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Trends and Developments in Complex Dynamics

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Abstract. We will focus on five aspects of holomorphic dynamics with recent and substantial progress: holomorphic dynamical systems with more than one active or free critical point, dynamics in several complex variables, Thurston's topological characterization of rational functions, dynamics of transcendental functions, and renormalization, rigidity, and a priori bounds. The organising principle of the workshop is to present the state of the art of each aspect, to evaluate the progress achieved and to formulate new questions.

Mathematics Subject Classification (2000): 37F15, 37F45, 37F10, 37F30.

Introduction by the Organisers

Holomorphic dynamics is a thriving field which has experienced tremendous progress over the last 25 years, involving mathematicians such as Douady, Hubbard, McMullen, Milnor, Sullivan, Thurston, and Yoccoz. Holomorphic dynamics is gifted with the tools of conformal and hyperbolic geometry that allow a deep penetration into its nature, with many further applications to real dynamics.

Conformal dynamics is currently in a phase of transition. Previously, most of the successful and deep work has been focused on one-parameter model families: families of quadratic or unimodal polynomials and families of meromorphic transcendental functions with one singular orbit. The theory is particularly advanced in the case of the quadratic family; in fact, a complete measure-theoretic picture of the dynamics was obtained in the real quadratic family (Lyubich, Avila, de Melo, Moreira, and others).

During the past two years, significant breakthroughs have appeared in two important directions: rigidity in polynomial dynamics, and Julia sets of positive measure. These results answer major long-standing open questions and simultaneously open up new perspectives. Both problems have served as landmarks in the field, and both have a long history going back to Fatou (and more recently, to Ahlfors, Smale and Mostow). The recent results have brought us very close to a full understanding of the model families, and in particular of the real unimodal dynamics.

Time is now ripe for holomorphic dynamics to move on from the study of the special model cases to a general theory of holomorphic dynamics, delivering on the promise that the model cases have impact in greater generality. Holomorphic dynamics is a vast area of research giving impetus to this transition from many directions. Given the time and number of participants we can not cover all of them within one workshop. We have thus chosen to focus on 5 such aspects of holomorphic dynamics with recent and substantial progress.

The organising principle of the workshop is to present the state of the art of each aspect, to evaluate the progress achieved and to formulate questions which may guide the transition.

Workshop: Trends and Developments in Complex Dynamics

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Abstracts

Monday Session: Holomorphic dynamical systems with more than one active or free critical point

Carsten Petersen

The opening session Monday focused on the understanding of holomorphic dynamical systems with more then one active or free critical point. Cubic polynomials, quadratic rational maps and more generally bi-critical rational maps are natural first candidates to study. However several results go much beyond these and concern maps with an arbitrary number of critical points. The well charted area are always a good starting or anchor point for investigating the unknown. For holomorphic dynamics such anchor points are typically post critically finite maps and hyperbolic components in parameter spaces.

The opening lecture by Tan Lei presented using the language of tableaux's the recent resolution of the generalised Branner-Hubbard conjecture by Qui and Yin [3] and independently by Koslowski van Strien [4] both proofs relying on Shen's critical nest construction [6] and the Kanh-Lyubich Covering Lemma [2]. The B-H-conjecture concerns the structure of the Julia set of polynomials with at least one escaping and one non escaping critical point [1]. The second speaker was Pascale Roesch who spoke on her recent work with Yin on the boundaries of bounded Fatou components of polynomials [5]. The third speaker was Eva Uhre presenting a model for the lines $Per_1(\exp^{i2\pi p/q})$ with a persistent parabolic fixed point in the space of quadratic rational maps. The fourth speaker was Arnaud Chéritat who surveyed the known results and principal questions on cubic polynomials with an irrationally indifferent fixed point. The last speaker Monday was Vladen Timorin who spoke on a surgery on hyperbolic quadratic rational maps involving a generalised version of Thurston's theorem. For other talks directly relevant to the Monday aspect see also the talk by J. H. Hubbard on Epstein's rigidity theorem, Wednesday and by Lasse Rempe on Thursday.

The Monday program was concluded by a problem session in which the following problems were formulated:

(1) Is there any essentially new phenomena involving the interplay of two or more critical points?
(2) Devise a general puzzle construction for rational maps
(3) Devise a combinatorial description of the space of quadratic rational maps.
(4) Devise a combinatorial description of the space of bicritical rational maps of degree d.
(5) Devise a combinatorial description of the cubic connectedness locus.
continuation Devise a combinatorial description of the degree d polynomial connectedness locus.

(6) Find a combinatorial description of the Branner-Hubbard slices of the space of cubic polynomials.

(7) Is the spherical area of a Cantor Julia set for a rational map set ways zero?

Douady conjecture Is the Brunjo condition optimal for rational maps?

(8) Is the Brunjo condition optimal for entire transcendental maps?

(9) Is it optimal for relatively compact (in the domain of definition) maximal Siegel disks?

Hubbard Conjecture Is the area of the filled-in Julia set K_λ for $P_\lambda(z) = \lambda z + z^2$ bounded from below by some constant $C > 0$ when $|\lambda| \leq 1$?

(10) Is the boundary of a Siegel disk for a polynomial/rational map always a Jordan curve?

(11) Is the Zakeri set always a Jordan arc in $Per_1(\omega)$ for any $|\omega| = 1$ in the moduli space of cubic polynomials?

(12) Same question in the moduli space of quadratic rational maps? Or more generally bicritical rational maps?

(13) If $\omega = \exp^{i2\pi\theta}$ with θ of bounded type, is the Zakeri arc then a quasi arc?

Generalized tuning conjecture Can we always replace the closure of a $z \mapsto z^d$ basin Λ for a rational map R by the filled-in Julia set K_P of any degree d polynomial P?

Continuation If the tuning is possible and both K_P and the Julia set for R are locally connected, is the resulting Julia set then locally connected?

(14) For any one-complex analytic parameter family of polynomials $P_a(z) = z^d + a_{d_2}(a)z^{d-2} + \ldots + a_0(a)$ is it true that for harmonic measure almost all values a in the bifurcation locus there is at least one critical point of P_a whose orbit is dense in the Julia set?

(15) Given a topological mating. When is the resulting space a topological sphere?

continuation If so is it then a faithful model?

A. Epstein Are there formal matings with arbitrarily long rays connections?

(16) Is the closure of the rigidity locus of cubic polynomials equal to the Misiurewicz locus?

(17) Let C_3 denote the connectedness locus of cubic polynomials, is $\overline{\mathring{C}_3} = C_3$?

(18) Is it true that any wandering connected component of any rational maps is eventually either a point or a topological circle?

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Branner-Hubbard conjecture

Tan Lei

The Branner-Hubbard conjecture has been recently proved in its full generality:

Theorem 1 (Qiu-Yin, Kozlovski-van Strien)**.** *Assume that g is a polynomial. Denote by K_g its filled Julia set. Assume that every component of K_g containing a critical point is aperiodic under the iteration of g. Then K_g is a Cantor set.*

In this talk we will illustrate the new techniques involved in the proof of this conjecture, using entirely in the tableaux language of Branner-Hubbard. We show that a particular double sequence of critical puzzle nests, constructed by Kozlovski-Shen-Strien, has a positive lower bound on their moduli, using Kahn-Lyubich covering lemma.

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The boundary of bounded Fatou components

Pascale Roesch

(joint work with Y.Yin)

Let f be a polynomial of abitrary degree. We look at the boundary of Fatou components and more precisely we are interested in its topology.

The boundary of the unbounded Fatou component is the whole Julia set $J(f)$. It can have a very complicate topology since in the presence of Cremer points it is not locally connected and contains the picture of a "hedgehog" (see [PM]); it can also have positive Lebesgue measure (see [BC]). However, the *bounded Fatou components* have a very nice boundary :

Theorem 1. *The bounded Fatou components, which are not Siegel disks, are Jordan domains (i.e. disks with Jordan curve boundary).*

One may naturally ask the following :

Conjecture 1. *Every bounded Fatou component (for a polynomial) is a Jordan domain.*

As a corollary of Theorem 1, we obtain that the model of the dynamics on the Fatou component extends to the boundary (by Caratheodory's Theorem) and also the following description of part of the filled-in Julia set $K(f)$:

Theorem 2. *Let U be a bounded Fatou component and denote by K_U the connected component of $K(f)$ containing U. It can be described as*

$$K_U = \overline{U} \cup \bigcup_{t\in\mathbb{R}/\mathbb{Z}} L_t$$

where each limb L_t intersects U in exactly one point. Moreover, every L_t which is not a point is eventually mapped to some L_{t_0} containing a critical point.

The picture that one can have in mind is a very digited curve, nothing can accumulate on a sub-arc of it but there might be combs (facing the basin of ∞) attached to it by one point. It is not known whether the Julia set is locally connected at every point of the boundary of bounded Fatou components nevertheless this holds for non eventually periodic points.

The proof of Theorem 1 can be sketched as follows:
Let U be a bounded Fatou component. We can assume up to replacing f by some f^k that U is fixed. Then, by classical surgery procedure (see [DoHu2, McMu]) we can reduce to the case where $J(f)$ is connected and U contains only one critical point. Therefore rays and equipotentials are well defined in U and in the basin of ∞. Note that if U is a parabolic domain, we have to use parabolic rays (see [PR]). The equipotentials, internal "rays" and external rays can be used in a standard way (see [R]) to construct a "forward" invariant graph Γ. The graph that one can have in mind is formed by the cycle of internal rays obtained from the angle $1/(d^k-1)$ where d is the local degree in U, together with the cycle generated by an external ray landing at the landing point in ∂U of previous internal ray, to which one adds any internal and external equipotential. It defines a puzzle as follows: Let Γ_n be the graph $f^{-n}(\Gamma)$, the puzzle pieces of depth n are the connected components of $\mathbb{C}\setminus\Gamma_n$. They have the nice property to cut ∂U (as well as $J(f)$) in connected sets. So the aim of the work now is the following: for any point $x\in\partial U$ if $P_n(x)$ is the puzzle piece of depth n containing x, prove that the diameter of $P_n(x)\cap\partial U$ goes to 0.

Step 1. *If x is eventually periodic, either the intersection $\bigcap_{n\in\mathbb{N}} \overline{P_n(x)}$ reduces to $\{x\}$ or there are two external rays landing at x and separating this intersection $\left(\bigcap_{n\in\mathbb{N}} \overline{P_n(x)}\right)$ from $\overline{U}\setminus\{x\}$. Therefore, $\bigcap_{n\in\mathbb{N}} \overline{P_n(x)}\cap\partial U=\{x\}$.*

Proof. We look at the fixed points of the external class of f on $X = \bigcap_{n\in\mathbb{N}} \overline{P_n(x)}$ following J. Kiwi's idea in [10].

Step 2. *If the orbit of x accumulates an eventually periodic point, the diameter of $P_n(x)$ goes to* 0.

Proof. If the orbit of x accumulates a fixed point y which is not parabolic, there exists a sequence (n_i) with $f^{n_i}(x) \in P_1(y)$. Moreover the annulus $P_0(y) \setminus \overline{P}_1(y)$ is non degenerate, so considering the first entrance in the puzzle piece, one can surround x by annuli of the same modulus.

If the orbit of x accumulates an eventually parabolic periodic point y, using distortion properties one can prove that the diameter of $P_n(x)$ goes to 0 by thickening the annulus containing the parabolic point in its boundary.

Step 3. *If the orbit of x does not accumulate (for the topology of the puzzle pieces) an eventually periodic point, one can find for every point z in the accumulation of the orbit of x a non degenerate annulus of the form $P_n(z) \setminus \overline{P_k(z)}$.*

Proof. If all consecutive puzzle pieces containing z themselves, they do at an eventually periodic point y of ∂U. But for such puzzle pieces we know from Step1 that $\bigcap_{n\in\mathbb{N}} \overline{P_n(y)} \cap \partial U = \{y\}$. Since the orbit of x is in ∂U and accumulates in these puzzle pieces, it follows that x accumulates the eventually periodic point y. Contradiction.

Step 4. *Let K_n, K'_n denote the KSS nest constructed in* [KSS, QY, TY] *starting from $P_k(c)$ where c is a critical point of the accumulation of the orbit of x. It is enough to look at such critical points whose orbits accumulate themselves. The intersection $\cap K_n$ reduces to the point $\{c\}$ and therefore the sequence $P_n(x)$ is a basis of neighbourhoods of $\{x\}$.*

Proof. By the work of [KSS, QY] the difference between the depths of the puzzle pieces K_n and K'_n tends to infinity. Therefore, for large n the annulus $K'_n \setminus \overline{K}_n$ contains a "pull-back" of previous $P_n(c) \setminus \overline{P_k(c)}$ so is non degenerate. Then using the covering Lemma of [KL] and still the results of [KSS, QY, TY] one obtains that the modulus of these annuli is bounded from below. The result follows for x by looking at the first entrance of the orbit of x in the puzzles $P_n(c)$.

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Obstructions to Parabolic Quadratic Rational Maps

Eva Uhre

Let Rat_2 denote the space of quadratic rational maps, and let $\mathcal{M}_2 = Rat_2/Rat_1$ denote the moduli space of Rat_2, modulo möbius conjugacy. Following Milnor, we consider loci:

$$Per_1(\lambda) = \{[f] \in \mathcal{M}_2 : f \text{ has a fixed point with multiplier } \lambda\}.$$

A quadratic rational map has two distinct critical points and three fixed points, counted with multiplicity. We denote the critical points c_1 and c_2, and the fixed point multipliers μ,λ and γ. The triple $\langle \mu, \lambda, \gamma \rangle$ uniquely determines an equivalence class $[f] \in \mathcal{M}_2$.
For the elementary symmetric functions of the multipliers:

$$\sigma_1 = \lambda + \mu + \gamma,$$
$$\sigma_2 = \lambda\mu + \lambda\gamma + \mu\gamma,$$
$$\sigma_3 = \lambda\mu\gamma,$$

the holomorphic fixed point index formula implies $\sigma_3 = \sigma_1 - 2$. The map

$$[f] \mapsto (\sigma_1(f), \sigma_2(f)) : \mathcal{M}_2 \to \mathbb{C}^2,$$

is biholomorphic and hence defines coordinates on $\mathcal{M}_2$. The locus $Per_1(\lambda)$ is a complex line in these Milnor–coordinates, and the map

$$[f] \mapsto \sigma(f) = \mu\lambda : Per_1(\lambda) \to \mathbb{C},$$

is an isomorphism, and defines a coordinate on $Per_1(\lambda)$. [M93]

Here we shall focus on parabolic slices $Per_1(\omega)$, with $\omega = e^{2\pi i p/q}$, $p/q \neq 0/1$. In such a slice $Per_1(\omega)$, one critical point c_1 is necessarily in the immediate basin for the parabolic fixed point, so there is one free critical point c_2. The whole slice $Per_1(\omega)$ is in the bifurcation locus in $\mathcal{M}_2$, and it is a natural place to look for the "birth" of new dynamical behavior, involving the interplay between two critical points. In $Per_1(\omega)$ all Julia sets are connected, so we characterize the dynamics according to the behavior of the free critical point.

We will call c_1 and c_2 *Fatou related* if they belong to the same grand–orbit of Fatou components. Further we say that c_1 and c_2 are related if $\omega(c_1) = \{z_0\} = \omega(c_2)$, where $\omega(c_i)$ is the omega–limit set of the critical point c_i, and z_0 is the (persistent) parabolic fixed point. It is clear that Fatou related critical points are also related in this more general sense.

We define the Fatou relatedness locus:

$$\mathcal{R}^\omega = \{[f] \in Per_1(\omega) : c_1, c_2 \text{ are Fatou related}\},$$

its complement

$$M^\omega = Per_1(\omega) \setminus \mathcal{R}^\omega = \{[f] \in Per_1(\omega) : c_1, c_2 \text{ are not Fatou related}\},$$

and an extended relatedness locus

$$\hat{\mathcal{R}}^\omega = \{[f] \in Per_1(\omega) : \omega(c_1) = \{z_0\} = \omega(c_2)\}. \tag{1}$$

Obviously, these loci can be defined for any $\lambda \in \mathbb{C}$. In the case of quadratic polynomials, i.e. in $Per_1(0)$, M^0 is just the Mandelbrot set M and $\hat{\mathcal{R}}^0 = \mathcal{R}^0$ is its complement $\mathbb{C} \setminus M$.

The set M^ω is connected, whereas the set $\mathcal{R}^\omega$ is not, in fact it consists of countably many open, simply connected components. On the other hand, $\hat{\mathcal{R}}^\omega$ is connected, but neither open nor closed.

Whereas there are for quadratic rational maps four types of hyperbolic components [R90], [M93], there are in the slices $Per_1(\omega)$ essentially three types of parabolic components, which we will call

- *Relatively hyperbolic* components, $\mathcal{H}$, of maps with an attracting cycle. These components are in M^ω.
- *Parabolic bitransitive* components, $\mathcal{B}$, of maps where both critical points are in the same cycle of components in the immediate basin of the parabolic fixed point, but not in the same component. These components are in $\mathcal{R}^\omega$.
- *Parabolic capture* components, $\mathcal{C}$, of maps where both critical points are in the basin of the parabolic fixed point, but only one is in the immediate basin. These components are in $\mathcal{R}^\omega$.

The p/q–limb of the Mandelbrot set is:

$$L_{p/q} = \{c \in M : P_c \text{ has a fixed point with comb. rotation number } p/q\}.$$

Conjecturally, $Per_1(\omega)$ should be understood as the mating of $M \setminus L_{-p/q}$ and a modified version (see [U08] for a description of this modification) of the filled–in Julia set for the corresponding quadratic polynomial $P_\omega(z) = \omega z + z^2$.

Let Λ^ω denote the parabolic basin of 0 for P_ω and let

$$\hat{K}_\omega = \{z \in \hat{\mathbb{C}} : \omega(z) = 0\} = \Lambda^\omega \cup \{z \in \hat{\mathbb{C}} : \exists n \geq 0, P_\omega^n(z) = 0\}.$$

There are q external rays landing at 0, dividing $\hat{\mathbb{C}}$ into q components, let S_p denote the component containing the critical value $P_\omega(-\omega/2)$.

Then, for all $g_\sigma \in \hat{\mathcal{R}}^\omega$, there exists $\eta_{\sigma,\omega} : U_\sigma \to \hat{K}_\omega$, a holomorphic conjugacy of g_σ to P_ω, where U_σ contains the maximal attracting flower for g_σ, both critical values $v_1 = g_\sigma(c_1)$ and $v_2 = g_\sigma(c_2)$, but not the free critical point c_2. The conjugacy obeys the relation $\phi_\omega \circ \eta_{\sigma,\omega} = \phi_\sigma$, where ϕ_σ, ϕ_ω are Fatou coordinates for P_ω and g_σ respectively, appropriately normalized.

We use this conjugacy to construct from $\hat{K}_\omega$ a model of $\hat{\mathcal{R}}^\omega$, which records the position of the free critical value v_2 in the basin Λ^σ [U08]. In particular, to do this we show:

Theorem 1. *For all $g_\sigma \in \hat{\mathcal{R}}^\omega$, $\eta_{\sigma,\omega}(v_2) \notin \hat{K}_\omega \cap S_p$, i.e. these configurations can not be realized by any parabolic, quadratic rational map.*

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On the linearization of degree three polynomials and Douady's conjecture

Arnaud Chéritat

(joint work with Xavier Buff)

This talk was mostly a survey.

Consider a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$, with an irrationnally indifferent fixed point at the origin:

$$P(0) = 0, \quad P'(0) = e^{2i\pi\theta}, \quad \theta \in \mathbb{R} \setminus \mathbb{Q}.$$

We focus on two questions:

(1) When is the fixed point linearizable?
(2) What is roughly the inner radius of the Siegel disk with respect to its center?

This amounts to evaluate the radius of convergence of linearizing series, or equivalently the conformal radius of the Siegel disk.

For any analytic function with $f(0) = 0$ and $f'(0) = e^{2i\pi\theta}$, Brjuno proved that if θ satisfies some diophantine condition, then f is automatically linearizable at 0. For degree 2 polynomials, Yoccoz proved that Brjuno's condition is necessary for linearizability. The following conjecture is still open.

Conjecture 2 (Douady)**.** *The Brjuno condition is also necessary for polynomials of degree $d \geq 3$.*

The Bjuno sum is the quantity $B(\theta) = \sum_{n=0}^{+\infty} \frac{\log q_{n+1}}{q_n}$ where p_n/q_n is the sequence of continued fraction convergents of θ (the Brjuno numbers are those for which $B(\theta) < +\infty$). Concerning the conformal radius $\mathrm{rad}(f)$ of the Siegel disk of an analytic map f fixing 0 with derivative $e^{2i\pi\theta}$, Yoccoz proved that under the asumption that f is univalent on the unit disk, then

$$\mathrm{rad}(f) \geq e^{-B(\theta)}/C$$

for some universal constant $C > 1$, and that this estimate is somehow optimal. For degree 2 polynomials, Buff and Chéritat proved that this optimality holds, i.e. that $\mathrm{rad}(e^{2i\pi\theta}z + z^2) \leq C' e^{-B(\theta)}$ for some universal $C' > 1$.

Let us focus on degree 3 polynomials. Then P is affinely conjugated to a polynomial of the form

$$P_{\theta,a}(z) = e^{2i\pi\theta}(z + az^2 + z^3)$$

for some $a \in \mathbb{C}$, and the only affine conjugacies in this family are $P_{\theta,a} \sim P_{\theta,-a}$. In particular a^2 is unique. For a fixed θ we have a one parameter family for which we can draw bifurcation loci on the computer.

There are three special cases:

- $a^2 = 4$: one critical point is mapped to the fixed point 0, the other is free.
- $a^2 = 3$: the two critical points merged into a critical point of local degree 3 (it is affine conjugated to $z^3 + c$ for some not necessarily unique c).
- $a^2 = 0$: then P has the symmetry $P(-z) = -P(z)$. It is semi-conjugated to $P_{2\theta,2}$, i.e. to the first special case but with a doubled angle 2θ: $D \circ P_{\theta,0} = P_{2\theta,2} \circ D$ where $D(z) = z^2$.

Perez-Marco proved a theorem that implies that for a non-Bjruno number θ, the set of $a \in \mathbb{C}$ such that $P_{\theta,a}$ is not linearizable at 0 is polar. In particular it is either empty (Douady's conjecture) or very small.

When θ is a non-Brjuno irrational, Lukas Geyer gave a sufficient condition for a polynomial to be non-linearizable: that the number of infinite critical orbit tails within J is equal to the number of irrational cycles (it is always at least this number by the Fatou-Shishikura inequality).

Geyer's theorem applies to two special cases $a^2 \in \{3, 4\}$. Thus Bjruno's condition is necessary for them. By the semiconjugacy, Brjuno's condition is also necessary in the case $a^2 = 0$.

Theorem 1 (Buff, Chéritat)**.** *Under some technical condition, more restrictive than Geyer's, on a compact family $\mathcal{F}$ of polynomials fixing* 0 *with indifferent multiplier, there exists $C > 0$ depending on $\mathcal{F}$ such that $\forall P \in \mathcal{F}$,*

$$\mathrm{rad}(P) \leq Ce^{-B(\theta)}$$

where θ is the rotation number of P at 0*. In other words, using Yoccoz's lower bound, $-B(\theta)$ approximates* $\log(\mathrm{rad})$ *up to a bounded additive term.*

For instance, it applies to $a^2 = 3$, $a^2 = 4$, but not to $a^2 = 0$.

Corollary 1. *($a^2=0$) There exists $C>1$ such that $\forall\theta$,*

$$\frac{1}{C}\exp\big(-\frac{1}{2}B(2\theta)\big)\le\log(\operatorname{rad}P_{\theta,0})\le C\exp\big(-\frac{1}{2}B(2\theta)\big).$$

This implies: $\exists C>1$, $\forall\theta>1$

$$\frac{1}{C}\exp\big(-B(\theta)\big)\le\log(\operatorname{rad}P_{\theta,0})\le C\exp\big(-\frac{1}{2}B(\theta)\big).$$

The second inequality is weaker, but it depends on $B(\theta)$ instead of $B(2\theta)$. Note that there are sequences θ_n such that $B(\theta_n)\longrightarrow+\infty$ and $B(2\theta_n)/2B(\theta_n)\longrightarrow 1$; and there are sequences θ_n such that $B(\theta_n)\longrightarrow+\infty$ and $B(2\theta_n)/B(\theta_n)\longrightarrow 1$. Thus the second set of inequalities is not far from being optimal for some values of θ.

We conjecture that this is the worst case:

Conjecture 3. *For $d\ge 3$, there exists $C=C(d)\in\mathbb{R}$ such that for all polynomial f of degree d with an indifferent fixed point at the origin,*

$$\log\operatorname{rad}(f)\le-\frac{1}{d-1}B(\theta)+\log\min|c_i|+C$$

where the c_i are the critical points of f and θ is the rotation number at the origin.

Zakeri studied the parameter slice consisting of the $P_{\theta,a}$ with a varying and θ fixed, of *bounded type*. He proved that the set $\mathcal{Z}$ of parameters such that both critical points belong to the boundary of the Siegel disk, is a Jordan arc from $a=-\sqrt{3}$ to $a=\sqrt{3}$ (he used in fact a slightly different parameterization of the family, for which the corresponding $\mathcal{Z}$ is a Jordan curve).

Note that for θ a Bjuno number, minus the logarithm of the conformal radius of the Siegel disk is a subharmonic function $f_\theta(a)$ of a, and that for θ of bounded type it is harmonic outside the Jordan arc $\mathcal{Z}$. The Laplacian of f_θ is a measure of mass 1. By looking at pictures, and in particular parabolic implosion pictures, we are tempted to make the following set of conjectures.

Conjecture 4. *For all bounded type numbers, the image of the Jordan arc by $a\mapsto a^2$ is naturally parameterized by the angle at the origin between the two internal rays landing on the two critical points.*
For all Brjuno number θ, the support of the Laplacian of f_θ is a Jordan arc.
For non-Bjruno irrationnals θ, there exists a function $\Phi_\theta(a)$ such that for all sequence of Brjuno numbers $\theta_n\longrightarrow\theta$, $-\log(\text{conf rad }P_{\theta_n,a})-Y(\theta_n)\longrightarrow\Phi_\theta(a)$, and the support of the Laplacian of $\Phi_\theta(a)$ is a Jordan arc. Here, $Y(\theta)$ denotes Yoccoz's version of the Brjuno sum.

We may even conjecture that this Jordan arc is always the set of a such that both critical points have the same ω-limit set, which may be a Siegel disk boundary or a hedgehog (or something else?); however, no computer pictures support this claim: we still do not have a good algorithm to draw hedgehogs. Also, there should be a notion of internal angle between the two critical points even when there is a hedgehog.

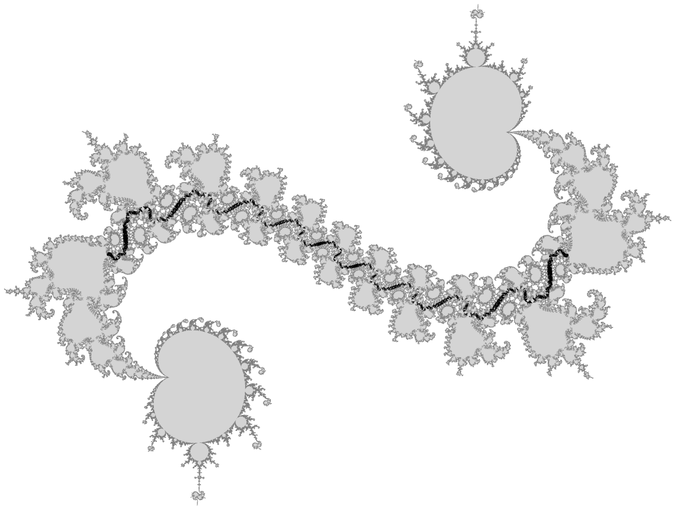

FIGURE 1. Example for $\theta = 1/(10 + 1/(10 + \cdots))$. In white: the non-connectedness locus. In dark gray: the bifurcation locus. In light gray: the rest. In black, we used an algorithm to highlight, a bit poorly, the Zakeri jordan arc $\mathcal{Z}$.

Regluing of rational functions

VLADLEN TIMORIN

1. Regluing. We study relations of the form $\Phi \circ f = g \circ \Phi$, where f and g are rational functions, and Φ is a certain partially defined map. Assume e.g. that Φ cuts the Riemann sphere along countably many curves and reglues them in a different way. Such map is called a *regluing*. An example of regluing is the map

$$j(z) = \sqrt{z^2 - 1}$$

defined on the complement to $[-1, 1]$. It reglues the segment $[-1, 1]$ into $[-i, i]$.

Suppose we reglue some curve. Then we need to reglue all its iterated preimages, thus we must deal with regluings of countably many curves.

2. Quadratic polynomials. The Julia set of $z \mapsto z^2 - 3$ is a Cantor subset of $\mathbb{R}$. Reglue all complementary segments. We obtain the map $z \mapsto z^2 - 2$, whose Julia set is a segment! More generally, let f be a quadratic polynomial $z \mapsto z^2 + c$, where c is the landing point of an external parameter ray R. Suppose that the Julia set of f is locally connected, and all periodic points are repelling. Also, consider a quadratic polynomial g, for which the corresponding parameter value belongs to R. Thus the Julia set of g is disconnected. Then $\Phi \circ f = g \circ \Phi$ for a regluing Φ.

3. Quadratic rational functions. The dynamical behavior of a rational function is determined by the behavior of its critical orbits. A quadratic rational function has two critical points. Thus, to simplify the problem, one puts restrictions on the dynamics of one critical point, say, makes it periodic of period k. For $k = 1$, we obtain quadratic polynomials. Suppose now that $k > 1$, and f is a quadratic rational function with a k-periodic critical point c_1 and a free critical point c_2. Recall that f is *hyperbolic rational function of type B* if c_2 lies in the immediate basin of c_1 (but necessarily not in the same component). The function f is *a hyperbolic rational function of type C* if c_2 lies in the full basin of c_1, but not in the immediate basin. The set of hyperbolic rational functions with a k-periodic critical point splits into *hyperbolic components*. We say that a hyperbolic component is of type B or C if it consists of hyperbolic rational functions of this type, see [1, 3].

Theorem 1 (see [5]). *If f is on the boundary of a type C hyperbolic component, but not on the boundary of a type B hyperbolic component, then $\Phi \circ f = h \circ \Phi$, where h is the center of a type C hyperbolic component, whose boundary contains f, and Φ is a regluing.*

Topological models for type C hyperbolic quadratic rational functions are known [4]. Thus the theorem above gives topological models for most functions on the boundaries of type C components.

4. Existence of regluings. Let X be a compact metric space, and $\mathcal{A}$ a set of compact subsets of X. We say that $\mathcal{A}$ is *contracted* if for every $\epsilon > 0$, there are only finitely many elements of $\mathcal{A}$, whose diameter exceeds ϵ. It is not hard to see that this property is topological, i.e. does not depend on the choice of metric.

Theorem 2 (see [5]). *Let $\mathcal{A}$ be a contracted set of disjoint simple curves in S^2. There exists a homeomorphism $\Phi : S^2 - \bigcup \mathcal{A} \to S^2 - \bigcup \mathcal{B}$ regluing $\mathcal{A}$ into another set $\mathcal{B}$ of disjoint simple curves.*

The statement of the theorem may seem intuitively obvious. Note, however, that the set $\bigcup \mathcal{A}$ may be everywhere dense in the sphere. The proof uses Moore's theory [2].

5. Holomorphy. Let Z be a countable union of disjoint simple curves. Assume that Z has zero Lebesgue measure. We say that a map $\Phi : \mathbb{C} - Z \to \mathbb{C}$ is *holomorphic modulo Z* if there is a function $\Psi : Z \to \mathbb{C}$ such that

$$\int_{\mathbb{C}-Z} \Phi \, \overline{\partial}\omega = \int_Z \Psi \, \omega$$

for every smooth (1,0)-form ω on $\mathbb{C}$ with compact support. Intuitively, this definition says that the distributional differential $\overline{\partial}\Phi$ must be a sum of countably many δ-like (0,1)-currents supported in Z. A regluing surgery is in many cases holomorphic in this sense [6].

6. A sequence of approximations. We will use the following version of Thurston's algorithm for partially defined functions. Let U be an open subset of the Riemann sphere, and let a function $f_1 : U \to \overline{\mathbb{C}}$ be a ramified covering over its image of degree 2 with two critical points. Assume that the critical orbits of f_1 are defined (hence, they lie in U). Consider a quadratic rational function R_1, whose critical values coincide with those of f_1. Then the multivalued function $R_1^{-1} \circ f_1$ splits into two single valued branches. Let j_1 denote one of the branches. The domain of j_1 coincides with the domain of f_1 (i.e. with U). Define $f_2 = j_1 \circ R_1$. The domain of f_2 is $R_1^{-1}(U)$. It is easy to see that $j_1 \circ f_1 \circ j_1^{-1} = f_2$ wherever the left-hand side is defined, hence the Möbius conjugacy class of f_2 does not depend on a particular choice of the branch j_1. The critical orbits of f_2 are also defined, and coincide with the j_1-images of the critical orbits of f_1. Repeating this procedure, we obtain a sequence f_n of partially defined functions, together with functions R_n and j_n satisfying the following relations:

$$f_{n+1} = j_n \circ R_n, \quad f_n = R_n \circ j_n.$$

The functions R_n are quadratic rational functions.

Now assume that the function f_1 is holomorphic throughout its domain. Then so are all the functions f_n and j_n. If we set $\Psi_n = j_n \circ \cdots \circ j_1$, then $f_{n+1} = \Psi_n \circ f_1 \circ \Psi_n^{-1}$ wherever the right-hand side is defined, and the functions Ψ_n are holomorphic on their domains. The main question is: *under what assumptions on U and f_1 do the maps Ψ_n converge uniformly on the intersection of their domains*? If they do converge to some map Ψ, then Ψ provides a topological surgery that makes f_1 into a quadratic rational function. Moreover, this surgery will be in a sense holomorphic on its domain.

7. Generalized regluing. Our particular choice of f_1 is the following. We start with a quadratic rational function f with critical points c_1 and c_2. Let $\alpha : [-1, 1] \to \overline{\mathbb{C}}$ be a simple path such that $f \circ \alpha(t) = f \circ \alpha(-t)$ (it follows that $\alpha(0)$ is a critical point of f, say, $\alpha(0) = c_1$). Assume that the forward orbits of c_1 and c_2 are disjoint from $\alpha[-1, 1]$. Then we can define j to be a branch of $R^{-1} \circ f$ over $\overline{\mathbb{C}} - \alpha[-1, 1]$, where R is a quadratic rational function, whose critical values are $f \circ \alpha(1)$ and $f(c_2)$. The branch is well defined because $\alpha[-1, 1]$ contains all ramification points of $R^{-1} \circ f$. Note that j cuts the Riemann sphere along α, and reglues α in a different way. Set $f_1 = j \circ R$.

Every function Ψ_n will be defined and holomorphic on the complement to finitely many simple curves. The intersection of the domains of all Ψ_n is then the complement to a countable union of simple curves. Note that, by a Baire category argument, such set is dense in the sphere. If Ψ_n converge uniformly on this set, then the limit Ψ provides a certain cut-and-glue surgery, which we call *generalized regluing*. The way one needs to cut and reglue can be made very explicit. A curious question is the following: which quadratic polynomials can be obtained from the Chebyshev polynomial $z \mapsto z^2 - 2$ by generalized regluing? Potentially, it should be possible to use generalized regluing to build topological models for many quadratic rational functions f with simple dynamical behavior: if we know that $f = \Psi^{-1} \circ g \circ \Psi$ on the domain of the right-hand side, where g is a rational function, for which an explicit topological model is known, and Ψ is a generalized regluing, then we obtain a topological model for f.

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Tuesday Session: Dynamics in Several Complex Variables

John Smillie

The talks on Tuesday involved dynamics in two complex variables. An important underlying theme was the connection between dynamics in one variable and dynamics in two variables. Listed below are four questions about dynamics for which it is interesting to compare the one and two dimensional situations.

- Which complex manifolds admit endomorphisms or automorphisms of infinite order?

  Complex manifolds with interesting dynamics are special. In dimension one $\mathbb{C}$, $\mathbb{P}^1$ and the torus are the complex one-manifolds with interesting dynamics. In two dimensions we have $\mathbb{C}^2$, $\mathbb{P}^2$, tori, $K3$ surfaces and the interesting class of surfaces birationally equivalent to $\mathbb{P}^2$. Eric Bedford discussed new examples of rational surfaces with interesting dynamics.
- How do topological invariants of maps behave under iteration?

  The fundamental topological invariant in one dimension is degree. In two dimensional dynamics there are two degrees to consider – a one dimensional degree and a two dimensional degree. The connection between the growth rates of these degrees was discussed in Romain Dujardin's second talk.
- How are periodic points and stable manifolds distributed?

  In one dimension we know that periodic points are dense in the Julia set. We also know that if you take inverse images of points they tend to the Julia set. These are basic parts of the one dimensional theory. In two dimensions we can ask how the iterates of one dimensional submanifolds behave and how periodic points behave. The state of our knowledge about these and similar questions was addressed by Romain Dujardin in his first talk.
- Can we understand parameter space combinatorially?

  We have an excellent symbolic picture of the Mandelbrot set. A key tool in this picture is the behavior of the critical point for quadratic polynomials. There is also interesting information about the cubic parameter locus which comes from looking at monodromy of horseshoes and the homomorphism from the fundamental group of the horseshoe locus into the automorphism group of the one sided 3-shift. The question of whether the monodromy map might be the key to understanding the parameter space of quadratic polynomial automorphisms of $\mathbb{C}^2$ (Hénon maps) was addressed in John Hubbard's talk.

The following problems were proposed in the Tuesday problem session.

- Does there exist a Siegel ball with boundary homeomorphic to the 3-sphere?
- Does there exist a rotation domain analytically equivalent to the product of an annulus and a complex plane?
- What does the domain of convergence of the linearizing map for a two dimensional rotation domain look like?
- Consider a complex Hénon map with J connected. There is a natural map ϕ^+ from $J^- - K$ to the complex solenoid. When is this map one-to-one?
- Consider a complex Hénon map with J connected. When is J the quotient of a solenoid? When J is the quotient of a solenoid which equivalence relations occur?
- Find a good characterization of connectedness of the Julia set for Hénon like maps.

Dynamics of Meromorphic Maps with Small Topological Degree

Romain Dujardin

(joint work with Jeffrey Diller, Vincent Guedj)

Let $f : X \to X$ be a dominant meromorphic mapping on a compact complex surface; we furthermore assume X is projective. We want to investigate the dynamics of f. A first basic observation is that the existence of a self map with nontrivial dynamics imposes severe restrictions on the surface X, which must be rational (that is 'resembling' the projective plane) or a torus or a K3 surface.

Famous examples of such maps are given by polynomial automorphisms of $\mathbb{C}^2$ and holomorphic endomorphisms of the projective plane $\mathbb{P}^2$, whose dynamical properties are dramatically different (see [BLS, FS, BrDu]). To distinguish between these cases, it is useful to introduce the following numbers (the *dynamical degrees*):

- the *topological degree* d_t is the number of preimages of a generic point;
- the *(first) dynamical degree* λ is the growth rate of degrees (or volumes) of iterated curves.

A dynamically nontrivial polynomial automorphism of $\mathbb{C}^2$, of degree d, has $d_t = 1$ and $\lambda = d$, while an endomorphism of $\mathbb{P}^2$ of degree d has $d_t = d^2$, $\lambda = d$. Recall that the degree of a rational mapping on $\mathbb{P}^2$ equals $\deg(f^{-1}(L))$, where L is a generic line.

A first connection between the dynamical degrees and the dynamics is the following famous inequality of Gromov's $h_{\text{top}}(f) \leq \log\max(d_t, \lambda)$ (h_{top} denotes topological entropy). Gradually, there has emerged a clear conjectural picture concerning the ergodic behavior of mappings with $d_t \neq \lambda$. It is in particular conjectured that the Gromov inequality is an equality and that there is a canonical measure of maximal entropy, with several interesting properties.

In the case of large topological degree ($d_t > \lambda$) this conjecture was fully proven in [G] (following important steps in [BrDu, DS]). Given such a f and a generic point x, the sequence of measures

$$\sum_{\{y, f^n y = x\}} \delta_y$$

converges to a probability measure μ, which is mixing, is the unique measure of maximal entropy, and describes the distribution of saddle orbits. This is very much analogous to the 1-dimensional situation, where the corresponding results were obtained in [Ly, FLM].

Our purpose here is to study the case of mappings with small topological degree ($\lambda > d_t$). The hope is to arrive at an interesting invariant measure by choosing two generic curves $C, C' \subset X$ and considering something like the sequence of measures

$$\frac{f^{-n}(C) \wedge f^n(C')}{\lambda^{2n}},$$

where the meaning of the wedge product here is summing Dirac masses at intersection points. The reader will not be surprised that the geometric aspects of the problems become more involved in this case.

An important subclass is that of birational mappings ($d_t = 1$), where a (almost) successful approach to the conjecture was carried out along the following lines:

Step 1 understand precisely the growth of the degrees of f^n, by constructing a "good birational model" [DF];

Step 2 analyze the action on cohomology and construct invariant currents with special geometric properties [DF];

Step 3 give a reasonable meaning to the wedge product $T^+ \wedge T^-$, both from the analytic and the geometric points of view [BeDi]; This results in a positive measure μ.

Step 4 study the dynamical properties of μ [Du].

This is only almost a success because Step 3 requires a technical assumption (which is generically satisfied, but there are counterexamples).

In the general case where $1 \leq d_t < \lambda$ is arbitrary, several new difficulties appear. In this generality the extension of Step 1 is wide open (despite recent progress, see [BFJ]). On the other hand, recent work of Favre and Jonsson [FJ] provide a rather satisfactory answer for the notable case of polynomial mappings of $\mathbb{C}^2$. In a joint work with Diller and Guedj, we have recently extended Steps 2 to 4 in a series of papers [DDG1, DDG2, DDG3].

[DDG1] deals with Step 2. Assuming the existence of a good birational model, there are two natural invariant closed positive currents T^+ (the "stable" current) and T^- (the "unstable" one), with good attraction and geometric properties. Even if this type of result has become quite classical now, this is the first place where this is proven in full generality (what is important is that we make no further assumption on X nor on the invariant cohomology classes).

In [DDG2] we deal with Step 3. Unfortunately, as for the case $d_t = 1$, we can only construct $\mu = T^+ \wedge T^-$ under a technical assumption on f ("finite dynamical energy"). This assumption holds for polynomial maps. Actually there is no known example of a map violating it. We can also prove that μ can be described as the "geometric intersection") of the geometric structures of $T^{\pm}$. Arrived at this point we know that if C, C' are generic curves, $\frac{1}{\lambda^{2n}} f^{-n}(C) \wedge f^n(C')$ converges weakly to μ. We also know that μ is invariant and mixing.

In [DDG3] we study the ergodic properties of μ, under the finite dynamical energy condition. We give optimal bounds for the Lyapunov exponents of μ, show that it has maximal entropy and (if furthermore $\log ||df|| \in L^1(\mu)$) that saddle orbits are equidistributed towards μ – the uniqueness of the measure of maximal entropy remains an open question. The core of the paper is to understand the quite delicate interplay between the geometry of the invariant currents and the "geometry" of the natural extension of f.

It is to be noticed that recently De Thlin and Vigny [DV] have found an alternate approach to the computation of entropy and Lyapunov exponents of μ (still, under a certain technical assumption).

Altogether, this gives a good picture of the ergodic theory of a wide class of (presumably all) mappings with small topological degree, including all polynomial mappings of $\mathbb{C}^2$.

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Automorphisms of rational surfaces with positive entropy

Eric Bedford

(joint work with Kyounghee Kim)

We consider compact projective manifolds of complex dimension 2. Such a complex surface $\mathcal{X}$ is said to be *rational* if it is birationally equivalent to the projective plane $\mathbb{P}^2$. We are interested in finding automorphisms (biholomorphisms) of $\mathcal{X}$ with positive entropy. By a theorem of Nagata, any such $\mathcal{X}$ can be obtained by performing a finite number of blowups of $\mathbb{P}^2$. This blowup procedure might involve iterated blowups, which means that centers of successive blowups might lie in blowup fibers. We start with mappings of the form

$$f(x,y) = \left(y, -x + \sum_{j=1}^{k-1} \frac{a_j}{y^{2j}} + \frac{1}{y^{2k}} \right)$$

and we show that f induces an automorphism of a manifold $\mathcal{X}$ which is constructed by performing a suitable family of $4k+1$ blowups over the 2 points where the x- and y-axes intersect the line at infinity. The surprising aspect of this family is that it depends on $k-1$ arbitrarily chosen complex parameters $a_1, \dots a_{k-1}$.
We also consider the map

$$h(x,y) = \left(y, -\delta x + cy + \frac{1}{y} \right)$$

where δ and c are suitably chosen. Again, we may construct a manifold $\mathcal{Y}$ by blowing up $\mathbb{P}^2$ at certain points on the line at infinity. (This needs a tree of 3 blowups over each point.) This leads to certain phenomena. If $\delta > 1$, then the line at infinity is an attractor, and the attracting basin has full measure in $\mathcal{Y}$. If $|\delta| = 1$, then the line at infinity is the center of a rotation domain. Thus the line at infinity contains a complex 1-parameter family of Siegel disks, which vary holomorphically.

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Monodromy of Hénon Horseshoes

John H. Hubbard

(joint work with Sarah C. Koch and Chris Lipa)

Quadratic Hénon mappings are written

$$H_{a,c} : (x, y) \mapsto (x^2 + c - ay, x).$$

For $a \neq 0$ these are polynomial diffeomorphisms of $\mathbb{C}^2$.

For Jacobian positive and c sufficiently negative, the Hénon map is a standard horseshoe map. Let $\mathcal{H}$ be the component of the hyperbolicity locus containing these horseshoes. The set $\mathcal{H}$ is naturally the base of a locally trivial bundle of Cantor sets, where the fibers are conjugate to the full 2-shift, S_2.

This gives a representation

$$\pi_1(\mathcal{H}, p_0) \to \mathrm{Aut}(S_2).$$

We give a conjectural description of infinitely many automorphisms that are images of loops in $\mathcal{H}$.

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Wednesday Session: On Thurston's characterization of rational functions

Kevin M. Pilgrim

This lecture is an overview of Thurston's theorem on the characterization of rational functions: its statement, its significance to complex dynamics and other areas of mathematics, and its relation to current research.

A *Thurston map* is an orientation-preserving, continuous, branched covering map of the two-sphere S^2 to itself, of degree at least two, such that the set $P_F = \cup_{n>0} F^{\circ n}(B_F)$ is finite, where B_F denotes the set of branch points of F, and $F^{\circ n}$ denotes the n-fold composition of F with itself. For example: let $F(z) = z^2 - 1$ map the extended complex plane $\overline{\mathbb{C}}$ to itself. Then $P_F = \{0, -1, \infty\}$.

Two Thurston maps F, G are *equivalent* if there are orientation-preserving homeomorphisms $h_0, h_1 : (S^2, P_F) \to (S^2, P_G)$ satisfying $h_0 \circ F = G \circ h_1$ and h_0, h_1 are homotopic through maps agreeing on P_F. This is an equivalence relation which may loosely be thought of as topological conjugacy, up to isotopy relative to the set P_F.

There is a natural forgetful map from the set of rational Thurston maps, modulo conjugation by Möbius transformation, to the set of Thurston maps, modulo equivalence. *Thurston's Characterization and Rigidity Theorem* [2]

- gives a combinatorial characterization of the image of this map, and
- asserts that the fibers are either points, or comprise a one-complex dimensional family of so-called flexible Lattès examples.

Since equivalence of Thurston maps invovles homotopy-theoretic conditions, one expects that equivalence can be phrased in the language of algebraic topology. This was accomplished by Kameyama [7], whose constructions were reinterpreted by the author [15], [16]. Nekrashevych [14] introduced a broader algebraic framework, and with Bartholdi [1] showed that new algebraic invariants and constructions could be used to solve Hubbard's *Twisted Rabbit Problem*.

The rigidity portion of Thurston's theorem can be interpreted as a transversality result, as discussed in Hubbard's lecture. If $z_i, z_j \in \overline{\mathbb{C}}$ and f is a rational map sending z_i to z_j with local degree d_i, then the coefficients of f and the points z_i, z_j satisfy a system of d_i polynomial equations. The Riemann-Hurwitz relations imply that if one tries to find a rational Thurston map by solving equations, then the number of equations equals the number of unknowns.

The following phenomenon is an example of an obstruction. Suppose F is a rational Thurston map and $A \subset \overline{\mathbb{C}} - P_F$ is an essential annulus containing, in each complementary component, at least two points of P_F. Suppose $F^{-1}(A)$ contains connected components A_i which are essential subannuli of A mapping by degree δ_i onto A. The Grötzsch inequality asserts that $\sum_i \operatorname{mod}(A_i) \le \operatorname{mod}(A)$ with equality iff the A_i are right Euclidean subcylinders in the canonical Euclidean metric on A. This implies that $\sum_i \frac{1}{\delta_i} \le 1$ with equality if and only if F is a flexible Lattès map. This gives conditions on the degrees δ_i which must be satisfied by a rational map. Clearly, this condition can be phrased in terms of how homotopy classes of simple closed curves behave under backward iteration of a Thurston map.

Thurston's theorem was first formulated and proved by Milnor and Thurston for unimodal quadratic maps of the interval to itself [12]. This was then used to show that the entropy of a map $x \mapsto \lambda x(1-x)$, $x \in [0,1], \lambda \in (0,4]$ is monotone increasing in the parameter λ. In the complex quadratic family, conjecturally, every parameter c for which the map $p_c(z) = z^2 + c$ is infinitely renormalizable is a limit of parameters c_n corresponding to Thurston maps encoding the successive renormalizations of the polynomial p_c [11]. Kiwi [8] used Thurston's theorem to characterize which real laminations (models for the Julia set) arise from polynomials with connected Julia set and with no indifferent cycles. In summary, Thurston maps play a role in complex dynamics akin to the role played by rationals in the reals. By exploiting this idea, one builds maps with desired properties by limiting arguments, as Sorensen did [17] to produce quadratic maps p_c with non-locally connected Julia sets. By using such a map p_c as one factor in the intertwining surgery of Epstein and Yampolsky [4], Henriksen [6] proved that the rational part of the lamination associated to a cubic polynomial is not enough to determine its topological dynamics.

Since the category of Thurston maps is flexible, one can define combination, decomposition, and surgery procedures using them, and then try to interpret these results for rational maps. *Mating*, an example of a such a combination procedure, was used to explain the structure of certain quadratic rational functions [13]. Generalizations [18] now treat non-postcritically finite maps, with an eye towards understanding, eventually, the structure of general parameter spaces.

At present, there is little extant literature on noninvertible dynamics with branching in more than one real dimension. So, Thurston maps, if regarded as topological dynamical systems, are a natural class to investigate. Equivalence classes of Thurston maps contain uncountably many distinct topological conjugacy classes. However, two Thurston maps which are both expanding and equivalent are, by standard arguments, necessarily topologically conjugate, and so an equivalence class contains at most one topological conjugacy class of expanding map. Thus, it makes sense to speak of the dynamics of an expanding Thurston map.

The proof of Thurston's theorem proceeds via iteration of a map σ on Teichmüller space. As reinterpreted in [1], the inverse of σ sometimes descends to a map on moduli space, and Koch [9] used this to create interesting critically finite holomorphic endomorphisms of higher-dimensional complex projective spaces, as was discussed in her lecture.

There is interest in Thurston maps outside of dynamical systems as well. The iterated monodromy groups associated to Thurston maps provide examples of so-called selfsimilar groups; some such groups satisfy a finiteness property known as *contraction*. It can be difficult to verify when a selfsimilar group is contracting. The groups arising from expanding Thurston maps are contracting [14]; see also [2]. These groups can be interpreted as Galois groups.

McMullen, Sullivan, and Thurston developed a "dictionary" between rational maps and Kleinian groups regarded as dynamical systems on the sphere [10]. In this dictionary, Thurston's Rigidity Theorem is akin to Mostow rigidity. With Haissinsky, the author developed a map analog of a Gromov hyperbolic group [5]. There are at least four reasonable methods of identifying the Julia set of a Thurston rational map as a boundary at infinity of a Gromov hyperbolic space. The general theory shows the they are all quasi-isometric. The techniques used can be applied to prove that expanding Thurston maps possess a canonical quasisymmetry class of metric, and hence tools from metric space analysis and coarse geometry can be used to analyze and classify such spaces and dynamical systems.

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Thurston's Pullback Map

Sarah C. Koch
(joint work with Xavier Buff, Adam Epstein, and Kevin Pilgrim)

Let S^2 be the topological 2-sphere, and let $f : S^2 \to S^2$ be a branched covering map of degree $d \geq 2$. A particular case of interest is when S^2 can be equipped with an invariant complex structure for f. In that case, $f : \Sigma \to \Sigma$ is conjugate to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$.

According to the Riemann-Hurwitz formula, the map f has $2d-2$ critical points, counting multiplicities. We denote Ω_f the set of critical points and $V_f := f(\Omega_f)$ the set of critical values of f. The *postcritical set* of f is the set

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f).$$

The map f is *postcritically finite* if P_f is finite. Following the literature, we refer to such maps simply as *Thurston maps*.

Two Thurston maps $f : S^2 \to S^2$ and $g : S^2 \to S^2$ are *equivalent* if there are homeomorphisms $h_0 : (S^2, P_f) \to (S^2, P_g)$ and $h_1 : (S^2, P_f) \to (S^2, P_g)$ for which $h_0 \circ f = g \circ h_1$ and h_0 is isotopic to h_1 through homeomorphisms agreeing on P_f. In particular, we have the following commutative diagram:

$$\begin{tikzcd} (S^2, P_f) \arrow[r, "h_1"] \arrow[d, "f"'] & (S^2, P_g) \arrow[d, "g"] \\ (S^2, P_f) \arrow[r, "h_0"] & (S^2, P_g). \end{tikzcd}$$

In [3], Douady and Hubbard, following Thurston, give a complete characterization of equivalence classes of rational maps among those of Thurston maps. The characterization takes the following form.

A branched covering $f : (S^2, P_f) \to (S^2, P_f)$ induces a holomorphic self-map

$$\sigma_f : \mathrm{Teich}(S^2, P_f) \to \mathrm{Teich}(S^2, P_f)$$

of Teichmüller space. Since it is obtained by lifting complex structures under f, we will refer to σ_f as the *pullback map* induced by f. The map f is equivalent to a rational map if and only if the pullback map σ_f has a fixed point. By a generalization of the Schwarz lemma, σ_f does not increase Teichmüller distances.

For most maps f, the pullback map σ_f is a contraction, and so a fixed point, if it exists, is unique.

In this talk, we give examples showing that the contracting behavior of σ_f near this fixed point can be rather varied.

Theorem 1. *There exist Thurston maps f for which σ_f is contracting, has a fixed point τ and:*

(1) *the derivative of σ_f is invertible at τ, the image of σ_f is open and dense in* $\mathrm{Teich}(\mathbb{P}^1, P_f)$ *and* $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \sigma_f\big(\mathrm{Teich}(\mathbb{P}^1, P_f)\big)$ *is a covering map,*

(2) *the derivative of σ_f is not invertible at τ, the image of σ_f is equal to* $\mathrm{Teich}(\mathbb{P}^1, P_f)$ *and* $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ *is a ramified Galois covering map,*

or

(3) *the map σ_f is constant.*

The first example of a Thurston map f satisfying (3) was found by Curt McMullen.

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Transversality according to Adam Epstein

John H. Hubbard

Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with critical set Ω_f, $X \subset \mathbb{P}^1$ a finite set and Y a finite subset such that $X \cup f(X) \cup f(\Omega_f) \subseteq Y$.

We define a "Teichmüller space " $\mathrm{Def}(f, X)$ of rational functions that have the dynamical features of f that are "visible in X." In this setting we can differentiate deformations of these features. The resulting infinitesimal variations are naturally expressed in terms of the polar parts of meromorphic quadratic differentials. As an application of these ideas, and their connection to Thurston Rigidity, we prove the Fatou-Shishikura inequality.

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Thursday Session: Dynamics of transcendental functions

Lasse Rempe

Thursday morning saw a session of talks on transcendental dynamics. Lasse Rempe gave an introductory talk that highlighted connections between the dynamics of polynomials and that of transcendental entire functions. He discussed a theorem (with Rottenfußer, Rückert and Schleicher) that shows, for a certain class of entire functions, the existence of curves that can be seen as analogs of *dynamic rays* of polynomials. He also discussed a theorem that can be seen as introducing an analog of the *Böttcher map* into transcendental dynamics, and touched upon ways in which these ideas have begun to feed back into polynomial dynamics. Following on, Walter Bergweiler gave a more detailed discussion of escaping sets. He discussed recent results with Karpińska and Stallard on the dimension of escaping sets of entire functions of small infinite order. Also given were results regarding the escaping set of a quasiregular self-map of $\mathbb{R}^3$, originally constructed by Zorich, that can be seen as an analog of a complex exponential function. Finally, Schleicher spoke about joint work with Hubbard and Shishikura that generalizes Thurston's theorem on branched coverings of the sphere to *topological exponential maps.* Open problems on transcendental dynamics were later discussed in Friday's open problem session:

- Suppose f is a transcendental entire function with wandering domain U. Can the limit functions of $\{f^n|_U\}$ be bounded? Suppose the postsingular set of f is bounded; can f have wandering domains? What if we suppose all critical points are non-recurrent?
- Suppose $f_{\lambda,k}(z) = \lambda z e^z + k$, where $\lambda = e^{2\pi i\theta}$. Assume that the critical value and the singular value are non-recurrent but the asymptotic value a is contained in the ω-limit set of the critical point. Can f have a Siegel disk? If a is not in the ω-limit set, there is no Siegel disk.
- Find a topological model (reflecting the dynamics) of the filled Julia set of a quadratic polynomial with a Cremer fixed point of high type, and make a picture.
- Let P be a polynomial of degree 3 of bounded type, with a fixed Siegel disk. Assuming both critical points belong to the boundary of the Siegel disk, what is the conformal radius?
- Let $\mathcal{F}$ be a family of rational maps of degree d. Is the complement of the hyperbolicity locus precompact in

$$\mathrm{Rat}_d/\text{conjuagtion by Möbius transformations?}$$

- In the family $f_\lambda(z) = \lambda e^z$, is it possible to characterize the set of parameters λ such that
  - the set of non-escaping points of f_λ is connected?
  - the set of non-escaping points union $\{\infty\}$ is connected?
- Can Thurston's algorithm be implemented on the order of

$$(\deg(f) + |\Omega_f|)^n?$$

  That is, given a Thurston map f, can we find an algorithm assigning a starting point in $\mathrm{Teich}(S^2, P_f)$ such that after iterating $C(\deg(f) + |\Omega_f|)^n$ times, we are sufficiently close to a fixed point of σ_f, or we have found a Thurston obstruction?
- Suppose that f is a hyperbolic entire function with $P(f) \subset\subset F(f)$. Then is

$$\dim_{\mathrm{hyp}}(f) = \dim(J(f) - I(f)) < 2?$$

- Are there parapuzzles for cubics with connected Julia set?

Postsingularly finite exponential maps and limits of quadratic differentials

Dierk Schleicher

(joint work with John H. Hubbard, Mitsuhiro Shishikura)

In the theory of iterated rational maps, the easiest maps to understand are postcritically finite: maps whose critical orbits are all periodic or preperiodic. These maps are also the most important maps for understanding the combinatorial structure of parameter spaces of rational maps.

We know a lot about postcritically finite rational maps. The main result is a theorem of Thurston [DH] which gives a purely topological criterion for whether or not a given postcritically finite branched covering map $f : S^2 \to S^2$ of the two-sphere to itself is equivalent (in the sense of Thurston) to a rational map. Either a postcritically finite branched cover is equivalent to an essentially unique rational function or there is a "Thurston obstruction". Such an obstruction is a collection of simple closed curves such that a certain associated matrix has leading eigenvalue at least 1.

Thurston's theorem has two limitations. One is that the criterion is not easy to check, even though it is purely combinatorial-topological. More relevant to the present paper is the fact that the degree of the map enters in an essential way in the proof; the proof just does not go through for transcendental functions.

The simplest non-trivial transcendental maps are exponential maps

$$z \mapsto E_\lambda(z) = \lambda \exp(z)$$

with $\lambda \in \mathbb{C}^* := \mathbb{C} - \{0\}$. These have been investigated by many people; see for example [1, 3, DGH, S1, RS1] and the references in these papers. Exponential maps have no critical values, but the unique singular value 0 plays an analogous role.

Postsingularly finite exponential maps are those for which the orbit of 0 is preperiodic. There are countably many such parameters. Bergweiler (unpublished) has used value distribution theory to estimate their density with respect to $|\lambda|$. There are no exponential maps with periodic singular orbits (but there are countably many hyperbolic components in exponential parameter space; these are completely classified in [S2]).

A *topological exponential map* is a covering map $f : S^2 - \{\infty\} \to S^2 - \{0, \infty\}$; this bears the same relation to exponentials as branched coverings $S^2 \to S^2$ bear to rational functions. Our Main Theorem is the analog of Thurston's characterization theorem: we show that a postsingularly finite topological exponential map is either equivalent to a holomorphic exponential map, or it admits a (degenerate) Levy cycle. As with Thurston's result, the complete classification of postsingularly finite maps is a separate step; we refer to [LSV] for details.

In the mid-1980's, [DGH] gave a conjectural description for postsingularly finite exponential maps in analogy to and as a limit of results for polynomials $\lambda(1+z/d)^d$ with a single finite critical point as $d \to \infty$. The theory of spiders [HS] was developed in the process. Our results confirm the conjecture in [DGH].

We use the same machinery for our proof as Thurston: given a postsingularly finite topological exponential map $f : S^2 \to S^2$, we set up a *Thurston map*

$$\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$$

in an appropriate Teichmüller space $\mathcal{T}_f$, and show that either σ_f has a fixed point, in which case the topological exponential map is equivalent to a holomorphic exponential map, or the iteration of σ_f diverges in Teichmüller space, and there is a degenerate Levy cycle.

As mentioned above, the proof given in [DH] for Thurston's result on rational maps depends in an essential way on the fact that rational maps have finite degree; it does not work for exponentials. That paper shows that, depending on the initial point of the iteration of σ_f, there is a subset of Teichmüller space (with compact projection to moduli space) such that as soon as the iteration leaves this subset, the existence of a Thurston obstruction follows. A key ingredient in this proof is an estimate about how moduli of annuli on the finitely-punctured Riemann sphere increase when erasing the points in $f^{-1}(P_f) - P_f$; the cardinality of this set is bounded by $d|P_f|$, which diverges when $d \to \infty$.

We use a different strategy to relate the failure of convergence to the existence of a Thurston obstruction. The cotangent space $T^*_\tau \mathcal{T}_f$ to Teichmüller space at $\tau \in \mathcal{T}_f$ is a certain space $Q^1(\tau)$ of integrable meromorphic quadratic differentials on $\mathbb{C}$, with at most as many poles as the length of singular orbit (plus possibly a pole at ∞). The dual of the L^1-norm on the space $Q^1(\tau)$ space defines the infinitesimal Teichmüller metric on $\mathcal{T}_f$. The analytic map $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ is weakly contracting for this metric in the sense that $\|d\sigma_f\| = \|(d\sigma_f)^*\| < 1$. This isn't surprising: all analytic maps are non-expanding; but the norm may tend to 1 as we iterate σ_f.

More precisely, if the sequence

$$\tau_0, \tau_1 = \sigma_f(\tau_0), \ldots, \tau_{n+1} = \sigma_f(\tau_n), \ldots$$

does not converge in $\mathcal{T}_f$, then there must exist $q_n \in Q^1(\tau_n)$ with $||q_n||_{\mathbb{C}} = 1$ such that $\lim_{n\to\infty} ||(d\sigma_f)^*(q_n)||_{\mathbb{C}} = 1$.

In this case, the q_n cannot converge in $L^1(\overline{\mathbb{C}})$. In fact, poles must coalesce, and very fat annuli in the complement of the poles of the q_n must develop; the core curves of these annuli will present us with the needed Levy cycle.

Proving this requires understanding how the mass of degenerating quadratic differentials is distributed. We prove a "thick-thin" decomposition theorem which describes this distribution in considerable detail.

Although in this paper we use this decomposition only for integrable meromorphic quadratic differentials on the Riemann sphere, it is proved for integrable quadratic differentials on an arbitrary Riemann surface; moreover, the constants that appear are independent of the topology. As such it may have many other applications.

Exponential maps are of course rather special transcendental entire maps. However, we believe that our methods should help to prove a similar result for larger classes of transcendental maps.

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Interplay between polynomial and transcendental entire dynamics

Lasse Rempe

In this lecture, we explore some aspects of the dynamics of polynomials and of transcendental entire functions that are related to each other.

There have been a number of influences of polynomial dynamics on the study of transcendental entire functions. Some of these are in the form of results and proofs that carry over directly. However, more often (and more interestingly), polynomial concepts that cannot be translated directly to the transcendental case may nonetheless inspire useful analogies.

The relationship between the two fields is not one-sided, however. As the understanding of transcendental entire dynamics has progressed, features and connections with some intriguing aspects of polynomial dynamics have also become apparent, and transcendental techniques are now beginning to be applied to study these.

In the main part of the talk, I will report on results in transcendental dynamics that have a distinct flavor of polynomial dynamics. Much of this work has previously been reported at Oberwolfach in 2007 [R2, Sch], although with a somewhat different emphasis. At the end of the talk, I will report on recent developments where transcendental methods are having an important impact on the understanding of polynomial dynamics, although this is still very much a nascent field.

Dynamic rays. If f is a transcendental entire function, then the *Fatou set* $F(f)$ is defined, as in the rational case, as the set of points $z \in \mathbb{C}$ that have a neighborhood where the iterates f^n form a normal family; the *Julia set* is the complement $J(f) := \mathbb{C} \setminus F(f)$. In our talk, we will also encounter the *escaping set*,

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\};$$

Eremenko [E] proved that $J(f) = \partial I(f)$.

Among the cornerstones of polynomial dynamics are *dynamic rays*, i.e. external rays for the filled Julia set. For transcendental entire functions, the escaping set is no longer open, and often has no interior at all, hence this concept does not generalize directly to this setting.

Nonetheless, it has been known since the 1980s that the escaping set of some entire functions consists of curves, and that these curves can be thought of as analogs of the dynamic rays of polynomials (see e.g. [DGH]). Until recently, such results where known only relatively restricted classes of entire functions, which nonetheless include many explicit examples such as the *exponential family* $z \mapsto \exp(z) + \kappa$ or the *cosine family* $z \mapsto a\exp(z) + b\exp(-z)$.

However, in joint work with Rottenfußer, Rückert and Schleicher [R^{3}S], we have been able to extend the class of functions with this type of behavior considerably. To state this result, we define the *Eremenko-Lyubich class*

$$\mathcal{B} := \{f \text{ transcendental, entire} : S(f) \text{ is bounded}\}.$$

Here $S(f)$ is the set of *singular values*; that is, the smallest closed set such that $f : f^{-1}(\mathbb{C} \setminus S(f)) \to \mathbb{C} \setminus S(f)$ is a covering map. Equivalently, $S(f)$ is the closure of the set of all critical and asymptotic values of f. Eremenko and Lyubich showed that $f \in \mathcal{B}$ implies $I(f) \subset J(f)$. Also, functions in this class have, in some sense, a natural combinatorial structure near infinity, so this is a natural setting for trying to apply ideas from the polynomial theory.

We also note that an entire function f is said to have *finite order* if there is some C such that

$$|f(z)| \leq e^{|z|^C}$$

for all sufficiently large z.

Theorem 1 ([R^{3}S, Theorem 1.2])**.** *Suppose that f is an entire function that can be written as a composition of finitely many finite-order functions in $\mathcal{B}$. Then every escaping point $z \in I(f)$ can be connected to ∞ by a curve consisting of escaping points.*

Barański [B] independently proved this result in the case where f has finite order and is of *disjoint type*, i.e. the singular set is contained in a single immediate attracting basin. This result, together with Theorem 3 below, can be used to give an alternative proof of Theorem 1 in the finite-order case.

The functions covered by this theorem include the above-mentioned explicit families, but also many more. As an example (suggested by Adam Epstein in the 1990s as a type of entire function to study), we consider the linearizing map associated to a repelling fixed point of a polynomial. More precisely, if p is a polynomial with $p(0) = 0$ and the multiplier $\mu := f'(0)$ has modulus greater than one, then there is an entire function $\psi : \mathbb{C} \to \mathbb{C}$ such that $p(\psi(z)) = \psi(\mu z)$ for all $z \in \mathbb{C}$. This function is called a *Poincaré* function, and it is easy to check that ψ has finite order. Moreover, the set of singular values $S(\psi)$ coincides with the postsingular set of p, so if p has connected Julia set, then ψ belongs to the Eremenko-Lyubich class.

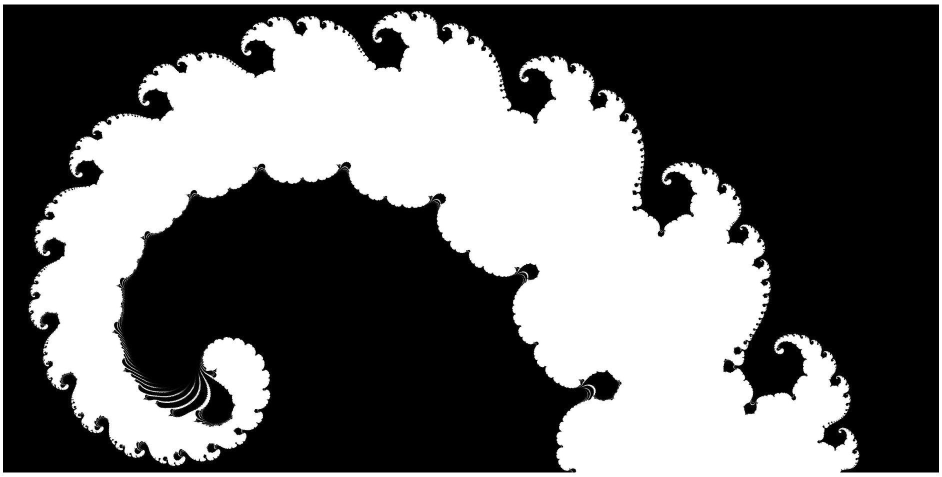

FIGURE 1. The Julia set of a Poincaré function ψ. Here ψ was normalized in a way that ensures that the Fatou set, shown in white, is a single immediate basin of attraction. The polynomial p is the center of the 1/15-limb of the Mandelbrot set.

If the multiplier μ is non-real, then the set where ψ is large will "spiral" towards infinity. (See Figure 1.) By our theorem, the escaping set of this function nonetheless consists entirely of curves.

Theorem 1 gives a partial positive answer to a question of Eremenko [E], who asked whether the conclusion of the theorem holds for *all* transcendental entire function. We show that this is not the case. (Recall that $I(f) \subset J(f)$ when $f \in \mathcal{B}$.)

Theorem 2 ([R^{3}S, Theorem 1.1]). *There exists a function $f \in \mathcal{B}$ (of disjoint type) such that $J(f)$ contains no curves to infinity.*

(That is, $J(f) \cup \{\infty\}$ is a compact, connected set such that the path-connected component of ∞ is a point.)

For any $\epsilon > 0$, this function can be chosen such that

$$\log \log |f(z)| < (\log |z|)^{1+\epsilon} \tag{*}$$

when $|z|$ and $|f(z)|$ are sufficiently large.

In fact, one can even construct a function $f \in \mathcal{B}$ for which $J(f)$ contains *no* nontrivial curves at all (this example, however, grows considerable more quickly than (*)).

Böttcher maps. We have seen that the analogy with external rays of polynomials does not extend to all functions, even in class $\mathcal{B}$. It is a reasonable question whether there is still some hope of studying families of "pathological" examples as in Theorem 2. The next result is an analog of yet another idea from polynomial dynamics: the Böttcher map. It holds in perhaps surprising generality. Let us say that two entire functions f and g are *quasiconformally equivalent near infinity* if there are quasiconformal maps $\phi, \psi : \mathbb{C} \to \mathbb{C}$ such that

$$\psi(f(z)) = g(\phi(z)) \tag{1}$$

whenever $|f(z)|$ and $|g(z)|$ are large enough.

Theorem 3 ([R1]). *Let $f, g \in \mathcal{B}$ be quasiconformally equivalent near infinity. Then there exists a quasiconformal map $\theta : \mathbb{C} \to \mathbb{C}$ such that $\theta \circ f = g \circ \theta$ on*

$$J_R(f) := \{z : |f^n(z)| \geq R \text{ for all } n \geq 1\}$$

Furthermore, θ has zero dilatation on $\{z \in A_R : |f^n(z)| \to \infty\}$.

Thus, even though the structure of the escaping set can change dramatically within class $\mathcal{B}$, it will stay constant within any given quasiconformal equivalence class. So any function quasiconformally equivalent to the function from Theorem 2 will also have essentially the same pathological behavior near infinity. This is all the more surprising as other properties, such as the order of growth, *can* change within quasiconformal equivalence classes. We also note that Theorem 3 is false without the assumption that $f, g \in \mathcal{B}$: take e.g. $f(z) = z + 1 + \exp(-z)$ and $g(z) = z - 1 + \exp(-z)$.

Theorem 3 also leads to results about the *rigidity* of escaping dynamics, and from there on to theorems on density of hyperbolicity in certain parameter spaces of real transcendental entire functions (joint with van Strien). See [R2] for further details and statements of results.

Transcendental influences in polynomial dynamics. Recent years have revealed some intriguing similarities between transcendental dynamics and features of various types of "infinitely renormalizable" polynomial dynamics (where high-degree iterates influence the dynamical behavior). More precisely, features reminiscent of the dynamics of exponential maps appear in the study of *parabolic and near-parabolic renormalization*; this was first observed by Shishikura in his proof that the boundary of the Mandelbrot set has Hausdorff dimension 2 [Sh]. Also, recent work of Avila and Lyubich [AL] on the hyperbolic dimension of constant-type Feigenbaum quadratics resonates e.g. with work of Urbanski and Zdunik [UZ] on the measurable dynamics of exponential maps.

Connections in the case of near-parabolic maps have recently begun to become much more explicit. Indeed, Shishikura has shown that, for many quadratic polynomials with a Cremer fixed point, any *Perez-Marco hedgehog* is path-connected: this result and its proof are in direct analogy to Theorem 1 for exponential maps.

In fact, Buff and Chéritat suggested a topological model for hedgehogs of Cremer quadratics; in the following, let us refer to these as "nonlinearizable arithmetical hedgehogs". Using the same ideas as in the proof of Theorem 3, we were recently able to give a complete description of the topology of these hedgehogs, in joint work with Buff, Chéritat and Inou (see Figure 2).

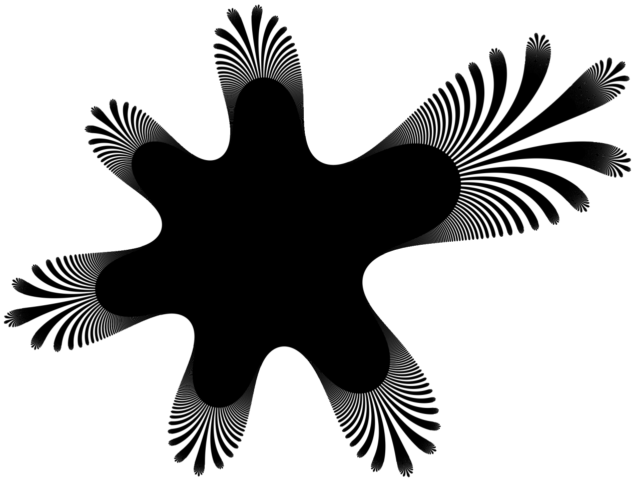

FIGURE 2. A nonlinearizable arithmetical hedgehog (picture courtesy of Arnaud Chéritat)

Theorem 4 (Hedgehogs)**.** *Every nonlinearizable arithmetical hedgehog K is a* Cantor bouquet*. That is, K is homeomorphic to $J(f) \cup \{\infty\}$, where f is the exponential map $f(z) = \exp(z) - 2$.*

It seems likely that connections between polynomial and transcendental dynamics such as the ones I described here will yield further fruitful insights in the future.

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Devaney hairs and Karpińska's paradox in dimension three

Walter Bergweiler

The *Julia set* $J(f)$ of a (non-linear) entire function $f : \mathbb{C} \to \mathbb{C}$ is the set where the iterates of f do not form a normal family. If ξ is an attracting fixed point of f, then

$$A(\xi) = \left\{ z : \lim_{k\to\infty} f^k(z) = \xi \right\}$$

is called the *attracting basin* of ξ. It is a standard result of complex dynamics that $\partial A(\xi) = J(f)$.

If $0 < \lambda < 1/e$, then the function $E_\lambda(z) = \lambda e^z$ has an attracting fixed point $\xi \in \mathbb{R}$. Devaney and Krych [1] proved that $J(E_\lambda) = \mathbb{C}\backslash A(\xi)$ and that $J(E_\lambda)$ is a "Cantor set of curves" for such λ. To put this in a precise form we say that a subset H of $\mathbb{C}$ (or $\mathbb{R}^n$) is a *(Devaney) hair* if there exists a homeomorphism $\gamma : [0, \infty) \to H$ such that $\gamma(t) \to \infty$ as $t \to \infty$. We call $\gamma(0)$ the *endpoint* of the hair H. With this terminology we obtain the following result from the work of Devaney and Krych.

Theorem A. *If $0 < \lambda < 1/e$, then $J(E_\lambda)$ is an uncountable union of pairwise disjoint Devaney hairs.*

We denote by $\dim S$ the Hausdorff dimension of a subset S of $\mathbb{C}$ (or $\mathbb{R}^n$). The following result is due to McMullen [7].

Theorem B. *If $\lambda \neq 0$, then $\dim J(E_\lambda) = 2$.*

In the situation of Theorem A the union of the Devaney hairs thus has Hausdorff dimension 2. Karpińska [5] proved the surprising and seemingly paradoxical result that this changes if one removes the endpoints of the hairs.

Theorem C. *Let $0 < \lambda < 1/e$ and let C_λ be the set of endpoints of the hairs that form $J(E_\lambda)$. Then $\dim(J(E_\lambda)\backslash C_\lambda) = 1$.*

Of course, it follows from Theorems B and C that $\dim C_\lambda = 2$. This had been proved before also by Karpińska [4].

An important example of a quasiregular map $F : \mathbb{R}^3 \to \mathbb{R}^3$ was given by Zorich [10]; see [8] for an introduction to quasiregular maps. Zorich's map can be considered as a three-dimensional analogue of the exponential function. To describe the map, we follow [3] and consider the square

$$Q := \left\{ (x_1, x_2) \in \mathbb{R}^3 : |x_1| \leq 1, |x_2| \leq 1 \right\}$$

and the upper hemisphere

$$U := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1,\ x_3 \geq 0\right\}.$$

Let $h : Q \to U$ be a bilipschitz map and define $F : Q \times \mathbb{R} \to \mathbb{R}^3$,

$$F(x_1, x_2, x_3) = e^{x_3} h(x_1, x_2).$$

Then F maps the "infinite square beam" $Q \times \mathbb{R}$ bijectively onto the upper half-space. By repeated reflection along the sides of square beams and the (x_1, x_2)-plane we obtain a map $F : \mathbb{R}^3 \to \mathbb{R}^3$. It turns out that this map F is quasiregular. Its dilatation is bounded in terms of the bilipschitz constant of h. We denote the norm of $Df(x)$ by $|Df(x)|$ and put $\ell(Df(x)) := \inf_{|h|=1} |Df(x)(h)|$.

If $DF(x_1, x_2, 0)$ exists, then

$$DF(x_1, x_2, x_3) = e^{x_3} DF(x_1, x_2, 0).$$

It follows that there exists $\alpha, m, M \in \mathbb{R}$ with $0 < \alpha < 1$ and $m < M$ such that

$$|DF(x_1, x_2, x_3)| \leq \alpha \quad \text{a.e. for } x_3 \leq m$$

while

$$\ell(DF(x_1, x_2, x_3)) \geq \frac{1}{\alpha} \quad \text{a.e. for } x_3 \geq M.$$

We choose

$$a \geq e^M - m$$

and consider the map $f_a : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f_a(x) = F(x) - (0, 0, a).$$

The following result can be seen as a three-dimensional analogue of Theorems A, B and C.

Theorem 1. *Let f_a be as above. Then there exists a unique fixed point $\xi = (\xi_1, \xi_2, \xi_3)$ satisfying $\xi_3 < m$. The set*

$$J := \left\{x \in \mathbb{R}^3 : f^n(x) \not\to \xi\right\}$$

consists of uncountably many pairwise disjoint hairs. The set C of endpoints of these hairs has Hausdorff dimension 3 while $J \backslash C$ has Hausdorff dimension 1.

The set C_λ of endpoints of the hairs in $J(E_\lambda)$, with $0 < \lambda < 1/e$, can also be characterized as the set of points which are accessible from the basin of attraction $A(\xi)$; see [2] and also [5]. The following result shows that the situation is different for Zorich maps.

Theorem 2. *Let f_a and J be as in Theorem 1. Then all points of J are accessible from $\mathbb{R}^3 \backslash J$.*

We mention that results similar to Theorems A and C have been obtained for general parameter values λ by Schleicher and Zimmer [9]. Besides the methods of [1, 5, 7], the proof of Theorem 1 also uses some of their techniques.

Remark 1. Instead of the square $Q = \{(x_1, x_2) \in \mathbb{R}^3 : |x_1| \le 1, |x_2| \le 1\}$ we could have taken any rectangle $Q = \{(x_1, x_2) \in \mathbb{R}^3 : |x_1| \le c_1, |x_2| \le c_2\}$, with $c_1, c_2 > 0$, in the construction of F and f_a. In particular, we may take $c_1 = \frac{1}{2}\pi$ and we may choose the function $h : Q \to U$ such that $h(x_1, 0) = (\sin x_1, 0, \cos x_1)$. Then $F(x_1, 0, x_3) = (e^{x_3} \sin x_1, 0, e^{x_3} \cos x_1)$. The function F thus leaves the (x_1, x_3)-plane invariant and its restriction to this plane is conjugate to the exponential function in the plane. This underlines that Zorich maps can be seen as three-dimensional analogues of the exponential function.

Remark 2. In contrast to the exponential function, Zorich maps have branch points. In fact, the branch points of F are the edges of the square beam $Q \times \mathbb{R}$ and the lines obtained from these edges by reflection. We mention that a theorem of Zorich [10] says that if $n \ge 3$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasiregular and not injective, then f has branch points.

Remark 3. A quasiregular map f is called *uniformly quasiregular* if the dilatation of the iterates f^k has an upper bound which does not depend on k. For uniformly quasiregular maps $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$, where $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, an iteration theory in the spirit of Fatou and Julia has been developed by Hinkkanen, Martin, Mayer and others; see [3, section 21] for an introduction. In principle it would also be possible to develop such a theory for uniformly quasiregular maps $f : \mathbb{R}^n \to \mathbb{R}^n$. However, for $n \ge 3$ no examples of such maps which do not extend to quasiregular self-maps of $\overline{\mathbb{R}^n}$ are known.

Remark 4. Zorich maps may also be defined in $\mathbb{R}^n$ for $n \ge 4$; see [6]. While it seems that the methods of this paper extend to this more general case, we have restricted to the case $n = 3$ for simplicity. We note that Iwaniec and Martin [3], whose presentation we have followed in the definition of Zorich maps, also confine themselves to the case $n = 3$. Restriction to this case thus allows to use their results directly.

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Friday Session: Renormalization, rigidity, and a priori bounds in holomorphic dynamics

Mikhail Lyubich

The final day of the conference was dedicated to the themes of renormalization, rigidity, and a priori bounds in holomorphic and one-dimensional dynamics. Jeremy Kahn presented a proof of the Covering Lemma, which is a recently discovered new analytical tool that has immediately found applications to several long-standing problems (e.g., to the the problem of local connectivity of the higher degree Mandelbrot set). Artur Avila outlined a proof of the exponential contraction property of the renormalization horseshoe for higher degree unicritical maps based on a new version of the Schwarz Lemma. Even in degree two, this proof is substantially more efficient than the previously known arguments. In the afternoon, Mitsuhiro Shishikura presented his work with Inou on almost parabolic renormalization that has recently played a crucial role in constructing a Julia set of positive area. He then applied this renormalization theory to prove rigidity of the hedgehogs of polynomial-like Cremer maps. The program concluded with a problem session, with a lively discussion of possible directions of further research.

The Schwarz Lemma

Artur Avila
(joint work with Mikhail Lyubich)

Let $d \geq 2$ be even, let $p_c(z) = z^d + c$, and let $\mathcal{M}$ be the corresponding Multibrot set, the set of parameters for which p_c has a connected Julia set. If $c \in \mathbb{R} \cap \mathcal{M}$, p_c restricts to a unimodal map in an interval $I = I_c$. In his fundamental work [L3], Lyubich established the fundamental "regular or stochastic" dichotomy for the real quadratic family. It states that, for $d = 2$, almost every parameter $c \in \mathbb{R} \cap \mathcal{M}$ corresponds either to a regular map, in the sense that it admits an attracting periodic cycle, or to a stochastic map, in the sense that it admits an absolutely continuous invariant measure (in fact "stochastic" was later refined to "Collet-Eckmann" in [AM], which is a condition on the critical orbit which implies a much more precise statistical description of the dynamics). By then it was known that regular maps correspond to an open and dense set of parameters ([GS], [L2]) while stochastic maps correspond to a positive measure set of parameters [J].

The understanding of the quadratic family achieved in [L3] was instrumental to subsequent results about more general analytic families of unimodal maps (but still with a quadratic critical point) obtained in [ALM]. However, the quadratic case has several specific features that do not persist for higher criticality. For instance, in the quadratic case any topological attractor is a measure-theoretical attractor (in the sense of Milnor), see [L1], while for sufficiently large criticality, wild (measure-theoretical, but not topological) attractors appear for certain parameters c [BKNS]. Moreover, at the technical level, several key results in the "quadratic theory" were developed around the "exponential decay of geometry", or "linear moduli growth" properties, which do not hold in any higher criticality.

Only recently technical tools were developed to overcome such hurdles for all criticalities, with [KSS] establishing density of regular parameters, and [KL1], [KL2] and [AKLS] developing the "Yoccoz theory" of finitely renormalizable (complex) parameters (we note that, even if ultimately interested only on real parameter values, the understanding of certain complex parameters is still necessary for several results).

Our goal is to extend the "regular or stochastic" result to higher criticality (extensions in the line of [AM] follow as well). Indeed, in [ALS], it is established that almost every parameter value is either regular, stochastic or infinitely renormalizable (which are neither regular or stochastic), eliminating for instance the possibility that "wild attractors" could correspond to a positive measure set of parameters. This work reduced the problem to proving that infinitely renormalizable parameters have zero Lebesgue measure.

According to the approach of [L3], such a result is a consequence of hyperbolicity of the renormalization operator in certain spaces of polynomial-like maps: hyperbolicity implies that renormalization windows inherit a definite proportion of regular parameter values, which prevents the existence of density points of infinitely renormalizable parameters. This argument combines usual hyperbolicity arguments with the fact that the space of polynomial-like maps with connected Julia set is foliated by hybrid classes, and this foliation, being codimension-one, has necessarily quasiconformal holonomy.

Given the results obtained in [ALS], the fundamental difference in establishing the hyperbolicity of renormalization for higher criticality turns out to lie in the proof of exponential contraction of the renormalization operator along hybrid classes. Indeed this result was established in the quadratic case by splitting in several cases: "bounded type", where the technique of McMullen towers (with geometric flavor) can be applied, "essentially bound type", where "parabolic towers" can be developed, and "high type", where macroscopic contraction towards polynomials is transparent in view of "moduli growth" (the Schwarz Lemma in Banach spaces yielding then the infinitesimal statement).

In higher criticality, several other possibilities arise (related to the lack of decay of geometry), and the approach taken in [L3] appears to be inadequate. In this talk, we describe a simpler, unified, approach to the contraction of renormalization, replacing the "geometric ingredients", whose specifics depend on the degree, by soft analysis. We discuss a proof of exponential contraction given "beau bounds" in the sense of Sullivan, which is based on a suitable generalization of the Schwarz Lemma (to the Caratheodory metric in path holomorphic spaces).

To some extent, this can be seen as the realization of the original Sullivan approach to contraction of renormalization. His "hybrid classes" consisted of polynomial-like maps considered up to a less strict equivalence relation, which allowed him to treat them as "Teichmüller spaces of Riemann surface laminations". Proving contraction of holomorphic maps in Teichmüller spaces is of course a recurring theme in holomorphic dynamics, however the weird Teichmüller spaces Sullivan needed to deal with were much more delicate to analyze than the usual ones, and this approach yielded only somewhat less satisfactory results [MS].

Of course in our approach, the "beau bounds" are the "hard fact" that still needs to be established before exponential contraction can be concluded. Here beau bounds means a strong "precompactness statement", that if one renormalizes enough (depending on the quality of the original polynomial-like map), one obtains polynomial-like maps with fixed quality. It turns out to be quite subtle to establish such control throughout entire hybrid classes, but it is much easier to achieve this if one restricts considerations to real-symmetric maps (for the bounded type case this is a result of Sullivan [MS], and in general it is due to [LS] and [LY]). In our work we also establish that "beau bounds for real maps" imply the full "beau bounds" throughout the hybrid classes of real maps, necessary for our approach, through soft analysis (with a topological ingredient). Besides allowing the development of the renormalization theory in higher degree, this in fact simplifies the current approach to the contraction of renormalization even in the quadratic case.

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Degenerate Complex Structures and the Covering Lemma

Jeremy Kahn
(joint work with Mikhail Lyubich)

We prove (in abstract) the covering lemma of [1] using degenerate complex structures. If $X \subset Y$ are disks, then we let $\mod(Y, X)$ denote the modulus of the annulus $Y \setminus X$. We first state the covering lemma:

Theorem 1. *Suppose that $A \subset A' \subset U$ and $B \subset\subset B' \subset\subset V$ are disks, and $f : U \to V$ is proper and holomorphic, and $f : A \to B$ and $f : A' \to B'$ are also proper. Let $D = \deg f$ and $d = \deg f|_{A'}$. Then for any $\alpha, \eta > 0$, if*

$$\mod(B', B) \geq \eta \quad \mod(U, A)$$

and

$$\mod(U, A) \leq \epsilon(\alpha, \eta, D)$$

then

$$\mod(V, B) \leq \left(\frac{4}{3} + \epsilon\right)\frac{d^2}{\eta} \quad \mod(U, A).$$

We let $\mod^+(B', B)$ denote the modulus of the largest annulus immersed in $V \setminus B$ that avoids the critical values of f lying in $V \setminus B'$. Clearly $\mod^+(B', B) \geq \mod(B', B)$.

We define a degenerate complex structure (of the first kind) on a topological surface S as a finite union T of disjoint embedded paths $(\gamma_\alpha)_{\alpha \in A}$ along with a "reference metric" ρ. If $X \subset Y$ are disks, we define $\mod(Y, X; (T, \rho))$ as

$$\sup_{f : T \to \mathbb{R}^+} \frac{(\inf \int_\gamma f\rho)^2}{\int_T f^2 \rho},$$

where the infimum is taken over the finite set of paths γ such that $\gamma \subset T$ and γ crosses $Y \setminus X$. Returning to the setting of $f : (A, A', U) \to (B, B', V)$, where U and V are now topological surfaces, and f is a branched cover, we can prove that if (T, ρ) is a degenerate complex structure on V, then for any $\eta > 0$, if

$$\mod^+(B', B; (T, \rho)) \geq \eta \quad \mod(U, A; (T, \rho))$$

then

$$\mod(V, B; (T, \rho)) \leq \frac{4d^2}{3\eta} \quad \mod(U, A; f^*(T, \rho)).$$

Thus we have proven the covering lemma for degenerate complex structures, with the constant $4/3 + \alpha$ replaced with $4/3$ (which is sharp), and the requirement that $\mod(U, A)$ is sufficiently small has been removed.

Now let (σ_i) be a sequence of complex structures such that $\mod(U, A)$ goes to zero (when computed with regard to $f^*\sigma_i$). After passing to a subsequence, we can find a sequence $\lambda_i \to \infty$ and a degenerate complex structure (T, ρ) on V such that

$$\lim_{i\to\infty} \lambda_i \quad \operatorname{mod}\,(V, B; \sigma_i) = (V, B; (T, \rho)), \tag{1}$$

$$\lim_{i\to\infty} \lambda_i \quad \operatorname{mod}^{+}(B', B; \sigma_i) = \quad \operatorname{mod}^{+}(B', B; (T, \rho)), \tag{2}$$

and

$$\lim_{i\to\infty} \lambda_i \quad \operatorname{mod}\,(U, A; f^*\sigma_i) \geq \quad \operatorname{mod}\,(U, A; f^*(T, \rho)). \tag{3}$$

This suffices to prove the covering lemma.

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Applications of near-parabolic renormalization

Mitsuhiro Shishikura

A fixed point z_0 of a holomorphic function $f(z)$ is called *irrationally indifferent*, if $f'(z_0) = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Irrationally indifferent fixed points are a rich source of interesting problems, such as linearization problem (solved by Siegel, Brjuno and Yoccoz), non-locally connected Julia sets (Douady-Sullivan), questions of the boundary of Siegel disks, local invariant sets (Perez-Marco's hedgehogs) and construction of quadratic Julia set with positive area (carried out by Buff-Chéritat). Our approach is to study irrationally indifferent fixed points of *high type* (defined below) via near-parabolic (or cylinder) renormalization. A croissant-shped fundamental region is constructed for such an f and the quotient of the region by f gives a Riemann surface isomorphic to $\mathbb{C}/\mathbb{Z}$, which in turn isomorphic (by $\mathrm{Exp}^\sharp(z) = e^{2\pi i z}$) to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The first return map to the fundamental region induces a holomorphic function $\mathcal{R}f$ which is defined near 0 (and near ∞).

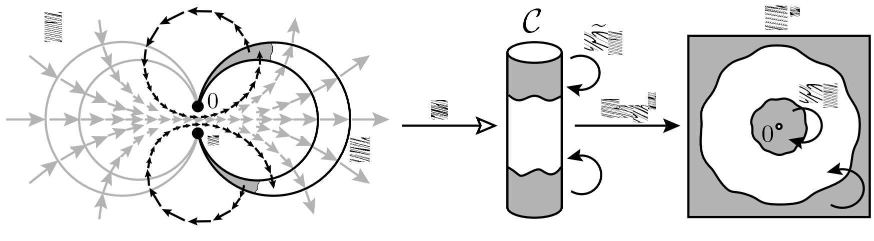

Figure 1. Definition of the first return map to the croissant-shaped fundamental region S and the near-parabolic renormalization $\mathcal{R}$

An irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ can be written as an accelerated continued fraction of the form:

$$\alpha = a_0 + \cfrac{\epsilon_0}{a_1 + \cfrac{\epsilon_1}{a_2 + \cfrac{\epsilon_2}{\ddots}}}, \quad \text{where} \quad \begin{array}{l} a_n \in \mathbb{Z}, \quad \epsilon_n = \pm 1 \ (n = 0, 1, 2, \dots), \\ a_n \geq 2 \ (n \geq 1). \end{array}$$

By *high type*, we mean that $a_i \geq N$ with a large N.

The key is the study of near-parabolic renormalization is the following theorem, which serves as an a priori bound for the renormalizations $\mathcal{R}f, \mathcal{R}^2 f, \mathcal{R}^3 f, \dots$.

Theorem 1 (Inou-Shishikura)**.** *Let $P(z) = z(1+z)^2$ and V a certain domain containing 0 and $-\frac{1}{3}$ (a critical pt of P). Define*

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \left| \begin{array}{c} \varphi : V \to \mathbb{C} \textit{ is univalent} \\ \varphi(0) = 0, \ \phi'(0) = 1 \textit{ and} \\ \textit{has a quasiconformal extension to } \mathbb{C} \end{array} \right. \right\}.$$

(i) For $f \in \mathcal{F}_1$, $f(0) = 0$, $f'(0) = 1$, $f''(0) \neq 0$ and f has a unique critical point;
(ii) If $h = z + z^2$ or $h \in \mathcal{F}_1$ and α is of sufficiently high type (i.e., there exists an N such that the following holds for α as above with $a_i \geq N$),
then for $f(z) = e^{2\pi i \alpha} h(z)$, $\mathcal{R}^n f$ are well defined and have the form $\mathcal{R}^n(f) = e^{2\pi i \alpha_n} h_n$ with $h_n \in \mathcal{F}_1$ $(n = 1, 2, \dots)$;
(iii) If f and $\hat{f}$ are as in (ii) with the same α, then for corresponding h_n and $\hat{h}_n$, we have $d(h_n, \hat{h}_n) \to 0$ $(n \to \infty)$ exponentially fast, where d is a certain (complete) metric defined on $\mathcal{F}_1$.

For a map with a parabolic fixed point, one can color the points in the parabolic basin as follows. One can define in the parabolic basin the attracting Fatou coordinate which conjugates the map to the translation $T : z \mapsto z + 1$. The Fatou coordinate is normalized so that the unique critical point corresponds to the coordinate 0. Points in the basin is colored according to the integer part of the real part of attracting Fatou coordinate (modulo 3). Additional shading is given according to the imaginary part y $(y < -2, -2 < y < 0, 0 < y < 2, 2 < y)$. This coloring is called *Checkerboard Pattern*. In the proof of the above theorem, it was essential to realize that within the class $\mathcal{F}_1$ (even when the map is only partially defined), some portion of checkerboard pattern (called *Truncated Checkerboard Pattern* can be defined and it persists after a small perturbation. More precisely, for a near-parabolic map, Truncated Checkerboard Pattern should be cut off at a certain width and neighborhoods of its left boundary and right boundary must be glued together. This gluing ψ_f comes from the fact that after a perturbation, attracting and repelling cylinders are identified by the orbits going through a narrow "gate" between the two fixed points.

In order to apply the above theorem, we need to derive some conclusions on the dynamics of f from that of $\mathcal{R}f, \mathcal{R}^2 f, \mathcal{R}^3 f, \dots$. In other words, we need to know how the dynamics of f *appears* within that of f. For Feigenbaum-type

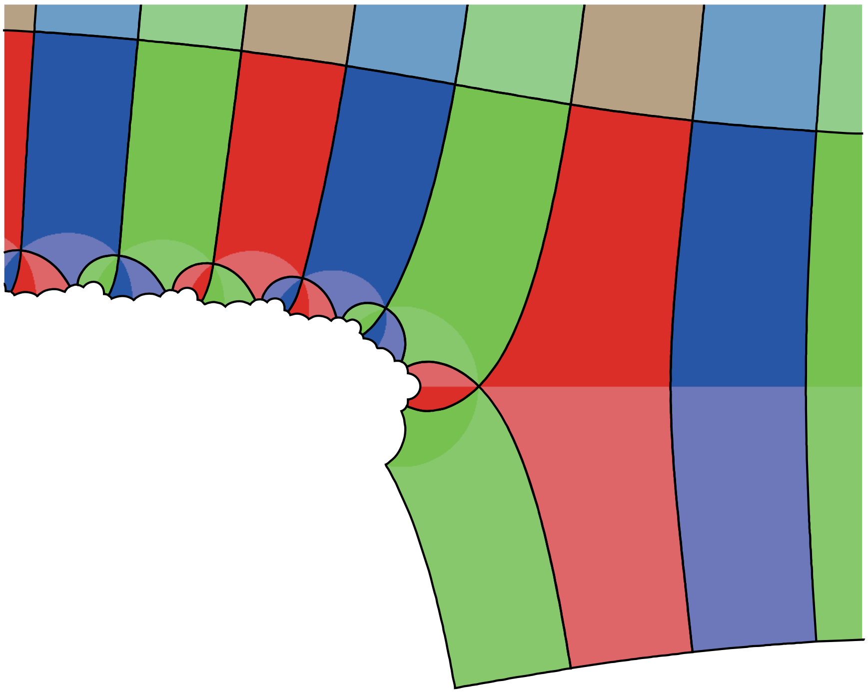

FIGURE 2. Truncated Checkerboard Pattern

renormalizations or circle map renormalizations, this is not a difficult problem because the renormalized dynamics come from disjoint intervals or intervals with disjoint interior. In our case, the construction involves choosing a fundamental region and identifying the boundary curves, whose choice is not so canonical. To overcome this difficulty, instead of looking at f itself, we focus on the "canonical dynamics" F_{can} defined on the canonical truncated checkerboard pattern Ω_{can} together with the gluing ψ_f between neighborhoods of left and right boundaries. Under this view, one can describe how the dynamics of $\mathcal{R}f$ appear in the dynamics of f. In fact, one can use copies of Ω_{can} and glue them by $\psi_{\mathcal{R}f}$ and $F_{can}^{-1} \circ \psi_{\mathcal{R}f}$ together with the map F_{can} and id on these copies correspond to a subdomain of the domain of definition of f. Repeating this procedure, we have the following.

Theorem 2 (structure)**.** *For $f(z) = e^{2\pi i\alpha}h(z)$ ($h \in \mathcal{F}_1$ or $h(z) = z + z^2$, α high type), there exist domains $\Omega^{(0)} \supset \Omega^{(1)} \supset \Omega^{(2)} \supset \ldots$ which correspond to the domains of definition for the renormalizations $f, \mathcal{R}f, \mathcal{R}^2 f, \ldots$, each $\Omega^{(k)}$ is a union of open sets $\Omega^{(k)}_{n_1,n_2,\ldots,n_k}$ and $\{0\}$, where $(n_1, \ldots, n_k)$ runs over a finite collection of k-tuple of integers, each $\Omega^{(k)}_{n_1,n_2,\ldots,n_k}$ is isomorphic to the truncated checkerboard pattern cut off at certain width, and $\overline{\Omega^{(k)}_{n_1,n_2,\ldots,n_k}} \subset \Omega^{(k-1)}_{n_1,n_2,\ldots,n_{k-1}} \cup$*

$\{0\}$. The intersection $\Lambda_f^{(\infty)} = \cap_{k=0}^{\infty} \Omega^{(k)}$ is a closed set containing 0 and the forward critical orbit. The map f is injective on this set.

As applications of this construction, we have the following.

Theorem 3 (rigidity)**.** *Let f and $\hat{f}$ be two maps as in Theorem 1, then there exists a quasiconformal map from a neighborhood of $\Lambda_f^{(\infty)}$ to a neighborhood of $\Lambda_{\hat{f}}^{(\infty)}$, which conjugates f to $\hat{f}$ on $\Lambda_f^{(\infty)}$, asymptotically conformal at $\Lambda_f^{(\infty)}$ and conformally differentiable at the critical orbit.*

Theorem 4 (hairs)**.** *Let f and $\Omega^{(k)}$, $\Omega^{(k)}_{n_1,n_2,\dots,n_k}$ be as in Theorem 1. For an "allowable" sequence $n_1, n_2, \dots$, the intersection $\cap_{k=1}^{\infty} \Omega^{(k)}_{n_1,n_2,\dots,n_k}$ is either empty or an arc tending to 0 (closed arc when 0 is added). The set of these arcs are cyclically permuted by f. In particular, there is an arc in $\Lambda_f^{(\infty)}$ from the critical point to 0.*

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Iterated Monodromy Groups and Linearizers

Laurent Bartholdi

(joint work with Dzmitry Dudko)

We describe the linearizer of a post-critically finite rational map at a repelling fixed point by relating its line graph to a graph constructed from the "iterated monodromy group" of the rational map.

1. Iterated monodromy groups

A construction by Nekrashevych associates a finitely generated group with any post-critically finite rational map [1], as follows.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map, with post-critical set P; assume for simplicity that P is finite. Set $\mathcal{M} = \widehat{\mathbb{C}} \setminus P$.

Choose a basepoint $*$ of $\mathcal{M}$; then $\pi_1(\mathcal{M}, *)$ acts naturally (by monodromy) on $T = \bigsqcup_{n\geq 0} f^{-n}(*)$. This set is the vertex set of a tree, if one connects $x \in f^{-n}(*)$ to $f(x) \in f^{1-n}(*)$.

The *iterated monodromy group* $G(f)$ of f is, by definition, the quotient of $\pi_1(\mathcal{M}, *)$ that acts faithfully on T. If one chooses a one-vertex cell complex $\mathcal{K} \subset \mathcal{M}$ (a "fat Hubbard complex") separating the post-critical values of f, then $G(f)$ is generated by the fundamental group of $\mathcal{K}$.

Furthermore, by choosing for each $x \in \Sigma := f^{-1}(*)$ a path ℓ_x from $*$ to x, one may identify T with the set of (finite) sequences over Σ.

The group G admits for all $n \geq 0$ an action on $f^{-n}(*)$, which we identify with Σ^n.

2. Limit spaces

A dynamical system may be recovered from the iterated monodromy group. Consider indeed $\mathcal{J}_G := \Sigma^{-\infty}/\sim$, where $(x_n) \sim (y_n)$ if for every $n \leq 0$ there exists $g_n \in G$ such that $g_n(x_n \dots x_0) = y_n \dots y_0$, and such that $\{g_n\}_{n\leq 0}$ is a finite subset of G.

A fundamental theorem by Nekrashevych is that, if f is a rational map, then $\mathcal{J}_{G(f)}$ is homeomorphic to the Julia set $\mathcal{J}_f$ of f, and that this homeomorphism intertwines the unilateral shift on $\mathcal{J}_{G(f)}$ with the action of f on $\mathcal{J}_f$.

This result has many avatars, one of them is the following. Fix a finite generating set S of $G(f)$. Consider for all n the finite graph $\mathcal{G}_n$ with vertex set Σ^n and edges given by the action of S. Metrize and scale this graph so that it has diameter 1. Then $\mathcal{G}_n$ is homeomorphic to the full preimage $f^{-n}(\mathcal{K})$ of the fat Hubbard complex. These graphs converge, in the Hausdorff topology, to the Julia set of f.

Note that the graphs $\mathcal{G}_n$ form an inverse sequence of graph coverings; their inverse limit gives a (pro-)graph structure on the boundary $\partial T = \Sigma^\infty$ of the tree T.

The graphs $\mathcal{G}_n$ also admit direct limits: let $\omega \in \Sigma^\infty$ be a ray in T, and let $\omega_n \in \Sigma^n$ be its length-n prefix. Then there is an unbounded increasing sequence (R_n) such that the ball of radius R_n around ω_n in $\mathcal{G}_n$ coincides with the same-radius ball around ω_N in $\mathcal{G}_N$, for all $N > n$. One may then define $\mathcal{G}_\omega$ as the limit of the balls $B(\omega_n, R_n) \subset \mathcal{G}_n$ under inclusion. This is a discrete, countable graph, and is in fact the graph structure on the leaf $G \cdot \omega \subset \partial T$.

3. Linearizers

Let again f be a rational map, and let p be, for simplicity, a repulsive fixed point of f. In particular, p belongs to the Julia set of f. Locally, f acts at p by multiplication by $\lambda \in \mathbb{C}$, with $|\lambda| > 1$.

The *linearizer map* at p is defined as follows. It is an analytic map $L : \mathbb{C} \to \widehat{\mathbb{C}}$; it is normalized by $L(0) = p$ and $L'(0) = 1$; and it satisfies

$$L(\lambda z) = f(L(z)).$$

It follows in particular from this equation that the critical values of L are precisely the post-critical values of f.

An analytic map such as L is conveniently described by its *line complex.* Let $\mathcal{K} \subset \widehat{\mathbb{C}}$ be a cell complex separating the critical values of L. The *line complex* of L is then defined as the full preimage $L^{-1}(\mathcal{K})$.

4. Main result

We announce the following description of the line complex of a linearizer; such a connection was asked us by Adam Epstein, and independently by Lasse Rempe.

Theorem 1. *Let f be a post-critically finite rational map; let p be a non-postcritical repelling periodic point for f; let $\omega \in \Sigma^{-\infty}$ be an encoding of p via the homeomorphism $\Sigma^{-\infty}/\sim \cong J_f$. Let L be the linearizer of f at p.*

Then the line complex of L is $\mathcal{G}_{\overleftarrow{\omega}}$, where $\overleftarrow{\omega} = (\omega_{-n})_{n\geq 0} \in \Sigma^{\infty}$.

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Straightening the square

Arnaud Chéritat

Take the square $[-1,1]^2 \subset \mathbb{R}^2 \equiv \mathbb{C}$. Put a Beltrami form on it that is 0 outside and constant inside, with value $a\, d\bar{z}/dz$ where $a = (K-1)/(K+1)$ for some $K > 1$. It corresponds to an ellipse field that is round outside and has vertical major axis inside, with dilatation ratio $= K$. Equivalently, to a complex structure identical to the standard complex structure outside the square, and preimage of the standard one by a real affine map inside, that transforms the ellipses into circles. The straightening ϕ of the Beltrami form is a quasiconformal map, conformal outside the square. It is unique if we require the normalization $\phi(z) = z + 0 + o(1)$ as $z \to \infty$.

(1) What is the image of the square?
(2) What happens when $K \to +\infty$?

If instead of a square, we take a Beltrami form that is constant on the unit disk, then unexpectedly there is an explicit and simple formula for the straightening ϕ: $\phi(z) = z + a/z$ outside and $\phi(z) = z + a\bar{z}$ inside. The circle is mapped to an ellipse which flattens down to a segment as $K \to +\infty$.

In the case of the square, even less expectedly, there is also an explicit formula, but only for ϕ^{-1}. It looks like the Schwarz-Christoffel formula: outside the image of the square by ϕ,

$$\phi^{-1}(z) = \int \left(\frac{(z-z_2)(z-z_4)}{(z-z_1)(z-z_3)} \right)^{\frac{\log K}{2i\pi}} dz,$$

for a well chosen primitive, where $z_1, \cdots, z_4$ are the image by ϕ of the corners of the square. The z_i are parameters to be determined, that depend on K. Inside, it should be post-composed with a real affine map. This formula is found as follows: straightening the Beltrami form is equivalent to uniformizing some $\mathbb{C}$-affine surface (like a Riemann surface, but with charts transition maps that are required to be $\mathbb{C}$-affine instead of just holomorphic). An affine surface structure compatible with a Riemann surface structure is completely characterized by a differential invariant: the distortion derivative (or nonlinearity). In the chart $\mathbb{C} \setminus \{z_1, \cdots, z_4\}$, it turns out to be a rationnal map whose polar parts at the z_i and behavior at ∞ are easy to determine. Therefore we know which rationnal map it is, whence the formula.

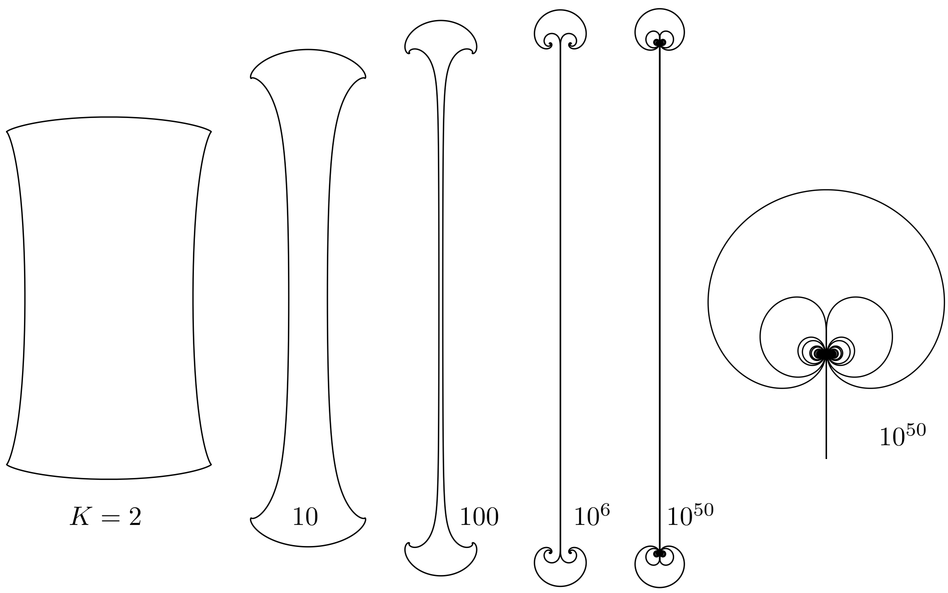

Figure 1. The image of the square by the straightening of the Beltrami form, for different values of K.

The log before the K in the formula allows to push K to extremely big values. We obtain the following pictures (rotated by 90): see figure 1. There seems to be a limit, and this limit is a surprise.

On figure 2 we show a flat model of an affine surface, which is probably the one behind the limit.

When the big axis of the constant ellipses field gets slanted, we obtain a less surprising limit: it is just the sewing of $\mathbb{C}$ minus the square under the identification of points of the boundary which have the same scalar product with a given vector parallel to the minor axis of the ellipses. The formula adapts to this setting and yields figures 3 and 4 where θ is the angle between the vertical axis and the major axis of the ellipses.

The affine surface approach can be used to uniformize any Beltrami form on $\mathbb{C}$ that is piecewise constant, on polygonal pieces, finitely many of them. We obtain a generalized Schwarz-Christoffel formula. It would be interesting to develop consequences of this.

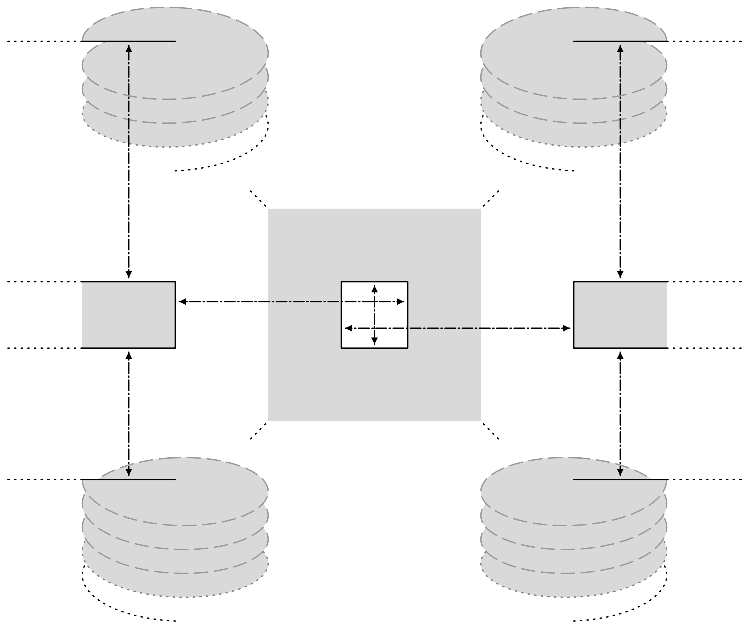

FIGURE 2. A model of the affine surface whose uniformization to $\mathbb{C}$ minus two points gives the limit shape of the image of the square.

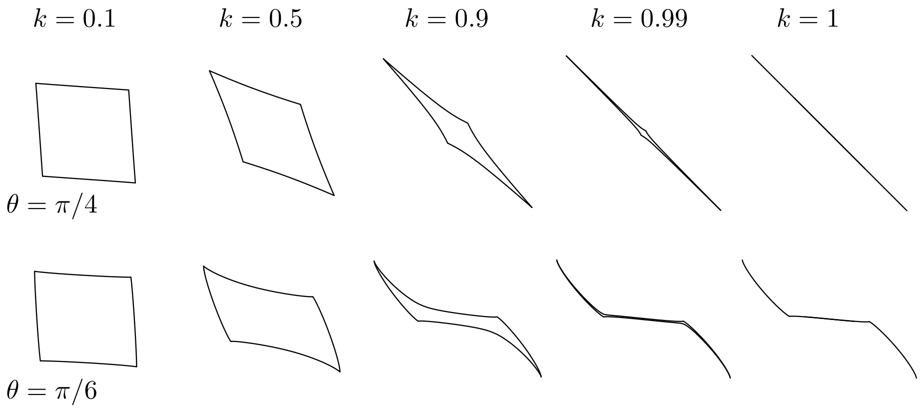

FIGURE 3. For two differents values of θ we show the limit as the ellipses flatten of the image of the square by the straightening ϕ.

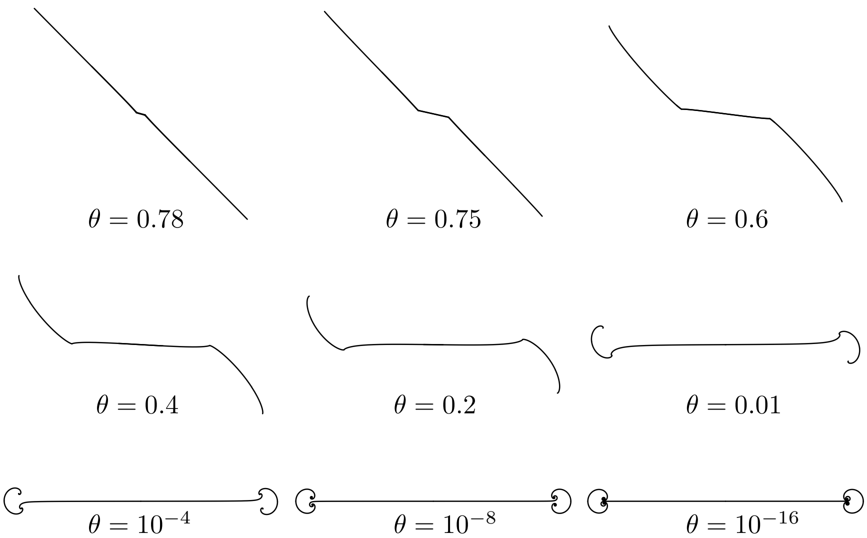

FIGURE 4. Different limits, for different values of θ. The limit as $\theta \to 0$ of the limit when $K \to +\infty$ is slightly different from the limit as $K \to +\infty$ of the case $\theta = 0$ illustrated in figure 1.

The Decoration Conjecture on the Mandelbrot set

DZMITRY DUDKO

The following conjecture concerns the size of decorations and is due to Carsten Lunde Petersen, Mikhail Lyubich and Dierk Schleicher:

Conjecture 5 (Lyubich, Petersen, Schleicher)**.** *Let $\mathcal{M}$ be the Mandelbrot set, and let $\mathcal{M}_1$ be a small copy of the Mandelbrot set. Then for any $\varepsilon > 0$, there are at most finitely many connected components of $\mathcal{M} \setminus \mathcal{M}_1$ with diameter greater than or equal to ε.*

We described a possible strategy for attacking this conjecture. The goal is to be in the case of this lemma:

Lemma 1. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots \mathcal{A}_n$ be compact sets in the bounded topological disk $\boldsymbol{D}$ such that for any $1 \le i \le n$:*

$$diam(\mathcal{A}_i) \ge \varepsilon_1,$$

and

$$mod(\boldsymbol{D} \setminus \bigcup_{j=1}^{i-1} \mathcal{A}_j,\ \mathcal{A}_i) \ge \varepsilon_2.$$

Then n is bounded in terms of ε_1, ε_2 and $\boldsymbol{D}$: $n \leq \mathbf{C}(\varepsilon_1, \varepsilon_2, \boldsymbol{D})$.

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Triviality of fibers for Misiurewicz parameters in the complex exponential family

Anna Benini

Among trascendental functions, the exponential family $e^z + c$ has been one of the most studied examples, mainly because it has only one singular value at c like the well known family of unicritical polynomials.
An articulated combinatorial study has been carried out mainly by L.Rempe, D.Schleicher and J. Zimmer, leading to a description of both parameter and dynamical plane in terms of parameter/dynamical rays respectively, in analogy with the polynomial family.
Among the many open problems in this field of exponential dynamics, it would be relevant to generalize some of the rigidity results in terms of fibers which have been one of the most active fields of research in polynomial dynamics during the last twenty years.
A general study about fibers and rigidity in connection with density of hyperbolicity has been carried out by Rempe and Schleicher, however Misiurewicz parameters (i.e. postcritically finite) are the first class of parameters for which it is actually possible to show triviality of fibers. The reason why those parameters turn out to be easier to treat is due to the rich combinatorial structure in terms of rays landing at them and to the classification following from a generalization of Thurston rigidity theorem, together with the linearizable dynamics in a neighborhood of the postcritical periodic orbit.
The talk will present a sketch of the proof that fibers of Misiurewicz parameters are trivial in the sense that given any Misiurewicz parameters it can be separated from any other parameter via a pair of parameter rays at periodic addresses landing together at some parabolic point.

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Classification of Newton maps

Yauhen Mikulich

Newton's method is a very old and perhaps the best known method for finding successively better approximations to the zeros (or roots) of a real-valued function. Starting with an initial guess x_1, one calculates the root $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ of the linear map tangent to f at x_1. This tangent linear function approximates f well near x_1, and it is reasonable to assume that x_2 will be a better approximation of ξ than x_1. Indeed it is a well know fact that if x_1 was sufficiently close to some root ξ of f, then the sequence obtained iteratively by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges to ξ. If it happens, we say that the starting value x_1 *finds* the roots ξ.

However, in many cases there exist open sets of starting values for Newton's method that do not find any roots.

Besides their application to root-finding, Newton maps form a class of functions that is interesting to study in its own right. From one hand, the space of Newton maps of polynomials forms a large enough and interesting sub-class of rational functions. On the other hand, it seems to have enough structure to make a classification possible. A classification of all Newton maps will suggest general methods and conjectures covering larger classes of rational functions. Hence a classification of Newton maps might provide an important intermediate step towards the major goal of the whole complex dynamics : a classification of all rational functions.

Consider the map $N_p(z) = z - \frac{p(z)}{p'(z)}$ used in the Newton method for a monic polynomial $p(z) = (z-a_1)^{m_1}(z-a_2)^{m_2}\dots(z-a_k)^{m_k}$ of degree d with complex coefficients. Such map is called a *Newton map*. By a theorem of Head [He], a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 3$ is a Newton map if and only if ∞ is a repelling fixed point of f, and for each fixed point $\xi \in \mathbb{C}$, there exists an integer $m \geq 1$ such that $f'(\xi) = (m-1)/m$. Let A_i denote the immediate basin of a_i. For simplicity, we will always assume that all fixed points of f in $\mathbb{C}$ are super-attracting, although for most of what we propose, this requirement is not essential. The space of holomorphic conjugacy classes of Newton maps of degree d has complex dimension $d-2$.

Compared to general rational functions, Newton maps have rather specific combinatorial structure that is defined by *accesses to infinity* from the centers of fixed attracting basins. On the other hand, Newton maps may have arbitrarily many critical points.

If $d = 2$, the dynamics of Newton's method is very easy: there is only one quadratic Newton map up to Moebius conjugation and the space of such maps reduces to a point. The case $d = 3$ is relatively well-understood [TL97]. However, very little is currently known in the case $d > 3$.

We discuss our ideas towards the combinatorial classification of higher degree Newton maps. We introduce the notion of the Newton graph, which are defined as follows:

Each immediate basin A_i has a global Böttcher coordinate $\phi_i : (D, 0) \to (A_i, a_i)$ with the property that $N_p(\phi_i(z)) = \phi_i(z^{k_i})$ for each $z \in D$, where $k_i - 1 \geq 1$ is the multiplicity of a_i as a critical point of N_p. The $k_i - 1$ internal rays in D, which are fixed under $z \to z^{k_i}$ map under ϕ_i to $k_i - 1$ pairwise disjoint, non-homotopic (with homotopies fixing the endpoints) injective curves $\Gamma_i^1, \Gamma_i^2, \dots, \Gamma_i^{k_i-1}$ in A_i that connect a_i to ∞ and are invariant under N_p. They represent all accesses to ∞ of A_i. In case $k_i \geq 3$, in other words there are at least two accesses to ∞ in A_i, we call the immediate basin A_i *multiple* and say that N_p has *multiple channels.* The union

$$\Delta = \bigcup_i \bigcup_{j=1}^{k_i-1} \overline{\Gamma_i^j}$$

forms a connected graph in $\bar{\mathbb{C}}$ that is called the *channel diagram.* The channel diagram records the mutual locations of the immediate basins of N_p and provides a first-level combinatorial information about the dynamics of the Newton map. For any $n \geq 0$, denote by Δ_n the connected component of $N_p^{-n}(\Delta)$ that contains Δ. The pair (Δ_n, N_p) of a graph Δ_n and a Netwon map N_p acting on it is called a *Newton graph* of N_p. Newton graphs give more precise combinatorial data than channel diagrams, which can be used as a combinatorial model for Newton maps.

In [Rü] J.Rueckert classified the so-called *postcritically fixed* Newton maps in terms of Newton graphs. While Newton graphs describe the dynamics of those critical points, which eventually fall into the graph, we are looking for a combinatorial object which would encode the dynamics of the rest critical points.

We discuss a possible candidate for such a model, which could lead to the classification of much larger class of Newton maps, such as postcritically finite Newton maps or even all Newton maps. Such a combinatorial classification is also closely related with the important longstanding question, asked by Smale in [Sm85], where he posed the problem of characterizing all possible combinatorics of attracting basins for a Newton map.

Namely we discuss how to obtain polynomial-like maps associated to each of the postcritically finite free critical point of N_p. One can construct an extended Hubbard tree which contains the forward orbit of the chosen free critical point and fixed points of the polynomial-like map, which can be connected to the Newton graph via curves analogous to fixed external rays. In the end one obtains the forward invariant finite graph, containing the whole postcritical set of N_p. Similar techniques as in [Rü] applied in this case could then give a classification of postcritically finite Newton maps in terms of extended Newton graphs.

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