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Komplexe Analysis

Organised by
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August 24th – August 30th, 2008

A . The main aim of this workshop was to discuss recent developments in several complex variables and complex geometry. The topics included: classification of higher dimensional varieties, Kähler geometry and moduli spaces.

Mathematics Subject Classification (2000): 14xx, 32xx.

Introduction by the Organisers

The workshop *Komplexe Analysis*, organised by Jean-Pierre Demailly (Grenoble), Klaus Hulek (Hannover), Ngaiming Mok (Hong Kong) and Thomas Peternell (Bayreuth) was held August 24th–August 30, 2008. This meeting was well attended with 46 participants from Europe, US, and the Far East. The participants included several leaders in the field as well as many young (non-tenured) researchers.

The aim of the meeting was to present recent important results in several complex variables and complex geometry with particular emphasis on topics linking different areas of the field, as well as to discuss new directions and open problems. Altogether there were nineteen talks of 60 minutes each, a programme which left sufficient time for informal discussions and joint work on research projects.

One of the topics at the center of the conference was the classification theory of higher dimensional varieties. Y. Kawamata lectured on the connections between the minimal model programme and derived categories; A. Corti discussed an approach to the finite generation of the canonical ring without minimal models, but still in connection with the seminal work which was presented by J. McKernan in the last Complex Analysis meeting in Oberwolfach 2006, where the finite generation of the canonical ring of varieties of general type was announced. Extension theorems, non vanishing and positivity result for certain direct image sheaves play a role in the global classification of complex manifolds.

This was largely discussed by M. Paun and B. Berndtsson. In their work analytic methods are central, whereas the talks by Kawamata and Corti were more of an algebraic nature. Also very much on the analytic side and connected to Berndtsson's talk, H. Tsuji lectured on generalised Kähler-Einstein metrics. Families of projective manifolds over higher-dimensional base spaces were considered in the talk by S. Kebekus. Direct images of coherent sheaves also play a central role in this context.

About five years ago, Campana introduced new variations on the concept of "orbifolds"; they were already the suject of talks in past sessions and have turned out to be of increasing interest – in the present session, new results on the hyperbolicity of orbifolds were presented in the talk by E. Rousseau.

As to varieties with special geometry, K. Oguiso spoke on non-algebraic hyperkähler manifolds and, with a rather different flavour, F. Catanese on complex and real threefolds fibered by rational curves, with a special emphasis on real algebraic geometry. J. Chen discussed the influence of terminal singularities in three-dimensional geometry, a more algebraic topic. On the analytic side, A. Teleman reported on recent progress in the classification of non-Kähler surfaces in the so called Kodaira class VII, using gauge-theoretical methods, and S. K. Yeung lectured on new results on fake projective planes. Group actions and envelopes of holomorphy were the topics of the talk by X. Zhou. S. Boucksom discussed equidistribution of Fekete points on complex manifolds, in relation with energy functionals for Monge-Ampère operators.

R. Lazarsfeld presented a very interesting new approach to study properties of linear systems and line bundles via convex geometry.

Overall, moduli spaces appeared to be a central theme in the workshop, and were discussed extensively in at least four talks: V. Gritsenko considered moduli spaces of K3-surfaces; S. Grushevsky spoke on intersection numbers of divisor on the moduli space of curves, and K. Ludwig and G. Farkas lectured on the moduli spaces of spin and Prym curves, their singularities, Kodaira dimension and enumerative geometry.

Workshop: Komplexe Analysis

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Abstracts

Positivity properties of twisted relative canonical bundles

B B
(joint work with Mihai Paun)

The object of the talk is to construct certain metrics on relative canonical bundles, using (generalized) Bergman kernels. In order to explain the method we will start with a discussion of such metrics on one fixed manifold, and then move on to the relative case.

Let Z be a compact complex manifold and let L be a holomorphic line bundle over Z, equipped with a (possibly singular) metric, ϕ. This metric will always be assumed to have positive curvature current, so that $i\partial\bar{\partial}\phi \geq 0$. This metric on L induces a natural L^2-metric on the space of sections to the adjoint bundle $K_Z + L$, through

$$||u||^2_\phi = ||u||^2 = \int_Z |u|^2 \varepsilon.$$

From this L^2-metric we get a *Bergman kernel* defined by

$$B_\phi(z) = B(z) = \sup |u(z)|^2,$$

with the supremum taken over all global holomorphic sections of $K_Z + L$ with L^2-norm at most 1. (In case there is no global holomorphic section of finite norm, we let the Bergman kernel be 0.) Here of course the pointwise value of $u(z)$ depends on the choice of a local frame, so B is not a function but defines a metric on $K_Z + L$, $\psi = \log B$. More precisely, we can let χ be an arbitrary smooth metric on $K_Z + L$, and define B by

$$B(z)e^{-\chi} = \sup |u(z)|^2 e^{-\chi}.$$

Jointly with the Bergman kernel, we shall also consider an analogous construction for twisted multiples of the canonical bundle. This is defined by first letting

$$||u||^2_m = \int_Z |u|^{2/m} e^{-\phi/m}.$$

Then one can imitate the definition of the Bergman kernel by putting

$$B_{\phi,m}(z) = \sup |u(z)|^2,$$

this time taking the supremum over all global holomorphic sections to $mK_Z + L$ of m-norm not exceeding 1(see [8], [6], [10]) . By construction, $B_{m,\phi}$ is a metric on $mK_Z + L$ with the property that any global section of this bundle having finite $L^{2/m}$-norm, is *pointwise* bounded with respect to this metric.

0.1. **The relative case.** Let next X and Y be projective manifolds and let p be a surjective holomorphic map. (More generally, we could allow a proper surjective map from X to an open manifold Y and assume that X has some holomorphic line bundle F with a metric of strictly positive curvature.) We consider the relative canonical bundle

$$K_{X/Y} = K_X - p^*(K_Y).$$

For generic y in Y, the fiber $p^{-1}(y) = X_y$ is then smooth and the restriction of $K_{X/Y}$ to X_y is (isomorphic to) K_{X_y}. Let L be a line bundle over X, and let ϕ be a metric with semipositive curvature current on L. Over the generic fibers we can then construct the Bergman kernel, and the m-Bergman kernel in the way described in the introduction. This way we get naturally defined metrics on $K_{X/Y} + L$ and $mK_{X/Y} + L$ over the Zariski open set of generic fibers. These metrics have no immediate regularity properties as the fibers vary, but it is not too hard to check (using normal families) that they are at least upper semicontinuous.

Our first result says that the (relative) Bergman kernel metric that we obtain in this way over the set of generic fibers has nonegative curvature current, and extends to a metric on $K_{X/Y}+L$ over all of X that also has nonnegative curvature. This holds under the assumption that the Bergman kernel is not identically equal to zero, i e that there is at least some fiber on which $K_{X_y} + L$ has a section with finite L^2-norm.(The first result in this direction, in a non-twisted situation, is the very influential theorem of Fujita, [4].)

After that we consider the (Zariski open) set of y:s in the base Y where the dimension of $H^0(X_y, K_{X_y} + L)$ is minimal. Over this set we have a naturally defined vector bundle with fibers $H^0(X_y, K_{X_y} + L)$, and this bundle has a naturally defined metric, namley the L^2-metric. Note that this is a singular metric, and that (just like what happens for singular metrics on line bundles), some sections may have infinite norm. We define a notion of positivity for such singular metrics which generalizes Griffiths positivity in the nonsingular case, and prove that the L^2-metric is positive in this sense. This follows in the same way as the positivity of the Bergman kernel metric, but is a stronger property.

Using these notions we finally prove analogous results for the m-Bergman kernel, generalizing to the twisted case Kawamata's positivity theorem, [7], for multiples of the canonical bundle.

Our first result is a fairly simple consequence of the main result from [1] on positivity of direct image bundles, if we assume that the metric on L is smooth of nonnegative curvature, and that moreover our surjective map p is a smooth fibration. The main point in the present work is the extension to nonsmooth metrics and general surjective maps. To overcome the difficulty coming from nonsmoothness of the metric, we work in a Zariski dense Stein manifold, where we can regularize our metric, and then extend. The difficulty coming from nonsmoothness of the fibration is handled via an a priori estimate where the Ohsawa-Takegoshi extension theorem is the key point.

Both these issues require new ideas in the case of the m-Bergman kernel. In particular, it is not enough to work in a Stein subdomain since divisors are not removable for $L^{2/m}$ if $m > 1$. This is where our use of nonsmooth metrics on *vector* bundles comes in. Instead of regularizing our metric on L we regularize the nonsmooth metric on the vector bundle with fiber $H^0(X_y, K_{X_y} + L)$, which is a much simpler, local problem.

One application of these results is a Bergman kernel proof of the Kawamata subadjunction theorem, [7], another is an estimate for restricted volumes due to Takayama and Hacon-McKernan, [9],[5].

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Convergence towards equilibrium on complex manifolds

S´ B

(joint work with Robert Berman, David Witt Nystrom)

1. T

Let L be a holomorphic line bundle over a compact complex manifold X of complex dimension n. Following [3], let (E, ϕ) be a *weighted subset*, that is a compact subset E of X together with the weight ϕ of a continuous Hermitian metric $e^{-\phi}$ on the restriction $L|_E$. Finally let μ be a probability measure supported by E.

The asymptotic study as $k \to \infty$ of the space of global sections $s \in H^0(X, kL)$ endowed with either the L^2 norm

$$||s||^2_{L^2(\mu,k\phi)} := \int_X |s|^2 e^{-2k\phi} d\mu$$

or the L^∞ norm

$$||s||_{L^\infty(E,k\phi)} := \sup_E |s|e^{-k\phi}$$

is a natural generalization of the classical theory of orthogonal polynomials. The latter indeed corresponds to the case

$$E \subset \mathbb{C}^n \subset \mathbf{P}^n =: X$$

endowed with the tautological ample bundle $\mathcal{O}(1) =: L$. It is of course well-known that $H^0(\mathbf{P}^n, \mathcal{O}(k))$ identifies to the space of polynomials of total degree at most k. The section of L cutting out the hyperplane at infinity induces a flat Hermitian metric on L over $\mathbb{C}^n$, so that a continuous weight ϕ on $L|_E$ is naturally identified with a function in $C^0(E)$. On the other hand, a psh function on $\mathbb{C}^n$ with at most logarithmic growth at infinity gets identified with the weight ϕ of a non-negatively curved (singular) Hermitian metric on L, which will thus be referred to as a *psh weight*.

Our geometric setting is therefore seen to be a natural (and more symmetric) extension of so-called *weighted potential theory* in the classical case. It also contains the case of *spherical polynomials* on the round sphere $S^n \subset \mathbb{R}^{n+1}$.

Indeed, the space of spherical polynomials of total degree at most k is by definition the image by restriction to S^n of the space of all polynomials on $\mathbb{R}^{n+1}$ of degree at most k. It thus coincides with (the real points of) $H^0(X, kL)$ with X being the smooth quadric hypersurface

$$\{X_1^2 + .. + X_n^2 = X_0^2\} \subset \mathbf{P}^{n+1}$$

endowed with the very ample line bundle $L := \mathcal{O}(1)|_X$. Here we take $E := S^n = X(\mathbb{R})$, and the section cutting out the hyperplane at infinity again identifies weights on L to certain functions on the affine piece of X.

In view of the above dictionary, one is naturally led to introduce the *equilibrium weight* of (E, ϕ) as

$$\phi_E := \sup\{\psi \text{ psh weight on } L,\ \psi \le \phi \text{ on } E\}, \tag{1.1}$$

whose upper semi-continuous regularization ϕ_E^* is a psh weight on L as soon as E is non-pluripolar, which will always be assumed.

The *equilibrium measure* of (E, ϕ) is then defined as the Monge-Ampère measure of ϕ_E^* normalized to unit mass:

$$\mu_{\mathrm{eq}}(E, \phi) := M^{-1}\mathrm{MA}\,(\phi_E^*).$$

This measure is concentrated on E, and we have $\phi = \phi_E^*$ a.e. with respect to it.

This approach is least technical when L is *ample*, but the natural setting appears to be the more general case of a *big* line bundle, which is the one considered in the present paper, following our preceding work [3]. As is shown there, the Monge-Ampère measure $\mathrm{MA}\,(\psi)$ of a psh weight ψ with minimal singularities, defined as the Beford-Taylor top-power $(dd^c\psi)^n$ of the curvature $dd^c\psi$ on its bounded locus, is well-behaved. Its total mass M is in particular an invariant of the big line bundle L, and in fact coincides with the *volume* $\mathrm{vol}(L)$, characterized by

$$N_k := \dim H^0(kL) = \mathrm{vol}(L)\frac{k^n}{n!} + o(k^n).$$

The main goal of the present paper is to give a general criterion involving spaces of global sections that ensures convergence of certain sequences of probability measures on E towards the equilibrium measure $\mu_{\mathrm{eq}}(E, \phi)$.

2. F

Let (E, ϕ) be a weighted subset as above. A *Fekete configuration* is a finite subset of points maximizing the determinant in the interpolation problem.

More precisely, let $N := \dim H^0(L)$ and

$$P = (x_1, ..., x_N) \in E^N$$

be a configuration of points in the given compact subset E. Then P is said to be a Fekete configuration for (E,ϕ) iff it maximizes the determinant of the evaluation operator

$$\mathrm{ev}_P : H^0(L) \to \oplus_{j=1}^N L_{x_j}$$

with respect to a given basis $s_1, ..., s_N$ of $H^0(L)$, that is the Vandermonde-type determinant

$$\left|\det(s_i(x_j))\right| e^{-(\phi(x_1)+...+\phi(x_n))}.$$

This condition is independent of the choice of the basis (s_j).

If $P = (x_1, ..., x_N) \in X^N$ is a configuration, then we let

$$\delta_P := \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j}.$$

Our first main result is an equidistribution result for Fekete configurations.

Theorem A. Let $P_k \in E^{N_k}$ be a Fekete configuration for $(E, k\phi)$. Then the P_k equidistribute towards the equilibrium measure, that is

$$\lim_{k\to\infty} \delta_{P_k} = \mu_{\mathrm{eq}}(E,\phi)$$

in the weak topology of measures.

Theorem A first appeared in the first two named authors' preprint [4]. It will be obtained here as a consequence of a more general convergence result (Theorem C below).

In the classical one-variable situation, this result is well-known. In the several-variable classical situation, this result has been conjectured for quite some time, probably going back to the pioneering work of Leja in the late 50's.

As explained above, the spherical polynomials situation corresponds to the round sphere S^n embedded in its complexification, the complex quadric hypersurface in $\mathbf{P}^{n+1}$. This special case of Theorem A thus yields:

Corollary A. Let $E \subset S^n$ be a compact subset of the round n-sphere, and for each k let $P_k \in E^{N_k}$ be Fekete configuration of degree k for E (also called *extremal fundamental system* in this setting). Then δ_{P_k} converges to the equilibrium measure $\mu_{\mathrm{eq}}(E)$ of E.

This is a generalization of the recent result of Morza and Ortega-Cerdà [8] on equidistribution of Fekete points on the sphere, which corresponds to the case $E = S^n$ whose equilibrium measure $\mu_{\mathrm{eq}}(S^n)$ is just the rotationally invariant probability measure on S^n.

3. B -M

Let again (E,ϕ) be a weighted subset, and let μ be a probability measure on E. The distorsion between the natural L^2 and L^∞ norms on $H^0(L)$ introduced above is locally acounted for by the *distorsion function* $\rho(\mu,\phi)$, whose value at $x \in E$ is defined by

$$\rho(\mu,\phi)(x) = \sup_{\|s\|_{L^2(\mu,\phi)}=1} |s(x)|^2_\phi, \tag{3.1}$$

the squared norm of the evaluation operator at x.

The function $\rho(\mu, \phi)$ is known as the *Christoffel-Darboux function* in the orthogonal polynomials literature. It sometimes also appears under the name *density of states function*, since the *probability* measure

$$\beta(\mu, \phi) := N^{-1}\rho(\mu, \phi)\mu, \tag{3.2}$$

which will be referred to as the *Bergman measure,* can be interpreted as a dimensional density for $H^0(L)$.

When μ is a smooth positive volume form on X and ϕ is smooth and strictly psh, the celebrated Bouche-Catlin-Tian-Zelditch theorem asserts that $\beta(\mu, k\phi)$ admits a full asymptotic expansion in the space of smooth volume forms, with $M^{-1}(dd^c\phi)^n$ as the dominant term.

As was shown by the first named author (in [1] for the $\mathbf{P}^n$ case and in [2] for the general case), part of this result still holds when the positive curvature asumption on ϕ is dropped. More specifically, the norm distorsion still satisfies

$$\sup_X \rho(\mu, k\phi) = O(k^n) \tag{3.3}$$

and the Bergman measures still converge towards the equilibrium measure:

$$\lim_{k\to\infty} \beta(\mu, k\phi) = \mu_{\mathrm{eq}}(X, \phi) \tag{3.4}$$

now in the weak topology of measures.

Both of these results fail when E, μ and ϕ are more general. However *sub-exponential* growth of the distorsion between $L^2(\mu, k\phi)$ and $L^\infty(E, k\phi)$ norms, that is

$$\sup_E \rho(\mu, k\phi) = O(e^{\varepsilon k}) \text{ for all } \varepsilon > 0, \tag{3.5}$$

appears to be a much more robust condition. Following a standard terminology, the measure μ will be said to be *Bernstein-Markov* for (E, ϕ) when (3.5) holds.

When $E = X$, *any* continuous measure is Bernstein-Markov for (X, ϕ) by the mean-value inequality.

Our second main result asserts that convergence of Bergman measures to equilibrium as in (3.4) holds for arbitrary Bernstein-Markov measure.

Theorem B. Let μ be a Bernstein-Markov measure for (E, ϕ). Then

$$\lim_{k\to\infty} \beta(\mu, k\phi) = \mu_{\mathrm{eq}}(E, \phi)$$

in the weak topology of measures.

In the classical one-variable setting, this theorem was obtained, using completely different methods, by Bloom and Levenberg [7]. A slightly less general version of Theorem B (dealing only with *stably* Bernstein-Markov measures) was first obtained in the first and third named author's preprint [5]. Theorem B will here be obtained as a special case of Theorem C below.

4. D ’ $\mathcal{L}$-

We now state our third main result, which is a general criterion ensuring convergence of Bergman measures to equilibrium in terms of $\mathcal{L}$-functionals, first introduced by Donaldson. This final result actually implies Theorem A and B above, as well as a convergence result for so-called *optimal measures* first obtained in [6].

The L^2 and L^∞ norms on $H^0(kL)$ introduced above are described geometrically by their unit balls, that will be denoted respectively by

$$\mathcal{B}^\infty(\mu, k\phi) \subset \mathcal{B}^2(E, k\phi) \subset H^0(kL).$$

We fix a reference weighted subset (E_0, ϕ_0), which should be taken to be the compact torus endowed with the standard flat weight in the classical $\mathbb{C}^n$ case. We can then normalize the Haar measure vol on $H^0(kL)$ by

$$\operatorname{vol} \mathcal{B}^\infty(E_0, k\phi_0) = 1,$$

and we introduce the following slight variants of Donaldson's $\mathcal{L}$-functional:

$$\mathcal{L}_k(\mu, \phi) := \frac{1}{2kN_k} \log \operatorname{vol} \mathcal{B}^2(\mu, k\phi)$$

and

$$\mathcal{L}_k(E, \phi) := \frac{1}{2kN_k} \log \operatorname{vol} \mathcal{B}^\infty(E, k\phi).$$

The main result of [3] can then be reformulated as

$$\lim_{k\to\infty} \mathcal{L}_k(E, \phi) = \mathcal{E}_{\mathrm{eq}}(E, \phi). \tag{4.1}$$

Here

$$\mathcal{E}_{\mathrm{eq}}(E, \phi) := \mathcal{E}(\phi_E^*)$$

denotes the *energy at equilibrium* of (E, ϕ) (with respect to (E_0, ϕ_0)), with $\mathcal{E}(\psi)$ standing for the *Aubin-Yau energy* of a psh weight ψ with minimal singularities, characterized as the primitive of the Monge-Ampère operator:

$$\frac{d}{dt}_{t=0_+} \mathcal{E}(t\psi_1 + (1-t)\psi_2) = \frac{1}{M} \int_X (\psi_1 - \psi_2) \mathrm{MA}\,(\psi_2)$$

normalized by

$$\mathcal{E}(\phi_{0,E_0}^*) = 0.$$

Note that we have actually divided the Aubin-Yau energy considered in [3] by the harmless constant $(n+1)M$ for convenience.

Since $\mathcal{L}_k(\mu, \phi) \geq \mathcal{L}_k(E, \phi)$ for any probability measure μ on E, (4.1) shows in particular that the energy $\mathcal{E}_{\mathrm{eq}}(E, \phi)$ at equilibrium is an *a priori* asymptotic lower bound for $\mathcal{L}_k(\cdot, \phi)$. Our final result describes what happens for asymptotically minimizing sequences:

Theorem C. Let μ_k be a sequence of probability measures on E such that

$$\lim_{k\to\infty} \mathcal{L}_k(\mu_k, \phi) = \mathcal{E}_{\mathrm{eq}}(E, \phi).$$

Then the associated Bergman measures satisfy

$$\lim_{k\to\infty} \beta(\mu_k, k\phi) = \mu_{\mathrm{eq}}(E, \phi)$$

in the weak topology of measures.

The condition bearing on the sequence (μ_k) in Theorem C is independent of the choice of the reference weighted subset (E_0, ϕ_0). In fact (4.1) shows that it can equivalently be written as the condition

$$\log \frac{\operatorname{vol} \mathcal{B}^2(\mu_k, k\phi)}{\operatorname{vol} \mathcal{B}^\infty(E, k\phi)} = o(kN_k),$$

which can be understood as a *weak Bernstein-Markov condition* on the sequence (μ_k), relative to (E, ϕ).

The proof of Theorem C is closely related to the generalization of Yuan's equidistribution theorem for generic points of asymptotically minimal heigh obtained in [3].

As a consequence of Theorem C, we also recover the main result of [6]. Following the latter paper, we say that a measure μ is *optimal* for (E, ϕ) if it realizes the minimum of $\mathcal{L}(\cdot, \phi)$ over the set $\mathcal{P}_E$ of all probability measures on E. This is equivalent to requiring that the norm distorsion $\sup_E \rho(\cdot, \phi)$ achieves its minimum over $\mathcal{P}_E$, to wit N, at μ. As a corollary to Theorem C, we get

Corollary C. If μ_k is an optimal measure for $(E, k\phi)$, then

$$\lim_{k\to\infty} \mu_k = \mu_{\mathrm{eq}}(E, \phi).$$

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Threefolds fibred by rational curves and the Nash conjecture

F C
(joint work with Frédéric Mangolte)

1. I

An established principle in complex algebraic geometry is that the Kodaira dimension of a smooth complex projective variety W of dimension n strongly influences the topology of the set $W(\mathbb{C})$ of its complex points. This principle is clearly manifest already in dimension 1, and related to other points of view, as the uniformization theorem, and the concept of curvature. This principle, although in a more difficult and complicated way, still goes on to hold in higher dimensions. Indeed the principle holds also in some way in real algebraic geometry.

Assume in fact that W is a smooth real projective variety and consider the topology of the set $W(\mathbb{R})$ of its real points. In dimension 1, the connected components are just diffeomorphic to the circle S^1, and the so called Harnack inequality says that their number m is bounded from above by $g + 1$, g being the genus of W.

In dimension 2, Comessatti proved in 1914 that if M is a connected component of the set $W(\mathbb{R})$ of real points of a (geometrically) rational real surface W, and M is orientable, then M is not of hyperbolic type. This means that either M is diffeomorphic to a sphere S^2 (a quadric of elliptic type), or to a torus $S^1 \times S^1$ (a quadric of parabolic type).

The theorem is sharp since in the non orientable case, every non orientable surface of Euler number $e = 2 - b_1$ (we consider here homology with coefficients in $\mathbb{Z}/2$) can be obtained by blowing up b_1 real points on the sphere S^2.

Unaware of Comessatti's work, John Nash in 1952, while showing that every compact differentiable manifold is diffeomorphic to a connected component M of the set $W(\mathbb{R})$ of real points of a smooth real algebraic variety W, asked whether the same could be true if one also requires W to be a rational variety.

As we saw, this is false by Comessatti' theorem.

Comessatti's result can today be easily understood and indeed extended to the case of real algebraic surfaces with geometric genus $p_g = 0$. Its proof is based on the following facts:

1) If M is orientable, its cohomology class is nontrivial and invariant for the involution determined by complex conjugation

2) the Algebraic Index Theorem (proved by Severi in 1913), shows that (since the hyperplane class is anti-invariant and with positive selfintersection) on the invariant part of the second cohomology group of $W(\mathbb{C})$ the intersection form is semi-negative definite

3) complex conjugation yields an isomorphism of the real normal bundle to M with its real tangent bundle, hence the self intersection of M equals $2 - 2g$, where g is the genus of M.

What happens of Nash's question in higher dimension ?

One may ask to which extent Kodaira dimension equal to $-\infty$ poses strong restrictions on the topology of a connected component N of $W(\mathbb{R})$. Or, ask the same question under the more stringent condition that W be rationally connected.

In an extremely interesting series of four papers Janos Kollár used the recent progress on the minimal model program for threefolds in order to understand the topology of the connected components $N \subset W(\mathbb{R})$ in the case where W has Kodaira dimension $-\infty$, especially in the case where the minimal model program yields a conic bundle fibration or a del Pezzo fibration.

In joint work with Mangolte ([1] [2]) we answered in the positive four questions posed by Kollár in the paper [4, Theorem 1.1].

The situation is as follows: let $f\colon W \to X$ be a real smooth projective threefold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable. Then, by [4, Theorem 1.1], a connected component $M \subset W(\mathbb{R})$ is obtained from a Seifert fibred manifold N' or from a connected sum N' of lens spaces by taking connected sums with a finite number a of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number b of copies of $S^1 \times S^2$, and one may assume that the number $a + b$ be maximal, and this decomposition is unique by a theorem of Milnor [5].

Consider the integers $k := k(N)$ and $n_l := n_l(N)$, $l = 1 \dots k$ defined as follows (and again well defined by Milnor's theorem):

(i) if $g\colon N' \to F$ is a Seifert fibration, k denotes the number of multiple fibres of g and $2 \le n_1 \le n_2 \le \cdots \le n_k$ denote the respective multiplicities;

(ii) if N' is a connected sum of lens spaces, k denotes the number of lens spaces and $n_1 \le n_2 \le \cdots \le n_k$, $n_l \ge 3$, $\forall l$, the orders of the respective fundamental groups (thus we have a decomposition $N' = \#_{l=1}^{k}(L(n_l, q_l)$ for some $1 \le q_l < n_l$ relatively prime to n_l).

Theorem 1. *Let $W \to X$ be a real smooth projective threefold fibred by rational curves over a geometrically rational surface X (these assumptions are equivalent to: W rationally connected and fibred by rational curves). Suppose that $W(\mathbb{R})$ is orientable. Then, for each connected component $N \subset W(\mathbb{R})$, $k(N) \le 4$ and $\sum_l (1 - \frac{1}{n_l(N)}) \le 2$. Furthermore, if N' is Seifert fibred over $S^1 \times S^1$, then $k(N) = 0$.*

The above result should be viewed as an analogue of Comessatti's theorem in dimension three, since it asserts that, if the base of the Seifert fibration is orientable, then it is not an orbifold of hyperbolic type.

The proof of Theorem 1 goes by reducing the proof of the estimate for the integers $n_l(N)$ to an inequality depending on the indices of certain singular points of a real component M of the topological normalization of $X(\mathbb{R})$ (obtained by replacing the singular points of $X(\mathbb{R})$ by its local branches).

Recall that a real surface singularity will be said to be *of type A^+_μ* if it is real analytically equivalent to

$$x^2 + y^2 - z^{\mu+1} = 0,\ \mu \ge 1\ ;$$

and *of type A^-_μ* if it is real analytically equivalent to

$$x^2 - y^2 - z^{\mu+1} = 0,\ \mu \ge 1\ .$$

In the above mentioned process, the number $k(N)$ can be made to correspond to the number of real singular points on M which are of type A^+_μ, and globally separating when μ is odd; each number $n_l(N) - 1$ corresponds to the index μ_l of the singularity $A^+_{\mu_l}$ of M.

One of the main technical results of the second paper is the following.

Main Theorem. *Let X be a projective surface defined over $\mathbb{R}$. Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of the topological normalization $\overline{X(\mathbb{R})}$ contains at most 4 singular points x_l of type $A^+_{\mu_l}$ which are globally separating for μ_l odd. Furthermore, their indices satisfy*

$$\sum(1 - \frac{1}{\mu_l + 1}) \leq 2\ .$$

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Three dimensional terminal singularities, Riemann-Roch formula and its applications to birational geometry

J A C

(joint work with Meng Chen, Christopher Hacon)

Three dimensional terminal singularises are classified due to the work od Reid and Mori. Basically, they are isolated singularities, cyclic quotient of compound DuVal singularities, usually denoted by cDV/μ_r. Each singularises can be deformed into a collection of cyclic quotient singularities of type $\frac{1}{r}(1, -1, b)$. The collection of these cyclic quotient singularities coming from the singularties of X is called the *basket* of X.

Miles Reid derived a Riemann-Roch form for threefolds with canonical singularities, by considering the contribution from singularities. It turns out that there is a formula for Euler characteristics depending on basket od X.

We study the baskets and Riemann-Roch in a more systematical way. We obtained a method which allows us to solve for baskets with given Euler characteristic. This method gives various application in birational geometry.

For example, if X is a minimal threefold of general type. Suppose that $\chi(mK_X) \geq 2$ for some $2 \leq m \leq 12$, then one can obtained a lower bound on $Vol(X)$ by some geometric method. Our method allows us to classified baskets with $\chi(mK_X) \leq 2$ for all $2 \leq m \leq 12$. Combining all theses, we prove that:

Theorem 1. Let X be a threefold of general type. Then the following holds.

(i) $P_{12} > 0, P_{24} > 1$.
(ii) $Vol(X) \geq 1/2660$.
(iii) The pluricanonical map φ_m is birational for all $m \geq 77$.

Similar technique can be applied to weak $\mathbb{Q}$-Fano threefolds as well.

Theorem 2. Let X be a weak $\mathbb{Q}$-Fano threefold. Then the following holds.

(i) $P_{-6} > 0, P_{-2k} > 1$ for all $k \geq 4$.
(ii) $-K^3 \geq 1/330$. This bound is sharp.

Moreover, we consider basket of X as an invariant of X and then study its behavior under some elementary birational map. Using this, we are able to give an effective termination of flips.

A final remark is that we also derived some new inequalities between Euler characteristics. One can check out [1] for a brief introduction and [2, 3, 4, 5] for more details.

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Towards finite generation without Minimal Models (work of V. Lazić)

A C

The finite generation of the canonical ring of (nonsingular, projective) algebraic varieties in characteristic 0 is now a theorem [1]. In this talk I propose a new direct approach to the proof, based on a sort of hyperplane section principle and induction on dimension.

Let X be a nonsingular projective variety, Λ a finitely generated semigroup and $D\colon \Lambda \to \operatorname{Div} X$ an additive map to the space of (integral, say, or rational) divisors on X. A *divisorial algebra* on X is an algebra of the form

$$R(X, D) = \oplus_{\lambda\in\Lambda} H^0(X, D(\lambda))$$

A divisorial algebra is *adjoint* if

$$D(\lambda) = r(\lambda)(K + \Delta(\lambda))$$

for some additive map $r\colon \Lambda \to \mathbb{Q}_+$ and $\Delta\colon \Lambda \to \operatorname{Div} X$ such that the pair $(X, \Delta(\lambda))$ is klt.

The finite generation conjecture states that a divisorial adjoint algebra is always finitely generated; as I said, this is now a theorem.

Property P. Fix a general small ample $\mathbb{Q}$-divisor A on X. Consider a snc divisor $B = \sum B_i \subset X$; denote by $\mathcal{B}$ the "box" $\{\Theta = \sum b_i B_i \mid 0 \leq b_i \leq 1\}$. We say that property P holds if for every component G of $\mathcal{B}$:

(i) $\mathcal{P}^G_A = \{\Theta \in \mathcal{B} \mid G \not\subset \mathbf{B}(K+A+\Theta)\}$ is a rational polyhedron. (Where, for a divisor D, $\mathbf{B}(D)$ denotes the stable base locus.)
(ii) $\Theta \in \mathcal{P}^G_A(\mathbb{Q})$ if and only if the 'Lelong number:'

$$\nu_G|K + A + \Theta| := \lim_{n\to\infty} \frac{1}{n}\text{mult}_G\, |K + A + \Theta| = 0.$$

In the talk I explain some ideas in the proof the following somewhat tentative statement:

Theorem (Lazić). Assume Property P. Then, if finite generation holds in dimension $n-1$, then finite generation holds in dimension n.

The proof is a transparent induction on the dimension. (I should say that this is work in progress and the statement just given is still provisional.) I believe that Property P is within reach of the analytic methods in nonvanishing theory, *see* for example the work of Mihai Paun. Hence, these ideas constitute a new approach to finite generation not relying on the detailed machinery of the minimal model program.

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The birational type of moduli spaces of curves with level structure

G F

The main aim of this work is to determine the birational type of two moduli spaces of curves with level two structure, the moduli space $\overline{\mathcal{R}}_g$ of Prym curves and the moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves.

First we study the moduli stack $\mathcal{R}_g$ classifying pairs (C, η) where $[C] \in \mathcal{M}_g$ is a smooth curve of genus g and $\eta \in \mathrm{Pic}^0(C)[2]$ is a torsion point of order 2 giving rise to an étale double cover of C. We denote by $\pi : \mathcal{R}_g \to \mathcal{M}_g$ the natural projection forgetting the point of order 2 and by $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$ the Prym map given by

$$P(C, \eta) := \mathrm{Ker}\{f_* : \mathrm{Pic}^0(\widetilde{C}) \to \mathrm{Pic}^0(C)\}^0,$$

where $f : \widetilde{C} \to C$ is the étale double covering determined by η. It is known that P is generically injective for $g \geq 7$ (cf. [FS]), hence one can view $\mathcal{R}_g$ as a birational model for the moduli stack of Prym varieties of dimension $g-1$. If $\overline{\mathcal{R}}_g$ denores the normalization of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_g$, then it is known that $\overline{\mathcal{R}}_g$ is isomorphic to the coarse moduli stack of Beauville admissible double covers (cf. [B]), and also to the stack of Prym curves in the sense of [BCF], that is, $\overline{\mathcal{R}}_g = \overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_2)$. It is known that the space $\mathcal{R}_g$ is unirational for $g \leq 6$ (cf. [D]). Verra has recently announced a proof of the unirationality of $\mathcal{R}_7$. The main result (obtained jointly with K. Ludwig) is the following:

Theorem 0.1. The moduli space $\overline{\mathcal{R}}_g$ is of general type for all $g > 13, g \neq 15$.

The strategy of the proof is similar to the one used by Harris and Mumford for proving that $\overline{\mathcal{M}}_g$ is of general type for large g (cf. [HM]). One first computes the canonical class $K_{\overline{\mathcal{R}}_g}$ in terms of the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)$ and then shows that $K_{\overline{\mathcal{R}}_g}$ is effective for $g > 13$ by explicitly computing the class of a specific effective divisor on $\overline{\mathcal{R}}_g$ and comparing it to $K_{\overline{\mathcal{R}}_g}$. The divisors we construct are of two types, dpending on whether g is even or odd. We also show that for $g \geq 4$ any pluricanonical form on $\overline{\mathcal{R}}_{\mathrm{g,reg}}$ automatically extends to any desingularization. This is a key ingredient in carrying out the program of computing the Kodaira dimension of $\overline{\mathcal{R}}_g$.

In the odd genus case we we set $g = 2i + 1$ and consider the vector bundle Q_C defined by the exact sequence

$$0 \longrightarrow Q_C^\vee \longrightarrow H^0(K_C) \otimes \mathcal{O}_C \to Q_C \longrightarrow 0.$$

(If other words, Q_C is the normal bundle of C embedded in its Jacobian). It is well-known that Q_C is a semi-stable vector bundle of rank $g-1$ on C of slope $\nu(Q_C) = 2 \in \mathbb{Z}$, so it makes sense to look at the theta divisors of its exterior powers. Recall that

$$\Theta_{\wedge^i Q_C} = \{\xi \in \mathrm{Pic}^{g-2i-1}(C) : h^0(C, \wedge^i Q_C \otimes \xi) \geq 1\},$$

and the main result from [FMP] identifies this locus with the difference variety $C_i - C_i \subset \mathrm{Pic}^0(C)$.

Theorem 0.2. For $g + 2i + 1$, the locus E_i consisting of those points $[C, \eta] \in \mathcal{R}_{2i+1}$ such that $\eta \in \Theta_{\wedge^i Q_C}$, is an effective divisor on $\mathcal{R}_{2i+1}$. Its class on $\overline{\mathcal{R}}_{2i+1}$ is given by the formula

$$E_i \equiv \frac{2}{i}\binom{2i-2}{i-1} \cdot \Big((3i+1)\lambda - \frac{i}{2}\delta_0^u - \frac{2i+1}{4}\delta_0^r - (\text{ higher boundary divisors})\Big).$$

This proves our main result in the odd genus case. The divisors we consider for even genus are of Koszul type in the sense of [F].

Theorem 0.3. For $g = 2i + 6$, the locus D_i of those $[C, \eta] \in \mathcal{R}_{2i+6}$ such that the Koszul cohomology group $K_{i,2}(C, K_C + \eta)$ does not vanish (or equivalently, $(C, K_C + \eta)$ fails the Green-Lazarsfeld property (N_i)), is a virtual divisor on $\mathcal{R}_{2i+6}$. Its class on $\overline{\mathcal{R}}_{2i+6}$ is given by the formula:

$$D_i \equiv \frac{1}{2}\binom{2i+2}{i}\Big(\frac{6(2i+7)}{i+3}\lambda - 2\delta_0^u - 3\delta_0^r - \cdots\Big).$$

In both Theorems 0.2 and 0.3, $\lambda \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ denotes the Hodge class and $\pi^*(\delta_0) = \delta_0^u + 2\delta_0^r$ (that is δ_0^r is the ramification divisor of π whereas δ_0^u is the complement of the ramification divisor in the pull-back of the boundary divisor δ_0 from $\overline{\mathcal{M}}_g$.). The boundary divisors δ_0^u and δ_0^r have clear modular description in terms of Prym curves and the same holds for the higher boundary divisors.

We have similar results for moduli spaces of spin curves. We mention the following theorem cf. [F1]:

Theorem 0.4. The compact moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves of genus g is of general type for $g > 8$.

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Moduli spaces and Automorphic forms

V G
(joint work with Klaus Hulek and Gregory Sankaran)

In my talk I give a review of our joint project (see [3]–[7]) on the geometry of modular varieties of orthogonal type. The basic example of such varieties is *the moduli space* $\mathcal{F}_{2d}$ *of polarised* K3 *surfaces of degree* $2d$.

Let L be an integral lattice of signature $(2, n)$. The lattice L determines the hermitian homogeneous domain of type IV

$$\mathcal{D}(L) = \{Z \in \mathbb{P}(L \otimes \mathbb{C}) \,|\, (Z, Z) = 0,\ (Z, \overline{Z}) > 0\}^+$$

($^+$ denotes a connected component). $\mathrm{O}^+(L)$ is the subgroup of index 2 of the integral orthogonal group fixing $\mathcal{D}(L)$. We define the stable integral orthogonal group $\widetilde{\mathrm{O}}^+(L) = \{g \in \mathrm{O}^+(L) \mid g|_{A_L} = \mathrm{id}\}$ where $A_L = L^\vee/L$ is the discriminant group. The main object to study is a quasi-projective modular variety

$$\mathcal{F}(L) = \widetilde{\mathrm{O}}^+(L) \setminus \mathcal{D}(L).$$

Examples. 1) Let be $L_{2d} = 2U \oplus 2E_8(-1) \oplus < -2d >$ where U is the hyperbolic plane. Then $\mathcal{F}(L_{2d}) = \mathcal{F}_{2d}$ according to the global Torelli theorem for the polarised K3 surfaces.

2) Let be $L_{2,2d} = L_{2d} \oplus < -2 >$. Then $\mathcal{F}(L_{2,2d})$ is the periodic domain of the split-polarised irreducible symplectic 4-folds deformationally equivalent to $\mathrm{K3}^{[2]}$ (see [6]). We note that $\dim \mathcal{F}(L_{2,2d}) = 20$.

The programm on the K3 surfaces and their moduli spaces was formulated by A. Weil in 1956. In the next twenty years all questions were solved except the problem on the birational type of the moduli spaces of polarised K3 surfaces. Our main result is the following.

Main Theorem (see [4], [6]). *The moduli space $\mathcal{F}_{2d}$ of* K3 *surfaces with a polarisation of degree* $2d$ *is of general type for any* $d > 61$ *and for* $d = 46, 50, 54, 57, 58$ *and* 60. *If* $d \geq 40$ *and* $d \neq 41, 44, 45$ *or* 47 *then the Kodaira dimension of* $\mathcal{F}_{2d}$ *is non-negative.*

The moduli space of polarised deformation $\mathrm{K3}^{[2]}$ *manifolds with polarisation of degree* $2d$ *and split type is of general type if* $d \geq 12$.

We note that Mukai proved that the space $\mathcal{F}_{2d}$ is unirational if $d \leq 10$ and $d = 12, 17, 19$ (see [8] and the references there).

The proof of the main theorem is based on the three general principles.

Principle of high rank (see [4]). *Let L be a lattice of signature* $(2, n)$ *with* $n \geq 9$. *Then there exists a toroidal compactification* $\overline{\mathcal{F}}(L)$ *having only canonical singularities. There are no fixed divisors in the boundary. The branch divisors of* $\mathcal{D}(L) \to \mathcal{F}(L)$ *are induced by elements* $g \in \widetilde{\mathrm{O}}^+(L)$ *such that* $\pm g$ *is a reflection with respect to a vector in L.*

In fact, if $6 \leq \operatorname{rank}(L) \leq 8$ then non-canonical singularities are rather rare and there is a possibility to describe all of them for any fixed L.

Automorphic principle (see [4]). *Let L be an integral lattice of signature $(2,n)$, $n \geq 9$. The modular variety $\mathcal{F}(L)$ is of general type if there exists a non-zero cusp (with zero of order one at infinity) form $F_a \in S_a(\widetilde{\mathrm{O}}^+(L),\chi)$ of small weight $a < n$ that vanishes along the branch divisor of the projection $\pi\colon \mathcal{D}(L) \to \mathcal{F}(L)$.*

We note that the character χ of $\widetilde{\mathrm{O}}^+(L)$ in the last principle is usually equal to determinate. This is explained by the next theorem.

Theorem (see [7]). *Let L be an even integral lattice containing at least two hyperbolic planes, such that* $\operatorname{rank}_2(L) \geq 6$ *and* $\operatorname{rank}_3(L) \geq 5$. *Then*

$$\widetilde{\mathrm{O}}^+(L)/[\widetilde{\mathrm{O}}^+(L), \widetilde{\mathrm{O}}^+(L)] \cong \mathbb{Z}/2\mathbb{Z}.$$

For such L the orthogonal group $\widetilde{\mathrm{O}}^+(L)$ has only one non-trivial character det.

As a corollary we obtain that if $L = 2U \oplus L_0$ is a lattice of signature $(2,n)$ and F is a modular form with character det or trivial character for $\widetilde{\mathrm{O}}^+(L)$, then the order of vanishing of F along any boundary component of $\mathcal{D}(L)$ is an integer.

The branch divisor of the projection $\pi\colon \mathcal{D}(L) \to \mathcal{F}(L)$ determines the main obstruction for continuation of the pluri-canonical differential forms on a smooth compact model of $\mathcal{F}(L)$. If the branch divisor would be smaller, then using the automorphic forms from [2] we get a much better result than in the main theorem.

Theorem (see [4]). *The moduli space $\mathcal{SF}_{2d} = \widetilde{\mathrm{SO}}^+(L_{2d})\backslash\mathcal{D}(L_{2d})$ of* K3 *surfaces of degree $2d$ with a spin structure is of general type if $d \geq 3$.*

For the orthogonal group $\widetilde{\mathrm{O}}^+(L)$ the branch divisor is much larger and we use

E_8-principle. *Let assume that there exists an embedding of a lattice M in the even unimodular lattice E_8 such that the number of roots in E_8 orthogonal to M is positive and is smaller than $12 + 2(\operatorname{rank} M)$. Then the modular variety $\mathcal{F}(2U \oplus 2E_8(-1) \oplus M(-1))$ is of general type.*

We did not formulate the E_8-principle in our papers but it was one of the basic point of [4] and [6]. To make it a theorem we have to add some technical conditions on the lattice M (a condition on the discriminant group and on the rank of M) but in principle it works. The main technical tool in this part is the Borcherds modular form Φ_{12} of the (singular) weight 12 with character det on $\mathcal{D}(2U \oplus 3E_8(-1))$ (see, e.g. [1]). In fact the E_8-principle gives us a cusp form of a small weight (smaller than the canonical weight) with a big divisor containing the branch divisor of the modular projection.

In order to apply the E_8-principle to the cases of the moduli spaces of polarised K3 surfaces and the irreducible symplectic 4-folds we want to know for which $2d > 0$ there exists a vector

$$l \in E_8,\ l^2 = 2d,\ l \ \text{ is orthogonal to at least 2 and at most 12 roots}$$

(the case of the polarised K3 surfaces) and

$$l \in E_7,\ l^2 = 2d,\ l \ \text{ is orthogonal to at least 2 and at most 14 roots}$$

(the case of the polarised symplectic 4-folds).

Theorem (see [4], [6]). *Such a vector l in E_8 does exist if*

$$4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d)$$

and such a vector l in E_7 does exist if

$$30N_{A_1\oplus D_4}(2d) + 16N_{A_5}(2d) < 5N_{D_6}(2d)$$

where $N_L(2d)$ denotes the number of representations of $2d$ by the lattice L.

To calculate the number $N_L(2d)$ for a lattice of odd rank we use a new variant of the Siegel formula in terms of the Cohen–Zagier L-function (see [6]). As a corollary we proved that the last inequalities are true for $d \geq 144$ or $d \geq 20$ respectively. We obtain the remaining vales of d in the main theorem considering some special vectors.

We note that using the three principles given above we can prove that many modular varieties of dimension $19 \leq n \leq 25$ are of general type. For example we have a result on the moduli spaces of dimension 21 of the O'Grady exceptional irreducible symplectic manifolds of dimension 10 with a polarisation.

In order to study modular varieties with $\dim \mathcal{F}(L) > 25$ we can use the Mumford-Hirzebruch proportionality principle together with automorphic results of [2]. The exact formula for the Mumford–Hirzebruch volume (an analogue of the Euler–Poincare characteristic) of any indefinite orthogonal group was found in [3]. This method works perfectly for modular varieties of big dimensions.

Theorem (see [5]). *Let L be an even unimodular lattice of signature $(2, n)$. Then $\mathcal{F}(L)$ is of general type if $n \geq 42$.*

Analysing the results of [5] I can formulate the following conjecture.

Conjecture. Let L be an even integral lattice of signature $(2, n)$.
1) The modular variety $\mathcal{F}(L)$ is of general type if n is big enough.
2) This is true for $n \geq 36$.

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Intersection numbers of divisors on $\overline{\mathcal{A}_g}$

S G
(joint work with Cord Erdenberger, Klaus Hulek)

Abstract. In this talk we report on our joint work with Cord Erdenberger and Klaus Hulek on intersection numbers of divisors on toroidal compactifications of the moduli space $\mathcal{A}_g$ of principally polarized complex abelian varieties. We compute all intersections numbers of divisors for the first and second Voronoi compactifications of $\mathcal{A}_4$, and for arbitrary g compute those intersection numbers of the Hodge and boundary divisor on the first Voronoi compactification that are supported away from the stratum which lies over the closure of $\mathcal{A}_{g-3}$ in the Satake compactification. The results of this work are presented in papers [2, 3].

The moduli stack $\mathcal{A}_g$ of g-dimensional complex principally polarized complex abelian varieties is the set of isomorphisms classes of pairs (A, Θ), where A is a g-dimensional complex abelian variety, and Θ is a principal polarization on A, i.e. an ample line bundle such that $h^0(A, \Theta) = 1$. There in fact exists (as a stack) the universal family $\pi : \mathcal{X}_g \to \mathcal{A}_g$, with the fiber over $[A] \in \mathcal{A}_g$ being the ppav A itself. The Hodge vector bundle is the rank g vector bundle on $\mathcal{A}_g$ given by $\mathbb{E} := \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$, and we denote $\lambda_i := c_i(\mathbb{E})$ its Chern classes.

The stack $\mathcal{A}_g$ is one of the classical central objects in algebraic geometry and number theory, and its geometric invariants are of obvious interest. Similarly to the case of the moduli space of curves $\mathcal{M}_g$, computing the entire homology and Chow rings is presumably extremely hard, and one can instead study its tautological ring: the subring of the Chow ring $CH^*(\mathcal{A}_g)$ generated by λ_i. In [4] van der Geer proves that the only relations in the tautological ring of $\mathcal{A}_g$ are $\lambda_g = 0$ and

$$(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1 \qquad (*).$$

The stack $\mathcal{A}_g$ is not compact, and a compactification needs to be considered for the intersection theory to make sense. From general theory it follows that $L := \det\mathbb{E}$ is an ample line bundle on $\mathcal{A}_g$, and thus a sufficiently high power of it defines an embedding of $\mathcal{A}_g$ into a projective space. By definition the Satake-Baily-Borel compactification $\mathcal{A}_g^{\mathrm{Sat}} \supset \mathcal{A}_g$ is the closure of the image of this embedding; as a set, $\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_1$.

In the 1970s the toroidal compactifications $\overline{\mathcal{A}_g}$ of $\mathcal{A}_g$ were constructed. Any such compactification admits a contraction $\pi : \overline{\mathcal{A}_g} \to \mathcal{A}_g^{\mathrm{Sat}}$, and we denote $\beta_i := \pi^{-1}(\mathcal{A}_{g-i}^{\mathrm{Sat}})$ the boundary strata. The Hodge bundle $\mathbb{E}$ extends to a vector bundle over any $\overline{\mathcal{A}_g}$, and it was also shown in [4] that the only relation in tautological subring of $CH^*_{\mathbb{Q}}(\overline{\mathcal{A}_g})$ is $(*)$ above. However, this subring captures very little information about $\overline{\mathcal{A}_g}$ and, similarly to the case of $\overline{\mathcal{M}_g}$, it is natural to try to determine the subring of $CH^*(\overline{\mathcal{A}_g})$ generated by the classes λ_i and β_i.

This ring may of course depend on the choice of a toroidal compactification. Two common choices of toroidal compactifications are the so-called first and second Voronoi compactifications, denoted $\mathcal{A}_g^F$ and $\mathcal{A}_g^S$. Alexeev [1] showed that there exists a universal family $\overline{X_g} \to \mathcal{A}_g^S$. However, $\lim_{g\to\infty} \operatorname{rk} \operatorname{Pic}_{\mathbb{Q}}(\mathcal{A}_g^S) = \infty$ (see [8]: for example $E := \beta_4 \subset \mathcal{A}_4^S$ is a divisor) and thus $CH^*(\mathcal{A}_g^S)$ is likely very complicated. On the other hand, the boundary $D \subset \mathcal{A}_g^F$ is an irreducible divisor, and thus $\operatorname{Pic}_{\mathbb{Q}}(\mathcal{A}_g^F) = \mathbb{Q}L \oplus \mathbb{Q}D$ for $g > 1$; moreover $\operatorname{codim}_{\mathcal{A}_g^F} \beta_i = i$. Shepherd-Barron [9] showed that $\mathcal{A}_g^F$ is the canonical model for $\mathcal{A}_g$ for $g \geq 12$, and thus also a natural compactification to study.

The Chow rings and intersection theory on $\mathcal{A}_1$ and $\mathcal{A}_2$, and compactifications are known classically. The Chow ring and the intersection theory of $\mathcal{A}_3^F = \mathcal{A}_3^S$ was computed by van der Geer in [5]. The resulting intersection numbers of divisors are

L^6	L^5D	L^4D^2	L^3D^3	L^2D^4	LD^5	D^6
$\frac{1}{181440}$	0	0	$\frac{1}{720}$	0	$-\frac{203}{240}$	$-\frac{4103}{144}$

.

The explicit solution to the Schottky problem in genus 4 is known: $\mathcal{M}_4 \subset \mathcal{A}_4$ is given by the Schottky modular form, and the class of its closure in $\mathcal{A}_4^F$ and $\mathcal{A}_4^S$ is computed by Harris and Hulek [7]. Using this, we computed the intersection numbers on $\mathcal{A}_4^F$ using the known intersection theory on $\overline{\mathcal{M}_4}$.

Theorem (Erdenberger, Grushevsky, Hulek, [2])**.** The intersection numbers of divisors on $\mathcal{A}_4^F$ are

L^{10}	L^9D	L^8D^2	L^7D^3	L^6D^4	L^5D^5	L^4D^6	L^3D^7	L^2D^8	LD^9	D^{10}
$\frac{1}{907200}$	0	0	0	$-\frac{1}{3780}$	0	0	$-\frac{1759}{1680}$	0	$\frac{1636249}{1080}$	$\frac{101449217}{1440}$

We also determined the intersection theory of divisors on $\mathcal{A}_4^S$: a toroidal computation is used to compute E^{10}, from which all the other numbers can be obtained.

The many zero intersection numbers in the tables above have naturally led us to make the following

Conjecture (Erdenberger, Grushevsky, Hulek, [3]).)**.** The intersection number $\langle L^n D^{\frac{g(g+1)}{2}-n}\rangle_{\mathcal{A}_g^F}$ is zero unless $n = \frac{k(k+1)}{2}$ for some $0 \leq k \leq g$.

To approach this conjecture, we first recall that L is a pullback of a line bundle on $\mathcal{A}_g^{\mathrm{Sat}}$ under the blowdown map $\pi : \mathcal{A}_g^F \to \mathcal{A}_g^{\mathrm{Sat}}$, and thus for dimension reasons $L^{\frac{(g-i)(g-i+1)}{2}+1} = 0 \in CH^*(\beta_i)$. We now start computing the intersection numbers for n large.

The top self-intersection number of L can be computed by the Hirzebruch-Mumford proportionality principle. For any $n > \frac{g(g-1)}{2}$ we see that $L^n = 0 \in CH^*(\beta_1)$, and thus the intersection number $\langle L^n D^m\rangle_{\mathcal{A}_g^F} = \langle L^n (D|_D)^m\rangle_D = 0$; so we get the first $g-1$ zeroes for the conjecture (this is essentially already present in [5]).

Next, for $\frac{g(g-1)}{2} \geq n > \frac{(g-1)(g-2)}{2}$, the corresponding power of L is zero on β_2. Since it is known that $D|_D = -2\Theta + L$, where Θ is the universal theta divisor on $X_{g-1} = \beta_1 \setminus \beta_2$, we can compute

$$\langle L^n D^m\rangle_{\mathcal{A}_g^F} = \langle L^n D^m\rangle_{\mathcal{A}_g^F \setminus \beta_2} = \langle L^n (D|_D)^{m-1}\rangle_{\beta_1\setminus\beta_2}$$
$$= \langle L^n(-2\Theta + L)^{m-1}\rangle_{X_{g-1}} = \langle L^n \pi_*((-2\Theta + L)^{m-1})\rangle_{\mathcal{A}_{g-1}}.$$

In [4] van der Geer uses an argument essentially due to Mumford to show that these pushforwards are zero for $m > g$, so that $\langle L^{\frac{g(g-1)}{2}-k} D^{g+k}\rangle_{\mathcal{A}_g^F} = 0$ for $k = 1 \dots g-2$, while the only non-zero pushforward gives

$$\langle L^{\frac{g(g-1)}{2}} D^g\rangle_{\mathcal{A}_g^F} = (-2)^{g-1}(g-1)!\langle L^{\frac{g(g-1)}{2}}\rangle_{\mathcal{A}_{g-1}}.$$

To deal with the intersection numbers $\langle L^{\frac{(g-3)(g-2)}{2}+k} D^{3g-3-k}\rangle_{\mathcal{A}_g^F}$, for $1 \le k \le g-2$ we first note that they are supported away from β_3. The fiber of the map π over some $B \in \mathcal{A}_{g-2}$ can be identified with the universal family of semiabelian varieties over B, and thus the intersection theory techniques for the Poincaré bundle developed in [6] can be applied to determine the pushforward of the relevant powers of the theta divisor for that case. To finish the computation of this intersection number, one needs to understand the combinatorics of the intersections of the boundary components of the level cover of $\mathcal{A}_g^F$, and to apply the singular version of the Grothendieck-Riemann-Roch formula for the pushforward map $\beta_1 \setminus \beta_3 \to \mathcal{A}_{g-1} \sqcup \mathcal{A}_{g-2}$, which has singular fibers. The result is the following

Theorem (Erdenberger, Grushevsky, Hulek, [3])**.** The conjecture above holds for $n > \frac{(g-3)(g-2)}{2} = \dim \mathcal{A}_{g-3}$. Moreover, explicit formulas are given in [3] for the non-zero numbers in this range: the non-trivial one is $\langle L^{\frac{(g-2)(g-3)}{2}} D^{2g-1}\rangle_{\mathcal{A}_g^F}$, and the formula for it involves a finite hypergeometric sum.

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Minimal Model Program and Semi-Orthogonal Decompositions of Derived Categories

Y K

Bondal and Orlov found a close parallelism between the operations in the minimal model program (Mori fiber spaces, divisorial contractions and flips) and the semi-orthogonal decompostions of bounded derived categories of coherent sheaves ([1]). But, although the MMP works for singular and logarithmic varieties, the derived categories behave nicely only for smooth varieties. The reason is that the derived category $D^b(\mathrm{Coh}(X))$ for a smooth projective varity X satisfies the following 3 nice properties: (1) it is of finite type, (2) it has a Serre functor, (3) it is saturated.

If the variety has a singularity, then the derived category is no more of finite type. The Serre duality holds only between $D^b(\mathrm{Coh}(X))$ and the subcategory $\mathrm{Perf}(X)$ of perfect complexes. The latter is of finite type, but is not saturated. The question is to find something between $D^b(\mathrm{Coh}(X))$ and $\mathrm{Perf}(X)$ for a singular variety X which satisfies the above 3 properties, like the intersection homology which lies between the homology and the cohomology.

If the variety has only quotient singularities, then the associated Deligne-Mumdord stack has a nice derived category. By using this "crepant resolution", one can prove that the derived category corresponding to a projective $\mathbf{Q}$-factrorial toric variety is generated by an exceptional collection consisting of sheaves ([3]).

As a variant of this result, we can prove the following:

Theorem 1. Let $f : X \to Y$ be a birational morphism between projective $\mathbf{Q}$-factrorial toric varieties. Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Deligne-Mumford stacks associated to X and Y respectively. Assume that an inequality $f^*K_Y \geq K_X$ holds. Then there exists an exceptional collection in $D^b(\mathrm{Coh}(\mathcal{Y}))$ such that its semi-orthogonal complement is equivalent to $D^b(\mathrm{Coh}(\mathcal{X}))$.

The assumption is satisfied for example by minimal resolutions and maximal resolutions of singularities for Y.

Next we consider a terminal singularity in dimension 3 which is non-toric. Namely we consider a variety X having an odd Pagoda singularity defined by an equation $xy + z^2 + w^{2n+1}$. By blowing-up at points n times, we obtain a resolution $f : Y \to X$. There are n exceptional divisors, where the first $n-1$ divisors $E_1, \dots, E_{n-1}$ are minimal ruled surfaces of degree 2 over $\mathbf{P}^1$, and the last one E_n is a singular quadric cone. Correspondingly, we have an exceptional collection of length n in $D^b(\mathrm{Coh}(Y))$. Let $\mathcal{D}$ be its semi-orthogonal complement. We claim that $\mathcal{D}$ is a desired "categorical crepant resolution" ([5]):

Theorem 2. The category $\mathcal{D}$ is a minimal saturated subcategory of $D^b(\mathrm{Coh}(Y))$ which contains $f^*\mathrm{Perf}(X)$. The right orthogonal subcategory $f^*\mathrm{Perf}(X)^\perp$ is generated by objects $c_1, \dots, c_n$ such that the first $n-1$ objects $c_1, \dots, c_{n-1}$ are 2-spherical objects, and the last one c_n satisfies $\mathrm{Hom}^*(c_n, c_n) \cong k[t]/t^3$ as graded rings, where $\deg(t) = 1$. The objects c_i define autoequivalences of $\mathcal{D}$, called twistings, which leave $f^*\mathrm{Perf}(X)$ invariant. The Serre functor $S_\mathcal{D}$ satisfies $S_\mathcal{D}(c_i) \cong c_i[2]$ for all i.

We note that the objects $c_1, \dots, c_{n-1}$ are 2-spherical instead of 3-sperical. The object c_n is similar to a $\mathbf{P}^2$ object ([2]) except that the degree of the generator t is 1 instead of 2.

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The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties

S K

(joint work with Sándor J. Kovács)

Let Y° be a quasi-projective manifold that admits a morphism $\mu : Y^\circ \to \mathfrak{M}$ to the moduli stack of canonically polarized varieties. Generalizing the classical Shafarevich hyperbolicity conjecture [6], Viehweg conjectured in [7] that Y° is necessarily of log general type if μ is generically finite. Equivalently, if $f^\circ : X^\circ \to Y^\circ$ is a smooth family of canonically polarized varieties, then Y° is of log general type if the variation of f° is maximal, i.e., $\mathrm{Var}(f^\circ) = \dim Y^\circ$. We refer to [4] for the relevant notions, for detailed references, and for a brief history of the problem, but see also [5].

Viehweg's conjecture was confirmed for 2-dimensional manifolds Y° in [4] using explicit surface geometry. In this talk, we employ recent extension theorems for logarithmic forms to study families over threefolds. If $\dim Y^\circ \leq 3$, we establish a strong relationship between the moduli map μ and the logarithmic minimal model program of Y°: in all relevant cases, any logarithmic minimal model program necessarily terminates with a fiber space whose fibration factors the moduli map. This allows us to prove a much refined version of Viehweg's conjecture for families over surfaces and threefolds, and give a positive answer to the conjecture even for families of varieties with only semi-ample canonical bundle. If Y° is a surface we recover the results of [4] in a more sophisticated manner. In fact, going far beyond those results we give a complete geometric description of the moduli map in those cases when the variation cannot be maximal.

The proof of our main result is rather conceptual and independent of the argumentation of [4] which essentially relied on combinatorial arguments for curve arrangements on surfaces and on Keel-McKernan's solution to the Miyanishi conjecture in dimension 2, [3]. Many of the techniques introduced here generalize well to higher dimensions, most others at least conjecturally.

We work over the field of complex numbers.

Main results. The main results are summarized in the following theorems which describe the geometry of families over threefolds.

Theorem 1 (Viehweg conjecture for families over threefolds)**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of varieties with semi-ample canonical bundle, over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. If f° has maximal variation, then Y° is of log general type. In other words,*

$$\mathrm{Var}(f^\circ) = \dim Y^\circ \;\Rightarrow\; \kappa(Y^\circ) = \dim Y^\circ.$$

□

For families of *canonically* polarized varieties, we can say much more. The following much stronger theorem gives an explicit geometric explanation of Theorem 1.

Theorem 2 (Relationship between the moduli map and the MMP)**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. Let Y be a smooth compactification of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.*

Then any run of the minimal model program of the pair (Y, D) will terminate in a Kodaira or Mori fiber space whose fibration factors the moduli map birationally. □

Remark 3. If $\kappa(Y^\circ) = 0$ in the setup of Theorem 2, then any run of the minimal model program will terminate in a Kodaira fiber space that maps to a single point. Since this map to a point factors the moduli map birationally, Theorem 2 asserts that the family f° is necessarily isotrivial if $\kappa(Y^\circ) = 0$.

Remark 4. Neither the compactification Y nor the minimal model program discussed in Theorem 2 is unique. When running the minimal model program, one often needs to choose the extremal ray that is to be contracted.

In the setup of Theorem 2, if $\kappa(Y^\circ) \geq 0$, then the minimal model program terminates in a Kodaira fiber space whose base has dimension $\kappa(Y^\circ)$. The following refined version of Viehweg's conjecture is therefore an immediate corollary of Theorem 2.

Corollary 5 (Refined Viehweg conjecture for families over threefolds, cf. [4])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold Y° of dimension $\dim Y^\circ \leq 3$. Then either*

i) $\kappa(Y^\circ) = -\infty$ *and* $\mathrm{Var}(f^\circ) < \dim Y^\circ$*, or*
ii) $\kappa(Y^\circ) \geq 0$ *and* $\mathrm{Var}(f^\circ) \leq \kappa(Y^\circ)$. □

As a further application of Theorem 2, we describe the family $f^\circ : X^\circ \to Y^\circ$ explicitly if the base manifold Y° is a surface and the variation is not maximal.

Outline of the proof. The proof of Theorems 1 and 2 relies heavily on the minimal model program, on results of Viehweg and Zuo concerning the existence of pluri-forms on the base of a family, and on extension theorems for differential forms. For convenience, we summarize these results first.

Theorem 6 (Existence of pluri-differentials on the base of a family, [8])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold Y°. Let Y be a smooth compactification of Y° such that $D := Y \setminus Y^\circ$ is a divisor with simple normal crossings.*

Then there exists a number $m \in \mathbb{N}$ and an invertible subsheaf

$$\mathscr{A} \subset \operatorname{Sym}^m \Omega^1_Y(\log D)$$

such that $\kappa(\mathscr{A}) \geq \operatorname{Var}(f^\circ)$. □

Theorem 7 (Extension theorem for log canonical pairs, [2])**.** *Let Z be a normal variety of dimension n and $\Delta \subset Z$ a reduced divisor such that the pair (Z, Δ) is log canonical. Let $\pi : \tilde{Z} \to Z$ be a log resolution, and set*

$$\tilde{\Delta}_{\text{lc}} := \textit{largest reduced divisor contained in } \pi^{-1}(\Delta \cup \textit{centers of log canonicity}).$$

If $p \in \{n, n-1, 1\}$, then the sheaf $\pi_* \Omega^p_{\tilde{Z}}(\log \tilde{\Delta}_{\text{lc}})$ is reflexive. □

One corollary of Theorem 7 is the following generalization of the well-known Bogomolov-Sommese vanishing theorem for snc pairs, cf. [1].

Theorem 8 (Bogomolov-Sommese vanishing for log canonical threefolds and surfaces, [2])**.** *Let Z be a normal variety of dimension $\dim Z \leq 3$ and let $\Delta \subset Z$ be a reduced divisor such that the pair (Z, Δ) is log canonical. Let $\mathscr{A} \subset \Omega^{[p]}_Z(\log \Delta)$ be a reflexive subsheaf of rank one. If $\mathscr{A}$ is $\mathbb{Q}$-Cartier, then $\kappa(\mathscr{A}) \leq p$.* □

In order to prove Theorem 2, we use the existence of the sheaf $\mathscr{A}$ to prove that the tangent sheaf of a minimal model (Y_λ, D_λ) of the pair (Y, D) is unstable in all relevant cases. The sheaf of reflexive differentials $\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)$ is also unstable, with maximally destabilizing subsheaf $\mathscr{B}$, of rank $p < \dim Y$. We obtain a subsheaf $\det \mathscr{B} \subset \Omega^{[p]}_{Y_\lambda}(\log D_\lambda)$ which, by Theorem 8, must have small Kodaira-Iitaka dimension. Using that Y_λ is minimal, a detailed and rather involved analysis of possible cases gives the result.

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Convex bodies associated to linear series

R L
(joint work with Mircea Mustaţă)

Let X be a smooth projective variety, and let D be a big divisor on X. Inspired by a construction introduced in passing by Okounkov [6], [7] in the classical setting of ample divisors, we associate to D a convex body $\Delta(D) \subseteq \mathbf{R}^d$. We use these to recover and extend many facts about the asymptotic properties of linear series. We give here a quick invitation to this work, borrowed from the Introduction to [4].

Okounkov's construction depends on the choice of a fixed flag of subvarieties:

$$Y_\bullet \ : \ X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_{d-1} \supseteq Y_d = \{\text{pt}\},$$

where Y_i is a smooth irreducible subvariety of codimension i in X. This flag determines in a natural way a valuation-like function

$$(*) \quad \nu = \nu_{Y_\bullet} = \nu_{Y_\bullet, D} \ : \ (H^0(X, \mathcal{O}_X(D)) - \{0\}) \longrightarrow \mathbf{Z}^d \ , \ s \mapsto \nu(s) = (\nu_1(s), \ldots, \nu_d(s)).$$

on the non-zero sections of any big divisor D. For example, when $X = \mathbf{P}^d$ and $Y_\bullet$ is a flag of linear spaces, $\nu_{Y_\bullet}$ is essentially the lexicographic valuation on polynomials. Write

$$\nu(D) = \operatorname{Im}\big((H^0(X, \mathcal{O}_X(D)) - \{0\}) \xrightarrow{\nu_Y} \mathbf{Z}^d \big)$$

for the set of valuation vectors of non-zero sections of $\mathcal{O}_X(D)$. It is not hard to check that

$$\# \, \nu(D) = h^0(X, \mathcal{O}_X(D)).$$

Then finally set

$$\Delta(D) = \Delta_{Y_\bullet}(D) = \text{ closed convex hull}\Big(\bigcup_{m \geq 1} \tfrac{1}{m} \cdot \nu(mD) \Big).$$

Thus $\Delta(D)$ is a convex body in $\mathbf{R}^d = \mathbf{Z}^d \otimes \mathbf{R}$, which we call the *Okounkov body* of D (with respect to the fixed flag $Y_\bullet$).

As one might suspect, the standard Euclidean volume of $\Delta(D)$ in $\mathbf{R}^d$ is related to the rate of growth of the groups $h^0(X, \mathcal{O}_X(mD))$. In fact, Okounkov's arguments in [7, §3] – which are based on results of Khovanskii – go through without change to prove

Theorem A. *If D is any big divisor on X, then*

$$\operatorname{vol}_{\mathbf{R}^d}(\Delta(D)) = \frac{1}{d!} \cdot \operatorname{vol}_X(D).$$

The quantity on the right is the *volume* of D, defined as the limit

$$\operatorname{vol}_X(D) =_{\text{def}} \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d / d!}.$$

In the classical case, when D is ample, $\operatorname{vol}_X(D) = \int c_1(\mathcal{O}_X(D))^d$ is just the top self-intersection number of D. In general, the volume is an interesting and delicate invariant of a big divisor, which has lately been the focus of considerable work.

One of our main results involves the variation of these bodies as functions of D. It is not hard to check that $\Delta(D)$ depends only on the numerical equivalence class of D, and that $\Delta(pD) = p \cdot \Delta(D)$ for every positive integer p. It follows that there is a naturally defined Okounkov body $\Delta(\xi) \subseteq \mathbf{R}^d$ associated to every rational numerical equivalence class $\xi \in N^1(X)_{\mathbf{Q}}$, and as before $\mathrm{vol}_{\mathbf{R}^d}(\Delta(\xi)) = \frac{1}{d!} \cdot \mathrm{vol}_X(\xi)$. We prove:

Theorem B. *There exists a closed convex cone*

$$\Delta(X) \ \subseteq\ \mathbf{R}^d \times N^1(X)_{\mathbf{R}}$$

characterized by the property that in the diagram

$$\begin{array}{ccc} \Delta(X) & \subseteq & \mathbf{R}^d \times N^1(X)_{\mathbf{R}} \\ & \searrow \quad \swarrow_{\mathrm{pr}_2} & \\ & N^1(X)_{\mathbf{R}}, & \end{array}$$

the fibre $\Delta(X)_\xi \subseteq \mathbf{R}^d \times \{\xi\} = \mathbf{R}^d$ of $\Delta(X)$ over any big class $\xi \in N^1(X)_{\mathbf{Q}}$ is $\Delta(\xi)$.

The image of $\Delta(X)$ in $N^1(X)_{\mathbf{R}}$ is the so-called pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ of X, i.e. the closure of the cone spanned by all effective divisors: its interior is the big cone $\mathrm{Big}(X)$ of X. Thus the theorem yields a natural definition of $\Delta(\xi) \subseteq \mathbf{R}^d$ for any big class $\xi \in N^1(X)_{\mathbf{R}}$, viz. $\Delta(\xi) = \Delta(X)_\xi$.

Theorem B renders transparent several basic properties of the volume function vol_X established by the first author in [5, 2.2C, 11.4.A], and independently by Boucksom [1] in the analytic setting. First, since the volumes of the fibres $\Delta(\xi) = \Delta(X)_\xi$ vary continuously for ξ in the interior of $\mathrm{pr}_2(\Delta(X)) \subseteq \mathbf{N}^1(X)_{\mathbf{R}}$, one deduces that the volume of a big class is computed by a continuous function

$$\mathrm{vol}_X : \mathrm{Big}(X) \longrightarrow \mathbf{R}.$$

Moreover $\Delta(\xi) + \Delta(\xi') \ \subseteq\ \Delta(\xi + \xi')$ for any two big classes $\xi, \xi' \in N^1(X)_{\mathbf{R}}$, and so the Brunn-Minkowski theorem yields the log-concavity relation

$$\mathrm{vol}_X(\xi + \xi')^{1/d} \ \geq\ \mathrm{vol}_X(\xi)^{1/d} + \mathrm{vol}_X(\xi')^{1/d}$$

for any two such classes.

The Okounkov construction also reveals some interesting facts about the volume function that had not been known previously. For instance, let $E \subseteq X$ be a very ample divisor on X that is general in its linear series, and choose the flag $Y_\bullet$ in such a way that $Y_1 = E$. Now construct the Okounkov body $\Delta(\xi) \subseteq \mathbf{R}^d$ of any big class $\xi \in \mathrm{Big}(X)$, and consider the mapping

$$\mathrm{pr}_1 : \Delta(\xi) \longrightarrow \mathbf{R}$$

obtained via the projection $\mathbf{R}^d \longrightarrow \mathbf{R}$ onto the first factor, so that pr_1 is "projection onto the ν_1-axis." Write $e \in N^1(X)$ for the class of E, and given $t > 0$ such that $\xi - te$ is big, set

$$\Delta(\xi)_{\nu_1 = t} \ =\ \mathrm{pr}_1^{-1}(t) \ \subseteq\ \mathbf{R}^{d-1} \ \ , \ \ \Delta(\xi)_{\nu_1 \geq t} \ =\ \mathrm{pr}_1^{-1}([t, \infty)) \ \subseteq\ \mathbf{R}^d.$$

We prove that

$$\Delta(\xi)_{\nu_1 \geq t} =_{\text{up to translation}} \Delta(\xi - te)$$
$$\mathrm{vol}_{\mathbf{R}^{d-1}}(\Delta(\xi)_{\nu_1 = t}) = \frac{1}{(d-1)!} \cdot \mathrm{vol}_{X|E}(\xi - te).$$

Here $\mathrm{vol}_{X|E}$ denotes the restricted volume function from X to E studied in [3]: when D is integral, $\mathrm{vol}_{X|E}(D)$ measures the rate of growth of the subspaces of $H^0(E, O_E(mD))$ consisting of sections that come from X. Since one can compute the d-dimensional volume of $\Delta(\xi)$ by integrating the $(d-1)$-dimensional volumes of its slices, one finds:

Corollary C. *Let $a > 0$ be any real number such that $\xi - ae \in \mathrm{Big}(X)$. Then*

$$\mathrm{vol}_X(\xi) - \mathrm{vol}_X(\xi - ae) = d \cdot \int_{-a}^{0} \mathrm{vol}_{X|E}(\xi + te)\, dt.$$

Consequently, the function $t \mapsto \mathrm{vol}_X(\xi + te)$ is differentiable at $t = 0$, and

$$\frac{d}{dt}\left(\mathrm{vol}_X(\xi + te)\right)|_{t=0} = d \cdot \mathrm{vol}_{X|E}(\xi).$$

This leads to the fact that vol_X is C^1 on $\mathrm{Big}(X)$. Corollary C was one of the starting points of the interesting work [2] of Boucksom–Favre–Jonsson, who found a nice formula for the derivative of vol_X in any direction, and used it to answer some questions of Teissier.

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Singularities of the moduli space of spin curves and Prym curves

K L

(joint work with Gavril Farkas)

Both moduli spaces considered, the moduli space S_g of spin curves and the moduli space R_g of Prym curves, parametrise pairs (C, L) of a smooth curve C of genus g and a line bundle L on C. In the case of spin curves, the line bundle is a square root of the canonical bundle ω_C, in the case of Prym curves, it is a non-trivial square root of the trivial bundle O_C. There are natural forgetful morphisms $\pi : \star_g \to M_g$, $\star \in \{S, R\}$, sending the isomorphism class $[C, L]$ to $[C]$.

Since the Kodaira dimension of $\star_g$ is, by definition, that of a smooth projective model $\widetilde{\star}_g$, one needs a compactification $\overline{\star}_g \supset \star_g$ and a desingularisation $\widetilde{\star}_g \to \overline{\star}_g$. A geometrically meaningful compactification $\overline{S}_g$ of S_g over the Deligne-Mumford compactification $\overline{M}_g$ was given by M. Cornalba [3] in terms of line bundles on so called quasistable curves. E. Ballico, C. Casagrande and C. Fontanari [1] constructed an analogous compactification $\overline{R}_g$ of R_g and proved that it is isomorphic to the compactification via admissible covers by A. Beauville [2]. The compactifications are coarse moduli spaces for the following objects.

Definition 1. A spin resp. Prym curve of genus $g \geq 2$ is a triple (X, L, b), where X is a quasistable curve of genus g, i.e. there exists a stable curve C and a blow up $\beta : X \to C$ of C at a set $N \subset \operatorname{sing} C$ of nodes, $L \in \operatorname{Pic}(X) \setminus \{\mathcal{O}_X\}$ is of degree $g-1$ resp. 0 and $b : L^{\otimes 2} \to \beta^* \omega_C$ resp. $b : L^{\otimes 2} \to \mathcal{O}_X$ is a homomorphism such that for every exceptional component E of β we have $L_{|E} = \mathcal{O}_E(1)$ and b is non-zero at the generic point of every non-exceptional component of X.

An automorphism of (X, L, b) is a pair (σ, γ) where $\sigma \in \operatorname{Aut} X$ and $\gamma : \sigma^* L \to L$ is an isomorphism compatible with the homomorphisms b and $\sigma^* b$.

The moduli spaces $\overline{\star}_g$ are normal and have quotient singularities. Locally at a point $[X, L, b] \in \overline{\star}_g$ the moduli space is isomorphic to the quotient of the versal deformation space $\mathbb{C}_\tau^{3g-3}$ of (X, L, b) by the linear action of the automorphism group $\operatorname{Aut}(X, L, b)$. Studying the automorphisms acting as quasireflections, i.e. having 1 as an eigenvalue of multiplicity $3g-4$, gives the following characterisations of the smooth locus.

Proposition 2 ([4, 6])**.** *Let $g \geq 4$.*

$[X, L, b] \in \overline{R}_g$ *is smooth if and only if* $\operatorname{Aut}(X, L, b)$ *is generated by elliptic tail involutions, i.e. automorphisms such that there exists an irreducible component C_1 of X of genus* 1 *meeting the rest of the curve in exactly one node such that σ is the involution on C_1 fixing the node and the identity on $\overline{X \setminus C_1}$.*

$[X, L, b] \in \overline{S}_g$ *is smooth if and only if the image of the natural homomorphism* $\operatorname{Aut}(X, L, b) \to \operatorname{Aut} C$ *is generated by elliptic tail involutions and a certain graph* $\Sigma(X)$ *is tree-like, i.e. removing all loops of* $\Sigma(X)$ *gives a tree. Here* $\Sigma(X)$ *has a vertex for every connected component of the partial normalisation of C at N and an edge for every exceptional component E of the blow up β.*

Quotienting out the subgroup of $\operatorname{Aut}(X, L, b)$ generated by quasireflections gives a description of the quotient singularity at $[X, L, b]$ as $\mathbb{C}_u^{3g-3}/K$ where K contains no quasireflections, hence the Reid–Shepherd-Barron–Tai criterion is applicable. A careful study of the occuring quotients gives the following

Proposition 3 ([4, 6])**.** *Let $g \geq 4$.* $[X, L, b] \in \overline{\star}_g$ *is a non-canonical singularity if and only if X has an elliptic tail C_1 of j-invariant* 0 *such that $L_{|C_1} = \mathcal{O}_{C_1}$.*

With this detailed local information we can prove the following global result.

Theorem 4 ([4, 6])**.** *For $g \geq 4$ every pluricanonical form ω on the smooth locus $\overline{\star}_g^{reg}$ extends holomorphically to a desingularisation $\widetilde{\star}_g$, i.e.*

$$H^0(\overline{\star}_g^{reg}, mK_{\overline{\star}_g}) = H^0(\widetilde{\star}_g, mK_{\widetilde{\star}_g}).$$

Remark 5. This implies that the Kodaira dimension of $\overline{\bigstar}_g$ can be computed on the moduli space $\overline{\bigstar}_g$ itself without refering to a smooth model $\widetilde{\bigstar}_g$. See the article of G. Farkas in this report for the results on the Kodaira dimensions.

idea of proof. Let ω be any pluricanonical form on $\overline{\bigstar}_g^{\mathrm{reg}}$ and $[X, L, b] \in \overline{\bigstar}_g$. If $[X, L, b]$ is a canonical singularity, the form ω extends locally to a desingularisation. If $[X, L, b]$ is a general non-canonical singularity, the stable model C has two irreducible components C_1 and C_2 meeting in one node, C_2 is a general smooth curve of genus $g - 1$, C_1 is an elliptic curve with j-invariant 0 and $L_{|C_1} = \mathcal{O}_{C_1}$. Deforming the elliptic tail C_1 gives a projective curve in $\overline{\bigstar}_g$ through $[X, L, b]$. We prove that there exists an open neighbourhood S of this curve such that ω extends holomorphically to a desingularisation of S. The basic idea is to contract a divisor containing the singularity to a codimension two locus in a smooth variety S_0, where the form naturally extends.

Now let $[X, L, b]$ be any non-canonical singularity. For every elliptic tail of j-invariant 0 such that the restriction of L is trivial consider a deformation to the general spin/Prym curve having this elliptic tail. The corresponding point in $\overline{\bigstar}_g$ is a general non-canonical singularity. The above considerations then give an open subset of a neighbourhood of $[X, L, b]$ fulfilling the conditions of a generalised Reid–Shepherd-Barron–Tai criterion by Harris and Mumford [5] which implies that ω lifts to a desingularisation. □

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Non-algebraic hyperkähler manifolds

K O

(joint work with Frédéric Campana, Thomas Peternell)

Let X be a compact Kähler manifold. Then, by the Moishezon criterion, X is projective if and only if it is algebraic in the sense that $a(X) = \dim X$. Here $a(X)$ is the algebraic dimension of X, i.e., the transcendental degree of the meromorphic function field of X. Also, by the famous criterion of Kodaira, X is projective if $H^2(X, \mathcal{O}_X) = 0$, or equivalently, by the Hodge symmetry, if $H^0(X, \Omega_X^2) = 0$.

A hyperkähler manifold is in some sense the simplest class of manifolds which do *not* satisfy $H^0(X, \Omega_X^2) = 0$:

Definition. *A simply connected compact Kähler manifold is called a holomorphic irreducible symplectic manifold or a hyperkähler manifold, if X admits an everywhere non-degenerate holomorphic* 2*-form σ_X such that $H^0(X, \Omega_X^2) = \mathbf{C}\sigma_X$. (Hence of an even dimension, say $2n$, and $\wedge^n \sigma_X$ is a $2n-$form without zeroes.)*

By the famous Bogomolov decomposition theorem [Be83], hyperkähler manifolds, Calabi-Yau manifolds of dimension ≥ 3 and complex tori form three important building blocks among all compact Kähler manifolds of vanishing first Chern class. Among these three, Calabi-Yau manifolds are always projective, and the set of algebraic dimensions of complex tori of dimension $n(\geq 2)$ is $\{a \in \mathbf{Z} | 0 \leq a \leq n\}$.

By Fujiki [Fu83], both projective and non-projective hyperkähler manifolds are in fact dense in the Kuranishi space of X. We are then interested in hyperkähler manifolds, particularly with their algebraic dimensions $a(X) < 2n$ and their algebraic reductions $f : X \dashrightarrow B$, which are unique up to bimeromorphic modification of B [Ue75]. These two are the most fundamental invariants in the classification of non-algebraic manifolds.

Before discussing hyperkähler case, we recall the case of surfaces. Let S be a compact smooth surface. Then, the intersection form (cup product) on the Néron-Severi group $NS(S)$ is of signature either $(1, 0, \rho(S)-1)$ in which case we say that $NS(S)$ is hyperbolic, $(0, 1, \rho(S)-1)$ ($NS(S)$ is parabolic), or $(0, 0, \rho(S))$ ($NS(S)$ is elliptic). According to these three cases, $a(S) = 2, 1, 0$. Moreover, if $a(S) = 1$, then we have a holomorphic algebraic reduction $f : S \longrightarrow C$ whose general fiber is an elliptic curve [BHPV04].

For a hyperkähler manifold X, we have the Beauville-Bogomolov-Fujiki's form

$$q_X : H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \to \mathbf{Z} .$$

This is a bilinear symmetric form of signature $(3, 0, b_2(X) - 3)$ [Be83] (see also [Bo75], [Fu87]). In many aspects, the Beauville-Bogomolov-Fujiki's form plays a very similar role to the interesection form on a surface. For instance, it induces a symmetric bilinear form on the Néron-Severi group $NS(X)$, and the signature is either $(1, 0, \rho(X) - 1)$ (hyperbolic), $(0, 1, \rho(X) - 1)$ (parabolic), or $(0, 0, \rho(X))$ (elliptic).

Example. (1) Let S be a K3 surface. Then $S^{[n]}$, the Hilbert scheme of n points on S, is a hyperkähler manifold of dimension $2n$. This is due to Fujiki [Fu83] and Beauville [Be83]. We have $a(S^{[n]}) = 0, n, 2n$ according to $a(S) = 0, 1, 2$. In addition, when $a(S) = 1$, the algebraic reduction map $S \longrightarrow \mathbf{P}^1$ induces a natural surjective morphism $S^{[n]} \longrightarrow \mathbf{P}^n$. This is the algebraic reduction of $S^{[n]}$ and it is also Lagrangian.

(2) Let T be a 2-dimensional complex torus. The generalized Kummer manifold $K_n(T)$ is also a hyperkähler manifold of dimension $2n$ [Be83]. One can also check that $a(K_n(T)) = 0, n, 2n$ according to $a(T) = 0, 1, 2$. In addition, when $a(T) = 1$, the algebraic reduction map $f : T \longrightarrow E$ induces a natural morphism $S^{n+1}T \longrightarrow S^{n+1}E$, which is compatible with $S^{n+1}T \longrightarrow T$, $S^{n+1}E \longrightarrow E$ (natural addition maps), and f. From this, one obtains a surjective morphism $K_n(T) \longrightarrow \mathbf{P}^n$. This morphism is nothing but the algebraic reduction of $K_n(T)$ and it is again Lagrangian.

In much deeper level, we have the following fundamental:

Theorem (Huybrechts [Hu99]). *A hyperkähler manifold X is projective if and only if $NS(X)$ is hyperbolic.*

Theorem (Matsushita [Ma99]). *Let $f : X \longrightarrow B$ be a surjective holomorphic map from a hyperkähler manifold X to a normal projecive variety B with $0 < \dim B < \dim X$. Then f is necesarily Lagraingian, that is, $\dim B = \dim X/2$ and $\sigma_X|F \equiv 0$ for a general fiber of f.*

Motivated by these, we formulated the following:

Conjecture ([COP08]). *Let X be a hyperkähler manifold of dimension $2n$. Then its algebraic dimension takes only the values $0, n, 2n$. Moreover, if $a(X) = n$, then the algebraic reduction has a holomorphic model $f : X \longrightarrow B$ with B a normal projective variety of dimension n (in particular, f is Lagrangian).*

In this conjecture, we also expected that *if $NS(X)$ is parabolic, then $a(X) = \dim X/2$*. However, we have no answer for this stronger assertion except for the examples discussed above.

At the workshop, I have reported the following answer toward the conjecture, obtained in our joint work, with some idea of proof:

Theorem ([COP08]). (1) *If $\dim X = 4$, then the conjecture above is true.*

(2) *Let X be a non-algebraic hyperkähler manifold of dimension $2n$. Then $0 \le a(X) \le 2n$. More precisely we have:*

(i) *If $NS(X)$ is elliptic, then $a(X) = 0$.*

(ii) *If $NS(X)$ is parabolic, then $0 \le a(X) \le n = \dim X/2$.*

(iii) *Assume that any compact Kähler manifold Y of $\dim Y \le 2n-1$, of algebraic dimension $a(Y) = 0$ and of Kodaira dimension $\kappa(Y) = 0$ and with effective canonical divisor K_Y, has a minimal model with numerically trivial canonical divisor. Then the conjecture above is true.*

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Relative critical exponents and non-vanishing

M P˘

The main goal of our talk was to give a general outline of the proof of the following result see [3].

Theorem 0.1. *Let X be a smooth projective manifold, and let B be an $\mathbb{R}$-divisor such that:*

(i) *The pair (X, B) is klt, and B is big ;*

(ii) *The adjoint bundle $K_X + B$ is pseudoeffective.*

Then there exist an effective $\mathbb{R}$-divisor $\sum_{j=1}^{N} \nu^j [Y_j]$ numerically equivalent with $K_X + B$. □

Firstly, we would like to mention that the above result generalizes the classical "non-vanishing" theorems of V. Shokurov and Y. Kawamata.

Secondly, the above result is not new : it was established by C. Birkar, P. Cascini, C. Hacon and J. McKernan as a by-product of their fundamental work [1], by using the minimal model program and characteristic p techniques.

One important aspect of our proof is that is Char p-free ; moreover we avoid the *explicit* use of the minimal model program algorithm. A theorem similar to 0.1 was established by Y.-T. Siu in [5], pages 31-46. Even if the hypothesis in his statement are much more restrictive than above, a substantial part of the arguments from his work are used here. Most of the subtle points in our arguments are equally observable in the algebraic geometry proof mentioned above, as it was kindly explained to us by J. McKernan and S. Druel; it would be very interesting to have a precise comparison between the two approaches. □

We comment here few aspects of the proof. If the dimension of $K_X + B$ is equal to zero, then the theorem 0.1 is a consequence of a result due to N. Nakayama (generalized by S. Boucksom). If this is not the case, we use the numerical positivity of $K_X + B$, together with (a version of) the usual log-canonical threshold, in order to identify a hypersurface S (the *minimal center*) of some modification of X such that by restriction to S we reproduce the same context as in 0.1, except that the dimension drops. This part of our proof could be seen as a generalization of the classical arguments used in the Fujita conjecture literature (see [5], [6] and the references therein).

During the restriction to the minimal center process, we will use in an essential manner the regularization techniques of J.-P. Demailly (see [2]) ; a diophantine approximation argument is also involved, to reduce to the case where the geometric objects we are dealing with are rational (see also [1]). Finally, we use the extension techniques of Y.-T. Siu ([4]) and C. Hacon-J. McKernan adapted to the present situation. The main technical point in our proof is an *ad hoc* version of the invariance of plurigenera : this is the part where the difference between the classical approach (Shokurov, Kawamata,...) and our arguments is quite important.

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Hyperbolicity of geometric orbifolds

E R

F. Campana has introduced in [1] orbifold structures, namely pairs (X/Δ) with X a complex manifold and a divisor $\Delta = \sum_i (1 - \frac{1}{m_i}) Z_i$ where the Z_i are distinct irreducible divisors and $m_i \in \mathbb{N} \cup \{\infty\}$, as a new frame for the classification of compact Kähler manifolds. These structures appeared naturally for fibrations $f : X \to Y$. Indeed the multiple fibres of f lead to the definition of the orbifold base of f, $(Y/\Delta(f))$ where

$$\Delta(f) := \sum_{D \subset Y} \left(1 - \frac{1}{m(f, D)}\right) D$$

$m(f, D)$ being the multiplicity of the fiber of f above the generic point of D. A new class of varieties was then introduced, the *special varieties*, as the varieties which do not admit fibrations of general type i.e with an orbifold base of general type. Campana [1] proves the existence for every complex algebraic manifold X of a fibration $c_X : X \to C(X)$, the core of X, such that its general fibers are special and if X is not special, c_X is of general type.

These geometric orbifolds should be considered as true geometric objects as one can define for them differential forms, fundamental groups, Kobayashi pseudo-distance... Here we study the hyperbolic aspects of these objects. An important conjecture of Campana [1] is that X is special if and only if the Kobayashi pseudo-distance d_X vanishes identically on $X \times X$. This is known only for curves, projective surfaces not of general type and rationally connected manifolds.

This conjecture then implies that d_X should be the pull-back by c_X of the Kobayashi pseudo-distance δ_X of the orbifold base of the core.

The study of the hyperbolic aspects of one-dimensional orbifolds has been done in [3]. In this work we study hyperbolicity of higher dimensional orbifolds following the philosophy of Campana that one should study these objects generalizing the tools we use for manifolds without orbifold structures or logarithmic manifolds.

First, we define the classical and non-classical Kobayashi hyperbolicity for orbifolds. Then we illustrate these notions in the case of orbifold curves. We compute explicitly the orbifold Kobyashi pseudo-distance for

$$(X/\Delta) = (D/(1 - \frac{1}{n})\{0\}), 0 < n \in \mathbb{N} \cup \{\infty\},$$

where D is the unit disk. This answers a question of Campana and Winkelmann (see [3]) and enables us to recover as a corollary the equivalence of classical and non-classical hyperbolicity for orbifold curves. Finally, we show that this is not the case in higher dimension giving an example of an orbifold surface which is classically hyperbolic but not hyperbolic.

Then, we define and use orbifold jet differentials. The main applications are algebraic degeneracy statements for entire curves with ramification in situations where no Second Main Theorem is known from value distribution theory. Namely, we prove

Theorem 1. *Let (X/Δ) be a smooth projective orbifold surface of general type where Δ has the following decomposition into irreducible components, $\Delta = \sum_{i=1}^{n}(1 - \frac{1}{m_i})C_i$. Suppose that $g_i := g(C_i) \geq 2$, $h^0(C_i, O_{C_i}(C_i)) \neq 0$ for all i and that the logarithmic Chern classes of $(X, \lceil\Delta\rceil)$ verify*

$$\overline{c_1}^2 - \overline{c_2} - \sum_{i=1}^{n} \frac{1}{m_i}(2g_i - 2 + \sum_{j \neq i} C_i C_j) > 0, \tag{0.1}$$

then there exists a proper subvariety $Y \subset X$ such that every entire curve $f : \mathbb{C} \to X$ which is an orbifold morphism, i.e ramified over C_i with multiplicity at least m_i, verifies $f(\mathbb{C}) \subset Y$.

This result can be seen as an orbifold version of results of McQuillan [5] (see also [6] and [4] for the logarithmic case) on the Green-Griffiths-Lang conjecture which can be generalized to the orbifold setting

Conjecture 2. *Let (X/Δ) be a smooth projective orbifold of general type. Then there exists a proper subvariety $Y \subset X$ such that every orbifold morphism $f : \mathbb{C} \to (X/\Delta)$ verifies $f(\mathbb{C}) \subset Y$.*

The methods used also enable us to generalize a result of Campana and Paun on weakly-special manifolds [2].

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Moduli spaces of holomorphic bundles over non-kählerian surfaces and applications

A T

1. Moduli spaces of stable and polystable bundles. Let X be a compact complex n-dimensional manifold. A Hermitian metric g on X is called Gauduchon if its Kähler form ω_g satisfies $\partial\bar\partial(\omega_g^{n-1}) = 0$ [6]. In every conformal class of Hermitian metrics on X there exists a Gauduchon metric (which is unique up to constant factor if $n \geq 2$), so there is no obstruction to the existence of Gauduchon metrics. A Gauduchon metric on X defines a degree map $\deg_g : \mathrm{Pic}(X) \to \mathbb{R}$, given by $\deg_g(\mathcal{L}) := \int_X c_1(\mathcal{L}, h) \wedge \omega_g^{n-1}$, where $c_1(\mathcal{L}, h)$ denotes the first Chern form of the Chern connection of any Hermitian metric on $\mathcal{L}$ (a representative of its first Chern class in Bott-Chern cohomology). For an arbitrary coherent sheaf $\mathcal{F}$ one puts as usually $\deg_g(\mathcal{F}) := \deg_g(\det(\mathcal{F}))$, $\mu_g(\mathcal{F}) := \deg_g(\det(\mathcal{F}))/\mathrm{rk}(\mathcal{F})$ (defined for non-trivial torsion-free sheaves) and introduces the stability and semi-stability condition in the usual way, by requiring the same inequalities as in the classical Mumford-Takemoto theory for bundles on projective manifolds. Similarly, a bundle $\mathcal{E}$ on X is called polystable if it is either stable, or isomorphic to a direct sum of stable bundles of same slope.

Consider now a C^∞ rank r-bundle E over the Gauduchon manifold (X, g), and fix a holomorphic structure $\mathcal{D}$ on the determinant line bundle $D := \wedge^r(E)$. We denote by $\mathcal{M}^{\mathrm{s}}_{\mathcal{D}}(E)$, $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$, $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ the moduli sets of equivalence classes of simple (respectively stable, polystable) holomorphic structures on E which induce $\mathcal{D}$ on D. $\mathcal{M}^{\mathrm{s}}_{\mathcal{D}}(E)$ has a natural structure of (in general non-Hausdorff) finite dimensional complex space, and $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ is a (in general g-dependent) Hausdorff open subset of this space, hence it inherits a natural Hausdorff complex space structure [8]. In order to put a natural topology on the larger moduli set $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ in the non-Kählerian framework one needs the Kobayashi-Hitchin correspondence [4], [2], [9], [8]. Suppose for simplicity $n = 2$. We fix a Hermitian metric h on E and denote by a the Chern connection of the pair $(\mathcal{D}, \det(h))$. The Kobayashi-Hitchin correspondence yields a bijection $\mathcal{M}^{\mathrm{ASD}}_a(E) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ which maps $\mathcal{M}^{\mathrm{ASD}}_a(E)^*$ onto $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$. Here $\mathcal{M}^{\mathrm{ASD}}_a(E)$ stands for the moduli space of projectively anti-selfdual Hermitian connections A on E which induce a on D, and $\mathcal{M}^{\mathrm{ASD}}_a(E)^*$ denotes the open subspace of irreducible such connections. The restriction $\mathcal{M}^{\mathrm{ASD}}_a(E)^* \to \mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ is a real analytic isomorphism [8]. In this way we get a natural Hausdorff topology on $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ (induced from the topology of $\mathcal{M}^{\mathrm{ASD}}_a(E)$) with respect to which $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ is open. Note however that, in general, on non-Kählerian manifolds, the complex space structure of $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ cannot be extended to $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$. This shows that in the non-Kählerian framework there cannot exist a coherent way to define moduli spaces of S-equivalence classes of semistable bundles within the complex geometric category. Moreover, the local structure of $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ around a split polystable bundle cannot be described with complex geometric methods; one has to study the Kuranishi local model of the corresponding reducible instanton with gauge-theoretical techniques [5], [15], [16].

For $n = 2$ the isomorphism $\mathcal{M}^{\mathrm{ASD}}_a(E) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ plays a crucial role in Donaldson theory: it was used by Donaldson as a tool to compute instanton moduli spaces with complex geometric methods [4], [5]. Unfortunately, on non-algebraic surfaces, describing the complex geometric term $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ becomes very difficult because the appearance of non-filtrable bundles. These bundles are stable with respect to any Gauduchon metric, but there exists no method to classify them.

Class VII surfaces. The Enriques-Kodaira classification of complex surfaces is not yet complete. The main obstacle is the Kodaira class VII. According to the modern terminology a class VII surface is a compact complex surface X having $b_1(X) = 1$, $\mathrm{kod}(X) = -\infty$. The subclass $VII^{b_2=0}$ of class VII surfaces with $b_2 = 0$ is completely classified: such a surface is biholomorphic either to a Hopf surface or to an Inoue surface [1], [10], [14]. It remains to classify the class $VII^{b_2>0}_{\min}$ of minimal class VII surfaces with $b_2 > 0$, which is a difficult, long-standing problem. The standard conjectures concerning this classification are:

C1. *Any surface $X \in VII^{b_2>0}_{\min}$ has b_2 rational curves.*

C2. *Any surface $X \in VII^{b_2>0}_{\min}$ contains a cycle of rational curves.*

By the fundamental result of Dloussky-Oeljeklaus-Toma [3] any surface $X \in VII^{b_2>0}_{\min}$ with b_2 rational curves is biholomorphic to a Kato surface (i.e. a surface with global spherical shell). Kato surfaces are well understood [7], [13], so (if true) C1 would solve the classification problem for class VII surfaces completely.

On the other hand, by a fundamental result of Nakamura [11] we know that any surface $X \in VII^{b_2>0}_{\min}$ containing a cycle of rational curves is a degeneration of a 1-parameter family of blown-up primary Hopf surfaces; therefore (if true) the weaker conjecture C2 would solve the classification problem for class VII surfaces up to deformation equivalence. Therefore, the main problem in understanding class $VII^{b_2>0}_{\min}$ surfaces is to prove *the existence of (sufficiently many) rational curves* on these surfaces.

The class $VII^{b_2>0}$ is interesting from a differential topological point of view: the intersection form $q_X : H^2(X,\mathbb{Z})/\mathrm{Tors} \times H^2(X,\mathbb{Z})/\mathrm{Tors} \to \mathbb{Z}$ of such a surface is negative definite, so by Donaldson's first theorem, it is standard over $\mathbb{Z}$ i.e. there exists a basis $(e_1, \dots, e_b)$ of $H^2(X,\mathbb{Z})/\mathrm{Tors}$ satisfying $q_X(e_i, e_j) = -\delta_{ij}$ (with $b := b_2(X)$). Taking into account that $c_1(X)^2 = -b$ and that $k := -c_1(X)$ is a characteristic class, we see that, replacing some of the e_i by their opposite if necessary, one can assume that $k = \sum e_i$, and a basis with this property is unique up to order.

Existence of curves. In [15], [16] we showed that one can use a combination of complex geometric and gauge theoretical techniques to make progress in the classification of class VII surfaces, namely to prove existence of curves.

Theorem: *C1 is true for $X \in VII^{b_2=1}_{\min}$ and C2 is true for $X \in VII^{b_2=2}_{\min}$.*

Let $X \in VII^{b_2>0}_{\min}$. The fundamental object coming up in the proof is the moduli space $\mathcal{M} := \mathcal{M}^{\mathrm{pst}}_{\mathcal{K}}(E)$, where E is a differentiable rank 2 bundle with $c_2(E) = 0$ and $\det(E) = K$ (the underlying C^∞ bundle of $\mathcal{K}$). Any *filtrable* bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) \simeq \mathcal{K}$ is an extension of $\mathcal{K} \otimes \mathcal{L}^{-1}$ by $\mathcal{L}$, where $\mathcal{L}$ is a line bundle with $c_1(\mathcal{L}) = e_I := \sum_{i \in I} e_i$, for a subset $I \subset \{1, \dots b\}$. $\mathcal{M}$ is always compact, and it is a b_2-dimensional complex space in the complement of the reductions (split polystable bundles). Since $H^1(\mathcal{K}) \simeq \mathbb{C}$, we obtain a non-trivial extension $0 \to \mathcal{K} \to \mathcal{A} \to \mathcal{O} \to 0$. The bundle $\mathcal{A}$ is stable if $\deg_g(\mathcal{K}) < 0$ (which can be assured by choosing g suitably) and X contained no cycle. Let $\mathcal{N} \subset \mathcal{M}$ be the union of connected components containing split polystable bundles. The proof starts with the question: *Does $\mathcal{A}$ belong to $\mathcal{N}$?* If yes, one can prove that X must contain a cycle. If not, the connected component Y of $\mathcal{A}$ in $\mathcal{M}$ is a smooth, compact n-fold contained in $\mathcal{M}^{\mathrm{st}}_{\mathcal{K}}(E)$ consisting generically of non-filtrable points. For $b_2 \in \{1, 2\}$ the appearance of such a component in the moduli space leads to a contradiction. For instance, for $b_2 = 1$, Y would be a Riemann surface, and the contradiction comes from the fact that Y is algebraic, whereas $a(X) = 0$.

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Generalized Kähler-Einstein metrics

H T

In 2006, I proved that the normalized limit of the dynamical system of Bergman kernels constructed in [T0] is nothing but the canonical Kähler-Einstein current on a smooth projective variety of general type ([T1]). The importance of this discovery is that this implies the plurisubharmonicity of the logarithm of the relative canonical Kähler-Einstein volume form on a projective family of varieties of general type ([T1]) by the result of Berndtsson on the variation of Bergman kernels ([B, T1]).

Since the dynamical systems had already been constructed not only for varieties of general type, but also for varieties with pseudoeffecitve canonical divisors, I expected a similar theory also in this case. However I could not find it for a year. But once I looked at the canonical bundle formula in [F-M, p.183, Theorem 5.2], I immediately realized that the corresponding metric satisfies the **log-version** of the Kähler-Einstein equation on the base of the Iitaka fibration with the boundary term coming from the curvature of the Hodge metric of the Hodge $\mathbb{Q}$ line bundle.

Let X be a smooth projective variety with nonnegative Kodaira dimension and let $f : X - \cdots \to Y$ be the Iitaka fibration associated with the complete linear system $|m_0!K_X|$ for some sufficiently large positive integer m_0. By taking a suitable modifications, we may assume the followings:

(i) Y is smooth and f is a morphism.
(ii) $f_*\mathcal{O}_X(m_0!K_{X/Y})^{**}$ is a line bundle on Y, where $**$ denotes the double dual.

We define the $\mathbb{Q}$ line bundle L (independent of m_0) on Y by $L := \frac{1}{m_0!} f_*\mathcal{O}_X(m_0!K_{X/Y})^{**}$. L carries a natural singular hermitian metric (the Hodge metric) h_L (independent of m_0) defined by

$$h_L^{m_0!}(\sigma,\sigma) = \left(\int_{X_y} |\sigma|^{\frac{2}{m_0!}}\right)^{m_0!} \qquad (\sigma,\sigma' \in m_0!L_y).$$

Let a be positive integer such that $f_*\mathcal{O}_X(aK_{X/Y}) \neq 0$. Then we see that

$$H^0(X,\mathcal{O}_X(maK_X)) \simeq H^0(Y,\mathcal{O}_Y(ma(K_Y + L)))$$

holds for every $m \geq 0$ and $\mathrm{Kod}(X) = \dim Y$ holds, where $\mathrm{Kod}(X)$ denotes the Kodaira dimension of X. Hence we see that $K_Y + L$ is big. Let A be a very ample line bundle on Y such that for every pseudoeffective singular hermitian line bundle (F, h_F) on Y, $\mathcal{O}_Y(K_Y + A + F) \otimes \mathcal{I}(h_F)$ is globally generated. Let h_A be a C^∞ hermitian metric on A with strictly positive curvature. We shall construct a sequence of Bergman kernels $\{K_m\}$ and a sequence of singular hermitian metrics $\{h_m\}_{m\geq 1}$ as follows. First we set

$$K_1 := \begin{cases} K(Y, K_Y + A, h_A), & \text{if } a > 1, \\ \\ K(Y, K_Y + L + m_0!(K_Y + L), h_L \cdot h_A), & \text{if } a = 1, \end{cases}$$

where for a singular hermitian line bundle (F, h_F) $K(Y, K_Y + F, h_F)$ denotes (the diagonal part of) the Bergman kernel of $H^0(Y,\mathcal{O}_Y(K_Y + F) \otimes \mathcal{I}(h_F))$ with respect to the L^2-inner product:

$$(\sigma,\sigma') := (\sqrt{-1})^{n^2} \int_Y h_F \cdot \sigma \wedge \bar{\sigma}',$$

where n denotes $\dim Y$. Then we set $h_1 := (K_1)^{-1}$. We continue this process. Suppose that we have constructed K_m and the singular hermitian metric h_m on $\lfloor \frac{m}{a} \rfloor a(K_Y + L) + (m -$

$\lfloor \frac{m}{a} \rfloor a)K_Y$, where for a real number λ, $\lfloor \lambda \rfloor$ denotes the largest integer less than or equal to λ. Then we define

$$K_{m+1} := \begin{cases} K(Y, (m+1)K_Y + \lfloor \frac{m+1}{a} \rfloor aL + A, h_m) & \text{if } m+1 \not\equiv 0 \mod a, \\ \\ K(Y, (m+1)(K_Y + L) + A, h_L^a \otimes h_m) & \text{if } m+1 \equiv 0 \mod a \end{cases}$$

and $h_{m+1} := (K_{m+1})^{-1}$. Thus inductively we construct the sequences $\{h_m\}_{m\geq 1}$ and $\{K_m\}_{m\geq 1}$. This inductive construction is essentially the same one originated by the author in [T0]. The following theorem asserts that the above dynamical system yields the canonical Kähler current on Y.

Theorem 1. *(*[T2]*) Let X be a smooth projective variety of nonnegative Kodaira dimension and let $f : X \longrightarrow Y$ be the Iitaka fibration as above. Let m_0 and $\{h_m\}_{m\geq 1}$ be as above and let n denote* $\dim Y$. *Then $h_\infty := \liminf_{m\to\infty} \sqrt[m]{(m!)^n \cdot h_m}$ is a well defined singular hermitian metric on $K_Y + L$ such that*

(i) *h_∞ is an AZD of $K_Y + L$, i.e., $\sqrt{-1}\,\Theta_{h_\infty}$ is a closed positive current on Y and $H^0(Y, \mathcal{O}(am(K_Y + L) \otimes \mathcal{I}(h_\infty^{am})) \simeq H^0(Y, \mathcal{O}(am(K_Y + L)))$ holds for every $m \geq 1$.*

(ii) *We set $\omega_Y := \sqrt{-1}\,\Theta_{h_\infty}$. Then there exists a nonempty Zariski open subset U such that $\omega_Y|U$ is a C^∞ Kähler form on U and it satisfies the equation:*

$$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\Theta_{h_L} = \omega_Y.$$

(iii) *We define the volume form $d\mu_{can}$ on X by $d\mu_{can} := f^*(\frac{1}{n!}\omega_Y^n \cdot h_L^{-1})$. Then $d\mu_{can}^{-1}$ is an AZD of K_X.*

□

$d\mu_{can}$ is unique and independent of A and h_A. $d\mu_{can}$ is said to be **the canonical measure on** X. And $\omega_X := -\mathrm{Ric}\, d\mu_{can}$ is said to be **the canonical semipositive current on** X. These are birationally invariant. We note that $d\mu_{can}$ as constructed independently by Song and Tian ([S-T]) in different context.

Theorem 2. *(*[T2]*) Let $f : X \longrightarrow S$ be a projective family such that X, S are smooth and f has connected fibers. Suppose that $f_*\mathcal{O}_S(mK_{X/S}) \neq 0$ for some $m > 0$. Then there exists a singular hermitian metric h_K on $K_{X/Y}$ such that*

(i) *$\omega_{X/S} := \sqrt{-1}\,\Theta_{h_K}$ is semipositive on X.*

(ii) *For a general smooth fiber $X_s := f^{-1}(s)$, $h_K|X_s$ is an AZD of K_{X_s} and $h_K^{-1}|X_s$ is the canonical measure on X_s.*

□

Theorems 1 and 2 generalize the results in [T1], where $X = Y$ and L is trivial.

Theorem 2 strengthen the semipositivity of the direct image of relative multicanonical bundles due to Kawamata ([K]). Also we may prove a similar theorem for a projective family of KLT pairs. I would like to propose the following conjecture:

Conjecture 3. *Let $f : X \longrightarrow Y$ be an algebraic fiber space, i.e., X, Y are smooth projective and f is surjective with connected fibers. Then every sufficiently large $m >> 1$, $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated outside of the discriminant locus of f.* □

I have proved the following partial answer to Conjecture 3.

Theorem 3. *Let $f : X \longrightarrow Y$ be an algebraic fiber space. Then $f_*O_X(mK_{X/Y})$ is almost globally generated as m tends to infinity outside of the discriminant locus D of f, in the sense that for there exists a nonempty Zariski open subset U of Y such that for every $y \in U$,*

$$\limsup_{m\to\infty} Q(am, y) = 1$$

holds, where a is a positive integer such that $f_*O_X(aK_{X/Y}) \neq 0$ and

$$Q(am, y) := \frac{\text{rank Image}\{H^0(Y, f^*O_X(amK_{X/Y})) \to f_*O_X(amK_{X/Y}) \otimes \mathbb{C}_y\}}{\text{rank } f_*O_X(amK_{X/Y})}.$$

□

The proof uses Theorem 2. In fact $\sqrt{-1}\,\Theta_{h_K}$ in Theorem 2 defines a (singular) Monge-Ampère foliation on the total space X which descends to a Monge-Ampère foliation on Y. Then we see that the leaf of the Monge-Ampère foliation corresponds to the fiber of the moduli map to the moduli space of pairs of the bases of the relative Iitaka fibration and the Hodge line bundles with the Hodge metrics. Then the desired sections can be constructed by the pull back of the sections on the moduli space. We can generalize the above results to the case of KLT pairs without any efforts. Theorem 4 implies the inequality : $\text{Kod}(X) \geq \text{Kod}(Y) + \text{Kod}(X/Y)$, where $\text{Kod}(X/Y)$ denotes the Kodaira dimension of a general fiber of $f : X \longrightarrow Y$.

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Fake projective planes and arithmetic fake compact hermitian symmetric spaces

S -K Y

The main theme of the talk is to present a joint project of Gopal Prasad and the author on classification and construction of fake projective planes and their higher dimensional analogues.

A fake projective plane is a smooth complex surface which has the same Betti numbers as $P^2_{\mathbb{C}}$ but which is not biholomorphic to $P^2_{\mathbb{C}}$. It is special in the sense that it has the smallest Euler number among smooth surfaces of general type. Furthermore, a fake projective plane turns out to be a quotient of the complex two ball by a torsion-free discrete subgroup of $PU(2,1)$. Hence it is a Shimura surface and carries rich geometric and arithmetic structures.

The first example of fake projective plane was constructed by Mumford [6], utilizing p-adic uniformization. Two more examples were later found by Ishida and Kato in [2], utilizing related methods. More recently, Keum constructed a fake projective plane with an order 7 automorphism in [3], starting with Ishida's analysis on Mumford's example. The main purpose of the project of Gopal Prasad and myself is to classify and construct examples of fake projective planes, as well as their higher dimensional analogues in arithmetic fake compact hermitian symmetric spaces.

As mentioned above, a fake projective plane M is uniformized by the complex ball (complex hyperbolic space of complex dimension 2), a consequence of Bogomolov-Miyaoka-Yau Inequality and results of Aubin and Yau on complex Monge-Ampere equation. Hence we may write $M = \Pi \backslash PU(2,1)/P(U(2)\times U(1))$, with Π a cocompact torsion-free subgroup of $PU(2,1)$. It is proved independently in the work of Klingler [4] and myself [11] that Π is an arithmetic lattice in $PU(2,1)$. Both of the approaches rely on analysis related to harmonic maps into Bruhat-Tits buildings associated to Π. As arithmetic lattices have been classified and are listed in [13], the classification problem is reduced to classification of arithmetic lattices with restricted topological invariants. This is the approach taken by Prasad and the author in [8].

Crucial to the results of [8] is the volume formula of Prasad in [5]. Equipped with the volume formula, we set out to list all arithmetic lattices Γ of $PU(2,1)$ with Euler number $\chi(\Gamma) \leq 3$. This is done with the help of various techniques in analytic number theory, which allow us to derive a reasonably sharp bound on the discriminants of the defining number fields. Once we are reduced to a small list of examples, we either construct examples with the help of Bruhat-Tits theory and number theory, or eliminate by conditions imposed on the values of associated Dedekind zeta and L functions. Here is the main result of [8].

Theorem 1. *(a) There are twenty-six non-empty classes of fake projective planes.*
(b) There can at most be five more classes of fake projective planes, corresponding to very specific number fields.

We remark that according to a conjecture of Rogawski, there should not be any fake projective planes of the type listed in (b). In fact, Cartwright and steger [1] have been able to eliminate three of the five classes in (b) as possible candidates for fake projective planes. For each of the twenty-six non-empty classes in (a), we have constructed at least one example. Very recently, Cartwright and Steger [1] succeeded in listing all examples in 23 classes above.

Theorem 2. *(Cartwright-Steger) There are precisely forty-five fake projective planes among twenty-three classes in Theorem 1(a).*

A potential application of the research of [8] and [1] is that they provide a list of projective algebraic surfaces equipped with a finite non-trivial automorphism group and small Chern numbers that may be useful in constructing new interesting surfaces to chart geography of surfaces of general type. In fact, quite a few of the list of examples in

(a) have non-trivial finite group actions whose quotients give rise to interesting algebraic surfaces after resolving singularities. In particular, it has been verified by Cartwright and Steger that such resolutions include examples of simply connected surfaces of general type with $K^2 = 3$, $q = 0 = p_g$, surfaces which have been sought after by algebraic geometers. This is parallel to a completely different recent construction of examples due to H. Park, J. Park and D. Shin [7].

As a generalization of the notion of fake projective planes in complex dimension two, Prasad and the author study arithmetic fake compact hermitian symmetric spaces in [9], [10]. Let $\overline{G}$ be a connected semi-simple real algebraic group of adjoint type. Let X be the symmetric space of $\overline{G}(\mathbb{R})$ and X_u be the compact dual of X. We shall say that the quotient X/Π of X by a cocompact torsion-free arithmetic subgroup Π of $\overline{G}(\mathbb{R})$ is an arithmetic fake X_u if its Betti numbers are same as that of X_u; X/Π is an irreducible arithmetic fake X_u if, further, Π is irreducible (i.e., no subgroup of Π of finite index is a direct product of two infinite normal subgroups). The main results of [9], [10] are the followings.

Theorem 3. *(a) There exists no arithmetic fake projective space of dimension different from* 2 *and* 4.
(b) There are at least four classes of arithmetic fake projective spaces in dimension 4.
(c) There are at least four distinct arithmetic fake $\mathbf{Gr}_{2,5}$ *and at least five irreducible arithmetic fake* $P^2_{\mathbb{C}} \times P^2_{\mathbb{C}}$

Theorem 4. *There is no arithmetic fake Hermitian symmetric space of type* B_n C_n, D_n *with* $n > 4$, E_6 *or* E_7.

We may define a fake compact hermitian symmetric space to be a Kähler manifold which has the same Betti numbers as a hermtian symmetric space of compact type of the same dimension. A natural geometric problem is to decide when a fake compact hermtian symmetric space is an arithmetic fake compact hermitian symmetric space. The two notions are the same for fake projective planes, but the problem is much more complicated and essentially open in higher dimensions. In particular, it is not true for fake projective spaces of odd dimension, where there are the examples of hyperquadrics. Hence the problem is interesting for fake projective spaces only in even dimensions. The following result in [12] is a positive result in this direction.

Theorem 5. *A fake projective four space has to be an arithmetic fake projective four space if any of the following conditions is satisfied.*
(i) $c_1^4(M) \neq 225$,
(ii) $H^4(M, \mathbb{Z})$ *modulo torsion is generated by* $\theta \cup \theta$, *where* θ *is a generator of* $H^2(M, \mathbb{Z})$ *modulo torsion, or*
(iii) The cycle corresponding to the canonical line bundle K_M *is not a generator of the Neron-Severi group.*

It will be interesting to clarify the situations in other cases.

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Rigidity and envelopes of holomorphy in group actions

X -Y Z

We discuss the rigidity property for automorphism groups of invariant domains in Stein manifolds which are homogenous under the complex reductive Lie groups.

Let $D \subset\subset (\mathbb{C}^*)^n$ be a Reinhardt domain. The automorphism group $Aut(D)$ of D obviously contains the n-dimensional torus group T^n. Are there additional positive dimensional symmetries? In one dimensional case, it's well-known that for the annulus, the automorphism group is just $T \rtimes Z_2$. For higher dimensional case, there are many works about this problem for the rigidity property of $Aut(D)$, the answer says that $Aut(D)$ is compact and the identity component $Aut(D)_0$ of $Aut(D)$ is exactly T^n. This result was established in several papers by different methods, see [2], [1], [5], [7].

Let K be a connected compact Lie group and L be a closed subgroup of K, $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ be (universal) complexifications of K and L, then $X = K/L$ is a compact homogenous space and $X_{\mathbb{C}} = K_{\mathbb{C}}/L_{\mathbb{C}}$ is a complexification of X which is a Stein manifold. There is a natural holomorphic action of $K_{\mathbb{C}}$ on $X_{\mathbb{C}}$ given by the left translation. Let $D \subset X_{\mathbb{C}}$ be a K-invariant domain. Throughout this report, a domain means a connected open set.

In [13], Zhou proved the following result.

Theorem ([13]). Let $D \subset\subset K_{\mathbb{C}}/L_{\mathbb{C}}$ be a K-invariant domain, then $Aut(D)$ is compact.

Under more assumption that (K, L) is a symmetric pair, the result is due to Fels and Geatti [4].

We present our new results based on [11]. Without loss of generality, we may assume that K acts effectively on $X = K/L$. In this setting, for a K-invariant domain D, it's easy to see that $Aut(D)$ obviously contains K. We may ask if there are not additional positive dimensional symmetries. It should be noted that it's not the case for general homogeneous space K/L. However, our results show that for some homogeneous spaces including symmetric spaces it's the case.

Theorem. Let D be an orbit connected K-invariant domain in $X_{\mathbb{C}} = K_{\mathbb{C}}/L_{\mathbb{C}}$. Let W be a connected compact subgroup of $Aut(D)$ containing K, then W can be naturally realized as a subgroup of the isometry group $Iso(X,\ g)$ of $(X\ g)$, where $X = K/L$ and g is some K-invariant Riemannian metric on X.

As an immediate corollary, we have the following: let (K, L) be a symmetric pair, K be semisimple, then $Iso(X,\ g) = K$ for any K-invariant Riemannian metric on X. Let D be a relatively compact K-invariant domain in $X_{\mathbb{C}}$. Then $Aut(D)_0 = K$.

The above corollary can be extended to the isotropy irreducible homogeneous spaces. A homogeneous space K/L is said to be isotropy irreducible if the adjoint representation of L is irreducible on the vector space $k/\,l$, where k and l are Lie algebras of K and L; to be strongly isotropy irreducible if the adjoint representation of the identity component L_0 of L is irreducible on the vector space $k/\,l$. These spaces are classified, and the isometry groups of the spaces are explicitly given and just equal to K for effective action of K except a couple of cases, see [10, 9]. Consequently we have $Aut(D)_0 = K$ for these spaces.

In the proof, a result of Zhou's about the univalence of the envelope of holomorphy of invariant domains plays a key role.

Theorem ([12]). Let M be a Stein manifold, $K_{\mathbb{C}}$ holomorphically act on M. Let $D \subset M$ be a K-invariant orbit connected domain. Then the envelope of holomorphy $E(D)$ of D is schlicht and orbit convex if and only if the envelope of holomorphy $E(K_{\mathbb{C}} \cdot D)$ of $K_{\mathbb{C}} \cdot D$ is schlicht. Furthermore, in this case, $E(K_{\mathbb{C}} \cdot D) = K_{\mathbb{C}} \cdot E(D)$.

This result unifies and extends many known results. In particular, we have the following theorem which is essentially used in the proof.

Theorem ([12]). Let K be a connected compact Lie group and L be a closed subgroup of K. If L is connected, then any K-invariant domain D in $X_{\mathbb{C}} = K_{\mathbb{C}}/L_{\mathbb{C}}$ has schlicht envelope of holomorphy.

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