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Dynamics of Patterns

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Abstract. This workshop focused on the dynamics of nonlinear waves and spatio-temporal patterns, which arise in functional and partial differential equations. Among the outstanding problems in this area are the dynamical selection of patterns, gaining a theoretical understanding of transient dynamics, the nonlinear stability of patterns in unbounded domains, and the development of efficient numerical techniques to capture specific dynamical effects.

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Introduction by the Organisers

Ideas from dynamical systems have had a profound impact on the way we think about pattern formation. Bifurcation theory, for instance, has helped tremendously in explaining pattern selection in experiments, including Rayleigh–Benard convection and Belousov–Zhabotinsky reactions. However, these results can typically only describe patterns with a given prescribed periodic lattice structure on the plane. Amplitude equations go beyond this limitation: They allow us to investigate the dynamics of slowly varying amplitude modulations of a fixed spatially homogeneous state over large, but finite, time intervals.

Over the past few years, the focus has shifted to situations where neither bifurcation theory nor the amplitude-equation formalism can give enough insight into the formation and the dynamics of patterns. Examples are the dynamical selection of patterns, extracting and describing transient dynamics, the nonlinear stability of patterns in unbounded domains, and the development of efficient numerical techniques to capture specific dynamical effects and behaviours. This workshop brought together researchers who work on these questions from different perspectives and with different techniques, ranging from dynamical systems theory, qualitative analysis of partial differential equations, and bifurcation theory to spectral analysis and numerical methods for patterns.

During the workshop, 25 presentations were given. In addition, three PhD students discussed their projects in shorter talks of 15 minutes length. On Tuesday afternoon, no talks were scheduled. Instead, the attendees had the opportunity to discuss more specialized topics in smaller groups. We now describe briefly the main outcomes and new directions that emerged during the workshop.

The formation and interaction of pulses in one space dimension were one central theme of the workshop. Recent efforts aim to describe the interaction of localized pulses analytically and to compute interacting pulses efficiently, using numerical means. Progress was made in particular for pulses that are only weakly localized: in certain circumstances, it is then still possible to capture the interaction of such pulses analytically.

Over the past few years, the *freezing method* has been investigated thoroughly from both analytical and numerical viewpoints. This method computes pulses numerically by separating the shape dynamics from the dynamics on the underlying symmetry group. These developments were discussed together with applications to spiral waves and to propagating pulses in partial differential equations of mixed type.

With all these successes, it became clear, however, that both analytic and numerical understanding of the evolution and interaction of two-dimensional localized spatio-temporal patterns is still very rudimentary.

Spatially extended patterns and their dynamics constituted a further focus. Much recent work has centered on explaining specific phenomena that have been observed experimentally: examples include turbulent stripe patterns in fluid flows, Liesegang precipitation patterns, planar hexagon patches, and vortex dynamics in flows past cylinders. Significant progress was also made in proving spectral and nonlinear stability of spatially extended waves such as rotating waves, spatially homogeneous oscillations, and spatially periodic structures. Furthermore, techniques to assess spectral stability for multi-dimensional fronts were discussed.

Systems with delay form an important class of infinite-dimensional systems that exhibit interesting dynamical patterns. State-dependent delays allow the delay of the system to depend on the prehistory state of that system itself. Hysteresis is one well-known example. State-dependent delays are motivated by important applications, generate a plethora of new dynamical patterns, and present formidable obstacles to analysis. Progress reports included patterns of periodicity in hysteresis, implicitly defined delays, and singularly perturbed equations.

Special Tuesday sessions. No talks were scheduled on Tuesday afternoon to give participants an additional opportunity for discussion in smaller groups. We briefly report on two group meetings that took place in this setting.

Poster discussion: More on pulses, shocks, and their interactions. The main intention of the posters during the work session was to discuss among a group of specialists the existence, stability and bifurcation of nonlinear waves that are either periodic in time or in space. The group discussed progress to these kinds of problems that involved various different approaches such as singular perturbation techniques, modulation equations, and pointwise Green's function estimates. Specific topics and the corresponding contributors were:

- Busse balloons and bifurcations of spatially periodic patterns (Arjen Doelman, with Harmen von der Ploeg, Jens Rademacher and Sjors van der Stelt);
- Interfaces between rolls in the Swift–Hohenberg equation (Mariana Haragus with Arnd Scheel);
- Delayed bifurcation in a simple reaction-diffusion equation (Tasso Kaper with Peter De Maesschalck and Nikola Popovic);
- Stability of time-periodic viscous shocks (Björn Sandstede with Margaret Beck and Kevin Zumbrun).

Another intention of the poster discussion was to have interactions between this group and junior researchers who were given the possibility to present more details (in particular numerical results) than during their short talks:

- Freezing waves in hyperbolic PDEs (Jens Rottmann-Matthes);
- Numerical decomposition of multistructures (Sabrina Selle).

Global parabolic dynamics. Progress and discussion addressed two aspects of the global dynamics of semilinear parabolic partial differential equations, mainly on a circle domain. These aspects are the Morse–Smale or Kupka–Smale property, on the one hand, and the characterization of global attractors, on the other hand. The Kupka–Smale property asserts hyperbolicity of all equilibria and periodic orbits, as well as transversality of their stable and unstable manifolds to hold for generic (i.e., for "most") nonlinearities. The Morse–Smale property asserts, in addition, the absence of any recurrence beyond periodicity. In such situations, it is conceivable, but still a formidable task, to study the detailed spatio-temporal structure of the patterns arising in the global attractor.

More precisely, we considered the following reaction-diffusion equations on the circle S^1:

$$u_t(x,t) = u_{xx}(x,t) + f(x, u(x,t), u_x(x,t)), \quad (x,t) \in S^1 \times \mathbf{R}^+,$$

where f is a regular function from $S^1 \times \mathbf{R}^2$ into $\mathbf{R}$. First, we have recalled that these equations satisfy the Poincaré–Bendixson property [2]. We have also stated the recent result of Czaja and Rocha [1], who showed that the stable and unstable manifolds of hyperbolic periodic orbits intersect transversally.

Geneviève Raugel presented a proof of the Morse–Smale property for the above equation. She first showed genericity (with respect to the non-linearity) of the hyperbolicity of all equilibria and periodic orbits [3]. The main ingredients are the non-increase of the zero number and Sard–Smale theorems. She also showed automatic transversality of the stable and unstable manifolds of equilibria with different Morse indices, the generic non-existence (with respect to the non-linearity) of orbits connecting two equilibria with the same Morse index, etc. [4]. These properties allow to show that, generically with respect to the non-linearity f, the above reaction-diffusion equation on the circle S^1 indeed possesses the Morse–Smale property.

In the second part of this discussion, we have considered genericity (with respect to the non-linearity) of hyperbolicity of all equilibria and periodic orbits in the case of a scalar reaction-diffusion equation in higher dimension and explained the results already obtained in a work in progress by P. Brunovskỳ, R. Joly, and G. Raugel. Geneviève Raugel noticed that the same types of arguments and techniques should lead them to the proof of the Kupka–Smale property for parabolic PDEs in the near future.

For x-independent nonlinearities $f = f(u, u_x)$ on the circle domain, the Morse–Smale property enters the description of all generic global attractors given in [5]. Carlos Rocha indicated how to characterize the set of 2π-periodic solutions of planar Hamiltonians of the form $u'' + g(u) = 0$ and obtained a useful tool for the description of the associated global attractors. He discussed a permutation characterization for the periodic solutions of the corresponding stationary problems. Essentially, the permutation describes the braid formed by the stationary solutions and traveling waves of the semilinear parabolic equation [6]. Extending this result to equilibria in the x-reversible case $f(u, -p) = f(u, p)$, this characterization indeed extends to describe the precise heteroclinic structure of the above parabolic partial differential equation in this case.

References

[1] R. Czaja and C. Rocha, *Transversality in scalar reaction-diffusion equations on a circle*, J. Diff. Eqs. **245** (2008), 692–721
[2] B. Fiedler and J. Mallet-Paret, *A Poincaré–Bendixson theorem for scalar reaction-diffusion equations*, Arch. Rational Mech. Analysis **107** (1989), 325–345.
[3] R. Joly and G. Raugel, *Generic hyperbolicity of equilibria and periodic orbits of the parabolic equation on the circle*, to appear in Transactions of the AMS.
[4] R. Joly and G. Raugel, *Generic Morse–Smale property for the parabolic equation on the circle*, manuscript.
[5] B. Fiedler, C. Rocha, M. Wolfrum, *Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle*, J. Diff. Eqs. **201** (2004), 99–138.
[6] B. Fiedler, C. Rocha, M. Wolfrum, *A permutation characterization of periodic orbits in autonomous Hamiltonians or reversible systems*, Preprint (2009).

Workshop: Dynamics of Patterns

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Abstracts

Temporal patterns in solutions of state-dependent delay-differential equations

John Mallet-Paret
(joint work with Roger Nussbaum)

We study a class of state-dependent delay-differential equations of the form

$$\varepsilon\dot{x}(t) = g(x(t-r_1), x(t-r_2), \ldots, x(t-r_m)), \qquad r_i = r_i(x(t)), \tag{1}$$

where $g : \mathbf{R}^m \to \mathbf{R}$ is a given nonlinearity, $r_i : \mathbf{R} \to (0, \infty)$ are time delays, and $\varepsilon > 0$ is a singular perturbation parameter. Such equations arise in a variety of scientific applications. Numerical simulations indicate that even for very simple (linear or affine) functions g and r_i, highly stable periodic solutions seem to exist. Moreover, in many cases the graph of the solution has a complicated structure involving multiple critical points, and which settles to a nontrivial singular solution for the limit $\varepsilon \to 0$.

It is a fundamental question to predict the limiting shape as $\varepsilon \to 0$ of such solutions from g and r_i. A second fundamental question is to show that given a singular solution (composed of inner and outer solutions) for $\varepsilon = 0$, then there exists a true solution for small ε. It is also fundamental to understand the characteristic multipliers and stability of such solutions.

In this lecture we discuss progress and new results for these problems. Most results have dealt with the case $m = 1$ of a single delay, for systems of negative feedback of the form of a generalized Mackey-Glass equation

$$\varepsilon\dot{x}(t) = -x(t) + f(x(t-r)), \qquad r = r(x(t)). \tag{2}$$

Here $f(0) = 0$, $uf(u) < 0$ for $u \neq 0$, $f'(0) < -1$, and a boundedness condition on f holds (one such condition being that $f(f(u))$ has sublinear growth at infinity). Under such conditions, a basic result from [1] is the following.

Theorem 1. *For every small $\varepsilon > 0$, equation (2) possesses at least one slowly oscillating periodic solution.*

By a slowly oscillating periodic solution (SOPS) $x(t)$ we mean a solution with consecutive zeros t_n satisfying

$$\dot{x}(t_n) \neq 0, \qquad t_{n+1} - t_n > r(0), \qquad t_{n+2} - t_n = p,$$

where p is the minimal period of the solution. Fixed point theorems and degree theory for cone maps are used to prove Theorem 1.

In many cases the limiting shape of SOPS's can be explicitly given in terms of f and r. A first step in this direction is the following result for equation (2) under the above conditions.

Theorem 2. *Assume that either (A) $r(u) \equiv r(0) > 0$; or (B) $r'(0) \neq 0$; or (C) $r'(0) = 0$ and $r''(0) > 0$; or (D) $r'(0) = 0$ and $r''(0) < 0$. Then there exists $K > 0$ and $\varepsilon_0 > 0$ such that*

$$\sup_{t \in \mathbf{R}} |x(t)| \geq K$$

for all SOPS's of equation (2) with $0 < \varepsilon < \varepsilon_0$.

The above theorem ensures that the limit of SOPS's for small ε is nontrivial. Formally, the result is no surprise, due to the instability assumption $f'(0) < -1$. One could think of the result as stating that the unstable manifold of the origin is of uniform size (i.e., does not shrink) as $\varepsilon \to 0$. But interestingly, the proofs for the four cases are all completely different, and technically rather complicated and nonituitive. Case (A) was proved in [1]; cases (B) and (C) in [3]; and case (D) very recently [6].

Under an additional monotonicity condition on f, the precise limiting shape of SOPS's as $\varepsilon \to 0$ can often be explicitly determined in terms of eigensolutions of a max-plus operator; see [4], [5]. More recently [6], finer details of the asymptotics along with stability information has been obtained. For example, one has for the period $p = p(\varepsilon)$ in three of the above cases that

(A) $p = 2r(0) + \varepsilon p_1 + o(\varepsilon)$ for some $p_1 > 0$;
(B) $p = p_0 + \varepsilon |\log \varepsilon| p_1 + \varepsilon p_2 + o(\varepsilon)$ for some p_i with $p_1 > 0$;
(D) $p = p_0 + \varepsilon^{4/3} p_1 + o(\varepsilon^{4/3})$ for some p_i with $p_1 > 0$.

Case (C) awaits analysis. The quantity p_1 in case (A) is determined by a global Mel'nikov analysis involving a connecting orbit for an associated system of delay equations; for cases (B) and (D) explicit but complicated formulas for p_1 are given; in case (D) in particular, this formula involves the leading zero of the Airy function.

For the model case of equation (2) where $f(u) = -ku$ with $k > 1$, and $r(u) = 1 + u$, there is known to be an SOPS $x(t)$ which asymptotically has a sawtooth shape with limiting period $k + 1$. Namely, the limiting graph of this solution has diagonally sloping pieces near the lines $x = t - 1 - n(k+1)$ for $t \in [n(k+1), (n+1)(k+1)]$, and transition layers near the vertical lines $t = n(k+1)$ for $x \in [-1, k]$, for each $n \in \mathbf{Z}$. (The earlier assumptions on f and r do not hold here, however, appropriate modifications of the methods can be made to allow for this variance.) It is further known [6] that this SOPS exhibits "superstability," namely, all of its nontrivial characteristic multipliers μ satisfy $\mu = O(\varepsilon)$ as $\varepsilon \to 0$.

Generally, the techniques used in proving the above asymptotic results are scaling results, with judicious choices of scaling obtained through geometric singular perturbation theory and center manifold analysis.

Results for multiple ($m \geq 2$) delays are in their infancy, but it is clear from numerical simulations that a rich array of solutions and asymptotics awaits. For the equation

$$\varepsilon \dot{x}(t) = -x(t) + f(x(t - r_1), x(t - r_2), \ldots, x(t - r_m)), \qquad r_i = r_i(x(t)),$$

with conditions similar to those assumed for equation (2), no general analog of Theorem 1 has been known except [2] in the special (and artificial) case where

$$r_1(0) = r_2(0) = \cdots = r_m(0) > 0.$$

And even here, no analog of Theorem 2 is known, and no analog of the max-plus analysis is known. However, very recently [6], the model two-delay equation

$$\begin{aligned} \varepsilon\dot{x}(t) &= -x(t) - k_1 x(t - r_1) - k_2 x(t - r_2), \\ r_1 &= 1 + x(t), \qquad r_2 = a + cx(t), \end{aligned} \tag{3}$$

was studied, where here k_i, a, and c are fixed positive constants. Under the assumptions

(i) $0 < a - c < k + 1$;
(ii) $0 < a + ck < k + 1$;
(iii) $k_1 + ck_2 > 1$;
(iv) $(1 - c)(k_2 k - k_1) < 1 - a < (1 - c)(k_1 k - k_2)$;
(v) $(k_1 + k_2)k > k_2(a - c) + 1$,

where $k = \dfrac{k_1 + ak_2}{1 + (1 - c)k_2}$, there exists for small $\varepsilon > 0$ a sawtooth-shaped periodic solution, with the same limiting shape as for the one-delay example above. This solution is obtained by means of a fixed point theorem.

While conditions (i)–(v) may seem artificial, they are roughly (but in a sense that can be made precise) the necessary and sufficient conditions for such a limiting sawtooth-shaped solution to exist. Moreover, it seems clear that the approach taken for the model equation (3) here can be extended to more general equations (1) with multiple delays.

References

[1] J. Mallet-Paret and R.D. Nussbaum, Global continuation and asymptotic behavior for periodic solutions of a differential–delay equation, *Ann. Mat. Pura ed Applicata* **145** (1986), 33–128.

[2] J. Mallet-Paret, R.D. Nussbaum, and P. Paraskevopoulos, Periodic solutions for functional differential equations with multiple state-dependent time lags, *Top. Meth. Nonlin. Anal.* **3** (1994), 101–162.

[3] J. Mallet-Paret and R.D. Nussbaum, Boundary layer phenomena for differential delay equations with state dependent time lags, II, *J. Reine Angew. Math.* **477** (1996), 129–197.

[4] J. Mallet-Paret and R.D. Nussbaum, A basis theorem for a class of max-plus eigenproblems, *J. Diff. Eq.* **189** (2003), 616–639.

[5] J. Mallet-Paret and R.D. Nussbaum, Boundary layer phenomena for delay-differential equations with state dependent time lags: III, *J. Diff. Eq.* **189** (2003), 640–692.

[6] J. Mallet-Paret and R.D. Nussbaum, in preparation.

Semiflows for differential systems with state-dependent delays: Implicitly given delay, and neutral equations

Hans-Otto Walther

Consider an autonomous feedback system which reacts to its present state $x = x(t) \in \mathbb{R}$ only after a delay time $d \in [0,h]$ which depends on the present state: $d = d(x(t))$. The differential equation for this reads

$$x'(t + d(x(t))) = f(x(t)). \tag{1}$$

It can be rewritten in the more familiar form of a delay differential equation

$$x'(u) = f(x(u + r(u)))$$

coupled to the algebraic equation

$$0 = d(x(u + r(u)) + r(u).$$

More generally, we consider algebraic-delay differential systems of the form

$$x'(t) = g(r(t), x_t), \tag{2}$$

$$0 = \Delta(r(t), x_t). \tag{3}$$

Here $x_t : [-h, 0] \to \mathbb{R}^n$ is given by $x_t(a) = x(t + a)$, $g(s, \phi) \in \mathbb{R}^n$ for $s \in \mathbb{R}^k$ and $\phi : [-h, 0] \to \mathbb{R}^n$, and $\Delta(s, \phi) \in \mathbb{R}^k$. These systems cover also several other types of differential equations with state-dependent delay: Equations with explicit delay

$$x'(t) = f(x(t), x(t - R)),\ \ R = R(x(t))$$

as in [14, 10, 11, 12, 13, 6], equations with a threshold condition

$$\int_s^0 K(\phi(0), \phi(u))du = \theta$$

for the delay s [1, 7], and signal delays s [18, 19] which are given by an equation

$$c\,s = \phi(s) + \phi(0) + 2w.$$

A specific example WBC with a delayed reaction as described initially models the regulation of the density of white blood cells [8].

For threshold and transmission delays the modelled situation often suggests natural hypotheses which guarantee that the algebraic equation corresponding to Eq. (3) uniquely determines the delay $r(t)$ as a function of the state x_t. This reduces the algebraic delay differential system to a single delay differential equation. A general delayed reaction as in Eq. (1), which we studied in [20], does not offer such hypotheses. It is here that a need for a more general theory of the system (2-3) arises.

Let C and C^1 denote the Banach spaces $C^1([-h, 0], \mathbb{R}^n)$ and $C^1([-h, 0], \mathbb{R}^n)$, respectively, with the usual norms. Guided by earlier work on equations without the algebraic component [16, 17, 3] we consider the system (2-3) for continuously differentiable maps $g : \mathbb{R}^k \times C^1 \supset U \to \mathbb{R}^n$ and $\Delta : \mathbb{R}^k \times C^1 \supset U \to \mathbb{R}^k$. Under mild additional smoothness conditions, which include existence of linear extensions

$D_e g(s,\phi) : \mathbb{R}^k \times C \to \mathbb{R}^n$ and $D_{2,e}\Delta(s,\phi) : C \to \mathbb{R}^k$ of derivatives $Dg(s,\phi)$ and $D_2\Delta(s,\phi)$, respectively, we obtain that the set

$$M = \{(s,\phi) \in U : \phi'(0) = g(s,\phi), 0 = \Delta(s,\phi), \det\, D_1\Delta(s,\phi) \neq 0\}$$

is a continuously differentiable submanifold of codimension $k+n$ in $\mathbb{R}^k \times C^1$, and that the system (2-3) generates a continuous semiflow F on M with continuously differentiable solution operators $F(t,\cdot)$ and F continuously differentiable for $t > h$.

Solutions with $r(t), x_t) \notin M$ are also possible, but outside M we have no uniqueness for the initial value problem. We shall come back to this below.

The second part of the lecture, about the model WBC, addresses a phenomenon which is not seen in differential equations with constant delay. Now the r-components of solutions are scalar, and one may ask whether for a given solution (r,x) the associated delayed argument function

$$\tau : t \mapsto t + r(t)$$

(which in WBC appears on the right hand side of Eq. (2)) is increasing or not.

Often in differential equations with state-dependent delay the delayed argument functions are *increasing*, see the survey paper [3]. This property may be felt natural and seems to facilitate the analysis.

A *decrease* of τ, on the other hand, means that the system reacts to states $\xi_1 = x(\tau(t_1))$, $\xi_2 = x(\tau(t_2))$ in the past, with $\tau(t_1) < \tau(t_2)$, in reverse temporal order, namely by reactions $x'(t_1)$ to ξ_1 and $x'(t_2)$ to ξ_2 at $t_2 < t_1$.

In the experiment by Libet et al. [4] on unconscious brain activity before the moment of awareness and voluntary action it was found that awareness of certain external stimuli in short time intervals arises in reverse temporal order. This may be taken as an indication that decreasing delayed argument functions have a counterpart in biological reality.

For WBC the manifold M decomposes into an open subset $M_+ \neq \emptyset$ formed by flowlines with strictly increasing delayed argument function, and into another open subset M_- formed by flowlines with strictly decreasing delayed argument function. The lifespan of the flowlines in M_- is bounded by h, and $M_- = \emptyset$ for constant delay.

It is proven that for any solution the set of t with $(r(t), x_t) \in M = M_+ \cup M_-$ is open and dense, and the delayed argument function is injective. This excludes transients between both patterns, from M_+ via the separatrix given by

$$\det\, D_1\Delta(s,\phi) = 0,$$

into M_- or vice versa. Unless the delay is constant, there are flowlines in M_+ and in M_- which terminate at points on the separatrix with the same delay component r. In addition there exist points (s,ϕ) on the separatrix from which two solutions bifurcate, one into M_+ and the other one into M_-.

A report about most recent work on neutral equations is deferred to another occasion.

References

[1] Alt, W., *Periodic solutions of some autonomous differential equations with variable time delay.* In: *Functional Differential Equations and Approximation of Fixed Points, Bonn 1978*, pp. 16-31, Peitgen, H.O., and H.O. Walther eds., Lecture Notes in Math., vol. 730, Springer, Berlin 1979.

[2] Arino, O., Hadeler, K. P., and M. L. Hbid, *Existence of periodic solutions for delay differential equations with state-dependent delay.* J. Differential Eqs. **144** (1998), 263–301.

[3] Hartung, F., Krisztin, T., Walther, H. O., and J. Wu, *Functional differential equations with state-dependent delay: Theory and applications.* In *HANDBOOK OF DIFFERENTIAL EQUATIONS, Ordinary Differential Equations*, volume 3, pp. 435-545, Canada, A., Drabek., P. and A. Fonda eds., Elsevier Science B. V., North Holland, Amsterdam 2006.

[4] Libet, B., Wright, E. W., Feinstein, B., and D. Pearl, *Subjective referral of the timing for a conscious sensory experience: A functional role for the somatosensory specific projection system in man.* Brain **102** (1979), 193–224.

[5] Krisztin, T., *C^1-smoothness of center manifolds for differential equations with state-dependent delays.* In *Nonlinear Dynamics and Evolution Equations*, Fields Institute Communications **48** (2006), 213–226.

[6] Krisztin, T., and O. Arino, *The 2-dimensional attractor of a differential equation with state-dependent delay.* J. Dynamics and Differential Eqs. **13** (2001), 453–522.

[7] Kuang, Y., and H. L. Smith, *Slowly oscillating periodic solutions of autonomous state-dependent delay differential equations.* Nonlinear Analysis TMA **19** (1992), 855–872.

[8] Mackey, M. C., personal communication.

[9] Magal, P., and O. Arino, *Existence of periodic solutions for a state-dependent delay differential equation.* J. Differential Eqs. **165** (2000), 61–95.

[10] Mallet-Paret, J., and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags: I.* Archive for Rational Mechanics and Analysis **120** (1992), 99–146.

[11] Mallet-Paret, J., and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags: II.* J. für die reine und angewandte Mathematik **477** (1996), 129–197.

[12] Mallet-Paret, J., and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags: III.* J. Differential Eqs. **189** (2003), 640–692.

[13] Mallet-Paret, J., Nussbaum, R. D., and P. Paraskevopoulos, *Periodic solutions for functional differential equations with multiple state-dependent time lags.* Topological Methods in Nonlinear Analysis **3** (1994), 101–162.

[14] Nussbaum, R. D., *Periodic solutions of some nonlinear autonomous functional differential equations.* Annali di Matematica Pura ed Applicata IV Ser. **101** (1974), 263–306.

[15] Qesmi, R., and H. O. Walther, *Center-stable manifolds for differential equations with state-dependent delay*, Discrete and Continuous Dynamical Systems, to appear.

[16] Walther, H. O., *The solution manifold and C^1-smoothness of solution operators for differential equations with state dependent delay.* J. Differential Eqs. **195** (2003), 46–65.

[17] Walther, H. O., *Smoothness properties of semiflows for differential equations with state dependent delay.* J. Mathematical Sciences **124** (2004), 5193–5207.

[18] Walther, H. O., *Stable periodic motion of a system using echo for position control.* J. Dynamics and Differential Eqs. **15** (2003), 143–223.

[19] Walther, H. O., *On a model for soft landing with state-dependent delay.* J. Dynamics and Differential Eqs. **19** (2007), 593–622.

[20] Walther, H. O., *A periodic solution of a differential equation with state-dependent delay.* J. Differential Eqs. **244** (2008), 1910–1945.

Parabolic problems with hysteresis

Pavel Gurevich
(joint work with Willi Jäger, Alexander Skubachevskii)

1. Preliminaries

We consider parabolic equations in bounded domains coupled with ODEs whose right-hand side is a nonlinear *hysteresis* operator. These models describe thermocontrol processes in biological cells, chemical reactors, and various climate control systems.

Thermocontrol models similar to ours were originally proposed in [2, 3], where the existence of solution was proved. The question whether *periodic* solutions exist turns out to be much more difficult. The one-dimensional case was treated in [1, 9, 4]. Large-time behavior of solutions for parabolic problems with hysteresis but without coupling with ODE was considered in [6, 7].

Periodicity and large-time behavior of solutions for the above coupled systems in the multidimensional case is generally an unsolved problem. We give a survey of recent results in this direction and formulate some open questions.

Note that, in spite of the fact that we study particular problems, they are of general interest, since evolution equations involving hysteresis nonlinearities arise in many applied fields, however, standard techniques cannot be used.

2. Setting of the problem

Consider the following example. Let $Q \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary Γ. Let $w(x,t)$ be the temperature at the point $x \in Q$ at the moment $t \geq 0$ satisfying

$$w_t(x,t) = \Delta w(x,t) - p(x)w(x,t) \quad ((x,t) \in Q_T), \tag{1}$$

$$w(x,0) = \varphi(x) \quad (x \in Q), \tag{2}$$

where $Q_T = Q \times (0,T)$, $p \in C^\infty(\mathbb{R}^n)$, $p(x) \geq 0$.

The boundary condition contains a control function $u(t)$ which regulates the heat flux through the boundary:

$$\frac{\partial w}{\partial \nu} = K(x)(u(t) - u_c) \quad ((x,t) \in \Gamma_T), \tag{3}$$

where $\Gamma_T = \Gamma \times (0,T)$, ν is the outward normal to Γ_T at the point (x,t), $u_c \in (0,1)$.

We introduce the "mean" temperature $w_m(t)$ as

$$w_m(t) = \int_Q m(x)w(x,t)\,dx,$$

where $m \in L_\infty(Q)$ is a given (nonnegative) function.

We assume that u is a solution of the following Cauchy problem:

$$u'(t) + au(t) = H(w_m)(t) \quad (t \in (0,T)), \tag{4}$$

$$u(0) = u_0, \tag{5}$$

where $a > 0$, $u_0 \in \mathbb{R}$, and w is the function satisfying relations (1)–(3). The operator $H(w_m)(t)$ is a hysteresis operator, which acts as follows (see [8, 10] for precise definitions). There are two given temperature thresholds $w_1 < w_2$. If $w_m(t) \leq w_1$, then the operator H "switches" to 1. If $w_m(t) \geq w_2$, then the operator H "switches" to 0. If $w_m(t)$ is in the interval (w_1, w_2), then the value of H at the moment t is the same as its value at the moment "just before" t.

Thus, we have a parabolic system (1)–(3) coupled with the Cauchy problem (4), (5) for the ODE.

3. Results

- We prove the *existence and uniqueness of solution* for problem (1)–(5) in appropriate Sobolev spaces.
- We show that a *periodic solution exists*, provided that a mean-periodic solution exists, which is a pair $(w(x,t), u(t))$ such that the "mean" temperature $w_m(t)$ and the control function $u(t)$ are both periodic in time with the same period.

  For example, this is the case if $p(x) \equiv 0$ and $m(x) \equiv \text{const}$, i.e., the "uniform" distribution of thermal sensors inside the domain is assumed. The mean-periodic solution $(w_m(t), u(t))$ forms a limit-cycle trajectory.
- We consider the model where the discontinuous hysteresis operator H is replaced by the *continuous Preisach operator* and Eq. (1) is replaced by

  $$w_t(x,t) = \Delta w(x,t) - p(x)w(x,t) + f(x,t,w,u) \quad ((x,t) \in Q_T).$$

  We find conditions implying that *there exists a T-periodic solution* if f is T-periodic.
- The existence of a *global B-attractor* for the system with the Preisach operator is proved.

Some of the above results are published in [5] without proofs. The full version will appear in *SIAM J. Math. Anal.*

4. Some open questions

- Given the discontinuous hysteresis operator:
  - (a) Find periodic solutions in the general case and study their stability property.
  - (b) Study large-time behavior of solutions (global attractors).
- Given the Preisach operator:
  - (a) It is known that stationary solutions exist in some cases. Check whether there are periodic solutions different from stationary solutions in this case.
  - (b) Investigate the structure of the global attractor.

References

[1] A. Friedman, L.-S. Jiang, *Periodic solutions for a thermostat control problem*, Commun. Partial Differential Equations **13 (5)** (1988), 515–550.

[2] K. Glashoff, J. Sprekels, *An application of Glicksberg's theorem to set-valued integral equations arising in the theory of thermostats*, SIAM J. Math. Anal. **12** (1981), 477–486.

[3] K. Glashoff, J. Sprekels, *The regulation of temperature by thermostats and set-valued integral equations*, J. Integral Equ. **4** (1982), 95–112.

[4] I. G. Götz, K.-H. Hoffmann, A. M. Meirmanov, *Periodic solutions of the Stefan problem with hysteresis-type boundary conditions*, Manuscripta Math. **78** (1983), 179–199 .

[5] P. L. Gurevich, W. Jäger, A. L. Skubachevskii, *On the existence of periodic solutions to some nonlinear thermal control problems*, Doklady Akademii Nauk **418**, No. 2 (2008), 151–154. English transl.: Dokl. Math. **77**, No. 1 (2008).

[6] N. Kenmochi, M. Primicerio, *One-dimensional heat conduction with a class of automatic heat-source controls*, IMA J. Appl. Math. **40** (1988), 205–216.

[7] N. Kenmochi, A. Visintin, *Asymptotic stability for nonlinear PDEs with hysteresis*, European J. Appl. Math. **5**, No. 1 (1994), 39–56.

[8] M. A. Krasnosel'skii, A. V. Pokrovskii, *Systems with Hysteresis*, Springer-Verlag, Berlin–Heidelberg–New York (1989). Translated from Russian: *Sistemy s Gisterezisom*, Nauka, Moscow, (1983).

[9] J. Prüss, *Periodic solutions of the thermostat problem*, Proc. Conf. "Differential Equations in Banach Spaces," Bologna, July 1985. Lecture Notes Math. **1223**, Springer-Verlag, Berlin–New York (1986), 216–226.

[10] A. Visintin, *Differential Models of Hysteresis*, Springer-Verlag, Berlin–Heidelberg (1994).

The tumbling universe: dynamics of Bianchi models in the Big-Bang limit

Stefan Liebscher

(joint work with M. Georgi, J. Häerterich, K. Webster)

We consider cosmological models of Bianchi type. They yield spatially homogeneous, anisotropic solutions $g_{\alpha\beta}$ of the Einstein field equations,

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}.$$

Here $R_{\alpha\beta}$ denotes the Ricci curvature and R the scalar curvature of the Lorenzian metric $g_{\alpha\beta}$ whereas $T_{\alpha\beta}$ denotes the stress energy tensor of an ideal non-tilted fluid.

Representing the spatial homogeneity by a three-dimensional Lie algebra, the problem can be reduced to a five-dimensional system of ordinary differential equations in expansion-normalized variables, see for example [6, 5]. For unimodal Lie algebras, Bianchi class A, the reduced system in terms of the spatial curvature

Bianchi Class	N_1	N_2	N_3
I	0	0	0
II	$+$	0	0
VI_0	0	$+$	$-$
VII_0	0	$+$	$+$
VIII	$-$	$+$	$+$
IX	$+$	$+$	$+$

Table 1. Bianchi classes given by the signs of the spatial curvature variables N_i. Remaining cases are related by equivariance.

variables N_i and the shear variables $\Sigma_\pm$ reads

$$\begin{aligned} N_1' &= (q - 4\Sigma_+)N_1, \\ N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \\ N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \\ \Sigma_+' &= -(2-q)\Sigma_+ - 3S_+, \\ \Sigma_-' &= -(2-q)\Sigma_- - 3S_-. \end{aligned} \tag{1}$$

The abbreviations

$$\begin{aligned} q &= 2\left(\Sigma_+^2 + \Sigma_-^2\right) + \frac{1}{2}(3\gamma - 2)\Omega, \\ \Omega &= 1 - \Sigma_+^2 - \Sigma_-^2 - K, \\ K &= \frac{3}{4}\left(N_1^2 + N_2^2 + N_3^2 - 2\left(N_1N_2 + N_2N_3 + N_3N_1\right)\right), \\ S_+ &= \frac{1}{2}\Big(\left(N_2 - N_3\right)^2 - N_1\left(2N_1 - N_2 - N_3\right)\Big), \\ S_- &= \frac{1}{2}\sqrt{3}\left(N_3 - N_2\right)\left(N_1 - N_2 - N_3\right). \end{aligned} \tag{2}$$

include the deceleration parameter q, the density parameter Ω, and the curvature parameter K. The fixed parameter $\frac{2}{3} < \gamma \leq 2$, given by the equation of state of the ideal fluid, describes the uniformly distributed matter. For example, a value $\gamma = 1$ corresponds to dust, whereas $\gamma = 4/3$ corresponds to radiation.

Equivariances are given by permutations of $\{N_1, N_2, N_3\}$ together with appropriate linear transformations of Σ_+, Σ_- corresponding to a representation of S_3 on $\mathbb{R}_2$. Together with the reflection $(N_1, N_2, N_3) \mapsto (-N_1, -N_2, -N_3)$, the system yields a $S_3 \times \mathbb{Z}_2$ equivariance group.

Note the classification of restrictions of the dynamical system to the various invariant regions, that corresponds to the Bianchi classification of Lie algebras, see table 1.

The invariant set $\{\Omega = 0\}$ of (1) corresponds to the 4-dimensional vacuum model. The Kasner circle $\mathcal{K} = \{N_1 = N_2 = N_3 = 0, \Omega = 0\}$, Bianchi class I, consists of equilibria. The attached spheres $\mathcal{H}_k = \{N_k \neq 0,\ N_l = N_m = 0,\ \Omega = 0\}$,

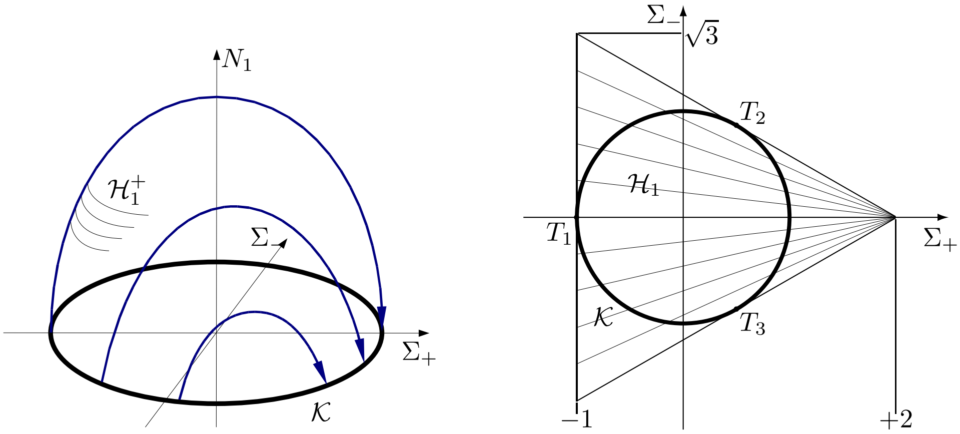

FIGURE 1. Heteroclinic caps $\mathcal{H}$ of vacuum Bianchi II solutions to the Kasner circle $\mathcal{K}$.

$\{k, l, m\} = \{1, 2, 3\}$, Bianchi class II, consist of heteroclinic orbits to equilibria on the Kasner circle, see figure 1. The projections of the trajectories of Bianchi class-II vacuum solutions onto the $\Sigma_\pm$-plane yield straight lines through the point $(\Sigma_+, \Sigma_-) = (2, 0)$ in the cap $\{N_1 \neq 0,\ N_2 = N_3 = 0\}$. The projections of the other caps are given by the equivariance.

Away from the tangential points, T_k, $k = 1, 2, 3$, the Kasner circle $\mathcal{K}$ is normally hyperbolic with 2-dimensional center-stable manifold given by the incoming heteroclinic orbits.

The Kasner map $\Phi : \mathcal{K} \to \mathcal{K}$ is defined as follows: for each point $q_+ \in \mathcal{K} \setminus \{T_1, T_2, T_3\}$ there exists a Bianchi class-II vacuum heteroclinic orbit $q(t)$ converging to q_+ as $t \to \infty$. This orbit is unique up to reflection $(N_1, N_2, N_3) \mapsto (-N_1, -N_2, -N_3)$. Its unique α-limit defines the image of q_+ under the Kasner map

$$\Phi(q_+) := q_- \tag{3}$$

Including the three fixed points, $\Phi(T_k) := T_k$, this construction yields a continuous map, $\Phi : \mathcal{K} \to \mathcal{K}$. In fact Φ is a non-uniformly expanding map and $\Phi(\mathcal{K})$ is a double cover of $\mathcal{K}$, see figure 2.

The α-limit, $t \to -\infty$, of the full system (1) corresponds to the big-bang singularity of the cosmological model. The dynamics in this limit, however, is not yet understood. It has been conjectured [3, 1] that the dynamics follows the (formal) Kasner map (3).

At least for Bianchi class-IX solutions the Bianchi attractor formed by the union of the Kasner circle and its heteroclinic orbits has been proven to indeed be a (global) attractor for trajectories to generic initial data under the time-reversed flow [4].

Therefore, as the first step towards a rigorous description of the α-limit dynamics of the Bianchi system, we describe the set of initial conditions near the

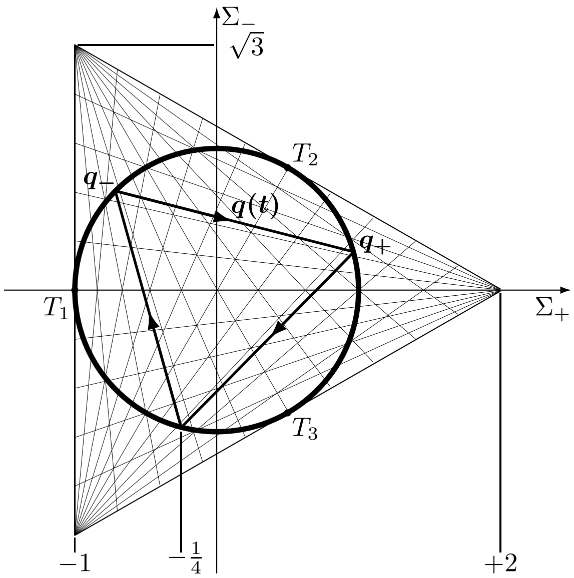

Figure 2. Kasner circle with Kasner map and its heteroclinic 3-cycle.

Bianchi attractor that follow the (up to equivariance) the unique period-3 heteroclinic cycle of the Kasner map, see figure 2. In fact we prove that this set forms a codimension-one Lipschitz manifold [2].

References

[1] M.H. Bugalho, A. Rica da Silva, and J. Sousa Ramos, *The order of chaos on a Bianchi-IX cosmological model*, Gen. Relativ. Gravitation **18** (1986), 1263–1274.

[2] M. Georgi, J. Härterich, S. Liebscher, K. Webster, *Ancient dynamics in Bianchi models. I: The three-cycle* (2009), in preparation.

[3] C.W. Misner, *Mixmaster universe*, Phys. Rev. Lett. **22** (1969), 1071–1074.

[4] H. Ringström, *The Bianchi IX attractor*, Ann. Henri Poincaré **2**(3) (2001), 405–500.

[5] J. Wainwright and G.F.R. Ellis (editors), *Dynamical systems in cosmology.* Cambridge University Press, Cambridge, 2005.

[6] J. Wainwright and L. Hsu, *A dynamical systems approach to Bianchi cosmologies: orthogonal models of a class A*, Classical Quantum Gravity **6**(10) (1989), 1409–1431.

Semi-strong front interaction by the renormalization group method

Peter van Heijster
(joint work with A. Doelman, T.J. Kaper, K. Promislow)

The 3-component reaction-diffusion system

$$\begin{cases} U_t = U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma), \\ \tau V_t = \dfrac{1}{\varepsilon^2} V_{\xi\xi} + U - V, \\ \theta W_t = \dfrac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W \end{cases}$$

introduced in [2] has become a paradigm model in pattern formation. It exhibits a rich variety of dynamics of fronts, pulses, and spots. The front and pulse interactions range in type from weak, in which the localized structures interact only through their exponentially small tails, to strong interactions in which they annihilate or collide and in which all components are far from equilibrium in the domains between the localized structures. Intermediate to these two extremes sits the semi-strong interaction regime, in which the activator component of the front is near equilibrium in the intervals between adjacent fronts, but both inhibitor components are far from equilibrium there, and hence their concentration profiles drive the front evolution. In this article, we focus on dynamically-evolving N-front solutions in the semi-strong regime. The primary result is to use a renormalization group method to rigorously derive the system of N coupled ODEs that governs the positions of the fronts. The operators associated to the linearization about the N-front solutions have N small eigenvalues, and the N-front solutions may be decomposed into a component in the space spanned by the associated eigenfunctions and a component projected onto the complement of this space. This decomposition is carried out iteratively at a sequence of times. The former projections yield the ODEs for the front positions, while the latter projections are associated to remainders that we show stay small in a suitable norm during each iteration of the renormalization group method. Our results also help extend the application of the renormalization group method from the weak interaction regime for which it was initially developed to the semi-strong interaction regime. The second set of results that we present is a detailed analysis of this system of ODEs, providing a classification of the possible front interactions in the cases of $N = 1, 2, 3, 4$, as well as how front solutions interact with the stationary pulse solutions studied earlier in [1, 3]. Moreover, we present some results on the general case of N-front interactions.

References

[1] A. Doelman, P. van Heijster, T.J. Kaper, *Pulse dynamics in a three-component system: existence analysis*, J. Dyn. Diff. Eq. online, DOI: 10.1007/s10884-008-9125-2 (2008).

[2] C.P. Schenk, M. Or-Guil, M Bode, H.-G. Purwins, *Interacting Pulses in Three-component Reaction-Diffusion Systems on Two-Dimensional Domains*, PRL **7**8(19) (1997), 3781–3784.

[3] P. van Heijster, A. Doelman, T.J. Kaper, *Pulse dynamics in a three-component system: stability and bifurcations*, Physica D **2**37(24) (2008), 3335–3368.

Freezing stable multipulses and multifronts

Sabrina Selle
(joint work with Wolf-Jürgen Beyn, Vera Thümmler)

We consider time dependent reaction diffusion systems in one space dimension that have multiple pulse or multiple front solutions, i.e. solutions that look like a finite superposition of several waves, see [1]. The systems are of the form

$$(1) \qquad u_t = Au_{xx} + f(u),\ x \in \mathbb{R},\ t \geq 0, \quad u(x,0) = u_0(x),\ x \in \mathbb{R},\ u(x,t) \in \mathbb{R}^m,$$

where $A \in \mathbb{R}^{m,m}$ is positive definite and $f : \mathbb{R}^m \to \mathbb{R}^m$ is sufficiently smooth. Furthermore, we assume that (1) has $N \geq 2$ traveling wave solutions of the form

$$u_j(x,t) = w_j(x - c_j t), \quad j = 1, \ldots, N,$$

with different speeds c_j and with limits $w_j^{\pm} = \lim_{\xi \to \pm\infty} w_j(\xi)$. Assume that the left and right limits of the single waves match in the sense that

$$w_j^+ = w_{j+1}^-, \quad j = 1, \ldots, N-1.$$

Consider the superposition

$$(2) \qquad W(x,t) = \sum_{j=1}^{N} \hat{w}_j(x - c_j t), \quad \hat{w}_j(\xi) = w_j(\xi) - \tilde{w}_j^-, \quad \tilde{w}_j^- = \begin{cases} 0, & j = 1 \\ w_j^-, & j \geq 2, \end{cases}$$

where we have subtracted left limits so that the modified profiles $\hat{w}_j$ fit together upon summation. We are interested in solutions $u(x,t)$ that have the shape of W for large times.

We present a numerical method for decomposing solutions of the Cauchy problem (1) into a superposition of functions $v_j(\cdot, t)$ that asymptotically assume the shape of the shifted waves $\hat{w}_j$. Our decomposition of the solution is analogous to (2) and has the form

$$(3) \qquad u(x,t) = \sum_{j=1}^{N} v_j(x - g_j(t), t).$$

Here g_j denotes the position of the pattern v_j at time t and N denotes the number of pulses or fronts. The functions v_j, g_j are unknowns and will be uniquely determined by the numerical process if we add extra phase conditions such that the decomposition (2) holds in an asymptotic sense.

The approach extends the method of freezing single pulses, which was developed in [4] to study the stability of single traveling waves, see [2, 3]. The freezing method allows to compute a moving coordinate frame in which, for example, a traveling wave becomes stationary.

We insert the ansatz (3) into equation (1), we introduce new coordinates $\xi = x - g_j(t)$ and use a positive bump function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ to generate a time-dependent partition of unity. A computation shows that u from (3) is a solution of (1) if $(v_j, g_j, \mu_j), j = 1, \ldots, N$ solves the following system with nonlocal couplings for $j = 1, \ldots, N$

(4) $$v_{j,t}(\xi, t) = Av_{j,\xi\xi}(\xi, t) + v_{j,\xi}(\xi, t)\mu_j(t) + f(\tilde{w}_k^- + v_j(\xi, t)) + F_j(v, g)(\xi, t)$$

and the simple set of ODEs

$$g_{j,t} = \mu_j(t), \quad j = 1, \ldots, N,$$

where $v = (v_1, \ldots, v_N)$, $g = (g_1, \ldots, g_N)$, $\xi_{kj}^g = \xi - g_k + g_j$ and

$$F_j(v,g)(\xi,t) = \frac{\varphi(\xi)}{\sum_{k=1}^N \varphi(\xi_{kj}^{g(t)})}\left[f\left(\sum_{k=1}^N v_k(\xi_{kj}^{g(t)}, t)\right) - \sum_{k=1}^N f\left(\tilde{w}_k^- + v_k(\xi_{kj}^{g(t)}, t)\right)\right].$$

The system is completed by initial data for v_j, g_j and by phase conditions

$$\langle v_j - \hat{v}_j, \hat{v}_{j,\xi}\rangle_{\mathcal{L}^2} = 0, \quad j = 1, \ldots, N$$

with given reference functions $\hat{v}_j$. Note the difference to [1] in the nonlinear terms of (4).

We present a stability theorem for multipulse and multifront solutions which states that the shifted traveling waves $\hat{w}_j$ are asymptotic stable solutions of (4) in a slightly weigthed space $\mathcal{H}^1$. This implies that the solution (3) of (1) converges to W with suitably shifted waves.

Example: Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-a), \quad x \in \mathbb{R}, t \geq 0, a = 0.25$$

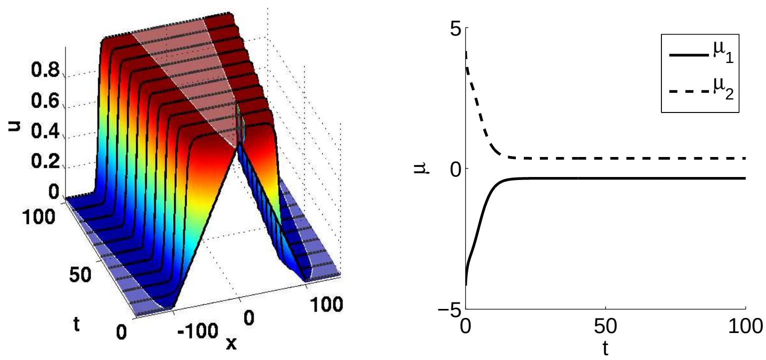

FIGURE 1. Two fronts moving in opposite directions in the Nagumo equation, evolution of superposition and velocities μ_1, μ_2, supports of v_1, v_2 are shaded

References

[1] W.-J. Beyn, S. Selle, V. Thümmler, *Freezing Multipulses and Multifronts*, SIAM Journal on Applied Dynamical Systems, Number 2 (Volume 7) Pages 577-608, 2008.

[2] V. Thümmler, *The effect of freezing and discretization to the asymptotic stability of relative equilibria*, Journal of Dynamics and Differential Equations, Number 2 (volume 20) Pages 425-477, 2008.

[3] V. Thümmler, *Numerical approximation of relative equilibria for equivariant PDEs*, SIAM Journal on Numerical Analysis, Number 6 (Volume 46) Pages 2978-3005, 2008.

[4] W.-J. Beyn, V. Thümmler, *Freezing solutions of equivariant evolution equations*, SIAM Journal on Applied Dynamical Systems, Number 2 (Volume 3) Pages 85-116, 2004.

Spiral waves in excitable media

Sebastian Hermann

(joint work with Georg Gottwald)

Spiral waves are generic solutions in two-dimensional excitable media. Our studies concentrate on rigidly rotating spiral waves in the large core limit. Depending on the chosen excitability ϵ a finger-like initial condition will either start to curl and subsequently develop into a rotating spiral wave or retract. Determining the critical excitability ϵ_c where spiraling fails is a non-trivial task from a numerical point of view since in the large core limit the spiral tip is moving on a large circle around the spiral core requiring computationally expensive large domain sizes. To overcome the restrictions of finite domains we employ a freezing method introduced by Beyn and Thümmler [1] which makes use of the underlying equivariance of excitable media with respect to the Euclidean group action of translation and rotation in the plane. By performing a symmetry reduction the dynamics of the full PDE can be split into two parts, one describing the dynamics on the group orbits (i.e. rotation and translation) and one, the so called base dynamics, describing the shape of the solution. This allows calculations to be performed on a relatively small domain with a small number of grid points.

Here we consider the Barkley model [2]. It turns out that freezing exhibits severe problems for this system and other excitable media where the inhibitor is non-diffusive. This causes the resulting symmetry reduced system to be of mixed hyperbolic-parabolic type which therefore causes numerical instabilities. Furthermore, the standard Neumann boundary conditions have a strong impact on the freezing procedure since they do not respect the underlying symmetries of the system. This symmetry breaking introduces large errors at the boundary which consequently propagate inwards and prevent numerically stable results.

To solve the numerical problems that arise at the boundary, we introduce (so far completely numerical) transparent spiral boundary conditions for polar and cartesian coordinates respectively. They respect the shape of the spiral much better by following contourlines across the boundary. It proves very useful in diminishing artificial oscillations occurring at the boundary during the process of freezing in polar coordinates. For the simulation of unfrozen large spirals in cartesian coordinates we also obtain very good improved results. However it cannot prevent the freezing method from failing in this setting due to missing information at the corners of the used box. We report on an application for this transparent boundary condition to remove boundary induced spiral drift [3].

References

[1] W.-J. Beyn, V. Thümmler, *Freezing Solutions of Equivariant Evolution Equations*, SIAM J.Applied Dynamical Systems, Vol. **3**, 2 (2004)

[2] D. Barkley, *A model for fast computer simulation of waves in excitable media*, Physica D **49**, 61-70 (1991)

[3] V.N. Biktashev, A.V. Holden, *Resonant drift of autowave vortices in two dimensions and the effects of boundaries and inhomogenities*, Chaos, Solitons & Fractals **5**, 575-622 (1995)

The Eckhaus scenario in delay differential equations with large delay

Matthias Wolfrum

(joint work with Serhiy Yanchuk)

Delay-differential equations (DDEs) play an important role in many applied problems including e.g. economy, neuroscience, and optoelectronics. In particular, semiconductor lasers with optical feedback or coupling show a huge variety of complex dynamical behavior, induced by the transmission delay of the feedback signal. In many cases this delay, which is caused by the finite speed of propagation of the light, has even to be considered as large compared to the time scale of the internal processes in the laser, being in the range of picoseconds. For a laser with delayed optical feedback the Lang-Kobayashi model has extensively been used in the physical literature to investigate the dynamical behavior related to the large delay, showing the coexistence of many periodic solutions with different stability properties, high dimensional chaos and other.

Starting from these phenomena, the following mathematical questions arise:

- How can the singular limit of delay $\tau \to \infty$ be used to study general systems with large delay
- What types of dynamics can arise typically in DDEs with large delay

In the following, we will briefly summarize some recent results related to these questions.

Scaling properties of the spectrum. We consider here an autonomous system of DDEs with a single fixed delay τ. Since the limit $\tau \to \infty$ is a singular limit, one has to expect dynamical phenomena on different timescales. On the level of linear stability analysis, this leads to different scaling behavior of the eigenvalues. To meet the standard notation, we introduce the small parameter $\varepsilon = \frac{1}{\tau}$. The characteristic equation for the spectrum of the linearized system at a stationary solution has in general the form

$$\chi(\lambda) = \det(\lambda \mathrm{Id} - A - e^{-\frac{\lambda}{\varepsilon}} B) = 0, \tag{1}$$

where the matrices A and B are the Jacobians with respect to the instantaneous and the delayed argument, respectively. It turns out that generically, two types of eigenvalues exist. (1) Strong instabilities, given in leading order by eigenvalues λ of A with positive real part. (2) Pseudo-continuous spectrum (PCS), which is obtained by introducing the scaling

(2) $$\lambda = \varepsilon + i\omega$$

into equation (1) and omitting higher order terms. In this way, we obtain

(3) $$\det(i\omega \mathrm{Id} - A - e^{-\gamma} e^{-i\Phi} B) = 0,$$

where $\Phi = \frac{\omega}{\varepsilon}$. The solutions of equation (3) defines a family of curves in the (γ, ω)-plane, which are parametrized by Φ. Since Φ is a rapidly oscillating term, these curves will for $\varepsilon \to 0$ more and more densely be filled with approximated eigenvalues, i.e the true eigenvalues will accumulate densely along these curves. Taking into account the scaling (2), we call instabilities originating from PCS with positive real part *weak instabilities*. It is evident that classical bifurcation theory can give only a very rough picture of the resulting dynamical scenarios, since immediately a large number of eigenvalues is involved in the destabilization. Instead, a description by spatially extended systems, i.e. amplitude equations, seems to be more adequate. We will now use this approach to study an example of an oscillatory instability under the influence of large delay.

The Stuart-Landau oscillator with delayed feedback. As the most simple example for an oscillatory instability under the influence of large delay, we investigate the Stuart-Landau oscillator with delayed feedback

$$z' = (\alpha + i\beta)z - z|z|^2 + e^{i\phi} z_\tau$$

for the complex variable z and large delay τ. Due to the phase-shift equivariance, periodic solutions can be calculated here explicitely. They emerge from the trivial solution at supercritical Hopf-bifurcations that are located on the circle

(4) $$\alpha^2 + (\omega - \beta)^2 = 1,$$

where ω denotes the frequency. The number of Hopf points along this curve is proportional to the delay τ, such that for large τ there can be found a large number of periodic solutions in the frequency band $\omega \in [\beta - 1, \beta + 1]$. In Figure 1 the region of existence of these periodic solutions is shown in grey. Note that omitting the feedback term, the system shows a single supercritical Hopf bifurcation at $\alpha = 0$, leading to a strong instability for $\alpha > 0$. Including the delay term we can calculate the PCS

$$\gamma(\omega) = -\frac{1}{2} \ln\left(\alpha^2 + (\omega - \beta)^2\right).$$

and note that it is unstable exactly within the circle (4). Concerning the stability of the bifurcating periodic solutions, we have obtained the following results: The primary branch, bifurcating closest to $\alpha = -1$ and $\omega = \beta$ is stable everywhere. All other branches emerge unstable from the trivial solution and undergo a sequence of Hopf-bifurcations. After that, the branches emerging for $\alpha < 0$ become stable. The asymptotic location of this stability boundary can be calculated explicitely in the following way. Due to symmetry, each periodic solution becomes a family of

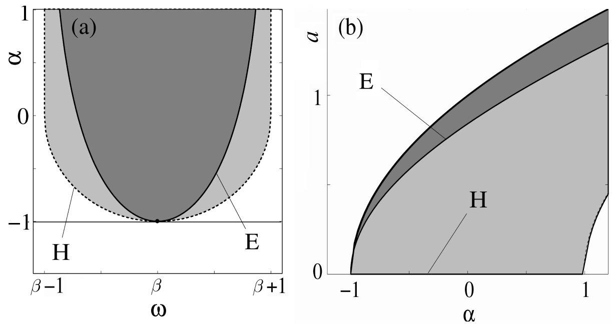

FIGURE 1. Regions with periodic solutions: existence (grey), stability (dark), torus bifurcations (hatched). Panel (a): bifurcation parameter α versus frequency ω. Panel (b): amplitude a versus α.

stationary solutions in a respectively corotating frame. Calculating for them again the PCS, one can observe that, caused by symmetry, one of the two branches $\gamma_1(0)$ touches the imaginary axis at 0. At the points where the curvature $\gamma_1''(0)$ becomes positive, the periodic solution undergoes a so called modulational instability. Their location can be calculated as

$$\alpha = \frac{2(\beta-\omega)^2 - 1}{\sqrt{1-(\beta-\omega)^2}}. \tag{5}$$

This curve coincides in leading order with the classical Eckhaus parabola for the stability boundary of periodic spatial patterns close to a Turing bifurcation. In Figure 1, it is depicted by the curve (E), separating the region of torus bifurcations from the region of coexisting stable periodic solutions. In analogy to the classical Eckhaus scenario in the case of a large but finite domain (see [2]), we obtain here a similar structure of a large number of coexisting periodic solutions interacting in a global picture with a universal stability boundary.

References

[1] M. Wolfrum, S. Yanchuk, *Eckhaus instability in systems with large delay*, Phys. Rev. Lett. **96** (2006), 220201.

[2] L. Tuckerman, D. Barkley, *Bifurcation analysis of the Eckhaus instability*, Physica D **46** (1990), 57–86.

M. Muster, *Computing other invariants of topological spaces of dimension three*, Topology **32** (1990), 120–140.

Stability in nonlinear hyperbolic PDEs

Jens Rottmann-Matthes

We consider an abstract hyperbolic semilinear partial differential equation

$$u_t = Bu_x + g(u), \qquad t \in \mathbb{R}_+,\ x \in \mathbb{R},\ u(x,t) \in \mathbb{R}^m. \tag{PDE}$$

We present a proof of the asymptotic stability with asymptotic phase of a stationary solution $\underline{u}$ under certain spectral assumptions, see [3] for a different proof. For parabolic problems the result is classic and a proof using analytic semigroup theory can be found in [2].

For simplicity we consider smooth functions g, and assume that B is a constant diagonal matrix with pairwise different diagonal entries. We make the following assumptions:

H1: $\underline{u} \in \mathcal{C}^3,\ \underline{u}_x \in H^2$, is a stationary solution,

H2: $C(x) := g_u(\underline{u}(x))$ satisfies

$$C_{\pm jj} = \lim_{x\to\pm\infty} C_{jj}(x) \le -2\delta < 0,\ j = 1,\dots,m,$$

H3: for $\omega \in \mathbb{R}$ holds:

$$s \in \sigma(i\omega B + C_+) \cup \sigma(i\omega B + C_-)\ \Rightarrow \mathrm{Re} \le -\delta.$$

By P we denote the linearization of (PDE) about the steady state $\underline{u}$:

$$Pu = Bu_x + g_u(\underline{u})u,$$

note that 0 is always an element of the spectrum of P due to the autonomy.

Our main result is as follows.

Theorem 1 (Main Theorem). *Under the assumptions from above and furthermore the assumption that* 0 *is a simple eigenvalue of* P*, there are constants* $\rho > 0,\ C > 0$ *such that for all initial data* $u_0 \in \underline{u} + H^2$, $\|u_0 - \underline{u}\|_{H^2} < \rho$ *the solution* u *exists for all positive times and is an element of* $\mathcal{C}^1(\mathbb{R}_+, \underline{u} + L^2) \cap \mathcal{C}^0(\mathbb{R}_+, \underline{u} + H^1)$. *Moreover there is a phaseshift* $\varphi_\infty \in \mathbb{R}$ *with*

$$\|u(t) - \underline{u}(\cdot - \varphi_\infty)\|_{H^1} \le Ce^{-\frac{\delta}{2}t}\ \forall t \ge 0.$$

To deal with the unknown phaseshift we use nonlinear coordinates to write the solution u as

$$u = \underline{u}(\cdot - \varphi) + w,$$

where w is in H^1 and φ in $\mathbb{R}$. These nonlinear coordinates increase the dimension of the problem by one and we pose a suitable algebraic constraint, a so called *phase-condition* which we write as $\langle \psi, w\rangle = 0$, to obtain a well-posed problem again. This leads to a reformulation of (PDE) as a partial differential algebraic equation (PDAE) of the form

$$\text{(PDAE)}\qquad \begin{aligned} w_t &= Pw + \varphi_t \underline{u}_x + G(\varphi, w), \\ \langle \psi, w\rangle &= 0, \end{aligned}$$

where G is an at least quadratic function in φ and w. We assume that ψ from the algebraic condition satisfies $\psi \in H^1 \cap L^1$ with $\langle \psi, \underline{u}_x\rangle_{L^2} \ne 0$.

The following Lemma shows that this reformulation really is equivalent to the original system and hence it suffices to analyze the (PDAE) for the proof of Theorem 1.

Lemma 1. *There exist* $\rho_0, \rho_1 > 0$ *such that a function* $u \in \mathcal{C}^1([0,T]; \underline{u} + L^2) \cap$

$\mathcal{C}^0([0,T];\underline{u}+H^1)$ *is a solution of* (PDE) *with* $\|u-\underline{u}\|_{L^2} < \rho_0$ *for all* t *if and only if* $u = \underline{u}(\cdot - \varphi) + w$, *where*

$$\varphi \in \mathcal{C}^1([0,T];\mathbb{R}),\ |\varphi| < \rho_1,\ \text{and } w \in \mathcal{C}^1([0,T];L^2) \cap \mathcal{C}^0([0,T];H^1)$$

solve (PDAE).

The effect of the reformulation is that the asymptotic stability with asymptotic phase for (PDE) becomes a classical Lyapunov stability for the reformulation (PDAE).

To show the stability for the partial differential algebraic equation we use the Laplace-transform technique for vector-valued functions [1] and directly analyze the spectral properties of the resulting resolvent equation

$$\begin{pmatrix} sI - P & -\underline{u}_x \\ \langle \psi, \cdot \rangle_{L^2} & 0 \end{pmatrix} \begin{pmatrix} \widehat{w} \\ \widehat{\varphi_t} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ 0 \end{pmatrix}.$$

These are shown to imply exponential stability for the linear PDAE. Carefully analyzing the nonlinearity, also nonlinear stability of (PDAE) under the assumptions of Theorem 1 can be concluded:

Theorem (Stability of (PDAE)). *There exists* $\varepsilon > 0$ *such that for all initial data*

$$(\varphi_0, w_0) \text{ with } |\varphi_0| < \varepsilon \text{ and } \|w_0\|_{H^2} < \varepsilon$$

the partial differential algebraic equation (PDAE) *has a unique solution* (φ, w) *for all* $t \geq 0$ *which satisfies*

$$\varphi \in \mathcal{C}^1([0,\infty);\mathbb{R}) \text{ and } w \in \mathcal{C}^1([0,\infty);L^2) \cap \mathcal{C}^0([0,\infty);H^1).$$

Moreover there are $\varphi_\infty \in \mathbb{R}$ *and* $C = C(\varepsilon) > 0$ *such that*

$$|\varphi(t) - \varphi_\infty| \leq Ce^{-\frac{\delta}{2}t},$$
$$\|w(t)\|_{H^1} \leq Ce^{-\frac{\delta}{2}t},\ \forall t \geq 0.$$

Together with Lemma 1 this result implies Theorem 1.

References

[1] W. Arendt, C.J.K. Batty, M. Hieber,F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser Verlag, 2001.

[2] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer Verlag, 1981.

[3] G. Kreiss, H.-O. Kreiss, N.A. Petesson, *On the convergence to steady state solutions of nonlinear hyperbolic-parabolic systems*, SIAM J. Numer. Anal. **31** (1994), 1577–1604.

Shooting and exit manifolds in planar reaction diffusion equations

J. Douglas Wright

If a system possesses spatially localized traveling wave solutions which are stable under perturbations, then one expects that such a system will also possess solutions which are roughly the linear superposition of multiple pulses, at least until such a time as those pulses pass close by one another. In this talk, we discuss how one can rigorously prove this assertion for planar reaction-diffusion equations

$$u_t = \mathcal{D}\Delta u + F(u). \tag{1}$$

(See [1, 4, 5] for similar work done on the real line.) It is assumed that exponentially localized traveling pulses $Q\left(R(\theta)(\mathbf{x} - ct\mathbf{i})\right)$ exist ($R(\theta)$ is a rotation matrix) and that the spectrum of linearization (denoted A) about such a pulse is sectorial and stable, apart from the triple eigenvalue which arises from translation and rotation invariance of the equation.

The first difficulty is that one cannot put (1) into a moving reference frame which renders the linear piece of the problem autonomous (as one can do for single pulses). This problem is circumvented by considering the larger system

$$\begin{aligned} U_t &= \mathcal{D}\Delta U + F(U) + \chi_1(x,t)(F(U+V) - F(U) - F(V)),\\ V_t &= \mathcal{D}\Delta V + F(V) + \chi_2(x,t)(F(U+V) - F(U) - F(V)). \end{aligned} \tag{2}$$

Here χ_1 and χ_2 are a partition of unity subordinate to half planes containing each pulse. Notice that $u = U + V$ solves (1). Making the substitution

$$U = Q\left(R(\theta_1)(\mathbf{x} - ct\mathbf{i} - \mathbf{x}_1)\right) + V_1(R(\theta_1)(\mathbf{x} - ct\mathbf{i} - \mathbf{x}_1), t),$$

$$V = Q\left(R(\theta_2)(\mathbf{x} - ct\mathbf{i} - \mathbf{x}_2)\right) + V_2(R(\theta_2)(\mathbf{x} - ct\mathbf{i} - \mathbf{x}_2), t),$$

one arrives at an equation of the form

$$\mathbf{V}_t = \mathbf{A}\mathbf{V} + \mathbf{B}(t)\mathbf{V} + \mathbf{H}(x,t) + N(\mathbf{V}),$$

where $\mathbf{V} = (V_1, V_2)$, $\mathbf{A} = \mathrm{diag}(A, A)$, $N(\mathbf{V}) = O(|\mathbf{V}|^2)$ and the inhomogeneous piece $\mathbf{H}(x,t) = O(\exp(-K\rho(t))$ where $\rho(t)$ is the distance between the two pulses. Since we assume that the operator A is sectorial we know $\mathbf{A}$ is sectorial as well (with a six-dimensional kernel). Thus, if we knew that $\mathbf{B}(t)$ was small, we could modify the proof that a single pulse is stable and show that $\|\mathbf{V}(t)\|$ decays exponentially quickly provided the pulses do not come to close together.

However, $\mathbf{B}(t)$, while linear, is an $O(1)$ operator (it consists of terms like Q_2R_1, for instance). This is the second difficulty in the problem. We prove a key lemma which says that (**a**) $\mathbf{B}(t)$ has a small operator norm provided the function $|\mathbf{V}(\mathbf{x},t)| \le C\exp(-\beta|\mathbf{x}|)$ and (**b**) that $|\mathbf{B}(t)\mathbf{V}(\mathbf{x},t)| \le C\exp(-\beta|\mathbf{x}|)$ even if $\mathbf{V}$ itself exhibits no such decay for large $\mathbf{x}$. Finally, we prove that these two properties are sufficient to treat $\mathbf{B}$ as if it were a small perturbation of $\mathbf{A}$. In this way, we can prove our main results, for instance:

Theorem (Stability of the Exit Manifold): *Let*

$$(3)\quad M_{exit} := \{Q(R(\theta_1)(\cdot - \mathbf{x}_1)) + Q(R(\theta_2)(\cdot - \mathbf{x}_2)) : \\ \theta_1 \neq \theta_1 \text{ and } \inf_{t\geq 0} |\mathbf{x}_1 - \mathbf{x}_2 + ct(R(\theta_1)\mathbf{i} - R(\theta_1)\mathbf{i})| \geq L\}.$$

There exists a neighborhood of M_{exit} (in $W^{1,p}$) such that solutions of (1) *with initial data in this neighborhood converge exponentially quickly to M_{exit} as $t \to \infty$. Specifically, the solution converges to the superposition of two pulses which are asymptotically the linear superposition of two pulses.*

References

[1] W.-J. Beyn, S. Selle, V. Thümmler, *Freezing multipulses and multifronts*, SIAM J. Appl. Dyn. Syst. **7**, No. 2 (2008), 577–608.

[2] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin-New York, 1981. iv+348 pp. ISBN: 3-540-10557-3

[3] A. Mielke, S. V. Zelik, *Infinite-dimensional hyperbolic sets and spatio-temporal chaos in reaction-diffusion systems in* $\mathbb{R}^n$, J. Dynam. Differential Equations **19**, No. 2 (2007), 333–389.

[4] A. Scheel, J. D. Wright, *Colliding dissipative pulses — the shooting manifold*, J. Differential Equations **245**, No. 1 (2008), 59–79.

[5] J. D. Wright, *Separating dissipative pulses — the exit manifold*, Preprint.

A role of network of unstable patterns in dissipative systems

Yasumasa Nishiura
(joint work with Takashi Teramoto, Yuan Xiaohui, and Kei-Ichi Ueda)

The issue here is to study the dynamics of moving spatially localized patterns when they interact with external environments. Hereafter such a localized patterns is called particle pattern. Two interesting cases are considered here: one is a colision with other particle ones, and the other is the response of pattern when the media is changed from homogeneous to heterogeneous, typically jump or bump heterogeneity. In both cases it turns out that unstable patterns play a crucial role to understand their dynamics, although those ones are only observable transiently.

1. Collision dynamics

Particle patterns mean any spatially localized structures sustained by the balance between inflow and outflow of energy-material which arise in the form of chemical blob, discharge pattern, morphological spot, and binary convection cell. These are modeled by typically three-component reaction diffusion systems or a couple of complex GL equations with concentration field. Strong interaction such as collision among particle patterns is a big challenge, since dissipative systems do not have many conservative quantities. Unlike weak-interaction through tails of those objects, there are so far no systematic methods to handle them because of large deformation of patterns during the collision process. We present a new approach to clarify a backbone structure behind the complicated transient collision process. A key ingredient lies in a hidden network of unstable solutions called scattors which play a crucial role to understand the input-output relation for collision process (namely the relation of two dynamics before and after collision). More precisely, the associated network of scattors via heteroclinic connections forms a backbone for the whole collisional dynamics. It should be noted that collision dynamics for traveling breathers depends the phase differnce of those waves (see [4]). The viewpoint of scattor network seems quite useful for a large class of model systems arising in gas-discharge phenomena, chemical blobs, and binary fluid convection. For references, see [2, 3, 5, 6, 1].

2. Dynamics in heterogeneous media

Localized waves are one of the main carriers of information and the effect of heterogeneity of the media in which it propagates is of great importance for the understanding of signaling processes in biological and chemical problems. A typical and simple heterogeneity is a spatially localized bump or dent in 1D or 2D, which in general creates associated defects in the media. One of the main issues is how the geometry of heterogeneity influences over the dynamics of waves. Here the geometry means slope, height, size, curvature and so on. Localized waves are sensitive to those factors and in fact present a variety of dynamics including rebound, pinning, splitting, and traveling motion around the defect. A reduction method to finite-dimensional system is presented, which clarifies the mathematical structure for those dynamics. In the reference below we mainly focus on a class of one-dimensional traveling pulses the associated parameters of which are close to drift and/or saddle-node bifurcations. The great advantage to study the dynamics in such a class is two-fold: firstly it gives us a perfect microcosm for the variety of outputs in general setting when pulses encounter heterogeneities. Secondly it allows us to reduce the original PDE dynamics to tractable finite dimensional system. Such pulses are sensitive when they run into the heterogeneities and show rich responses such as annihilation, pinning, splitting, rebound as well as penetration. The reduced ODEs explain all these dynamics and the underlying bifurcational structure controlling the transitions among different dynamic regimes. It turns out that there are hidden ordered patterns associated with the critical points of ODEs which play a pivotal role to understand the responses of the pulse. We mainly focus on a bump and periodic types of heterogeneity, however our approach is also applicable to general case. It should be noted that there appears spatio-temporal chaos for periodic type of heterogeneity when its period becomes comparable with the size of the pulse. For references, see [7, 8, 9].

3. Behaviors of amoeba (Physarum plasmodium) in heterogeneous environments

We report here a new kind of behavior that seems to be "findecisive" in an amoeboid organism, the Physarum plasmodium of true slime mold. The plasmodium migrating in a narrow lane stops moving for a period of time (several hours but the duration differs for each plasmodium) when it encounters the presence of a chemical repellent, quinine. After stopping period, the organism suddenly begins to move again in one of three different ways as the concentration of repellent increases: going through the repulsive place (penetration), splitting into two fronts of going through it and turning (splitting) and turning back (rebound). In relation to the physiological mechanism for tip migration in the plasmodium, we found that the frontal tip is capable of moving further although the tip is divided from a main body of organism. This means that a motive force of front locomotion is produced by a local process at the tip. Based on this finding, a mathematical model for front locomotion is considered in order to understand the dynamics for both the long period of stopping and three kinds of behavior. A model based on reaction-diffusion equations succeeds to reproduce the experimental observation. The origin of long-time stopping and three different outputs may be reduced to the hidden instabilities of internal dynamics of the pulse, which may be a skeleton structure extracted from much more complex dynamics imbedded in the Physarum plasmodium. For references, see [10, 11].

References

[1] M. Iima and Y. Nishiura, *Unstable periodic solution controlling collision of localized convection cells in binary fluid mixture*, Phsyica D, in press.

[2] Y. Nishiura, T. Teramoto and K.-I. Ueda, *Scattering and separators in dissipative systems*, Phys. Rev. E **67** (2003) 056210.

[3] Y. Nishiura, T. Teramoto and K.-I. Ueda, *Dynamic transitions through scattors in dissipative systems*, Chaos **13**(3) (2003), 962-972

[4] T. Teramoto, K.-I. Ueda and Y. Nishiura, *Phase-dependent output of scattering process for traveling breathers*, Phys. Rev. E **69**(4) (2004), 056224.

[5] Y. Nishiura, T. Teramoto and K.-I. Ueda, *Scattering of traveling spots in dissipative systems*, Chaos **15** (2005), 047509-047519.

[6] T. Teramoto, K.-I. Ueda and Y. Nishiura, *Breathing Scattors in Dissipative Systems*, Progress of Theoretical Physics Supplement **161**(2006), 364-367.

[7] Y. Nishiura, T. Teramoto, X. Yuan and K-I. Ueda, *Dynamics of traveling pulses in heterogeneous media*, Chaos **17**(3), (2007), 037104

[8] X. Yuan, T. Teramoto, and Y. Nishiura, *Heterogeneity-induced defect bifurcation and pulse dynamics for a three-component reaction-diffusion system*, Physical Review E **75**(3) (2007), .036220.

[9] Y. Nishiura, Y. Oyama, and K.-I. Ueda, *Dynamics of traveling pulses in heterogeneous media of jump type*, Hokkaido Math. J. **36**(1)(2007), 207-242.

[10] S. Takagi, Y. Nishiura, T. Nakagaki, T. Ueda and K.-I. Ueda, *Indecisive behavior of amoeba crossing an environmental barrier*, Proceedings of Int. Symp. on Topological Aspects of Critical Systems and Networks, World Scientific Publishing Co. (2007) 86-93.

[11] T. Nakagaki, M. Iima, T. Ueda, Y. Nishiura, T. Saigusa, A. Tero, R. Kobayashi, and K. Showalter, *Minimum-Risk Path Finding by an Adaptive Amoebal Network*, PRL **99**, (2007), 068104.

Localised two-dimensional patterns

David J. B. Lloyd

(joint work with Daniele Avitabile, John Burke, Alan R. Champneys, Edgar Knobloch, Juergen Knobloch, Björn Sandstede, Thomas Wagenknecht)

Localized stationary structures play an important role in many biological, chemical and physical processes. Such structures have been observed in a variety of experiments ranging from vertically vibrated granular materials, liquid crystals, binary-fluid convection, autocatalytic chemical reactions such as the Belousov–Zhabotinsky system, electrochemical systems, and localized micro-structures in solidification to nonlinear optical devices. Localized patterns have also been found in many nonlinear models such as those derived from magnetohydrodynamics, flame fronts, lasers, vibrated granular materials, neural networks, and cellular buckling as well as in the Swift–Hohenberg equation, which often serves as a paradigm for general pattern-forming systems.

In my talk, I consider stationary solutions of the Swift–Hohenberg equation

$$u_t = -(1+\Delta)^2 u - \mu u + \nu u^2 - u^3 \tag{1}$$

where $x \in \mathbb{R}$ for the 1D version and $(x, y) \in \mathbb{R}^2$ in the planar case. We focus on the region $\nu \geq 0$ since the case $\nu < 0$ is then recovered upon replacing u by $-u$. The trivial state $u = 0$ is stable for $\mu > 0$ and destabilizes at $\mu = 0$ with respect to perturbations that have nonzero finite spatial wavelength. At $\mu = 0$, hexagons bifurcate in a transcritical bifurcation from $u = 0$ for each $\nu > 0$, while rolls bifurcate in a subcritical pitchfork bifurcation from $u = 0$ provided $\nu > \nu_{\mathrm{r}} := \sqrt{27/38}$. While the bifurcating hexagons and rolls are initially unstable for $\mu > 0$, they stabilize in a subsequent saddle-node bifurcation, leading to a region of bistability between the nontrivial patterns and the trivial state for $\mu > 0$. This bistability region of trivial and patterned states opens up the possibility of finding fully localized stationary patches of hexagons or rolls.

In the talk, we will look at recent results pertaining to localised 2D patterns; see [2, 4, 3, 1]. We start by looking at the link between heteroclinic orbits (connecting the trivial state and a periodic orbit) with localised patterns in reversible, Hamiltonian ODEs. Geometric analysis allows one to predict the bifurcation structure of localised patterns from knowledge of the heteroclinic orbit. Also, asymmetric (ladder) localised states connecting symmetric branches of localised patterns are also predicted. This analysis is extended to localised patterns on the cylinder where we compare the geometric predicts with numerical investigations.

We then examine the shape of the hexagon patches along the snaking curve. We find their interfaces resemble planar hexagon fronts with different orientations with respect to a fixed hexagonal lattice. The saddle-node bifurcations of the localized hexagon patches are aligned with saddle-nodes of planar hexagon fronts.

By looking at planar hexagon interfaces and their interfacial energies, we present a direction of research that may allow us to analyse and predict the dynamics of large hexagon patches. This direction is based on the ideas of spatial-dynamics and variational analysis.

Finally, several open problems are presented ranging from analysis of localised 2D patches and oscillating localised patterns.

References

[1] D. Avitabile, J. Burke, E. Knobloch, D. J. B. Lloyd, and B. Sandstede, *To snake or not to snake in the planar swift-hohenberg equation.* In preparation, 2008.
[2] M. Beck, J. Knobloch, D. J. B. Lloyd, B. Sandstede, and T. Wagenknecht, *Snakes, ladders, and isolas of localised patterns.* Accepted SIAM J. Math. Anal., 2008.
[3] D. J. B. Lloyd and B. Sandstede, *Localized radial solutions of the Swift–Hohenberg equation.* Accepted Nonlinearity, 2008.
[4] D. J. B. Lloyd, B. Sandstede, D. Avitabile, and A. R. Champneys, *Localized hexagon patterns of the planar Swift–Hohenberg equation*, SIAM J. Appl. Dynam. Syst., (2008). (accepted).

Grassmannian spectral shooting and the stability of multi-dimensional travelling waves

Simon J.A. Malham
(joint work with Veerle Ledoux, Jitse Niesen and Vera Thümmler)

We will present a new numerical method for computing the pure-point spectrum associated with the linear stability of coherent structures. In the context of the Evans function shooting and matching approach, all the relevant information is carried by the flow projected onto the underlying Grassmann manifold. We show how to numerically construct this projected flow in a stable and robust manner. In particular, the method avoids representation singularities by, in practice, choosing the best coordinate patch representation for the flow as it evolves. The method is analytic in the spectral parameter and of complexity bounded by the order of the spectral problem cubed. For large systems it represents a competitive method to those recently developed that are based on continuous orthogonalization. We demonstrate this by first comparing the two methods in three finite-dimensionsal applications: Boussinesq solitary waves, autocatalytic travelling waves and Ekman boundary layer. Second we then consider the linear stability of multi-dimensional travelling fronts to nonlinear parabolic systems. Transverse to the direction of propagation we project onto a finite Fourier basis. This generates a large, linear, one-dimensional system of equations for the longitudinal Fourier coefficients. We also compare the two methods with standard projection methods that directly project the spectral problem onto a finite multi-dimensional basis satisfying the boundary conditions. As a model application, we study the stability of two-dimensional wrinkled front solutions to a cubic autocatalysis model system.

Turing-type pattern and non-Turing-type pattern in mathematical biology

Angela Stevens
(joint work with Ivano Primi, Juan J. L. Velázquez)

Cells can detect chemical and mechanical information by signal specific receptors on the cell surface. Cells signal to interact with their environment and with neighboring cells, e.g. by
- diffusive signals,
- spatially localised signals, which are bound to the extra cellular matrix,
- cell surface bound signals.

The reactions of cells to external signals often result in macroscopic structure formation on the population level. The understanding of pattern formation on this level, in wildtype populations as well as in mutant populations, thus can reveal basic underlying principles of cellular signaling, motion, and growth.

A question of major interest is as follows: *Can possible relevant mechanisms be ruled out or detected from the pattern we see?*

In Turing's famous work [1] diffusion driven instabilities for symmetry breaking and pattern formation in cellular systems were introduced. The following necessary conditions resulted from his theoretical analysis
- two or more chemicals,
- with different rates of diffusion.
- Chemical interaction of activator-inhibitor type.

For the chemicals C_1, C_2 two reaction diffusion equations were considered - for simplicity we have a look at the linear one-dimensional situation

$$\partial_t \tilde{C}_1 = D_1 \partial_{xx} \tilde{C}_1 + a_{11}\tilde{C}_1 + a_{12}\tilde{C}_2,$$
$$\partial_t \tilde{C}_2 = D_2 \partial_{xx} \tilde{C}_2 + a_{21}\tilde{C}_1 + a_{22}\tilde{C}_2.$$

Calculating the characteristic equation for the system without diffusion, the conditions for stability (*no pattern*) are

$$a_{11} + a_{22} < 0,$$
$$a_{11}a_{22} - a_{12}a_{21} > 0.$$

With diffusion the crucial condition for diffusion driven instabilities is

$$a_{11}D_1 + a_{22}D_2 > 2\sqrt{D_1 D_2 (a_{11}a_{22} - a_{12}a_{21})} > 0.$$

So it is necessary to have a short range acting activator with a smaller diffusion coefficient, and a long range inhibitor with a larger diffusion coefficient. Then a characteristic wavelength driven by diffusion can be found and a characteristic pattern is in principal possible.

In this context a natural next question is *Which pattern can result from local interactions, like e.g. from direct cell-cell contact, and not from cell growth and death?*

An example for such a kind of phenomenon are the counter-migrating traveling population waves of myxobacteria, which occur before their final aggregation under starvation conditions takes place, [2]. The cells align in a nearly one-dimensional fashion. If countermigrating cells come into direct contact, they exchange a so-called C-signal, which makes them move into the opposite direction. Looking at a simple one-dimensional model, one aims to find nonlinearities with a suitable structure to obtain rippling patterns. Let u, v be the counter-migrating species, then

$$\begin{aligned}\partial_t u + \partial_x u &= -T(u,v)u + T(v,u)v,\\ \partial_t v - \partial_x v &= T(u,v)u - T(v,u)v.\end{aligned}$$

In this model the steady states are not isolated, like in the Turing case. A one-dimensional curve of steady states exists. Linearization does not show instabilities with a defined wavelength. Without symmetry, three equations of this type are sufficient to obtain a pattern with a defined wavelength. Since in the given biological context symmetry is important, we have a look at the following system

$$\begin{aligned}\partial_t u_1 + \partial_x u_1 &= -T_1(u_1,u_2,v_1,v_2) + T_2(v_1,v_2,u_1,u_2),\\ \partial_t u_2 &= T_1(u_1,u_2,v_1,v_2) - T_2(u_1,u_2,v_1,v_2),\\ \partial_t v_1 - \partial_x v_1 &= T_2(u_1,u_2,v_1,v_2) - T_1(v_1,v_2,u_1,u_2),\\ \partial_t v_2 &= T_1(v_1,v_2,u_1,u_2) - T_2(v_1,v_2,u_1,u_2).\end{aligned}$$

Here T_1, T_2 are supposed to be positive. Examples of such systems for which a defined wavelength can be found, need suitable dependencies for T_1, T_2, e.g.,

$$\begin{aligned}T_1 &= F_1(u_1+u_2+v_1+v_2, u_1, v_1, v_2),\\ T_2 &= F_2(u_1+u_2+v_1+v_2).\end{aligned}$$

If u_2, v_2 do move, they need a different speed than u_1, v_1, and inhibiting effects are necessary to obtain a defined wavelength. This assumption is not reasonable in the context of rippling in myxobacteria, but interesting by itself. Only for six such equations all 'species' can be assumed to move with the same speed, and reasonable kinetics are possible to obtain a defined wavelength.

A good test experiment for the model is the following experiment. Wildtype cells are mixed with mutant cells, which do not produce the cell-surface bound C-signal. Upon contact of a wildtype cell with a countermigrating mutant, the wildtype cell does not change direction, whereas the mutant does. The more mutants exist in these mixed populations, the larger the observed wavelength is on the population level. Too many mutants make the rippling pattern disappear. To allow for the above mentioned phenomena, three types of wildtype cells moving in one direction are needed, as well as the respective mutants, so overall twelve equations. The species change from $u_1 \to u_2 \to u_3 \to v_1 \to v_2 \to v_3$. All are allowed to move with the same speed. Let

$$\lambda = u_1 + u_2 + u_3 + v_1 + v_2 + v_3 + \bar{u}_1 + \bar{u}_2 + \bar{u}_3 + \bar{v}_1 + \bar{v}_2 + \bar{v}_3$$

denote the total population density, where $\bar{u}_l, \bar{v}_l$, $l+1,2,3$ denote the respective mutants, which also move with the same speed. Then, for instance the following dependencies are needed in order to qualitatively observe the effects of the biological experiment.

$$T_1 = F_1(\lambda, u_1),\ T_2 = u_2 F_2(v_1 + v_2 + v_3),\ T_3 = f_3 u_3$$
$$\bar{T}_1 = F_1(\lambda, \bar{u}_1),\ \bar{T}_2 = \bar{u}_2 F_2(v_1 + v_2 + v_3),\ \bar{T}_3 = f_3 \bar{u}_3.$$

Details are given in the preprint [3]. Interestingly the nonlinearities have to be quite specific to allow for the rippling pattern, its increasing wavelength, and the loss of the pattern.

References

[1] A. Turing, *The chemical basis of morphogenesis*, Phil. Trans. Roy. Soc. B **237** (1952), 37–72.

[2] M. Dworkin, D. Kaiser (eds.), *Myxobacteria II*, American Society for Microbiology Press (1993).

[3] I. Primi, A. Stevens, J.J.L. Velázquez *Pattern forming instabilities driven by non-diffusive interaction*, Preprint (2009).

Fronts in Fermi–Pasta–Ulam chains

Jens D.M. Rademacher

(joint work with Michael Herrmann)

This abstract is a summary of the main results of [9]. We consider infinite chains of identical particles as plotted in Figure 1. These are nearest neighbour coupled in a convex potential $\Phi : \mathbb{R} \to \mathbb{R}$ by Newton's equations

$$\ddot{x}_\alpha = \Phi'(x_{\alpha+1} - x_\alpha) - \Phi'(x_\alpha - x_{\alpha-1}), \tag{1}$$

where $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$ is the time derivative, $x_\alpha(t)$ the atomic position, $\alpha \in \mathbb{Z}$ the index.

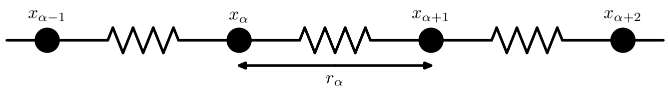

Figure 1. The atomic chain with nearest neighbour interaction.

Such chains model particles connected by springs in one dimension and serve as simplified models for crystals and solids. In their seminal paper [3] Fermi, Pasta and Ulam studied such chains assuming that the interaction potential Φ contains only cubic or quartic terms. We consider convex Φ with nonlinear force function Φ' and allow for turning points, i.e., points where $\Phi''' = 0$, but still refer to (1) as FPU chains. In fact, the existence of fronts, which is studied in this paper, requires that Φ' has at least one turning point, see [8], and this excludes, for instance, the famous Toda potential.

Fronts are travelling waves, i.e., solutions for which there exists a smooth profile that travels with constant speed and shape through the chain. The main types of travelling waves are periodic *wave trains*, homoclinic *solitons* (or *solitary waves*), and heteroclinic *fronts*. Rigorous existence proofs of such solutions are an important basic issue and during the last two decades a lot of research addressed the existence of solitons and wave trains: [6, 4, 7, 2] establish the existence of such waves by solving constrained optimization problems, [15, 13, 12, 16] apply the Mountain Path Theorem to the action integral for travelling waves, and [10] uses center manifold reduction with respect to the spatial dynamics.

In comparison, little is known rigorously for fronts. For (non-smooth and non-convex) double-well potentials composed of the same quadratic parabolas, the existence of fronts connecting oscillatory states has been recently shown in [17]. Such fronts can be interpreted as phase transitions and more physical results can be found for instance in [14, 18].

In these cases the connection between fronts and shocks in the naive continuum limit of (1) was crucial. This limit is the so-called p-system formed by the hyperbolic conservation laws for mass and momentum. Shocks come in different types given by the relation of their speed and the sound speed of the asymptotic states, though in the case of quadratic parabolas there is only one sound speed. Shocks that are faster (slower) than these sound speeds are called supersonic (subsonic). The fronts found in [17] for the quadratic double-well case correspond to subsonic shocks when taking the average of the asymptotic oscillations.

The connection to p-system shocks is also crucial for our result, but we solely consider *convex* potentials and the fronts we find are *supersonic* and *monotone* with constant asymptotic states, see Figure 2. For such potentials, the only previous result concerning fronts we are aware of is the bifurcation result by Iooss in [10] for supersonic fronts of small amplitude connecting constant states near a convex-concave turning point of Φ'.

The analytical investigations in this paper are motivated by the numerical simulations of atomistic Riemann problems which have recently been studied by the authors in [8]. We observed fronts in numerical simulations of (1) even for initial data that are far from the data of a front, see Figure 2. Hence, fronts are dynamically stable and provide fundamental building blocks for atomistic Riemann solvers.

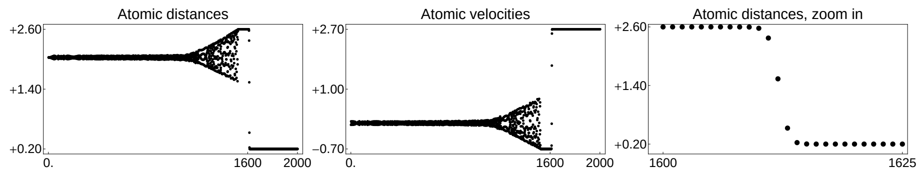

FIGURE 2. Fronts appear in the numerical simulations of FPU chains: snapshots of the atomic data with monotone front around particle $\alpha = 1610$.

Solving the p-system in the case of forces with turning points requires non-classical hyperbolic theory, see e.g., [11], and energy conservation provides the necessary "kinetic relation." The arising non-classical shocks conserve energy, while classical shocks typically dissipate energy. As expected classical shocks do not correspond to fronts in FPU, but the chain generates microscopic oscillations, see Figure 2, that form so-called dispersive shocks and lead to measure-valued macroscopic solutions, see, e.g., [8] for a review in this context.

The main result is the following theorem.

Theorem 1. *For all convex and twice continuously differentiable potentials Φ the following assertions are satisfied:*

(1) *Each front in the chain corresponds to a conservative shock in the p-system.*
(2) *For each supersonic conservative shock in the p-system there exists a corresponding monotone front in the chain.*

The first part of Theorem 1 is fairly straightforward and was proven in a different way in [1]. The second part is new and uses the convexity of Φ as well as the supersonic front speed in various fundamental steps. We next give an overview of the key ideas for the proof.

(1) We use the a priori knowledge of the front speed in order to reformulate the problem as a nonlinear fixed point equation for a suitably normalised profile.
(2) We identify an action functional for the deviation from the discontinuous shock profile such that the fixed point equation is the corresponding Euler-Lagrange equation.
(3) We use the invariance of the cone of monotone profiles under the gradient flow of the Lagrangian to connect stationary points in this cone with fronts.
(4) We establish bounds for the action functional and use the direct approach to show that the Lagrangian attains its minimum in the cone.

References

[1] W. Dreyer, M. Herrmann, and J. Rademacher, *Pulses, traveling waves and modulational theory in oscillator chains*, Analysis, Modeling and Simulation of Multiscale Problems (A. Mielke, ed.), Springer, 2006.
[2] W. Dreyer, M. Herrmann, and A. Mielke, *Micro-macro transition for the atomic chain via Whitham's modulation equation*, Nonlinearity **19** (2006), no. 2, 471–500.
[3] E. Fermi and J. Pasta and S. Ulam, *Studies on nonlinear problems*, Los Alamos Scientific Laboraty Report LA–1940 (1955).
[4] A.-M. Filip and S. Venakides, *Existence and modulation of traveling waves in particle chains*, Comm. Pure Appl. Math. **51** (1999), no. 6, 693–735.
[5] G. Friesecke and R.L. Pego, *Solitary waves on FPU lattices. I. Qualitative properties, renormalization and continuum limit*, Nonlinearity **12** (1999), no. 6, 1601–1627.
[6] G. Friesecke and J.A.D. Wattis, *Existence theorem for solitary waves on lattices*, Comm. Math. Phys. **161** (1994), no. 2, 391–418.
[7] M. Herrmann, *Ein Mikro-Makro-Übergang für die nichtlineare atomare Kette mit Temperatur*, Phd thesis, Humboldt-Universität zu Berlin, 2005.
[8] M. Herrmann and J. Rademacher, *Riemann solvers and undercompressive shocks in FPU chains*, Submitted.

[9] M. Herrmann, J.D.M. Rademacher, *Heteroclinic travelling waves in convex FPU-type chains*, Submitted, preprint in ArXiv.
[10] G. Iooss, *Travelling waves in the Fermi-Pasta-Ulam lattice*, Nonlinearity **13** (2000), 849–866.
[11] P.G. LeFloch, *Hyperbolic systems of conservation laws*, Lectures in Mathematics: ETH Zürich, Birkhäuser, Basel, 2002.
[12] A. Pankov, *Traveling waves and periodic oscillations in fermi-pasta-ulam lattices*, Imperial College Press, London, 2005.
[13] A. Pankov and K. Pflüger, *Traveling waves in lattice dynamical systems*, Math. Meth. Appl. Sci. **23** (2000), 1223–1235.
[14] L. Slepyan, A. Cherkaev, and E. Cherkaev, *Transition waves in bistable structures. II. Analytical solution: wave speed and energy dissipation*, J. Mech. Phys. Solids **53** (2005), 407–436.
[15] D. Smets and M. Willem, *Solitary waves with prescribed speed on infinite lattices*, J. Funct. Anal. **149** (1997), 266–275.
[16] H. Schwetlick and J. Zimmer, *Solitary waves for nonconvex FPU lattices*, J. Nonlinear Sci. **17** (2007), no. 1, 1–12.
[17] ———, *Existence of dynamic phase transitions in a one-dimensional lattice model with piecewise quadratic interaction potential*, submitted, 2008.
[18] L. Truskinovsky and A. Vainchtein, *Kinetics of martensitic phase transitions: lattice model*, SIAM J. Appl. Math. **66** (2005), 533–553.

Patterns in pipe flow

Dwight Barkley
(joint work with D. Moxey)

There is now good evidence to suggest that in shear flows, such as channels and pipes, there is a generic transition from a state of uniform turbulence, in which turbulence fills the system, to a state in which turbulent and laminar flow coexist - *turbulent-laminar patterns*. Such patterns have been observe in very large-aspect-ratio Couette systems (Prigent *et al.* 2002; Barkley and Tuckerman, 2005), rotor-stator flow (Cros and Le Gal, 2002), pressure driven channel flow (Tsukahara *et al.* 2005), and in connection to turbulent puffs in pipe flow (Moxey and Barkley, 2009). These fascinating states were not appreciated until recently because they develop on very long length scales. in terms of channel heights or pipe diameters. The onset of such pattern occurs as the Reynolds number is decreased from large values toward the lower limit, Re_c for which turbulence in sustained. It appears that turbulent-laminar patterns are in fact inevitable intermediate states on the route from turbulent to laminar dynamics in large aspect shear flows.

We report on numerical simulations of flow in pipes at Reynolds numbers from 2500 down to 2000 - near the minimum Reynolds numbers that supports turbulence. The computational domains are periodic in the streamwise direction with lengths up to 150 pipe diameters. We find both intermittent and equilibrium puffs. More particularly we find that, just as with other shear flows near the transition to turbulence, there are well defined transitions between uniform turbulence, intermittent states of turbulent and laminar flow, and spatially periodic states of turbulent and laminar flow.

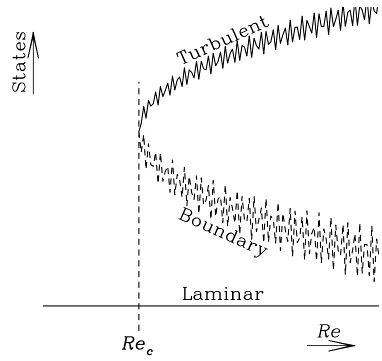

FIGURE 1. Schematic diagram for the discontinuous (subcritical) transition to turbulence typical of shear flows. States are represented as a function of non-dimensional control parameter, the Reynolds number Re. For small Re the flow is always laminar. For Re larger than the critical value, Re_c, turbulent flow is also possible. If laminar flow is perturbed beyond the boundary then the flow becomes and remains turbulent.

The interest and importance of this problem are the following: At the most basic level, understanding how laminar fluid flows become turbulent in pipes and channels is important for numerous practical engineering applications. Turbulent-laminar patterns are rather unique flow states connecting turbulent and laminar states of fluid motions. One would like to understand how the flow maintains an equilibrium in which fluid parcels continually move between turbulent and laminar motions. Why does this occur near Re_c in so many shear flows and why is the length scale of the pattern so large? Answering these questions should shed light on the transition from laminar to turbulent flow.

Even more fundamentally, turbulent-laminar patterns represent a new type of symmetry breaking and pattern formation in nonlinear dynamical systems since the patterns involve highly dynamical states and are only steady in some appropriately defined average sense. This problem thus has high potential for novel mathematics and will likely have applications far outside of fluid dynamics.

References

[1] A. Prigent, G. Gregoire, H. Chaté, O. Dauchot, and W. van Saarloos, *Large-scale finite-wavelength modulation within turbulent shear flows*, Phys. Rev. Lett. **89** (2002) 014501.

[2] D. Barkley and L.S. Tuckerman, *Computational study of turbulent laminar patterns in Couette flow*, Phys. Rev. Lett. **94** (2005) 014502.

[3] A. Cros and P. Le Gal, *Spatiotemporal intermittency in the torsional Couette flow between a rotating and a stationary disk*, Phys. Fluids **14** (2002) 3755-3765.

[4] T. Tsukahara, Y. Seki, H. Kawamura, and D. Tochio, *DNS of turbulent channel flow at very low Reynolds numbers*, In Proc. 4th Intl Symp. on Turbulence and Shear Flow Phenomena, (2005) 935-940.

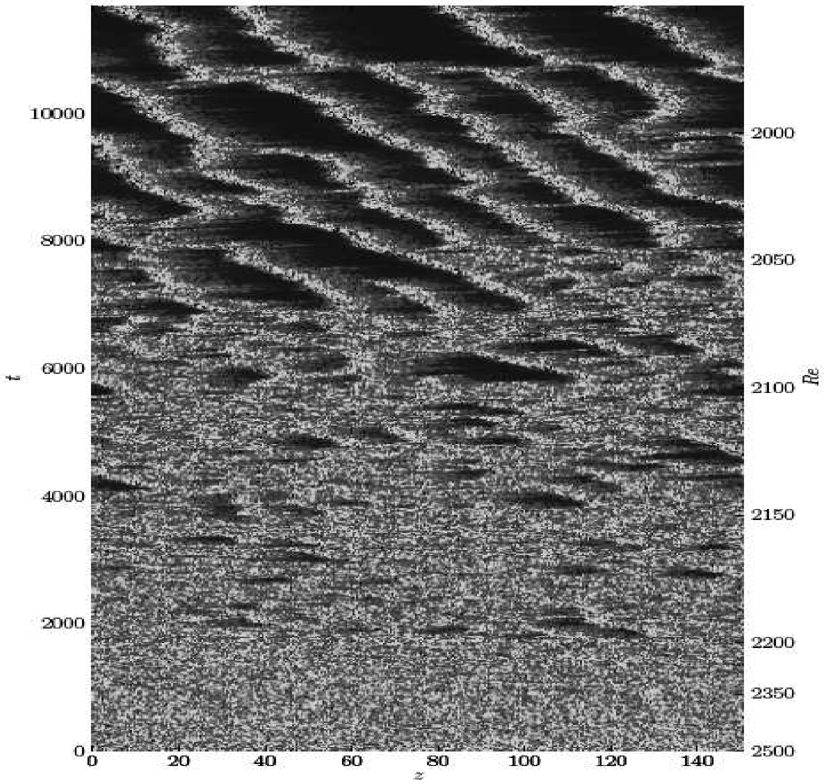

FIGURE 2. Space time diagram for turbulent flow in a pipe. Numerical simulations are shown from $Re = 2500$ down to $Re = 2000$. The state of the turbulence in the pipe is shown as transverse kinetic energy in grey scale with light indicating turbulence and dark indicating laminar flow. The horizontal axis is the axis of a pipe, which here is 150 diameters long. The vertical axis shows the time evolution as well as changes in Re. The figure shows clearly the emergence of a patterned state of turbulent and laminar flow as Re is decreased.

[5] D. Moxey and D. Barkley (to be published).

Full center manifold discretizations for near-onset convection patterns in the spherical Bènard problem

Klaus Böhmer
(joint work with G. Dangelmayr)

We use this problem for demonstrating the power of the methods in [2, 3]. Large dynamical systems are often obtained as discretizations of parabolic PDEs with nonlinear elliptic parts, either equations or system of order 2 or $2m,\ m > 1$. Space and time discretization methods, so called full discretizations, are necessary to determine the dynamics on center manifolds. We report that, allowing stable and center manifolds for the standard space discretization methods, e.g., the standard methods used in nonlinear elliptic PDEs (cf. [2, 3]), the space discrete center manifolds converge to the original center manifolds in the following sense (cf. [1, 3]). The coefficients of the Taylor expansion of a discrete center manifold and its normal form converge to those of the original center manifold. Then standard, e.g., Runge–Kutta or geometric time discretization methods can be applied to the discrete center manifold system of small dimension of ordinary differential equations.

These results are applied to near-onset convection patterns in the spherical Bènard problem in the Earth mantle. The governing dimensionless parameters are Rayleigh and Prandtl numbers R, P. Decomposing the velocity field into toroidal and poloidal scalar field $\Phi,\ \Psi$, yields with $\lambda \equiv R - R_c$

$$S\frac{\partial}{\partial t}u = G(u,\lambda) = (L + \lambda N_{11})u + N_{20}[u,u]. \tag{1}$$

Here $u = \big(\Phi, \Psi, \theta\big)$, the linear operators S and L are given by

$$S = \begin{pmatrix} \frac{1}{P}\nabla^2\mathcal{L}^2 & 0 & 0 \\ 0 & \frac{1}{P}\mathcal{L}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} \nabla^4\mathcal{L}^2 & 0 & -R_c g(r)\mathcal{L}^2 \\ 0 & \nabla^2\mathcal{L}^2 & 0 \\ \tau(r)\mathcal{L}^2 & 0 & \nabla^2 \end{pmatrix},$$

$$N_{11} = \begin{pmatrix} 0 & 0 & -g(r)\mathcal{L}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$N_{20}[u,u]$ is an extremely complicated bilinear operator (cf. [6]), and $\mathcal{L}^2 = -\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\sin\theta\dfrac{\partial}{\partial\theta} - \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}$ is the angular part of the full Laplacian ∇^2. The operators $\mathcal{L}^2$, $\nabla^2\mathcal{L}^2$, and $\nabla^4\mathcal{L}^2$ are of order 2, 4, and 6, respectively.

We aim for the second bifurcation point with

$$l_2 = 2 : \mathcal{N} := \mathcal{N}(G_u(u_0 = 0,\lambda)) = \{\varphi_m \big| -2 \le m \le 2\},\ \varphi_m \equiv f_2^0(r)Y_{2m}(\theta,\phi), \tag{2}$$

with the spherical harmonics $Y_{2m}(\theta,\phi)$, and the unknown radial null eigenvector f_2^0 of L defined by $L(f_2^0 Y_{2m}) = 0$. Then the center manifold, with $u = v + w$, is

$$\mathcal{W}^c := \{u = v + w;\ (v,w); v \in \mathcal{N}, w = W(v) \in \mathcal{M} = \mathcal{R}(L) \perp \mathcal{N}(L^*)\}, \tag{3}$$

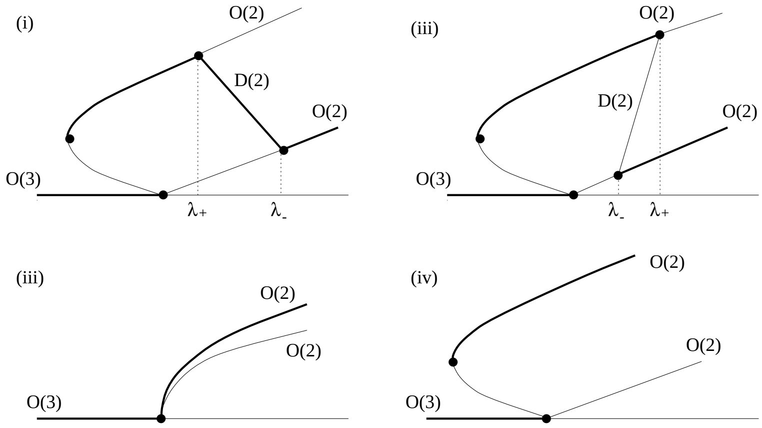

FIGURE 1. Bifurcation diagrams for *(i)* $\alpha > 0$, $b < 0$; *(ii)* $\alpha > 0$, $b > 0$; *(iii)* $\alpha = 0$; *(iv)* $\alpha < 0$. Bold lines indicate stable solution branches.

where

$$v = z\varphi = \sum_{m=-2}^{2} z_m(t)\varphi_m \in \mathcal{N}, z_m \in \mathbb{C}, z_{-m} = (-1)^i \bar{z}_m.$$

This problem is 5-determined, so we need the center manifold, instead of a Liapunov–Schmidt technique. The numerical method has to inherit the equivariance, so the spherical harmonics and the $\mathcal{L}^2$ remain unchanged, the Δ^h replaces Δ. Thus, the f_2^0 are approximated by a Chebyshev collocation spectral method,

$$L^h(f_2^{0,h} Y_{2m}) = 0.$$

Instead of the exact we obtain the approximate $v^h = z^h\varphi = \sum_{m=-2}^{2} z_m h(t)\varphi_m$ and the corresponding discrete normal form, where we determine the first terms in

$$\dot{z}^h = g^h(z^h, \lambda) = g_{111}^h \lambda z + \sum_{i\geq 2} \sum_{j=1}^{m_2^i} \sum_{k\geq 0} g_{ijk}^h \lambda^k Z_2^{ij}(z).$$

With the universal topological unfolding parameter α and the the modal parameter $b = \text{sgn}(g_{310}^h) g_{111}^h g_b / g_a^{2,h}$, we obtain the bifurcation diagrams in Figure 1 for solutions with $O(2)$ and $D(2)$ symmetries. A retransformation of the dynamical scenarios w.r.t. the parametres in the original problem is presented in [4, 5, 3].

REFERENCES

[1] K. Böhmer, *On hybrid methods for bifurcation studies for general operator equations.* In B. Fiedler, editor, *Ergodic theory, Analysiss, and Efficient Simulation of Dynamical Systems*, pp. 73–107, Berlin, Heidelberg, New York (2001). Springer.

[2] K. Böhmer, *Numerical Methods for Nonlinear Elliptic Differential Equations, a Synopsis* (2009). Preliminary Version, ca 650 p.

[3] K. Böhmer, *Numerical Methods for Bifurcation and Center Manifolds in Nonlinear Elliptic and Parabolic Differential Equations* (2010). Preliminary Version, ca 550 p.
[4] K. Böhmer, D. Janovská, V. Janovský, *Computer aided analysis of imperfect bifurcation diagrams, ii pitchfork bifurcation*, East-West Journal Numer. Math. **6**(3) (1998), 207–222.
[5] K. Böhmer, D. Janovská, V. Janovský. *On the numerical analysis of the imperfect bifurcation*, SIAM J. Numer. Anal. **40**(2) (2002), 416–430.
[6] C. Geiger, J. D. Rodriguez, G. Dangelmayr, *$o(3)-$equivariant bifurcation in the $l = 4$ representation* (1996) Preprint.

Orbital stability of spatially periodic waves to the generalized KdV equation

Todd Kapitula
(joint work with Bernard Deconinck)

In this paper we generalize previous work on the stability of waves for infinite-dimensional Hamiltonian systems to include those cases for which the skew-symmetric operator $\mathcal{J}$ is singular. We assume that $\mathcal{J}$ restricted to the orthogonal complement of its kernel has a bounded inverse. With this assumption and some further genericity conditions we show that the spectral stability of the wave implies its orbital stability, provided there are no purely imaginary eigenvalues with negative Krein signature. We use our theory to investigate the (in)stability of spatially periodic waves to the generalized KdV equation for various power nonlinearities when the perturbation has the same period as that of the wave. Different solutions of the integrable modified KdV equation are studied analytically in detail, while numerical computations come to our aid for the nonintegrable cases with a fifth- and sixth-order nonlinearity. The stability question for KdV has been answered when the period of the perturbation is the same as that of the underlying cnoidal wave. However, by using the integrable structure associated with KdV we are able to affirmatively settle the question of the orbital stability of these waves with respect to periodic perturbations whose period is an integer multiple of the wave period.

References

[1] J. Angulo, J. Quintero. *Existence and orbital stability of cnoidal waves for a 1D Boussinesq equation*, Int. J. Math. and Math. Sci. 2007:52020 (2007).
[2] J. Angulo, J. Bona, and M. Scialom. *Stability of cnoidal waves*, Adv. Diff. Eq. **11** (2006), 1321–1374.
[3] T. Benjamin, *Lectures on nonlinear wave motion*. In *Nonlinear wave motion (Proc. AMS-SIAM Summer Sem., Clarkson Coll. Tech., Potsdam, N.Y., 1972)*. 3–47. Lectures in Appl. Math. **15**. Amer. Math. Soc., Providence, R.I. (1974).
[4] T. Benjamin *The stability of solitary waves*, Proc. Roy. Soc. (London) Ser. A **328** (1972), 153–183.
[5] J. Bona, P. Souganidis, W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. R. Soc. London A **411** (1987), 395–412.
[6] N. Bottman, B. Deconinck, *KdV cnoidal waves are linearly stable*, Preprint (2008).
[7] J. Bronski, M. Johnson *The modulational instability for a generalized KdV equation*, arXiv:0809.4402v3 (2008).

[8] J. D. Carter, B. Deconinck, F. Kiyak, J. N. Kutz, *SpectrUW: a laboratory for the numerical exploration of spectra of linear operators*, Math. Comp. Simul. **74** (2007), 370–379.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. American Math. Soc. **16**(3) (2003), 705–749.

[10] B. Deconinck, J.N. Kutz. *Computing spectra of linear operators using Hill's method*, J. Comp. Physics **219** (2006), 296–321.

[11] A. Doelman, R. Gardner, T. Kaper, *Stability analysis of singular patterns in the 1-D Gray-Scott model I: a matched asymptotics approach*, Physica D **122**(1–4) (1998), 1–36.

[12] A. Doelman, R. Gardner, T. Kaper, *A stability index analysis of the 1-D Gray-Scott model*, Memoirs AMS **155**(737) (2002).

[13] F. Finkel, A. González-López, M. Rodríguez, *A new algebraization of the Lamé equation*, J. Phys. A: Math. Gen. **33** (2000), 1519–1542.

[14] T. Gallay, M. Hărăguş, *Orbital stability of periodic waves for the nonlinear Schrödinger equation*, J. Dyn. Diff. Eqns. **19** (2007), 825–865.

[15] T. Gallay, M. Hărăguş, *Stability of small periodic waves for the nonlinear Schrödinger equation*, J. Diff. Eq. **234** (2007), 544–581.

[16] R. Gardner, *Spectral analysis of long wavelength periodic waves and applications*, J. Reine Angew. Math. **491** (1997), 149–181.

[17] M. Grillakis, J. Shatah, W. Strauss, *Stability theory of solitary waves in the presence of symmetry, II*, Journal of Functional Analysis **94** (1990), 308–348.

[18] M. Hărăguş, T. Kapitula, *On the spectra of periodic waves for infinite-dimensional Hamiltonian systems*, Physica D **237**(20) (2008), 2649–2671.

[19] T. Kapitula, *On the stability of N-solitons in integrable systems*, Nonlinearity **20**(4) (2007), 879–907.

[20] T. Kapitula, P. Kevrekidis, B. Sandstede, *Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems*, Physica D **195**(3&4) (2004), 263–282.

[21] T. Kapitula, J. N. Kutz, B. Sandstede, *The Evans function for nonlocal equations* Indiana U. Math. J. **53**(4) (2004), 1095–1126.

[22] T. Kapitula, P. Kevrekidis, B. Sandstede, *Addendum: Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems*, Physica D **201**(1&2) (2005), 199–201.

[23] P. Lax, *Periodic solutions of the KdV equation*, Comm. Pure Appl. Math. **28** (1975), 141–188.

[24] J. Maddocks, R. Sachs, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math. **46** (1993), 867–901.

[25] W. Magnus, S. Winkler, *Hill's Equation*, volume 20 of *Interscience Tracts in Pure and Applied Mathematics*. Interscience Publishers, Inc. (1966).

[26] H. P. McKean, *Stability for the Korteweg-de Vries equation*, Comm. Pure Appl. Math. **30** (1977), 347–353.

[27] R. Pego, M. Weinstein, *Eigenvalues, and instabilities of solitary waves*, Phil. Trans. R. Soc. Lond. A **340** (1992), 47–94.

[28] J. Rademacher, B. Sandstede, A. Scheel, *Computing absolute and essential spectra using continuation*, Physica D **229**(1&2) (2007), 166–183.

[29] B. Sandstede, A. Scheel, *On the stability of travelling waves with large spatial period*, J. Diff. Eq. **172** (2001), 134–188.

Nonlinear stability of localized rotating patterns

Wolf-Jürgen Beyn
(joint work with Jens Lorenz)

We consider reaction-diffusion equations for a vector function $U(x,t) \in \mathbb{R}^m$

$$U_t = A\Delta U + f(U), \quad x \in \mathbb{R}^2, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is a positive definite matrix and $f : \mathbb{R}^m \mapsto \mathbb{R}^m$ is sufficiently smooth. In the talk we present a theorem on nonlinear stability with asymptotic phase in the Sobolev space $H^2 = H^2(\mathbb{R}^2, \mathbb{R}^m)$ for a rotating pattern of the form

$$U(x,t) = u_*(R_{-ct}x),\ x \in \mathbb{R}^2,\ t \in \mathbb{R}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2}$$

Here $c \neq 0$ denotes the rotational velocity and $u_* : \mathbb{R}^2 \mapsto \mathbb{R}^m$ is a smooth function. We discuss the main result and refer to [2] for details of the proof.

We assume that the pattern is localized in the following sense:
Assumption 1: For some $u_\infty \in \mathbb{R}^m$ we have $u_* - u_\infty \in H^2$ and

$$\sup_{|x|\geq R, 1\leq|\alpha|\leq 2} (|u_*(x) - u_\infty| + |D^\alpha u_*(x)|) \to 0 \quad \text{as} \quad R \to \infty.$$

Our second assumption concerns stability of the pattern in the far field, i.e., of the ODE obtained by linearizing (1) at $x = \infty$.
Assumption 2: The matrix $B_\infty = f'(u_\infty)$ is negative definite, more precisely $B_\infty + B_\infty^T \leq -4\beta I$ for some $\beta > 0$.

By a simple shift we may assume $u_\infty = 0$. It is well known that the operator $F(u) = A\Delta u + f(u), u \in H^2$, on the right-hand side of (1) is equivariant with respect to the action of the Euclidean group $SE(2)$; see [3]. That is, for the action given by $[a(\gamma)u](x) = u(R_{-\theta}(x - \eta))$, $\gamma = (\eta, \theta) \in SE(2) = \mathbb{R}^2 \ltimes S^1$, the following relation holds:

$$F(a(\gamma)u) = a(\gamma)F(u), \quad \text{for } u \in H^2, \gamma = (\eta, \theta) \in SE(2).$$

As a consequence, equation (1) has a three-dimensional solution manifold obtained by replacing u_* in (2) by any element of the group orbit $G(u_*) = \{a(\gamma)u_* : \gamma \in SE(2)\}$. We also note that a rotating wave is a special type of a relative equilibrium for general equivariant evolution equations (cf. [3],[6]).

Transforming (1) to a corotating frame via $U(x,t) = u(R_{-ct}x, t)$ leads to

$$u_t = A\Delta u + cD_\phi u + f(u) \quad \text{where } D_\phi u = -x_2 D_1 u + x_1 D_2 u, \tag{3}$$

and equivariance implies that all elements of $G(u_*)$ are steady states of (3). Differentiating with respect to the group variables shows that the linearized operator

$$Lv = A\Delta v + cD_\phi v + f'(u_*)v \tag{4}$$

has eigenvalues 0 and $\pm ic$ on the imaginary axis with corresponding eigenfunctions $D_\phi u_*$ and $D_1 u_* \pm iD_2 u_*$.
Assumption 3: The functions $D_1 u_*, D_2 u_*, D_\phi u_*$ are nontrivial elements of the

space $H^2_{Eucl} = \{v \in H^2 : D_\phi v \in L^2(\mathbb{R}^2, \mathbb{R}^m)\}$ and the corresponding eigenvalues 0 and $\pm ic$ of L from (4) are algebraically simple.

Assumption 2 implies that the operator $L : H^2_{Eucl} \mapsto L^2$ has essential spectrum strictly to the left of the imaginary axis. Assumption 3 guarantees that the three known eigenvalues on the imaginary axis are simple. Our final assumption excludes further isolated eigenvalues with nonnegative real part.

Assumption 4: The operator $L : H^2_{Eucl} \to L^2$ has no eigenvalues $s \in \mathbb{C}$ with $\Re s \geq -2\beta$ except for the eigenvalues $0, \pm ic$ from Assumption 3.

Main Theorem Let Assumptions 1-4 hold. Then there exists $\varepsilon > 0$ such that for any solution of (1) satisfying $||U(0) - u_*||_{H^2} \leq \varepsilon, U(0) \in H^2_{Eucl}$, there is a C^1-function $\gamma(t) = (\theta(t), \eta(t)) \in SE(2)$ and some $(\theta_\infty, \eta_\infty) \in SE(2)$ so that we have for all $t \geq 0$:

$$||U(\cdot,t) - a(\gamma(t))u_*||_{H^2} + |\eta(t) - \eta_\infty| + |\theta(t) - (ct + \theta_\infty)| \leq Ce^{-\beta t}||U(0) - u_*||_{H^2}.$$

Remarks:

1. Our Main Theorem states nonlinear stability of the pattern in H^2 with asymptotic phase. Moreover, we have exponential convergence towards the pattern u_* and to some asymptotic phase depending on the initial values. Structurally, our approach follows Henry's method [5, Ch.5] for proving stability with asymptotic phase of traveling waves. We decompose the solutions of (3) as

$$u(\cdot,t) = a(\gamma(t))u_* + w(\cdot,t), \quad \gamma(t) \in SE(2),\ w(\cdot,t) \in W,$$

where $W \subset H^2_{Eucl}$ is an invariant subspace of L that is complementary to $\mathrm{span}\{D_1u_*, D_2u_*, D_\phi u_*\}$. Equation (3) may then be written as a system

$$\dot\gamma - E_c\gamma = r^{[\gamma]}(\gamma, w(\cdot,t)), \quad \dot w - Lw = r^{[w]}(\gamma, w(\cdot,t)), \quad E_c = \begin{pmatrix} -cR_{\frac{\pi}{2}} & 0 \\ 0 & 0 \end{pmatrix}$$

with suitable estimates for the remainders $r^{[w]}, r^{[\gamma]}$. In contrast to the situation considered in [5], the operator L generates only a C^0-semigroup e^{tL} on H^2, but not an analytic semigroup. Consequently, exponential decay estimates do not follow from the integral representation of e^{tL} and resolvent estimates.

2. Bates and Jones [1] set up an invariant manifold theory for C^0-semigroups that allows to conclude exponential decay towards traveling waves for certain mixed hyperbolic-parabolic systems in one space dimension. More generally, an abstract principle of reducing the dynamics near a relative equilibrium to a center manifold is derived in [6]. The authors also prove exponential attraction of the center manifold, which applies to the rotating waves considered here. However, stability with asymptotic phase is not discussed in [6].

3. The exponential estimate for e^{tL} is obtained via an abstract result on C^0-semigroups [2, Appendix] that states the following. Suppose the operator $A : D(A) \subset X \mapsto X$ generates a C^0 semigroup with bound $e^{t\omega}$ and $B : X \mapsto X$ is bounded such that $Be^{tA}, t > 0$, is compact and all eigenvalues of $A + B$ have real part $\leq \omega$. Then $A + B$ generates a semigroup with bound $e^{t\omega}$. Essentially, this

theorem is applied to $(Bu)(x) = (f'(u_*(x)) - f'(u_\infty))u(x)$ and $A = L - B$.
4. In [2, Ch.8] our Main Theorem is applied to spinning solitons found in the quintic-cubic Ginzburg Landau equation; see [4]. In this case, the essential spectrum forms a zig-zag structure to the left of the imaginary axis and Assumption 2 can be verified explicitly. The further assumptions are tested numerically. In particular, we find that in addition to the three eigenvalues on the imaginary axis there are eight pairs of isolated simple eigenvalues that lie between the zig-zag structure and the imaginary axis.

References

[1] P.W. Bates, C.K.R.T. Jones, *Invariant manifolds for semilinear partial differential equations*, pp. 1-38 in Dynamics Reported Vol. 2, (1989), Teubner.
[2] W.-J. Beyn, J. Lorenz, *Nonlinear stability of rotating patterns*, To appear in Dynamics of PDEs (2009).
[3] P. Chossat, R. Lauterbach, *Methods in equivariant bifurcations and dynamical systems*, Advanced Series in Nontlinear Dynamics Vol. 15, World Scientific Publishing (2000).
[4] L.-C. Crasovan, B.A. Malomed, D. Mihalache, *Stable vortex solitons in the two-dimensional Ginzburg-Landau equation*, Phys. Rev. E **63** (2001), 016605.
[5] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer, 1981.
[6] B. Sandstede, A. Scheel, C. Wulff, *Dynamics of spiral waves on unbounded domains using center-manifold reductions.* J. Differential Equations **141** (1997), 122–149.

Bifurcating tori in spatially extended systems

Guido Schneider
(joint work with Andreas Kirchhoff)

We are interested in the existence of bifurcating invariant tori in spatially extended dissipative systems with marginally stable background state. Examples of systems we are interested in are reaction-diffusion systems in $\mathbb{R}^d$ with spatially localized amplification terms or the flow around some obstacle problem. Such tori can bifurcate in case that simultaneously pairs of complex conjugate eigenvalues cross the imaginary axis and possibly contain quasiperiodic solutions. These tori are the second bifurcation in the Ruelle-Takens scenario [2] of the onset of turbulence. There is a serious difficulty in the construction of such tori according to the fact that the linearization around the trivial solution possesses continuous spectrum up to the imaginary axis for all values of the bifurcation parameter.

As a toy problem we consider

$$\partial_t U(x,t) = \Delta U(x,t) + \sum_{j=1}^{4} |v_j(t)|^4 v_j(t) e^{-x^2} - U(x,t)^3, \tag{1}$$

$$\partial_t v_m(t) = (\alpha + i\omega_m) v_m(t) - |v_m(t)|^2 v_m(t) + v_m(t) \int U(x,t) e^{-x^2} dx \tag{2}$$

with $\omega_m = -\omega_{-m}$ for $m \in \{-2,-1,1,2\}$ where $U : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, $v_j : \mathbb{R}^+ \to \mathbb{R}$ and ω_1, ω_2 rationally independent which can be seen as some kind of normal form of the mentioned systems. For all values of the bifurcation parameter α the system possesses the trivial solution $(U,V)=(0,0)$. The linearization around the trivial solution decouples and possesses essential spectrum $\{-|\xi|^2 : \xi=(\xi_1,\ldots,\xi_d)\in\mathbb{R}^d\}$ up to the imaginary axis for all values of the bifurcation parameter α. Moreover, for $\alpha=0$ simultaneously two pairs of complex conjugate eigenvalues $\alpha\pm i\omega_1$ and $\alpha\pm i\omega_2$ cross the imaginary axis from left to right. Our major goal is an existence result for an invariant torus associated to the modes $v_1=r_1e^{i\phi_1}=\overline{v_{-1}}$ and $v_2=r_2e^{i\phi_2}=\overline{v_{-2}}$. It turns out that $v_j=\mathcal{O}(\varepsilon)$ and $U=\mathcal{O}(\varepsilon^5)$ for $\varepsilon^2=\alpha\to 0$. Hence in polar coordinates the equations for the v_js are given by

$$\partial_t r_1=\alpha r_1-r_1^3+h.o.t.,\qquad \partial_t\phi_1=\omega_1,$$
$$\partial_t r_2=\alpha r_2-r_2^3+h.o.t.,\qquad \partial_t\phi_2=\omega_2.$$

Ignoring the h.o.t., we see that this system possesses an invariant torus if the first two equations possess a nontrivial fixed point with $r_1\neq 0$ and $r_2\neq 0$, here $(r_1^*,r_2^*)=\sqrt{\alpha}(1,1)$. With some hard implicit function theorem the persistence of the torus can be established in the full system, too. Hence, our result is as follows.

Theorem. *Let $d\geq 5$. Then there exists an $\alpha_0>0$ such that for all $\alpha\in(0,\alpha_0)$ there exists a two-dimensional torus $M=\{(r,U)=(r,U)(\phi)\mid\phi\in\mathbb{T}^2\}$ of size $\mathcal{O}(\sqrt{\alpha})$ which is invariant under the flow of the toy problem.*

The question is motivated by the flow around some obstacle problem leading to the same principal difficulties to be overcome. The difficulty of the proof comes from the essential spectrum up to the imaginary axis. In case that there is a spectral gap an application of the center manifold theorem easily would give the existence of an invariant torus for every $\alpha>0$ sufficiently small. However there is no spectral gap and so the center manifold theorem cannot be applied. For the construction of the invariant torus, we use its invariance which leads to some condition in differential form, namely $\partial_t r=\frac{\partial r}{\partial\phi}\partial_t\phi$, $\partial_t U=\frac{\partial U}{\partial\phi}\partial_t\phi$. Inserting the above equations for $\partial_t r$ and $\partial_t U$ yields for $w=r-r^*$ and U the PDEs

$$-2\varepsilon^2 w+h.o.t.=\frac{\partial w}{\partial\phi}(\Omega+h.o.t.),\qquad \Delta U+h.o.t.=\frac{\partial U}{\partial\phi}(\Omega+h.o.t)$$

where $\Omega=(\omega_1,\omega_2)$. Since h.o.t. also contains inhomogeneous terms, for $h.o.t.\neq 0$ the point $(w,U)=(0,0)$ is no longer a solution. In order to apply the implicit function theorem two serious difficulties have to be overcome. The first difficulty already occurs for the first equation. The spectrum of the linear operator $w\mapsto -2\varepsilon^2 w-\frac{\partial w}{\partial\phi}\Omega$ is given by $\{-2\varepsilon^2-in_1\omega_1-in_2\omega_2\mid n_1,n_2\in\mathbb{Z}\}$ and therefore possesses a bounded inverse of order $\mathcal{O}(\varepsilon^{-2})$ from $H^s(\mathbb{T}^2)$ to $H^s(\mathbb{T}^2)$. However, this operator is not smoothing, but the nonlinear terms $\frac{\partial w}{\partial\phi}(h.o.t)$ loose one derivative w.r.t. ϕ. Therefore, the usual implicit function theorem has to be replaced by the hard implicit function theorem or Nash-Moser theorem, cf. [3]. The iteration scheme to solve an equation $F(r) = 0$ is a smoothed Newton method

$$r_{n+1} = r_n - \left(\frac{\partial F}{\partial r}(r_n)\right)^{-1} S_n F(r_n),$$

where S_n is some smoothing operator with $S_n \to I$ for $n \to \infty$. The limit function $r^\infty \in H^s(\mathbb{T}^2, \mathbb{R}^2)$ satisfies $r^\infty = r^\infty - \left(\frac{\partial F}{\partial r}(r^\infty)\right)^{-1} IF(r^\infty)$, i.e. r^∞ solves $F(r^\infty) = 0$. The idea of the hard implicit function theorem is more or less as follows. First we have to show that $(\frac{\partial F}{\partial r}(r_n))^{-1}$ is a bounded operator from H^s to H^s with a bound independent of n. In order to avoid the loss of regularity of the nonlinear terms the smoothing operator S_n is added. Due to the quadratic convergence of Newton's method S_n can be chosen closer and closer to the identity such that for $n \to \infty$ the limit function r^∞ solves $F(r^\infty) = 0$. The major difficulty in applying the hard implicit function theorem is the proof of so called tame estimates

$$\|(F'(r))^{-1}\varphi\|_{H^s} \leq C_s\|\varphi\|_{H^{s+q}} + \|u\|_{H^{s+q}}\|\varphi\|_{H^{2q}} \qquad \text{for all } s \geq q$$

for the inverse for all r in an H^s-neighborhood of r^*. However, for our problem these estimates easily follow with energy estimates.

The second difficulty is the inversion of the linear operator $U \mapsto \Delta U - \frac{\partial U}{\partial \phi}\Omega$ in the second equation. The spectrum of this operator is given by $\{-k^2 - in_1\omega_1 - in_2\omega_2 \mid n_1, n_2 \in \mathbb{Z} \;\; k \in \mathbb{R}\}$. Hence we have essential spectrum coming arbitrary close to the origin. Nevertheless, this operator still can be controlled due to the fact that Δ is invertible from $L^1 \cap H^s$ into H^s for $d \geq 5$ and due to the Cauchy-Schwarz inequality which shows that nonlinear terms are bounded from H^s into $L^1 \cap H^s$. This is a consequence of

$$\begin{aligned}
\|\Delta^{-1}u\|_{L^2}^2 &= \||k|^{-2}\hat{u}\|_{L^2}^2 \leq \||k|^{-2}\chi_{|k|\leq 1}\hat{u}\|_{L^2}^2 + \||k|^{-2}\chi_{|k|>1}\hat{u}\|_{L^2}^2 \\
&\leq \||k|^{-2}\chi_{|k|\leq 1}\|_{L^2}^2\|\hat{u}\|_{L^\infty}^2 + \||k|^{-2}\chi_{|k|>1}\hat{u}\|_{L^2}^2 \\
&\leq C\|\hat{u}\|_{L^\infty}^2 + \|\hat{u}\|_{L^2}^2 \leq C\|u\|_{L^1}^2 + \|u\|_{L^2}^2,
\end{aligned}$$

where we used

$$\||k|^{-2}\chi_{|k|\leq 1}\|_{L^2}^2 = \int_{|k|\leq 1} |k|^{-4} dk \leq C \int_0^1 r^{d-5} dr < \infty,$$

which is true for $d \geq 5$.

The existence proof of quasiperiodic solutions would lead to some small divisor problem similar to KAM theory. For dissipative systems an overview about existence proofs of quasiperiodic solutions can be found for instance in [1]. By proving only the existence of an invariant torus the small divisor problem is avoided. The hard implicit function theorem which is usually used to solve this problems is used here for the above mentioned reasons.

Numerical experiments in order to illustrate our result in the general situation are hard to obtain due to $d \geq 5$. However, by restricting to the rotational symmetric situation the problem can be reduced to a problem on the real line, namely

$$(3)\qquad \partial_t U(x,t) = \partial_x^2 U(x,t) + \frac{d-1}{x}\partial_x U(x,t) + \sum_{j=1}^{4} |v_j(t)|^4 v_j(t) e^{-x^2} - U(x,t)^3$$

with Neumann boundary conditions at $x = 0$.

References

[1] H.W. Broer, G.B. Huitema, M.B. Sevryuk, *Quasi-periodic motions in families of dynamical systems. Order amidst chaos.* Lecture Notes in Mathematics. 1645. Berlin: Springer. 1997.

[2] D. Ruelle, F. Takens, *On the nature of turbulence.* Comm. Math. Phys. **20** (1971), 167–192.

[3] X. Saint Raymond, *A simple Nash-Moser implicit function theorem.* Enseign. Math., II. Ser. **35**, No. 3/4, (1989), 217–226.

A Hamiltonian analogue of the meandering transition

Claudia Wulff

The meandering transition in spiral wave dynamics is a transition from rigidly rotating to meandering and drifting spiral waves. In symmetry terms, it is a bifurcation from rotating waves to modulated rotating and modulated traveling waves in systems with SE(2)-symmetry. Here $SE(2) = SO(2) \ltimes \mathbb{R}^2$ is the special Euclidean group of motions of the plane. Rotating waves are solutions which become stationary in a corotating frame and are examples of relative equilibria. Modulated rotating and modulated traveling waves are solutions which become periodic in a corotating/comoving frame and are examples of relative periodic orbits (RPOs). In non-Hamiltonian systems, the meandering bifurcation corresponds, in a rotating frame, to a Hopf bifurcation induced by changing an external parameter. Typically the bifurcating relative periodic orbits are modulated rotating waves, and modulated traveling waves only occur at certain resonances. See for example [1, 2, 3] and the references therein.

The transition from rotating waves to modulated traveling waves occuring in the meandering transition is an example of resonance drift, as analyzed in [7]. Resonance drift occurs if there is a discontinuity of the average drift velocities of the bifurcating relative periodic orbits at the relative equilibrium. In the case of the meandering transition it is a discontinuous jump between a rotational and a translational velocity.

In this talk the first ever analysis of the Hamiltonian analogue of this meandering transition is presented (for more details see [8]). Examples of Hamiltonian systems where such a transition occurs are rotating point vortices on the plane [5] or rotating rigid bodies in ideal fluids [4]. In a Hamiltonian system it is natural to study the persistence and bifurcation of the rotating wave to nearby momentum levels since the momentum map is a conserved quantity and hence an internal parameter of the system. In the case of SE(2) symmetry, the components of the momentum map are angular and linear momentum.

The differential equations near Hamiltonian relative equilibria in symmetry-adapted local coordinates from [6] are used to study the transition from rotating waves to modulated rotating and modulated traveling waves on nearby momentum levels in Hamiltonian systems with SE(2)-symmetry. Thereby a Hamiltonian analogue of the meandering transition of spiral waves is obtained.

It is shown that, depending on the symmetry properties of the momentum map, either modulated traveling waves are typical near rotating waves, as momentum is varied, or that modulated traveling waves do not occur. The first scenario occurs in the case of momentum maps which are equivariant with respect to the coadjoint group action on the dual Lie algebra $\mathrm{se}(2)^*$ of the symmetry group SE(2). The second case occurs if the group action on $\mathrm{se}(2)^*$ has a non-trivial cocycle. Moreover, rotating waves and transitions to relative periodic orbits are continued in the cocycle parameter which determines the symmetry properties of the momentum map. These results hold under conditions which are generically satisfied.

The meandering transition is a transition from relative equilibria to relative periodic orbits. In non-Hamiltonian systems it is a Hopf bifurcation of the symmetry reduced dynamics. The Hamiltonian analogue of a Hopf bifurcation is a Lyapounov centre bifurcation. Lyapounov centre bifurcations for the reduced Hamiltonian system on the symplectic slice readily yield families of RPOs nearby elliptic relative equilibria. In the case of SE(2) symmetry these are families of MRWs with zero linear momentum. Lyapounov centre type theorems are also proved for the full symmetry reduced system which is a Poisson system and not a Hamiltonian system. It is shown that on the bifurcating family of RPOs which correspond to periodic orbits of the reduced dynamics outside the symplectic leaf of the original equilibrium resonance drift occurs. For systems with SE(2) symmetry the bifurcating RPOs are MTWs with non-vanishing linear momentum.

In [8] the Hamiltonian analogue of the meandering transition is also discussed for systems with spherical symmetry and for systems with the Euclidean symmetry group of three-dimensional space.

It remains a challenging open problem to extend these results to infinite dimensional Hamiltonian systems such as PDE models of vortex dynamics.

References

[1] D. Barkley, *Euclidean symmetry and the dynamics of rotating spiral waves*, Phys. Rev. Lett. **72** (1994), 164–167.

[2] B. Fiedler, B. Sandstede, A. Scheel, C. Wulff, *Bifurcation from relative equilibria to noncompact group actions: Skew products, meanders, and drifts*, Doc. Math. J. DMV **1** (1996), 479–505.

[3] M. Golubitsky, V. LeBlanc, I. Melbourne, *Meandering of the spiral tip – an alternative approach*, J. Nonl. Sc. **7** (1997), 557–586.

[4] N.E. Leonard, J.E. Marsden, *Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry*, Physica D **105** (1997), 130–162.

[5] P. Newton, *The N-Vortex Problem*, Volume 145 of Appl. Math. Sc., Springer-Verlag, 2001.

[6] R.M. Roberts, C. Wulff, J. Lamb, *Hamiltonian systems near relative equilibria*, J. Differential Equations **179** (2002), 562–604.

[7] C. Wulff, *Transition from relative equilibria to relative periodic orbits*, Doc. Math. J. DMV **5** (2000), 227-274.
[8] C. Wulff. *A Hamiltonian analogue of the meandering transition.* SIAM J. Appl. Dyn. Systems **7**(4) (2008), 1213–1246.

How robust are Liesegang patterns?

Arnd Scheel

We review Liesegang patterns as an elementary, yet puzzling pattern forming mechanism. Liesegang patterns exhibit precipiatation spikes at locations that obey a characteristic spatial scaling law. We argue that such patterns are untypical in reaction-diffusion systems. We then propose a restricted class of reaction-diffusion systems, based on the irreversibility of certain chemical reactions, in which Liesegang patterns are robust, that is, they occur for an open subset of kinetic, diffusion constants, and boundary conditions. The proof is constructive. We superimpose elementary building blocks such as spikes and boundary layers and control errors using a spatial dynamics homoclinic bifurcation analysis.

References

[1] A. Scheel, *Robustness of Liesegang patterns*, Nonlinearity, to appear.

Effective dynamics in nonlinear lattices

Jannis Giannoulis

Let us consider a macroscopic body of material, the internal microscopic structure of which we know exactly. In the case of an infinite monoatomic crystal in one dimension, the latter could be given by the rest positions $j \in \mathbb{Z}$ of the atoms and the interaction potentials V between neighbouring atoms as well as some background potential W acting on each atom independently of its neighbours. Then the displacement $x_j(t)$ at time $t \in \mathbb{R}$ of the j-th atom is given by Newton's equations of motion

$$(1) \qquad \ddot{x}_j(t) = V'\big(x_{j+1}(t) - x_j(t)\big) - V'\big(x_j(t) - x_{j-1}(t)\big) - W'(x_j(t)), \quad j \in \mathbb{Z}.$$

Given this complete microscopic description, we are able to determine exactly the dynamics within the object at hand, as for instance the propagation of a wave caused by an initial excitation $(x_j(0), \dot{x}_j(0))$ of the atoms. Doing so,we observe that for initial data of a macroscopically traceable shape — i.e. a shape varying with respect to a macroscopic space variable $y = \varepsilon j$, $0 < \varepsilon \ll 1$ — the ensuing wave, calculated by (1), displays at later times $t > 0$ also some macroscopic shape. If we are interested in the evolution of only this shape, the question arises whether a corresponding evolution equation can be derived from (1), which relates to the macroscopic initial data directly the macroscopic form at time $t > 0$, without the 'uninteresting' (from this point of view) microscopic information delivered automatically by the solutions of (1). Of course, we require from such an *effective* evolution equation that its solutions are comparable to the macroscopic shapes obtained from the solutions of (1), when starting from the same macroscopic initial data. More precisely, making the multiscale ansatz

$$x = X_\varepsilon^A + \mathcal{O}(\varepsilon^{a+1}), \quad (X_\varepsilon^A)_j(t) = \varepsilon^a A(\varepsilon^b t, \varepsilon(j-ct))\mathrm{e}^{\mathrm{i}(\omega(\vartheta)t+\vartheta j)} + \text{c.c.} \tag{2}$$

we are interested in the dynamics of the amplitude $A : [0,\infty)\times\mathbb{R}\to\mathbb{C}$. Here, $\mathbf{E}(j,t) = \mathrm{e}^{\mathrm{i}(\omega(\vartheta)t+\vartheta j)}$, as well as its complex conjugate (c.c.), is a plane wave solution to the linearization of (1), which means that its frequency $\omega(\vartheta)\in\mathbb{R}$ and wavenumber $\vartheta\in\mathbb{T}=\mathbb{R}/_{2\pi\mathbb{Z}}$ satisfy the dispersion relation $\omega^2(\vartheta) = 2v(1-\cos\vartheta)+w$, $v = V''(0)$, $w = W''(0)$, where, in order to guarantee stability of solutions, we assume $w, 4v+w > 0$. Hence, considering ϑ as fixed, $A\mathbf{E}$ + c.c. is a macroscopically (amplitude-)modulated pulse. Based on the macroscopic space scale $y = \varepsilon(j-ct)$ (thereby allowing for moving space-coordinate frames with velocity $c\in\mathbb{R}\setminus\{0\}$), we can choose the corresponding macroscopic time scale $\tau = \varepsilon^b t$ as well as the relative size of the amplitude by choosing a, b among certain values. This choice depends on the macroscopic phenomena we want to capture.

Having determined the form (2) of solutions we are looking for, and recalling that we want them to satisfy (1) as exact as possible, i.e. up to some residual terms of order $\mathcal{O}(\varepsilon^k)$, $k\in\mathbb{N}$, we obtain corresponding *necessary conditions* to be satisfied by A, by inserting (2) into (1), expanding the left and right hand side of the latter in terms of ε^k and $\mathbf{E}^n$ (recall here that (1) is nonlinear), and equating the respective coefficients of the two sides. In particular, for $a = 1,\ b = 2$, we obtain $c = -\omega'(\vartheta)$ from the equation for $\varepsilon^2\mathbf{E}$, and the *nonlinear Schrödinger equation*

$$\mathrm{i}\partial_\tau A = \frac{1}{2}\omega''(\vartheta)\partial_y^2 A + \rho|A|^2 A, \tag{3}$$

from the equation for $\varepsilon^3\mathbf{E}$, ρ depending on ϑ, V, W (cf. (2.12) in [1]). Equation (3) describes the dispersive deformation of the amplitude of a single pulse, which is observed by travelling with its group velocity c. The extremely slow time scale $\tau = \varepsilon^2 t$ (dispersive scaling) is needed in order to capture the deformation of A, since the dynamics of εA are close to linear. The justification of (3) is established by the following theorem, see [2].

Theorem 1. *Let $V, W \in \mathrm{C}^5(\mathbb{R})$ with $w, (16/3)v+w > 0$, $A : [0,\tau_0]\times\mathbb{R}\to\mathbb{C}$, $\tau_0 > 0$, the solution of* (3) *with $A(0,\cdot)\in\mathrm{H}^6(\mathbb{R})$, and X_ε^A the approximation* (2) *with $c = -\omega'(\vartheta)$. Then, for any $c > 0$ there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and any solution x of* (1)

$$\begin{aligned} &\|(x(0),\dot{x}(0)) - (X_\varepsilon^A(0),\dot{X}_\varepsilon^A(0))\|_{\ell^2\times\ell^2} \le c\varepsilon^{3/2} \\ &\implies \|(x(t),\dot{x}(t)) - (X_\varepsilon^A(t),\dot{X}_\varepsilon^A(t))\|_{\ell^2\times\ell^2} \le C\varepsilon^{3/2} \quad \textit{for all } t\in[0,\tau_0/\varepsilon^2]. \end{aligned} \tag{4}$$

The proof relies on a Gronwall-type argument, and is straight-forward for potentials V, W with absent cubic terms (cf. [1]), since the time is scaled by $\tau = \varepsilon^2 t$. In the presence of quadratic nonlinear terms in (1), one has first to make use of a normal form transformation, see [2].

As a second example for the above modulational approach let us briefly discuss the interaction of modulated pulses. In the simplest setting we make the ansatz

$$x_j(t) = \varepsilon \sum_{n=1}^{3} A(\varepsilon t, \varepsilon j)\mathbf{E}_n(j,t) + \mathcal{O}(\varepsilon^2) + \text{c.c.} \tag{5}$$

with three different plane-waves $\mathbf{E}_n(j,t) = \mathrm{e}^{\mathrm{i}(\omega(\vartheta_n)t+\vartheta_n j)}$, $\omega(\vartheta_n) > 0$, $n = 1, 2, 3$, which satisfy the *resonance condition*

$$\vartheta_1 + \vartheta_2 = \vartheta_3 \quad \text{in } \mathbb{T}, \quad \omega_1 + \omega_2 = \omega_3. \tag{6}$$

This means that the three pulses $A_n\mathbf{E}_n$ interact with each other. (One can imagine this as two pulses which collide and create a third.) We formally derive the evolution equations for the amplitudes A_n by inserting (5) into (1) and following the same steps as above. To this end we need the nonresonance conditions

$$(k\omega(\vartheta_1)+\ell\omega(\vartheta_2))^2 \neq \omega^2(k\vartheta_1+\ell\vartheta_2), \quad (k,\ell) = (2,0),(0,2),(2,2),(2,1),(1,2),(1,-1). \tag{7}$$

They guarantee that no pulses except those considered are generated by interaction. This yields the *three-wave-interaction equations*

$$\begin{cases} \partial_\tau A_1 - \omega'(\vartheta_1)\partial_y A_1 = \dfrac{\mathrm{i}\overline{c}}{\omega(\vartheta_1)} A_3\overline{A_2}, \\ \partial_\tau A_2 - \omega'(\vartheta_2)\partial_y A_2 = \dfrac{\mathrm{i}\overline{c}}{\omega(\vartheta_2)} A_3\overline{A_1}, \\ \partial_\tau A_3 - \omega'(\vartheta_3)\partial_y A_3 = \dfrac{\mathrm{i}c}{\omega(\vartheta_3)} A_1 A_2 \end{cases} \tag{8}$$

with $c = 4\mathrm{i}V'''(0)\sin\left(\frac{\vartheta_1}{2}\right)\sin\left(\frac{\vartheta_2}{2}\right)\sin\left(\frac{\vartheta_3}{2}\right) + \frac{1}{2}W'''(0)$. Equations (8) can be justified by the same method as above.

Theorem 2. *For $V, W \in \mathrm{C}^3(\mathbb{R})$ and the solution $A = (A_1, A_2, A_3) : [0, \tau_0]\times\mathbb{R} \to \mathbb{C}^3$, $\tau_0 > 0$, to* (8) *with $A(0,\cdot) \in (\mathrm{H}^3(\mathbb{R}))^3$, the approximation $X_\varepsilon^A = \varepsilon\sum_{n=1}^3 A_n\mathbf{E}_n$ for three pulses with* (6) *and* (7) *satisfies* (4) *for all $t \in [0, \tau_0/\varepsilon]$.*

Note, that the hyperbolic scaling $\tau = \varepsilon t$, $y = \varepsilon j$ used here 'fits' exactly to quadratic nonlinearities. For the full details of the proof as well a complete discussion of all possible resonance and nonresonance conditions and the corresponding macroscopic equations up to an arbitrary order of approximation in the case of arbitrary many pulses in multidimensional lattices with scalar displacement x, we refer the reader to [3].

We conclude by mentioning a different approach to effective dynamics. The macroscopic equations (3) and (8), corresponding to (2) and (5), were derived by inserting the latter into the microscopic system (1), which posseses Lagrangian and Hamiltonian structure (LHS). However, as in these examples, often one observes *a posteriori* that also the derived equations possess a (macroscopic) LHS. Thus, the question arises, whether the latter can be derived *directly* from the microscopic LHS of (1). Here, the interesting features of the problem are the discreteness of (1) as well as the inherent microscopic patterns of (2) and (5). Embedding (1) in a corresponding continuous system, and taking into account the microscopic patterns by introducing a corresponding number of phase variables $\phi \in \mathbb{T}^n$ (e.g. $n = 1$ for (2) and $n = 2$ for (5)), reveals the existence of additional integrals of motion inherent to the miroscopic (continuous) system which are hidden from (1). Only the consideration of these new integrals of motion allows for a correct, exact two-scale transformation of the microscopic LHS. Then, considering the Hamiltonian structure on the tangent bundle, one can expand the transformed LHS consistently with respect to ε, which gives immediately the relevant *reduced* macroscopic structures. The consistency of these expansions can be obtained only on the tangent (and *not* on the more familiar co-tangent) bundle, due to different scaling behaviour of velocities and momenta. For the full exposition of this reduction procedure and further examples, see [4].

Finally, for an overview of results and literature concerning the derivation of macroscopic continuum limits from nonlinear lattices we refer the reader to [5].

References

[1] J. Giannoulis, A. Mielke, *The nonlinear Schrödinger equation as a macroscopic limit for an oscillator chain with cubic nonlinearities*, Nonlinearity **17** (2), (2004) 551–565.

[2] J. Giannoulis, A. Mielke, *Dispersive evolution of pulses in oscillator chains with general interaction potentials*, Discrete Contin. Dyn. Syst. Ser. B **6** (3), (2006) 493–523.

[3] J. Giannoulis, *Interaction of modulated pulses in nonlinear lattices*, in preparation.

[4] J. Giannoulis, M. Herrmann, A. Mielke, *Lagrangian and Hamiltonian two-scale reduction*, J. Math. Phys. **49**, 103505, (2008), 42 pages.

[5] J. Giannoulis, M. Herrmann, A. Mielke, *Continuum descriptions for the dynamics in discrete lattices: derivation and justification,* in: A. Mielke (ed.), Analysis, Modeling and Simulation of Multiscale Problems, Springer, 2006, 435 – 466.

Diffusive stability of oscillations in reaction-diffusion systems

Thierry Gallay
(joint work with Arnd Scheel)

Synchronization of spatially distributed oscillators is a very common phenomenon which has been observed in a wide variety of physical systems. The aim of this work is to study the stability of synchronized oscillations in spatially extended systems under very general assumptions, without detailed knowledge of internal oscillator dynamics or coupling mechanisms.

To be specific, we consider the reaction-diffusion system

$$(1) \qquad u_t \;=\; D\Delta u + f(u), \qquad u = u(t,x) \in \mathbb{R}^N\,, \quad x \in \mathbb{R}^n\,, \quad t \ge 0\,,$$

with positive coupling matrix $D \in \mathcal{M}_{N\times N}(\mathbb{R})$, $D = D^T > 0$, and smooth kinetics $f \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$. We suppose that the ODE $\dot u = f(u)$ has a periodic solution $u_*(t)$ with minimal period $T > 0$ (of course, this is possible only if $N \ge 2$.) Assuming that this solution is asymptotically stable for the ODE dynamics, our goal is to investigate the stability of the spatially homogeneous, time-periodic solution $u(t,x) = u_*(t)$ of (1).

Linearizing (1) at u_* we obtain the time-periodic equation

$$u_t \;=\; D\Delta u + f'(u_*(t))u\,,$$

which (after Fourier transformation in space) is equivalent to the family of ODE's

$$\hat{u}_t \;=\; -k^2 D\hat{u} + f'(u_*(t))\hat{u}\,, \quad k \in \mathbb{R}^n\,. \tag{2}$$

For each fixed k we denote by $F_k(t,s)$ the two-parameter evolution operator associated to the linear time-periodic system (2), so that $\hat{u}(t) = F_k(t,s)\hat{u}(s)$ for any $t \geq s$. The k-dependent Floquet exponents $\lambda_1(k),\ldots,\lambda_N(k)$ are then classically defined by

$$\det\Big(F_k(T,0) - \mathrm{e}^{\lambda_j(k)T}\Big) \;=\; 0\,, \quad j = 1,\ldots,N\,,$$

and the set of all Floquet exponents is referred to as the Floquet spectrum. We assume that the periodic orbit u_* is spectrally stable in the following strict sense:

Hypothesis (Spectral stability)
(i) The Floquet spectrum in the closed half-space $\{\mathrm{Re}\,\lambda \geq 0\}$ *is nonempty only for* $k = 0$*, in which case it consists of the simple Floquet exponent* $\lambda_1 = 0$*;*
(ii) Near $k = 0$*, the neutral Floquet exponent continues as* $\lambda_1(k) = -d_0k^2 + \mathcal{O}(k^4)$ *for some* $d_0 > 0$*.*

We emphasize that these assumptions are satisfied for an open class of reaction-diffusion systems. In particular, since by (i) $\lambda_1 = 0$ is a simple Floquet exponent for $k = 0$, it is clear from (2) that $\lambda_1(k)$ satisfies an expansion of the form (ii) for some $d_0 \in \mathbb{R}$. Assuming $d_0 > 0$ is therefore robust.

Of course, a necessary condition for our spectral assumption to hold is that $u_*(t)$ be a stable periodic solution of the ODE $\dot{u} = f(u)$, but this hypothesis alone is not sufficient in general, except if the diffusion matrix is a multiple of the identity. Indeed, even if $N = 2$, one can find examples of periodic solutions which are asymptotically stable for the ODE dynamics, but become unstable if a suitable diffusion is added [3, 4]. One possible scenario, which is usually called *phase instability* or *sideband instability*, is that the coefficient d_0 be negative, in which case the periodic orbit is unstable with respect to long-wavelength perturbations. It may also happen that the Floquet spectrum is stable for k in a neighborhood of the origin, but that there exists an unstable Floquet exponent for some $k_* \neq 0$, and therefore for all k in a neighborhood of k_*. This mechanism is reminiscent of the *Turing instability* for spatially homogeneous equilibria.

Having assumed that u_* is spectrally stable, we now discuss the *nonlinear stability* of this periodic orbit as a solution of the PDE (1). Of course, the stability properties may depend on the class of admissible perturbations. The only result we have so far concerns the relatively simple situation where the perturbations are *spatially localized*. In that case, one can optimally exploit the properties of the heat semigroup to show that the perturbations decay diffusively to zero as $t \to +\infty$. Denoting $X = L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, our result can be stated as follows:

Theorem. *Assume that the periodic orbit* u_* *is spectrally stable as specified in the Hypothesis above. Then there are positive constants* C *and* δ *such that, for any* *initial data* $u(0,x) = u_*(t_0) + v_0(x)$ *with* $t_0 \in \mathbb{R}$ *arbitrary and* $\|v_0\|_X \le \delta$*, there exists a unique, smooth global solution* $u(t,x)$ *of* (1) *for* $t \ge 0$*. Moreover* $u(t,x)$ *converges to the periodic solution* u_* *in the sense that*

$$\sup_{x\in\mathbb{R}^n} \Big|u(t,x) - u_*(t_0+t)\Big| \le \frac{C\|v_0\|_X}{(1+t)^{n/2}}\,, \quad \text{for all } t \ge 0\,. \tag{3}$$

We emphasize in particular that the perturbations we consider, being localized in space, do not alter the overall phase t_0 of the periodic solution. We also observe that the decay rate in (3) is optimal. As a matter of fact, under the assumptions of the Theorem, one can show that the solution of $u(t,x)$ of (1) has the following asymptotic expansion as $t \to +\infty$:

$$\begin{aligned} u(t,x) &= u_*(t_0+t) + u_*'(t_0+t)\,\frac{\alpha_*}{(4\pi d_0 t)^{n/2}}\,\mathrm{e}^{-|x|^2/(4d_0t)} + \mathrm{o}(t^{-n/2}) \\ &= u_*\Big(t_0 + t + \frac{\alpha_*}{(4\pi d_0 t)^{n/2}}\,\mathrm{e}^{-|x|^2/(4d_0t)}\Big) + \mathrm{o}(t^{-n/2})\,, \end{aligned}$$

uniformly in $x \in \mathbb{R}^n$, for some $\alpha_* \in \mathbb{R}$. To leading order, the effect of the perturbation is thus a spatially localized modulation of the phase of the periodic solution.

The proof of the Theorem is relatively simple in high space dimensions. If we look for solutions of (1) of the form $u(t,x) = u_*(t) + v(t,x)$, we obtain for the perturbation v the equation

$$v_t = D\Delta v + f'(u_*(t))v + N(u_*(t),v)\,, \tag{4}$$

where $N(u_*,v) = f(u_*+v) - f(u_*) - f'(u_*)v = \mathcal{O}(v^2)$. Let $\mathcal{F}(t,s)$ be the two-parameter semigroup defined by the linear, time-periodic equation $v_t = D\Delta v + f'(u_*(t))v$. The Hypothesis above implies that the operator $\mathcal{F}(t,s)$ satisfies the same L^p–L^q estimates as the heat semigroup $\mathrm{e}^{(t-s)\Delta}$, namely

$$\|\mathcal{F}(t,s)v\|_{L^q(\mathbb{R}^n)} \le \frac{C}{(t-s)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}}\,\|v\|_{L^p(\mathbb{R}^n)}\,, \quad t > s\,, \tag{5}$$

for $1 \le p \le q \le \infty$. Using this observation, it is straightforward to show that small solutions of (4) in X stay bounded and decay diffusively to zero as $t \to +\infty$ *provided* that $n > 2$. If $n = 1$ or $n = 2$, the quadratic terms in the nonlinearity $N(u_*,v)$ are no longer "irrelevant" (in the terminology of [1]) and the naive approach breaks down.

To prove stability in low space dimensions, the main idea is to use a *normal form* transformation for the ODE dynamics which removes all "relevant" terms in the perturbation equation. This transformation is defined in a tubular neighborhood of the periodic orbit and takes the form $u = \Psi(\theta,v)$, where $\theta \in S^1$ is the phase variable and $v \in \mathbb{R}^{N-1}$ the transverse coordinate. The ODE $\dot{u} = f(u)$ becomes

$$\dot{\theta} = \omega\,, \quad \dot{v} = L(\theta)v + g(\theta,v)[v,v]\,, \tag{6}$$

where $\omega = 2\pi/T$, L is a linear operator in $\mathbb{R}^{N-1}$ depending on θ, and g is a quadratic form on $\mathbb{R}^{N-1}$ depending on θ and v. In particular, the periodic orbit

$u_*(t)$ corresponds to the trivial solution $\theta(t) = \omega t$, $v(t) = 0$ of (6). If we now apply the normal form transformation Ψ to the full equation (1), we obtain a quasilinear system of PDE's in which the reaction terms, describing the kinetics, have the simpler form (6). Moreover, the two-parameter semigroup corresponding to the linearization at the periodic orbit satisfies better estimates than (5), with a faster decay in time of the transverse variable v. These estimates are sufficient to control the nonlinear terms in the pertubation equation, in any space dimension $n > 0$. All details can be found in [2].

An important question is whether our spectral stability assumption is sufficient to ensure nonlinear stability with respect to perturbations in a larger class (for instance, bounded perturbations). This problem is left for future investigations.

References

[1] J. Bricmont, A. Kupiainen, and G. Lin. *Renormalization group and asymptotics of solutions of nonlinear parabolic equations.* Comm. Pure Appl. Math. **47** (1994), 893–922.

[2] Th. Gallay and A. Scheel, *Diffusive stability of oscillations in reaction-diffusion systems*, preprint available on `http://arxiv.org/abs/0806.4915`

[3] E. Risler, *Criteria for the stability of spatial extensions of fixed points and periodic orbits of differential equations in dimension 2.* Phys. D **146** (2000), 121–136.

[4] E. Risler, *A direct computation of the phase stability criterion for spatially homogeneous time periodic solutions close to a Hopf bifurcation.* Internat. J. Bifur. Chaos Appl. Sci. Engrg. **11** (2001), 2097–2103.

Pattern formation and partial differential equations

Felix Otto

In this talk, I discussed three partial differential equations (PDE) that model pattern formation. Numerical simulations reveal that solutions of these deterministic equations have indeed stationary or self-similar statistics, which are independent of the system size and of the details of the initial data. We show how PDE methods can be used to understand some aspects of this universal behavior.

The first PDE has the structure of a gradient flow (a feature on which the analysis relies), the second PDE has the structure of a *driven* gradient flow, whereas the third PDE is half-way between a conservative and a dissipative system.

1. Bounds on the coarsening rate in spinodal decomposition

The PDE — the Cahn–Hilliard equation — is given by

$$\partial_t u + \triangle(u(1-u^2) + \triangle u) = 0$$

with periodic boundary conditions in the spatial domain $(0, L)^d$. Here, u denotes the (renormalized) volume fraction of a binary mixture, which is quenched (slightly) below the critical temperature and thus wants to segregate.

Numerical simulations reveal that for generic initial data (e. g. small amplitude white noise) after an initial layer, $(0, L)^d$ divides into a convoluted domain where $u \approx 1$ and its complement where $u \approx -1$, separated by a characteristic interfacial layer of width $O(1)$. This domain configuration coarsens over time in a statistically self-similar way. More precisely, the average length scale of the domains behaves as $O(t^{1/3})$. This is reflected by the fact that the average energy per volume, i.e.,

$$E = L^{-d} \int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2 \, dx,$$

which is proportional to the total interfacial area per system volume, behaves as $O(t^{-1/3})$.

PDE analysis is only able to rigorously establish a much weaker result: In a joint work with R. V. Kohn [6] we prove that, in a time-averaged sense, $E \geq O(t^{-1/3})$. The proof makes use of the gradient flow structure of the evolution. This allows to translate a bound on the energy landscape (energy cannot decrease too fast as a function of the intrinsic distance to the reference configuration) into a bound on the steepest descent dynamics (energy cannot decrease too fast as a function of time).

2. Bounds on the Nusselt number in Rayleigh–Bénard convection

The system of PDEs is given by an advection-diffusion equation for the temperature T, and the Stokes equations with buoyancy for the fluid velocity u, i.e.,

$$\begin{aligned} \partial_t T + \nabla \cdot (Tu) - \triangle T &= 0, \\ -\triangle u + \nabla p &= T(0,0,1), \\ \nabla \cdot u &= 0 \end{aligned}$$

in the 3-d spatial domain $(0,L)^2 \times (0,H)$ with periodic boundary conditions in the two horizontal dimensions. The PDE is complemented by inhomogeneous (and thus driving) Dirichlet boundary conditions at the top and bottom boundaries

$$T = 1 \text{ for } z=0, \quad T = 0 \text{ for } z=H, \quad u = 0 \text{ for } z=0,H.$$

Experiments and numerical simulations for $H, L \gg 1$ show a chaotic velocity field u, with regions of high temperature $T \approx 1$ in form of mushrooms (plumes). This leads to a high upwards heat transport — much higher than the one mediated by diffusion allone. This upwards heat flux is given by the Nusselt number

$$Nu := \limsup_{T\uparrow\infty} T^{-1}L^{-2}H^{-1} \int Tu \cdot (1,0,0) dx.$$

Experiments and asymptotic analysis suggest that $Nu = O(1)$.

Again, PDE analysis is only able to rigorously establish a much weaker result: In a joint work with C. Doering and M. Reznikoff-Westdickenberg [4] we show that indeed $Nu \leq O(1)$ in $H \gg 1$ (up to the cube root of a logarithm). This slightly improves an earlier result by P. Constantin and C. Doering [3]. We use the background field method; our background temperature profile is non-monotone, and thus enjoys enhanced stability which allows to contain its boundary layers.

3. Bounds on the average dissipation in the Kuramoto–Sivashinsky equation

The PDE — the Kuramoto–Sivashinsky equation — is given by

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) + \partial_x^2 u + \partial_x^4 u = 0$$

with periodic boundary conditions on $(0, L)$. In one particular application, u denotes the slope $\partial_x h$ of a (one-dimensional) crystal surface. The Kuramoto-Sivashinsky equation describes the evolution of the crystal surface in the presence of slope selection, curvature regularization and strong deposition — in a regime where there is no coarsening of facets. It can also be seen as a toy model for the energy transfer from large wave lengths to small wave lengths in the Navier Stokes equations.

For $L \gg 1$, numerical simulations reveal that the solutions have an average length scale of $O(1)$, and an average amplitude of $O(1)$ and display spatio-temporal chaos. Moreover, numerical simulations of the power spectrum show "equipartition of energy".

Again, PDE analysis is only able to rigorously establish a much weaker result: in [8], we prove that the average dissipation rate, i.e.,

$$\lim_{T \uparrow \infty} T^{-1} L^{-1} \int_0^T \int_0^L (\partial_x^2 u)^2 dx dt,$$

is $O(1)$ in $L \gg 1$ (up to a logarithm). The argument relies on a new observation on the inhomogeneous inviscid Burger's equation

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = \partial_x f.$$

It improves earlier results by [7, 2, 1].

References

[1] J. C. Bronski, T. N. Gambill, *Uncertainty estimates and L_2 bounds for the Kuramoto–Sivashinsky equation*, Nonlinearity **19** (2006), 2023–2039.

[2] P. Collet, J.-P. Eckmann, H. Epstein, J. Stubbe, *A global attracting set for the Kuramoto–Sivashinsky equation*, Comm. Math. Phys. **152** (1993) 203–214.

[3] P. Constantin, C. Doering, *Infinite Prandtl number convection*, J. Statist. Phys. **94**, No. 1-2 (1999), 159–172.

[4] C. Doering, F. Otto, M. Reznikoff, *Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh–Bénard convection*, J. Fluid Mech. **560** (2006), 229–241.

[5] L. Giacomelli, F. Otto, *New bounds for the Kuramoto–Sivashinsky equation*, Comm. Pure Appl. Math. **58** (2005), 297–318.

[6] R. V. Kohn, F. Otto, *Upper bounds on coarsening rates*, Comm. Math. Phys. **229**, No. 3 (2002), 375–395.

[7] B. Nicolaenko, B. Scheurer, R. Temam, *Some global dynamical properties of the Kuramoto–Sivashinsky equations: Nonlinear stability and attractors*, Phys. D **16** (1985), 155–183.

[8] F. Otto, *Optimal bounds on the Kuramoto–Sivashinsky equation*, SFB 611, Preprint 386 (2008).

Infinite-dimensional hyperbolic attractors in a periodically driven Swift–Hohenberg equation

Dmitry Turaev
(joint work with Sergey Zelik)

We show that given any $\varepsilon > 0$ one can find a space- and time-periodic function $f(x,t)$ such that $\|f\| < \varepsilon$ and the equation

$$\partial_t u = -(1+\partial_{xx})^2 u + \alpha u + \beta u^2 - u^3 + f(x,t) \tag{1}$$

(here $x \in \mathbb{R}^1$) has, for some open region of values of α and $beta$, a local attractor Λ the flow on which is topologically conjugate to a suspension over a direct product of an infinite number of two-dimensional hyperbolic Plykin attractors. This result augments a previous result of [1] about the existence of an infinite-dimensional hyperbolic set within the maximal attractor of equation (1); namely, we show that such hyperbolic sets can be locally attracting.

References

[1] A. Mielke, S. Zelik, *Weak interaction in multi-pulse structures and space-time chaos in reaction-diffusion equations*, Memoirs of AMS, to appear

Traveling waves in heterogeneous media: pinning and homogenization

Karsten Matthies

Pinning. We analyze traveling and pinned fronts for reaction-diffusion equations of the form

$$\begin{aligned} u_\tau &= D(x/\epsilon)\Delta_{x,y} u + f(u, \nabla u, y, \lambda, x/\epsilon) \\ u(0) &= u_0 \in H^s_{loc}(\mathbb{R} \times \Omega, \mathbb{R}^n), \end{aligned} \tag{1}$$

with a domain in form of a strip $(x,y) \in \mathbb{R} \times \Omega$ with periodic boundary condition in the cross-section Ω. The nonlinearity is assumed to be an entire function of u, ∇u, y, and continuous in x/ϵ, ϵ. The dynamics of these fronts are compared with homogenized problems like

$$u_\tau = D\Delta u + \bar{f}(u, \nabla u, y, \lambda)$$

When looking for stationary solutions of (1), e.g. pinned waves, we obtain an equation of the form

$$\Delta_{x,y} u + \tilde{f}(u, \nabla u, x/\epsilon) = 0.$$

We rewrite this equation by using spatial dynamics. This is a way to construct special solutions to PDEs on unbounded domains. For this we let

$$U = \begin{pmatrix} u \\ u_x \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -\Delta_y & 0 \end{pmatrix}, \quad F(U, x/\epsilon) = \begin{pmatrix} 0 \\ -\tilde{f}(u, u_x, \nabla_y u, x/\epsilon) \end{pmatrix}.$$

Renaming x as time t for the spatial dynamics approach, we have the equation

$$U_t = AU + F(U, t/\epsilon), \tag{2}$$

which has the form of a rapidly forced evolution equation. The phase space X is a function space on the cross-section Ω like $X = H^{s+1}(\Omega, \mathbb{R}^n) \times H^s(\Omega, \mathbb{R}^n)$. Here the initial value problem is not well-posed. We compare this with a corresponding homogenized equation

$$\bar{U}_t = A\bar{U} + \bar{F}(\bar{U}) \tag{3}$$

Then we obtain in [3] that pinning can only occur exponentially small parameter intervals.

Theorem 1. *Assume a standing front for homogenized equation (3) connecting two t-independent equilibria for* $\lambda = \lambda_0$ *and some transversality and non-degeneracy conditions. Then there exist solutions* U_ϵ, λ_ϵ *of (2) and*

$$\|U_\epsilon - \bar{U}\| \leq C\epsilon, \qquad |\lambda_\epsilon - \lambda_0| \leq C\epsilon$$

for all $0 < \epsilon < \epsilon_0$.

Furthermore, other pinned solutions V_ϵ *can only exist in an exponentially small parameter interval. In other words, let* V_ϵ, $\tilde{\lambda}_\epsilon$ *be a solution of (2) nearby, i.e.,*

$$\|V_\epsilon - \bar{U}\| \leq C\epsilon_0, \qquad |\tilde{\lambda}_\epsilon - \lambda_0| \leq C\epsilon_0,$$

then

$$|\tilde{\lambda}_\epsilon - \lambda_\epsilon| \leq C\exp(-c\epsilon^{-1/2}).$$

A main ingredient in the proof is the homogenization of equation (2), here methods from [2] are used.

Variants. When considering traveling waves in heterogeneous media, the ansatz

$$u(x, y, t) = v(x - ct, y, x/\epsilon),$$

is used. The profile v of the traveling wave is changing periodically while moving through the periodic medium. This can be also formulated as a spatial dynamics problem, for details how to obtain homogenization results, see [4].

Future research should aim e.g. at the question of pinning of planar waves in $\mathbb{R}^2$. Here variants also include heterogeneities in the main part like in classical homogenization theory. A first example are second-order elliptic equations like

$$-\nabla \cdot (\mathcal{A}(x/\epsilon)\nabla u)(x)) = f(x)$$

with $\mathcal{A} \in L^\infty(T^d)$ symmetric and uniformly elliptic. Assuming periodic boundary conditions for x, exponential homogenization results could be obtained in [1]. This should be a first step to understand pinning in reaction-diffusion system multi-dimensional heterogeneous media.

References

[1] V. Kamotski, K. Matthies, and V. Smyshlyaev, *Exponential homogenization of linear second order elliptic problems with periodic coefficients*, SIAM J. Math. Anal. **38** (2007), 1565–1587.

[2] K. Matthies, *Homogenization of exponential order for elliptic systems in infinite cylinders*, Asymptot. Anal. **43**, 205–232, 2005.

[3] K. Matthies, and C.E. Wayne, *Wave pinning in strips*, Proc.Roy. Soc. Edinburgh A **136** (2006), 971–995.

[4] K. Matthies, G. Schneider, and H. Uecker. *Exponential averaging and traveling waves in rapidly varying periodic media*, Math. Nachr. **280** (2007), 408–422.

Traveling waves and spreading fronts in spatially heterogeneous media

Hiroshi Matano

A function is called ergodic if it has a certain averaging property. This class is wider than and is often more natural than the class of almost periodic functions when one studies front propagation under spatially heterogeneous environtments. In this talk, I have discussed the following subjects:

(1) speed of traveling waves in spatially ergodic media;
(2) long-time behavior of ergodically disturbed planar fronts in the Allen–Cahn equation;
(3) spreading fronts in spatially stratified diffusive media.

Here, by "ergodic", I mean *uniquely ergodic* with respect to the space variable. Note that I only consider deterministic models.

Before presenting the main results, let us clarify the meaning of some basic concepts. Let X be a metric space with $\mathbb{R}^m$ action. This means that there exists a family of homeomorphisms $T_a : X \to X$ $(a \in \mathbb{R}^m)$ satisfying $T_a \circ T_b = T_{a+b}$. Given an element $g \in X$, we define its *hull* $\mathcal{H}_g$ by

$$\mathcal{H}_g := \overline{\{ \, T_a g \mid a \in \mathbb{R}^m \, \}}^X,$$

where $\overline{A}^X$ stands for the closure of a set A in the X-topology. We say that an element $g \in X$ is *uniquely ergodic* if there is a unique probability measure on $\mathcal{H}_g$ that is invariant with respect to all T_a $(a \in \mathbb{R}^m)$. One can easily check that g is uniquely ergodic if and only if, for any continuous function Ψ on $\mathcal{H}_g$, the following limit exists uniformly in $a \in \mathbb{R}^m$:

$$\lim_{R\to\infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(T_x g)\, dx.$$

A typical situation we have in mind is when X is the space of uniformly continuous functions on $\mathbb{R}$ or $\mathbb{R}^n$ with the L^∞_{loc} topology, and T_a is the spatial translation $g(x) \mapsto g(x+a)$. Any almost periodic function is uniquely ergodic, but the converse is not true. For example, a function on $\mathbb{R}^2$ whose level sets have the Penrose tiling pattern is uniquely ergodic but not almost periodic in the sense of Bohr.

One can also consider the situation when the translation T_a is limited to a subspace of $\mathbb{R}^n$. For example, given a bounded uniformly continuous function $g(x, y)$ in $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, we say that g is uniquely ergodic in the x-direction if it is uniquely ergodic with respect to the translations $T_a : g(x, y) \mapsto g(x + a, y)$.

1. Topic 1: Speed of TW in spatially ergodic media

Consider the following simple model equation:

$$u_t = u_{xx} + b(x)f(u), \tag{1D}$$

where $f(u)$ is a nonlinear term satisfying $f(0) = f(1) = 0$, and $b(x)$ is a positive smooth function on $\mathbb{R}$. The classical notion of traveling wave does not apply to such an equation unless $b(x)$ is a constant. But one can naturally extend the notion of traveling waves using the hull of b. See [5] or [4] for details. The question then is whether the generalized traveling wave has a well-defined average speed. The following theorem answers this question.

Theorem 1 ([5]). *In problem* (1D)*, the traveling wave has an average speed if $b(x)$ is uniquely ergodic.*

The same result holds for a more general class of equations. For example, the recent paper [4] considers traveling waves in a two-dimensional infinite cylinder whose boundary undulates ergodically. It is shown that the traveling wave has an average speed, and the homogenization limit of this traveling wave is determined. A similar study was made in an earlier paper [6] for periodically undulating cylinders.

2. Topic 2: Ergodically disturbed planar fronts

Consider the Allen–Cahn equation on $\mathbb{R}^n$:

$$u_t = \Delta u + f(u), \tag{AC}$$

where f is a bistable nonlinearity with -1 and $+1$ being its stable zeros. A solution $u(x, y, t)$ with $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is called a planar wave if it is written in the form $u(x, y, t) = \Phi(y - ct)$, where c is a constant representing the speed of this planar wave. Most of the previous studies on the asymptotic stability of planar waves is limited to small perturbations. Recently, [8] proved the stability with asymptotic phase for perturbations that are almost periodic in the x-direction. The following theorem extends this result:

Theorem 2 ([7]). *A planar wave of* (AC) *is stable with asymptotic phase under perturbations that are uniquely ergodic in the x-direction.*

Note that smallness of the initial perturbation is assumed here, but that some mild condition near $y = \pm\infty$ is imposed, as in [8]. This theorem is proved by using the following key observations:

(a) $u(x, y, t)$ can be approximated by $\Phi(y - \gamma(x, t))$ for all large t, where $\gamma(x, t)$ is the zero-level surface of u;
(b) $\gamma(x, t)$ can be approximated by a solution of the mean curvature flow with a drift term;
(c) the ergodicity of the initial value $u_0(x, y)$ is inherited by the solution, therefore $\gamma(x, t)$ remains ergodic in x for every large t;

(d) solutions of the mean curvature flow with uniquely ergodic initial value converges to a drifting hyperplane uniformly as $t \to \infty$.

3. Topic 3: Spreading fronts in spatially stratified media

By a spreading front we mean a solution that starts from a non-negative compactly supported initial data.

Here we consider a KPP type diffusion equation on $\mathbb{R}^2$ of the form

$$u_t = u_{xx} + u_{yy} + b(x)f(u), \qquad \text{(KPP)}$$

where f is a KPP type monostable nonlinearity, and $b(x)$ is positive. In [3], assuming that $b(x)$ is L-periodic, we have considered a variational problem of finding the optimal $b(x)$ that maximizes the spreading speed of the front in each direction, under the integral constraint $\langle b \rangle = \alpha$, where $\langle b \rangle$ denotes the mean of $b(x)$ and α is any given constant. We have shown that the maximizing b is not a function but a periodically arrayed line measure, and that the spreading front has a parabolic shape when L is very large. The result is proved by using a result in the earlier paper [2], in which a similar variational problem was considered in one space dimension.

Front propagation for a system of equations is much harder, but we have recently obtained partial results for the following epidemic model:

$$\begin{cases} S_t = \delta S - S + rI \\ H_t = -\alpha(x)HS \\ I_t = \alpha(x)HS - \beta I. \end{cases}$$

More precisely, what we have obtained is the existence of near planar waves in each direction. This is a first step for analysing the asymptotic shape of spreading fronts, but whether such an asymptotic shape exists for this system of equations is completely open, largely because lack of comparison principle.

Coming back to the equation (KPP), where $b(x)$ is no longer periodic, the question of spreading front is largely open. In particular, it is not even known whether or not there exists a near-planar wave in every direction, except for the special case where $b(x)$ is a uniform limit of periodic functions.

References

[1] A. Ducrot and H. Matano, *Plant diseases propagation over a field with periodic structures*, preprint.

[2] X. Liang, X. Lin and H. Matano, *A variational problem associated with the minimal speed of travelling waves for spatially periodic reaction-diffusion equations*, to appear in Trans Amer. Math. Soc.

[3] X. Liang, X. Lin and H. Matano, *Spreading fronts for a KPP equation in periodically stratified media*, in preparation.

[4] B. Lou, H. Matano and K.-I. Nakamura, *Recurrent traveling waves in a two-dimensional saw-toothed cylinder and their homogenization Limit*, preprint.

[5] H.Matano, *Traveling waves in spatially recurrent media*, in preparation.

[6] H. Matano, K.-I. Nakamura and B. Lou, *Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit*, Networks and Heterogeneous Media **4** (1) (2006), 537–568.
[7] H. Matano and M. Nara, *Large time behavior of disturbed planar fronts in the Allen–Cahn equation*, preprint.
[8] H. Matano, M. Nara and M. Taniguchi, *Stability of planar waves in the Allen–Cahn equations*, to appear in Comm. Partial Differential Equations.

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