ has (via crystalline realizations) $\{x \in \mathrm{Lie}(\mathcal{G})[\frac{1}{p}] | \varphi(x) = x\}$ as its Lie algebra and (ii) for each prime $l \neq p$, its extension to $\mathbb{Q}_l$ is (via l-adic realizations and the isomorphisms η_{l^n} with $n \in \mathbb{N}$) a subgroup of $G_{\mathbb{Q}_l}$.

(c) We say the *unramified CM lift property* holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, if there exists $h \in \mathfrak{E}_0(y)$ for which the morphism $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ factors through $\mathcal{N}$ as in (a) and for which there exists a lift $z(h) : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ of it whose generic fibre is a special point.

4. Main Theorem. *We assume that (G, X) has compact factors and each simple factor of (G^{ad}, X^{ad}) is of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ type. Let Θ be the Frobenius endomorphism of $\mathbb{F}$ that fixes the residue field of v. Then we have:*

(a) *The isogeny, the endomorphism, and the unramified CM lift properties hold for all points $y : Spec(\mathbb{F}) \to \mathcal{N}$.*

(b) *The Langlands–Rapoport conjecture of* [4] *and* [5] *holds i.e., there exists a bijection of $\mathbb{Z}\Theta \times G(\mathbb{A}_f^{(p)})$-equivariant sets between $\mathcal{N}(\mathbb{F})$ and $\bigsqcup_{[\varphi:\mathfrak{P}\to\mathfrak{G}]} X_p(\varphi) \times X^p(\varphi)$, where $\mathfrak{P}$ is the pseudo-motivic $\overline{\mathbb{Q}}/\mathbb{Q}$ groupoid of motives over $\mathbb{F}$, where $\mathfrak{G}$ is the $\overline{\mathbb{Q}}/\mathbb{Q}$ groupoid defined by G, where $\varphi : \mathfrak{P} \to \mathfrak{G}$ is an admissible homomorphism of $\overline{\mathbb{Q}}/\mathbb{Q}$ groupoids, where $X_p(\varphi)$ and $X^p(\varphi)$ are $\mathbb{Z}\Theta$- and $G(\mathbb{A}_f^{(p)})$-equivariant sets naturally associated to φ, and where $[\varphi]$ is the $G(\overline{\mathbb{Q}})$-conjugacy class of φ.*

(c) *We have $\mathcal{N} = \mathcal{N}^{cl}$.*

The proof of the Main Theorem relies heavily on the Basic Theorem 1.3 (c) as well as on the following basic result.

5. Basic Theorem. *Let (G_0, H_0) be an arbitrary simple, adjoint Shimura pair of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ Shimura type. We assume that the group $G_{0,\mathbb{Q}_p}$ is unramified (i.e., it extends to a reductive group scheme over $\mathbb{Z}_p$). Then there exists an injective map $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ of Shimura pairs such that the following two properties hold (for each prime v of $E(G, X)$ that divides p):*

(a) *We have $(G_0, X_0) = (G^{ad}, X^{ad})$, G^{der} is simply connected, and there exists a $\mathbb{Z}$-lattice L of W such that the property (*) holds.*

(b) *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$ be a basic point (i.e., a point such that all Newton polygon slopes of $(Lie(\mathcal{G})[\frac{1}{p}], \varphi)$ are 0). If (G_0, X_0) is of B_n, C_n, or $D_n^{\mathbb{R}}$ type, then the abelian variety A is supersingular and thus the classical Tate conjecture holds for A; moreover the Conjecture 2 holds for $y : Spec(\mathbb{F}) \to \mathcal{N}$.*

References

[1] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Lecture Notes in Math., Vol. **244**, 123–165, 1971.
[2] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions, Part 2, 247–289, Proc. Sympos. Pure Math., Vol. **33**, 1979.
[3] P. Deligne, *Hodge cycles on abelian varieties*, Hodge cycles, motives, and Shimura varieties, Lecture Notes in Math., Vol. **900**, 9–100, Springer-Verlag, 1982.
[4] R. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. reine angew. Math. **378** (1987), 113–220.
[5] J. S. Milne, *The points on a Shimura variety modulo a prime of good reduction*, The Zeta functions of Picard modular surfaces, 153–255, Univ. Montréal, Montreal, Quebec, 1992.
[6] A. Vasiu, *Integral canonical models of Shimura varieties of preabelian type*, Asian J. Math., Vol. **3** (1999), no. 2, 401–518.
[7] A. Vasiu, *Projective integral models of Shimura varieties of Hodge type with compact factors*, J. reine angew. Math. **618** (2008), 51–75.
[8] A. Vasiu, *Integral canonical models of unitary Shimura varieties*, Asian J. Math., Vol. **12** (2008), no. 2, 151–176.
[9] A. Vasiu, *A motivic conjecture of Milne*, http://arxiv.org/abs/math/0308202.
[10] A. Vasiu, *CM lifts for isogeny classes of Shimura F-crystals over finite fields*, preprint, MPI, Bonn, Germany, July 2007.
[11] A. Vasiu, *Good reductions of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic, Parts I and II*, http://xxx.arxiv.org/abs/0707.1668 and http://arxiv.org/abs/0712.1572.

Modularity of 2-adic Galois representation

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(joint work with Chandrashekhar Khare)

In this joint work with Chandrashekhar Khare ([4]), We extend Wiles Taylor-Wiles modularity theorem to 2-adic Galois representations. We rely on endhancements of the method of Wiles Taylor-Wiles ([9],[8]) by Diamond ([1]) Fujiwara ([3]) and Kisin ([5]). As Dickinson, who proved some particular cases of our theorem ([2]), we use the whole adjoint Galois representation instead of the representation in the matrices of trace 0.

Let us state our result. $p = 2$, $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_2})$ a continuous odd Galois representation. Let $\overline{\rho}$ be the reduction of ρ which is defined up to semisimplification.

One supposes that $\overline{\rho}(G_{\mathbb{Q}})$ have non-solvable image (then it is isomorphic to $\mathrm{SL}_2(\mathbb{F}_{2^r})$ for $r \geq 2$).

One supposes that $\overline{\rho}$ is modular. Let us be more precise. Following Serre ([6]), we define $k(\overline{\rho})$ to be 2 if the restriction of $\overline{\rho}$ to the decomposition group D_2 comes from a finite flat group over $\mathbb{Z}_2$, and $k(\overline{\rho}) = 4$ if not (then the restriction of $\overline{\rho}$ to D_2 has up to unramified twist trivial semisimplified reduction and is "très ramifiée"). Then $\overline{\rho}$ is the reduction of the Galois representation associated to a primitive form g in $S_2(\Gamma_1(N))$ with $k = 2$ and 2 does not divide N if $k(\overline{\rho}) = 2$ and 2 divides exactly N if $k(\overline{\rho}) = 4$ (that if $\overline{\rho}$ comes from a modular form then it comes from such a g is a particular case of the theorem that weak form of Serre's conjecture implies strong form).

One supposes that ρ is unramified outside a finite set of primes and the restriction of ρ to D_2 is crystalline of Hodge-Tate weights $(0,1)$ if $k(\overline{\rho}) = 2$ and semistable of Hodge-Tate weights $(0,1)$ if $k(\overline{\rho}) = 4$.

Then ρ is modular, meaning that there exists N' and a primitive form f in $S_2(\Gamma_1(N'))$ such that ρ is isomorphic to the Galois representation associated to f. Here the 2 part of N' is the same as the one of N, and the prime to 2 part of N' is the prime to 2 part of the conductor of ρ.

For the proof we need to introduce a solvable totally real field F. We prove that $\rho_{|G_F}$ is modular, *i.e.* arises from an Hilbert modular form, or from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ which at infinity is the discrete series corresponding to parallel weight 2 Hilbert modular forms. Then, by Arthur-Clozel solvable base change, we will know that ρ is modular.

With F, we gain in particular following advantages. First, we can suppose that $\rho_{|G_F}$ is unramified outside primes over 2 and primes in a finite set of primes Σ' where at $v \in \Sigma'$, ρ_{D_v} is semistable. We call $\Sigma = \Sigma'$ if $k(\overline{\rho}) = 2$ and $\Sigma = \Sigma' \cup V_2$ if $k(\overline{\rho}) = 4$ (V_2 is the set of primes of F above 2). Furthermore, we have an automorphic representation π' such that $\overline{\rho}_{|G_F}$ arises from π', and π_v is unramified outside Σ and is unramified twist of Steinberg at $v \in \Sigma$. We also have that π' comes from an automorphic representation π'' of $(D \otimes_F \mathbb{A}_F)^*$ by Jacquet-Langlands correspondence, where D is the quaternion algebra of center F and ramified exactly at infinity and primes in Σ.

Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_2$ that is supposed to be big enough. Let ψ be the character $\det(\rho)\chi_2^{-1}$ where χ_2 is the 2-adic cyclotomic character. We also call ψ the character of $(\mathbb{A}_F^f)^*/F^* \to \mathcal{O}^*$ associated to the totally even Galois character ψ. For $U = \prod U_v$ an open subgroup of the finite adeles $(D \otimes_F \mathbb{A}_F^f)^*$, we define the space of modular forms $S_\psi(U, \mathcal{O})$ with coefficients in $\mathcal{O}$ with central character ψ. It has the following combinatorial description : it is the $\mathcal{O}$-module of functions $g \mapsto f(g)$ of $D^*\backslash(D \otimes_F \mathbb{A}_F^f)^*$ to $\mathcal{O}(\gamma)$ such that $f(gu) = u^{-1}f(g)$ for $u \in U$, $f(gz) = \psi(z)f(g)$ for $z \in (\mathbb{A}_F^f)^*$, γ being a character of $U(\mathbb{A}_F^f)^*$ coming from the twists occuring in the twists of Steinberg.

We take U_v to be $\mathrm{GL}_2(O_v)$ if $v \notin \Sigma$ and U_v to be the multiplicative group of the completion D_v of D at $v \in \Sigma$. We call $T_\psi(U)$ the Hecke algebra acting on $S_\psi(U, \mathcal{O})$ generated by the Hecke operators T_v and S_v at primes $v \notin \Sigma \cup V_2$. The automorphic form π'' defines an eigenform f hence a morphism $T_\psi(U) \to \mathcal{O}$ sending T_v to the eigenvalue of T_v on f and S_v to $\psi(\omega_v)$, where ω_v is a uniformizer of F_v. By reduction it defines a maximal ideal m of $T_\psi(U)$. Let $T_\psi(U)_\mathsf{m}$ be the completion. By Deligne and Carayol, we get a Galois representation we call $\rho_T : G_F \to \mathrm{GL}_2(T_\psi(U)_\mathsf{m})$ that lifts $\overline{\rho}$. It satisfies the following condition : $\det(\rho_T) = \psi\chi_2$, it is unramified outside $\Sigma \cup V_2$, it is finite if $v \in V_2$ if $k(\overline{\rho}) = 2$ and semistable if $k(\overline{\rho}) = 4$, it is semistable at $v \in \Sigma$. Let $S = V_\infty \cup V_2, \cup \Sigma$. We call $\overline{R}_S^\psi$ the $\mathrm{CNL}_\mathcal{O}$-algebra whose formal spectrum classifies the deformations of $\overline{\rho}$ that satisfies these conditions. By a $\mathrm{CNL}_{\mathcal{O}}$-algebra we mean a Complete Noetherian Local $\mathcal{O}$-algebra with residue field the coefficient field $\mathbb{F}$ of $\overline{\rho}$.

We get a surjective map $\overline{R}_S^{\psi} \to T_{\psi}(U)_{\mathfrak{m}}$. To get our theorem, we prove that this map is bijective after inverting 2, or that $\overline{R}_S^{\psi}[1/2]$ acts faithfully on $S_{\psi}(U, \mathcal{O})_{\mathfrak{m}}[1/2]$

For that, we use the idea of Kisin to consider framed deformations of $\overline{\rho}$: a framed deformation with values in a $\mathrm{CNL}_{\mathcal{O}}$ algebra A is the data of a deformation with values in A and of a basis of the underlying space, or equivalently a morphism with values in $\mathrm{GL}_2(A)$ that lifts $\overline{\rho}$. A deformation is a lift modulo conjugation by the completion of GL_2 at the identity of the special fiber. We indicate by a $\square$ when we consider framed deformations.

We use the technique of auxiliary primes. For each integer $n \geq n_0$, we have a finite set of primes Q_n of primes of F : we impose to Q_n to be disjoint of S, that $v \in Q_n$ to be such that $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues and $N(v) \equiv 1 \mathrm{mod}\ 2^n$. We consider modular forms space $S_{\psi}(U_{Q_n}, \mathcal{O})$ where we allow level at Q_n. It is defined by the open subgroup U_Q of U, where $(U_Q)_v = (U)_v$ for $v \notin Q_n$ and for $v \in Q_n$

$$(U_Q)_v = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (\pi_v), ad^{-1} \to 1 \in \Delta_v\},$$

with $\Delta_v = \Delta'_v / 2^a \Delta'_v$ for an integer a independent of n and Δ'_v the biggest 2 quotient of the multiplicative group k_v^* of the residue field. We have corresponding $T_{\psi}(U_{Q_n})$ Hecke algebra. We can choose a maximal ideal $\mathfrak{m}$ of $T_{\psi}(U_{Q_n})$ and consider the completion $S_{\psi}(U_{Q_n}, \mathcal{O})_{\mathfrak{m}}$ of the space of modular forms. $S_{\psi}(U_{Q_n}, \mathcal{O})$ has a $\Delta_{Q_n} = \prod_{v \in Q_n} \Delta_v$ by Diamond operators. We have the following control theorem : $S_{\psi}(U_{Q_n}, \mathcal{O})_{\mathfrak{m}}$ is free over $\mathcal{O}[\Delta_{Q_n}]$-module and its coinvariants by Δ_{Q_n} is $S_{\psi}(U, \mathcal{O})_{\mathfrak{m}}$. Its proof relies in particular on the combinatorial description of the spaces of modular forms.

We have the framed deformation ring $\overline{R}_{S \cup Q_n}^{\square, \psi}$ where we impose the same condition as above outside Q_n and no condition for $v \in Q_n$. It has an action of Δ_{Q_n} coming from the action of (tame) inertia at $v \in Q_n$, with coinvariants $\overline{R}_S^{\square, \psi}$, and a surjective map $\overline{R}_S^{\square, \psi} \to T_{\psi}(U_{Q_n})$.

We consider $\overline{R}_{S \cup Q_n}^{\square, \psi}$ as an algebra on $\overline{R}_S^{\square, \mathrm{loc}, \psi}$ that is $\hat{\otimes}_{v \in S} \overline{R}_v^{\square}$, where $\overline{R}_v^{\square}$ is ring of framed deformations of the restriction of $\overline{\rho}$ at the decomposition group D_v that satisfies the current conditions for $v \in S$. We know from the study of local deformation rings that $\overline{R}_{S \cup Q_n}^{\square, \psi}$ is a domain, flat over $\mathcal{O}$ of relative dimension $3|S|$ and regular after inverting 2.

We have to control the number of generators of $\overline{R}_{S \cup Q_n}^{\square, \psi}$ as an algebra on $\overline{R}_S^{\square, \mathrm{loc}, \psi}$. By deformation theory, it is controled by the dimension $h^1_{L_v}(S \cup Q_n, \mathrm{ad}^0(\overline{\rho}))$ of the H^1, L_v meaning that we consider $c \in H^1$ such that the image c_v by the localization map in $H^1(D_v, \mathrm{ad}^0(\overline{\rho}))$ is in L_v. Here L_v is (0) if $v \in S$ and $H^1(D_v, \mathrm{ad}^0(\overline{\rho}))$ for

$v \in Q_n$. The formula of Wiles for $M = \mathrm{ad}^0(\overline{\rho})$ allows this control :

$$\frac{|H^1_{\{L_v\}}(S \cup V, M)|}{|H^1_{\{L_v^\perp\}}(S \cup V, M^*(1))|} = \frac{|H^0(G_F, M)|}{|H^0(G_F, M^*(1))|} \prod_{v \in S} \frac{|L_v|}{|H^0(D_v, M)|} \prod_{v \in V} \frac{|L_v|}{|H^0(D_v, M)|}$$

where $\{L_v^\perp\}$ is the dual of L_v in Tate's duality. For $p \neq 2$ one can finds Q_n such that $h^1_{\{L_v^\perp\}}(S \cup Q_n, \mathrm{ad}^0(\overline{\rho})^*(1)) = 0$, where $|Q_n| = h^1_{\{L_v^\perp\}}(S, \mathrm{ad}^0(\overline{\rho})^*(1))$. This is possible as for every irreducible subspace $V \subset (\mathrm{ad}^0)$, there is a $\sigma \in G_{F(\mu_{p^n})}$ which have eigenvalue not equal to 1 in (ad^0) but has eigenvalue 1 on V (lemma 2.5. of [7]). This last assertion is not true for $p = 2$. But, as Dickinson uses, for $p = 2$, we can find a set Q_n with $|Q_n| = h^1(S, \mathrm{ad}(\overline{\rho})) - 2$ such that $H^1_{\{L_v^\perp\}}(S \cup Q_n, \mathrm{ad}(\overline{\rho}))$ has dimension 2 (it coincides with the image of $H^1(F(\mu_{2^n})/F, \mathbb{F})$).

We come to the patching argument. We take a projective limit of quotients of finite length of the $\overline{R}^{\square,\psi}_{S \cup Q_n}$ and the modules $S_\psi(U_{Q_n}, \mathcal{O})_{\mathfrak{m}} \otimes_{\overline{R}^{\psi}_{S \cup Q_n}} \overline{R}^{\square,\psi}_{S \cup Q_n}$.

We obtain a ring R_∞ and a module M_∞ with action the Iwasawa algebra $\mathcal{O}[[y_1, \ldots, y_h]]$ with $h = |Q_n|$ and control theorem saying that M_∞ is finite free over the Iwasawa algebra with coinvariants $S_\psi(U, \mathcal{O})_{\mathfrak{m}} \otimes_{\overline{R}^{\psi}_S} \overline{R}^{\square,\psi}_S$. To get the theorem, we prove that M_∞ is a faithful R_∞-module that is finite free after inverting 2.

To prove the first statement, we need to prove that the relative dimension of R_∞ is $\leq$ to $h + 4|S| - 1$ ($4|S| - 1$ comes from the framing). For that we also to consider a patch R'_∞ of deformation rings for representations which satisfies current properties at $v \in S$ and that are unramified outside $S \cup Q_n$ (we do not impose determinant only at places in S). Wiles formula allows to control the number of generators of R'_∞. At the end, one has to prove the codimension of $\mathrm{Specf}(R_\infty)$ in $\mathrm{Specf}(R'_\infty)$ is $t := h^1_{S-\mathrm{split}}(Q_n, \mathbb{F})$ (split at S unramified outside Q_n). One sees applying Wiles formula for $M = \mathbb{F}$ that $t = 2 - |S| + |Q_n|$. One proves that one can impose to Q_n to be such that the Galois group G_n of the maximal abelian extension of F which is of order a power of 2 which is split at S and unramified outside Q_n has a quotient G'_n isomorphic to $(\mathbb{Z}/2^{n-2}\mathbb{Z})^t$. The proof uses the Wiles formula for $M = \mathbb{Z}/2^n\mathbb{Z}$ and the construction of an explicit class in $H^1_{Q_n-\mathrm{split}}(S, \mu_{2^n})$ which has order divisible by 2^{n-1}. The action of twists by characters of G'_n patches to an action of the formal split torus T of dimension t on $\mathrm{Specf}(R'_\infty)$. This action is free (that follows from that $\overline{\rho}$ has non solvable image). One proves that $\mathrm{Specf}(R_\infty)$ is a torsor of group $(\mu_2)^t$ on $\mathrm{Specf}(R^{\mathrm{inv}}_\infty)$, where R^{inv}_∞ is the ring of invariants of R'_∞ under the action of the torus T. In particular R_∞ is finite over R^{inv}_∞ and this gives the needed inequality of dimensions. Then an easy argument gives that R_∞ acts faithfully on M_∞.

The proof of the freeness of the module after inverting 2 one uses the usual argument of Daimond and Fujiwara which uses Auslander-Buchsbaum theorem.

References

[1] Fred Diamond. The Taylor-Wiles construction and multiplicity one. Invent. Math 128 (1997). 2, 379-391.

[2] Mark Dickinson. On the modularity of certain 2-adic Galois representations. Duke Math. J. 109 (2001), no. 2, 319–382.

[3] Kazuhiro Fujiwara. Galois deformations and arithmetic geometry of Shimura varieties International Congress of Mathematicians. Vol. II, 347–371, Eur. Math. Soc., Zürich, 2006.

[4] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture (II). Preprint. Available at http://www-irma.u-strasbg.fr/ wintenb/.

[5] Mark Kisin. Moduli of finite flat group schemes, and modularity. *preprint* (2004).

[6] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.

[7] Richard Taylor. On the meromorphic continuation of degree two L-functions. Documenta Math. Extra Volume: John H. Coates' Sixtieth Birthday (2006) 729–779.

[8] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[9] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

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