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New Perspectives in Stochastic Geometry

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Abstract. The workshop was devoted to the discussion and exploration of recent developments in stochastic geometry. Two main themes were new results and methodology in classical stochastic geometry and allocation and matching procedures for point processes and random measures.

Mathematics Subject Classification (2000): 60xx, 51xx.

Introduction by the Organisers

The principal aim of stochastic geometry is the mathematical analysis of random geometric structures. Fundamental examples of such structures are point processes of geometric objects, random tessellations of space into convex or non-convex regions, random systems of non-overlapping balls (or more general convex bodies), or excursion and level sets of Gaussian random fields. The workshop was devoted to the discussion and exploration of recent advances in stochastic geometry and its related areas. Among the 23 participants were many leading figures in the field as well as some very promising young scientists.

One main theme of the workshop concerned new results and methodology in classical stochastic geometry. In recent years it has become possible not only to make conclusions about mean values of geometric quantities, but also to derive distributional properties, to prove limit theorems and large deviation results, and to explore higher-order moment properties. Examples include: Hug, Reitzner (approximation of polytopes), Calka (visibility properties of the Boolean model), Schreiber (polygonal Markov fields), Penrose (normal approximation in random geometric graphs), Yukich (limit theorems), Reitzner (random tessellations), Baccelli (point processes and information theory).

A second main theme of the workshop concerned allocation and matching procedures for point processes and random measures. The study of such allocations has resulted in some remarkable progress in the understanding of invariance properties of Palm probability measures and associated transport and coupling questions. There are deep and striking relationships with random tessellations, classical optimal transport theory, potential theory, and complex analysis. Examples include: Peres (fair allocations, optimal matchings, gravitational allocation, ...), Sturm (optimal transportation of measures on metric spaces), Thorisson (transport kernels and mass-stationarity), Last (Cox and Bernoulli transports).

Three series of lectures formed the organizational backbone of the workshop. They were delivered by Matthias Reitzner on "Recent results in stochastic geometry", by Yuval Peres on "Fair allocations" and by Theo Sturm on "New trends in optimal transportation on Riemannian and singular spaces". Many of the contributed talks were related to one or more issues covered by these lectures. Another integral part of the programme was provided by three discussion sessions "Probability models with geometric flavour" organized by Jesper Møller and Sergei Zouev, "Asymptotics in stochastic geometry" organized by Peter Mörters and Mathew Penrose, and "From convex to metric and fractal geometry" organized by Martina Zähle and Wolfgang Weil. These sessions provided space for participants to bring up new ideas and discuss open problems in an informal manner.

Many participants of the workshop co-authored a collection of papers "New Perspectives in Stochastic Geometry" edited by W.S. Kendall and I. Molchanov that will be published by the Oxford University Press in 2009.

Workshop: New Perspectives in Stochastic Geometry

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Abstracts

A Palm Theory Approach to Capacity and Error Exponents in Information Theory

François Baccelli
(joint work with Venkat Anantharam)

Let μ^n be a sequence of stationary ergodic marked point processes. We assume that for all n, μ^n is a simple point process in $\mathbb{R}^n$. Let λ_n denote the intensity of the point process μ^n. We denote by $\{T_k^n\}$ the points of μ^n. The mark of the point T_k^n is a pair (X_k^n, C_k^n), where X_k^n is an vector in $\mathbb{R}^n$ and C_k^n is a measurable subset of $\mathbb{R}^n$. The countable collection of sets $\{C_k^n\}$ is assumed to form a decomposition of $\mathbb{R}^n$. We call X_k^n the *displacement vector* associated to the point T_k^n and C_k^n the *decoding region* associated to the point T_k^n.

Each point T_k^n of μ^n may be thought of as a codeword. When this codeword is transmitted, the channel adds to it the displacement vector X_k^n, so that the received signal is $T_k^n + X_k^n$. We would want the received signal to land in the correct decoding region. With this interpretation, we define the *probability of error* as

$$p_e(n) = \lim_{W\to\infty} \frac{\sum_k \mathbb{1}_{T_n^k \in B(0,W)} \mathbb{1}_{T_k^n + X_k^n \notin C_k^n}}{\sum_k \mathbb{1}_{T_n^k \in B(0,W)}}, \tag{1}$$

with $B_n(x,r)$ the ball with center x and radius r in the n-dimensional Euclidean space $\mathbb{R}^n$. The *probability of success* is defined as $p_s(n) = 1 - p_e(n)$. The limit in (1) exists almost surely from the assumption that the marked point process μ^n with marks (X_k^n, C_k^n) is stationary and ergodic.

Let $\mathbb{P}_n^0$ denote the Palm probability of μ^n. The pointwise ergodic theorem implies that

$$p_e(n) = \mathbb{P}_n^0 \left(X_0^n \notin C_0^n\right) \ . \tag{2}$$

Gaussian Case.

Capacity. Let us restrict attention to the case where the displacement vectors are independent of the points, each displacement vector being Gaussian with i.i.d. coordinates having mean zero and variance σ^2 and with the displacement vectors being i.i.d. from point to point. Further, C_k^n is the Voronoi cell of T_k^n in the point configuration μ^n. Write λ_n as e^{nR_n}. Then we can prove:

Theorem 1 *For any subsequence $n_k \to \infty$ such that $\liminf_{k\to\infty} R_{n_k} > \frac{1}{2}\ln\frac{1}{2\pi e\sigma^2}$, we have $\lim_{k\to\infty} p_e(n_k) = 1$.* □

Further, we can prove:

Theorem 2 *Let μ^n be a Poisson process of intensity $\lambda_n = e^{nR_n}$. For any subsequence $n_k \to \infty$, if $\limsup_{k\to\infty} R_{n_k} < \frac{1}{2}\ln\frac{1}{2\pi e\sigma^2}$, we have $\lim_{k\to\infty} p_e(n_k) = 0$.* □

Together these theorems are the analog of the Shannon–Poltyrev capacity theorem for stationary ergodic point process in the case of Gaussian noise.

Error Exponent. We have the following representation for the probability of success in this Gaussian case:

$$p_s(n) = \int\limits_{r\geq 0}\int\limits_{\vec{v}\in\mathbb{S}^{n-1}} \mathbb{P}_n^0(\mu^n(B(r\vec{v},r))=0)\frac{g_n^\sigma(r)}{A_{n-1}}d\vec{v}dr\ , \tag{3}$$

with $S^{n-1}(r)$ the sphere of radius r in $\mathbb{R}^n$ centered at the origin, A_{n-1} the area of $S^{n-1}(1)$ and

$$g_n^\sigma(r) = 1_{r>0}e^{-\frac{r^2}{2\sigma^2}}\frac{1}{2^{n/2}}\frac{r^{n-1}}{\sigma^n}\frac{2}{\Gamma(n/2)}.$$

Using this formula for Poisson and Mattérn point processes, we were able to recover the Poltyrev error exponent [4].

Additive noise capacity of a stationary point process. Starting with point processes μ^n of rate $\lambda_n = e^{nR_n}$ in $\mathbb{R}^n$, with the displacement vectors independent of the points, each displacement vector having i.i.d. coordinates having some density with differential entropy h, the displacement vectors being i.i.d. from point to point, we can prove the following:

Theorem 3 *For any subsequence $n_k \to \infty$ such that $\liminf_{k\to\infty} R_{n_k} + h > 0$, and* any *choice C_k^n for the points T_k^n of the process (that are subsets of $\mathbb{R}^n$ jointly stationary with the points and the displacements, forming a decomposition of $\mathbb{R}^n$), we have $\lim_{k\to\infty} p_e(n_k) = 1$.* □

Further, we can prove:

Theorem 4 *Let μ^n be a Poisson process of intensity $\lambda_n = e^{nR_n}$. For any subsequence $n_k \to \infty$, if $\limsup_{k\to\infty} R_{n_k} + h < 0$, it is possible to choose C_k^n for the points T_k^n of the process (that are subsets of $\mathbb{R}^n$ jointly stationary with the points and the displacements, forming a decomposition of $\mathbb{R}^n$), such that $\lim_{k\to\infty} p_e(n_k) = 0$.* □

Together these results give a kind of capacity theorem for stationary point process perturbed by additive noise. This result is closely related to the concept of *information theoretic sphere packing* coined by Loeliger [3].

Suggestions for further work. The viewpoint described here suggests an approach to attack the gap between the best currently known upper and lower bounds on the error exponent of the AWGN channel. For instance by considering sequences of stationary point processes μ^n in $\mathbb{R}^n$ with intensity $\lambda_n = e^{nR}$ for fixed

$R = -\frac{1}{2}\ln 2\pi e\alpha^2\sigma^2$, with $\alpha \geq 1$, specifically those having a repulsive structure between the points. This research program appears to offer a novel viewpoint to attack a classical problem in information theory.

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Maximal visibility in a Boolean model

Pierre Calka

(joint work with Julien Michel, Sylvain Porret-Blanc)

Let X be a homogeneous Poisson point process of intensity one. For a fixed distribution μ charging $[R_*, R^*] \subset (0, +\infty)$, we consider a collection of i.i.d. μ-distributed variables R_x, $x \in \mathbf{X}$. We then construct the associated Boolean model [4] whose occupied phase is $\mathcal{O} = \bigcup_{x \in \mathbf{X}} B_d(x, R_x)$, where $B_d(x,r)$ denotes the ball centered at x and of radius $r > 0$. Moreover, the process is conditioned on the event $A = \{O \notin \mathcal{O}\}$.

We aim at studying the maximal visibility $\mathcal{V}$ of an observer placed at the origin O, i.e. the length of the largest segment emanating from the origin and contained in the unoccupied phase. We show that an explicit calculation of the distribution of $\mathcal{V}$ and precise tail estimates can be obtained in dimension two. These results can be extended to some extent in two directions: in dimension two when the disks are replaced by a rotation-invariant random convex body and in dimension three with deterministic balls.

In [5] G. Polya introduced the question of the visibility in a forest such that identical trees (discs with constant radius R) are situated at all but one point of the regular square lattice. He showed that in order to see in a fixed direction at a distance r the radius R should be (asymptotically when r is large) taken as $1/r$ (see also [3]). Very recently, I. Benjamini, J. Jonasson, O. Schramm & J. Tykesson [1] have considered the problem of maximal visibility in the hyperbolic disk. In particular, they have obtained a critical intensity for the almost-sure visibility to infinity. Such a behaviour will not occur in the two and three-dimensional Euclidean spaces.

Let us restrict to the two-dimensional case. Our key result is the connection between the distribution function of the maximal visibility $\mathcal{V}$ and a covering probability of the circle. Indeed, let $P(\nu, n)$ be the probability of covering the circle with perimeter one by the union of n independent identically distributed random arcs with uniformly distributed centres and with lengths distributed according to the probability distribution ν. We have

$$\mathbb{P}\{\mathcal{V} \geq r\} = e^{-\pi(2rR+r^2)} \sum_{n \geq 0} \frac{(\pi(2rR + r^2))^n}{n!}(1 - P(\nu_r, n))$$

where ν_r is the probability measure

$$\nu_r(du) = \frac{\pi r}{rR + \frac{r^2}{2}} \mathbf{1}_{[0, \frac{1}{\pi} \arctan \frac{R}{r}]}(u) \left(r \sin(2\pi u) + \frac{\sin(\pi u)(R^2 + r^2 \cos(2\pi u))}{\sqrt{R^2 - r^2 \sin^2(\pi u)}} \right) du$$
$$+ \frac{\pi R^2}{rR + \frac{r^2}{2}} \mathbf{1}_{[\frac{1}{\pi} \arctan(\frac{R}{r}), \frac{1}{2}]}(u) \frac{\cos(\pi u)}{\sin^3(\pi u)} du.$$

In particular, an explicit formula for $P(\nu_r, n)$ has been provided by A. F. Siegel and L. Holst [6].

In order to get lower and upper bounds for the probability $\mathbb{P}\{\mathcal{V} \geq r\}$, it seems natural to try to replace the covering probability $P(\nu_r, n)$ with a more elementary one. As a matter of fact, it can be shown in the same spirit as in [2] that for any probability distributions ν_1 and ν_2 on $[0, 1]$ satisfying $\nu_1 \leq_{\mathrm{cv}} \nu_2$ (where $\leq_{\mathrm{cv}}$ denotes the convex order), we have $P(\nu_1, n) \leq P(\nu_2, n)$. In particular, denoting by m_r the mean of ν_r, we can compare ν_r with δ_{m_r} and $\big((1 - 2m_r)\delta_0 + 2m_r\delta_{1/2}\big)$. Consequently, for r large enough, there exist two positive constants C_1 and C_2 such that

$$\mathbb{P}\{\mathcal{V} \geq r\} \geq C_1 r \exp(-\pi m_r(2\mathbb{E}\,(R)r + r^2))$$

and

$$\mathbb{P}\{\mathcal{V} \geq r\} \leq C_2 r^2 \exp(-\pi m_r(2\mathbb{E}\,(R)r + r^2)).$$

Moreover, we have

$$\lim_{r \to +\infty} \frac{1}{r} \log \mathbb{P}\{\mathcal{V} \geq r\} = -2\mathbb{E}\,(R)$$

where R is distributed according to μ.

In a more general setting, let $\mathbf{K}$ be a random convex body which is rotation-invariant. When the balls of the Boolean model are replaced by i.i.d. copies of $\mathbf{K}$, it can be shown that the maximal visibility $\mathcal{V}$ satisfies

$$\lim_{r \to +\infty} \frac{1}{r} \log \mathbb{P}\{\mathcal{V} \geq r\} = -\mathbb{E}\,(b(\mathbf{K}))$$

where $b(\mathbf{K})$ denotes the mean width of $\mathbf{K}$.

Other minor results concern the distribution of $\mathcal{V}$ conditionally on the distance from the origin to the first obstacle and asymptotic results in the three-dimensional case with deterministic balls. Some of the open problems related to our work are the following:

- What can be said about the visibility in one particular direction or the maximal visibility when the obstacles of the Boolean model absorb only a fraction of the "light", which will be proportional to the length of the portion of the ray inside the obstacle?

- The set $\mathbf{T}$ of all points which are visible from the origin is finite almost surely and the first moment of its area can be easily calculated. The distributions of the radii of the largest disk centered at the origin included in $\mathbf{T}$ (spherical contact distribution) and of the smallest disk centered at the origin and containing $\mathbf{T}$ (maximal visibility) are known. Can the boundary of $\mathcal{T}$ be described in more details?
- The question of maximal visibility can also be considered in other random media, such as bicolored random tessellations.

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From Palm Theory to Mass-Transport-Principles

Daniel Gentner

(joint work with Günter Last)

By *mass-transport-principles* we mean mass-conservation laws in expectation for certain random transports between certain random measures. To motivate such principles we start with the purely deterministic special case. Take a set S and imagine some mass being distributed over the points of S. By *transport* we simply mean a map $m : S \times S \to [0, \infty]$ with the interpretation that $m(s,t)$ denotes the mass being transported from s to t. This gets interesting if we consider some group G operating on S and require that m fulfills the invariance condition

$$m(gs, gt) = m(s,t), \quad g \in G, s, t \in S.$$

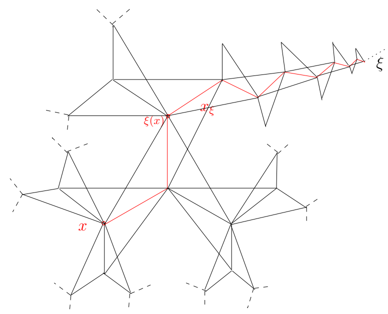
As a concrete example consider $\mathbb{Z}^2$ operating on itself via translation and a map $m : \mathbb{Z}^2 \times \mathbb{Z}^2 \to [0, \infty]$ that fulfills the invariance condition $m(g+s, g+t) = m(s,t)$ for all $g, s, t \in \mathbb{Z}^2$. Then one may easily verify that the total mass transported out of some fixed $b \in \mathbb{Z}^2$ equals the total mass being transported into some possibly different $b' \in \mathbb{Z}^2$, that is we have

$$\sum_{s \in \mathbb{Z}^2} m(b,s) = \sum_{s \in \mathbb{Z}^2} m(s,b').$$

A similar, but different equation arises for non-transitive operations. For instance consider $SO(2)$ operating naturally on $\mathbb{R}^2$. Denoting Haar measure on $SO(2)$ by λ the corresponding equation for any two representatives b and b' of orbits under this operation is $\int m(b, \varphi(b'))\lambda(d\varphi) = \int m(\varphi(b), b')\lambda(d\varphi)$ which may be written as

$$\int m(b, s)\mu_{b'}(ds) = \int m(s, b')\mu_b(ds)$$

using the push-forwards $\mu_s := \lambda \circ \pi_s^{-1}$ of the Haar measure under the projections $\pi_s : G \to S, \pi_s(g) = gs$. The topological structure of the operating group turns out to play a crucial role for such balance equations. Consider for instance the ξ-grandfather graph $\xi(T_3)$ of the regular 3-tree T_3 with respect to some fixed end ξ of T_3 (this example is taken from [5]):



This graph is constructed starting from T_3 by inserting for each vertex x of T_3 an edge connecting x and the second vertex (which we call the ξ-grandfather of x and denote by $\xi(x)$) on the unique ray in ξ starting in x. Denote the vertex set of $\xi(T_3)$ by V.

One easily verifies that any graph automorphism φ of $\xi(T_3)$ leaves ξ fixed and hence that the transport $m(s,t) := \mathbf{1}\{t = \xi(s)\}$ is $\mathbf{Aut}(\xi(T_3))$-invariant. Here $\sum_{s \in V} m(b, s)$ is the number of ξ-grandfathers of b, which is 1, while $\sum_{s \in V} m(s, b)$ is the number of ξ-grandchildren of b, which is 4. It turns out that the modular function of the operating group (here $\mathbf{Aut}(\xi(T_3))$) is the right tool to rescue a mass-conservation principle: We have

$$\sum_{s \in V} m(b, s)\Delta(\varphi_{b,s}^{-1}) = \sum_{s \in V} m(s, b'), \quad b, b' \in V,$$

where for $s, t \in V$ $\varphi_{s,t}$ is an element of $\mathbf{Aut}(\xi(T_3))$ with the property that $\varphi_{s,t}(s) = t$. This version of a deterministic MTP finds applications in percolation theory, see [1]. A more general deterministic situation is to consider two G-invariant measures μ, ν on S and to transfer the modular function from G to $S \times S$ by use of some function $k > 0$ on S satisfying $\mu_s k < \infty$, $s \in S$, (such functions exist if and only if the operation of G on S is *proper* in the sense that the pushforwards μ_s, $s \in S$, are uniformly σ-finite) by setting $\Delta_k(s,t) = \frac{\mu_t k}{\mu_s k}$. Then we have

$$(1) \qquad \iint m(s,t)\Delta_k(s,t)\mathbf{1}_B(s)\nu(ds)\mu(dt) = \iint m(s,t)\mathbf{1}_B(t)\nu(ds)\mu(dt).$$

for any $B \in \mathcal{S}$ satisfying the following symmetry condition with respect to k:

$$\frac{\mu_s B}{\mu_s k} = \frac{\mu_t B}{\mu_t k}, \quad s,t \in S.$$

This condition is crucial for the mass-conversation (1) to hold and means that the set B consists of constant k-weighted proportions from each orbit. Note that k may be chosen identically 1 iff G is compact. This equation contains all previous examples as special cases and points the way to the following generalization from G-invariant deterministic measures to jointly G-stationary random measures ξ and η and random G-invariant transports $m : \Omega \times S \times S \to [0,\infty]$ in expectation (a G-stationary random measure is a random measure ξ whose distribution is a G-invariant measure on the space of all measures on S, i.e. fulfills $\mathbb{P}(\xi \in gA) = \mathbb{P}(\xi \in A)$ for all measurable subsets A of the space of all measures on S and $g \in G$.) :

(2)

$$\mathbb{E} \iint m(\mathrm{id}_\Omega, s,t)\Delta_k(s,t)\mathbf{1}_B(s)\eta(ds)\xi(dt) = \mathbb{E} \iint m(\mathrm{id}_\Omega, s,t)\mathbf{1}_B(t)\eta(ds)\xi(dt),$$

where B satisfies the same symmetry condition as above. Proving this requires an advanced Palm theoretical machinery for G-stationary random measures under arbitrary (possibly non-transitive and non-unimodular) group operations which is developed in [2] and relies heavily on results of Olav Kallenberg in [3]. (Classical Palm theory focusing exclusively on the (transitive) case of groups operating on themselves via left-translation and random measures on such groups does not suffice.)

Here is a brief sketch: For a random measure ξ on some measurable space $(S,\mathcal{S})$ the expected value of integrals with respect to ξ of random functions is of interest, i.e. expressions of the form $\mathbb{E}\int f(\mathrm{id}_\Omega, s)\xi(ds) =: C_\xi f$ where f is an arbitrary measurable function on $\Omega \times S$. The induced measure C_ξ on $\Omega \times S$ is called the Campbell measure of ξ and any invariant desintegration (ν, Q) from S to Ω (i.e. ν is an invariant supporting measure of ξ and Q an invariant kernel from S to Ω such that $C_\xi = \nu \otimes Q$) is called *invariant Palm pair* of ξ. Now consider some group G operating measurably on S. Results of Olav Kallenberg (see [3]) justify the following convenient model of G-stationarity: Replace the underlying probability space by a σ-finite measure space $(\Omega, \mathcal{A}, \mathbb{P})$, let G also operate on Ω, assume $\mathbb{P}$ to be G-invariant and adapt G-stationary random elements τ in some space $(E,\mathcal{E})$ (on which G operates) to the operation on Ω by replacing them by versions satisfying $\tau(g\omega) = g\tau(\omega)$, $g \in G, \omega \in \Omega$. In particular this means a G-stationary random measure can be represented by a G-covariant version satisfying $\xi(g\omega, A) = \xi(\omega, g^{-1}A)$ for $g \in G, \omega \in \Omega$ and $A \in \mathcal{S}$. In this setting the following special case of a strong generalization of Neveus classical exchange formula (see [6] for the classical version) to the case of general G-covariant random measures ξ

and η on a space S (see [2]) leads to a short proof of (2). Namely if (ν_ξ, Q_ξ) and (ν_η, Q_η) are invariant Palm pairs of ξ and η respectively then

$$\int \mathbb{E}_{Q_{\eta,b}} \int m(\theta_e, b, s)\Delta^*(s)\xi(ds)\nu_\eta^*(db) = \int \mathbb{E}_{Q_{\xi,b}} \int m(\theta_e, s, b)\eta(ds)\nu_\xi^*(db),$$

where, fixing any measurable system of representatives O of the orbits in S, $\Delta^*(s) := \Delta(g_s^{-1})$, g_s such that $g_s Rep(s) = s$ and for a G-invariant measure ν on S the measure ν^* denotes the unique measure concentrated on O satisfying $\nu = \int \mu_b(\cdot)\nu^*(db)$.

We hope for manifold applications of (2). A first quickly derived application of it deals with Voronoi tesselations of isotropic point processes in $\mathbb{R}^d$. Here we have for any $SO(d)$-invariant set A and a set B containing constant proportions from each orbit the following result for the expected volume of $V(B)$, the union of all Voronoi cells induced by the isotropic point process with centers in B:

$$\mathbb{E}\,[\lambda^d(A \cap V(B))] = \lambda^d(A \cap B).$$

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Random polytopes: geometric aspects

Daniel Hug

(joint work with Károly J. Böröczky, Ferenc Fodor, Lars Hoffmann)

The theory of random polytopes has been subject to a dramatic development in the last decade. Various classical results concerning the convergence of mean values have been generalized for instance to sharp estimates of higher moments, thus leading to central limit theorems. Moreover, several results which were only available in the plane and under strong smoothness and curvature assumptions are understood much better now. This progress is due to new techniques coming from probability and also to new geometric arguments. Here we mainly describe progress on the geometric side. In particular, geometric duality arguments in combination with integralgeometric transformations are applied for the study of random polyhedral sets.

1. Random polytopes

Let $K \subset \mathbb{R}^d$ be a convex body (compact convex set with nonempty interior). In K we choose random points $x_1, \dots, x_n$ independently and according to the uniform distribution. The convex hull $K_{(n)}$ of these random points is a random polytope in K. As the number n of points increases, at least on the average the random polytope $K_{(n)}$ will provide an increasingly better approximation of K. The degree of approximation can be measured by comparing, for suitable geometric functionals F of convex bodies, $F(K)$ and the expected value $\mathbb{E}\, F(K_{(n)})$ of $F(K_{(n)})$. An appropriate class of functionals is provided by the intrinsic volumes V_i, $i = 0, \dots, d$, which can be defined via the Steiner formula

$$V_d(K + \lambda B^d) = \sum_{i=0}^{d} \lambda^{d-i} \kappa_{d-i} V_i(K), \qquad \lambda \geq 0,$$

where V_d is the volume functional, B^d is the d-dimensional unit ball B^d, and κ_d is its volume. Also note that $W = 2\kappa_{d-1}/(d\kappa_d)V_1$ is the mean width functional and V_{d-1} is proportional to the surface area.

Bounds for the mean volume deficit, due to Bárány and Larman [1], are

$$c \cdot \frac{\ln^{d-1} n}{n} \leq \mathbb{E}\, \left(V_d(K) - V_d(K_{(n)})\right) \leq C \cdot n^{-\frac{2}{d+1}}$$

with positive constants c, C. A general asymptotic result is

$$\lim_{n\to\infty} \left(\frac{n}{V_d(K)}\right)^{\frac{2}{d+1}} \mathbb{E}\, \left(V_d(K) - V_d(K_{(n)})\right) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx), \tag{1}$$

where ∂K is the boundary of K, $\kappa(x)$ is the generalized Gauss-Kronecker curvature (defined for almost all boundary points), $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and c_d is an explicitly known constant. This result is stated in [8] and generalized substantially in [3], where a new approach is developed.

Theorem 1 ([3])**.** *For a convex body K in $\mathbb{R}^d$, a probability density function ϱ on K, and an integrable function $\lambda : K \to \mathbb{R}$ such that, on a neighborhood of ∂K relative to K, λ and ϱ are continuous and ϱ is positive,*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\, \mathbb{E}_{\,\varrho,K} \int_{K \setminus K_{(n)}} \lambda(x)\, dx = c_d \int_{\partial K} \varrho(x)^{\frac{-2}{d+1}} \lambda(x) \kappa(x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(dx) \tag{2}$$

where $K_{(n)}$ is the convex hull of random points chosen according to the density ϱ ($\mathbb{E}_{\,\varrho,K}$ indicates this dependence on ϱ, K).

Observe that (1) is recovered by choosing constant functions λ, ϱ. Let $f_0(P)$ denote the number of vertices of a polytope P. Applying an argument due to Efron, we obtain the following consequence.

Corollary 2 ([3])**.** *For a convex body K in $\mathbb{R}^d$, and for a probability density function ϱ on K which is continuous and positive in a neighborhood of ∂K relative* *to K,*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\,\mathbb{E}_{\varrho,K}(f_0(K_{(n)})) = c_d \int_{\partial K} \varrho(x)^{\frac{d-1}{d+1}}\kappa(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

The case of uniform random points is well known.

For the other intrinsic volumes, results similar to (1) have been obtained by Bárány [2] and Reitzner [7] under additional smoothness and curvature assumptions. In the special case of the mean width, these assumptions are relaxed in [5]. Generalizing the concepts of a Macbeath region and a convolution body and introducing some new approximation arguments, an extension of the result in [5] to all intrinsic volumes is achieved in [4].

Theorem 3 ([4])**.** *Let $K \subset \mathbb{R}^d$ be a convex body in which a ball rolls freely, and let $j \in \{1,\dots,d-1\}$. Then*

$$\lim_{n\to\infty}\left(\frac{n}{V(K)}\right)^{\frac{2}{d+1}} \mathbb{E}\,\left(V_j(K) - V_j(K_{(n)})\right) = c_{d,j}\int_{\partial K} \sigma_{d-1}(x)^{\frac{1}{d+1}}\sigma_{d-j}(x)\,\mathcal{H}^{d-1}(dx)$$

where $c_{d,j}$ is a constant and $\sigma_{d-j}(x)$ is the normalized elementary symmetric function of order $d-j$ of the principal curvatures of K at $x \in \partial K$.

For $j = 1$ (mean width), an extension of Theorem 3 including a suitable weight function can be proved. This turns out to be useful for the study of random polyhedral sets.

2. Random polyhedral sets

Polytopes can be defined as (bounded) intersections of halfspaces. This remark leads to random polyhedral sets which provide an alternative model of a random polytope (if bounded). More specifically, given a convex body K, we consider n random hyperplanes $H_1,\dots,H_n$ which intersect $K_1 := K + B^d$, but not the interior of K. The intersection of the closed halfspaces, bounded by the given hyperplanes and containing K, is a random polyhedral set $K^{(n)}$. The distribution of the random hyperplanes is defined by restricting the motion invariant Haar measure on the space of hyperplanes to the hyperplanes hitting K_1, but not the interior of K; see [6] for recent related work.

The following asymptotic result for the mean width is established in [3]. The proof is based on duality arguments and on an application of Theorem 1.

Theorem 4 ([3])**.** *Let K be a convex body in $\mathbb{R}^d$ and $\omega_d := d\kappa_d$. Then*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}}\,\mathbb{E}\,(W(K^{(n)}\cap K_1) - W(K)) = 2\,c_d\,\omega_d{}^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

The intersection with K_1 can be avoided if on restricts the expectation to bounded realizations of $K^{(n)}$ (as $n \to \infty$, unbounded realizations become increasingly unlikely).

An analogue of Corollary 2 for the number of facets $f_{d-1}(P)$ of a polyhedral set P is available.

Theorem 5 ([3]). *If K is a convex body in $\mathbb{R}^d$, then*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\,\mathbb{E}\,(f_{d-1}(K^{(n)})) = c_d\,\omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

The preceding two theorems have been proved more generally for random polyhedral sets which are derived from not necessarily uniform hyperplane distributions.

3. Polarity

For a convex body $K \subset \mathbb{R}^d$ with $o \in \operatorname{int}(K)$, let K^* denote the polar body of K. With a suitable choice of a density ϱ (concentrated on K^*), the random polyhedral sets $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distribution, where $K^*_{(n)}$ is defined in terms of ϱ. This connection can be exploited to show that the limit in Theorem 4 is an integral over ∂K^* involving the Gaussian curvature κ^* of K^*. Then one has to transform this curvature integral into one over ∂K, involving the Gaussian curvature κ of K. This is accomplished by the following result. A major difficulty is the lack of any smoothness assumptions on K.

Proposition 6 ([3]). *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int}(K)$. Let $f : [0,\infty)\times S^{d-1} \to [0,\infty)$ be measurable and $\tilde{f}(x) := f\left(\|x\|^{-1}, \|x\|^{-1}x\right)$ for $x \in \partial K^*$. Then*

$$\int_{\partial K^*} \tilde{f}(x)\frac{\kappa^*(x)^{\frac{1}{d+1}}}{\|x\|^{-1}}\,\mathcal{H}^{d-1}(dx) = \int_{\partial K} f(h(K,\sigma_K(x)),\sigma_K(x))\kappa(x)^{\frac{d}{d+1}}\,\mathcal{H}^{d-1}(dx).$$

An analogue of Theorem 4 for the volume functional can be established by duality methods. Here the assumption that K is a summand of a ball seems to be needed.

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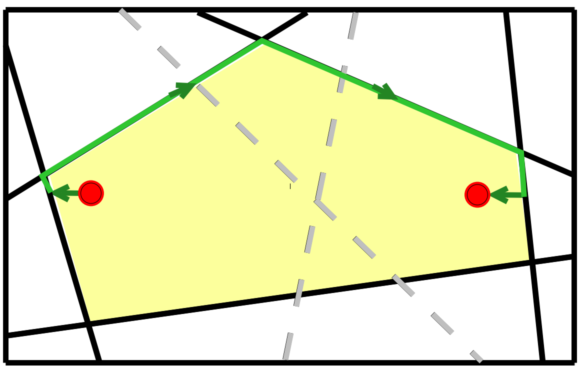
Networks and Poisson line patterns: fluctuation asymptotics

Wilfrid Kendall
(joint work with David Aldous)

In [1] it was shown how to construct networks connecting arbitrary configurations $\mathbf{x}^n$ of n cities in a square of area n which for example (i) involve only εn more total network length than the Euclidean Steiner tree connecting all n cities, and yet (ii) establish a network connection length between two randomly chosen cities which is on average only $O(\log n)$ more than average Euclidean connection length [1, Theorem 3, version (b)]. Moreover, under a certain quantitative equidistribution condition on the city locations $\mathbf{x}^n$ (which can be phrased either analytically or in terms of a truncated Wasserstein coupling between a randomly chosen city and the uniform distribution on the square), a complementary result shows that for $O(n)$ total connection length the average network connection length must have an excess over the average Euclidean connection length of at least $\Omega(\sqrt{\log n})$ [1, Theorem 5]. The methods of proof involve stochastic geometry: in the case of the lower bound [1, Theorem 5] the idea is to associate the uniform choice of two cities with approximately uniform random lines, and then to use simple ideas from stereology. In the case of the upper bound [1, Theorem 3] one augments the Euclidean Steiner tree by a sparse Poisson line process, to obtain good long-distance communication, and adds an additional relatively infinitesimal amount of additional connectivity, to ensure efficient passage from Steiner tree to line process network.

The key calculation for [1, Theorem 3] concerns construction of an augmented Poisson line network connecting two planar points at $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$. Let $\mathcal{C}_n$ be the cell containing the two planar points formed by the tessellation Π_n^* of all lines from a unit intensity Poisson line process Π which do not separate $(-\frac{n}{2}, 0)$ from $(\frac{n}{2}, 0)$. We can consider connecting $(-\frac{n}{2}, 0)$ to $(\frac{n}{2}, 0)$ as in the figure, by first proceeding from $(-\frac{n}{2}, 0)$ towards $(-\infty, 0)$ until the perimeter $\partial\mathcal{C}_n$ is met, then proceeding around $\partial\mathcal{C}_n$ either clockwise or anti-clockwise (according to taste), until encountering the ray from $(n, 0)$ to $(\infty, 0)$, then finally proceeding back to $(\frac{n}{2}, 0)$ along this ray.



Line process properties and Palm distribution theory are used in [1] to express the mean length of the perimeter $\partial\mathcal{C}_n$ as a double integral: analysis shows that this is asymptotic to $2n + \frac{8}{3}(\log n + \gamma + \frac{5}{3}) + o(1)$ (this agrees with similar higher-dimensional results in [2, Theorem 1.3], also compare [6, Satz 5]). The result provides an asymptotic upper bound for the mean network length between $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$ in the network formed by augmenting Π by two Exponential(1) random segments required to connect $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$ to Π as above.

This augmented Poisson line network has an intrinsic interest, and various natural questions arise. For example, how far will the clockwise path following $\partial\mathcal{C}_n$ deviate laterally from the Euclidean path running directly from $(-\frac{n}{2}, 0)$ to $(\frac{n}{2}, 0)$? and where will the maximum lateral deviation occur? How much random variation in the length of the path should one expect to see? To what extent will the true network geodesic deviate from one of the two paths running clockwise or anti-clockwise around the cell?

The question of vertical deviation is best addressed by using the methods used to evaluate the mean length of $\partial\mathcal{C}_n$. Almost surely a point (x, y) of $\partial\mathcal{C}_n$ of maximal y-coordinate must be an intersection of two lines of Π^*_n, for which one line has positive and the other has negative slope, and for which no further lines of Π^*_n separate (x, y) from $(-\frac{n}{2}, 0)$ and $(\frac{n}{2}, 0)$. Almost surely there is exactly one such point, and Palm distribution arguments then show that its probability density is as follows, for $-\infty < x < \infty$ and $y > 0$:

$$(1)\quad \frac{1}{4}\left(\sin\alpha + \sin\beta - \sin(\alpha+\beta)\right) \times \\ \times \exp\left(-\tfrac{1}{2}\left(\sqrt{(x-\tfrac{n}{2})^2 + y^2} + \sqrt{(x+\tfrac{n}{2})^2 + y^2} - n\right)\right) \mathrm{d}x\mathrm{d}y\,.$$

Here α $\beta \in (0, \pi)$ are the interior angles at $(-\frac{n}{2}, 0)$, and $(\frac{n}{2}, 0)$ of the triangle formed by (x, y), $(-\frac{n}{2}, 0)$, and $(\frac{n}{2}, 0)$. Using new coordinates $u = \frac{2}{n}x$ and $v = y/\sqrt{n}$, it follows that the limiting density for large n in (u, v) coordinates is given by

$$(2)\qquad \frac{v^3}{(1-u^2)^2}\exp\left(-\frac{v^2}{1-u^2}\right)\mathrm{d}u\mathrm{d}v\,.$$

Hence asymptotically the point of maximal y-coordinate has x-coordinate distributed uniformly over $(-\frac{n}{2}, \frac{n}{2})$, and has y-coordinate which has conditional distribution the length of a Gaussian 4-vector of zero mean and variance parameter $\frac{n}{2}(1 - \frac{4x^2}{n^2})$.

Questions of random variation can be addressed by reformulating the problem in terms of a simple growth process. Slightly abusing notation, let Π^*_∞ be the tessellation obtained from the Poisson line process by deleting all lines intersecting the positive x-axis. Construct the cell $\mathcal{C}_\infty$ containing the origin formed by Π^*_∞, and consider the path from the origin to $(\infty, 0)$ formed as above, and proceeding clockwise round the cell. The y-coordinate of this path grows to infinity, and we can understand the asymptotic random variation of the length of $\partial\mathcal{C}_n$ by investigating the stochastic dynamics of this growth.

Let the path be parametrized by τ, the excess of arc-length S over x-coordinate X. Let Θ be the angle that the path makes with the positive x-axis, so that $\Theta_0 = \pi$ and Θ decreases with increasing τ. Poisson line process computations show that in τ-time the angle Θ changes at instants of a Poisson point process of intensity $\frac{1}{2}$; moreover the jump in angle $\Theta - \Theta_-$ is such that

$$\mathbb{P}\left[\Theta_- - \Theta \leq \phi \mid \Theta_-\right] \quad = \quad \frac{1-\cos\phi}{1-\cos\Theta_-} \qquad \text{for } 0 \leq \phi \leq \Theta_- \,. \tag{3}$$

Using Rebolledo's martingale central limit theorem [5] for a compensated version of $-\log\Theta$ as a function of excess τ, and applying $\mathrm{d}\tau = (\sec\Theta - 1)\mathrm{d}X$, we may obtain asymptotic expressions using Brownian motion B. Omitting analytical details which actually require careful attention, the argument runs as follows,

$$X_\tau \quad \approx \quad 2\int_0^\tau \exp\left(\tfrac{3}{2}u - \sqrt{7}B_u\right)\mathrm{d}u \,. \tag{4}$$

This leads to the existence of a field of Brownian motions $\{\widetilde{B}^\tau_u : u \geq 0\}$ (related to B by time-reversal) parametrized by τ and such that the excess τ at x-coordinate X satisfies

$$\tau \;\approx\; \tfrac{2}{3}\left(\log X_\tau + \sqrt{7}B_\tau - \log 2 - \log\int_0^\infty \exp\left(-\tfrac{3}{2}u + \sqrt{7}\widetilde{B}^\tau_u\right)\mathrm{d}u\right)\,. \tag{5}$$

Hence it follows that the mean excess of τ at distance x is asymptotically $\frac{2}{3}\log x$ (in fact suggesting yet another approach to the asymptotics of [1]) and the excess has random variation of order $O(\sqrt{\log x})$ when $X = x$, agreeing with simulations.

Note that the integral $\int_0^\infty \exp\left(-\frac{3}{2}u + \sqrt{7}\widetilde{B}^\tau_u\right)\mathrm{d}u$ is of Dufresne type, hence is proportional to the reciprocal of a Gamma random variable [4, 7]; see also [3]. However approximation errors will be of the same order as the contribution of this term.

A fully rigorous argument, with applications to the behaviour of true geodesics in the augmented network (the third of the questions mentioned above), will be published elsewhere.

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Bernoulli and Cox transports of stationary random measures

Günter Last
(joint work with Hermann Thorisson)

Let ξ be a random measure on a locally compact Abelian group G and X a random element which G acts on, for instance a random field indexed by G. The pair (X, ξ) is *mass-stationary* if the origin is a typical location for (X, ξ) in the mass of ξ. This concept was introduced in [2] as an extension to random measures of *point-stationarity*, which in turn was introduced in [3] for simple point processes in $\mathbb{R}^d$ having a point at the origin. Point-stationarity formalizes the intuitive idea that the point at the origin is a typical point of the point process In [3] and [1] it was shown that point-stationarity is an intrinsic characterization of Palm versions of stationary point processes, and the same is proved in [2] for mass-stationarity and random measures. In this talk we discuss further characterizations of mass-stationarity based on Bernoulli and Cox randomizations.

An *allocation* τ is a map taking each location $s \in G$ to another location $\tau(s) \in G$ depending on $\xi(\cdot - s)$, and τ is *preserving* if the image of ξ under τ is ξ itself. An allocation τ is a *matching* if τ is its own inverse. The term 'Bernoulli transport' refers to a randomized allocation that allows staying at a location s with a probability $p(s)$ depending on $\xi(\cdot - s)$, and otherwise chooses another location according to a (non-randomized) allocation. This makes it possible to preserve discrete point-masses even if there are point-masses of different sizes. Our first result shows that mass-stationarity of discrete random measures can be reduced to distributional invariance under shifts of the origin by preserving Bernoulli transports.

A Cox process ζ is a Poisson process with a random intensity measure ξ. Such a process can be thought of as a collection of points scattered independently over the space G according to the mass distribution of ξ, so these points are at typical locations in the mass of ξ. Thus if ξ is mass-stationary and we add a point at the origin to the Cox process to obtain $\zeta^0 := \zeta + \delta_0$, then also the points of ζ^0 are at typical locations in the mass of ξ. In fact, one might expect that the new point at the origin is a typical point of ζ^0, in other words that ζ^0 is point-stationary, and even that the pair (ξ, ζ^0) is point-stationary. Actually, one might expect that the pair (ξ, ζ^0) is point-stationary *if and only if* ξ is mass-stationary. We show that this is indeed the case. The term 'Cox transport' refers to applying allocations for point processes to general random measures through the Cox process. In particular, mass-stationarity of ξ then reduces to point-stationarity with respect to ζ^0. Also, it follows that mass-stationarity is characterized by applying preserving Bernoulli transports to the Cox process. In the special case of a diffuse random measures mass-stationarity is characterized by applying matchings to the Cox process.

The application of an allocation to the Cox process ζ^0 can also be interpreted as a *transport kernel* redistributing the mass of ξ in an invariant way. We show that the distribution of a mass-stationary ξ is invariant under such kernels. Whether or not this property is even characteristic for mass-stationarity is an interesting open problem.

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Probability models with geometric flavour

Jesper Møller

There is a lack of models in stochastic geometry with interaction and tractable likelihoods. We discuss some exceptions and open problems.

First, a germ-grain model with interaction is presented. The model is viewed as the result of a spatio-temporal process of growing germs, with constant speed along all rays, and where the germs are started to grow at different times after a certain dependent thinning of a primary Poisson point process has happened. The likelihood has a closed form expression and maximum likelihood estimates may easily be derived.

Second, a flexible model for points along linear structures is presented, considering again a spatio-temporal construction of independent primary Poisson point processes, which results in secondary point processes for "independent" and "dependent" cluster points. Here a simulation-based Bayesian analysis is feasible.

A class of random measures and their fractal geometry

Peter Mörters

We show that a wide class of random measures have very similar fractal geometry and argue that this can be traced back to their similar local hitting, scaling and conditioning behaviour. This is a contribution to, and significant extension of, an exciting research programme initiated by Kallenberg in [4].

Typical random measures Ξ belonging to our class are

- occupation measures of stable subordinators with stability index $0 < \alpha < 1$
- states of a Dawson-Watanabe superprocesses in $\mathbb{R}^d$, $d \geq 2$,
- intersection local times of two Brownian paths in $\mathbb{R}^d$, $d = 2, 3$.

Very roughly, with some modification in the critical cases $d = 2$, the following basic common properties of these examples can be identified:

- The *local hitting* properties are related to the *local intensity*, i.e. for some *scaling index* $\alpha > 0$ we have,

$$\varepsilon^{\alpha}\, \mathbb{P}\{\Xi B_\varepsilon(x) > 0\} \sim \mathbb{E}\Xi(B_\varepsilon(x)).$$

- Given that Ξ charges a small ball B, its neighbourhood looks like a translation of the *Palm distribution* $\mathbb{P}^0$ associated with a stationary version of the process.
- Given that Ξ charges two balls with distance of larger order than their size, the behaviour of Ξ inside these balls is (up to constant factors) *conditionally independent*.
- *Local self-similarity* holds with scaling index α,

$$\Xi(r\,\cdot) \approx r^{\alpha}\Xi(\,\cdot\,) \qquad \text{under } \mathbb{P}^0.$$

- There is a finite *annular lacunarity index* ξ such that

$$\mathbb{P}^0\{\Xi(B_1 \setminus B_r) = 0\} \approx r^{\xi} \qquad \text{as } r \downarrow 0.$$

The indices associated with our examples are the stability index α and $\xi = 2\alpha$ in the case of stable subordinators; $\alpha = 2$, $\xi = 4$ for the superprocess example; and in the intersection example $\alpha = 2$, $\xi = \frac{35}{12}$ if $d = 2$, $\alpha = 1$, $1 < \xi < 2$ unknown if $d = 3$. The lacunarity index in the planar case of the intersection example goes back to the seminal work of Lawler, Schramm and Werner.

Coming to the fractal geometry, in all our examples, the measure Ξ can be approximated by the Lebesgue measure on ε-neighbourhoods of the support. More precisely, let

$$S(\varepsilon) = \big\{x \in \mathbb{R}^d \colon \Xi(B_\varepsilon(x)) > 0\big\}.$$

Then, at least in probability, as $\varepsilon \downarrow 0$,

$$\phi(\varepsilon)\,\mathrm{Leb}\big(\,\cdot\, \cap S(\varepsilon)\big) \longrightarrow \Xi$$

for a suitable function of the form $\phi(\varepsilon) = \varepsilon^{\alpha-d}\,L(\varepsilon)$, where L is a slowly varying correction required in the critical cases. See [6] for the subordinators, [8] for intersections, and [5] for the superprocess case. In the subordinator case the result was probably known to the pioneers of local time, like Paul Lévy, as early as the 1940s.

All our examples have an interesting *multifractal spectrum* that does not conform to the classical multifractal spectrum of statistical physics. While

$$\liminf_{r\downarrow 0} \frac{\log \Xi(B_r(x))}{\log r} = \alpha \quad \text{for all } x \in S,$$

we have variations of the limsup behaviour. For every $\alpha \le a \le \frac{\xi\alpha}{\xi-\alpha}$,

$$\dim\Big\{x \in S\colon \limsup_{r\downarrow 0} \frac{\log \Xi(B_r(x))}{\log r} = a\Big\} = \alpha - \xi + \frac{\xi\alpha}{a}\,.$$

This is shown in [3] for subordinators, [13] for superprocesses and [7] for intersections. Note that the latter paper includes intersections of Brownian paths in the critical dimension $d = 2$, but the critical case for superprocesses is still open.

An *average density*, as introduced by Bedford and Fisher [1], can be defined in the non-critical cases as

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\Xi(B_r(x))}{r^\alpha}\, \frac{dr}{r} = D_2 \quad \text{for } \Xi\text{-almost every } x.$$

In the critical cases this order-two average diverges, but an order-three average

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \frac{\Xi(B_r(x))}{r^\alpha L(r)}\, \frac{dr}{r\log(1/r)} = D_3 \quad \text{exists for } \Xi\text{-almost every } x.$$

See [2] for subordinators, [10] for intersections and [12] for superprocesses.

Finally, and only in the non-critical cases, we have an integral test for the *packing measures* of the support S,

$$\mathcal{P}^\psi(S) = \begin{cases} 0 \\ \infty \end{cases} \qquad \text{iff} \qquad \int_{0+} r^{-1-\xi}\psi(r)^{\frac{\xi}{\alpha}}\, dr \begin{cases} < \infty, \\ = \infty. \end{cases}$$

See [14] for subordinators, [9] for superprocesses, and [11] for intersections.

At this moment, proofs rely on specific features of the examples, in particular on the Markov property. It is an interesting challenge for the future to provide proofs that follow directly from the hitting, scaling and conditioning properties of the random measures, and to add further examples of different flavour.

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Multivariate probability and geometry

Ilya Molchanov

The talk highlights some relationships between distributions of random vectors on one side and convex and star-shaped sets on the other one. In particular, it aims to relate the cumulative distribution function and the characteristic function of a random vector ξ to some functions that are common in convex geometry, namely the *support function* defined as

$$h_K(u) = \sup\{\langle u, x\rangle :\ x \in K\}\,, \quad u \in \mathbb{R}^d\,,$$

for a bounded set K and the *Minkowski functional*

$$\|u\|_F = \inf\{s \geq 0 :\ u \in sF\}$$

defined for a star-shaped set F. If F is also convex, then $\|\cdot\|_F$ defines a *norm* on $\mathbb{R}^d$. In this case, $\|u\|_F = h_K(u)$, where F and K are *polar* to each other.

It will be seen that possible candidates for sets K and F stem from expectations of random sets. Recall that the expectation of a random convex compact set X is defined by

$$\mathbf{E} h_X(u) = h_{\mathbf{E}X}(u) \quad \text{for all} \quad u \in \mathbb{R}^d\,,$$

see [1]. Its L_p-variant $\mathbf{E}_p X$ comes from $[\mathbf{E}(h_X(u))^p]^{1/p} = h_{\mathbf{E}_p X}(u)$ with $p \in [1, \infty)$.

A *zonoid* Z is an expectation of a random segment, correspondingly an L_p-zonoid is the L_p-expectation. Up to a translation, the random segment can be chosen to be centred or have one end-point at the origin. For instance, if $X = [0, \xi]$, then

$$h_Z(u) = \mathbf{E}\max(0, \langle u, \xi\rangle) = \mathbf{E}(\langle u, \xi\rangle)_+.$$

It is known that this zonoid does not determine uniquely the distribution of ξ, see [5]. The distribution of $\xi \in \mathbb{R}^d$ is determined uniquely by the *lift zonoid* defined in the space $\mathbb{R}^{d+1}$ with the support function

$$h_{\hat{Z}_\xi}(u_0, u) = \mathbf{E}(u_0 + \langle u, \xi\rangle)_+\,, \quad u_0 \in \mathbb{R}\,,\ u \in \mathbb{R}^d\,.$$

This support function admits an obvious financial interpretation as the basket option price, where the extra added coordinate represents the bond. For $d = 1$ one obtains the classical call and put options.

Symmetry properties of financial options form an important financial topic. In particular, the central symmetry property of lift zonoids translates into the call-put parity. The planar symmetries of lift zonoids is equivalent to the put-call *symmetry*, i.e. the symmetry of the function

$$f(u_0, u_1, \ldots, u_d) = \mathbf{E}(u_0 + u_1\xi_1 + \cdots + u_d\xi_d)_+$$

with respect to permutations of its arguments, see [4]. This property is stronger than the exchangeability of the coordinates of ξ. For instance, in the log-normal case it holds if and only if $\log \xi$ has the covariance matrix

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{pmatrix}$$

and the expectation $\mu = \sigma^2(-\frac{1}{2}, \dots, -\frac{1}{2})$.

Another relationship between multivariate probability distributions and stable laws comes from the representation of *strictly stable* random vectors, namely those which satisfy

$$a^{1/\alpha}\xi_1 \oplus b^{1/\alpha}\xi_2 \stackrel{d}{=} (a+b)^{1/\alpha}\xi$$

for all $a, b > 0$, where ξ_1 and ξ_2 are iid copies of ξ. The operation $\oplus$ may be the arithmetic sum or coordinatewise maximum or any other semigroup operation on $\mathbb{R}^d$. If the exponent α equals 1, $\oplus$ is the arithmetic addition and ξ is symmetric (i.e. $S1S$ law), then

$$\phi(u) = \mathbf{E}e^{i\langle u, \xi\rangle} = e^{-h_Z(u)}\,,$$

where Z is a zonoid. If the main operation is the coordinatewise maximum on $\mathbb{R}^d_+$, then

$$\mathbf{P}\{\xi \le u\} = e^{-h_M(u^{-\alpha})}\,, \quad u \in \mathbb{R}^d_+\,,$$

where the power of u is understood coordinatewisely and M is a *max-zonoid*, i.e. the expectation of a random crosspolytope, see [3]. For general $\alpha \in (0, 2]$, symmetric ξ and the arithmetic addition ($S\alpha S$ law),

$$\phi(u) = e^{-\|u\|_F^\alpha}\,, \tag{1}$$

where F is an L_α-ball, meaning that $(\mathbb{R}^d, \|\cdot\|_F)$ is isometrically embeddable in $L_\alpha([0,1])$. If $\alpha \in [1, 2]$, then F is the polar set to L_α-zonoid Z, see [2]. If ξ is a non-negative strictly stable vector with $\alpha \in (0, 1)$, then

$$\mathbf{E}e^{-\sum u_i\xi_i} = e^{-h_Z(u^\alpha)}\,,$$

where Z is the expectation of randomly rescaled ℓ_q-ball with $q = 1/(1-\alpha)$, e.g. expectation of an ellipsoid if $\alpha = \frac{1}{2}$, see [2].

Representations of $S\alpha S$-laws using star-shaped sets can be useful, for instance, to identify *sub-Gaussian* laws as those corresponding to *ellipsoids* F. Among others, this leads to a conclusion that a probability distribution on $\mathbb{R}^d$ is $\mathbb{S}\alpha S$ if and only if it can be approximated by sums of independent sub-Gaussian laws.

Furthermore, it is possible to use the representation (1) to calculate various characteristics of multivariate $\mathbb{S}\alpha S$ laws. For instance, the value of the density of ξ at the origin is given by

$$f(0) = \frac{1}{(2\pi)^d}\,\Gamma(1 + \frac{d}{\alpha})\mathrm{Vol}_d(F)\,,$$

and the moments of the norm of ξ are

$$\mathbf{E}\|\xi\|^\lambda = \frac{2^{\lambda-1}}{\pi^{d/2}}\Gamma(\frac{d+\lambda}{2})\frac{\Gamma(1-\frac{\lambda}{\alpha})}{\Gamma(1-\frac{\lambda}{2})}\int_{\mathbb{S}^{d-1}}\|u\|_F^\lambda du$$

for $\lambda \in (-d, \alpha)$. Furthermore, integrals of the density f over linear subspaces are related to the volumes of intersection of F with the corresponding orthogonal subspaces, that establishes links to the Busemann problem from convex geometry.

Among open problems one can mention geometric representation of not necessarily symmetric and not totally skewed strictly stable laws. The geometric representation of general infinitely divisible or self-decomposable distributions is also open.

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Shapes, Projections, Proteins

Victor M. Panaretos

What can be said about an unknown density function on $\mathbb{R}^3$ given a finite collection of two-dimensional marginals at random and unknown orientations?

This question arises in single particle electron microscopy, a powerful method that biophysicists employ to learn about the structure of biological macromolecules. The method images unconstrained particles -as opposed to particles fixed on a lattice (crystallography)- and yields random profiles of the particle potential density from unknown directions (see Glaeser [1]).

Let $\rho : \Delta \to [0, \infty)$ be a centred and bounded probability density function defined on a compact domain $\Delta \subset \mathbb{R}^3$. We define a *random projection* of ρ, $\Pi\{\rho\}$, as

$$(\Pi\{\rho\}(A))(x, y) := \int_{-\infty}^{+\infty} A\rho(x, y, z)dz,$$

where $A\rho(\mathbf{x}) := \rho(A^T\mathbf{x})$ and A is a random element of $SO(3)$, distributed according to Haar measure. Then, given a realisation of n independent copies of the random field $\Pi\{\rho\}$ – interpreted as a *stochastic Radon transform*– we ask what can be inferred about the density ρ. Since the random rotations generating the projections are not observable, this problem is qualitatively different from the classical problem of tomography, where inversion of a Radon transform crucially depends on the observation of the projection orientations (e.g. Helgason [2]).

The right invariance of Haar measure implies that ρ cannot be recovered, except perhaps up to a rotation/reflection. However, it is seen that the *shape* of the density $[\rho] := \{A\rho : A \in O(3)\}$ is identifiable, i.e. uniquely determined by the law of the random field $\Pi\{\rho\}$: we can potentially statistically invert this *stochastic Radon transform* without observing the corresponding projection angles. The random shape of the field $\Pi\{\rho\}$ is seen to be a sufficient statistic for the infinite dimensional parameter $[\rho]$, suggesting that statistical inference should be modular both in terms of the parameter as well as in terms of the sample space.

An explicit statistical inversion is then carried out in the case when ρ can be finitely expanded in a radial basis, exploiting the interface with Kendall's Euclidean shape theory (Kendall et al. [3]; Kendall & Le [4]), via the framework introduced in Panaretos [5, 6]. The radial basis expansion allows for a finite-dimensional representation of the shapes $[\rho]$ and $[\Pi\{\rho\}]$ through a global coordinate system induced by the Gram matrix of the centres of the basis functions, and the expansion coefficients. A direct connection between the mean projected shape and the original shape is established using these Euclidean coordinates, and a consistent Method of Moments / Maximum Likelihood hybrid estimator is constructed and studied. The ill-posedness of the inversion problem manifests itself through a delicate deconvolution step required in the inversion procedure. More details can be found in Panaretos [7].

Radial basis densities are especially appropriate as coarse approximations to biological particles. Such "low-resolution" approximations can be used as initial estimates of the potential density corresponding to the unknown particle, in order to initialise an iterative procedure which at each step estimates the projection angles given the current model , and then updates the model via traditional tomographic techniques. Obtaining these coarse approximations is a challenging problem in practice. The results described can potentially be of practical use in the construction of objective data-dependent initial models for particle reconstruction.

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Normal approximation in geometrical and combinatorial probability

Mathew D. Penrose
(joint work with Larry Goldstein)

Let $B_1, \dots, B_n$ be interpenetrating unit balls, independently uniformly randomly scattered in a cube C_n of volume n in d-space (with periodic boundary conditions). Define the variables

$$V_n := \text{Volume}\left(\cup_{i=1}^n B_i\right),$$
$$S_n := \sum_{i=1}^n \mathbf{1}\left\{B_i \cap (\cup_{j \neq i} B_j) = \emptyset\right\}.$$

Thus V_n is the total volume covered by the balls, and S_n is the number of isolated balls. Alternatively, S_n may be thought of as the number of singletons in a random geometric graph on n uniform vertices in C_n with distance parameter 2. Such coverage processes and random geometric graphs are fundamental objects of study in stochastic geometry; see [3, 5, 8].

Let θ denote the volume of the unit ball in $\mathbb{R}^d$. It is easy to see that as $n \to \infty$,

$$\mathbb{E}\, V_n \sim n(1 - e^{-\theta}); \qquad \mathbb{E}\, S_n \sim n e^{-2^d \theta},$$

and it is also straightforward to show that there are constants c_1, c_2 such that

$$\text{Var}(V_n) \sim c_1 n; \qquad \text{Var}(S_n) \sim c_2 n,$$

and to give formulae for c_1 and c_2. It is not so clear from the formulae for c_1 and c_2 that their values are non-zero for all d and all choices of radius (our choice of unit radius was arbitrary).

Let $Z \sim N(0,1)$ be a standard normal random variable, and let $\Longrightarrow$ denote convergence in distribution. Let $\text{SD}(\cdot)$ denote standard deviation. In the *thermodynamic limit* of $n \to \infty$ (with the radii of the balls fixed at 1), we have the following central limit theorems:

$$(V_n - \mathbb{E}\, V_n)/\text{SD}(V_n) \Longrightarrow Z; \tag{1}$$
$$(S_n - \mathbb{E}\, S_n)/\text{SD}(S_n) \Longrightarrow Z. \tag{2}$$

The first result (1) is due to Moran [4], and both results, and also the proof that c_1 and c_2 are strictly positive (an issue apparently not addressed in [4]) can be obtained via general results of Penrose and Yukich [7].

It is natural to ask about the rate of normal approximation in (1) and (2). Let d_K denote the Kolmogorov distance between probability distributions, i.e. $d_K(F, G) = \sup_{t \in \mathbb{R}} |F(t) - G(t)|$. In the work described in this talk, we provide explicit Berry-Esséen type error bounds which show that (writing $d_K(X, Y)$ for $d_K(F_X, F_Y)$) as $n \to \infty$ we have

$$d_K\left((V_n - \mathbb{E}\, V_n)/\text{SD}(V_n), Z\right) = O(n^{-1/2}); \tag{3}$$
$$d_K\left((S_n - \mathbb{E}\, S_n)/\text{SD}(S_n), Z\right) = O(n^{-1/2}). \tag{4}$$

Since S_n is integer valued, it is not hard to show that there is a lower bound on the Kolmogorov distance to the normal with the same rate of decay in the case of S_n, i.e., that we can change the right hand side of (4) to $\Theta(n^{-1/2})$. The same ought to be true in the case of (3) but we do not have a proof of this.

Chatterjee [1] obtains similar bounds to (3) and (4) for the Kantorovich-Wasserstein (rather than the Kolmogorov) distance between probability distributions (but also states that new ideas are needed for Kolmogorov distance bounds). In the Poissonized case with a Poisson point process of unit intensity on C_n, rather than exactly n points as considered here, the Kolmogorov distance bounds corresponding to (3) and (4) were already known (see [6] and references therein). However, it is not clear that the proof of error bounds in the Poissonized setting, using spatial independence properties of the Poisson process, is of any use in deriving (3) and (4).

Our proof uses the idea of *size biasing*. For a nonnegative random variable Y with distribution F and finite mean μ, the size biased distribution of Y is defined to be the distribution $\tilde{F}$ with $d\tilde{F}(x) = xdF(x)/\mu, x \geq 0$. We prove (3) and (4) using a result of Goldstein [2] which says, loosely speaking, that if one can closely couple a random variable (in this case V_n or S_n) to another variable with the size-biased distribution of the original variable, then one may be able to obtain a good bound on its Kolmogorov distance from the normal.

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Lectures on fair allocations

Yuval Peres

(joint work with several coauthors)

In the first lecture we deal with the "extra head" problem. Our starting point is the beautiful "extra head" result of Thorisson that, in particular, shows one can shift a Poisson process to obtain its Palm version. Determining explicit "extra head" rules is equivalent to finding fair allocations, i.e. shift-invariant partitions of space into cells of equal volume, matched to the points of the process. This has been called a stable marriage of Poisson and Lebesgue. The Gale-Shapley stable marriage algorithm suggests how to do this. The lecture is based on joint work [4] with A. Holroyd, motivated also by earlier work of Liggett.

In the second lecture we discuss estimates for the stable allocation, and Krikun's connected allocation in the plane. Maxim Krikun found a beautiful allocation of Lebesgue measure to a Poisson process by combining the minimal spanning forest, stable allocation ideas and the Riemann mapping theorem. But does Krikun's method yield bounded cells? The tail estimates we discuss are based on works [2] with Hoffman, Holroyd, Pemantle and the late Oded Schramm.

The topic of the third lecture are gravitational allocation to Poisson points. Motivated by an idea of Sodin and Tsirelson, we partition space into domains of attraction for Newtonian gravity. The expression for the total gravity force converges in dimensions 3 and higher, as discovered by Chandrasekar (1942). The equal volume property of the partition follows from properties of Newtonian potentials and a surprising identity obtained by changing the order of summation. Ideas from dependent percolation allow a bound on the diameters of the cells. The lecture is based on joint work [3] with S. Chatterjee, R. Peled and D. Romik.

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Lévy bases and Cox processes

Michaela Prokešová

(joint work with Eva B. Vedel Jensen, Gunnar Hellmund)

In the talk we discuss the notion of a Lévy basis and use this object to define two new classes of spatial Cox point processes. In the paper [7] was introduced and studied the notion of independently scattered and infinitely divisible random measure, i.e. a collection of real-valued random variables $L = \{L(A), A \in \mathcal{A}\}$, where $\mathcal{A}$ is the δ-ring of bounded Borel subsets of $\mathcal{R} \subseteq \mathbb{R}^d$, which have infinitely divisible distributions and $L(\cup_n A_n) = \sum_n L(A_n)$ a.s. for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{A}$, provided $\cup_n A_n \in \mathcal{A}$. Thus actually this object does not have to be a random measure in that respect, that the equality holds a.s. for each sequence of sets separately, but does not have to hold for all sequences simultaneously a.s. A short terminology of the Lévy basis was introduced in [1, 2] where it was succesfully used for turbulence modelling.

Even though Lévy bases does not have to be random measures (like e.g. nonatomic Gaussian random measures) they do include random measures in the strict sense like e.g. Poisson random measures, mixed Poisson random measures as well as so-called $G-$measures [3] and it is possible to define integration with respect to them [7]. Thus having in mind the construction of the shot noise Cox processes [5] the second step in defining the driving intensity $\Lambda(\cdot)$ of the spatial Cox process should be a kernel smoothing of the Lévy basis

$$\Lambda(\xi) = \int k(\xi, \eta) L(d\eta), \quad \xi \in W \subset \mathbb{R}^d,$$

where k is a kernel (weight) function. By this operation we "smooth out" the possibly purely atomic Lévy basis to get the driving field Λ and also introduce spatial dependencies in Λ which were not present in the Lévy basis L. By this we arrive at the definition of the Lévy driven Cox processes – i.e. Cox processes with the random driving intensity function defined by an integral of a weight function with respect to a Lévy basis, which were introduced in [4].

Using the Lévy-Khintchine representation of the characteristic function of the Lévy basis it is possible to derive close formulas for product densities of the Lévy driven Cox process which depend only on this characterization and the kernel function k. This kind of representation also allows effective introduction of different forms of inhomogeneities into the model producing point patterns with different geometrical properties.

The second "Lévy based" Cox process model is obtained by defining the driving intensity as the exponential of a kernel smoothing of a Lévy basis (now allowing for non-positive weight functions and non-positive Lévy bases). Such processes were called log Lévy driven Cox processes in [4] and (under regularity conditions) their driving field is of the form $\Lambda = \Lambda_1 \Lambda_2$, where Λ_1 and Λ_2 are independent, Λ_1 is a log Gaussian field and Λ_2 is a log shot noise field. When $\Lambda_2 = 1$ we get the well known log Gaussian Cox processes [6], when $\Lambda_1 = 1$ we get the log shot noise Cox processes. Here too, we are able to derive close formulas for the n-th order product densities of the point processes.

What is still an open question is the development of the statistical inference for the (log) Lévy driven Cox processes. Of course when having the closed formulas for product densities we can use moment methods for parameter estimation, but it remains to investigate to what degree the known procedures for other classes of Cox processes, based on summary statistics, likelihood or Bayesian reasoning, can be adjusted to deal with the Lévy driven Cox processes.

Further interesting question would be the extension of the Lévy driven Cox process models to spatio-temporal setting.

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Random Polytopes and Random Polyhedra

Matthias Reitzner

Assume $X_1, \dots, X_n$ is a random sample of n iid points chosen according to some distribution in $\mathbb{R}^d$. The *random polytope* P_n is the convex hull of these points: $P_n = [X_1, \dots, X_n]$. In the first two lectures we are interested in the $\boldsymbol{f}$-vector of P_n, where $\boldsymbol{f}(P_n) = (f_0(P_n), \dots f_{d-1}(P_n))$, and $f_\ell(P_n)$ is the number of ℓ–dimensional faces of P_n, and further in the volume $V_d(P_n)$. Most results concern random points chosen either uniformly in a convex body or according to the Gaussian distribution. In the third lecture we are interested in Poisson hyperplane tessellations and Poisson Voronoi mosaics. Particular interest is in distributional results concerning the number of faces and the intrinsic volumes of the zero cell and the typical cell of the mosaic. For recent surveys see [2], and [15].

1. Geometric probabilities: an introduction to random polytopes

Choose the random points $X_1, \dots, X_n$ according to the uniform distribution in a given convex set K. The expectation of the $\boldsymbol{f}$-vector was investigated in a series of papers (see e.g. [1], [3], [13]). If K is a smooth convex body (with boundary of differentiability class $\mathcal{C}^2$ and positive Gaussian curvature), then $\mathbb{E}\boldsymbol{f}(P_n) = \boldsymbol{c}_d\, \Omega(K)\, n^{(d-1)/(d+1)}(1+o(1))$ as $n \to \infty$ where $\boldsymbol{c}_d$ is a constant vector. Here $\Omega(K)$ denotes the affine surface area of the convex body K. If K is a polytope, then $\mathbb{E}\boldsymbol{f}(P_n) = \overline{\boldsymbol{c}}_d\ T(K) \ln^{d-1} n(1 + o(1))$. Here $T(K)$ denotes the number of chains $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ of i–dimensional faces F_i of K.

Of high interest is the question to determine the extremal convex sets minimizing or maximizing the mean values mentioned above. In particular it would be of interest to prove that among all convex sets the simplex is an extremal body for $\mathbb{E}f_0(P_n)$ which is known only for $d = 2$. Interest in this question stems from its connection to the hyperplane conjecture and the isotropic constant of a convex set K (see [10]). Related recent research [9] asks for uniform bounds for the isotropic constants of random polytopes.

2. Distributional aspects of random polytopes

The last years have seen several new results on the asymptotic distribution of the random variables $V_d(P_n)$ and $f_s(P_n)$, see [12], [17], and [4]. If K is either smooth or a polytope, then the random variables $V_d(P_n)$ and $f_\ell(P_n)$ satisfy a central limit theorem.

Recently, large deviation inequalities have been proved in important cases, see [16] and [5]. If K is smooth, then for $Z_n = f_0(P_n)$ and $Z_n = V_d(P_n)$

$$\mathbb{P}\left(\left|\frac{Z_n - \mathbb{E}Z_n}{\sqrt{\mathrm{Var} Z_n}}\right| \geq t\right) \leq 2e^{-ct^2} + e^{-cn^{\frac{d-1}{3d+5}}},$$

for $t^2 \leq n^{\frac{(d-1)(d+3)}{(d+1)(3d+5)}}$. If K is a polytope, similar slightly weaker results have been proved in [16] and [11]. All these results follow from large deviation inequalities for general convex sets.

An interesting new direction of research, with important applications coming from linear error correcting codes, concerns Gaussian polytopes where the points $X_1, \ldots, X_n$ are chosen according to the Gaussian distribution. In particular the connection between Gaussian polytopes and projections of high dimendional simplices and cross-polytopes turns out to be of importance, see [6] and [7].

3. Random polyhedra and random mosaics

A Poisson hyperplane process in $\mathbb{R}^d$ tessellates space into bounded convex polytopes, the cells of the Poisson hyperplane tessellations. Of interest are the number of faces and the volume of the zero cell Z_0 and the typical cell Z. As for the zero cell it was proved in [14] that $\mathbb{E}V_d(Z_0) = 2^{-d}d!\lambda^{-d}V_d(\Pi^o B)$ and $\mathbb{E}f_0(Z_0) = 2^{-d}d!V_d(\Pi B)V_d(\Pi^o B)$. The question to determine $\mathbb{E}f_\ell(Z_0)$ for $\ell \geq 1$ seems to be open. Large deviation inequalities for $V_d(Z_0)$ follow from the solution of Kendall's conjecture: 'Given $V_2(Z_0)$ is large, show that the shape of Z_0 is close to a circle'. This problem was settled in large generality in [8].

As for Poisson Voronoi mosaics the mean values $\mathbb{E}V_d(Z)$ and $\mathbb{E}f_0(Z)$ for the typical cell are known, but results for the zero cell seem to be missing at all.

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Polygonal Markov fields in the plane: graphical constructions and geometry of higher order correlations

Tomasz Schreiber

Polygonal Markov fields, originally introduced and studied by Arak, Clifford & Surgailis [1, 2, 3, 4], are random ensembles of non-intersecting polygonal contours in the plane, interacting by hard-core exclusions, with a variety of additional possible terms entering the Hamiltonian, including length and area elements. In our talk we discuss recent developments in this area. We argue that the polygonal fields share a number of essential features with the two-dimensional Ising model, prominent examples including the presence of an Ising-like phase transition [8, 9] as well as low temperature phase separation and Wulff droplet creation [10]. For these reasons, the polygonal Markov fields are sometimes regarded as continuum counterparts of the Ising (and Potts) model. We mention that in many aspects the polygonal fields are *exactly tractable*, especially in the so-called *consistent regime* falling into the supercritical temperature region. In particular, at the consistency point we know the exact value of the partition function as well as the first and second order characteristics of the field (T.Arak & D.Surgailis, P.Clifford). Further, we discuss a number of new results about the higher order correlations, including certain exact formulae [11] and martingale (random walk) representations (TS'08, not yet published). Another striking feature of polygonal Markov fields is that they admit a number of particularly convenient algorithmic constructions – graphical representations [2, 3, 9, 10, 11] which are in fact the main tool for establishing of the afore-mentioned results. The geometric ingredient in these considerations is so predominant that in many cases no supplementary calculations are needed.

The class of graphical constructions we developed for polygonal fields have also found their applications in Bayesian image processing (joint work with M.N.M. van Lieshout and R. Kluszczynski) where we used them to generate image segmentations [5, 6, 7, 12]. Experimenting with various black-white and grayscale images we already obtained promising results, further algorithms are a subject of our ongoing research in progress.

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Asymptotics of scan statistics and complete spatial randomness

Evgeny Spodarev

(joint work with Pavel Grabarnik, Zakhar Kabluchko)

Let $\{\xi(t), t \geq 0\}$ be a Lévy process. A *Lévy noise* $\mathcal{Z}$ is an independently scattered homogeneous random Lévy measure on $\mathbb{R}^d$, i.e., a stochastic process $\{\mathcal{Z}(R), R \in \mathcal{B}(\mathbb{R}^d)\}$, indexed by the collection $\mathcal{B}(\mathbb{R}^d)$ of Borel sets in $\mathbb{R}^d$, such that

(a) $\mathcal{Z}(R)$ has the same distribution as $\xi(|R|)$, where $|R|$ is the Lebesgue measure of a Borel set R.

(b) If $R_1, \ldots, R_n$ are disjoint Borel subsets of $\mathbb{R}^d$ then $\mathcal{Z}(R_1), \ldots, \mathcal{Z}(R_n)$ are independent and $\mathcal{Z}(\cup_{i=1}^n R_i) = \sum_{i=1}^n \mathcal{Z}(R_i)$.

Assume that

$$\mathbb{E}\,\mathcal{Z}(R) = \mu|R|, \quad \mathrm{Var}\mathcal{Z}(R) = \sigma^2|R|, \qquad R \in \mathcal{B}(\mathbb{R}^d)$$

for some $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. A natural problem is how to detect inhomogeneities, e.g. locations of unusually large mass of the observed random measure. To this end, consider the *scan statistic* T_n of $\mathcal{Z}$ defined as

$$T_n = \sup_{R \in \mathcal{R}(n)} \mathcal{Z}(R),$$

where $\mathcal{R}(n)$ is the collection of all cubes contained in $[0, n]^d$ (confer [2] for more details on scan statistics).

In this talk, we partially report about the recent results of [3] that the limiting behaviour of T_n depends on the sign of μ in the following way:

Theorem 1. *Let $\{\mathcal{Z}(R), R \in \mathcal{B}(\mathbb{R}^d)\}$ be a Lévy noise defined above.*

(i) *If $\mu > 0$ then the distribution of $\sigma^{-1} n^{-d/2} \left(\sup_{R \in \mathcal{R}(n)} \mathcal{Z}(R) - n^d \mu \right)$ converges as $n \to \infty$ to the standard normal distribution.*

(ii) *If $\mu = 0$ then the distribution of $\sigma^{-1} n^{-d/2} \sup_{R \in \mathcal{R}(n)} \mathcal{Z}(R)$ converges as $n \to \infty$ to the distribution of $\sup_{R \in \mathcal{R}(1)} \mathcal{W}(R)$, where $\mathcal{W}$ is the Brownian sheet on $[0,1]^d$.*

(iii) *If $\mu < 0$ and under additional assumptions on the distribution of $\xi(1)$ it holds*

$$\lim_{n\to\infty} \mathbb{P}[\sup_{R \in \mathcal{R}(n)} \mathcal{Z}(R) \le u_n(\tau)] = \exp\{-e^{-\tau}\}$$

for the proper choice of the constant $u_n(\tau)$; see [3] for more details.

To prove the non-trivial part (iii) of the above Theorem, we use the method of double sums introduced by Pickands e.g. in [4].

Since the scan statistic is designed to detect unusual clusters, it can not be successfully used e.g. to test the complete spatial randomness hypothesis (CSR). This hypothesis states that $\mathcal{Z}$ is a spatially homogeneous Poisson counting measure. More generally, we would like to test a hypothesis

$$H_0: \text{ observed random measure } \mathcal{Z} = \mathcal{Z}_0 \text{ vs. } H_1: \mathcal{Z} \neq \mathcal{Z}_0,$$

where $\mathcal{Z}_0$ is a particular Lévy noise.

Assume that $\mathcal{Z}$ and $\mathcal{Z}_0$ are non–negative measures. Let $\mathcal{Z}$ be observed within cubic scanning windows $R_x = x + [0,r]^d \subset [0,n]^d$ of a fixed size $r > 0$ that are located at lattice points $x \in \lambda\mathbb{Z}^d \cap [0,n]^d$. Here $\lambda > 0$ is the resolution of the lattice. Let $X_i = \mathcal{Z}(R_{x_i})$, $i = 1, \ldots, m$ be the available observations of $\mathcal{Z}$. In order to construct a test statistic for H_0 which uses the information of all scans (and not only of the maximal ones), form order statistics $X_{(i)}$ out of the above scans and consider their *Lorenz curve*

$$\left(\frac{i}{m}, \sum_{j=1}^{i} X_{(j)} / \sum_{j=1}^{m} X_j \right), \qquad i = 0, \ldots, m.$$

Its continuous counterpart is the curve

$$\left(u, \frac{1}{\mu r^d} \int_o^u F_\xi^{-1}(v)\, dv \right), \qquad u \in [0,1],$$

where F_ξ^{-1} is the quantile function of the distribution of $\mathcal{Z}([0,r]^d) = \xi(r^d)$ and $\mu > 0$. Take the area under the theoretical Lorenz curve and its empirical counterpart:

$$S_r = \frac{1}{\mu r^d} \int_o^1 \int_o^u F_\xi^{-1}(v)\, dv\, du, \qquad \widehat{S}_r = \sum_{i=0}^{m} \sum_{j=1}^{i} X_{(j)} / \sum_{j=1}^{m} X_j$$

and consider the statistic $T_r = 2S_r$ and $\widehat{T}_r = 2\widehat{S}_r$ for all $0 < r \leq n$, $0 \leq T_r, \widehat{T}_r \leq 1$. It is known from simulation experiments that the test statistic

$$T = \int_{r_o}^{r_1} (T_r - \widehat{T}_r)^2 \, dr$$

performs very well testing the CSR hypothesis for the proper choice of constants r_o and r_1. A matter of ongoing research is to study the asymptotical properties of $\widehat{S}_r$ and T as $n \to \infty$ and $\lambda \to 0$ which would allow the construction of asymptotical tests of hypotheses H_0 vs. H_1.

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New Trends in Optimal Transportation on Riemannian and Singular Spaces

Karl-Theodor Sturm

Lecture I: The Wasserstein Space of Probability Measures and its Riemannian Structure. In this lecture, we discuss the geometric and Riemannian structure of the space of probability measures on a metric (or Riemannian, resp.) space. We begin with an introduction to the optimal transportation problems of Monge and Kantorovich and then present basic results concerning topology and geometry of the space of probability measures equipped with the L^2-Wasserstein distance.

The main results will be the representation results of Brenier and McCann for the optimal transport maps in the Euclidean and Riemannian setting. As a corollary, we deduce the Riemannian structure (tangent space, scalar product, exponential map, equation for geodesics) on the Wasserstein space.

Calculating the Wasserstein gradient of the relative entropy (and related functionals) will be an explicit application of these results.

Lecture II: Optimal Transportation, Gradient Flows and Functional Inequalities. The main topics of the second lecture are Otto's gradient flow aspect for various PDEs and – as a consequence of it – the re-interpretation due to Otto & Villani of functional inequalities like logarithmic Sobolev inequalities in terms of differential inequalities on the Wasserstein space.

We sketch the proof for the fact that the heat equation is the gradient flow for the entropy. Moreover, we discuss in detail convexity properties of the entropy and related functionals under optimal transports. Applications to interacting particle systems and their scaling limits will be presented.

In the Riemannian case, we point out the role of lower bounds for the Ricci curvature in optimal transportation problems and indicate recent links between optimal transportation and Ricci flow.

Lecture III: Ricci Bounds for Metric Measure Spaces. In the third lecture, we present the concept of generalized lower Ricci curvature bounds for metric measure spaces (M, d, m), introduced by Lott, Villani and the author. These curvature bounds are defined in terms of optimal transportation, more precisely, in terms of convexity properties of the relative entropy $Ent(.|m)$ regarded as function on the Wasserstein space of probability measures on the given space M. For Riemannian manifolds, $Curv(M, d, m) \geq K$ if and only if $Ric_M \geq K$ on M. Other important examples covered by this concept are Finsler manifolds.
One of the main results is that these lower curvature bounds are stable under (e.g. measured Gromov-Hausdorff) convergence.

Moreover, we introduce a curvature-dimension condition $\mathrm{CD}(K, N)$ being more restrictive than the curvature bound $Curv(M, d, m) \geq K$. For Riemannian manifolds, $\mathrm{CD}(K, N)$ is equivalent to $\mathrm{Ric}_M(\xi, \xi) \geq K \cdot |\xi|^2$ and $\dim(M) \leq N$.
Condition $\mathrm{CD}(K, N)$ implies sharp version of the Brunn-Minkowski inequality, of the Bishop-Gromov volume comparison theorem and of the Bonnet-Myers theorem.
Extension of this curvature concept to discrete spaces and infinite dimensional spaces will be indicated, e.g. for the Wiener space $Curv(M, d, m) = 1$.

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Balancing Transports for Random Measures: Stationarity and Mass-Stationarity

Hermann Thorisson
(joint work with Günter Last)

Consider a locally compact Abelian group G, eg. $G = \mathbb{R}^d$, acting on a sample space Ω. Let λ denote the Haar measure on G and θ_s the shift of $\omega \in \Omega$ by $s \in G$. A random measure ξ on G is invariant if $\xi(\theta_s, s + \cdot) = \xi(\theta_0, \cdot)$, $s \in G$. A *transport-kernel* is a Markovian kernel T that redistributes mass over G and depends on both $\omega \in \Omega$ and a location $s \in G$. The number $T(\omega, s, B)$ is the proportion of mass *transported* from location s to the set B. The kernel T is *invariant* if it is invariant under joint shifts of all three arguments. If ξ and η are random measures such that $\xi T = \eta$ then T is (ξ, η)-*balancing*. In particular, if $\xi T = \xi$ then T is ξ-*preserving*. Sometimes T can be reduced to an *allocation* τ (depending on $\omega \in \Omega$) that maps each location s to a new location $\tau(s)$, $T = \delta_\tau$. In fact, we might think of a transport-kernel T as the conditional distribution of a randomized allocation.

Liggett [4] presented the following surprising result. Consider a doubly-infinite sequence of i.i.d. coin tosses. Move the origin to a head as follows. If there is a head at the origin, stay there. If there is a tail at the origin, move to the right counting heads and tails until you have more heads than tails. Then you are at a head. If you ignore that head, the rest of the coin tosses turn out to be i.i.d. (so you have found an extra head!). This procedure thus gives a *shift-coupling* of the i.i.d. coin tosses and its Palm version: they are the same up to a shift of the origin. Note that if Liggett's rule is applied to all locations (not only the origin) then it generates an invariant allocation τ, transporting counting measure on the integers to the Bernoulli $(1/2)$ random measure with intensity 1. Liggett also treated a general Bernoulli parameter p and the Poisson process on the line.

Triggered by Liggett's paper the case when $G = \mathbb{R}^d$ (or $G = \mathbb{Z}^d$), ξ is Lebesgue measure, and η is a stationary ergodic point process of intensity 1, has received considerable attention in recent years; see [3] for references. In particular, Holroyd and Peres [2] presented the following beautiful allocation of Lebesgue measure to the Poisson process with intensity 1. Partition space by associating to each point a region of exact size 1. For this purpose place a small ball around each point of the Poisson process and expand the balls simultaneously until they reach size 1. If a ball hits a region that has already been taken by another point, let it continue to grow passively until it again finds space that has not yet been taken by another point. In this way $\mathbb{R}^d$ is partitioned into (not necessarily connected, but bounded) regions of size 1, each containing one point of the Poisson process. Now transport each location to the point of its region. This reshapes Lebesgue measure into the Poisson process in an invariant way. And if we shift the origin to the point of its region, we obtain a shift-coupling of the stationary Poisson process and its Palm version. Holroyd and Peres also treated the case when η is a stationary ergodic point process.

Actually, an abstract shift-coupling result for groups due to Thorisson [5] already implied that shift-coupings exist in the above cases, see [6]. Last and Thorisson [3] use that result to prove the following theorem which gives a necessary and sufficient condition for the existence of balancing invariant transport-kernels in the positive finite intensity case:

Theorem 1. *Let $\mathbb{P}$ be a stationary σ-finite measure on Ω, i.e. $\theta_s \stackrel{d}{=} \theta_0$, $s \in G$, under $\mathbb{P}$. Let ξ and η be invariant random measures with positive and finite intensities. Then there exists a $\mathbb{P}$-a.e. (ξ,η)-balancing invariant transport-kernel if and only if*

$$\exists B \in \mathcal{G},\ 0 < \lambda(B) < \infty : \qquad \mathbb{E}_{\mathbb{P}}[\xi(B)|\mathcal{I}] = \mathbb{E}_{\mathbb{P}}[\eta(B)|\mathcal{I}] \quad \mathbb{P}\text{-a.e.}$$

where $\mathcal{I}$ is the invariant σ-field.

Mass-stationarity w.r.t. a random measure ξ means, informally, that the origin is a typical location in the mass of ξ, just like stationarity means that the origin is a typical location in the space G. Mass-stationarity is an extension of the concept of point-stationarity introduced in Thorisson [6] in the case of simple point processes on $\mathbb{R}^d$. (Think of the stationary Poisson process on the line with an extra point at the origin: that point is a typical point because the intervals remain i.i.d. exponential if we shift the origin to the n^{th} point on the right, or to the n^{th} on the left.) The formal definition in that paper can be loosely phrased as follows: a probability measure $\mathbb{Q}$ on Ω is *point-stationary* w.r.t. an invariant simple point-process ξ if it is invariant under shifts induced by invariant ξ-preserving allocations *against any independent stationary background* (the shifts were allowed to depend not only on ξ but also on any independent stationary random field obtained by extending the underlying probability space). The main result of [6] was that point-stationarity is a characterizing property of Palm versions of stationary $\mathbb{P}$. The question whether the 'independent stationary background' could be removed from the definition of point-stationarity inspired considerable research activity; see [3] for references. Finally, Heveling and Last [1] showed that this can be done, that is, no external randomization is needed. In a subsequent paper they extended this result to Abelian G.

It is natural to attempt to extend the above definition to random measures ξ by demanding that the probability measure $\mathbb{Q}$ be invariant under shifts induced by invariant preserving allocations. However, Last and Thorisson [3] show that here external randomization would be needed if the property is to characterize Palm versions of stationary $\mathbb{P}$. One could allow external randomization in the definition (i.e. add an independent stationary background, or apply transport-kernels rather than only allocations), but it is still an open problem whether this would suffice.

Instead of using the above condition as a definition, Last and Thorisson [3] use the following condition, which in the point process case was proved in [6] to be equivalent to point-stationarity: say that a σ-finite measure $\mathbb{Q}$ on Ω is *mass-stationary* w.r.t. an invariant random measure ξ if for all relatively compact λ-continuity sets $C \in G$ such that $\lambda(C) > 0$, it holds that

$$(\theta_V, U + V) \stackrel{d}{=} (\theta_0, U) \quad \text{under } \mathbb{Q}$$

where U and V are defined on an extension of Ω in such a way that U is uniformly distributed (according to λ) on C and independent of θ_0, and the conditional distribution of V given (θ_0, U) is $\xi(\cdot|C-U)$. In [3] it is shown that mass-stationarity defined in this way is indeed an intrinsic characterization of Palm versions of stationary $\mathbb{P}$:

Theorem 2. *There exists a σ-finite stationary measure $\mathbb{P}$ on Ω such that $\mathbb{Q}$ is the Palm measure of $\mathbb{P}$ w.r.t. a random measure ξ iff $\mathbb{Q}$ is mass-stationary w.r.t. ξ.*

Further, [3] shows that mass-stationarity is equivalent to distributional invariance under shifts induced by certain kernels T that can be non-Markovian. As mentioned above, the question whether this can be narrowed down to transport-kernels is still an open problem. See Last's contribution in this volume for further characterizations of mass-stationarity.

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Limit theory for convex hulls

Joseph E. Yukich
(joint work with Tomasz Schreiber)

We show in [5] that the random point measures induced by the vertices in the convex hull of a Poisson sample on the unit ball, when properly scaled and centered, converge to a mean zero Gaussian field. We establish limiting variance and covariance asymptotics in terms of the density of the Poisson sample. Similar results hold for the point measures induced by the maximal points in a Poisson sample. The approach involves introducing a generalized spatial birth growth process allowing for cell overlap, showing that the spatial birth growth process stabilizes, and then appealing to general results on limit theorems for stabilizing functionals.

Let us now state our results more precisely. Recall that B_d denotes the unit radius ball centered at the origin of $\mathbb{R}^d$ and let ∂B_d denote its boundary. Let $\rho : B_d \to \mathbb{R}^+$ be a continuous density on B_d. We shall assume that

$$\rho(x) = \rho_0(x/|x|)(1-|x|)^\delta(1+o(1))$$

for some $\delta \geq 0$ and that $\rho_0 : \partial B_d \to \mathbb{R}^+$ is continuous and bounded away from 0. Let $P_{\lambda\rho}$ be a Poisson point process on B_d with intensity measure $\lambda\rho(x)dx$ and let $\mu_{\lambda\rho}$ be the random measure obtained by putting a unit point mass at each vertex of the convex hull of $P_{\lambda\rho}$. Let $\bar{\mu}_{\lambda\rho} := \mu_{\lambda\rho} - \mathbb{E}\,\mu_{\lambda\rho}$.

The results of Schreiber and Yukich [5] yield the following limit theory for the random measures $\mu_{\lambda\rho}$. Let $N(0,1)$ denote the standard normal random variable. Let $C(B_d)$ be the continuous functions on B_d and for each $f \in C(B_d)$ let $\langle f, \mu_{\lambda\rho}\rangle$ denote the integral of f with respect to $\mu_{\lambda\rho}$.

Theorem 1. *There are constants $M := M(d,\delta)$ and $V := V(d,\delta)$ such that for all $f \in C(B_d)$*

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d-1+2(1+\delta))}\mathbb{E}\left[\langle f, \mu_{\lambda\rho}\rangle\right] = M \int_{\partial B_d} f(s)\rho_0^{(d-1)/(d-1+2(1+\delta))}(s)ds$$

and

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d-1+2(1+\delta))}\mathrm{Var}[\langle f, \mu_{\lambda\rho}\rangle] = V \int_{\partial B_d} f^2(s)\rho_0^{(d-1)/(d-1+2(1+\delta))}(s)ds.$$

Moreover, the finite-dimensional distributions

$$\lambda^{-(d-1)/2(d-1+2(1+\delta))}(\langle f_1, \bar{\mu}_{\lambda\rho}\rangle, \ldots, \langle f_k, \bar{\mu}_{\lambda\rho}\rangle),$$

$f_i \in C(B_d)$, of $(\lambda^{-(d-1)/2(d-1+2(1+\delta))}\bar{\mu}_{\lambda\rho})$ converge as $\lambda \to \infty$ to those of a mean zero Gaussian field with covariance kernel

$$(f,g) \mapsto V \int_{\partial B_d} f(s)g(s)\rho_0^{(d-1)/(d-1+2(1+\delta))}(s)ds, \quad f,g \in C(B_d).$$

Additionally, if $\delta = 0$, then for all $f \in C(B_d)$

$$\sup_t \left| P\left[\frac{\langle f, \bar{\mu}_{\lambda\rho}\rangle}{\sqrt{\mathrm{Var}\langle f, \bar{\mu}_{\lambda\rho}\rangle}} \leq t\right] - P[N(0,1) \leq t]\right| = O\left(\lambda^{-(d-1)/2(d+1)}(\log\lambda)^{3+2(d-1)}\right)$$

Remarks. (i) Taking $f_1 \equiv 1$ (and all other $f_i \equiv 0,\ i = 2, ..., k$) provides a central limit theorem for the cardinality of the number of vertices in the convex hull of $P_{\lambda\rho}$.

(ii) Theorem 1 adds to the work of the following authors: (a) Groeneboom [2] and Cabo and Groeneboom [1], who prove a central limit theorem for the cardinality of the number of vertices in the convex hull of $P_{\lambda\rho}$, when ρ is uniform and when $d = 2$, (b) Reitzner [3] who considers the one dimensional central limit theorem and who establishes a rate of convergence $O(\lambda^{-(d-1)/2(d+1)}(\log \lambda)^{2+2/(d+1)})$ to the normal for ρ uniform (whence $\delta = 0$ in our setting), without giving asymptotics for the limiting variance and covariance, and (c) Van Vu [4], who proves a central limit theorem for the cardinality of the number of vertices of the convex hull of $\{X_i\}_{i=1}^n$, X_i i.i.d. uniform, but who also does not consider limiting covariances. Concerning rates, we believe that the power on the logarithm, namely $3 + 2(d - 1)$ can be reduced to $2(d - 1)$ but we have not tried for this sharper rate.

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Global and local notions of curvatures for self-similar random sets

Martina Zähle

Self-similar random fractals were first considered independently by Falconer, Graf and Mauldin /Williams in 1986/87. In the following period their multifractal measure and dimension properties have been studied extensively. In 2000 Gatzouras [1] proved that they are Minkowski measurable, i.e., the rescaled volumes of small tubular neighborhoods a.s. converge (in the average) to a limit, the so-called *Minkowski content.*

In order to distinguish fractal sets with equal Hausdorff or Minkowski dimension, but different geometric and topological features, curvature parameters appear to be useful. Such properties have first been investigated by my student Winter [2] for the case of deterministic self-similar sets with polyconvex neighborhoods. He obtained limit results for rescaled *curvature measures* of parallel sets and extended Gatzouras' deterministic version for the Minkowski content. Moreover, he established a formula for calculating the total values, which leads to numerical results in concrete examples.

Under some additional geometric conditions, which are not restrictive in $\mathbb{R}^d$ with

$d \leq 3$, we generalize these results to *Lipschitz-Killing curvatures* $C_k(F)$ of *stochastically self-similar random sets* F with *Hausdorff dimension* D and non-polyconvex ε-neighborhoods $F(\varepsilon)$. The following limits exist:

$$\overline{C_k}(F) := \lim_{\delta\to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-k}\, \mathbb{E}C_k(F(\varepsilon)) \frac{1}{\varepsilon}\, d\varepsilon$$

and

$$C_k(F) := \lim_{\delta\to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-k} C_k(F(\varepsilon)) \frac{1}{\varepsilon}\, d\varepsilon = X\, \overline{C_k}(F) \quad \text{a.s.}$$

for a determined random variable X independent of $k = 0, \ldots, d$. If the distribution of the logarithmic contraction ratios of the random similarities generating F is non lattice, then averaging over $\varepsilon \to 0$ in the limits is not necessary ([3]).
Like in the case of the Minkowski content the classical renewal theorem for the expectations and a renewal theorem for general random walks for a.s. convergence play an essential role.
Using an associated dynamical system and Birkhoff's ergodic theorem local versions of these results in terms of *curvature densities* can be proved. This suggests the possibility of determining curvatures also for other classes of (random) fractals.

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Locally interacting sequential processes

Sergei Zuyev
(joint work with Ilya Molchanov)

Many observed in real life phenomena evolve sequentially when existing objects (particles, organisms, etc.) create new objects (offspring) in a way depending on their surroundings. Although suggested dynamics is locally defined, the overall behaviour of the system depends on whether this local dependence propagate through the whole system or stays localised. We consider the following theoretical framework.

Given an initial configuration of particles in some space, such as the d-dimensional Euclidean space, new particles are added sequentially according to the following construction rule. One particle is randomly chosen from the existing ones and then a new particle is generated from a distribution which depends on the geometry of the chosen particle and its neighbours. More precisely, the distribution of a new point depends on the stopping set related to the chosen particle and the configuration. A stopping set is a random set generalising the classical notion of a stopping time to spatial processes, see, e.g., [4] and [5]. Therefore, particles in such a sequential process exhibit a certain local interaction, suggesting the name locally interacting sequential process. This model was motivated by an attempt to model population density turned out to be a very rich model that can be used to describe e.g. (1) fragmentation such as stellar fragments and meteoroids in Astrophysics, fractures and earthquakes in Geophysics, crushing in the mining industry, breaking of crystals in Crystallography, degradation of large polymer chains in Chemistry, fission of atoms in Nuclear Physics, as well as fragmentation of a hard drive or files in Computer Science, and (2) classical stochastic models such as the Chinese restaurant process, Dirichlet process, and generalised Polya urn scheme.

Although application driven models are usually multidimensional, already one dimensional case exhibit very rich and nontrivial behaviour. To illustrate this statement, we consider a few particular models. In all models we start from initial configuration of a finite number of points in $[0,1]$. Assume that X_n is configuration of particles at step n. In Model 1, having chosen uniformly a 'parent' particle ξ_n on n-th step, we add a new particle x_{n+1} uniformly distributed in the segment $S(\xi_n, X_n) = [\xi_n, d^+(\xi_n, X_n)]$, where $d^+(\xi, X_n)$ is the distance from x to the closest particle of X_n to the right from ξ_n (or to 1, if the chosen particle ξ_n is the rightmost). The segment $S(\xi_n, X_n)$ is a stopping set and the distribution of x_{n+1} depends on it. In Model 2, we take the stopping set to be $S(\xi_n, X_n) = [\xi_n - d(\xi_n, X_n), \xi_n + d(\xi_n, X_n)]$, where $d(\xi, X_n)$ is the distance from x to the closest particle of X_n (both to the right or to the left from ξ_n) and the distribution of x_{n+1} is still uniform on $S(\xi_n, X_n)$. In Model 3, the stopping set is the same as in Model 2, but the distribution of x_{n+1} has a form $\xi_n + d(\xi, X_n)\eta_n$, where $\{\eta_n\}$ is a sequence of i.i.d. random variables. A common feature of all these models is a kind of density reinforcement: the more density of particles in a certain area, the more often a particle from this area is chosen as a parent. But since the density is high, the corresponding stopping set is small so the new generated particle will be close to the parent thus increasing the density even more. This behaviour is analogous to Blackwell-MacQueen construction [1] which leads to construction of a Dirichlet process (a random measure), see, e.g., [3], and shows that the limit μ of a normalised sequence of sample measures $\mu_n(\cdot) = n^{-1}\sum_{i=1}^n \mathbf{1}\{x_i \in \cdot\}$ is, generally, a random measure.

We show that the limiting measure in Model 1 is, however, not a Dirichlet process, which is atomic, but equivalent to the so-called Dubbins-Freedman class of random distributions [2]. Each realisation of its p.d.f. is a strictly growing function though a fractal curve in $[0,1]^2$. In contrast, seemingly small change in the definition of the stopping set in Model 2 leads to significant change of the limiting measure: its p.d.f. now becomes piecewise constant and the support of the measure is a fractal. Moreover, employing methods of Lyapunov functions and stochastic stability, we demonstrate that the support now can be larger than $[0,1]$ but with probability one stays compact.

Finally, in Model 3 the support of μ may become infinite for heavy-tailed distributions of η or may even not exist. The situation is yet not clear here and represents a challenge the author is currently working at.

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