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Discrete Geometry

Organised by
Martin Henk (Magdeburg)
Jiří Matoušek (Praha)
Emo Welzl (Zürich)

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Abstract. A number of remarkable recent developments in many branches of discrete geometry have been presented at the workshop, some of them demonstrating strong interactions with other fields of mathematics (such as harmonic analysis or topology). A large number of young participants also allows us to be optimistic about the future of the field.

Mathematics Subject Classification (2000): 52Cxx.

Introduction by the Organisers

Discrete Geometry deals with the structure and complexity of discrete geometric objects ranging from finite point sets in the plane to more complex structures like arrangements of n-dimensional convex bodies. Classical problems such as Kepler's conjecture and Hilbert's third problem on decomposing polyhedra, as well as classical works by mathematicians such as Minkowski, Steinitz, Hadwiger and Erdős are part of the heritage of this area. By its nature, this area is interdisciplinary and has relations to many other vital mathematical fields, such as algebraic geometry, topology, combinatorics, computational geometry, convexity, and probability theory. At the same time it is on the cutting edge of applications such as geographic information systems, mathematical programming, coding theory, solid modelling, computational structural biology and crystallography.

The workshop was attended by 40 participants. There was a series of 12 survey talks giving an overview of developments in Discrete Geometry and related fields:

- Keith Ball: *A sharp discrete geometric version of Vaaler's Theorem*
- Marcus Schaefer: *Hanani–Tutte and related results*
- Frank Vallentin: *Fourier analysis, linear programming and distance avoiding sets in* $\mathbb{R}^n$

- Nathan Linial: *What is high-dimensional combinatorics?*
- Gábor Tardos: *Conflict free coloring of rectangles*
- Matthias Beck: *Recent results on Ehrhart series of lattice polytopes*
- Assaf Naor: *Embeddings of discrete groups and the speed of random walks*
- János Pach: *Beyond planarity—Geometric intersection patterns*
- Günter M. Ziegler: *On the number of simplicial 4-polytopes and* 3*-spheres with* N *facets*
- Alex Iosevich: *Discrete geometry and Fourier analysis in discrete, continuous and finite field settings*
- Alexander Barvinok: *Random matrices with prescribed row and column sums*
- Jesús De Loera: *How to integrate a polynomial over a polytope*

In addition, there were 18 shorter talks and an open problem session chaired by János Pach on Wednesday evening—a collection of open problems resulting from this session can be found in this report. The program left ample time for research and discussions in the stimulating atmosphere of the Oberwolfach Institute.

Workshop: Discrete Geometry

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Abstracts

A sharp discrete-geometric version of Vaaler's Theorem

Keith Ball
(joint work with Maria Prodromou)

In [2] Vaaler proved that for every d and n, every d-dimensional central section of the cube $[-1,1]^n$ has volume at least 2^d. His result provided a sharp version of Siegel's Lemma in the geometry of numbers and was used by Bombieri and Vaaler himself [1] for applications in Diophantine approximation. Vaaler's theorem is obviously sharp since the sections by d-dimensional coordinate subspaces are cubes of volume 2^d.

If $(\epsilon_i)_1^d$ are IID choices of sign and $x = (x_i)$ is a vector in $\mathbb{R}^d$ then

$$\mathrm{E}\left(\sum x_i\epsilon_i\right)^2 = \sum x_i^2.$$

Thus, if P is the uniform probability measure on the corners of the cube $[-1,1]^d$ then the quadratic form

$$\int_{[-1,1]^d} v \otimes v \,\mathrm{d}P(v)$$

is the identity on $\mathbb{R}^d$. In this talk we prove the following sharp discrete-geometric version of Vaaler's Theorem.

Theorem 1. *Let K be a d-dimensional subspace of $\mathbb{R}^n$. Then, there is a probability measure P on $[-1,1]^n \cap K$, with*

$$\int_{[-1,1]^n\cap K} v \otimes v \,\mathrm{d}P \geq I_K \tag{1}$$

where the dominance is in the sense of positive definite operators.

Thus, each section of the cube not only has large volume but it is also "fat in all directions" in the same way as a cube.

Observe that if we start with the uniform probability on the corners of the n-dimensional cube and project it orthogonally onto the subspace K, we will obtain a probability measure that yields the identity (in the above sense). However, for most subspaces, the support of this projected measure will extend far outside the section $[-1,1]^n \cap K$ so it will not be a suitable choice in the theorem.

Coordinate subspaces show that Theorem 1 is sharp in the sense that we cannot guarantee to beat a larger multiple of the identity. What is more surprising is that lower-dimensional cubes do not provide the only extreme cases. For example, the section of the 3-dimensional cube perpendicular to its main diagonal is a regular hexagon whose corners are points like $(1,-1,0)$ which are at distance $\sqrt{2}$ from the origin. If we take the traces of the operators appearing in equation (1) we obtain

$$\int_{[-1,1]^n\cap K} |v|^2 \geq \dim(K).$$

So the probability measure guaranteed by Theorem 1 must be supported on the corners of the hexagon and we cannot beat any multiple of the identity larger than 1. A similar argument works for the diagonal section of the cube in any odd dimension. The existence of a large family of subspaces for which the inequality is sharp makes it highly unlikely that we could write down the measure we wish to find in any reasonably explicit way. Our argument builds the probability as the end result of a sequence of linked optimisation problems.

It is a simple (and pretty well-known) fact that if $(x_i)_1^d$ is a sequence of unit vectors in $\mathbb{R}^d$ then there is a unit vector v in $\mathbb{R}^d$ for which

$$|\langle v, x_i\rangle| \le \frac{1}{\sqrt{d}}, \quad \text{for all} \quad i = 1, \ldots, n.$$

It follows from Theorem 1 that this fact can be generalised:

Theorem 2. *Let $(x_i)_1^n \subset \mathbb{R}^d$ be a sequence of vectors that satisfy $\sum_{i=1}^n |x_i|^2 = d$. Then there exists a unit vector $v \in \mathbb{R}^d$, such that*

$$|\langle v, x_i\rangle| \quad \le \quad \frac{1}{\sqrt{d}}, \quad \textit{for all} \quad i = 1, \ldots, n. \tag{2}$$

In this talk Theorem 1 and Theorem 2 are both obtained from the following.

Theorem 3. *Let $(u_i)_{i=1}^n$ be a sequence of vectors in $\mathbb{R}^d$ that satisfies $\sum_{i=1}^n u_i \otimes u_i = I_d$ and Q a positive semi-definite quadratic form on $\mathbb{R}^d$. Then there is a vector $w \in \mathbb{R}^d$, such that $|\langle w, u_i\rangle| \le 1$, for all $i = 1, \ldots, n$ and*

$$w^T Q w \ge tr(Q).$$

The most intriguing feature of the proof is that it is to some extent constructive. In view of the importance in Diophantine approximation of finding vectors with small inner product with a given sequence, it is natural to ask whether there is a lattice version of Theorem 2 that can be proved without the averaging technique implicit in the proof of the Dirichlet-Minkowski box principle.

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Hanani–Tutte and related results

MARCUS SCHAEFER

(joint work with Michael Pelsmajer, Daniel Štefankovič, Despina Stasi)

Theorem 1 (Hanani–Tutte [3, 10])**.** *If a non-planar graph is drawn in the plane, then the drawing contains two non-adjacent edges that cross an odd number of times.*

We give a direct geometric proof of this classical result which, in turn, is based on a strengthening of a result by Pach and Tóth [6]. Call an edge in a drawing of a graph *even* if it intersects every other edge an even number of times.

Theorem 2 (Pelsmajer, Schaefer, Štefankovič [9])**.** *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane so that no edge in E_0 is involved in a crossing and there are no new pairs of edges that cross an odd number of times.*

The proof of this result relies on a geometric redrawing idea, using rotations in an essential way. An extension of this result can be used to show that $\operatorname{cr}(G) \leq \binom{2\operatorname{iocr}(G)}{2}$, where $\operatorname{iocr}(G)$ is the *independent odd crossing number* of G, the smallest number of pairs of non-adjacent edges that cross in a drawing of G. In particular, this means that all the standard crossing numbers (excluding the rectilinear crossing number, which behaves differently): crossing number, odd crossing number, pair-crossing number, independent odd crossing number are within a square of each other (answering an open question of Pach).

We were recently able to establish a Hanani–Tutte theorem for the projective plane:

Theorem 3 (Pelsmajer, Schaefer, Stasi)**.** *If a non-projective planar graph is drawn in the projective plane, then the drawing contains two non-adjacent edges that cross an odd number of times.*

The proof does not use the same approach as the planar case (indeed, Theorem 2 fails on the projective plane). It remains open whether the result is true for any other surfaces. However, if we drop the requirement that the two edges that cross be non-adjacent, we get a "weak Hanani–Tutte" theorem which is true for all surfaces (a result first shown for orientable surfaces by Cairns and Nikolayevsky [1]).

A graph is a *thrackle* if it can be drawn such that any pair of edges intersects exactly once, where a common endpoint of two edges counts as an intersection of these two edges. A *generalized thrackle* is a graph that can be drawn such that any pair of edges intersects an odd number of times (again counting endpoints). Our proof techniques allow us another simple proof of the following result.

Theorem 4 (Cairns, Nikolayevsky [2]; Pelsmajer, Schaefer, Štefankovič [8])**.** *If G is a bipartite, generalized thrackle on a (orientable or non-orientable) surface, then G can be embedded on that surface.*

The planar case of the theorem was first proved by Lovász, Pach, and Szegedy [5]: if a bipartite graph is a generalized thrackle, then it is planar. Cairns and Nikolayevsky established the result for orientable surfaces [2].

Since in the proofs of all of these results the rotation system of a graph plays a role, it seems natural to ask about the complexity of standard graph drawing problems, such as the crossing number, in the presence of a rotation system. We can show that computing the crossing number (or odd-crossing number or pair-crossing number) of a graph with rotation system is **NP**-hard [7]. As a corollary we obtain Hliněný's result [4] that computing the crossing number of a 3-connected, cubic graph (without rotation system) and computing the minor-monotone crossing number is **NP**-complete, and we can show that the independent odd crossing number, Tutte's algebraic alternative to crossing number, is **NP**-complete. This is unfortunate (if not unexpected), since as mentioned above, the independent odd crossing number is within a square of the crossing number.

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Fourier analysis, linear programming, and densities of distance avoiding sets in $\mathbb{R}^n$

Frank Vallentin

(joint work with Fernando M. de Oliveira Filho)

This is an extended abstract of [12] *which steps on previous work* [2].

Let $d_1, \ldots, d_N$ be positive real numbers. We say that a subset A of the n-dimensional Euclidean space $\mathbb{R}^n$ *avoids the distances* $d_1, \ldots, d_N$ if the distance between any two points in A is never $d_1, \ldots, d_N$. We define the *upper density* of a Lebesgue measurable set $A \subseteq \mathbb{R}^n$ as

$$\overline{\delta}(A) = \limsup_{T\to\infty} \frac{\mathrm{vol}(A \cap [-T,T]^n)}{\mathrm{vol}[-T,T]^n}.$$

In this expression $[-T,T]^n$ denotes the regular cube in $\mathbb{R}^n$ with side $2T$ centered at the origin. We denote the *extreme density* a measurable set in $\mathbb{R}^n$ that avoids the distances $d_1, \ldots, d_N$ can have by

$$m_{d_1,\ldots,d_N}(\mathbb{R}^n) = \sup\{\, \overline{\delta}(A) \;:\; A \subseteq \mathbb{R}^n \text{ measurable and avoids distances } d_1, \ldots, d_m \,\}.$$

Our main objective is to derive upper bounds for the extreme density from the solution of a linear programming problem.

To formulate the main theorem we consider the function Ω_n, which is the spherical average of an exponential function, given by

$$\Omega_n(t) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{t}\right)^{\frac{1}{2}(n-2)} J_{\frac{1}{2}(n-2)}(t),$$

where $J_{\frac{1}{2}(n-2)}$ is the *Bessel function of the first kind* with *parameter* $(n-2)/2$.

Theorem. *Let $d_1, \ldots, d_N$ be positive real numbers. Let $A \subseteq \mathbb{R}^n$ be a measurable set which avoids the distances $d_1, \ldots, d_N$. Suppose there are real numbers z_0, z_1, $\ldots$, z_N which sum up to at least one and which satisfy*

$$z_0 + z_1\Omega_n(td_1) + z_2\Omega_n(td_2) + \cdots + z_N\Omega_n(td_N) \geq 0$$

for all $t \geq 0$. Then, the upper density of A is at most z_0.

For the proof in we make essential use of Fourier analysis.

We apply the main theorem in a variety of situations: sets avoiding one distance, and sets avoiding many distances. For the history behind these kinds of problems we refer to the surveys by Székely [14] and Székely and Wormald [6] and the references therein.

Sets avoiding one distance have been extensively studied by combinatorialists because of their relation to the measurable chromatic number of the Euclidean space. This is the minimum number of colors one needs to color all points in $\mathbb{R}^n$ so that two points at distance 1 receive different colors and so that points receiving the same color form Lebesgue measurable sets. Since every color class of a coloring provides a measurable set which avoids the distance 1, we have the inequality

$$m_1(\mathbb{R}^n) \cdot \chi_m(\mathbb{R}^n) \geq 1. \tag{1}$$

For the plane it is only known that $5 \leq \chi_m(\mathbb{R}^2) \leq 7$, where the lower bound is due to Falconer [8] and the upper bound comes e.g. from a coloring one constructs using a tiling of regular hexagons with circumradius slightly less than 1. Erdős conjectured that $m_1(\mathbb{R}^2) < 1/4$ so that (1) would yield an alternative proof of Falconer's result. So far the best known results on $m_1(\mathbb{R}^2)$ are the lower bound $m_1(\mathbb{R}^2) \geq 0.2293$ by Croft [5] and the upper bound $m_1(\mathbb{R}^2) \leq 12/43 \approx 0.2790$ by Székely [13]. We compute new upper bounds for $m_1(\mathbb{R}^n)$ for dimensions $n = 2$, $\ldots$, 24 based on a strengthening of our main theorem by so-called clique inequalities, e.g. we show $m_1(\mathbb{R}^n) \leq 0.2683$. These new upper bounds for $m_1(\mathbb{R}^n)$ imply by (1) new lower bounds for $\chi_m(\mathbb{R}^n)$ in dimensions 3, $\ldots$, 24.

If one considers sets which avoid more than one distance one can ask how N distances can be chosen so that the extreme density becomes as small as possible: What is the value of $\inf\{\, m_{d_1,\ldots,d_N}(\mathbb{R}^n) : d_1, \ldots, d_N > 0 \,\}$ for fixed N?

Recently, Bukh [3], using harmonic analysis and ideas resembling Szémeredi's regularity lemma, showed that $\inf\{ m_{d_1,\ldots,d_N}(\mathbb{R}^n) : d_1, \ldots, d_N > 0 \}$ drops exponentially to zero in N. This implies a theorem of Furstenberg, Katznelson, and Weiss [11] that for every subset A in the plane which has positive upper density there is a constant d so that A does not avoid distances larger than d. Their original proof used tools from ergodic theory and measure theory. Alternative proofs have been proposed by Bourgain [1] using elementary harmonic analysis and by Falconer and Mastrand [10] using geometric measure theory. Bukh's result also implies that $m_{d_1,\ldots,d_N}(\mathbb{R}^n)$ becomes arbitrarily small if the distances $d_1, d_2, \ldots, d_N$ become arbitrarily small. This is originally due to Bourgain [1] and Falconer [9]. We give a short proof of the exponential decay using our main theorem.

The idea of linear programming bounds for packing problems of discrete point sets in metric spaces goes back to Delsarte [7] and it has been successfully applied to a variety of situations. Cohn and Elkies [4] were the first who were able to set up a linear programming bound for packing problems in non-compact spaces; by then 30 years since Delsarte's fundamental contribution had gone by. Our main theorem can be viewed as a continuous analogue to their linear programming bounds.

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Forbidden order types

Gyula Károlyi
(joint work with József Solymosi and Géza Tóth)

Throughout this report we will always assume that every point set is in general position in the plane, that is, no three points of the configuration are collinear. Two such configurations are said to be of the same *order type*, if there is a one-to-one correspondence between them which preserves the orientation of each triple. Thus, order types are equivalence classes of configurations. We will say that the order type $\mathcal{T}$ contains the order type $\mathcal{S}$ if some (hence any) configuration in $\mathcal{T}$ contains a subset which belongs to $\mathcal{S}$. Ramsey theoretic aspects of order types have been studied by Nešetřil and Valtr in [11]. Order types play an important role in canonical versions of the Erdős–Szekeres theorem [3]. A connection was first established via the so called 'same type lemma' by Bárány and Valtr [1].

For any integer $n \geq 3$ the vertex set of any convex n-gon belongs to the same order type we denote by $\mathcal{C}_n$. According to the Erdős–Szekeres theorem, there is an integer N_0 such that every order type $\mathcal{T}$ with $|\mathcal{T}| \geq N_0$ contains $\mathcal{C}_n$. Denoting the smallest such number by $F(n)$, it is known [4, 12] that

$$2^{n-2} + 1 \leq F(n) \leq \binom{2n-5}{n-2} + 1,$$

the lower bound conjectured to be tight. This is a truly Ramsey-type result whose relation to Ramsey's theorem is widely explored in e.g. [10]. Motivated by a conjecture of Erdős and Hajnal [2] in graph Ramsey theory, Gil Kalai [6] suggested the following problem. For a fixed order type $\mathcal{T}$, define $F_{\mathcal{T}}(n)$ as the smallest integer N_0 such that any order type of size at least N_0 that does not contain $\mathcal{T}$ necessarily contains $\mathcal{C}_n$. Is it always true that $F_{\mathcal{T}}(n)$ is bounded above by a polynomial function of n? Somewhat surprisingly, the analogue with graph Ramsey theory breaks here. In [7] we have shown the existence of an order type $\mathcal{T}$ with $F_{\mathcal{T}}(n) > 2^{n-2}$, in contrast with the original Erdős–Hajnal problem where a sub-exponential upper bound is known [2].

Our proof however was based on a general result of Nešetřil and Valtr from which it is not easy to extract a concrete order type $\mathcal{T}$ with the above property. One novelty in this report is the exhibition of explicit order types $\mathcal{T}$ for which $F_{\mathcal{T}}(n)$ is exponentially large. Such an order type of size 6 can be obtained, for example, by putting an extra point at the centre of a regular pentagon. Large order types containing neither this, nor $\mathcal{C}_n$ can be constructed by a doubling process similar to the one found in [9].

An other result concerns a complete characterization of order types whose convex hull is a triangle, according to the behavior of the function $F_{\mathcal{T}}(n)$. They each fall in one of the following three categories:

(i) $F_{\mathcal{T}}(n)$ is bounded by a linear function in n;
(ii) $F_{\mathcal{T}}(n)$ is at least quadratic in n but bounded by a polynomial in n;
(iii) $F_{\mathcal{T}}(n)$ is exponentially large in n.

Part of this result originates in [7]. Besides that and the methods involved therein the most crucial element is a new construction that we obtain via modification of Horton's well-known example [5]. The details will be explained in the forthcoming paper [8].

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Sets of unit vectors with small subset sums

Konrad J. Swanepoel

Let $d \geq 2$, $m \geq 2$ and $2 \leq k \leq m-2$. Define $m(k,d)$ to be the largest m such that there exists a d-dimensional normed space with a family $\{x_1, \dots, x_m\}$ of m unit vectors satisfying the *k-collapsing condition*:

$$\forall I \in \binom{\{1,\dots,m\}}{k}, \quad \|\sum_{i\in I} x_i\| \leq 1.$$

Also define $m_0(k,d)$ to be the largest m such that there exists a d-dimensional normed space with a family $\{x_1, \dots, x_m\}$ of m unit vectors satisfying the k-collapsing condition as well as the *balancing condition*:

$$x_1 + \cdots + x_m = o.$$

Balancing and collapsing conditions on sets of unit vectors arise in the study of the local structure of Steiner minimal trees in normed spaces [3, 4, 5, 6, 8]. For a related condition see [7].

The space ℓ_∞^d together with the set of $2d$ unit vectors $\{\pm e_1, \ldots, \pm e_d\}$ shows that $m_0(k,d) \le m(k,d) \le 2d$ for all $k \in \{2, \ldots, m-2\}$.

In [3] it is shown that $1.02^d < m(2,d) < 3^d$ and the upper bound was improved in [5] to $m(2,d) \le 2^{d+1}+1$. The latter result uses the Brunn–Minkowski inequality. We generalise this result to $k \ge 3$ by combining the Brunn–Minkowski inequality with the Hajnal–Szemerédi theorem from graph theory.

Theorem 1. $m(k,d) \le k(1+\frac{2}{k})^d + k$.

The following result is an asymptotic improvement of Theorem 1, but can be seen to be worse for $k \le 7$ when considering the value of the constant c.

Theorem 2. *If* $2 \le k \le \sqrt{m}$, *then* $m(k,d) \le (1+\frac{c \log k}{k^2})^d$ *for some universal* $c > 0$.

For larger k, up to $m - \sqrt{m}$, we obtain almost optimal results.

Theorem 3.

(1) If $\sqrt{m} \le k \le m/4$ *then* $m(k,d) \le 3d$.
(2) If $m/4 < k \le m - \sqrt{m}$ *then* $m(k,d) = 2d$.

For values of k larger than $m - \sqrt{m}$ we obtain the following.

Theorem 4. *If* $k = m - f(m)$ *where* $f(m)$ *is a function going to* ∞ *with* m, *then* $f(m(k,d)) \le \sqrt{d/2}$.

The proofs of the above three theorems use a reduction to $m \times m$ matrices which are in a weak sense perturbations of the identity, together with results on lower bounds of the ranks of such matrices [1, 2]. In order to apply these lower bounds we also have to solve a certain convex optimization problem.

The last theorem above gives no information when $m - k = O(1)$. When $m - k$ is small, we also show the following:

Theorem 5. *If* $k = m - f(m)$, *then* $m(k,d) \le cd^{d+2} + f(m)$.

This gives a better bound than Theorem 4 when $f(m) \le c\sqrt{\frac{\log m}{\log \log m}}$. However, this is still very far from the lower bound of $2d$. Its proof is similar to that of Theorem 1, and uses some further tools from convexity, in particular the separation theorem and the theorem of Carathéodory.

The following lower bound almost matches the first bound in Theorem 2. Its proof uses a simple random construction of sets of Euclidean unit vectors that are almost orthogonal.

Theorem 6. $m(k,d) \ge (1+\frac{8}{k^2})^d$.

The following theorem matches the upper bound in Theorem 2 when $k = \Theta(m^{1/s})$, $s = 2,3,4,\ldots$. Its proof uses the existence of certain codes, such as the Kerdock code.

Theorem 7. *Let* $s \in \{2,3,4,\ldots\}$ *and* $k = \Theta(m^{1/s})$. *Then* $m(k,d) \ge cd^{s/2}$.

For the quantity $m_0(k,d)$, we have the following complete answer, which follows from rank arguments and convex optimization.

Theorem 8. *For any* $k = 2, \ldots, m-2$, $m_0(k,d) = 2d$.

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Allowable double permutation sequences, arrangements of double pseudolines of Michel Pocchiola and their applications to planar families of convex sets

Richard Pollack

(joint work with J. E. Goodman)

1. Introduction

Approximately thirty years ago in [7], Goodman and Pollack introduced a combinatorial encoding of planar point configurations designed to open problems on configurations to purely combinatorial investigation. This encoding, which assigned to each planar configuration of n points, $\{p_1, ..., p_n\}$, a circular sequence of permutations of $1, \ldots, n$, [defined by projecting the set of points on a directed line L and recording the permutation of the indices and then rotating the line L (counterclockwise) and recording each new permutation (occurring when the rotating line L passed through a direction orthogonal to a connecting line of the set] has been used in a number of papers since then, in dual form (as an encoding of line arrangements) as well as in primal form; because the same object encodes pseudoline arrangements as well, it has also been used to derive results on pseudoline arrangements. For a survey of results obtained by this technique, see, e.g., [6]. Recent applications include [18] and [20].

In the present talk, we review the extension of the encoding of point configurations by circular sequences of permutations to an encoding of planar families of disjoint compact convex sets by circular sequences of what we call "double permutations." It turns out that this encoding applies as well to a more general class of objects: families of compact *connected* sets in the plane with a specified arrangement of pairwise tangent and pairwise separating *pseudolines*, and thereby permits us to extend combinatorial results from convex sets to these more general objects. we also present a brief overview of the ideas and some of the results in [12] and in [13].

To fix our ideas, we adopt Grünbaum's model from [15]: the affine plane is represented by the interior of a closed disk Δ, and a pseudoline by an arc joining a pair of antipodal points on $\partial\Delta$.

If C is a point set in the plane, we say that a pseudoline T is *tangent* to C if T meets C, and if C is contained in one of the two closed (pseudo)halfplanes determined by T. If C_1 and C_2 are two disjoint sets, we call T a *double tangent* to the pair C_1, C_2 if T is tangent to both; it is *externally tangent* if C_1 and C_2 lie in the same closed halfplane determined by T; *internally tangent* otherwise.

We will often work with directed pseudolines, i.e, we will specify one of the endpoints of a pseudoline as its terminal point. The new encoding works as follows. Given a planar family $\mathcal{C} = \{C_1, \ldots, C_n\}$ of mutually disjoint compact convex (connected) sets together with a family $\mathcal{A}$ of pseudolines which contain, for each $1 \leq i < j \leq n$, a pseudoline $S_{i,j}$ which separates C_i from C_j in the sense that C_i, C_j lie in each of the two open (pseudo)halfplanes determined by $S_{i,j}$ and four common pseudoline common tangents to C_i, C_j (2 internal and 2 external. We further assume that $\mathcal{C}, \mathcal{A}$ are in "general position" by which we mean; for any $\{i, j\} \neq \{i', j'\}$, no common tangent to C_i, C_j is equal or parallel (we say that 2 pseudolines are parallel if they share common endpoints) to a tangent common to $C_{i'}, C_{j'}$, we project the sets onto a directed line L and denote the endpoints of each projected set C_i by i, i', according to their order on L. This gives, in general, a permutation of the $2n$ indices $1, \ldots, n, 1', \ldots, n'$. We then rotate L counterclockwise, and record the circular sequence consisting of all the "double permutations" of $1, 1', \ldots, n, n'$ that arise in this way. (Notice that if the convex sets are points and we identify i and i', this encoding reduces to the encoding by circular sequences of permutations in the sense of [7].)

It turns out that this simple-minded encoding is strong enough to capture all of the features of the family $\mathcal{C}$ that are essential in determining the (partial and complete) transversals that $\mathcal{C}$ possesses, and that it extends in a natural way to the case where the sets C_i are merely connected.

For recent surveys in geometric transversal theory, see [3, 9, 21]; for recent work on pseudoline arrangements, see [2, Chap. 6] and [6].

2. The sequence of double permutations

Definition 1. *If $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a family of mutually disjoint compact convex sets, we associate to the family $\mathcal{C}$ a circular sequence of permutations of the symbols $1, 1', 2, 2', \ldots, n, n'$, as follows. Project the sets in $\mathcal{C}$ orthogonally onto a* *directed line L, and record the order in which the endpoints of the intervals constituting the projections of the sets C_i occur. This gives a permutation of the indices $1, 1', \ldots, n, n'$ (where the primed indices correspond to the right-hand endpoints of the intervals). If L rotates counterclockwise, the permutation changes every time L becomes orthogonal to an (undirected) double tangent, and we obtain a periodic sequence of permutations of $1, 1' \ldots, n, n'$ which we call the* circular sequence of double permutations of $C_1, \ldots, C_n$. *The* move *from each term to the next is simply the switch of two adjacent indices such as i, j or i', j or i, j' (but not i, i').*

(Notice that instead of thinking of each term of the sequence as defined by projection onto L, we can think of it as arising by sweeping a line orthogonal to L from left to right, and recording the order in which the sweepline enters and then leaves the various sets.) If we adopt the convention that each switch is written in the order in which the indices appear *after* they have switched, then it is easy to see that

(i) the switch ij corresponds to a left-left ij tangent,
(ii) the switch $i'j'$ corresponds to a right-right ij tangent,
(iii) the switch $i'j$ corresponds to a left-right ij tangent,
(iv) the switch ij' corresponds to a right-left ij tangent,

It is also easy to see that, knowing the sequence of ordered switches, we can reconstruct the double-permutation sequence itself: Any term can be reconstructed from the half-period of switches following it (since these are compatible with only one possible order among the indices $1, 1', \ldots, n, n'$), and the remaining terms can be then be written down by successively applying the switches that follow.

This allows us to define the double-permutation sequence in greater generality:

Definition 2. *For a family $\mathcal{C}$ of connected sets, together with a given arrangement $\mathcal{A} = \mathcal{A}_T \cup \mathcal{A}_S$ of double tangent pseudolines and separators, we generate the circular sequence $\mathcal{S}(\mathcal{C}, \mathcal{A})$ of double permutations as follows. We first record the periodic sequence of ordered switches (which is read off from the directions of the $8\binom{n}{2}$ ordered double tangents), and we then (re)construct the double permutations themselves by the method described just above.*

A very natural question arises;

Question 1. *Is every allowable double permutation sequence the double permutation sequence arising from a family of connected sets $\mathcal{C}$ together witha family $\mathcal{A}$ of pseudoline separotors and common tangents?*

This question has been answered affirmatively by Habert and Pocchiola in [12] and in [13] along the following lines. They first extend Ringels theorem [19] that arrangements of pseudolines are connected by triangle switches (also called mutations) by having a pseudoline pass over an adjacent vertex, to double pseudoline arrangements. They then show that given families $\mathcal{C}, \mathcal{A}$ of connected sets and pseudoline separators and common tangents and an $\mathcal{A}'$ which differs from Af by a single triangle switch how to change $\mathcal{C}$ to $\mathcal{C}'$ so that $\mathcal{C}', \mathcal{A}'$ is also a family of connected sets and pseudoline separators and common tangents. Hence if we simply connect a $\mathcal{C}, \mathcal{A}$ consisting of disjoint convex sets $\mathcal{C}$ in general position together with their family of straight line common tangents and separating lines to each pair from $\mathcal{C}$ to the $\mathcal{A}$ which produces the given allowable double permutation sequence.

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What is high-dimensional combinatorics?

Nati Linial

Several key advances have brought a change in the standing of combinatorics within the mathematical sciences: (i) The adoption of the asymptotic perspective. (ii) The development of extremal combinatorics. (iii) The emergence of the probabilistic method. (iv) The computational perspective of the field. What is the next frontier we should pursue in this historical progress of the field? The main thesis of my talk is that we should "go up in dimension". We first recall the notion of a simplicial complex.

Definition 1. *Let V be a finite set of* vertices*. A collection of subsets $X \subseteq 2^V$ is called a* simplicial complex *if it satisfies the following condition: $A \in X$ and $B \subseteq A \Rightarrow B \in X$. A member $A \in X$ is called a* simplex *or a* face *of dimension $|A| - 1$. The dimension of X is the largest dimension of a face in X.*

In theoretical computer science simplicial complexes were used in (i) The study of the evasiveness conjecture, starting with [Kahn, Saks and Sturtevant '83]. (ii) Impossibility theorems in distributed asynchronous computation (starting with [Herlihy, Shavit '93] and [Saks, Zaharoglou '93]).

In combinatorics: (i) Lovász's proof of A. Frank's conjecture on graph connectivity 1977. (ii) Lower bounds on chromatic numbers of Kneser's graphs and hypergraphs (starting with [Lovász '78]). (iii) The study of matching in hypergraphs (starting with [Aharoni, Haxell '00]).

The major challenges that we raise are: (i) To start a systematic attack on topology from a combinatorial perspective, using the extremal/asymptotic paradigm. In particular we hope to introduce the probabilistic method into topology. In the other direction we suggest to use ideas from topology to develop new probabilistic models (random lifts of graphs offer a small step in this direction). We also hope to introduce ideas from topology into computational complexity.

Can we develop a theory of random complexes, similar to random graph theory? Specifically we seek a higher-dimensional analogue to $G(n, p)$. To fix ideas we consider the simplest possible case: Two-dimensional complexes with a full one-dimensional skeleton. Namely, we start with a complete graph K_n and add each triple (=simplex) independently with probability p. This probability space of two-dimensional complexes is denoted by $X(n, p)$.

We recall from Erdős and Rényi's work:

Theorem 2 (ER '60)**.** *The threshold for graph connectivity in $G(n, p)$ is*

$$p = \frac{\ln n}{n}.$$

We next ask when a simplicial complex should be considered connected. Unlike the situation in graphs, this question has many (in fact infinitely many) meaningful answers, i.e.: (i) The vanishing of the first homology (with any ring of coefficients). (ii) Being simply connected (vanishing of the fundamental group).

Theorem 3 (Linial and Meshulam '06)**.** *The threshold for the vanishing of the first homology in $X(n,p)$ over $GF(2)$ is*

$$p = \frac{2\ln n}{n}.$$

This extends to d-dimensional simplicial complexes with a full $(d-1)$-st dimensional skeleton and also, for other coefficient groups. (Most of this was done by Meshulam and Wallach). We still do not know, however:

Question 2. *What is the threshold for the vanishing of the homology with integer coefficients?*

On the vanishing of the fundamental group we have:

Theorem 4 (Babson, Hoffman, Kahle '09)**.** *The threshold for the vanishing of the fundamental group in $X(n,p)$ is near $p = n^{-1/2}$.*

We next move on to some extremal problems and recall:

Theorem 5 (Brown, Erdős, Sós '73)**.** *Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a (triangulation of the) two-sphere. The bound is tight.*

and state:

Conjecture 6. *Every n-vertex two-dimensional simplicial complex with $\Omega(n^{5/2})$ simplices contains a (triangulation of the) torus.*

We can show that if true this bound is tight. This may be substantially harder than the BES theorem, where one actually finds a bi-pyramid. We suspect that such a "local" triangulation of the torus need not exist. With Ehud Friedgut we showed that $\Omega(n^{8/3})$ simplices suffice.

Here is another classical theorem of Erdős in extremal graph theory:

Theorem 7. *For every two integers g and k there exist graphs with girth $\geq g$ and chromatic number $\geq k$.*

With L. Aronshtam (work in progress) we can show:

Theorem 8. *For every two integers g and k there exist two-dimensional complexes with a full one-dimensional skeleton, such that for every vertex x, the link of x is a graph with girth $\geq g$ and chromatic number $\geq k$.*

Much more remains to be done here.

Even very elementary subjects in combinatorics take on a new life when you think high-dimensionally. E.g.: A permutation can be viewed as an $n \times n$ array of zeros and ones where every line (i.e., a row or a column) contains exactly a single 1. What about higher dimensional analogues? Let's start with dimension three. Namely, we consider an $n \times n \times n$ array of zeros and ones A where every line (now with three types of lines) contains exactly a single 1. This is easily seen to be equivalent to the notion of a Latin square. It is an ancient problem to determine or estimate $\mathcal{L}_n$, the number of $n \times n$ Latin squares.

Currently the best known bound is:

Theorem 9 (Van Lint and Wilson)**.** $(\mathcal{L}_n)^{1/n^2} = (1 + o(1))\frac{n}{e^2}$.

The (fairly easy) proof uses two substantial facts about permanents: The proof of the van der Waerden conjecture and Brégman's Theorem. This raises several challenges: (i) Improve this bound (which only determines $\mathcal{L}_n$ up to $e^{o(n^2)}$). (ii) Solve the even higher dimensional cases. (iii) Factorials are, of course, closely related to the Gamma function. Are there higher dimensional analogues of Γ?

Finally, a few words on tensors. Recall

Proposition 1. *The rank of a matrix M is the least number of rank-one matrices whose sum is M.*

Likewise, a three-dimensional tensor A has rank one iff there exist vectors x, y and z such that $a_{ijk} = x_i y_j z_k$. We define the rank of a three-dimensional tensor Z as the least number of rank-one tensors whose sum is Z.

We mention that the following question is still open:

Question 3. *What is the largest rank of an $n \times n \times n$ real tensor.*

It is only known (and easy) that the answer is between $\frac{n^2}{3}$ and $\frac{n^2}{2}$. With A. Shraibman we have constructed a family of examples which suggests

Conjecture 10. *The answer is $(1 + o(1))\frac{n^2}{2}$*

Our ignorance may be somewhat justified since tensor rank is NP-hard to determine (Håstad '90).

Conflict-free coloring of points and rectangles

Gábor Tardos

Consider a finite family S of geometric regions. The geometric hypergraph determined by S has S as it vertex set and has the sets $S_p = \{R \in S \mid p \in R\}$ for the points p in the plane with $|S_p| \geq 2$ as its hyperedges. In this survey talk the best known results were presented about the maximal *chromatic number, conflict-free chromatic number* and minimal *independence number* of a a geometric hypergraph determined by n axis-parallel rectangles.

Recall that a subset of the vertices of a hypergraph is called *independent* if it does not contain a hyperedge and the size of the largest independent set is the *independence number.* A *proper coloring* of a hypergraph is partitioning its vertex set into independent sets and minimal number of parts (color classes) for which this is possible is called the *chromatic number* of the hypergraph. The *conflict-free* *chromatic number* is similarly the minimum number of colors used in a conflict-free coloring, where a vertex coloring is *conflict-free* if each hyperedge H contains a vertex $v \in H$ whose color is different from the color of any other vertex $w \in H$, $w \neq v$. These parameters are connected by obvious relations and therefore the two maximization and the one minimization problem mentioned above are very closely related.

Smorodinsky [4] proved that the chromatic number of the geometric hypergraph determined by n axis-parallel rectangles is $O(\log n)$. As a corollary he obtains an $\Omega(n/\log n)$ lower bound on the independence number of this hypergraph and an $O(\log^2 n)$ upper bound on the conflict-free chromatic number.

With J. Pach we proved [3] that the first two bounds of Smorodinsky are tight op to the constant factor. In particular, we gave an explicit construction of n axis-parallel rectangles that determine a geometric hypergraph with independence number $O(n/\log n)$ and thus its chromatic number is $\Omega(\log n)$. Unfortunately, the construction gives a geometric hypergraph whose conflict-free chromatic number is also $\Theta(\log n)$ (which had been previously achieved by simpler constructions), making the best bound possible for this parameter an open problem. Given an integer parameter $r \geq 2$ we also construct a families S of n axis-parallel rectangles with the following stronger property: if a subset $S' \subseteq S$ of at least $Cr \log r(n/\log n)$ rectangles are selected then there is a point contained in exactly r rectangles in S, all of them in S'. Here C is an absolute constant.

The problem of estimating the same parameters for the following dual hypergraph was also studied. Here a finite set P of points in the plane determine the hypergraph with vertex set P and with hyperedges of the form $P \cap R$, where R is an axis-parallel rectangle with $|P \cap R| \geq 2$.

Although the combinatorial structure of this dual problem seems to be simpler (among other thing one can ignore hyperedges of size at least three and concentrate on a graph coloring problem) the known bounds are very far apart.

Ajwani, Elbassioni, Govindarajan and Ray [1] proved that the (conflict-free) chromatic number of the hypergraph determined n points in the plane is $O(n^{.382}$. In other words this many colors are enough to color the points in such a way that the minimal axis-parallel rectangle determined by any two points of the same color contains a third point of different color. This improves the trivial upper bound of $O(\sqrt{n})$.

From the other side, instead of a construction, the best result is based on properties of a random point set. With Chen, Pach and Szegedy [2] we proved that if n points are selected uniformly at random from the unit square, then with high probability the independence number of the hypergraph they determine is $O(n \log^2 \log n/\log n)$, and therefore the chromatic number is $\Omega(\log n/\log^2 \log n)$. This bound is in enormous distance from the bounds in the other direction. It seems to be hard to figure out whether the maximal chromatic number of such a (hyper)graph is $O(n^{\varepsilon})$ for every $\varepsilon > 0$. Unfortunately, random point sets will not help in deciding this, as the upper bound above on their independence number is almost tight: with high probability it is $\Omega\left(\frac{n \log\log n}{\log n \log\log\log n}\right)$.

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Asymptotics of Ehrhart series of lattice polytopes

Matthias Beck

(joint work with Alan Stapledon)

Fix a positive integer d. If $h(t) = h_0 + h_1\, t + \cdots + h_d\, t^d$ is a nonzero polynomial of degree at most d with nonnegative integer coefficients and $h_0 \geq 1$, then

$$\frac{h(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} g(m) t^m,$$

where $g(m) = \sum_{i=0}^{d} h_i \binom{m+d-i}{d}$ is a polynomial of degree d with rational coefficients. For every positive integer n, define $\mathrm{U}_n h(t)$ to be the polynomial of degree at most d with integer coefficients satisfying

$$\frac{\mathrm{U}_n h(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} g(nm)\, t^m,$$

and write $\mathrm{U}_n h(t) = h_0(n) + h_1(n)\, t + \cdots + h_d(n)\, t^d$. The (Hecke) operator U_n was studied by Gil and Robins in a more general setting [6] and more recently by Brenti and Welker, who proved the following theorem [2, Theorem 1.4].

Theorem 1 (Brenti–Welker)**.** *For any positive integer d, there exists real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{d-1} < \alpha_d = 0$ such th at, if $h(t) = h_0 + h_1\, t + \cdots + h_d\, t^d$ is a polynomial of degree at most d with nonnegative integer coefficient s and $h_0 \geq 1$, then for n sufficiently large, $\mathrm{U}_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ and $\beta_i(n) \to \alpha_i$ as $n \to \infty$.*

A sequence of positive integers $(a_0, \ldots, a_d)$ is *strictly log concave* if $a_i^2 > a_{i-1}a_{i+1}$ for $1 \leq i \leq d-1$ and is *strictly unimodal* if $a_0 < a_1 < \cdots < a_j$ and $a_{j+1} > a_{j+2} > \cdots > a_d$ for some $0 \leq j \leq d$. One easily verifies that if $(a_0, \ldots, a_d)$ is strictly log concave then it is strictly unimodal. An induction argument implies that if the polynomial $a_0 + a_1 t + \cdots + a_d t^d$ has negative real roots then the sequence $(a_0, \ldots, a_d)$ is strictly log concave and hence strictly unimodal.

Let $w = (w_1, \ldots, w_d)$ be a permutation of d elements. A *descent* of w is an index $1 \leq j \leq d-1$ such that $w_{j+1} < w_j$. If $A(d,i)$ denotes the number of permutations of d elements with $i-1$ descents, then the polynomial $A_d(t) = \sum_{i=1}^{d} A(d,i)\, t^i$ is called an *Eulerian polynomial* and the roots of $\frac{A_d(t)}{t}$ are simple, real and negative [3, p. 292, Exercise 3]. The real content of our main result below is the statement that the lower bounds only depend on d.

Theorem 2. *Fix a positive integer d and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exists positive integers m_d and n_d such that, if $h(t)$ is a polynomial of degree at most d with nonnegative integer coefficients and $h_0 \geq 1$, then for $n \geq n_d$, $\mathrm{U}_n h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\beta_i(n) \to \rho_i$ as $n \to \infty$, and the coefficients of $\mathrm{U}_n h(t)$ are positive, strictly log concave, and satisfy $h_i(n) < m_d h_d(n)$ for $0 \leq i \leq d$. Furthermore, if $h_0 + \cdots + h_{i+1} \geq h_d + \cdots + h_{d-i}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor - 1$, with at least one of the above inequalities strict, then we may choose n_d such that, for $n \geq n_d$,*

$$h_0 = h_0(n) < h_d(n) < h_1(n) < \cdots < h_i(n) < h_{d-i}(n) < h_{i+1}(n) < \cdots$$
$$< h_{\lfloor \frac{d+1}{2} \rfloor}(n) < m_d\, h_d(n)\,.$$

Our main motivating example comes from the theory of lattice point enumeration of polytopes. More specifically, let N be a lattice of rank n and set $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. A *lattice polytope* $P \subset N_{\mathbb{R}}$ is the convex hull of finitely many points in N. Fix a d-dimensional lattice polytope $P \subset N_{\mathbb{R}}$ and, for each positive integer m, let $f_P(m) := \#(mP \cap N)$ denote the number of lattice points in the m'th dilate of P. A famous theorem of Ehrhart [5] asserts that $f_P(m)$ is a polynomial in m of degree d, called the *Ehrhart polynomial* of P. Equivalently, the generating series of $f_P(m)$ can be written in the form

$$\frac{\delta_P(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} f_P(m)\, t^m\,,$$

where $\delta_P(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$ is a polynomial of degree at most d with integer coefficients, called the *δ-polynomial* of P. We call $(\delta_0, \delta_1, \ldots, \delta_d)$ the *(Ehrhart) δ-vector* of P; alternative names in the literature include *Ehrhart h-vector* and *h^*-vector* of P. If $h(t) = \delta_P(t)$ then the assumptions of the Theorem 2 hold by a result of Hibi [7], and we deduce the following corollary.

Corollary 3. *Fix a positive integer d and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exists positive integers m_d and n_d such that, if P is a d-dimensional lattice polytope and $n \geq n_d$, then $\delta_{nP}(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\beta_i(n) \to \rho_i$ as $n \to \infty$, and the coefficients of $\delta_{nP}(t)$ are positive, strictly log concave, and satisfy*

$$1 = \delta_0(n) < \delta_d(n) < \delta_1(n) < \cdots < \delta_i(n) < \delta_{d-i}(n) < \delta_{i+1}(n) < \cdots < \delta_{\lfloor \frac{d+1}{2} \rfloor}(n)$$
$$< m_d\, \delta_d(n)\,.$$

We can apply Theorem 2 also to Veronese subrings of graded rings.

It is an open problem to determine the optimal choices for the integers m_d and n_d in Theorem 2 and Corollary 3. In this direction, we can show that for any positive integer d and $n \geq d$, if $h(t)$ satisfies certain inequalities, then $h_{i+1}(n) > h_{d-i}(n)$ for $i = 0, \dots, \lfloor \frac{d}{2} \rfloor - 1$. In particular, this holds when $h(t) = \delta_P(t)$.

We now explain our original motivation for this paper. A triangulation τ of the polytope P with vertices in N is *unimodular* if for any simplex of τ with vertices $v_0, v_1, \dots, v_d$, the vectors $v_1 - v_0, \dots, v_d - v_0$ form a basis of N. While every lattice polytope can be triangulated into lattice simplices, it is far from true that every lattice polytope admits a *unimodular* triangulation (for an easy example, consider the convex hull of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$). The following theorem, however, says that we can obtain a unimodular triangulation if we allow our polytope to be dilated.

Theorem 4 (Knudsen–Mumford–Waterman [8]). *For every lattice polytope P, there exists an integer n such that nP admits a unimodular triangulation.*

There are several conjectures that would strengthen this theorem, most notably, that n only depends on the dimension of P [4]. A recent theorem of Athanasiadis–Hibi–Stanley [1, Theorem 1.3] gives certain inequalities for the δ-vector of P that hold if P admits a regular triangulation. One may hope to use the Athanasiadis–Hibi–Stanley theorem to construct a counter-example to some of the Knudsen–Mumford–Waterman conjectures. However, a consequence of Corollary 3 and its proof is that this approach can not possibly work.

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Distances and eigenvalues

Van Vu

I discussed an identity, found with T. Tao in [1], which gives a connection between the distances from vertices of a simplex (in R^n) to the opposite hyperplanes and the singular values of the matrix formed by this simplex.

Technically speaking, let A be an n by n matrix and d_i the distance from the ith row vector to the subspace formed by the remaining $n-1$ row vectors and $\sigma_1 \geq \cdots \geq \sigma_n$ the singular values of A. Then

$$\sum_{i=1}^{n} d_i^{-2} = \sum_{i=1}^{n} \sigma_i^{-2}.$$

The same result holds for a rectangular matrix. This identity appears useful in situations when one side of the identity is easier to compute than the other. For example, if the matrix has random independent entries, then d_i are easier to control and by this we gain a good estimate for the singular values. For example, if the entries of A are iid Bernoulli random variables (taking value ± 1 with probability half), then one can show that a.s.

$$\sigma_{n-k} = \Omega(k/\sqrt{n})$$

for $k \geq \log n$. This type of result was an important tool in the proof of the Circular Law Conjecture in [1].

Another application concerning concentration of the determinant of a random matrix was discussed, motivated by recent joint work with K. Costello [2]. We proved that if the entries of A are $c_{ij}\xi_{ij}$, where $1 \leq c_{ij} \leq C = O(1)$ are constants, and ξ_{ij} are iid Bernoulli random variables, then a.s, $|\det A| = E(|\det A|)\exp(O(n^{2/3})$. I conjecture that $\exp(O(n^{2/3}))$ can be replaced by $n^{O(1)}$. A connection to the problem of computing determinant was also mentioned.

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On the relation between the shape of an oval K and the geometric permutations of a family F_K of disjoint translates of K

Helge Tverberg

In 2003 A. Asinowski et al. published a paper [1] on that relation. Here an *oval* is a compact convex set in $\mathbb{R}^2$. A *geometric permutation* (GP) of a finite family $F_K = \{c_1 + K, \ldots, c_n + K\}$ is given by a permutation $i_1, \ldots, i_n$ for which there is a straight line meeting the translates in the order $c_{i_1} + K, \ldots, c_{i_n} + K$. Note that $i_1, \ldots, i_n$ is identified with $i_n, \ldots, i_1$.

In [1] it was described how an F_K, if it admits a GP, belongs to one out of 6 Types, depending on the nature of the at most three GP's it admits. Some relations between the shape of K and the Types which an F_K can belong to were given. If an F_K can be of Type x, K was said to admit Type x, $x = 1, \ldots, 6$. A parallellogram, for instance, does not admit Type 4, which means that no three translates of it has three GP's. This property characterizes the parallellograms among the ovals, in fact. I raised the problem of finding a similar characterization of the parallellepipeds in higher dimensions.

Another example is that of an ellipse K, which does not admit Type 4. This means that an F_K can not have both GP's 1234 and 2413. But this does not characterize the ellipse, for any affine image of a K of constant width, or of a regular k-gon, with $k \not\equiv 0 (\text{mod } 4)$, has the same property. I raised the question whether a centrally symmetric K with a smooth boundary and having the property discussed would have to be an ellipse, but on second thoughts I find this to be unlikely even if one adds the condition of strict convexity.

It was also pointed out how useful Minkowski symmetrization is in this area, since F_K and F'_K (with $K' = 1/2(K - K)$) have the same GP's. When K is of constant width, K' is a circle, which explains one of the results just mentioned.

In the last part of the talk the problem of *x-goodness* was discussed. K is said to be *x-good* iff arbitrarily large F_K's of Type x exist. In [1] a nasc. for x-goodness was given, in each of the cases $x = 1, \dots, 5$, while for $x = 6$ one necessary and one sufficient condition were given. In the talk a weakening of the sufficient one was described: In [1] bd K was assumed to be smooth at both endpoints of a certain segment, but one can also assume smoothness only at one end, but now to a specified higher order.

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On the union of cylinders in three dimensions

Esther Ezra

Let $\mathcal{K}$ be a collection of n infinite cylinders in $\mathbb{R}^3$, and let $\mathcal{A}(\mathcal{K})$ denote the three-dimensional arrangement induced by the cylinder boundaries in $\mathcal{K}$, i.e., the decomposition of 3-space into vertices, edges, faces, and three-dimensional cells, each being a maximal connected set contained in the intersection of a fixed subcollection of the cylinder boundaries of $\mathcal{K}$ and not meeting any other cylinder boundary. Let $\mathcal{U} = \bigcup_{K \in \mathcal{K}} K$ denote the union of $\mathcal{K}$. The *combinatorial complexity* of $\mathcal{U}$ is the number of vertices, edges and faces of the arrangement $\mathcal{A}(\mathcal{K})$ appearing on the union boundary. The goal of this talk is to present a nearly-optimal bound on the combinatorial complexity of the union $\mathcal{U}$.

The problem of determining the combinatorial complexity of the union of simply-shaped bodies in d-space has received considerable attention in the past twenty years, although most of the earlier work has concentrated on the planar case.

The case involving *pseudodiscs* (that is, a collection of simply connected planar regions, where the boundaries of any two distinct objects intersect at most twice), arises for Minkowski sums of a fixed convex object with a set of pairwise disjoint convex objects (which is the problem one faces in translational motion planning of a convex robot), and has been studied by Kedem *et al.* [11]. In this case, the union has only linear complexity. Matoušek *et al.* [13, 14] proved that the union of n α-*fat* triangles (where each of their angles is at least α) in the plane has only $O(n)$ holes, and its combinatorial complexity is $O(n \log \log n)$. The constant of proportionality, which depends on the fatness factor α, has later been improved by Pach and Tardos [16]. Extending the study to the realm of curved objects, Efrat and Sharir [8] studied the union of planar convex fat objects. Here we say that a planar convex object c is *α-fat*, for some fixed $\alpha > 1$, if there exist two concentric disks, $D \subseteq c \subseteq D'$, such that the ratio between the radii of D' and D is at most α. In this case, the combinatorial complexity of the union of n such objects, such that the boundaries of each pair of objects intersect in a constant number of points, is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$. See also Efrat and Katz [7] and Efrat [6] for related (and slightly sharper) nearly-linear bounds.

In three and higher dimensions, Boissonnat *et al.* [4] proved that the maximum complexity of the union of n axis-parallel hypercubes in $\mathbb{R}^d$ is $\Theta\left(n^{\lceil d/2 \rceil}\right)$, and that the bound improves to $\Theta\left(n^{\lfloor d/2 \rfloor}\right)$ if all hypercubes have the same size. Pach *et al.* [15] showed that the combinatorial complexity of the union of n nearly congruent arbitrarily oriented cubes in three dimensions is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. Agarwal and Sharir [2] have shown that the complexity of the union of n congruent infinite cylinders is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. In fact, the more general problem studied in [2] involves the union of the Minkowski sums of n pairwise disjoint triangles with a ball (where congruent infinite cylinders are obtained when the triangles become lines), and the nearly quadratic bound is extended in [2] to this case as well. Aronov *et al.* [3] showed that the union complexity of n *κ-round* objects in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where an object c is κ-round if for each $p \in \partial c$ there exists a ball $B \subset c$ that touches p and its radius is at least $\kappa \cdot diam(c)$. The bound is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, for κ-round objects in $\mathbb{R}^4$. Finally, Ezra and Sharir [9] have recently shown that the complexity of the union of n *α-fat* tetrahedra (that is, tetrahedra, each of whose four solid angles at its four respective apices is at least α) of arbitrary sizes in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. This result immediately yields a nearly-quadratic bound on the complexity of the union of arbitrary cubes, and thus generalizes the result of Pach *et al.* [15], who showed this bound only for the case where the cubes have nearly equal size lengths. Each of the above known nearly-quadratic bounds (for the three-dimensional case) is almost tight in the worst case.

To recap, all of the above results indicate that the combinatorial complexity of the union in these cases is roughly "one order of magnitude" smaller than the complexity of the arrangement that they induce. While considerable progress has been made on the analysis of unions in three dimensions, the case of the union of infinite cylinders of arbitrary radii has so far been remained elusive.

In this study we make a significant progress on the problem of bounding the complexity of the union of infinite cylinders of arbitrary radii in 3-space, and show a nearly-quadratic bound on this complexity, thus settling a conjecture of Agarwal and Sharir [2], who showed this bound only for the case where the cylinders are (nearly) congruent. Our bound, which is the first known non-trivial bound for this general problem, is almost tight in the worst case.

Specifically, we show:

Theorem 1. *The complexity of the union of n infinite cylinders is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε. The bound is almost tight in the worst case.*

The analysis is based on some of the ideas presented in [2, 9], and on $(1/r)$-*cuttings* [5], in order to partition space into triangular-prism subcells, so that, on average, the overwhelming majority of the cylinders intersecting a subcell Δ are "good", in the sense that they behave as *functions* within Δ with respect to some direction ρ. Thus vertices of the union that are incident (only) to good cylinders appear on the boundary of the "sandwich region" enclosed between the "ρ-lower" envelope of a subset of these functions and the "ρ-upper" envelope of another subset of such functions, and, as shown in [1, 12], the complexity of this region is nearly-quadratic. It then only remains to analyze the number of other types of vertices (incident to some of the few "bad" cylinders that cross Δ), a task which is handled by the divide-and-conquer mechanism based on our cutting (see below for details).

The problem that we study is a generalization of the case where all the cylinders are equal radii - a problem that has been studied by Agarwal and Sharir [2]. In fact, a simple specialization of our analysis to that case yields the same asymptotic bound on the complexity of the union as above. The analysis, based on our approach, is significantly simpler than the analysis in [2], and can thus replace that of [2]. (Note that we use a variant of some of the ideas given in [2], however, most of the analysis steps taken in [2] are no longer needed.)

We extend our analysis to the case of "cigars" of arbitrary radii, that is, Minkowski sums of line-segments and balls, and show that the bound on the combinatorial complexity of the union is nearly-quadratic in this case as well. This problem has been studied in [2] for the restricted case where the cigars are equal-radii. Here too, our analysis is significantly simpler than that of [2], and, in particular, the original problem is much easier to extend to this case using our new approach than the approach of [2].

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Hardness of embedding simplicial complexes in $\mathbf{R}^d$

Uli Wagner
(joint work with Jiří Matoušek, Martin Tancer)

Does a given (finite) simplicial complex[1] K of dimension at most k admit an embedding into $\mathbf{R}^d$? We consider the computational complexity of this question, regarding k and d as fixed integers. To our surprise, this question has apparently not been explicitly addressed before (with the exception of $k = 1$, $d = 2$ which is graph planarity), as far we could find.

For algorithmic embeddability problems, we focus on *piecewise linear* (PL) embeddings, which can easily be represented on a computer. Let us remark that there are at least two other natural notions of embeddings of simplicial complexes in $\mathbf{R}^d$: *linear embeddings* (also called *geometric realizations*), which are more restricted than PL embeddings, and arbitrary *topological embeddings*, which give us more freedom than PL embeddings.[2]

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[1] We will formally regard a simplicial complex as a geometric object. That is, a simplicial complex is a (finite) collection K that is closed under taking faces and such that any two of the simplices intersect in a common face. The *dimension* of a simplicial complex K is the maximum of the dimensions of its simplices. The *polyhedron* of K, denoted by $|K|$, is the union of all simplices in K. Often we do not strictly distinguish between a simplicial complex and its polyhedron; for example, by an embedding of K in $\mathbf{R}^d$ we really mean an embedding of $|K|$.

A simplicial complex K' is a *subdivision* of K if $|K'| = |K|$ and each simplex of K' is contained in some simplex of K.

[2] Since $|K|$ is compact, we may define an embedding as a continuous map $f : |K| \to \mathbf{R}^d$ that is injective. For a linear embedding, we insist that f be linear on each simplex, so that it is completely specified by the images of the vertices. A PL embedding is one that is linear on some arbitrary subdivision of the original complex.

To illustrate the differences with a familiar example, consider embeddings of 1-dimensional simplicial complexes, i.e., simple graphs, into $\mathbf{R}^2$. For a topological embedding, the image of each edge can be an arbitrary Jordan arc, for a PL embedding it has to be a polygonal arc (made of finitely many straight segments), and for a linear embedding, it must be a single straight segment. For this particular case ($k = 1$, $d = 2$), all three notions happen to give the same class of embeddable complexes, namely, planar graphs (by Fáry's theorem). For higher dimensions, however, there are significant differences.[3]

We thus introduce the decision problem $\mathrm{EMBED}_{k\to d}$, whose input is a simplicial complex K of dimension at most k, and where the output should be YES or NO depending on whether K admits a PL embedding into $\mathbf{R}^d$.

We assume $k \le d$, since a k-simplex cannot be embedded in $\mathbf{R}^{k-1}$. For $d \ge 2k+1$ the problem becomes trivial, since it is well known that *every* finite k-dimensional simplicial complex embeds in $\mathbf{R}^{2k+1}$, even linearly (this result goes back to Menger). In all other cases, i.e., $k \le d \le 2k$, there are both YES and NO instances; for the NO instances one can use, e.g., examples of k-dimensional complexes not embeddable in $\mathbf{R}^{2k}$ due to Van Kampen [17] and Flores [6].

Let us also note that the complexity of this problem is monotone in k by definition, since an algorithm for $\mathrm{EMBED}_{k\to d}$ also solves $\mathrm{EMBED}_{k'\to d}$ for all $k' \le k$.

It is well known that $\mathrm{EMBED}_{1\to 2}$ (graph planarity) is solvable in linear time. Based on planarity algorithms and on a characterization of complexes embeddable in $\mathbf{R}^2$ due to Halin and Jung [9], it is not hard to come up with a polynomial-time decision algorithm for $\mathrm{EMBED}_{2\to 2}$.

Moreover, $\mathrm{EMBED}_{k\to 2k}$ is solvable in polynomial time for every $k \ge 3$. This is due to the fact that the cohomological *van Kampen obstruction* provides a complete characterization of embeddability.[4]

According to a celebrated result of Novikov ([18]; also see, e.g., [10] for an exposition), the following problem is algorithmically unsolvable: Given a d-dimensional simplicial complex, $d \ge 5$, decide whether it is homeomorphic to S^d, the d-dimensional sphere. By a simple reduction we obtain the following result:

Theorem 1. $\mathrm{EMBED}_{(d-1)\to d}$ *(and hence also* $\mathrm{EMBED}_{d\to d}$*) is algorithmically undecidable for every* $d \ge 5$.

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[3]For examples of complexes that are PL, but not linearly, embeddable, see [2, 4]. As for topological versus PL embeddability, it is known that they coincide for (k, d) whenever $d - k \ge 3$ [3], and also for $(k, d) = (2, 3)$. The latter follows from from Theorem 5 of Bing [1] together with the result by Papakyriakopoulos [11] ("Hauptvermutung" for 2-dimensional polyhedra) that any two 2-dimensional polyhedra that are homeomorphic are also PL homeomorphic. However, topological and PL embeddability do *not* always coincide: There is an example of a 4-dimensional complex (namely, the suspension of the Poincaré homology 3-sphere) that embeds topologically, but not PL, into $\mathbf{R}^5$. For this example we are indebted to Colin Rourke (private communication).

[4]This is based on ideas by Van Kampen [17], which were made precise by Shapiro [13] and by Wu [20]. Deciding whether the Van Kampen obstruction vanishes amounts to solving a linear system of equations over the integers, for which several polynomial-time algorithms are available in the literature, see for instance [16].

$k =$ \ $d =$	2	3	4	5	6	7	8	9	10	11	12	13
1	P	+	+	+	+	+	+	+	+	+	+	+
2	P	**?**	NPh	+	+	+	+	+	+	+	+	+
3		**?**	NPh	NPh	P	+	+	+	+	+	+	+
4			NPh	UND	NPh	NPh	P	+	+	+	+	+
5				UND	UND	NPh	NPh	**?**	P	+	+	+
6					UND	UND	NPh	NPh	NPh	**?**	P	+
7						UND	UND	NPh	NPh	NPh	**?**	**?**

TABLE 1. The complexity of $\text{EMBED}_{k\to d}$ (P = polynomial-time solvable, UND = algorithmically undecidable, NPh = NP-hard, + = always embeddable, **?** = no result known).

We remark that, by contrast, linear embeddability is always algorithmically decidable[5]. Our main result is hardness for cases where $d \geq 4$ and k is larger than roughly $\frac{2}{3}d$.

Theorem 2. $\text{EMBED}_{k\to d}$ *is* NP*-hard for every pair* (k, d) *with* $d \geq 4$ *and* $d \geq k \geq \frac{2d-2}{3}$.

Let us briefly mention where the dimension restriction $k \geq (2d-2)/3$ comes from. There is a certain necessary condition for embeddability of a simplicial complex into $\mathbf{R}^d$, called the *deleted product obstruction.* A celebrated theorem of Haefliger and Weber, which is a far-reaching generalization of the ideas of Van Kampen mentioned above, asserts that this condition is also *sufficient* provided that $k \leq \frac{2}{3}d - 1$ (these k are said to lie in the *metastable range*). The condition on k in Theorem 2 is exactly that k must be outside of the metastable range.

There are examples showing that the restriction to the metastable range in the Haefliger–Weber theorem is indeed necessary, in the sense that whenever $d \geq 3$ and $d \geq k > (2d-3)/3$, there are k-dimensional complexes that cannot be embedded into $\mathbf{R}^d$ but the deleted product obstruction fails to detect this. We use constructions of this kind, namely, examples due to Segal and Spież [14], Freedman, Krushkal, and Teichner [5], and Segal, Skopenkov, and Spież [15], as the main ingredient in our proof of Theorem 2.

The current complexity status of $\text{EMBED}_{k\to d}$ is summarized in Table 1. In our opinion, the most interesting currently open cases are $(k, d) = (2, 3)$ and $(3, 3)$. These fall outside the metastable range, and it took the longest to find an example showing that they are not characterized by the deleted product obstruction; see [7]. That example does not seem to lend itself easily to a hardness reduction, though.

[5]For k and d fixed, it even belongs to PSPACE, since since the problem can easily be formulated as the solvability over the reals of a system of polynomial inequalities with integer coefficients, which lies in PSPACE [12].

For all we know, $\text{EMBED}_{k\to d}$ might turn out to be undecidable in all cases except for those listed above as tractable, i.e., $d \leq 2$ or $d = 2k \geq 6$.

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Integer knapsacks: Average behavior of the Frobenius numbers

Iskander Aliev
(joint work with Martin Henk)

For a positive integral vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_N) \in \mathbb{Z}^N_{>0}$ with $\gcd(a_1, a_2, \ldots, a_N) = 1$ and a positive integer b the *knapsack polytope* $P = P(\boldsymbol{a}, b)$ is defined as

$$P = \{\boldsymbol{x} \in \mathbb{R}^N_{\geq 0} : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. The integer programming feasibility problem

(1) Does the polytope P contain an integer vector?

is called the *integer knapsack problem* and is well-known to be NP-complete.

Given the input vector $\boldsymbol{a} \in \mathbb{Z}^N$, the largest integral value b such that the instance of (1) is infeasible is called the *Frobenius number* of $\boldsymbol{a}$, denoted by $g_N = g_N(\boldsymbol{a})$. The Frobenius number plays an important role in the analysis of integer programming algorithms and, vice versa, integer programming algorithms are known to be an effective tool for computing the Frobenius number. The general problem of finding g_N has been traditionally refereed to as the *Frobenius problem.* There is a rich literature on the various aspects of this question. For an impressive list of references see Ramirez Alfonsin [5].

Computing g_N when N is not fixed is an NP-hard problem. For any fixed N the Frobenius number g_N can be found in polynomial time by a sophisticated algorithm due to Kannan. One should mention here that, due to its complexity, Kannan's algorithm has apparently never been implemented.

In the most interesting case $a_i \sim a_j$, $i, j = 1, \ldots, N$, all known upper bounds for $g_N(\boldsymbol{a})$ are of order $\|\boldsymbol{a}\|^2_\infty$, where $\|\cdot\|_\infty$ denotes the maximum norm. In general, one can show that the quantity $\|\boldsymbol{a}\|^2_\infty$ plays a role of a limit for estimating the Frobenius number g_N from above.

The next natural and important question is to derive a good upper estimate for the Frobenius number in average. This problem appears to be hard, and to the best of our knowledge it has firstly been systematically investigated by Arnold [1, 2, 3] and Bourgain and Sinai [4].

In this talk we show that the asymptotic growth of the Frobenius number in average is significantly slower than the growth of the maximum Frobenius number. More precisely, we prove that it does not essentially exceed $\|\boldsymbol{a}\|^{1+1/(N-1)}_\infty$.

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Embeddings of discrete groups and the speed of random walks

Assaf Naor

Let G be a group generated by a finite set S and equipped with the associated left-invariant word metric d_G. For a Banach space X let $\alpha^*_X(G)$ (respectively $\alpha^\#_X(G)$) be the supremum over all $\alpha \geq 0$ such that there exists a Lipschitz mapping (respectively an equivariant mapping) $f : G \to X$ and $c > 0$ such that for all $x, y \in G$ we have $\|f(x) - f(y)\| \geq c \cdot d_G(x, y)^\alpha$. In particular, the *Hilbert compression exponent* (respectively the *equivariant Hilbert compression exponent*) of G is $\alpha^*(G) := \alpha^*_{L_2}(G)$ (respectively $\alpha^\#(G) := \alpha^\#_{L_2}(G)$). We show that if X has modulus of smoothness of power type p, then $\alpha^\#_X(G) \leq \frac{1}{p\beta^*(G)}$. Here $\beta^*(G)$ is the largest $\beta \geq 0$ for which there exists a set of generators S of G and $c > 0$ such that for all $t \in \mathbb{N}$ we have $\mathbb{E}\big[d_G(W_t, e)\big] \geq ct^\beta$, where $\{W_t\}_{t=0}^\infty$ is the canonical simple random walk on the Cayley graph of G determined by S, starting at the identity element. This result is sharp when $X = L_p$, generalizes a theorem of Guentner and Kaminker, and answers a question posed by Tessera. We also show that if $\alpha^*(G) \geq \frac{1}{2}$ then $\alpha^*(G \wr \mathbb{Z}) \geq \frac{2\alpha^*(G)}{2\alpha^*(G)+1}$. This improves the previous bound due to Stalder and Valette. We deduce that if we write $\mathbb{Z}_{(1)} := \mathbb{Z}$ and $\mathbb{Z}_{(k+1)} := \mathbb{Z}_{(k)} \wr \mathbb{Z}$ then $\alpha^*(\mathbb{Z}_{(k)}) = \frac{1}{2-2^{1-k}}$, and use this result to answer a question posed by Tessera on the relation between the Hilbert compression exponent and the isoperimetric profile of the balls in G. We also show that the cyclic lamplighter groups $C_2 \wr C_n$ embed into L_1 with uniformly bounded distortion, answering a question posed by Lee, Naor and Peres. Finally, we use these results to show that edge Markov type need not imply Enflo type.

Beyond planarity: intersection patterns of curves

János Pach
(joint work with Jacob Fox)

According to Euler's formula, every planar graph with n vertices has at most $O(n)$ edges. How much can we relax the condition of planarity without violating the conclusion?

A graph drawn in the plane by possibly crossing curvilinear edges is called a *topological graph*. It is assumed for simplicity that no edge of a topological graph passes through a vertex, no *two* edges have infinitely many points in common, and no *three* edges pass through the same point. A topological graph is *k-quasiplanar* if it has no k pairwise crossing edges. It was shown in [3], [2], and [1] that, for $k = 3$ and 4, all k-quasiplanar graphs with n vertices have $O(n)$ edges. It is conjectured that the same is true for every fixed positive integer $k > 4$. The best known general upper bound was established in [6].

Theorem 1. *Every k-quasiplanar topological graph with n vertices has at most $n(\log n)^{c \log k}$ edges, where c is an absolute constant.*

Better bounds are known if we assume that any two edges cross at most once [14], [16] or that the edges are chords of a convex polygon [5].

Another relaxation of planarity was studied in [13]. Two ℓ-element collections of edges in a topological graph are said to form an *ℓ-grid* if every edge in the first collection crosses all edges in the second.

Theorem 2 ([13]). *For any ℓ, there exists a constant $c_\ell \approx 5^{5^\ell}$ such that every topological graph with n vertices and more than $c_\ell n$ edges contains and ℓ-grid.*

In [8], the dependence of the constant c_ℓ on ℓ was improved to roughly $\sqrt{\ell}$, which is essentially optimal. One can also prove that the existence of at least $c_\ell n \log^* n$ edges in a topological graph with n vertices always guarantees an ℓ-grid in which no pair of edges share an endpoint. Here $\log^* n$ stands for the iterated logarithm of n, and it can be conjectured that the statement remains true without this factor.

Most of the proofs of the above results heavily use the Crossing Lemma [4], [11] and various forms of an inequality of Leighton linking the crossing number and the bisection width of a graph [14], [15], [9]. We use these tools to establish the following statement, from which Theorem 2 can be easily deduced.

Theorem 3. *For any ℓ, there exists a constant c'_ℓ such every $K_{\ell,\ell}$-free intersection graph of n continuous curves in the plane has at most $c'_\ell n$ edges.*

Another important ingredient of the proof is the following variant of the Lipton–Tarjan Separator Theorem for planar graphs [12], [7].

Theorem 4. *Every intersection graph G of n continuous curves in the plane, which has e edges, permits a separator of size at most $ce^{3/4}\sqrt{\log e}$, where c is an absolute constant. In other words, one can remove at most this many curves so that the remaining curves can be divided into two groups of size at most $2n/3$ with no intersection between them.*

An alternative proof of Theorem 3, providing much worse constants c'_ℓ, can be given based on [10].

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On the number of simplicial 3-spheres and 4-polytopes with N facets

Günter M. Ziegler
(joint work with Bruno Benedetti)

1. Question

Is the number of combinatorial types of simplicial 3*-spheres on* N *facets bounded by an exponential function* C^N*?*

This question is fundamental for the construction of a partition functions for quantum gravity [1], where space is modelled by a 3-sphere glued from regular tetrahedra of edge lengths ε, and one is interested in the limit if $N \to \infty$, which corresponds to modelling space by triangulations by regular tetrahedra of edge lengths $\varepsilon \to 0$.

2. Related

A related question asks for the number of simplicial 3-spheres and 4-polytopes on n vertices. Here it is long known that there are only exponentially many polytopes [5], while there are more than exponentially many spheres [11].

3. Local constructibility

In the lower-dimensional case of simplicial 2-spheres, we have the same count for 2-spheres and for 3-polytopes with N facets, due to Steinitz' theorem. The answer is asymptotically of the order of $(\frac{256}{27})^{N/2}$, according to Tutte [12].

An elementary approach to this case, which also gives an exponential upper bound and invites for generalization to higher dimensions, first counts plane "trees of N triangles" (which correspond to triangulations of an $(2N+1)$-gon, so there are less than 2^{2N} of these), and then gluings on the boundary, which amounts to planar matchings in the exterior (which again yields a factor of 2^{2N}).

In 1995 Durhuus and Jonsson [3] introduced a concept that generalizes this approach: A simplicial 3-sphere is *locally constructible* (LC) if it can be obtained from a tree of tetrahedra by successive gluings of adjacent (!) boundary triangles. They showed that there are only exponentially-many LC 3-spheres.

4. Hierarchy

We link the LC concept with the notions of shellability and constructibility that were established in combinatorial topology [2], and thus obtain the following hierarchy for simplicial 3-spheres:

$$\text{polytopal} \Rightarrow \text{shellable} \Rightarrow \text{constructible} \Rightarrow \text{LC}.$$

5. Main Results

Theorem 1. *Every constructible simplicial sphere is LC.*

This result establishes the hierarchy above. We also have an extension to simplicial d-spheres, $d \geq 2$. It depends on a simple lemma, according to which gluing two LC d-pseudomanifolds along a common strongly-connected pure $(d-1)$-complex in the boundary yields an LC d-complex.

Theorem 2. *There are less than 2^{8N} LC simplicial 3-spheres on N facets.*

This result slightly sharpens an estimate by Durhuus and Jonsson. We also extend it to LC d-spheres.

Combination of Theorem 2 with the hierarchy (Theorem 1) yields that there are only exponentially-many simplicial 4-polytopes with a given number of facets. (This answers a question by Kalai; as pointed out by Fukuda at the workshop, this may as well be derived from the fact that there are only exponentially many simplicial 4-polytopes on n vertices by [5].)

More generally, for fixed d we get that there are only exponentially-many shellable d-spheres on N facets. This is interesting when compared with the studies of Kalai [7] and Lee [8], which showed that for $d \geq 4$, there are more than exponentially many shellable d-spheres on n vertices.

Theorem 3. *If a simplicial 3-sphere S contains a triangle L that is knotted such that the fundamental group of its complement in S has no presentation with 3 generators, then S is not LC.*

This result is derived from the fact that if S is an LC 3-sphere and Δ is a facet of S, then $S \setminus \Delta$ is collapsible. By a result by Lickorish [9] this implies that the fundamental group of $S \setminus L$ has a presentation with at most 3 generators.

Combined with the known constructions of simplicial 3-spheres with badly-knotted triangles (which go back to Furch [4]), this yields that not all simplicial 3-spheres are locally constructible. This solves a problem by Durhuus and Jonsson. More precisely, spheres with a knotted triangle are not constructible by [6], but if the knot is not complicated, they can be LC (this we derive from [10]).

The basic question about the number of simplicial 3-spheres with N facets remains, as far as we know, open.

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Geometric configurations and Fourier analysis

Alex Iosevich

The class of problems we study has seen much recent and not so recent activity in harmonic analysis, ergodic theory, geometric combinatorics and number theory. The basic idea is to show that a sufficiently large subset of a vector space contains a suitable copy of a given geometric configuration. The following results due to Furstenberg, Katznelson and Weiss ([7]), proved by ergodic methods, are interesting and influential examples of this phenomenon.

Theorem 1. *Let $E \subset \mathbb{R}^d$, of positive upper Lebesgue density in the sense that*

$$\limsup_{R\to\infty} \frac{\mathcal{L}^d\{E \cap [-R,R]^d\}}{(2R)^d} > 0, \tag{1}$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. Then there exists l_0 such that for $l > l_0$ one can find a pair of points $x, y \in E$ such that $|x-y| = l$.

Theorem 2. *Let $E \subset \mathbb{R}^2$, of positive upper Lebesgue density in the sense of (1). Let E_δ denote the δ-neighborhood of E. Then, given vectors u, v in $\mathbb{R}^2$, there exists l_0 such that for $l > l_0$ and any $\delta > 0$, there exists $\{x, y, z\} \subset E_\delta$ forming a triangle congruent to $\{\mathbf{0}, lu, lv\}$, where $\mathbf{0}$ denotes the origin in $\mathbb{R}^2$.*

Bourgain ([2]) proved that in Theorem 2, E_δ cannot in general be replaced by E. He showed that if $E = \{(x_1, x_2) \in \mathbb{R}^2 : \exists n \in \mathbb{Z} \text{ with } |x_1^2 + x_2^2 - n| < \frac{1}{10}\}$, then E has positive upper Lebesgue density, yet it is not in general possible to find a congruent copy of the configuration $\{\mathbf{0}, u, 2u\}$ inside E. Observe that Bourgain's example involves a degenerate triangle where all three vertices are on the same line and are arranged in an arithmetic progression of length three. On the other hand, Bourgain ([2]) used a Fourier analytic approach to prove the following result for non-degenerate simplexes.

Theorem 3. *Let $E \subset \mathbb{R}^d$, of positive upper Lebesgue density in the sense of (1) and suppose that $1 < k \leq d$. Then E contains a sufficiently large dilate of every non-degenerate k-simplex.*

Tamar Ziegler ([22]) has recently generalized Theorem 2 as follows.

Theorem 4. *Let $E \subset \mathbb{R}^d$, of positive upper Lebesgue density in the sense of (1). Let E_δ denote the δ-neighborhood of E. Let $V = \{\mathbf{0}, v^1, v^2, \dots, v^{k-1}\} \subset \mathbb{R}^d$, where $k \geq 2$ is a positive integer. Then there exists $l_0 > 0$ such that for any $l > l_0$ and any $\delta > 0$ there exists $\{x^1, \dots, x^k\} \subset E_\delta$ congruent to $lV = \{\mathbf{0}, lv^1, \dots, lv^{k-1}\}$.*

The following results can be viewed as discrete analogs of the configuration results of Bourgain, Furstenberg, Katznelson, Weiss, Ziegler and others we just described. Let $E \subset \mathbb{F}_q^d$, $d \geq 2$, where $\mathbb{F}_q$ is a finite field with q elements and $\mathbb{F}_q^d$ is the d-dimensional vector space over $\mathbb{F}_q$. The finite version of the k-point configuration problem that we consider here is the following.

Problem 5. *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$. Let $V = \{\mathbf{0}, v^1, \dots, v^{k-1}\}$, $v^j \in \mathbb{F}_q^d$, be non-degenerate in the sense that $\{v^1, \dots, v^{k-1}\}$ spans a $(k-1)$-dimensional sub-space of $\mathbb{F}_q^d$. Here $\mathbf{0}$ denotes the origin in $\mathbb{F}_q^d$. How large does E need to be to ensure that E contains a congruent copy of V in the sense that there exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$ such that $\tau + O(V) \subset E$?*

Here and throughout, $SO_d(\mathbb{F}_q)$ is the group of orthogonal d by d matrices with entries in $\mathbb{F}_q$. Observe that dilations are not used in Problem 5 because the lack of order in $\mathbb{F}_q$ makes the notion of a sufficiently large dilation meaningless. The following result is joint work with my Ph.D. student Derrick Hart.

Theorem 6 (Hart and Iosevich ([9])**.** *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$, such that $|E| \geq Cq^{d\frac{k-1}{k}+\frac{k-1}{2}}$ with a sufficiently large constant $C > 0$, where $|E|$ denotes the number of elements in E. Then, given $V = \{\mathbf{0}, v^1, \dots, v^{k-1}\}$, $v^j \in \mathbb{F}_q^d$, non-degenerate in the sense of Problem 5, there exists a congruent copy of V in E in the sense of Problem 5.*

Moreover, the number of distinct congruent copies of V contained in E is equal to

$$|E|^k q^{-\binom{k}{2}}(1 + o(1)). \tag{2}$$

We also have the following analog of Theorem 2 where we are able to consider triangles in the plane, not covered in Theorem 6, at the cost of only being able to realize a positive proportion of the triangles. The following result is joint work with Ignacio Uriarte-Tuero and my Ph.D. students, David Covert and Derrick Hart.

Theorem 7 (Covert, Hart, Iosevich and Uriarte-Tuero ([4]))**.** *Let $E \subset \mathbb{F}_q^2$ and let $T_3(E)$ denote the set of non-congruent three point configurations where congruence is defined in Problem 5 above. Suppose that $|E| = \rho q^2$, where $Cq^{-\frac{1}{2}} \leq \rho \leq 1$. Then $|T_3(E)| \geq c\rho q^3$.*

It is also interesting to mention the following distance set finite field model, previously obtained by Rudnev and me ([15]). It should be viewed as the $k = 2$ special case of Theorem 6 since fixing the distance, in the sense described below, determines a two-point configuration up to congruence. The sharpness aspects of this result were established in ([10]).

Theorem 8 (Rudnev and Iosevich ([15]))**.** *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$. Then if $|E| > 2q^{\frac{d+1}{2}}$, then $\Delta(E) = \{\|x - y\|, x, y \in E\} = \mathbb{F}_q$, where $\|x\| = x_1^2 + x_2^2 + \cdots + x_d^2$. Moreover, if the dimension d is odd, the exponent $\frac{d+1}{2}$ cannot be improved even if instead of asking for $\Delta(E) = \mathbb{F}_q$ we simply ask for $|\Delta(E)| \geq cq$.*

In two dimensions, my Ph.D. student Doowon Koh and I ([13]) recently proved that if $|E| \geq Cq^{\frac{4}{3}}$, then $|\Delta(E)| \geq cq$, thus matching the Euclidean exponent obtained by Wolff in ([21]) in the context of the Falconer problem in $\mathbb{R}^2$. This was accomplished using restriction theory for the sphere, developed in the finite field context by Koh and me (see [12] and the references contained therein) and using the discrete analog of the Mattila integral introduced in ([15]).

A *local version* of the question underlying Theorem 1 can be stated as follows. Let $E \subset [0,1]^d$. How large does the Hausdorff dimension of E need to be to ensure that the distance set $\Delta(E) = \{|x - y| : x, y \in E\}$ has positive one-dimensional Lebesgue measure, where $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$. This is known as the Falconer distance problem, introduced in [6]. This problem can be viewed as a continuous analog of the Erdős distance problem. See, for example, [16] and the references contained therein. It is shown in [6], using the set obtained by a suitable scaling of the thickened integer lattice that the best result we can hope for is the following.

Conjecture 9. *(Falconer distance conjecture) Let $E \subset \mathbb{R}^d$ with $dim_{\mathcal{H}}(E) > \frac{d}{2}$. Then $\mathcal{L}^1(\Delta(E)) > 0$, where $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure.*

The best known result in this direction, due to Wolff ([21]) in the plane and to Erdogan ([5]) in higher dimensions, says that $\mathcal{L}^1(\Delta(E))$ is indeed positive if the Hausdorff dimension of E, $dim_{\mathcal{H}}(E)$, is greater than $\frac{d}{2} + \frac{1}{3}$. The proofs are Fourier analytic in nature and rely, at least in higher dimensions, on bi-linear extension theory.

The theme of analytic bounds implying geometric and arithmetic results has a long and distinguished history in harmonic analysis. A particularly well-known example of this phenomenon is the fact that the restriction conjecture for the Fourier transform implies the Besicovitch-Kakeya conjecture which says that the Hausdorff dimension of a set in $\mathbb{R}^d$ containing a unit line segment in every direction is d. This stems from C. Fefferman's use of the Besicovitch construction to prove that the ball multiplier is not bounded for any $p \neq 2$. This implication is much more than a black box curiosity. While the implication cannot be completely reversed, the associated ideas have led to significant improvements in our state of knowledge of the restriction problem. See [19], [20] and the references contained therein.

As we mention above, the improvements on the $\frac{d+1}{2}$ exponent in the Falconer distance problem by Bourgain ([3]), Wolff ([21]) and Erdogan ([5]) were accomplished via the estimation of the Mattila integral. Moreover, in all of these papers, the estimation of the Mattila integral is accomplished via the point-wise estimation of the spherical average

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 \mathrm{d}\omega. \tag{3}$$

Furthermore, the structure of the sphere as such did not play a role in these arguments, only the fact that S^{d-1} is smooth, convex and has non-vanishing Gaussian curvature. In view of this, consider the following generalization of the Mattila integral, studied, for example, in ([1]) and ([11]): $\mathcal{M}_K = \int_1^\infty \left(\int_{\partial K} |\widehat{\mu}(t\omega)|^2 \mathrm{d}\omega_K \right)^2 t^{d-1} \mathrm{d}t$, where K is a bounded convex set with a smooth boundary ∂K with everywhere non-vanishing Gaussian curvature. One can check that the boundedness of the Mattila integral implies the positivity of the Lebesgue measure of the generalized distance set $\Delta_K(E) = \{\|x-y\|_K : x, y \in E\}$, where $\|\cdot\|_K$ denotes the norm induced by K. The analog of the spherical average here is $\int_{\partial K} |\widehat{\mu}(t\omega)|^2 \mathrm{d}\omega_K$, where $d\omega_K$ is the Lebesgue measure on ∂K. If μ is supported on a set of Hausdorff dimension s, it is not difficult to see that the best estimate we can expect is

$$\int_{\partial K} |\widehat{\mu}(t\omega)|^2 \mathrm{d}\omega_K \lessapprox t^{-s}, \tag{4}$$

where here and throughout, $X \lessapprox Y$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon t^\epsilon Y$.

The examples that show that this is not in general possible in [18] and [14], both constructed for the case when $\partial K = S^{d-1}$, may not be very relevant to the Falconer distance problem itself. The example in [18] uses a rather specific measure which is not a Frostman measure and replacing it with a different one allows for a much stronger estimate to go through. The examples in [14] use Frostman measures, but they do not contradict (4) when the Hausdorff dimension of the underlying set is close to $\frac{d}{2}$, the conjectured range of the Falconer conjecture.

Let us now focus on the point that all the efforts to date to bound the Mattila integral do not really distinguish between S^{d-1} and a smooth convex ∂K with non-vanishing curvature. The following result follows from the proof of Theorem 1.4 in [14] using the construction from sub-section 3.2 above.

Theorem 10 (Rudnev and Iosevich ([14]))**.** *Suppose that the bound (4) holds for every Frostman measure μ supported on a compact set E of Hausdorff dimension s, with s contained in an interval including $\frac{d}{2}$. Then for every compact smooth convex surface ∂K with everywhere non-vanishing Gaussian curvature,*

$$\#\{\mathbb{Z}^d \cap R\partial K\} \lessapprox R^{d-2}. \tag{5}$$

The estimate (5), for $d \geq 3$, is sometimes called the Schmidt conjecture ([17]). While it is known to hold in the case $\partial K = S^{d-1}$ via a classical and highly non-trivial result in analytic number theory ([8]), it is nowhere near resolution for general smooth convex surfaces with everywhere non-vanishing Gaussian curvature. Thus one cannot realistically expect to obtain anything resembling a sharp estimate for (3) without either coming to grips with the underlying difficulties in the Schmidt conjecture or by explicitly using the fact that one is working with S^{d-1} and not a general smooth convex surface with everywhere non-vanishing Gaussian curvature. When $d = 2$, the estimate (5) is not true. An example due to Konyagin shows that there exists a smooth convex curve Γ, with everywhere non-vanishing curvature, such that there exists a sequence $R_i \to \infty$ for which $\#\{R_i\Gamma \cap \mathbb{Z}^2\} \approx \sqrt{R_i}$. One can use this example and a modification of the proof of Theorem 10 to see that the estimate (4) does not in general hold and thus the Falconer conjecture cannot be proved in this way without distinguishing between the circle and a general smooth convex curve with non-vanishing curvature in a very non-trivial way.

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Random matrices with prescribed row and column sums

Alexander Barvinok

Let us fix a positive integer m-vector $R = (r_1, \ldots, r_m)$ and a positive integer n-vector $C = (c_1, \ldots, c_n)$ such that

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j = N.$$

Let us consider the set $\Sigma(R, C)$ of all $m \times n$ non-negative integer matrices $D = (d_{ij})$ with row sums R and column sums C (also known as *contingency tables*) as a finite probability space with the uniform measure. We are interested in what a random matrix $D \in \Sigma(R, C)$ is likely to look like. It turns out that asymptotically, for large m and n, a random matrix $D \in \Sigma(R, C)$ is very likely to be close to a particular non-negative matrix Z with row sums R and column sums C computed as follows.

Let us define a function

$$g(x) = (x+1)\ln(x+1) - x\ln x \quad \text{for} \quad x \geq 0.$$

Thus g is a strictly concave increasing function. Then Z is the unique matrix maximizing the value of

$$\sum_{ij} g(x_{ij})$$

on the set (transportation polytope) of all non-negative $m \times n$ matrices $X = (x_{ij})$ with row sums R and column sums C. Let us fix a set

$$S \subset \big\{(i,j) : \quad i = 1, \ldots, m; \ j = 1, \ldots, n\big\}$$

of indices. Then, if S is sufficiently large, that is, if $|S| \geq \delta mn$ for some fixed $\delta > 0$, then, as m and n grow, with overwhelming probability the sum of entries in S of a random table $D \in \Sigma(R,C)$ is very close to the sum of entries in S of matrix Z, see [3] for the precise statement. Interestingly, unless all the row sums r_i are equal or all the column sums c_j are equal, matrix Z differs from the "independence table" $Y = (y_{ij})$ defined by $y_{ij} = r_i c_j / N$.

Similarly, let us consider the set $\Sigma_0(R,C)$ of all $m \times n$ matrices $D = (d_{ij})$ with row sums R, column sums C, and entries $d_{ij} \in \{0,1\}$ for all i,j. We consider the set $\Sigma_0(R,C)$, if non-empty, as a finite probability space with the uniform measure. It turns out that a random $D \in \Sigma_0(R,C)$ is very likely to be close to the matrix Z maximizing the sum of entropies

$$H(X) = \sum_{ij} H(x_{ij}) \quad \text{where} \quad H(x) = x \ln \frac{1}{x} + (1-x)\ln\frac{1}{1-x} \quad \text{for} \quad 0 \leq x \leq 1$$

among all $m \times n$ matrices $X = (x_{ij})$ with row sums R, column sums C and entries $0 \leq x_{ij} \leq 1$ for all i,j, see [2] for the precise statement.

These results for random tables are based on the estimates of the cardinalities $|\Sigma(R,C)|$ and $|\Sigma_0(R,C)|$. The following result was proved in [1].

Let

$$\rho(R,C) = \min_{0 < x_i, y_j < 1} \left(\prod_{i=1}^{m} x_i^{-r_i}\right)\left(\prod_{j=1}^{n} y_j^{-c_j}\right)\left(\prod_{ij} \frac{1}{1 - x_i y_j}\right).$$

Then

$$\rho(R,C) \ \geq \ |\Sigma(R,C)| \ \geq \ N^{-\gamma(m+n)}\rho(R,C)$$

for some absolute constant $\gamma > 0$.

A similar result holds for 0-1 tables [2]. Let

$$\alpha(R,C) = \inf_{x_i, y_j > 0} \left(\prod_{i=1}^{m} x_i^{-r_i}\right)\left(\prod_{j=1}^{n} y_j^{-c_j}\right)\left(\prod_{ij}(1 + x_i y_j)\right).$$

Then

$$\alpha(R,C) \;\geq\; |\Sigma_0(R,C)| \;\geq\; \frac{(mn)!}{(mn)^{mn}} \left(\prod_{i=1}^{m} \frac{(n-r_i)^{n-r_i}}{(n-r_i)!}\right) \left(\prod_{j=1}^{n} \frac{c_j^{c_j}}{c_j!}\right) \alpha(R,C).$$

Stirling's formula implies that the ratio between the lower and the upper bounds is of the order of $(mn)^{-\gamma(m+n)}$ for an absolute constant $\gamma > 0$. Substitutions $x_i = e^{s_i}$, $y_j = e^{t_j}$ convert the problems of computing $\rho(R,C)$ and $\alpha(R,C)$ into convex optimization problems. The typical matrix Z is the solution to the corresponding convex dual problem.

Similar results hold for matrices with prescribed row and column sums and zeros in prescribed positions.

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How to integrate a polynomial over a polytope

Jesús A. De Loera

(joint work with V. Baldoni, N. Berline, M. Köppe, M. Vergne)

Let Δ be a d-dimensional rational simplex inside $\mathbb{R}^n$ and let $f \in \mathbb{Q}[x_1,\dots,x_n]$ be a polynomial with rational coefficients. We consider the problem of how to efficiently compute the *exact* value of the integral of the polynomial f over Δ, which we denote by $\int_\Delta f \mathrm{d}m$. We use here the *integral Lebesgue measure* $\mathrm{d}m$ on the affine hull of the simplex Δ.This normalization of the measure occurs naturally in Euler–Maclaurin formulas for a polytope P, which relate sums over the lattice points of P with certain integrals over the various faces of P. For this measure, the volume of the simplex and every integral of a polynomial function with rational coefficients are *rational numbers*.

The main goals of our work have been to discuss the computational complexity of the problem and to provide practical methods to do the computation that are both theoretically efficient and have reasonable performance in concrete examples.

Computation of integrals of polynomials over polytopes is fundamental throughout applications. We already mentioned summation over lattice points of a polytope. Simplices are the fundamental case to consider for integration since any convex polytope can be triangulated into finitely many simplices.

Before we can state our results let us understand better the input and output of our computations. Our output will always be the rational number $\int_\Delta f \mathrm{d}m$ in the usual binary encoding. The d-dimensional input simplex will be represented by its vertices $\mathbf{s}_1,\dots,\mathbf{s}_{d+1}$ (a V-representation) but note that, in the case of a simplex, one can go from its representation as a system o f linear inequalities (an H-representation) to a V-representation in polynomial time, simply by computing the inverse of a matrix.

The encoding size of Δ is given by the number of vertices, the dimension, and the largest binary encoding size of the coordinates among vertices. Computations with polynomials also require that one specifies concrete data structures for reading the input polynomial and to carry on the calculations. There are several possible choices. One common representation of a polynomial is as a sum of monomial terms with rational coefficients. Some authors assume the representation is *dense* (polynomials are given by a list of the coefficients of all monomials up to a given total degree r), while other authors assume it is *sparse* (polynomials are specified by a list of exponent vectors of monomials with non-zero coefficients, together with their coefficients). Another popular representation is by *straight-line programs.* A straight-line program which encodes a polynomial is, roughly speaking, a program without branches which enables us to evaluate it at any given point. General straight-line programs are *too compact* for our purposes, so instead we restrict to a subclass we call *single-intermediate-use (division-free) straight-line programs* or *SIU straight-line programs* for short. The reader should think that polynomials are represented as fully parenthesized arithmetic expressions involving binary operators + and $\times$.

Now we are ready to state our first result.

Theorem 1 (Integrating general polynomials over a simplex is hard)**.** *The following problem is NP-hard.*
Input:

- *numbers* $d, n \in \mathbb{N}$ *in unary encoding,*
- *affinely independent rational vectors* $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1} \in \mathbb{Q}^n$ *in binary encoding,*
- *an SIU straight-line program* Φ *encoding a polynomial* $f \in \mathbb{Q}[x_1, \ldots, x_n]$ *with rational coefficients.*

Output, in binary encoding:

- *the rational number* $\int_\Delta f \mathrm{d}m$*, where* $\Delta \subseteq \mathbb{R}^n$ *is the simplex with vertices* $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1}$ *and* $\mathrm{d}m$ *is the integral Lebesgue measure of the rational affine subspace* $langle\Delta\rangle$.

But we can also prove the following positive results.

Theorem 2 (Efficient integration of polynomials of fixed *effective* number of variables)**.** *For every fixed number* $D \in \mathbb{N}$*, there exists a polynomial-time algorithm for the following problem.*
Input:

- *integer numbers* d, n, M *in unary encoding,*
- *affinely independent rational vectors* $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1} \in \mathbb{Q}^n$ *in binary encoding,*
- *a polynomial* $f \in \mathbb{Q}[X_1, \ldots, X_D]$ *represented by either an SIU straight-line program* Φ *of formal degree at most* M*, or a sparse or dense monomial representation of total degree at most* M*,*

- *a rational matrix L with D rows and n columns in binary encoding, the rows of which define D linear forms $\mathbf{x} \mapsto \langle \ell_j, \mathbf{x} \rangle$ on $\mathbb{R}^n$.*

Output, in binary encoding:

- *the rational number $\int_\Delta f(\langle \ell_1, \mathbf{x} \rangle, \ldots, \langle \ell_D, \mathbf{x} \rangle) \mathrm{d}m$, where $\Delta \subseteq \mathbb{R}^n$ is the simplex with vertices $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1}$ and $\mathrm{d}m$ is the integral Lebesgue measure of the rational affine subspace $\langle \Delta \rangle$.*

In particular, the computation of the integral of a power of *one* linear form can be done by a polynomial time algorithm. This becomes false already if one considers powers of a quadratic form instead of powers of a linear form. Actually, we prove Theorem 1 by looking at powers Q^M of the Motzkin–Straus quadratic form of a graph.

Corollary 3 (Efficient integration of polynomials of fixed degree)**.** *For every fixed number $M \in \mathbb{N}$, there exists a polynomial-time algorithm for the following problem. Input:*

- *numbers $d, n \in \mathbb{N}$ in unary encoding,*
- *affinely independent rational vectors $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1} \in \mathbb{Q}^n$ in binary encoding,*
- *a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ represented by either an SIU straight-line program Φ of formal degree at most M, or a sparse or dense monomial representation of total degree at most M.*

Output, in binary encoding:

- *the rational number $\int_\Delta f(\mathbf{x}) \mathrm{d}m$, where $\Delta \subseteq \mathbb{R}^n$ is the simplex with vertices $\mathbf{s}_1, \ldots, \mathbf{s}_{d+1}$ and $\mathrm{d}m$ is the integral Lebesgue measure of the rational affine subspace $langle \Delta \rangle$.*

A full-length version of this work will be soon posted in the mathematics ArXiv.

Lower bounds on weak ϵ-nets

Boris Bukh
(joint work with Gabriel Nivasch, Jiří Matoušek)

A common theme in mathematics is approximation of large, complicated objects by simpler, smaller objects of lower complexity. An instance of that in discrete geometry are ϵ-nets. For a family of sets $\mathcal{F}$ in $\mathbb{R}^d$ and a finite set of points $P \subset \mathbb{R}^d$ a point set $N \subset P$ is said to be an *ϵ-net* if every set $F \in \mathcal{F}$ containing at least $\epsilon|P|$ of points from P contains at least one point from N. Though set P itself is a an ϵ-net, often P rather tiny ϵ-net. A notable example is when the set family $\mathcal{F}$ is of finite Vapnik-Chervonenkis dimension, or VC-dimension for short. If $\mathcal{F}$ has finite VC-dimension, then for every P, however large, there is always an ϵ-net of size $O(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$. As an example, VC-dimension is finite for the family of all convex polyhedra in $\mathbb{R}^d$ with at most n faces, for any fixed n.

The VC-dimension of the family of all convex sets in $\mathbb{R}^d$ is infinity, and in fact there is no function f such that every set has an ϵ-net of size at most $f(\epsilon)$.

However, as shown by Alon, Bárány, Füredi and Kleitman [1] in this case there are still nets of size bounded by a function of ϵ if one does away with the requirement that the net is a subset of the set it approximates. Precisely, call $N \subset \mathbb{R}^d$ a *weak ϵ-net* for P if every set $F \in \mathcal{F}$ containing $\epsilon|P|$ points of P contains at least one point of N. In [1] showed that there is always a weak ϵ-net for the family of convex sets in $\mathbb{R}^d$ of size bounded by function of only d and ϵ. We will denote by $f_d(\epsilon)$ the least integer such that for every $P \subset \mathbb{R}^d$ there is a weak ϵ-net for convex sets of size at most $f_d(\epsilon)$. In the sequel when we speak of weak ϵ-nets we will always mean weak ϵ-nets for convex sets.

The bounds that Alon, Bárány, Füredi and Kleitman established are still best known for two dimensions: $f(\epsilon) \leq \epsilon^{-2}$, but the best bound in higher dimensions $f(\epsilon) \leq \epsilon^{-d} \log^{c_d} \frac{1}{\epsilon}$ is due to Chazelle, Edelsbrunner, Grigni, Guibas, Sharir and Welzl[3]. The only lower bound for the size of weak ϵ-nets in any dimension is $1/\epsilon$. This bound follows from the observation that every set P can be partitioned into $1/\epsilon$ parts by parallel hyperplanes, and there has to be at least one point of a weak ϵ-net between each pair of adjacent hyperplanes. Alon, Kaplan, Nivasch, Sharir, Smorodinsky [2] came close to this lower bound when they showed that if $P \subset \mathbb{R}^2$ is in *convex position*, then there is always a weak ϵ-net for P of size $\frac{1}{\epsilon}\alpha(\frac{1}{\epsilon})$ where α is the inverse of the Ackerman function. As α grows extremely slowly, and finding ϵ-nets for sets in convex positions hardly seems easier than for arbitrary sets, it was a good evidence that $1/\epsilon$ might be the true order of magnitude for $f_2(\epsilon)$.

We establish the first superlinear lower bounds on weak ϵ-nets.

Theorem 1. *For every $d \geq 1$ and every $0 < \epsilon < 1/2$ the bound*

$$f_d(\epsilon) \geq c_d \frac{1}{\epsilon} \log^{d-1} \frac{1}{\epsilon}$$

holds. Here c_d is constant that depends only on the dimension d.

For $d = 2$ this bound is larger than the upper bound $\frac{1}{\epsilon}\alpha(1/\epsilon)$ for sets in convex position mentioned above. In particular, the worst set for constructing weak ϵ-nets is necessarily not in convex position.

The construction that establishes the theorem is very simple. For positive real numbers A and B let $A \ll B$ mean $f(A) < B$ for large and sufficiently quickly growing function f (for concreteness, one can take $f(x) = (d+1)!(x+1)^d$). Let $x_{i,j}$ for $i = 1, \ldots, d$ and $j = 1, \ldots, n$ be any numbers satisfying $x_{i_1,j_1} \ll x_{i_2,j_2}$ whenever (i_1, j_1) precedes (i_2, j_2) lexicographically (i.e. if either $i_1 < i_2$ or $(i_1 = i_1) \wedge (j_1 < j_2)$). Let $X_i = \{x_{i,j} : j = 1, \ldots, n\}$. The sets that yields the lower bound in the theorem above is $X = X_1 \times \cdots \times X_d \subset \mathbb{R}^d$.

The main fact that makes this construction work is that every convex set C can be approximated by a combinatorially simple set that behaves almost like C with respect to the points of X. These combinatorially simple sets are called *stairconvex* sets. Here are a couple of equivalent ways to define them:

Definition 2. Stairconvex hull *of two points $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$ and $q = (q_1, \ldots, q_d) \in \mathbb{R}^d$ is defined recursively as:*

(1) If $d = 1$, then $\operatorname{sconv}(p, q)$ *is the interval* $[p_1, q_1]$.

(2) If $d > 1$, and $p_d \le q_d$, then

$$\mathrm{sconv}(p,q) = \mathrm{sconv}((p_1,\dots,p_{d-1}),(q_1,\dots,q_{d-1})) \times \{q_d\} \cup (p_1,\dots,p_{d-1}) \times [p_d,q_d].$$

(3) If $d > 1$, and $q_d \le p_d$, then

$$\mathrm{sconv}(p,q) = \mathrm{sconv}((q_1,\dots,q_{d-1}),(p_1,\dots,p_{d-1})) \times \{p_d\} \cup (q_1,\dots,q_{d-1}) \times [q_d,p_d].$$

A set C is said to be stairconvex if C contains $\mathrm{sconv}(p,q)$ for every $p,q \in C$.

Definition 3. *In $\mathbb{R}^1$ the stairconvex sets are intervals. In $\mathbb{R}^d$ a set C is stairconvex if*

1. *(1) $C_x := \{(x_1,\dots,x_{d-1}) \in \mathbb{R}^{d-1} : (x_1,\dots,x_{d-1},x) \in C\}$ is stairconvex for every x, and*
2. *(2) $C_x \subset C_y$ whenever $x \le y$ and C_y is non-empty.*

Though first definition of stairconvex sets is the one which resembles the definition of convex sets more, it is the second definition, which is of most use. For example, it implies that an arbitrary union of boxes of the form $[x_1,a_1] \times \cdots \times [x_d,a_d]$ for a fixed $a \in \mathbb{R}^d$ and varying x's, is stairconvex. It is in sharp contrast with the situation for convex sets: a union of two convex sets is rarely convex. While enjoying nice combinatorial properties denied to convex sets, stairconvex sets retain many useful properties of convex set such as Helly's and Tverberg's theorems.

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Gale duality bounds for roots of polynomials with nonnegative coefficients

Julian Pfeifle

We bound the location of roots of polynomials that have nonnegative coefficients with respect to a fixed but arbitrary basis of the vector space of polynomials of degree at most d. For this, we interpret the basis polynomials as vector fields in the real plane, and at each point in the plane analyze the combinatorics of the Gale dual vector configuration. We apply our technique to bound the location of roots of Ehrhart and chromatic polynomials. Finally, we give an explanation for the clustering seen in plots of roots of random polynomials.

The Ehrhart polynomial of a d-dimensional lattice polytope Q is a real polynomial of degree d, which has the following two representations:

$$i_Q \;=\; i_Q(z) \;=\; \sum_{j=0}^{d} c_j z^j \;=\; \sum_{i=0}^{d} a_i \binom{z+d-i}{d}.$$

Here we chose the letter z for the independent variable in order to emphasize that we think of i_Q as a polynomial defined over the complex numbers. The coefficients c_0, c_{d-1} and c_d in the first representation are positive, while the others generally can vanish or take on either sign. In contrast, a famous theorem of Stanley [5] asserts that all coefficients a_i of i_Q in the latter representation are nonnegative, $a_i \geq 0$ for $0 \leq i \leq d$.

First bounds obtained in [1] on the location of the roots of i_Q were substantially improved by Braun [2] and Braun & Develin [3]. All of these papers use the nonnegativity of the a_i's, but Braun's crucial new insight is to think of the value $i_Q(z)$ at each $z \in \mathbb{C}$ as a linear combination with nonnegative coefficients of the $d+1$ complex numbers $b_i = b_i(z) = \binom{z+d-i}{d}$. In particular, for z_0 to be a zero of i_Q, there must be a nonnegative linear combination of the $b_i(z_0)$ that sums to 0.

We build on and generalize this approach in several directions. To see how, let $B = \{b_0, \dots, b_d\}$ be any basis of P_d, the $(d+1)$-dimensional vector space of real polynomials of degree at most d in one variable.

- We regard the given basis of P_d as a collection of *vector fields*: for each complex number $z \in \mathbb{C}$, the basis elements $b_0(z), \dots, b_d(z)$ define a configuration $\mathcal{B}(z) = \big(w_0(z), \dots, w_d(z)\big)$ of real vectors $w_j(z) = (\operatorname{Re} b_j(z), \operatorname{Im} b_j(z))^T$ in the plane $\mathbb{R}^2$. This point of view converts the algebraic problem of bounding the location of roots of a polynomial into a combinatorial problem concerning the discrete geometry of vector configurations.
- We analyze the combinatorics of the vector configuration $\mathcal{B}(z)$ in terms of the Gale dual configuration $\mathcal{B}^*(z)$. In particular, there exists a polynomial $f = \sum_{i=0}^{d} a_i b_i(z)$ with nonnegative coefficients $a_i \geq 0$ and a root at $z = z_0$ whenever the vector configuration $\mathcal{B}(z_0)$ has a nonnegative circuit, and this occurs whenever $\mathcal{B}^*(z_0)$ has a nonnegative cocircuit.

  The important point here is that we obtain a semi-explicit expression for $\mathcal{B}^*$ for *any* basis of P_d, not just the binomial coefficient basis. For four common bases of P_d, namely the power basis $b_i = z^i$, the rising and falling factorial bases $b_i = z^{\overline{i}}, z^{\underline{i}}$, and the binomial coefficient basis $b_i = \binom{z+d-i}{d}$ we can make the Gale dual completely explicit.
- In concrete situations one often has more information about the coefficients of f than just nonnegativity. Gale duality naturally allows to incorporate any linear equations and inequalities on the coefficients, and in some cases this leads to additional restrictions on the location of roots.

  As an illustration, we show how the inequality $a_d \leq a_0 + a_1$ that is valid for Ehrhart polynomials further constrains the location of the roots of i_Q. We also study the case of chromatic polynomials, for which Brenti [4] has shown the nonnegativity of the coefficients with respect to the binomial coefficient basis.
- In the case of the binomial coefficient basis, Braun & Develin [3] derive an implicit equation for a curve $\mathcal{C}$ bounding the possible locations of roots of $f = \sum_{i=0}^{d} a_i \binom{z+d-i}{d}$, and our method gives an explicit equation for a real algebraic curve whose outermost oval is precisely $\mathcal{C}$.

The fundamental ingredient of this program is the matrix

$$W \;=\; W(x,y) \;=\; \begin{pmatrix} R_0 & R_1 & \dots & R_d \\ I_0 & I_1 & \dots & I_d \end{pmatrix},$$

where $R_j = R_j(x,y)$ and $I_j = I_j(x,y)$ denote the real and imaginary part of the complex polynomial $b_j = b_j(x+iy)$. The rank of W is 2, so any matrix $\overline{W}$ that is Gale dual to W has size $(d+1)\times(d-1)$. The following proposition gives an explicit representative for $\overline{W}$ involving polynomials p_k, q_k, r_k that depend on the basis B. For four especially relevant bases, we will make the Gale dual $\overline{W}$ completely explicit. These bases are:

- The *power basis*, where $b_i = z^i$;
- the *falling power basis*, where $b_i = z^{\underline{i}} = z(z-1)\cdots(z-i+1)$;
- the *rising power basis*, where $b_i = z^{\overline{i}} = z(z+1)\cdots(z+i-1)$; and
- the *binomial coefficient basis*, where $b_i = \binom{z+d-i}{d}$.

Here $z^0 = z^{\underline{0}} = z^{\overline{0}} = 1$.

Proposition. A Gale dual matrix to W may be chosen to have exactly three non-zero diagonals

$$\overline{W} \;=\; \overline{W}(x,y) \;=\; \begin{pmatrix} p_0 & 0 & 0 & \dots & 0 \\ -q_0 & p_1 & 0 & \dots & 0 \\ r_0 & -q_1 & \ddots & & \vdots \\ 0 & r_1 & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & p_{d-2} \\ & & & r_{d-3} & -q_{d-2} \\ 0 & \dots & & 0 & r_{d-2} \end{pmatrix}.$$

Moreover, its entries may be chosen to lie in $\mathbb{R}[x,y]$. For the four bases considered, we may choose the following explicit values:

b_i	p_k	q_k	r_k
z^i	x^2+y^2	$2x$	1
$z^{\underline{i}}$	$(x-k)^2+y^2$	$2(x-k)-1$	1
$z^{\overline{i}}$	$(x+k)^2+y^2$	$2(x+k)+1$	1
$\binom{z+d-i}{d}$	$(x-k)^2+y^2$	$p_k + r_k - d(d-1)$	p_{k+1-d} .

Note that in the last row, $q_k = 2\big(x-(k-\frac{d-1}{2})\big)^2 + 2y^2 - \frac{d^2-1}{2}$.

Proof. We first prove that the matrix $\overline{W}$ can be chosen to have the displayed triple band structure regardless of the basis B chosen for P_d. For this, define the rational functions $g_k = \frac{b_{k+1}}{b_k} \in \mathbb{R}(z)$ for $0 \le k \le d-1$; specific values for g_k become apparent from the relations $z^{k+1} = z\cdot z^k$, $z^{\underline{k+1}} = (z-k)z^{\underline{k}}$, $z^{\overline{k+1}} = (z+k)z^{\overline{k}}$ and $\binom{z+d-k-1}{d} = \frac{z-k}{z+d-k}\binom{z+d-k}{d}$.

The triple (p_k, q_k, r_k) lists nontrivial coefficients of a real syzygy

$$p_k b_k + q_k b_{k+1} + r_k b_{k+2} \;=\; b_k\big(p_k + g_k q_k + g_k g_{k+1} r_k\big) \;=\; 0$$

whenever

$$\begin{pmatrix} 1 & \operatorname{Re} g_k & \operatorname{Re} g_k g_{k+1} \\ 0 & \operatorname{Im} g_k & \operatorname{Im} g_k g_{k+1} \end{pmatrix} \begin{pmatrix} p_k \\ q_k \\ r_k \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But the displayed matrix with entries in $\mathbb{R}(x, y)$, call it M, obviously has rank at least 1, and rank 2 whenever $\operatorname{Im} g_k(x + iy) \neq 0$, so that such triples certainly exist. Moreover, by multiplying with a common denominator we may assume $p_k, q_k, r_k \in \mathbb{R}[x, y]$, and so the relations $p_k b_k + q_k b_{k+1} + r_k b_{k+2} = 0$ imply that $\overline{W}$ is in fact a Gale dual of W. The concrete syzygies listed above arise by choosing explicit bases for $\ker M$. □

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What are higher-dimensional trees, and how do they look like?

Michael Joswig

(joint work with Sven Herrmann, Anders Jensen, Bernd Sturmfels)

A tree with non-negative edge lengths gives rise to a finite metric space on its set of leaves (or *taxa*) by adding up the edge lengths along the unique paths. A metric arising in this way is *tree-like.* We abbreviate $[n] := \{1, 2, \ldots, n\}$, and $\binom{[n]}{d}$ is the set of all d-element subsets of $[n]$. Our point of departure is a collection of results which we mash up into the following theorem.

Theorem 1. (Bandelt & Dress [1]; Kapranov [8]; Sturmfels & Yu [12]) *For a map $\delta : \binom{[n]}{2} \to \mathbb{R}$ the following are equivalent:*

(1) δ is a tree-like metric (up to translation and scaling),
(2) the tight span of δ is one-dimensional,
(3) the dual graph of $\Sigma_{-\delta}$ is a tree,
(4) $\Sigma_{-\delta}$ is a matroid decomposition.

Some explanations are in order. Bandelt & Dress [1] associate an unbounded polyhedron in $\mathbb{R}^n$ with each finite metric δ, and the polytopal complex formed by the bounded faces is the *tight span* of δ. The tight span is always contractible but, in general, it is higher-dimensional. Sturmfels & Yu suggested to read a metric δ (or rather its negative $-\delta$) as a lifting function on the vertices of $\Delta(2,n)$. The *hypersimplex* $\Delta(d,n)$ is the convex hull of all 0/1-vectors of length n with exactly d ones. The regular polytopal subdivision of $\Delta(2,n)$ induced by $-\delta$ is denoted as $\Sigma_{-\delta}$. The tight span of δ is dual to $\Sigma_{-\delta}$. Gel′fand, Goresky, MacPherson & Serganova [4] showed that a (d,n)-*matroid polytope* is a subpolytope of $\Delta(d,n)$ whose edges are parallel to differences of standard basis vectors. A polytopal decomposition of a hypersimplex into matroid polytopes is a *matroid decomposition.* We suggest to view matroid decomposition of hypersimplices as higher-dimensional analogs of (duals of) abstract trees. This answers the first question in the title.

The *Dressian* $\mathrm{Dr}(d,n)$ is the subfan of the secondary fan of $\Delta(d,n)$ which is induced by the matroid decompositions. It can be seen as spherical polytopal complex in $\mathbb{S}^{\binom{n}{d}-n}$. The Dressian $\mathrm{Dr}(d,n)$ can be seen as a combinatorial (outer) approximation to the *tropical Grassmannians* $\mathrm{Gr}(d,n)$ of Speyer & Sturmfels [11]. These are related, for instance, to questions concerning compactifications of Bruhat-Tits buildings of type $\tilde{A}_{n-1}$ [8, 9]. The Dressian $\mathrm{Dr}(2,n)$ is the space of phylogenetic trees with n labeled leaves studied in [10, §3.5]. Our main new result is the following characterization.

Theorem 2. ([5]) *The matroid decompositions of* $\Delta(3,n)$ *bijectively correspond to the equivalence classes of arrangements of* n *metric trees (on* $n-1$ *labeled taxa).*

An *arrangement of* n *metric trees* is a collection of tree-like metrics $\delta_1, \delta_2, \ldots, \delta_n$ satisfying

$$\delta_i(j,k) \;=\; \delta_j(k,i) \;=\; \delta_k(i,j)$$

A matroid decomposition of $\Delta(3,n)$ induces a matroid decomposition on each face. In particular, this holds for the *contraction facets*, which are isomorphic to $\Delta(2,n-1)$, and hence they yield (dual) trees. One application of our main result is an algorithm to actually compute Dressians which were beyond reach before. In particular, using `Gfan` [6] and `polymake` [3] we could show the following.

Theorem 3. ([5]) *The Dressian* $\mathrm{Dr}(3,7)$ *is a six-dimensional spherical polytopal complex with* f*-vector*

$$(616, 13860, 101185, 315070, 431025, 211365, 30)$$

and integral (reduced) homology

$$\tilde{H}_*\big(\mathrm{Dr}(3,7);\mathbb{Z}\big) \;=\; H_5\big(\mathrm{Dr}(3,7);\mathbb{Z}\big) \;=\; \mathbb{Z}^{7440}\,.$$

It remains to explain how matroid decompositions look like. To this end one can make use of the fact that each regular matroid decomposition of $\Delta(d,n)$ is induced by a regular subdivision of (any of) its vertex figures, which are isomorphic to the product of simplices $\Delta_{d-1} \times \Delta_{n-d-1}$. These are dual to configurations of $n-d$ points in the tropical torus $\mathbb{TA}^{d-1}$ [2, 7].

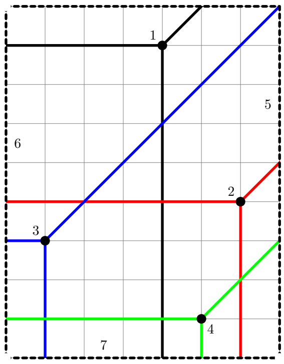

Figure 1. Arrangement of seven metric trees describing a matroid subdivision of $\Delta(3,7)$. This is equivalent to the tropical convex hull of four points in $\mathbb{TA}^2$.

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The sum product problem

József Solymosi

The *sumset* of a finite set of an additive group, A, is defined by

$$A + A = \{a + b : a, b \in A\}.$$

The *productset* and *ratioset* are defined in a similar way.

$$AA = \{ab : a, b \in A\},$$

and

$$A/A = \{a/b : a, b \in A\}.$$

A famous conjecture of Erdős and Szemerédi [4] asserts that for any finite set of integers, M,

$$\max\{|M + M|, |MM|\} \geq |M|^{2-\varepsilon},$$

where $\varepsilon \to 0$ when $|M| \to \infty$. They proved that

$$\max\{|M + M|, |MM|\} \geq |M|^{1+\delta},$$

for some $\delta > 0$. In a series of papers, lower bounds on δ were find. $\delta \geq 1/31$ [9], $\delta \geq 1/15$ [5], $\delta \geq 1/4$ [2], and $\delta \geq 3/14$ [12]. The last two bonds were proved for finite sets of real numbers.

1. Results

Our main result is the following.

Theorem 1. *Let A be a finite set of positive real numbers. Then*

$$|AA||A + A|^2 \geq \frac{|A|^4}{4\lceil \log |A| \rceil}$$

holds.

The inequality is sharp—up to the power of the log term in the denominator—when A is the set of the first n natural numbers. Theorem 1 implies an improved bound on the sum-product problem.

Corollary 2. *Let A be a finite set of positive real numbers. Then*

$$\max\{|A + A|, |AA|\} \geq \frac{|A|^{4/3}}{2\lceil \log |A| \rceil^{1/3}}$$

holds.

To illustrate how the proof goes, we are making two unjustified and usually false assumptions, which are simplifying the proof.

Suppose that AA and A/A have the same size, $|AA| \approx |A/A|$, and many elements of A/A have about the same number of representations as any other. This means that for many reals $s, t \in A/A$ the two numbers s and t have the same multiplicity, $|\{(a, b)|a, b \in A, a/b = s\}| \approx |\{(b, c)|b, c \in A, b/c = t\}|$. A geometric interpretation of the cardinality of A/A is that the Cartesian product $A \times A$ is covered by $|A/A|$ concurrent lines going through the origin. Label the rays from the origin covering the points of the Cartesian product anticlockwise by $r_1, r_2, \dots, r_m$, where $m = |A/A|$.

Our assumptions imply that each ray is incident to $|A|^2/|AA|$ points of $A \times A$. Consider the elements of $A \times A$ as two dimensional vectors. The sumset $(A \times A) + (A \times A)$ is the same set as $(A+A) \times (A+A)$. We take a subset, S, of this sumset,

$$S = \bigcup_{i=1}^{m-1} (r_i \cap A \times A) + (r_{i+1} \cap A \times A) \subset (A+A) \times (A+A).$$

Simple elementary geometry shows that the sumsets in the terms are disjoint and each term has $|r_i \cap A \times A||r_{i+1} \cap A \times A|$ elements. Therefore

$$|S| = |AA|(|A|^2/|AA|)^2 \leq |A+A|^2.$$

After rearranging the inequality we get $|A|^4 \leq |AA||A+A|^2$, as we wanted.

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Convex polygons are cover-decomposable

Géza Tóth
(joint work with Dömötör Pálvölgyi)

Let $P = \{ P_i \mid i \in I \}$ be a collection of planar sets. It is a *k-fold covering* if every point in the plane is contained in at least k members of P. A 1-fold covering is simply called a *covering*.

Definition. *A planar set P is said to be cover-decomposable if the following holds. There exists a constant $k = k(P)$ such that every k-fold covering of the plane with translates of P can be decomposed into two coverings. J. Pach proposed the problem of determining all cover-decomposable sets in 1980. For related problems, conjectures, see* [2], *Chapter 2.1.*

Conjecture (J. Pach)**.** *All planar convex sets are cover-decomposable.*

This conjecture has been verified in three special cases.

Theorem A. *(i)* [6] *Every centrally symmetric open convex polygon is cover-decomposable.*
(ii) [4] *The open unit disc is cover-decomposable.*
(iii) [10] *Every open triangle is cover-decomposable.*

In this note we verify the conjecture for open convex polygons.

Theorem. *Every open convex polygon is cover-decomposable.*

Just like in [6] and in [10], we formulate and solve the problem in its dual form. That is, suppose P is a polygon of n vertices and we have a collection $P = \{ P_i \mid i \in I \}$ of translates of P. Let O_i be the center of gravity of P_i. The collection P is a k-fold covering of the plane if and only if every translate of $\bar{P}$, the reflection of P through the origin, contains at least k points of the collection $O = \{ O_i \mid i \in I \}$.

The collection $P = \{ P_i \mid i \in I \}$ can be decomposed into two coverings if and only if the set $O = \{ O_i \mid i \in I \}$ can be colored with two colors, such that every translate of $\bar{P}$ contains a point of both colors.

Divide the plane into small regions, say, squares, such that each square contains at most one vertex of any translate of $\bar{P}$. If a translate of $\bar{P}$ contains sufficiently many points of O, then it contains many points of O in one of the little squares. We color the points of O separately in each of the squares. If we concentrate on points in just one of the little squares, then instead of translates of $\bar{P}$ we can consider translates of n different wedges, corresponding to the n vertices of P.

First we prove some results about coloring point sets with respect to translates of wedges. Then we formulate the problem precisely in the dual version, and apply the results to prove the Theorem.

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On triconnected and cubic plane graphs on given point sets

Pavel Valtr
(joint work with Alfredo García, Ferran Hurtado, Clemens Huemer, Javier Tejel)

A *geometric graph* G is a a simple finite graph whose vertex set $V(G)$ is a finite set of points in general position in the plane (i.e., no three of them are collinear), and each edge in $E(G)$ is a closed segment whose endpoints belong to $V(G)$. If $V(G) = S$ we also say that the geometric graph G *is on top of* S, or simply that G *is on* S. A geometric graph is a *plane graph* if no two edges cross. That is, two edges in a plane graph may intersect only at a common endpoint. It is also usual to use the expressions *non-crossing geometric graph* or *crossing-free geometric graph* as synonymous for *plane graph.* A (geometric) graph is *cubic*, if the degree of every vertex is three. A (geometric) graph on at least $k+1$ vertices is k*-connected* if it is connected and it remains connected whenever $k-1$ vertices are removed.

For any set S of n points in general position in the plane it is easy to construct a connected plane graph on top of S, even with the additional requirement that it has minimum possible number of edges, $n-1$. For example, we may take the minimum spanning tree of S or we may connect the points by a path visiting the points of S in lexicographically increasing order, say, of their coordinates. Similarly, it is also not difficult to construct a 2-connected plane graph on top of S with the minimum number, n, of edges: We can construct a *polygonization of* S, i.e., a simple polygon whose vertex set is S.

On the opposite direction, there are point sets that do not admit any 4-connected plane graph on top of them. Some examples are given by Dey et al. in the paper [1], where they also provide a necessary and sufficient condition for point sets whose convex hull consists of exactly three vertices. However, a general characterization of the sets of points admitting a 4- or 5-connected plane graph is not known [1].

For the case of 3-connectivity this characterization is quite obvious and was described in [1] as well. Let us recall that we say that a point set S is *in convex position* if each point of S is *extreme* (a vertex of the convex hull of S). If S is in convex position, then every plane triangulation of S contains vertices of degree two, therefore it is impossible to get any 3-connected plane graph on top of S. On the contrary, when S is not in convex position, it is easy to check that the following method produces a 3-connected plane graph on S: Let C be the cycle formed by the edges connecting consecutive vertices of the convex hull of S and let $v \in S$ be any point interior to the convex hull; join v to all the vertices in C and then insert iteratively the remaining points. At each step the point being inserted is connected to the three vertices of the triangular face it falls into.

Notice that in general this algorithm does not produce a 3-connected plane graph using as few edges as possible. In fact, it always produces a *triangulation of S*, i.e., a plane graph on S with the maximum number of edges, in which all faces are triangles with the only possible exception of the outer face.

In this paper we aim to the minimality of the construction, as was already known for 1- and 2-connectivity, and we describe a polynomial algorithm which, given a point set S not in convex position, finds a 3-connected plane graph on S with the minimum number of edges. Achieving good connectivity by adding as few edges as possible is a classic family of problems in graph theory.

Another natural and related problem that we consider here is that of characterizing the point sets that admit a cubic plane graph. Observe that a connected cubic graph on top of S is not necessarily 3-connected; therefore, a specific approach is required. The analogous problem of constructing 1- or 2-regular plane graphs is easily solved using a polygonization on S mentioned above—the edges of a simple polygon P on S form a 2-regular plane graph and, if n is even, taking every second segment in P (or in any plane Hamiltonian path on S) gives a 1-regular plane graph on S.

Throughout this abstract, S denotes a set of $n \geq 4$ points in general position in the plane, $H = H(S)$ denotes the set of vertices of the convex hull of S, $h = h(S)$ denotes the size of H, and $I = I(S) = S \setminus H$ denotes the set of interior points of S.

Here is our main result:

Theorem 1. *Let S be a set of n points in general position in the plane. Suppose that S is not in convex position. Then there is a 3-connected plane graph on S with* $\max\{\lceil 3n/2\rceil, n + h(S) - 1\}$ *edges, and it can be found in polynomial time. Moreover, there is no 3-connected plane graph on S with a smaller number of edges.*

Theorem 1 immediately gives the following characterization of sets admitting 3-connected cubic plane graphs:

Corollary 2. *Let S be a set of $n \geq 4$ points in general position in the plane. Then there is a 3-connected cubic plane graph on S if and only if n is even and $h(S) \leq n/2 + 1$.*

A (geometric) graph on at least $k + 1$ vertices is *k-edge-connected* if it is connected and it remains connected whenever $k - 1$ edges are removed. The above results hold also for 3-edge-connectivity:

Theorem 3. *The statements of Theorem 1 and Corollary 2 also hold when 3-edge-connectivity is considered instead of 3-connectivity.*

If we focus on connecting the points of the set S by a cubic plane graph, without the additional requirement of 3-connectivity, the situation changes substantially. Of course, we need that n, the number of points of S, is even. Our main result in this topic is as follows:

Theorem 4. *Let $n \geq 4$ be an even integer. Then, we have:*
(i) Any set S of n points in general position in the plane satisfying $h(S) \leq 3n/4$ admits a cubic 2-connected plane graph on S.
(ii) If h is an integer such that $3n/4 < h < n-1$, then among sets S of n points in general position with $h(S) = h$, at least one set admits a cubic 2-connected plane graph on S and at least one set admits no cubic plane graph on S.
(iii) Sets S of n points with $h(S) \geq n - 1$ admit no cubic plane graph on S, with the only exception the case $|S| = n = 4$ with $h(S) = n - 1 = 3$.

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The simultaneous packing and covering constants in the plane

Chuanming Zong

In 1950, C. A. Rogers introduced and studied two constants $\gamma(K)$ and $\gamma^*(K)$ for an n-dimensional *convex body* K. Explicitly, $\gamma(K)$ is the smallest positive number r such that there is a translative packing $K + X$ satisfying $E^n = rK + X$, and $\gamma^*(K)$ is the smallest positive number r^* such that there is a lattice packing $K + \Lambda$ satisfying $E^n = r^*K + \Lambda$, where E^n denotes the n-dimensional *Euclidean space* and Λ denotes an n-dimensional *lattice* in E^n. In some references, the two numbers are called the *simultaneous packing and covering constants* for the convex body. Clearly, these constants are closely related to the *packing densities* and the *covering densities* of the convex body, especially to the *Minkowski–Hlawka theorem.*

In 1970 and 1978, S. S. Ryškov and L. Fejes Tóth independently introduced and investigated two related numbers $\rho(K)$ and $\rho^*(K)$, where $\rho(K)$ is the largest positive number r such that one can put a translate of rK into every translative packing $K + X$, and $\rho^*(K)$ is the largest positive number r^* such that one can put a translate of r^*K into every lattice packing $K + \Lambda$.

Clearly, for every convex body K we have

$$\gamma(K) \le \gamma^*(K)$$

and

$$\rho(K) \le \rho^*(K).$$

As usual, let C denote an n-dimensional centrally symmetric convex body. Then, we also have

$$\gamma(C) = \rho(C) + 1$$

and

$$\gamma^*(C) = \rho^*(C) + 1.$$

Let B^n denote the n-dimensional unit ball. Just like the *packing density problem* and the *covering density problem*, to determine the values of $\gamma(B^n)$ and $\gamma^*(B^n)$ is important and interesting. However, so far our knowledge about $\gamma(B^n)$ and $\gamma^*(B^n)$ is very limited. We list the main known results in the following table.

n	2	3	4	5
$\gamma^*(B^n)$	$\sqrt{\frac{4}{3}}$	$\sqrt{\frac{5}{3}}$	$\sqrt{2\sqrt{3}(\sqrt{3}-1)}$	$\sqrt{\frac{3}{2}+\frac{\sqrt{13}}{6}}$
Author	Trivial	Böröczky	Horváth	Horváth

Let $\delta(K)$ and $\delta^*(K)$ denote the maximal *translative packing density* and the maximal *lattice packing density* of K, respectively. A fundamental problem in Packing and Covering is to decide if

$$\delta(K) = \delta^*(K)$$

holds for every convex body. It is easy to see that $\gamma^*(C) \ge 2$ will imply

$$\delta(C) \ge 2\delta^*(C), \tag{1}$$

which will give a negative answer to the previous problem. On the other hand, if $\gamma^*(C) \le 2 - \mu$ holds for a positive constant μ and for every centrally symmetric convex body C, then the Minkowski–Hlawka theorem can be improved to

$$\delta^*(C) \ge \frac{1}{(2-\mu)^n}. \tag{2}$$

In 1950, C. A. Rogers discovered a constructive method by which he deduced

$$\gamma^*(C) \le 3$$

for all n-dimensional centrally symmetric convex bodies (W. Banaszczyk, J. Bourgain and M. Henk did some related works). In 1972, via mean value techniques developed by C. A. Rogers and C. L. Siegel, the above upper bound was improved by G. L. Butler to

$$\gamma^*(C) \le 2 + o(1).$$

This result is fascinating, because it gives hopes to both (1) and (2).

In two and three dimensions, as one can imagine, the situation is much better. In 1978, based on an ingenious idea of I. Fáry, J. Linhart proved that

$$\gamma(K) = \gamma^*(K) \le \tfrac{3}{2}$$

holds for every two-dimensional convex domain, and the upper bound is attained by triangles only. However, just like the packing density problem, to determine the best upper bound for $\gamma^*(C)$ turns out to be much more challenging. Recently C. Zong and obtained

$$\gamma^*(C) \leq 1.2$$

for all two-dimensional centrally symmetric convex domains and

$$\gamma^*(C) \leq 1.75$$

for all three-dimensional centrally symmetric convex bodies. Needless to say, neither of them is optimal. In this paper we will prove the following theorem.

Theorem. *For every two-dimensional centrally symmetric convex domain C we have*

$$\gamma(C) = \gamma^*(C) \leq 2(2-\sqrt{2}) \approx 1.17157\ldots,$$

where the second equality holds if and only if C is an affinely regular octagon.

Universally optimal and balanced spherical codes

Achill Schürmann

(joint work with Henry Cohn, Noam Elkies, Abhinav Kumar and Grigoriy Blekherman, Brandon Ballinger, Noah Giansiracusa, Elizabeth Kelly)

Originally motivated in physics, many people have studied distributions of finitely many points on the unit sphere S^{n-1} (in $\mathbb{R}^n$) minimizing some potential energy. In particular, given a (continuous, decreasing) *potential function* $f\colon (0,4] \to \mathbb{R}$, one may ask for the minimum *f-potential energy*

$$E_f(\mathcal{C}) = \frac{1}{2} \sum_{\substack{x,y\in\mathcal{C} \\ x\neq y}} f\big(|x-y|^2\big).$$

of a *spherical code* $\mathcal{C} \subset S^{n-1}$ of cardinality $|C| = N$.

As f varies, optimal configuration usually vary as well. Sometimes they vary only in surprisingly simple families. The most striking case is when such a family is a single configuration and independent of f. Cohn and Kumar [5] introduced the notion of *universally optimal spherical codes*, for codes minimizing E_f for all *completely monotonic* f, i.e., for all infinitely differentiable f with $(-1)^k f^{(k)}(r) \geq 0$ for all $k \geq 0$ and $r \in (0,4)$. Important examples of completely monotonic functions are all *inverse power laws* $f(x) = 1/r^s$ with $s > 0$. Setting $s = n/2 - 1$ we obtain in particular the *harmonic potential function*, which generalizes the *Coulomb potential* studied in physics. As universally optimal spherical codes minimize f-potential energy for arbitrarily large s, they are also *optimal spherical codes* in the sense that they maximize the minimal distance among elements.

Generalizing *linear programming bounds* by Yudin [13], Kolushov and Andreev, Cohn and Kumar prove in [5] universal optimality for several spherical codes. They show that the regular 600-cell and all *sharp configurations* are universally optimal.

A sharp configuration is a spherical $(2m-1)$-design, in which at most m distinct distances occur. Recall that a code $\mathcal{C} \subset S^{n-1}$ is a *spherical t-design* if

$$(1/|S^{n-1}|) \int_{S^{n-1}} f(x)\mathrm{d}x \quad = \quad (1/|\mathcal{C}|) \sum_{x \in \mathcal{C}} f(x)$$

for every polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $\leq t$. A spherical code with at most m distinct distances, can be a spherical $2m$-design, but not a $2m+1$-design. So sharp configurations give almost highest possible designs among codes with their distance distribution. The full list of all known sharp configurations, respectively universal optima is given in [3].

As the 600-cell is an example of a non-sharp, but universally optimal configuration, it is naturally to ask whether or not there exist other non-sharp universal optima. It is easily seen that universal optima $\mathcal{C}$ have to be *balanced* in the sense of Leech (see [9]), that is they have to be in equilibrium under all possible force laws. So one way to prove completeness of the list of known universal optima is to classify balanced spherical codes. Leech classified them on S^2, by which the list of known universal optima is complete up to dimension 3.

What can be said about dimensions $n \geq 4$? Do there exist more than the known universal optima? In [3] we report on results from massive computer experiments aimed at finding potentially new universal optima. The starting point is a gradient descent search for configurations with minimum *harmonic energy*, that is, with respect to $f(r) = 1/r^{n/2-1}$. The search in dimensions $n \leq 32$, led to 56 balanced, conjectured harmonic optima with at most 64 and at least $2n+1$ points (see `http://aimath.org/data/paper/BBCGKS2006/` for coordinates). Note that for up to $2n$ points it is known that only the regular simplex and the regular crosspolytope are universally optimal. Among the obtained configurations we find only two possible universal optima: one with 40 points in $\mathbb{R}^{10}$ and one with 64 points in $\mathbb{R}^{14}$. Both codes were discovered before (see [11],[8], [10], [6]). Recently, Bannai, Bannai and Bannai [2] proved that these codes define unique 3-class and 4-class association schemes (see also Abdukhalikov, Bannai, Suda [1]). It remains an open problem to prove that these codes actually are examples of yet unknown universal optima.

However, all of the found configurations are interesting and beautiful objects and worth to be studied. We refer to [3] for many fascinating examples. There is hope that analyzing these spherical codes may lead to new insights. One question that naturally arose while studying the obtained balanced spherical codes, is whether or not they all have some special symmetry. A closer look at the known universal optima and at the conjectured harmonic optima $\mathcal{C}$ reveals that they are all *group balanced*, that is, for every $x \in \mathcal{C}$ the stabilizer of x in the automorphism group of $\mathcal{C}$ does fix no linear subspace except the line through $\{\pm x\}$.

Every group balanced code is easily seen to be balanced. So one is easily tempted to **conjecture** that spherical codes are balanced if and only if they are group balanced. By Leech's classification of balanced spherical codes, this conjecture is true for $n \leq 3$. However, as we show in [4], the conjecture is false for all

$n \geq 7$. For $n = 12$ we even find a balanced spherical code having a trivial symmetry group. One main difficulty when studying the above problem is to construct balanced configurations without symmetries. The following Lemma provides a possibility to obtain balanced configurations and the mentioned counterexamples (see [4] for a proof and further details).

Lemma. *Let $\mathcal{C}$ be a spherical t-designs in which each element has at most t distinct inner products other than ± 1. Then $\mathcal{C}$ is balanced.*

A class of configurations to which the lemma applies, comes from *spectral embeddings* of *strongly regular graphs*. Recall that a strongly regular graph with parameters (n, k, a, c) is a non-trivial k-regular graph with n vertices and precisely a (respectively c) common neighbors, for each pair of neighboring (respectively non-neighboring) vertices. It is well known that spectral embeddings of strongly regular graphs are 2-distance sets and 2-designs (see [7]). So by our lemma above these embeddings all give balanced spherical codes.

However, there exist strongly regular graphs without symmetries. One having the lowest dimensional embedding (see [12]) has parameters $(25, 12, 5, 6)$, and is one of the fifteen *Paulus graphs*. The lowest dimensional example of a balanced but not group balanced spherical code we know so far is given by the spectral 7-dimensional embedding of one of the four strongly regular graphs with parameters $(28, 12, 6, 4)$. We refer to [4] for further details and examples.

The initiated study of universally optimal and balanced spherical codes raises many interesting and challenging problems. We think in particular that balanced point configurations deserve further studies. Problems one might consider are for example:

- Find balanced, non group balanced spherical codes in dimension 4, 5 or 6 or prove that such do not exist.
- Classify balanced spherical codes on S^3.
- Study balanced configurations in Euclidean and hyperbolic space.

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Helly numbers and geometric permutations

Xavier Goaoc

Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}^d$ and denote by $\mathcal{T}_k(\mathcal{C})$ the set of k-transversals to $\mathcal{C}$, that is, of k-dimensional affine subspaces that intersect every member of $\mathcal{C}$. Helly's theorem asserts that if $\mathcal{C}$ consists of convex sets then $\mathcal{T}_0(\mathcal{C})$ is nonempty if and only if $\mathcal{T}_0(F)$ is nonempty for any subset $F \subset \mathcal{C}$ of size at most $d+1$. Whether Helly's theorem generalizes to other values of k is a natural question which has been investigated since the 1930's. The answer turns out to be negative in general but positive when the geometry of the objects is adequately constrained. The study of how the geometry of the objects in $\mathcal{C}$ determines the structure of $\mathcal{T}_k(\mathcal{C})$, and subsequent developments of similar flavor, is now designated as *geometric transversal theory* [5].

For the case of *line transversals*, i.e. $k = 1$, Helly-type theorems are known for a variety of objects' shapes: axis-parallel boxes in $\mathbb{R}^d$ [10], families of "thinly distributed" balls in $\mathbb{R}^d$ [6], disjoint translates of a compact convex set in $\mathbb{R}^2$ [11], or disjoint unit balls in $\mathbb{R}^d$ [3]. Most, if not all, proofs use the notion of *geometric permutation*; a geometric permutation of $\mathcal{C}$ is a pair of orderings, one reverse of the other, of the collection $\mathcal{C}$ induced by one of its line transversals. In certain situations, one can replace geometric assumptions by conditions on geometric permutations and still retain bounded Helly numbers. The main idea is to apply the following generalization of Helly's topological Theorem to sets of lines.

Theorem 1 (Matoušek [9])**.** *For any $d \geq 2$, $m \geq 1$ there exists a number $h(d,m)$ such that the following holds. Let $\mathcal{H}$ be a collection of sets in $\mathbb{R}^d$ such that the intersection of any nonempty finite sub-family of $\mathcal{H}$ has at most m path-connected components, each of them contractible. Then $\mathcal{H}$ has a point in common if and only if every $h(m,d)$ members have a point in common.*

Let $\mathcal{C}$ be a collection of disjoint convex sets in $\mathbb{R}^d$. We parameterize a non-horizontal line in $\mathbb{R}^d$ by its intersection points with the hyperplanes $x_d = 0$ and $x_d = 1$; this identifies the space of non-horizontal lines with $\mathbb{R}^{2d-2}$. For any set

$X \subset \mathcal{C}$ we denote by $T_1^+(X)$ the set of non-horizontal lines that intersect X, oriented along increasing x_d. In order to apply Theorem 1 to

$$\mathcal{H} = \left\{T_1^+(A) \mid A \in \mathcal{C}\right\},$$

one has to control both the number and the topology of the connected components of $T_1^+(X)$ for all subsets X of $\mathcal{C}$.

1. In the plane

Given an ordering $\prec$ on $\mathcal{C}$, denote by $\mathcal{K}_\prec(\mathcal{C})$ the set of directions, in $\mathbb{S}^{d-1}$, of oriented lines that intersect all the members of $\mathcal{C}$ in the order $\prec$. It follows from Helly's theorem that

$$\mathcal{K}_\prec(\mathcal{C}) = \bigcap_{X \in \binom{\mathcal{C}}{d}} \mathcal{K}_\prec(X),$$

where $\binom{\mathcal{C}}{d}$ denotes the set of subsets of C of size d. Thus, for $d = 2$ we have that $\mathcal{K}_\prec(\mathcal{C})$ consists of a single interval, contained in an open half-circle. The set of lines that intersect all the members of $\mathcal{C}$ in the order $\prec$ is homotopic to $\mathcal{K}_\prec(\mathcal{C})$ [3, Lemma 14]. It follows that for $d = 2$, the set of lines intersecting $\mathcal{C}$ in any fixed order $\prec$ is contractible. This implies that for any $X \subset \mathcal{C}$, $T_1^+(X)$ consists of contractible components, at most two per geometric permutation of X (see e.g. [4, Theorem 5.6]). Thus, the Helly number of sets of line transversals to members of a collection $\mathcal{C}$ of disjoint convex sets in $\mathbb{R}^2$ can be bounded in terms of the maximum number of geometric permutations of the subsets of $\mathcal{C}$. Let m denote a positive integer and call a family $\mathcal{C}$ of disjoint convex sets *m-regular* if every subset $\mathcal{F} \subset \mathcal{C}$ has at most m geometric permutations. Altogether, we have:

Theorem 2. *For any $m \geq 1$ there exists a number $g(m)$ such that the following holds. A m-regular collection of disjoint convex sets in $\mathbb{R}^2$ has a line transversal if every $g(m)$ of its members have a line transversal.*

2. In higher dimension

Theorem 2 does not extend to dimension $d \geq 3$, as there exists arbitrary large 1-regular families of disjoints translate of a convex set with no line transversal but all of whose proper subsets have a line transversal [8]. It does, however, generalize to collections of disjoint balls:

Theorem 3. *For any $m \geq 1$ and $d \geq 2$ there exists a number $f(m,d)$ such that the following holds. A m-regular collection of disjoint balls in $\mathbb{R}^d$ has a line transversal if every $f(m,d)$ of its members have a line transversal.*

In particular, $f(1,d) \leq 2d-1$, as Matoušek's Theorem can be replaced by Helly's topological Theorem; this immediately generalizes Grünbaum's bound on the Helly number for line transversals to families of thinly distributed balls, as such families are 1-regular. The main ingredient in this extension is the following observation on $\mathcal{K}_\prec(\mathcal{C})$.

Theorem 4 (Borcea et al. [1]). *The directions of all oriented lines intersecting a given finite family of disjoint balls in $\mathbb{R}^d$ in a specific order form a strictly convex subset of the sphere $\mathbb{S}^{d-1}$.*

3. A local version

Another immediate consequence of Theorem 4 is the following Helly-type theorem for isolated line transversals to disjoint balls [1, 3], in the flavor of dimensional versions of Helly's theorem [7].

Theorem 5. *If a line ℓ is an isolated line transversal to a collection $\mathcal{C}$ of disjoint balls in $\mathbb{R}^d$, then it is an isolated line transversal to a subset $\mathcal{P} \subset \mathcal{C}$ of size at most $2d-1$.*

The constant $2d-1$ is tight for all dimensions [2]. Similar statements can be proven for collections of convex polyhedra in $\mathbb{R}^3$, under the condition that the line ℓ is not coplanar with any polyhedron face, and extends to smooth convex semi-algebraic sets of finite complexity, under the condition that no two objects meet the line in the same point. Whether further generalization is possible, in particular to arbitrary collections of disjoint convex sets in $\mathbb{R}^d$, remains an open question.

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Kirchberger's theorem, coloured and multiplied

Imre Bárány
(joint work with J. Arocha, X. Bracho, R. Fabila, L. Montejano)

Finite sets $A, B \subset \mathbb{R}^d$ are *separated* if $\text{conv}A \cap \text{conv}B = \emptyset$. By the separation theorem this is equivalent to the following. Finite sets $A, B \subset \mathbb{R}^d$ are separated if and only if there is a hyperplane H with $A \subset H^+$ and $B \subset H^-$. Here H^+, H^- are the two open halfspaces bounded by H.

Kirchberger's theorem from 1903 [1] states that finite sets $A, B \subset \mathbb{R}^d$ are separated if and only if for every $X \subset A \cup B$ of size at most $d+2$, the sets $A \cap X$ and $B \cap X$ are separated. This means that being separated is a very finite property: it suffices to check small size subsets in order to decide whether two sets are separated or not. This phenomenon is similar to the one encountered in Helly's theorem or in Carathéodory's theorem.

A generalization of this result was proved by A. Pór in 1998 in his diploma thesis (unpublished). First we need a definition. Finite sets $A_1, \ldots, A_r \subset \mathbb{R}^d$ (where $r \geq 2$) are *separated* if $\bigcap_1^r \text{conv}A_i = \emptyset$. By the multiple separation theorem this is same as saying that there are open halfspaces $H_1, \ldots, H_r$ with $A_i \subset H_i$ for each i such that $\bigcap_1^r H_i = \emptyset$. Now Pór's theorem says that the finite sets $A_1, \ldots, A_r \subset \mathbb{R}^d$ are separated if and only if for every $X \subset \bigcup_1^r A_i$ of size at most $(r-1)(d+1)+1$ the sets $X \cap A_1, \ldots, X \cap A_r$ are separated.

In this talk a further generalization of this result was presented. Let $d \geq 1, r \geq 2$ be integers and set $n = (r-1)(d+1)+1$. For every $i \in \{1, \ldots, r\}$ and every $j \in \{1, \ldots, n\}$, there is a finite set $A_{i,j} \subset \mathbb{R}^d$. Let $C_i = \bigcup_{j=1}^n A_{i,j}$ be the "colours" and $G_j = \bigcup_{i=1}^r A_{i,j}$ be the "groups". (These sets may be multisets, actually.) We say that a set $X \subset \bigcup_{i=1}^n \bigcup_{j=1}^n A_{i,j}$ is *separated along the colours* if the sets $X \cap C_1, \ldots, X \cap C_r$ are separated. Finally, $T \subset \bigcup_{i=1}^n \bigcup_{j=1}^n A_{i,j}$ is a *transversal* of the sets system $A_{i,j}$ if $|T \cap G_j| \leq 1$ for every $j \in \{1, \ldots n\}$.

Theorem 1. *If every transversal of the sets system $A_{i,j}$ is separated along the colours, then one of the groups is also separated along the colours.*

This result contains, as a special case, Pór's theorem. To see this one should simply take $A_{i,j} = A_i$ for every i and j. The condition implies that every transversal of this set system is separated along the colours, so one of the groups is also separated along the colours. But each group consists of the same sets $A_1, \ldots, A_r$ which is then separated in the usual sense.

Another interesting special case is Tverberg's theorem [3] stating that every set $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ where $n = (r-1)(d+1)+1$ can be partitioned into r sets $X_1, \ldots, X_r$ with $\bigcap_1^r \text{conv}X_i \neq \emptyset$. This follows from our theorem by taking $A_{i,j} = \{x_j\}$ for every i and j. In this case, clearly, none of the groups is separated along the colours. So there is a transversal T which is not separated along the colours. This transversal is then one element from each G_j, which is just x_j together with a specification that it comes form the i-th colour. Then the sets

$X_i = T \cap C_i$ $(i = 1, \dots, r)$ form a partition of X into r classes and $\bigcap_1^r \mathrm{conv} X_i$ is nonempty.

The proof is based on the following fact which is a modification of a neat result of Sarkaria [2]. We need an artificial tool: let $v_1, \dots, v_r \in \mathbb{R}^{r-1}$ be vectors with a unique (up to a multiplier) linear dependence $v_1 + \cdots + v_r = 0$. Assume that $A_1, \dots, A_r \subset \mathbb{R}^d$ are finite sets, let $A = \bigcup_1^r A_i$. If $a \in A$, then $a \in A_i$ for a unique i and we define a^* as the tensor product of the vector $(a, 1) \in \mathbb{R}^{d+1}$ with v_i. Thus a^* is a $d + 1$ by $r - 1$ matrix.

Lemma 2. *Finite sets $A_1, \dots, A_r \subset \mathbb{R}^d$ are separated if and only if $0 \notin \mathrm{conv}\{a^* : a \in A\}$.*

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Open problems

collected by János Pach

1. Connectivity of weighted rank 3 matroids

(by Günter M. Ziegler with Grigory Mikhalkin)

Let M be a finite simple rank 3 matroid on n points, which have real weights a_i, where the sum of the weights $\sum_{i=1}^{n} a_i$ is non-negative. Is it true that the points of non-negative weight are connected by the lines of non-negative weight?

Notes.

(1) This arises on the way to a "Lefschetz hyperplane theorem for tropical varieties".
(2) This is just the rank 3 case of a more general question: For a rank r matroid, is the order complex given by the non-negative proper flats $(r-3)$-connected? Is it shellable?
(3) The special case where there is only one negative point follows from the shellability of geometric semi-lattices, established by Wachs and Walker.

2. Least-squares matching of point sets under rotation

(by Günter Rote)

This question goes back to Karel Zikan (1991).

For $2n$ points $A_1, \dots, A_n, B_1, \dots, B_n$ in the plane, the least-squares matching is the permutation π that minimizes

$$\sum_{i=1}^{n} \|A_i - B_{\pi(i)}\|^2 \tag{1}$$

It can be computed in $O(n^3)$ time by solving a weighted bipartite matching problem (an assignment problem).

If we allow the set B to be rotated to improve the matching, we can look at the function

$$f(\theta) := \min_{\pi} \sum_{i=1}^{n} \|A_i - R(\theta) \cdot B_{\pi(i)}\|^2,$$

where $R(\theta)$ denotes rotation around the origin by θ. We want to determine the minimum of f. The parameter range of θ is split into intervals, on which the optimum permutation is constant. The function $f(\theta)$ is a sine curve on each interval; between the intervals there are breakpoints where the slope changes. The function $f(\theta)$ can be computed by solving a *parametric* assignment problem, but the running time depends on the number of intervals.

(1) At most many intervals with different optimal permutations can the problem have? Is this number polynomially bounded?
(2) At most how many local minima can the function f have?
(3) Is there a polynomial-time algorithm for finding the best rotation?

(4) Is there a faster algorithm for optimizing (1)?

Notes.

(1) Empirically, the number of intervals is $\Theta(n^2)$ [Zikan 1991].
(2) The objective function can be expanded as follows

$$\sum_{i=1}^{n} \|A_i - B_{\pi(i)}\|^2 = \text{const} - \sum_{i=1}^{n} \langle A_i, B_{\pi(i)} \rangle.$$

The weights $c_{ij} = \langle A_i, B_j \rangle$ of this assignment problem form a rank-2 matrix. In the parametric problem, the weights have the form

$$c_{ij} = \cos\theta \cdot u_{ij} + \sin\theta \cdot v_{ij}. \tag{2}$$

For the parametric assignment problem of the form (2) with *arbitrary* coefficients u_{ij}, v_{ij}, there are examples which have a subexponential (but superpolynomial) number of optimal permutations [Carstensen 1983]. However, in these examples, the costs do not form a rank-2 matrix.

3. Blocking number

(by Chuanming Zong)

Conjecture 1 (C. Zong)**.** *Let K be an n-dimensional convex body and let $b(K)$ denote its blocking number. In other words, $b(K)$ is the smallest number of nonoverlapping translates $K + \mathbf{x}_i$ such that all of them touch K at its boundary and can block any other translate from touching it. Then we have*

$$2n \le b(K) \le 2^n.$$

Notes. Let $D(K)$ be the difference body of K. Like the kissing numbers, we have

$$b(K) = b(D(K)).$$

Let I^n be an n-dimensional parallelopiped. It can be shown that $b(I^n) = 2^n$. However, the conjecture is open even in E^3.

4. Tight polytope around the unit ball

(by Chuanming Zong)

Conjecture 2. *Let B^n be the n-dimensional unit ball centered at the origin and let P_{2n} be an n-dimensional polytope with $2n$ facets and containing B^n. Then*

$$\max_{\mathbf{x} \in P_{2n}} d(\mathbf{o}, \mathbf{x}) \ge \sqrt{n},$$

where equality holds if and only if P_{2n} is a cube circumscribed to B^n.

Notes. This conjecture is open for $n \ge 5$.

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5. Essential covers of the cube by hyperplanes

(by Nati Linial)

This problem is from [1]: We wish to cover the vertices of the discrete cube $\{0,1\}^n$ with as few affine hyperplanes $H_1, \dots, H_k$ as possible, subject to the following requirements:

(1) Every H_i is essential. Namely, for every i there is a point in the cube that only H_i covers.
(2) Every variable appears somewhere. Namely, for every $n \geq j \geq 1$ there is an index i such that the variable x_j appears with a nonzero coefficient in the equation defining the hyperplane H_i.

We ask for the smallest $k = f(n)$ for which this is possible. We can show that

$$n/2 + 1 \geq f(n) \geq \Omega(\sqrt{n}).$$

Can you close this gap?

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6. Additive bases of vector spaces over prime fields

(by Nati Linial)

This problem is from [1]: Let p be a prime. Is there a constant $r = r(p)$ such that the following holds? Let V be a finite-dimensional space over the field of order p and let $B_1, \dots B_r$ be any r bases for V. Then every vector in V can be expressed as a $0,1$ combination of vectors from $B_1, \dots B_r$.
This is already open for $p = 3$.

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7. Very high-dimensional Erdős–Szekeres

(by Jirí Matoušek)

Let $f(d)$ be the smallest integer such that every set of $f(d)$ points in general position in $\mathbb{R}^d$ contains a subset of d^2 points in convex position. Estimate the growth of $f(d)$. (Here d^2 is just for concreteness—$2d$ or d^{10} seem to be equally open.)

8. Representing projective planes by convex sets

(by Jirí Matoušek)

Let us call a set system $(X, \mathcal{F})$ *d-representable* if there are convex sets C_x, $x \in X$, in $\mathbb{R}^d$ such that for every $I \subseteq X$, we have $\bigcap_{x \in I} C_x \neq \emptyset$ if and only if I is contained in some set $F \in \mathcal{F}$. The problem is to decide whether there is a universal constant d_0 such that every finite projective plane, considered as a set system, is d_0-representable. This problem appears in [1]. More generally, we can ask whether all finite *almost disjoint* set systems are d_0-representable, where almost disjoint means that no two sets share two or more points.

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9. Permanent of doubly stochastic matrices with bounded Frobenius norm

(by Alexander Barvinok and Alex Samorodnitsky)

Recall that the *permanent* of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of all $n!$ permutations of the set $\{1, \dots, n\}$. A matrix A is called *doubly stochastic* if it is non-negative and have all row and column sums equal to 1:

$$\sum_{j=1}^{n} a_{ij} = 1 \text{ for } i = 1, \dots, n,$$
$$\sum_{i=1}^{n} a_{ij} = 1 \text{ for } j = 1, \dots, n,$$
$$a_{ij} \geq 0 \text{ for all } \quad i, j.$$

The famous van der Waerden conjecture proved by Falikman and Egorychev states that

$$\operatorname{per} A \ \geq \ \frac{n!}{n^n} \approx e^{-n}\sqrt{2\pi n}$$

if A is an $n \times n$ doubly stochastic matrix.

Problem. Let us fix a constant $\gamma \geq 1$. Is it true that

$$\operatorname{per} A = n^{O(1)} e^{-n}$$

provided A is an $n \times n$ doubly stochastic matrix such that

$$\sum_{ij} a_{ij}^2 \ \leq \ \gamma?$$

Notes. If we impose a stronger condition that

$$a_{ij} \ \leq \ \frac{\gamma}{n} \quad \text{for all} \quad i, j$$

the estimate follows by an inequality conjectured by Minc and proved by Bregman. In fact, it would be nice to be able to replace the ℓ^∞ norm by some smooth norm with better concentration properties, such as ℓ^p for some fixed p.

10. Are there always many more crossing-free spanning trees than Triangulations?

(by Emo Welzl)

Given a finite planar point set P, we denote by $\mathsf{tr}(P)$ and $\mathsf{st}(P)$ the number of triangulations of P and crossing-free spanning trees of P, respectively. The question is whether there is a real constant $c > 1$ such that for every P large enough,

$$\mathsf{st}(P) \geq c^{|P|} \cdot \mathsf{tr}(P) \ .$$

I do not even know how to prove the statement $\mathsf{st}(P) \geq \mathsf{tr}(P)$. If P is a set of n points in convex position, then $\mathsf{tr}(P)$ is roughly 4^n and $\mathsf{st}(P)$ is roughly 6.75^n ("roughly" means up to polynomial factors).

11. Number of vertices of edge-antipodal polytopes

(by Konrad Swanepoel)

Two vertices x and y of a d-polytope P are *antipodal* if there exist two parallel hyperplanes, one through x and one through y, such that P is contained in the closed slab bounded by the two hyperplanes. The polytope P is called *antipodal* if any pair of vertices of P are antipodal. Danzer and Grünbaum [5] proved that an antipodal d-polytope has at most 2^d vertices. Talata [8] introduced the following weaker notion. A d-polytope P is called *edge-antipodal* if the endpoints of any edge of P are antipodal.

Problem. Show that an edge-antipodal d-polytope has at most c^d vertices, for some absolute $c > 0$.

Talata has an example of an edge-antipodal d-polytope that is not antipodal for each $d \geq 4$ (see [4]). It is not immediately clear that there is an upper bound for the number of vertices that depends only on d. This was proved by Pór [6]. In [7] the explicit upper bound of $(\frac{d}{2}+1)^d$ is derived. The d-cube shows that $c \geq 2$ if it exists. A lot is known for the cases $d \in \{3, 4\}$ (see [1], [2], [3], [4]).

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12. Allowable double-permutation sequences

(by Richard Pollack)

An *allowable double-permutation sequence* on the indices $1, 1', \dots, n, n'$ is simply an infinite sequence in which;

(i) Each term is a permutation of the symbols $1, 1', \dots, n, n'$, with each pair i, i' appearing in that order in every term.

(ii) The move from each term to the next consists of a switch between two successive indices other than i, i'.

(iii) Each period is composed of two half-periods. Each term in the second half is the reversal of the corresponding term in the first half, with the primed and unprimed indices interchanged; for example, if the term $13543'5'1'22'4'$ occurs somewhere, then the term $422'1534'5'3'1'$ will occur exactly a half-period later.

(iv) The full period consists of precisely $8\binom{n}{2}$ double permutations.

Given an allowable double-permutation sequence on $1, 2, \dots, n, 1', 2', \dots, n'$, produce n not necessarily convex polygons, $P_1, P_2, \dots, P_n$, and $4\binom{n}{2}$ polygonal common tangents (piecewise linear curves, crossing once per pair, for which the initial and final half-lines have the same slope), 2 internal ($T_{i,j'}$ and $T_{i',j}$) and 2 external ($T_{i,j}$ and $T_{i',j'}$) tangents to each pair of polygons P_i and P_j, so that the sequence of slopes determine the corresponding switches $\{i, j'\}, \{i', j\}, \{i, j\}$, and $\{i', j'\}$ producing the terms following the term $\{1, 1', 2, 2', \dots, n, n'\}$.

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13. Union of lines tangent to three convex sets in $\mathbb{R}^3$

(by Xavier Goaoc)

A line is *tangent* (or *support*) to a convex set if it intersects the set and is contained in a plane that bounds a closed halfspace that contains the set. Given three convex sets A, B and C in $\mathbb{R}^3$, we denote by $\mathcal{S}(A,B,C)$ the union of the lines that are tangent to A, B and C. In other words, $\mathcal{S}(A,B,C)$ is the set of points in $\mathbb{R}^3$ through which passes a line tangent to all three sets.

The set $\mathcal{S}(A,B,C)$ is related to "visual events": as a moving observer crosses $\mathcal{S}(A,B,C)$ the topology of the apparent contour (or silhouette) of the set $A \cup B \cup C$ changes. These sets play a role in visibility questions, e.g aspect graphs in Computer Vision and shadow boundaries computation in Computer Graphics [1], and are usually referred to as "visual event surfaces". Yet, there doesn't seem to be a proof that $\mathcal{S}(A,B,C)$ is two-dimensional in general.

Conjecture 3. *For any pairwise disjoint convex sets A, B and C the set $\mathcal{S}(A,B,C)$ is contained in a countable union of 2-manifolds.*

A weaker version of this conjecture is the following:

Conjecture 4. *For any pairwise disjoint convex sets A, B and C the set $\mathcal{S}(A,B,C)$ is contained in a set of measure 0.*

For any pairwise disjoint convex sets A, B and C, $\mathcal{S}(A,B,C)$ is closed and has empty interior [1]. This implies conjectures 1 and 2 in certain particular cases, for instance if the convex sets are semi-algebraic sets of constant description complexity, but not in general: a closed set with empty interior may have nonzero measure, as the following example shows.

Let φ be a bijection from $\mathbb{N}$ to $\mathbb{Q}$ and U the following subset of $\mathbb{R}$:

$$U = \bigcup_{n \in \mathbb{N}} \left(\varphi(n) - \frac{1}{n^2}, \varphi(n) + \frac{1}{n^2} \right).$$

The set U is open, dense and has measure at most $\frac{\pi^2}{3}$. Thus, $\mathbb{R} \setminus U$ is closed, has empty interior, and infinite measure.

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Reporters: Jiří Matoušek, Helena Nyklová

Participants

Dr. Iskander Aliev
School of Mathematics
Cardiff University
23, Senghennydd Road
GB-Cardiff CF24 4AG

Prof. Dr. Boris Aronov
Department of Computer Science
Polytechnic Institute of NYU
333 Jay Street
Brooklyn , NY 11201
USA

Prof. Dr. Keith M. Ball
Department of Mathematics
University College London
Gower Street
GB-London WC1E 6BT

Prof. Dr. Imre Barany
Alfred Renyi Institute of
Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Prof. Dr. Alexandre I. Barvinok
Department of Mathematics
University of Michigan
Ann Arbor , MI 48109-1109
USA

Prof. Dr. Matthias Beck
Department of Mathematics
San Francisco State University
1600 Holloway Avenue
San Francisco , CA 94132
USA

Boris Bukh
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton , NJ 08544
USA

Prof. Dr. Jesus A. De Loera
Department of Mathematics
University of California, Davis
1, Shields Avenue
Davis , CA 95616-8633
USA

Dr. Ester Ezra
Department of Computer Science
Duke University
Box 90129
Durham , NC 27708-0129
USA

Prof. Dr. Komei Fukuda
Institute for Operations Research
ETH Zürich
CH-8092 Zürich

Prof. Dr. Xavier Goaoc
INRIA Lorraine
Technopole de Nancy-Brabois
615 rue de Jardin Botanique
F-54600 Villers-les-Nancy

Prof. Jacob E. Goodman
Department of Mathematics
The City College of New York
Convent Avenue at 138th Street
New York , NY 10031
USA

Prof. Dr. Martin Henk
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg

Prof. Dr. Alexander Iosevich
Department of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia , MO 65211
USA

Prof. Dr. Michael Joswig
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. Gyula Karolyi
Dept. of Algebra and Number Theory
Eötvös University Budapest
Pazmany Peter Setany 1/C
H-1117 Budapest

Dr. Jan Kyncl
Department of Applied Mathematics
Charles University
Malostranske nam. 25
118 00 Praha 1
CZECH REPUBLIC

Prof. Dr. Nathan Linial
School of Computer Science and
Engineering
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Jiri Matousek
Department of Applied Mathematics
Charles University
Malostranske nam. 25
118 00 Praha 1
CZECH REPUBLIC

Prof. Dr. Assaf Naor
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York , NY 10012-1110
USA

Prof. Dr. Janos Pach
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York , NY 10012-1110
USA

Dr. Julian Pfeifle
Departament de Matematica Aplicada
MA II; Universitat Politecnica de
Catalunya
c/Jordi Girona 1-3
E-08034 Barcelona

Prof. Dr. Richard M. Pollack
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York , NY 10012-1110
USA

Dr. Günter Rote
Institut für Informatik
Freie Universität Berlin
Takustr. 9
14195 Berlin

Prof. Dr. Marcus Schaefer
DePaul University
School of CTI
243 South Wabash, Ste 401
Chicago , IL 60604
USA

Dr. Achill Schürmann
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg

Prof. Dr. Raimund Seidel
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken

Prof. Dr. Jozsef Solymosi
Dept. of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver , BC V6T 1Z2
CANADA

Prof. Dr. Konrad Swanepoel
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz

Prof. Dr. Gabor Tardos
School of Computing Science
Simon Fraser University
8888 University Drive
Burnaby , B.C. V5A 1S6
CANADA

Dr. Csaba David Toth
Department of Mathematics and
Statistics
University of Calgary
2500 University Drive N.W.
Calgary Alberta T2N 1N4
CANADA

Prof. Dr. Geza Toth
Alfred Renyi Institute of
Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Prof. Dr. Helge Tverberg
Department of Mathematics
University of Bergen
Johs. Brunsgate 12
N-5008 Bergen

Dr. Frank Vallentin
CWI - Centrum voor Wiskunde en
Informatica
Postbus 94079
NL-1090 GB Amsterdam

Prof. Dr. Pavel Valtr
Department of Applied Mathematics
Charles University
Malostranske nam. 25
118 00 Praha 1
CZECH REPUBLIC

Prof. Dr. Van H. Vu
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway , NJ 08854-8019
USA

Dr. Uli Wagner
Institut für Informatik
ETH-Zürich
ETH-Zentrum
CH-8092 Zürich

Prof. Dr. Emo Welzl
Theoretische Informatik
ETH Zürich
CH-8092 Zürich

Prof. Dr. Günter M. Ziegler
Institut für Mathematik
MA 6-2
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Prof. Dr. Chuanming Zong
School of Mathematical Sciences
Peking University
Beijing 100871
P.R. CHINA