Mathematisches Forschungsinstitut Oberwolfach

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# Mini-Workshop: Mathematical Approaches to Collective Phenomena in Large Quantum Systems

Organised by
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August 31st – September 6th, 2008

Abstract. Despite enormous progress in the last couple of decades, collective phenomena still mark one of the basic challenges of mathematical statistical mechanics. A number of approaches have been developed that provide a rigorous control of classical systems; however only few are directly applicable to quantum systems and so many basic, but extremely important, quantum collective effects remain beyond the scope of present-day mathematics. The workshop brings together people of different backgorund working on collective phenomena of quantum systems.

*Mathematics Subject Classification (2000):* 82B10; 82B20; 82B21; 82B28; 82B26; 82C10;60G55; 60F10: 82C10; 81S40.

## Introduction by the Organisers

The workshop *Mathematical Approaches to Collective Phenomena in Large Quantum Systems*, organised by Stefan Adams (Warwick) and Robert Seiringer (Princeton) was held August 31st–September 6th, 2008. This meeting was well attended with 17 participants with broad geographic representation from all continents and with a good mixture of younger Postdocs/researchers (Boland, Bru, Crawford, Schlein, and Starr) and senior researchers. This workshop was a nice blend of researchers with various backgrounds ranging from analysis, probability theory, and functional integration. The fifteen talks are focused along the three general research directions developing mathematical theories of quantum phase transitions:

(1) Dilute gas limit and nonlinear effective theories
(2) Random walk representations via Feynman-Kac formula
(3) Field theory/semiclassical representations via coherent states

Area (1) is represented with new developments in talks by Lieb and Solovej focused on the Bogoliubov theory to derive second order correction to the ground state energy and in a talk by Yngvason on rotating trapped Bose gases. New developments for quantum spin systems have been reported by Nachtergaele and Starr. A new probabilistic approach has been introduced in a talk by König which shows a modelling of the free energy in the thermodynamic limit, and future will show if this approach goes beyond the existing theory on finite temperature Bose gases. An interesting area has been summarised by Ueltschi on random permutations. This area is recently under much focus in the probability community. The contribution from the youngest participant, the PhD-student Boland, showed an interesting result concerning the Bose condensate density and the density of so-called long cycles. Long cycles appear if probability mass of finite cycles gets lost in the thermodynamic limit, an area which is studied in the combinatorics community via shape measure analysis.

The schedule of the workshop allowed intensive discussion during the afternoons including Friday. The workshop has been closed with an informal discussion/meeting on Friday evening which was attended by the majority of the participants. The workshop showed interesting new developments, and it is the hope of the organisers that there will be more interdisciplinary collaborations, in particular between the analysis and the probability community.

## Mini-Workshop: Mathematical Approaches to Collective Phenomena in Large Quantum Systems

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# Abstracts

## Review of some mathematical results on the Bose gas

Elliott H. Lieb

The low temperature properties of quantum-mechanical many-body systems (bosons) at low density, $\rho$, can be examined experimentally and it is therefore important to verify mathematically some of the theories deduced about these systems by many authors over a period of six decades. For systems with two-body interaction potentials the experimental low temperature state and the ground state are nearly synonymous – and this concentration on the ground state is used in most modeling.

The Hamiltonian that is usually used to describe these gases of $N$ atoms or particles (with coordinates $x \in \mathbb{R}^d$) is

$$H_N = -\sum_{1 \leq j \leq N} \Delta_{x_j} + \sum_{1 \leq i < j \leq N} v(x_i - x_j) \tag{1}$$

with the two-body interaction $v$ taken to be non-negative in most rigorous work.

The first really serious mathematical, but non-rigorous treatment was by Bogolubov in 1947 [1] and this was reviewed in the talk. Many of the predictions of the theory are still deemed valid but unproved rigorously. He deduced the small $\rho$ leading asymptotic term in the ground state energy (= smallest eigenvalue) $E_0$:

$$E_0 \approx 4\pi N \rho\, a \tag{2}$$

for $n = 3$, where $a$ is the scattering length of the two-body, short-range potential $v$.

Owing to the delicate and peculiar nature of bosonic correlations (such as the strange $N^{7/5}$ law for charged bosons), four decades of research failed to establish this plausible formula (2) rigorously. In 1957 [2] Dyson derived an upper bound for $E_0$ that agreed with (2). The only previous lower bound for the energy was in [2] but it was 14 times too small. The more modern theory starts with the rigorous derivation [3] of this asymptotic formula (2). A different formula, postulated as late as 1971 by Schick, holds in two dimensions and this, too, was shown to be correct.

With the aid of the methodology developed to prove the lower bound for the homogeneous gas, several other problems have been successfully addressed. One is the proof [4] that, with the appropriate scaling $v(x) \to N^2 v(Nx)$, the *Gross-Pitaevskii equation* correctly describes the ground state density and energy in the 'traps' actually used in the experiments. For this system it is also possible to prove complete Bose-Einstein condensation and superfluidity [5, 6]. There is also a Gross-Pitaevskii equation for a rotating, trapped Bose gas, but it was not known if this equation correctly describes the ground state. The problem is that the absolute ground state of the Hamiltonian for a rotating gas need not be the same as the ground state in the class of symmetric (i.e., Bose) functions – as it is if rotation is absent. Purely physical reasoning could not quite decide the issue, but it turns out that the GP equation correctly describes the ground state of bosons [7].

Another recent development about the Gross-Pitaevskii equation is the analysis of its time dependent version, which ought to predict the density of a gas whose wave function $\psi$ evolves under the Schrödinger time evolution: $i\partial\psi/\partial t = H_N\psi$. This has been shown to be the case in the non-rotating case by Erdös et al [8] for suitable initial data.

Another topic is a proof that Foldy's 1961 calculation [9] (based on Bogolubov's theory) of a *one*-component, *high density* Bose gas of electrically charged particles correctly describes its ground state energy, even though the Coulomb electrostatic interaction is very long-ranged. He predicted a $-C_1N\rho^{1/4}$ law for $E_0$. A proof has also been given of the $E_0 \approx -C_2N^{7/5}$ formula for the ground state energy of the *two*-component charged Bose gas predicted by Dyson. Both power laws (1/4 and 7/5) were proved in [10] in 1988. The sharp constants $C_1$, $C_2$ were proved in [11, 12], thereby showing the correctness of the preceding predictions.

Many of these results were presented in a book which was an outgrowth of lectures [13] given at an Oberwolfach seminar.

Despite these partial successes there are still big open problems. Does Bose-Einstein condensation occur for homogeneous Bose gases in the thermodynamic limit (i.e., not in traps)? Can one go beyond the first term, eq. (2), in the asymptotics of the ground state energy? And what about the excitation spectrum, which is so important for understanding superfluidity? These, and others, are left for the future.

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## Lieb-Robinson Bounds for Quantum Spin Dynamics and Applications

Bruno Nachtergaele
(joint work with Yoshiko Ogata, Hillel Raz, Benjamin Schlein, Robert Sims)

Locality is a fundamental property of all current physical theories. Sets of observables can be associated with points or bounded regions in space or space-time and a relativistic dynamics will preserve this structure. There is a wide range of important physical systems, however, which we prefer to describe by very effective non-relativistic quantum theories with Hamiltonian dynamics. Even if the Hamiltonian has only finite-range interactions, the dynamics it generates generally does not preserve locality, i.e., there is no strict equivalent to the finite speed of light. However, locality still holds in an approximate sense, and there is an associated finite velocity, which is sometimes referred to as the group velocity. We call it the Lieb-Robinson velocity since Lieb and Robinson were the first to prove its existence and to obtain a bound for it [9]. They proved that to a high degree of accuracy locality is preserved by quantum spin dynamics in the sense that any local observable evolved for a time $t > 0$ remains localized in a region of space with diameter proportional to $t$, up to an arbitrarily small correction. This also means that spatial correlations between observables separated by a distance $d$ cannot be established faster than a time of order $d$.

The fundamental issue of locality may be sufficient motivation to extend the Lieb-Robinson bounds to more general situations, but there are other good reasons to try to generalize their result and to improve the estimates they obtained. We discuss in this talk how locality, or the approximate locality of the dynamics, has been shown to be responsible for a considerable number of other important properties relevant for models of many-body systems. In many situations, however, the implications of locality have yet to be fully explored.

We will begin by presenting a short proof of the new Lieb-Robinson bounds obtained successively in [12], [6],[14], and [10]. This improved result sharpens the bounds previously obtained in that the prefactor now only grows as the smallest surface area of the supports of the local observables. An application where this surface area dependence, rather than volume dependence, is important can be found in [3].

Lieb-Robinson bounds can be used to provide explicit estimates on the local structure of the time evolution. As a consequence, one easily derives bounds on, for example, multiple commutators and the rate at which spatial correlations can be established in normalized product states.

Lieb-Robinson bounds are an essential element in the proof of the so-called Exponential Clustering Theorem. In the relativistic context it has been known for a long time that a gap in the spectrum above the vacuum state implies exponential decay of spatial correlations in that state [4]. That a similar result should hold in the non-relativistic setting such as quantum spin systems was long expected and taken for granted by theoretical physicists. In [5], Hastings proposed to use Lieb-Robinson bounds to obtain such a result and a complete proof was recently given in [12, 6].

As a final application of these locality bounds, we describe a new proof of the Lieb-Schultz-Mattis theorem, see [5, 13]. These results can be traced back to [5] where Hastings introduced a new way to construct and analyze variational states for low-lying excitations of gapped Hamiltonians. He developed a notion of a *quasi-adiabatic evolution* [7] which he then used to present a multi-dimensional analogue of the celebrated Lieb-Schultz-Mattis theorem [8]. Such a theorem is applicable, for example, to the standard spin-1/2, anti-ferromagnetic Heisenberg model and yields an upper bound on the first excited state of order $c(\log L)/L$ for systems of size $L$. His arguments rely on Lieb-Robinson bounds and the Exponential Clustering Theorem in an essential way, and we have recently obtained a rigorous proof of this result which holds in a rather general setting, see [13].

We expect that the ideas currently emerging from recent applications of Lieb-Robinson bounds will continue to lead to interesting new results for quantum spin systems in the near future.

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## A Variational Formula for the Free Energy of a Many-Boson System

Wolfgang König
(joint work with Stefan Adams and Andrea Collevecchio)

(This is work in progress.)

We consider the $N$-particle Hamilton operator

$$\mathcal{H}_N = -\sum_{i=1}^{N} \Delta_i + \sum_{1\le i<j\le N} v(|x_i - x_j|),$$

in a centred box $\Lambda_N$ in $\mathbb{R}^d$ with periodic boundary condition. The volume $|\Lambda_N|$ of the box is equal to $N/\rho$, where $\rho \in (0,\infty)$ is the particle density. The pair-interaction potential $v$ is supposed to be nonnegative, explodes at zero and vanishes at infinity sufficiently fast. We want to study bosons at positive temperature $1/\beta \in (0,\infty)$ and consider the symmetrised trace of the operator $\mathrm{e}^{-\beta\mathcal{H}_N}$. It is our goal to derive an explicit variational formula for the limiting free energy, in the hope to derive a proof for Bose-Einstein condensation from this in future.

It is known since decades that the above mentioned trace can be represented in terms of a Feynman-Kac formula (an expectation over $N$ Brownian bridges with symmetrised initial-terminal conditions) and that this formula can be expanded in a combinatorial way according to the structure of the cycles. Our new input is the introduction of a marked random Poisson process such that this formula may be represented as an expectation over the stationary empirical field, $R_N$, of the Poisson process. Here the marks are the cycles. This enables us to write the trace in the form

$$\mathrm{Tr}_+\big(\mathrm{e}^{-\beta\mathcal{H}_N}\big) = \mathbb{E}\Big[\mathrm{e}^{-|\Lambda_N|F(R_N)} 1_{\{G(R_N)=\rho\}}\Big],$$

where $F$ expresses all the interactions between the marks of the process, and $G$ expresses the total length of all the cycles of the process in the unit box. This representation makes the study of the system amenable to a large-deviation analysis, in particular to the application of Varadhan's lemma, which formally turns the exponential rate of the right-hand side into the variational formula

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathrm{Tr}_+\big(\mathrm{e}^{-\beta\mathcal{H}_N}\big) = -\inf\{F(P) + I(P)\colon G(P) = \rho\},$$

where $I$ denotes the rate function for a large-deviation principle that is satisfied by $(R_N)_{N\in\mathbb{N}}$, according to work of Georgii and Zessin (1994), and $P$ runs through the set of all random marked stationary point fields. This assertion is only formal, since both $F$ and $G$ are not continuous neither bounded. Actually, only the upper bound with '=' replaced by '$\leq$', has been proved, and the question if the lower bound also holds is intimately connected with the presence of Bose-Einstein condensation (BEC). This famous effect is signalled by a significant appearance of the 'infinitely long cycles' (those cycles whose lengths increase to $\infty$ with $N$) in the formula, and these cycles do not explicitly appear in the formula. It is conjectured that the absence of BEC is equivalent to the validity of the lower bound, which describes the part of the system that comes only from 'small' cycles. Future work is devoted to the study of this relation and to the question for what values of the parameters $\beta$ and $\rho$ the two variational formulas coincide. Even more ambitious will be the search for a reformulation in a setting that is also able to describe all the 'infinitely long cycles'.

## Lieb-Robinson Bounds for Harmonic and Anharmonic Lattice Systems

Benjamin Schlein

(joint work with Bruno Nachtergaele, Hillel Raz, and Robert Sims)

We discuss locality properties of the dynamics of quantum harmonic and anharmonic lattice models. Since these are non-relativistic models there is no a priori bound on the speed of propagation of signals in these systems. In the case of quantum spin systems with finite-range interactions, Lieb and Robinson showed in [1] that there is nevertheless an upper bound on the speed of propagation in the sense that disturbances in the system remain confined in a "light" cone up to small corrections that decay exponentially fast away from the light cone.

More precisely, if $\tau_t$ denotes the time evolution (a one-parameter group of automorphisms on the algebra of observables) of a spin system with short range interaction, Lieb-Robinson proved that, for any observable $A$ supported in a set $X$ and every observable $B$ supported in $Y$, one has

$$\|[\tau_t(A), B]\| \leq C\|A\|\|B\| \, e^{-\mu(d(X,Y)-v|t|)} \tag{1}$$

where $d(X,Y) = \min_{x\in X, y\in Y} |x-y|$ and $|x| = \sum_{j=1}^{\nu} |x_j|$. The physical interpretation of this inequality is straightforward; if two observables $A$ and $B$ are supported in disjoint regions, then even after evolving the observable $A$, apart from exponentially small contributions, their supports remain essentially disjoint up to times $t \leq d(X,Y)/v$. In other words, this bound asserts that the speed of propagation of perturbations in quantum spin systems is bounded.

It seems natural to ask whether Lieb-Robinson bounds such as (1) can be extended to systems defined on infinite dimensional Hilbert spaces, and described by unbounded Hamiltonians. Although the constant $C$ and the velocity $v$ in (1) can be chosen independently of the dimension of the spin spaces (see [2, 3]), they depend on the operator norm of the interactions $\Phi(X)$, and thus the generalization is not simple. In [4], we prove that Lieb-Robinson bounds of the form (1) can be established for three different types of models with unbounded Hamiltonians:

1) systems with bounded interactions (described by Hamiltonians with bounded nonlocal terms and, possibly, unbounded local terms), 2) harmonic lattice systems, 3) anharmonic lattice systems (for a certain class of bounded anharmonicity).

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## On the Switching Lemma for the Transverse Ising Model

Nicholas Crawford
(joint work with Dmitry Ioffe)

We outline work in progress [6], [7] generalizing powerful techniques used to study the classical Ising model at the critical temperature $\beta_c$ to the transverse Ising model on a general graph $\mathcal{G}$, where

$$\hat{H}_{\mathcal{G}} = -\sum_{<ij>} J_{<ij>}\hat{\sigma}_i^{(z)}\hat{\sigma}_j^{(z)} - \sum_{i\in\mathcal{G}}(h\hat{\sigma}_i^{(z)} + \lambda\hat{\sigma}_i^{(x)}). \tag{1}$$

is the transverse Ising Hamiltonian on $\mathcal{G}$ acting on the $|\mathcal{G}|$ particle vector space

$$\mathcal{H}_{\mathcal{G}} := \otimes_{i\in\mathcal{G}}\mathbb{C}^2 \tag{2}$$

and $\hat{\sigma}_i^x, \hat{\sigma}_i^z$ denote the usual Pauli operators acting only on the $i$'th factor of $\mathcal{H}_G$. The first sum here is over all bonds of $\mathcal{G}$.

In [1], M. Aizenman introduced a graphical representation called the Random Current Representation (RCR), which is a variation of the usual high temperature expansion from classical statistical mechanics. Combining this representation with a combinatorial identity known as the Switching Lemma leads to a unified derivation of many (old and new) correlation inequalities which played a crucial role in the analysis of the Ising model on $\mathbb{Z}^d$, proving, among other things, the triviality of the scaling limit in $d \geq 5$ when the strength of the external field $h = 0$. The subsequent paper Aizenman and R. Fernandez [3] extended this analysis to the $h > 0$ regime, while Aizenman, D. Barsky, and Fernandez [2] used the Switching Lemma along with a certain random walk decomposition to derive differential inequalities which lead to a proof that the magnetic transition is sharp (i.e. there is no intermediate phase).

Our talk consists of two topics: Using the path integration techniques introduced in [5], we formulate a version of the RCR directly for the transverse Ising model and described how to recover a (weak) form of the Switching Lemma in this context. This sort of generalization should be expected on the basis of the representation of the transverse Ising model as a strong coupling limit classical Ising models [4], though it does not easily follow from this limiting procedure.

The second topic (work done with Ioffe and Y. Velenik) is a brief sketch of a relatively simple proof that the truncated correlation functions

$$\langle \hat{\sigma}_i^z ; \hat{\sigma}_j^x \rangle_{\mathcal{G}}, \langle \hat{\sigma}_i^z ; \hat{\sigma}_j^z \rangle_{\mathcal{G}} \langle \hat{\sigma}_i^x ; \hat{\sigma}_j^x \rangle_{\mathcal{G}}, \tag{3}$$

all decay exponentially in the graph distance $d_{i,j}$ under the assumptions that $h > 0$ and $J_{i,j} \geq J > 0$

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## Rotating Bose gases

Jakob Yngvason

Ultra-cold Bose gases display remarkable quantum phenomena under rotation, in particular quantized vortices related to superfluidity. The article [1] reviews the subject from the physics point of view, the monograph [2] is concerned with more mathematical aspects. Most of the theoretical work has been within the framework of the Gross-Pitaevskii (GP) equation for the wave function of the Bose-Einstein condensate. This non-linear Schrödinger equation is the variational equation obtained by minimizing the GP energy functional

$$\mathcal{E}^{\rm GP}[\psi] = \int_{\mathbb{R}^d} \left\{ |(\mathrm{i}\nabla + \mathbf{A})\psi|^2 + (V - \tfrac{1}{4}\Omega^2 r)|\psi|^2 + g|\psi|^4 \right\} \mathrm{d}\mathbf{x} \tag{1}$$

with $d = 3$ or 2 and the normalization condition $\int_{\mathbb{R}^d} |\psi|^2 = 1$. Here $\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{\Omega} \wedge \mathbf{x}$ with $\mathbf{\Omega}$ the angular velocity, $V$ the external confining potential and $r$ the distance from the rotation axis. The interaction between the gas particles is encoded in a single parameter, $g = 4\pi a N/L$, where $a$ is the scattering length of the interaction potential, $N$ the particle number and $L$ a length scale associated with the external potential. For rotating gases in their ground state the GP equation was derived in [3] and [4] from the quantum mechanical many-body Hamiltonian with repulsive, short range interactions and fixed values of $\Omega$ and $g$ as $N \to \infty$. The extension of this derivation to the case when $\Omega$ and $g$ tend to infinity (or $\Omega$ approaches a critical value in the case of harmonic traps) has not yet been completed, but the leading order asymptotics of the many-body ground state energy for large coupling and rotational velocity in an anharmonic trap was computed in [5].

Detailed results on the emergence of vortices as the rotational velocity is increased have been obtained within two-dimensional GP theory when the GP interaction parameter $g$ is large ('Thomas-Fermi' limit) and the rotational velocity $\Omega$ is of the order of the logarithm of this parameter, see [2] and [6] and references quoted there. In this case the number of vortices remains finite as the interaction parameter tends to infinity. For faster rotation, new effects that are briefly reviewed in [7] come into play.

It is customary to write $g = 1/\varepsilon^2$, so large $g$ corresponds to small $\varepsilon$. The dimensionless parameter $\varepsilon$ is a measure for the size of the vortices relative to the size of the confining trap. The main part of the lecture at the workshop was based on recent joint work [8] with Michele Correggi on the energy and vorticity of the minimizers of the functional (1) in the parameter range

$$|\log\varepsilon| \ll \Omega \ll 1/(\varepsilon^2|\log\varepsilon|). \tag{2}$$

We have proved the following

**Theorem 1** (**Energy to subleading order).** *Let* $E^{\rm GP}$ *denote the two-dimensional GP energy, i.e., the minimum of the GP energy functional* (1) *with* $d = 2$, *in a flat trap of radius 1 with Neumann boundary conditions. Let* $E^{\rm TF}$ *denote the minimal energy of the GP functional* without *the first (kinetic energy) term. If* $|\log\varepsilon| \ll \Omega \ll 1/\varepsilon$ *as* $\varepsilon \to 0$, *then*

$$E^{\rm GP} = E^{\rm TF} + (\Omega/2)|\log(\varepsilon^2\Omega)|(1 + o(1)). \tag{3}$$

*If* $1/\varepsilon \lesssim \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$ *then*

$$E^{\rm GP} = E^{\rm TF} + (\Omega/2)|\log\varepsilon|(1 + o(1)). \tag{4}$$

Previous results [9] in the parameter range (2) where limited to the the leading order contribution. The subleading terms in Eqs. (3) and (4) correspond to the energy of trial functions where vortices of degree one are distributed on a regular lattice with density $\Omega/(2\pi)$. For matching lower bounds the problem is reformulated in such a way that results from Ginzburg-Landau (GL) theory obtained in [10] and [11] can be employed. The strong inhomogeneity of the density in fast rotating condensates due to the centrifugal forces causes problems that make the reduction to the GL case not entirely straightforward.

When the rotational velocity reaches $O(1/(\varepsilon^2|\log\varepsilon|))$ a different trial function, with all the vorticity concentrated in a region where the density is small ('giant vortex'), gives a lower energy than the vortex lattice. In this region a lower bound matching the variational upper bound has not yet been derived and the details of the transition from the vortex lattice to the giant vortex merit further study. A further challenging problem is to derive the subleading term in the energy from the full quantum mechnical many-body problem.

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# Ground state properties of the weak coupling 2D Hubbard model on the honeycomb lattice

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(joint work with Vieri Mastropietro)

We consider a 2D Hubbard model on the honeycomb lattice, as a model for a single layer graphene sheet in the presence of screened Coulomb interactions. At half filling and for weak enough coupling, we construct the whole set of correlation functions via a resummed convergent series, stemming from a multiscale analysis based on constructive fermionic renormalization group. The result is that in the interacting theory the large distance asymptotic behavior of correlations is the same as in the absence of interactions, modulo a finite renormalization of the Fermi velocity and of the wave function. The large distance asymptotics is described by a massive QED theory in $2+1$ dimensions, in the presence of an ultraviolet cutoff. Remarkably, the $U(1)$ symmetry of QED is dynamically restored, thanks to a discrete rotational symmetry of the original Hubbard model.

In order to state our results in a more precise form, let us introduce the Hamiltonian of the 2D Hubbard model on the honeycomblattice at half filling in second quantized form:

$$
\begin{aligned}
H &= -\sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \sum_{\sigma=\uparrow\downarrow} \left( a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_i,\sigma} + b^+_{\vec{x}+\vec{\delta}_i,\sigma} a^-_{\vec{x},\sigma} \right) + \\
&+ \frac{U}{3} \sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \left( a^+_{\vec{x},\uparrow} a^-_{\vec{x},\uparrow} a^+_{\vec{x},\downarrow} a^-_{\vec{x},\downarrow} + b^+_{\vec{x}+\vec{\delta}_i,\uparrow} b^-_{\vec{x}+\vec{\delta}_i,\uparrow} b^+_{\vec{x}+\vec{\delta}_i,\downarrow} b^-_{\vec{x}+\vec{\delta}_i,\downarrow} \right) + \\
&+ U' \sum_{\substack{\vec{x}\in\Lambda\\ i=1,2,3}} \sum_{\sigma,\sigma'=\uparrow\downarrow} a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma} b^+_{\vec{x}+\vec{\delta}_i,\sigma'} b^-_{\vec{x}+\vec{\delta}_i,\sigma'}
\end{aligned}
$$

where:

(1) $\Lambda$ is a periodic triangular lattice, defined as $\Lambda = \mathbb{B}/L\mathbb{B}$, where $L \in \mathbb{N}$ and $\mathbb{B}$ is the triangular lattice with basis $\vec{a}_1 = \frac{1}{2}(3, \sqrt{3})$, $\vec{a}_2 = \frac{1}{2}(3, -\sqrt{3})$.

(2) The vectors $\vec{\delta}_i$ are defined as

$$\vec{\delta}_1 = (1, 0)\ , \quad \vec{\delta}_2 = \frac{1}{2}(-1, \sqrt{3})\ , \quad \vec{\delta}_3 = \frac{1}{2}(-1, -\sqrt{3})\ .$$

(3) $a^{\pm}_{\vec{x},\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec{x} \in \Lambda$, satisfying periodic boundary conditions in $\vec{x}$

(4) $b^{\pm}_{\vec{x}+\vec{\delta}_i,\sigma}$ are creation or annihilation fermionic operators with spin index $\sigma =\uparrow\downarrow$ and site index $\vec{x} + \vec{\delta}_i \in \Lambda + \vec{\delta}_1$, satisfying periodic boundary conditions in $\vec{x}$.

(5) $U$ is the strength of the on–site density–density interaction and $U'$ is the strength of the nearest neighbor density–density interaction; they can both be either positive or negative.

Note that the Hamiltonian above is particle-hole symmetric (i.e., it is invariant under the exchanges $a^{\pm}_{\vec{x},\sigma} \to a^{\mp}_{\vec{x},\sigma}$, $b^{\pm}_{\vec{x},\sigma} \to -b^{\mp}_{\vec{x},\sigma}$), so that in particular $< N >= 2|\Lambda|$, where $< \cdot >= \lim_{\beta\to\infty} \mathrm{Tr}\{e^{-\beta H}\}/\mathrm{Tr}\{e^{-\beta H}\}$.

We define the two component fermionic operators $\Psi^{\pm}_{\vec{x},\sigma} = \left(a^{\pm}_{\vec{x},\sigma}, b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}\right)$ and we write $\Psi^{\pm}_{\vec{x},\sigma,1} = a^{\pm}_{\vec{x},\sigma}$ and $\Psi^{\pm}_{\vec{x},\sigma,2} = b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}$. We also consider the operators $\Psi^{\pm}_{\mathbf{x},\sigma} = e^{Hx_0}\Psi^{\pm}_{\vec{x},\sigma}e^{-Hx_0}$ with $\mathbf{x} = (x_0, \vec{x})$ and $x_0 \in [0, \beta]$, for some $\beta > 0$; we shall call $x_0$ the time variable. We write $\Psi^{\pm}_{\mathbf{x},\sigma,1} = a^{\pm}_{\mathbf{x},\sigma}$ and $\Psi^{\pm}_{\mathbf{x},\sigma,2} = a^{\pm}_{\mathbf{x}+\boldsymbol{\delta}_1,\sigma}$, with $\boldsymbol{\delta}_1 = (0, \vec{\delta}_1)$.

We are interested in computing the specific ground state energy

$$e(U, U') = \lim_{\beta\to\infty} \lim_{|\Lambda|\to\infty} (\beta|\Lambda|)^{-1} \log \mathrm{Tr}\, e^{-\beta H}$$

and the zero temperature Schwinger functions:

$$S_n(\mathbf{x}_1, \varepsilon_1, \sigma_1, \rho_1; \ldots; \mathbf{x}_n, \varepsilon_n, \sigma_n, \rho_n) = \lim_{\beta\to\infty} \lim_{|\Lambda|\to\infty} \frac{\mathrm{Tr}\, e^{-\beta H}\mathbf{T}(\Psi^{\varepsilon_1}_{\mathbf{x}_1,\sigma_1,\rho_1} \cdots \Psi^{\varepsilon_n}_{\mathbf{x}_n,\sigma_n,\rho_n})}{\mathrm{Tr} e^{-\beta H}}$$

where $\mathbf{x}_i, \mathbf{y}_i \in [0, \beta] \times \Lambda$, $\sigma_i =\uparrow\downarrow$, $\varepsilon_i = \pm$, $\rho_i = 1, 2$ and $\mathbf{T}$ is the operator of fermionic time ordering. Our main result is the following.

**Theorem** [1] *For $U, U'$ small enough, the specific ground state energy and the zero temperature Schwinger functions are analytic functions of $U, U'$. In particular, if $S(\mathbf{x}-\mathbf{y})_{\rho,\rho'} = S_2(\mathbf{x}, -, \sigma, \rho; \mathbf{y}, +, \sigma, \rho')$ and $\vec{p}_F^{\,\pm} = (\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}})$, we can write*

$$S(\mathbf{x}-\mathbf{y}) = \sum_{\omega=\pm} e^{i\vec{p}_F^{\,\pm}(\vec{x}-\vec{y})} S_\omega(\mathbf{x}-\mathbf{y})$$

*where*

$$\hat{S}_\omega(k_0, \vec{k}')^{-1} = Z \begin{pmatrix} -ik_0 & -v_F(ik_1' + \omega k_2') \\ -v_F(-ik_1' + \omega k_2') & -ik_0 \end{pmatrix} \Big(\mathbf{1} + R(\mathbf{k}')\Big)\,,$$

*with: (i) $\mathbf{k}' = (k_0, \vec{k}')$, where $k_0 \in \mathbb{R}$ and $\vec{p}_F^{\,\omega} + \vec{k}' \in \mathbb{R}^2/\Lambda^*$, $\Lambda^*$ being the dual lattice of $\Lambda$; (ii) $Z$ and $v_F$ two real constants such that $|Z-1|, |v_F - 3/2| \le C(|U| + |U'|)$; $R(\mathbf{k}')$ is a higher order correction satisfying $||R(\mathbf{k}')|| \le C|\mathbf{k}'|^\vartheta$ as $|\mathbf{k}'| \to 0$, for some constants $C, \vartheta > 0$.*

The theorem above is proven by rigorous fermionic renormalization group methods, see [2] and references therein. The convergence of the (resummed) perturbation series is achieved by determinant bounds, and by a step by step modification of the propagator, combined with the fact that at half filling all quartic or higher order interactions are irrelevant in a renormalization group sense.

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## Long Cycles in the Infinite-Range-Hopping Bose-Hubbard Model with Hard Cores

Gerry Boland

(joint work with J.V. Pulé)

*We consider the relation between long cycles and Bose-Einstein Condensation (BEC) in the Infinite range Bose-Hubbard Model with a hard core interaction*[3]. *We calculate the density of particles on long cycles in the thermodynamic limit and find that the existence of a non-zero long cycle density coincides with the occurrence of BEC but this density is not equal to that of the condensate.*

In 1953, Feynman[5] analysed the partition function of an interacting Bose gas in terms of the statistical distribution of permutation cycles of particles and emphasised the roles of long cycles at the transition point. Penrose and Onsager[6], pursuing his arguments, observed that BEC occurs when the fraction of the total particle number belonging to long cycles is strictly positive. A precise formulation of the relation between BEC and long cycles was lacking until the work of Sütö [8] but its validity has been checked only in a few models: the free and mean field Bose gas[8] (see also [10]), and the perturbed mean field model of a Bose gas[4].

In these models it is shown that the density of particles on long cycles is equal to the condensate density.

We test the validity of the hypothesis in yet another model of a Bose gas, the Infinite-Range Bose-Hubbard Model. Its Hamiltonian is given by

$$H^{\mathrm{IR}} = \frac{1}{2V} \sum_{x,y=1\dots V} (a_x^* - a_y^*)(a_x - a_y) + \lambda \sum_{x=1\dots V} n_x(n_x - 1)$$

where $a_x^*$ and $a_x$ are the usual Bose creation and annihilation operators satisfying $[a_x^*, a_y] = \delta_{x,y}$ and $n_x = a_x^* a_x$. The properties of this model is studied in [2] and [1]. Here we choose a special case, introduced and analysed by Tóth [9], where $\lambda = +\infty$, that is complete single-site exclusion (hard-core). Penrose [7] obtained equivalent results and in addition, calculated the density of the BEC, $\rho_\beta^c$.

In the canonical ensemble, we re-formulate this problem by applying a hard-core projection $\mathcal{P}_n^{\mathsf{hc}}$ to the symmetrised $n$-particle Hilbert space with $V$ sites, $\mathcal{H}_{V,+}^{(n)}$, to exclude states violating the hard-core condition. We may write $H_{\lambda=+\infty}^{\mathrm{IR}}$ as

$$H_{n,V}^{\mathsf{hc}} := \mathcal{P}_n^{\mathsf{hc}} (n - P_V^{(n)}) \mathcal{P}_n^{\mathsf{hc}}$$

applied to $\mathcal{H}_{n,V,+}^{\mathsf{hc}} := \mathcal{P}_n^{\mathsf{hc}} \mathcal{H}_{V,+}^{(n)}$, where

$$P_V^{(n)} = P_V \otimes I \otimes \cdots \otimes I + I \otimes P_V \otimes I \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes P_V$$

and $P_V$ is the single particle ground state operator. $P_V^{(n)}$ can be thought of as representing the hopping of the particles. The structure of the proof is as follows:

*1.* Using a combinatorial argument, we express the density of particles on cycles of length $q$ for $n$ particles, $c_V^n(q)$, in terms of the trace of the exponential of the Hamiltonian for $n - q$ bosons and $q$ distinguishable particles (no statistics):

$$c_V^n(q) = \frac{1}{Z_\beta(n,V)} \frac{1}{V} \operatorname{trace}_{\mathcal{H}_{q,n,V}^{\mathsf{hc}}} \left[ U_q \mathrm{e}^{-\beta H_{n,V}^{\mathsf{hc}}} \right]$$

where $\mathcal{H}_{q,n,V}^{\mathsf{hc}} := \mathcal{P}_n^{\mathsf{hc}} (\mathcal{H}_V^{(q)} \otimes \mathcal{H}_{V,+}^{(n-q)})$, $\mathcal{H}_V^{(q)}$ is the unsymmetrised $q$-particle space, $U_q$ is the unitary representation of a $q$-cycle on $\mathcal{H}_V^{(q)}$, and $Z_\beta(n,V)$ is the canonical partition function. Summing this over all cycle lengths yields $\sum_{q=1}^n c_V^n(q) = n/V$. The long and short cycle densities are defined in the thermodynamic limit ($n, V \to \infty$ with $\frac{n}{V} = \rho$) as

$$\rho_{\mathrm{short}} = \lim_{Q \to \infty} \lim_{\substack{n,V \to \infty \\ n/V = \rho}} \sum_{q=1}^{Q} c_V^n(q) \qquad \text{and} \qquad \rho_{\mathrm{long}} = \rho - \rho_{\mathrm{short}}.$$

*2.* We prove that in the thermodynamic limit that we can neglect the hopping of the $q$ distinguishable particles. This is almost equivalent to the reduction of the lattice by $q$ sites. Moreover these particles are on a cycle of length $q$.

*3.* As these distinguishable particles may now be taken not to hop, but are cycled, due to the hard-core condition only cycles of unit length contribute. For short cycles, the thermodynamic limit and the sum over cycle lengths can be reversed.

Thus we obtain

$$\rho_{\rm long} = \rho - \lim_{\substack{n,V\to\infty \\ n/V=\rho}} c_V^n(1).$$

*4.* The one-cycle density, apart from some scaling, is the partition function for the boson system with one site and one particle removed, which can be calculated using a large-deviation argument.

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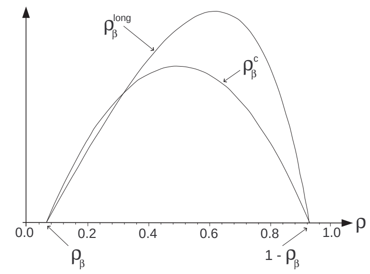


$\rho_\beta^c$ *and* $\rho_\beta^{\rm long}$ *for* $\beta > 2$

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## A simple solution of the Matsubara UV problem for many fermions

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(joint work with W. Pedra)

It is known that perturbation theory converges in fermionic field theory at weak coupling if the interaction and the covariance are summable and if certain determinants arising in the expansion can be bounded efficiently, e.g. if the covariance admits a Gram representation with a finite Gram constant. The covariances obtained from the grand canonical ensemble of quantum statistical mechanics do not fall into this class due to the slow decay of the covariance at large Matsubara frequency. This causes an ultraviolet problem in the integration over degrees of freedom with Matsubara frequencies larger than some $\Omega$ (usually the first step in a multiscale analysis). I discuss a simple solution of this problem, obtained in collaboration with Walter Pedra [PS]. We show that these covariances do not have Gram representations on any separable Hilbert space. We then prove a general bound for determinants associated to chronological products which is stronger than the usual Gram bound and which applies to the many–fermion case. This allows us to prove convergence of the first integration step in a rather easy way, for a short–range interaction which can be arbitrarily strong, provided $\Omega$ is chosen large enough. The method allows to give nonperturbative bounds on all scales for the case of scale decompositions of the propagator which do not impose cutoffs on the Matsubara frequency. Applications include an easy proof of $\ell^1$ clustering of truncated correlation functions in weakly interacting fermion systems, and ultraviolet smoothing of Hubbard–Stratonovich transformations.

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## Quantum Spin Systems at Positive Temperature

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(joint work with Marek Biskup, Lincoln Chayes)

It is a challenge to prove that spin systems possess phase transitions, for both classical and quantum spins. However, in a number of cases the classical phase transition was proved where the quantum phase transition is also expected but not proved. Taking $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ to be the spin magnitude of each site, one knows that the limit $j \to \infty$ is supposed to recover the classical model. Therefore, one may hope, at least, to prove a phase transition for the quantum model for large enough $j$, assuming one has proved a phase transition for the classical model. This is what we proved, in the paper [1], for models satisfying the following conditions:

(i) The quantum Hamiltonian is reflection positive for each $j$.
(ii) The classical model is also reflection positive and can be proved to have a phase transition via the Peierls argument, using the Chessboard estimates of Frohlich and Lieb [2] at inverse temperature $\beta$.
(iii) The upper and lower symbols of the spin-$j$ Hamiltonian converge, and converge to the classical Hamiltonian, in the $j \to \infty$ limit. More specifically, we require all the local interactions to converge in Lipschitz norm (i.e., to converge in the Hölder space $C^{0,1}$).
(iv) The parameter $j/\beta^2$ is sufficiently large.

Conditions (i) and (ii) are the most restrictive. But, on the other hand, there are a variety of models where these conditions are known, which we will describe at the end. Condition (iii) is guaranteed for any Hamiltonian whose interactions are polynomials in spin matrices by a calculation of Lieb [3], which covers essentially all models of interest. Condition (iv) is a technical restriction of the proof. It has the following unfortunate consequence: suppose by a Peierls argument that you can prove that the quantum model has a phase transition for all $\beta > \beta_0$. Then for each $\beta > \beta_0$ you can prove that the quantum model also has a phase transition for $j$ sufficiently large, but you cannot prove it for the same $j$ if you make $\beta$ much larger. Solving this problem would be an extension of considerable interest (but is open).

Coherent states are vectors in $\mathbb{C}^{2j+1}$, the spin-$j$ representation of SU(2), indexed by points on the sphere $\omega = (\omega^x, \omega^y, \omega^z) \in \mathbb{S}^2 \subseteq \mathbb{R}^3$ such that

$$(\omega^x S^x + \omega^y S^y + \omega^z S^z)\psi^j_\omega = j\psi^j_\omega\,.$$

This condition (and the normalization condition) uniquely characterizes the coherent state vector $\psi^j_\omega$ modulo a phase factor, which drops out of the coherent state projector $|\psi^j_\omega\rangle\langle\psi^j_\omega|$. An important formula is the resolution of the identity:

$$\mathbb{I} = \frac{2j+1}{4\pi} \int_{\mathbb{S}^2} d\omega\, |\psi^j_\omega\rangle\langle\psi^j_\omega|\,.$$

Given any continuous function $f : \mathbb{S}^2 \to \mathbb{C}$, the coherent states give a method to obtain an operator $\hat{f}^j : \mathbb{C}^{2j+1} \to \mathbb{C}^{2j+1}$ as

$$\hat{f}^j = \frac{2j+1}{4\pi} \int_{\mathbb{S}^2} d\omega\, f(\omega)\, |\psi^j_\omega\rangle\langle\psi^j_\omega|\,.$$

If $\hat{f}^j = A$ then one says that $f$ is an upper symbol for $A$. But there are many upper symbols for $A$, so the upper symbol is not unique. The fact that upper symbols exist for every operator $A$ on $\mathbb{C}^{2j+1}$ is also important and follows from calculations in Lieb's paper on the classical limit of quantum spin systems [3]. Let us write $\check{A}^j$ for the set of all upper symbols for the operator $A$. Lieb showed that for any reasonable Hamiltonian one can choose a sequence of upper symbols $f_j \in \check{H}^j$ such that $f_j$ converges, in the $j \to \infty$ limit to the classical Hamiltonian. The same is true for the lower symbol of $H$ which is the function $\omega \mapsto \langle\psi^j_\omega, H\psi^j_\omega\rangle$. For example, for the Heisenberg interaction $h_{\boldsymbol{r},\boldsymbol{r}'} = \boldsymbol{S}_{\boldsymbol{r}} \cdot \boldsymbol{S}_{\boldsymbol{r}'}/j^2$, one can choose the upper symbol to be $[1+j^{-1}]^2\omega_{\boldsymbol{r}} \cdot \omega_{\boldsymbol{r}'}$ and the lower symbol is $\omega_{\boldsymbol{r}} \cdot \omega_{\boldsymbol{r}'}$.

Lieb, and independently Berezin shortly before, proved a pair of inequalities known as the Berezin-Lieb inequalities:

$$\int_{\mathbb{S}^2} \frac{d\omega}{4\pi}\, e^{\langle\psi^j_\omega, A\psi^j_\omega\rangle} \leq \frac{\mathrm{Tr}(e^A)}{2j+1} \tag{1}$$

for any self-adjoint operator $A : \mathbb{C}^{2j+1} \to \mathbb{C}^{2j+1}$, and

$$\frac{\mathrm{Tr}(e^{\hat{f}^j})}{2j+1} \leq \int_{\mathbb{S}^2} \frac{d\omega}{4\pi}\, e^{f(\omega)} \tag{2}$$

for any real, continuous function $f(\omega)$. As a historical note, while Lieb's proof of (1) used essentially just Jensen's inequality, his original proof of (2) used an idea similar to the Golden-Thompson inequality, which is a truly quantum mechanical bound. But later, Lieb derived a proof of (2) also just using Jensen's inequality, and this is reported in Barry Simon's survey of coherent states [4]. Taking $A$ to be a quantum Hamiltonian – such as a polynomial in spin operators – and taking $f$ to be an upper symbol for the same Hamiltonian, Lieb was able to prove that the free energy of a quantum model converges to the free energy of the classical model in the $j \to \infty$ limit.

In order to prove a phase transition, one has to go beyond this. In our paper [1], we proved a bound which reads as follows.

**Theorem 1.** *Suppose that* $H^j_\Lambda = \sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle \subset \Lambda} h^j_{\boldsymbol{r},\boldsymbol{r}'}$ *is translation invariant and* $h^j_{\boldsymbol{r},\boldsymbol{r}'}$ *is a polynomial in* $S^x_{\boldsymbol{r}}/j, \dots S^z_{\boldsymbol{r}'}/j$ *(the same polynomial for all* $j$*) then for each* $M < \infty$*, there is a* $K < \infty$ *such that for all* $\beta \leq Mj^{1/2}$*,*

$$\left|\langle \psi^j_{\boldsymbol{\omega}}, e^{-\beta H^j_\Lambda} \psi^j_{\boldsymbol{\omega}'}\rangle\right| \leq e^{-\beta \check{H}^j(\omega) + d(\boldsymbol{\omega},\boldsymbol{\omega}') + K\beta j^{-1/2}|\Lambda|},$$

*where* $\psi^j_{\boldsymbol{\omega}}$ *is the tensor-product vector* $\bigotimes_{\boldsymbol{r}\in\Lambda}[\psi^j_{\omega_{\boldsymbol{r}}}]_{\boldsymbol{r}}$ *for each* $\boldsymbol{\omega} = (\omega_{\boldsymbol{r}})_{\boldsymbol{r}\in\Lambda}$ *and*

$$d(\boldsymbol{\omega},\boldsymbol{\omega}') = \sum_{\boldsymbol{r}\in\Lambda} \min\{j^{1/2}\|\omega_{\boldsymbol{r}} - \omega_{\boldsymbol{r}'}\|, j\|\omega_{\boldsymbol{r}} - \omega_{\boldsymbol{r}'}\|^2\}.$$

In this theorem $\check{H}^j(\omega)$ is an upper symbol for $H^j$, chosen appropriately according to the method of Lieb. The proof follows by a differential inequality, and then an application of Gronwall's theorem. Note that for $\beta = 0$ the theorem follows trivially. By differentiating with respect to $\beta$ and keeping careful track of all the terms, one can derive this bound. But one problem with it is that the error term in the exponent seems quite large since $K\beta j^{-1/2}|\Lambda|$ is of the order of the system size. It is here, however, that Frohlich and Lieb's chessboard estimate is particularly useful.

Denote by $\langle \cdots \rangle_{\beta H}$ the Gibbs state $\langle A\rangle_{\beta H} = \operatorname{Tr}(Ae^{-\beta H})/\operatorname{Tr}(e^{-\beta H})$. For a reflection-positive Hamiltonian, $H_\Lambda$, Frohlich and Lieb proved [2] that for an operator $A$ acting on a single spin site, $\boldsymbol{r}_0$,

$$|\langle A\rangle_{\beta H_\Lambda}| \leq \|A\|_{\beta H_\Lambda} := \left(\left\langle \prod_{\boldsymbol{r}\in\Lambda} \theta_{\boldsymbol{r}}(A)\right\rangle_{\beta H_\Lambda}\right)^{1/|\Lambda|},$$

where $\theta_{\boldsymbol{r}}$ is the appropriate reflection of the observable algebra for spin site $\boldsymbol{r}_0$ to $\boldsymbol{r}$, used in the definition of reflection positivity. This is their Chessboard estimate (or generalized Hölder's inequality). Here one has disseminated the operator $A$, acting on a single site, to the entire lattice so it is now acting on all sites. For the case that $A$ is a projector (or an indicator function in the classical case) Frohlich and Lieb call the disseminated operator $\prod_{\boldsymbol{r}\in\Lambda}\theta_{\boldsymbol{r}}(A)$ the "universal projector." Using Theorem 1, we see that for any measurable set of configurations for a single spin $E \subset \mathbb{S}^2$, and for each $M < \infty$ there is a $K < \infty$ such that

$$\|\hat{\mathbf{1}}_E\|_{\beta H^j_\Lambda} \leq \|\mathbf{1}_E\|_{\beta H^\infty_\Lambda} e^{K\beta j^{-1/2}},$$

for all $\beta \leq Mj^{1/2}$, where $\mathbf{1}_E$ is the indicator function of $E$ (the definition of $f \mapsto \hat{f}^j$ extends naturally to nonnegative measurable functions) and $H^\infty_\Lambda$ denotes the classical Hamiltonian which is also the limit of the upper and lower symbols for $H^j_\Lambda$ in the $j \to \infty$ limit. In particular, as long as $\beta j^{-1/2}$ is sufficiently small this inequality is effective. The same inequality works for finite blocks of spins instead of single sites. Using Frohlich and Lieb's version of the Peierls argument, with

Chessboard estimates, the small parameter in the expansion into Peierls contours is exactly $\|\mathbf{1}_E\|_{\beta H^\infty_\Lambda}$ where $E$ is a set of "bad" block events whose disseminated probability is small at sufficiently large $\beta$. Therefore, one can essentially run the same Peierls argument for the quantum model, assuming conditions (i) to (iv) are satisfied.

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## Local semicircle law and complete delocalization for Wigner matrices


László Erdős

(joint work with Benjamin Schlein and Horng-Tzer Yau)


The Wigner semicircle law states that the empirical density of the eigenvalues of a random matrix is given by the universal semicircle distribution. This statement has been proved for many different ensembles, in particular for the case when the distributions of the entries of the matrix are independent, identically distributed (i.i.d.). To fix the scaling, we normalize the matrix so that the bulk of the spectrum lies in the energy interval $[-2, 2]$, i.e., the average spacing between consecutive eigenvalues is of order $1/N$. We now consider a window of size $\eta$ in the bulk so that the typical number of eigenvalues is of order $N\eta$. In the usual statement of the semicircle law, $\eta$ is a fixed number independent of $N$ and it is taken to zero only after the limit $N \to \infty$. This can be viewed as the largest scale on which the semicircle law is valid. On the other extreme, for the smallest scale, one may take $\eta = k/N$ and take the limit $N \to \infty$ followed by $k \to \infty$. If the semicircle law is valid in this sense, we shall say that the *local semicircle law* holds. Below this smallest scale, the eigenvalue distribution is expected to be governed by the Dyson statistics related to sine kernels.

In this talk, I establish the local semicircle law up to logarithmic factors in the energy scale, i.e., for $\eta \sim N^{-1}(\log N)^8$. The result holds for any energy window in the bulk spectrum away from the spectral edges. Prior to our work the best result was obtained in [1] for $\eta \gg N^{-1/2}$. See also [4] and [5] for related and earlier results.

It is widely believed that the eigenvalue distribution of the Wigner random matrix and the random Schrödinger operator in the extended (or delocalized) state regime are the same up to normalizations. Although this conjecture is far from the reach of the current method, a natural question arises as to whether the eigenvectors of random matrices are extended. More precisely, if $\mathbf{v} = (v_1, \dots, v_N)$ is an $\ell^2$-normalized eigenvector, $\|\mathbf{v}\| = 1$, we say that $\mathbf{v}$ is *completely delocalized* if $\|\mathbf{v}\|_\infty = \max_j |v_j|$ is bounded from above by $CN^{-1/2}$, the average size of $|v_j|$. We prove that all eigenvectors with eigenvalues away from the spectral edges are completely delocalized (modulo logarithmic corrections) in probability. This result, in particular, answers (up to logarithmic factors) the question posed by T. Spencer that $\|\mathbf{v}\|_4$ should be of order $N^{-1/4}$.

Denote the $(i,j)$-th entry of an $N \times N$ hermitian matrix $H$ by $h_{ij} = \overline{h_{ji}}$. These matrices form a *Hermitian Wigner ensemble* if

$$h_{ij} = N^{-1/2}[x_{ij} + \sqrt{-1}\, y_{ij}], \quad (i < j), \quad \text{and} \quad h_{ii} = N^{-1/2} x_{ii}, \tag{1}$$

where $x_{ij}, y_{ij}$ $(i < j)$ and $x_{ii}$ are independent real random variables with mean zero. We assume that $x_{ij}, y_{ij}$ $(i < j)$ all have a common distribution $\nu$ with variance $1/2$ and we assume that it satisfies the logarithmic Sobolev inequality. The diagonal elements, $x_{ii}$, also have a common distribution that may be different from $\mathrm{d}\nu$. Let $\mathbb{P}$ denote the probability w.r.t the joint distribution.

Let $H$ be the $N \times N$ Wigner matrix with eigenvalues $\mu_1 \le \mu_2 \le \ldots \le \mu_N$. The Stieltjes transform of the empirical distribution function of the eigenvalues is

$$m_N(z) = \frac{1}{N} \mathrm{Tr}\, \frac{1}{H - z} \tag{2}$$

for any spectral parameter $z = E + i\eta \in \mathbb{C}$, $\eta > 0$. Its imaginary part is the normalized density of states of $H$ around energy $E$ and regularized on scale $\eta$.

The semicircle law is a distribution on the real line with density function

$$\varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,\mathbf{1}(|x| \le 2)\ .$$

Its Stieltjes transform is

$$m_{sc}(z) = \int_{\mathbb{R}} \frac{\varrho_{sc}(x)\mathrm{d}x}{x - z}$$

**Theorem 2.** *Let $z = E + i\eta$ with $|E| < 2$ and $\eta \ge (\log N)^8/N$. For any $\varepsilon > 0$, the Stieltjes transform $m_N(z)$ of $H$ satisfies*

$$\mathbb{P}\Big\{ |m_N(z) - m_{sc}(z)| \ge \varepsilon \Big\} \le e^{-c_\varepsilon (\log N)^2} \tag{3}$$

This result identifies the density of states away from the spectral edges in a window where the typical number of eigenvalues is of order bigger than $(\log N)^8$. The following theorem shows that all eigenfunctions are fully delocalized.

**Theorem 3.** *For any $\kappa > 0$ and $C$ large, there exists $c > 0$ such that*

$$\mathbb{P}\left\{ \exists\ \mathbf{v} \text{ with } H\mathbf{v} = \mu\mathbf{v},\ \|\mathbf{v}\| = 1,\ \mu \in [-2+\kappa, 2-\kappa] \text{ and } \|\mathbf{v}\|_\infty \ge \frac{C(\log N)^{9/2}}{N^{1/2}} \right\} \le C\, e^{-c(\log N)^2}\ .$$

The proofs of these theorems are found in [2, 3].

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## On the Brockett Flow Equations


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(joint work with V. Bach and M.W. Walser)


Flow equations for operators are defined by Brockett [1] and Wegner [2] as the following non-linear operator-valued first-order differential equations for positive time $t \geq 0$ :

$$\begin{cases} \partial_t Y_t = [Y_t, [Y_t, A_t]] \\ Y_{t=0} := Y_0 \end{cases} \tag{1}$$

with (possibly unbounded) operators $A_t$ and $Y_0$ acting on a Banach space $\mathcal{X}$ or Hilbert space $\mathcal{H}$, and with $[Y_t, A_t] := Y_t A_t - A_t Y_t$ being the commutator between the operators $Y_t$ and $A_t$.

The existence of a solution $Y_t$ of this differential equation is not obvious but interesting since it defines an isospectral flow on the fixed operator $Y_0$. Indeed, the flow equation (1) is closely related to non–autonomous evolution equations. For instance, let $U_{t,s}$ be an evolution operator. In other words, let $U_{t,s}$ be a jointly strongly continuous in $s$ and $t$ operator satisfying the "cycle condition" $U_{t,\tau}U_{\tau,s} = U_{t,s}$ for any $t \geq \tau \geq s \geq 0$ with $U_{t,t} = \mathrm{I}$ being the identity[1]. Take in particular a solution $U_{t,s}$ of the non-autonomous evolution equation[2]

$$\begin{cases} \partial_t U_{t,s} = -iG_t U_{t,s} \\ U_{t,t} := \mathrm{I} \end{cases} \tag{2}$$

with infinitesimal generator $G_t := i\,[A_t, Y_t]$. Then, the operator

$$Y_t = U_{t,s} Y_s U_{t,s}^{-1} = U_t Y_0 U_t^{-1} \tag{3}$$

would be a solution of (1), where by definition $U_t := U_{t,0}$ and $U_t^{-1}$ is its right inverse. In the context of self-adjoint operators $A_t$ and $Y_0$ on a Hilbert space $\mathcal{H}$, the flow equation (1) generates a family of unitarily equivalent operators $Y_t$. Now, the next question is to understand how this isospectral flow can be used. In fact,

[1] I is always the identity operator on the corresponding Hilbert or Banach space.

[2] $\partial_s U_{t,s} = iU_{t,s}G_s$ is a consequence of (2) combined with the "cycle condition".

a solution $Y_t$ of (1) should converge, at least for real symmetric matrices $Y_0$ and (time-independent) $A := A_t$, to a symmetric matrix $Y_\infty$ commuting with $A$.

Their mathematical foundations were missing until [3], which proves the well–posedness of the Brockett flow for bounded operators acting on a Banach space $\mathcal{X}$, as well as some asymptotic properties. The talk was devoted to these results and to a rigorous application of the Brockett flow to diagonalize (unbounded) quadratic bosonic Hamiltonians [4].

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## Rigorous Bethe Ansatz


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(joint work with Alexander Povolotsky, Maxim Samsonov)


The Hamiltonian for the isotropic Heisenberg chain with periodic boundary conditions was diagonalised by Bethe in 1931 using his famous Ansatz where the eigenfunctions are written in terms of a linear combination of waves with permuted wavenumbers $k_1, \dots, k_M$, where $M$ is the number of down-spins:

$$\psi(i_1, \dots, i_M) = \sum_{\sigma \in S_M} \exp\left[ i \sum_{j=1}^{M} k_{\sigma(j)} x_j + i \sum_{\substack{j<l \\ \sigma(j)>\sigma(l)}} \phi_{\sigma(j)\sigma(l)} \right]. \tag{1}$$

Here $i_1, \dots, i_M$ denote the positions of the down spins, and the phase factors $\phi_{jl}$ are related to the wavenumbers by $2\cot(\phi_{jl}) = \cot(k_j/2) - \cot(k_l/2)$. The wavenumbers $k_1, \dots, k_M$ have to satisfy the following set of nonlinear equations, called the Bethe Ansatz Equations (BAEs),

$$e^{ik_jN} = \prod_{\substack{l=1 \\ l\neq j}}^{M} \exp(i\phi_{jl}), \qquad (j = 1, \dots, M). \tag{2}$$

It was suggested already by Bethe that the solutions of these equations, when written in terms of the new variables

$$\Lambda_j = \cot(\tfrac{1}{2}k_j), \tag{3}$$

form $n$-tuples of complex numbers with equal real parts and with imaginary parts which are equally separated and centred about the real axis, up to an error which is exponentially small in $N$. This is known as the "*string hypothesis*". Assuming that this string hypothesis is valid, one can derive equations for the real parts of the strings, which involve integers (or half-odd integers) $I_\alpha^{(n)}$, where $n$ labels the length of the string and $\alpha$ numbers the different strings of length $n$. All $I_\alpha^{(n)}$ can be assumed to satisfy $|I_\alpha^{(n)}| < \frac{N-n}{2}$.

The Bethe Ansatz has since been applied to many other models in statistical mechanics and field theory, but the mathematical status of the string hypothesis is still unclear. (It was however proven by Tarasov and Varchenko [4] that the total set of solutions of (2) yields a complete basis of eigenstates. See also [5].)

In the talk I demonstrated that even in the case $M = 2$, the string hypothesis needs to be modified, but that it is essentially true, in that it holds for the *majority* of solutions (namely those with $|I| < N - \sqrt{N}\ln N$), and the error term is *not exponentially small* in $N$, but does vanish as $N \to \infty$. This claim can also been proven for higher values of $M$ in the case of real solutions as well as for a single string. In the case $M = 3$, for example, the solutions in the case of a single string are shown in the following figure:

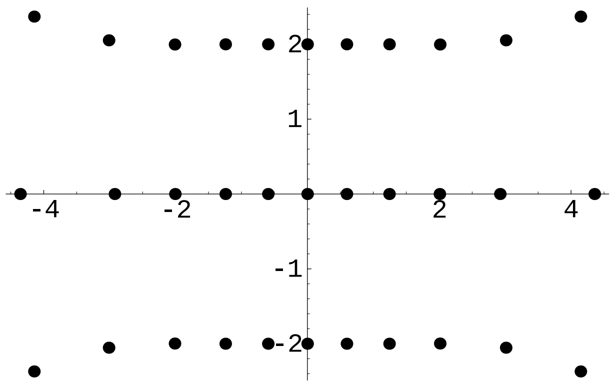


Figure 2. *Solutions of the BAEs for a string of length 3 with $N = 16$. Notice that the outer most points ($I = \pm 6$) are missing; these do not converge.*

It is clear from the figure that the hypothesis holds for solutions away from the edges. The general case of several strings is still under investigation.

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## Rigorous applications of the Bogolubov approximation

Jan Philip Solovej

In this talk I discuss an approach to making rigorous the Bogolubov approximation [1].

I will discuss the application of the Bogolubov approximation to three different examples.

*Example (a).* The dilute Bose gas with Hamiltonian

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1\leq i<j\leq N} V(x_i - x_j),$$

where $V$ is a sufficiently fast decaying non-negative potential and the Hilbert space is $\mathcal{H} = \bigotimes^N L^2(\Lambda)$. Here $\Lambda \subset \mathbb{R}^3$ is a cube and $-\Delta$ is the Laplacian with Dirichlet boundary conditions.

We are interested in the thermodynamic ground state energy

$$e(\rho) = \lim_{\substack{|\Lambda|\to\infty \\ N/|\Lambda|=\rho}} |\Lambda|^{-1} \inf \mathrm{spec}_{\mathcal{H}} H_N.$$

It is expected that Bogolubov theory gives the energy asymptotics in the dilute limit $\rho a^3 \to 0$, where $a$ is the scattering length of the potential $V$.

*Example (b).* The one-component charge Bose gas with Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\Delta_i - \rho \int_\Lambda |x_i - y|^{-1} dy) + \sum_{1\leq i<j\leq N} V(x_i - x_j) + \frac{1}{2}\rho^2 \iint_{\Lambda\times\Lambda} |x-y|^{-1} dx dy.$$

The Hilbert space is again $\mathcal{H} = \bigotimes^N L^2(\Lambda)$. Again the quantity of interest is the thermodynamic ground state energy

$$e(\rho) = \lim_{\substack{|\Lambda|\to\infty \\ N/|\Lambda|=\rho}} |\Lambda|^{-1} \inf \mathrm{spec}_{\mathcal{H}} H_N.$$

Bogolubov theory gives the energy asymptotics in the high density limit $\rho \to \infty$.

*Example (c).* The two-component charged Bose gas with Hamiltonian

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1\leq i<j\leq N} \frac{q_i q_j}{|x_i - x_j|}.$$

This time the Hilbert space is $\mathcal{H} = \bigotimes^N L^2(\mathbb{R}^3)$ We study the energy optimized over the charges $q_i = \pm 1$:

$$E(N) = \inf\{\inf \mathrm{spec}_{\mathcal{H}} H_N \mid q_i = \pm 1\}.$$

Bogolubov theory gives the energy asymptotics in the limit $N \to \infty$.

In case (b) the energy asymptotics based on the Bogolubov approximation was suggested by Foldy [3] and proved rigorously in [5, 7]. The result is

$$e(\rho) = \frac{4^{5/4}\Gamma(3/4)}{5\pi^{1/4}\Gamma(5/4)}\rho^{5/4} + o(\rho^{5/4})$$

as $\rho \to \infty$.

In case (c) the following result was conjectured by Dyson [2] and proved rigorously in [6, 7]

$$\lim_{N\to\infty} N^{-7/5}E(N) = \inf\left\{ \int_{\mathbb{R}^3} |\nabla\Phi|^2 - \frac{4^{5/4}\Gamma(3/4)}{5\pi^{1/4}\Gamma(5/4)} \int_{\mathbb{R}^3} \Phi^{5/2} \,\middle|\, 0 \leq \Phi,\ \int_{\mathbb{R}^3} \Phi^2 = 1 \right\}.$$

For the dilute gas in Example (a) the situation is more complicated. Recently, in joint work with Lieb the method in [5, 6, 7] has been extended to prove the asymptotics

$$e(\rho) = 4\pi\rho^2 a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3})\right)$$

under the assumption that $V(x) = R^{-3}v(x/R)$ and $(\rho a^3)^{1/2} \ll a/R < (\rho a^3)^{1/3-\varepsilon}$ for some small, but fixed, positive $\varepsilon$. In particular, this includes the case when $R \ll \rho^{-1/3}$ which implies, of course, that the gas is low density in the sense that particles rarely "see" each other. The opposite is true if $R \gg \rho^{-1/3}$.

In the case when $V$ is a cut-off Yukawa potential the asymptotics above has been announced by Giuliani and Seiringer in the narrow range $(\rho a^3)^{1/2} \ll a/R < (\rho a^3)^{1/2-\varepsilon}$. It is believed that the asymptotic formula originally derived in [4] should hold for all $(\rho a^3)^{1/2} \ll a/R$.

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## Feynman-Kac representation of the Bose gas and spatial random permutations

Daniel Ueltschi

Let $\Lambda \subset \mathbb{R}^d$ be a cube of size $L$ and volume $V = L^d$, and let $N \in \mathbb{N}$. The state space of the model of spatial permutations is

$$\Omega_{\Lambda,N} = \Lambda^N \times \mathcal{S}_N, \tag{1}$$

with $\mathcal{S}_N$ the symmetric group of permutations of $N$ elements. We are interested in the properties of permutations, and all our random variables are functions

$\theta : \mathcal{S}_N \to \mathbb{R}$. Their probability distributions depend on spatial variables in an indirect but essential way. Let $\ell_i(\pi)$ denote the length of the cycle that contains $i$, i.e. the smallest integer $n \geq 1$ such that $\pi^n(i) = i$. The most important random variable is the density of points in cycles of certain lengths. For $n, n' \in \mathbb{N}$, let

$$\boldsymbol{\varrho}_{n,n'}(\pi) = \frac{1}{V} \#\big\{i = 1, \dots, N : n \leq \ell_i(\pi) \leq n'\big\}. \tag{2}$$

The expectation of the random variable $\theta$ is defined by

$$E_{\Lambda,N}(\theta) = \frac{1}{Z(\Lambda, N)N!} \int_{\Lambda^N} \mathrm{d}\boldsymbol{x} \sum_{\pi \in \mathcal{S}_N} \theta(\pi)\, \mathrm{e}^{-H(\boldsymbol{x},\pi)}\,. \tag{3}$$

Here, the normalization factor $Z(\Lambda, N)$ is chosen so that $E_{\Lambda,N}(1) = 1$. The integral is over $N$ points in $\Lambda$, denoted $\boldsymbol{x} = (x_1, \dots, x_N)$.

We consider Hamiltonians of the form

$$H(\boldsymbol{x}, \pi) = \sum_{i=1}^{N} \xi(x_i - x_{\pi(i)}) + \sum_{1 \leq i < j \leq N} V(x_i, x_{\pi(i)}, x_j, x_{\pi(j)}), \tag{4}$$

with $\xi$ a spherically symmetric function $\mathbb{R}^d \to [0, \infty]$, and $V$ a translation invariant function $\mathbb{R}^{4d} \to \mathbb{R}$. We also suppose that $\xi$ is increasing and that $\xi(0) = 0$. One should think of typical permutations as involving finite jumps, i.e. $|x_i - x_{\pi(i)}|$ stays bounded as $\Lambda, N \to \infty$.

The major question concerns the occurrence of infinite cycles. It turns out that the distribution of cycles can be well characterized in the absence of interactions, with the potential $V \equiv 0$. We need a few hypotheses on $\xi$, namely that $\int \mathrm{e}^{-\xi} = 1$, and that $\mathrm{e}^{-\xi}$ has positive Fourier transform, which we denote $\mathrm{e}^{-\varepsilon(k)}$. The case of physical relevance is $\xi(x) = \frac{1}{4\beta}|x|^2$ with $\beta$ the inverse temperature, in which case $\varepsilon(k) = 4\pi^2\beta|k|^2$.

We define the *critical density* by

$$\rho_{\mathrm{c}} = \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{\mathrm{e}^{\varepsilon(k)} - 1}. \tag{5}$$

The critical density is finite for $d \geq 3$, but it can be infinite for $d = 1, 2$.

**THEOREM** [Sütő 2002; Betz, U 2008]

*Let $\xi$ satisfy the assumptions above. Then for any $0 < a < b < 1$, and any $s \geq 0$,*

(a) $$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{1,V^a}) = \begin{cases} \rho & \text{if } \rho \leq \rho_{\mathrm{c}}; \\ \rho_{\mathrm{c}} & \text{if } \rho \geq \rho_{\mathrm{c}}; \end{cases}$$

(b) $$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{V^a,V^b}) = 0;$$

(c) $$\lim_{V\to\infty} E_{\Lambda,\rho V}(\boldsymbol{\varrho}_{V^b,sV}) = \begin{cases} 0 & \text{if } \rho \leq \rho_{\mathrm{c}}; \\ s & \text{if } 0 \leq s \leq \rho - \rho_{\mathrm{c}}; \\ \rho - \rho_{\mathrm{c}} & \text{if } 0 \leq \rho - \rho_{\mathrm{c}} \leq s. \end{cases}$$

This theorem shows that infinite cycles occur above the critical density, and that they are *macroscopic*. The proof can be found in [1]. It extends an earlier result of Sütő for the ideal Bose gas [2].

Two-body interactions between quantum particles translate into many-body interactions for permutations. But we can perform an expansion and see that, to lowest order, we obtain a two-body interaction between permutation jumps. A computation reveals that the interaction between jumps $x \mapsto y$ and $x' \mapsto y'$ is given by

$$V(x,y,x',y') = \int \big[1 - \mathrm{e}^{-\frac{1}{4}\int_0^{4\beta} U(\omega(s))\mathrm{d}s}\big] \mathrm{d}\widehat{W}^{4\beta}_{x-x',y-y'}(\omega). \tag{6}$$

See [3]. The model of spatial permutations is simpler than the Feynman-Kac representation of the Bose gas, and is therefore better suited to Monte-Carlo simulations. It is expected, but not proved, that this model is *exactly* related to the original quantum boson model, to lowest order in the scattering length $a$.

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