Mathematisches Forschungsinstitut Oberwolfach

Report No. 01/2009

DOI: 10.4171/OWR/2009/01

Toric Geometry

Organised by
Klaus Altmann, Berlin
Victor Batyrev, Tübingen
Yael Karshon, Toronto

January 4th – January 10th, 2009

Abstract. *Toric Geometry* originated from investigations of torus actions on geometric and algebraic objects. It is addressed through algebraic geometry, symplectic geometry, equivariant topology, as well as the theory of convex polyhedra within discrete mathematics. In spite of using their own language these completely different disciplines often observe similar or even identical combinatorial phenomena. Thus toric geometry leads to a fascinating and fruitful interplay between these disciplines.

Mathematics Subject Classification (2000): 14-xx, 14M25, 16Gxx, 32-XX, 52-XX, 53Dxx.

Introduction by the Organisers

The workshop *Toric Geometry* was organised by Klaus Altmann (Berlin), Victor Batyrev (Tübingen), and Yael Karshon (Toronto). The meeting was attended by almost 50 participants from many European countries, Canada, and the USA. We are grateful for the Oberwolfach OWLG program and NSF grants, which enabled us to invite additional young researchers. The workshop program consisted of 23 talks, leaving ample room for informal discussions in small groups.

Toric geometry is a fascinating combination of polyhedral and algebraic, or of polyhedral and symplectic geometry. In contrast to the 2006 Oberwolfach workshop "Convex and Algebraic Geometry", we now tried to also combine the algebraic and the symplectic aspects of the theory. While we intended not to overemphasize polytope theory and tropical geometry, which have their own Oberwolfach workshops, these subjects were naturally part of the conference.

One of the main topics at the center of the conference was the amount of information that arbitrary varieties can inherit from morphisms to toric varieties, e.g. from different toric ambient spaces or from toric contractions or quotients. In this context, Jürgen Hausen, Diane Maclagan, and Jenia Tevelev focused on the structure of the Mori cone and the Cox ring, and Sam Payne represented the Berkovich analytification as an inverse limit.

Michel Brion presented vanishing theorems for generalizing the Hodge theory of complete intersections in toric varieties to other ambient manifolds. Greg Smith investigated the relation between defining varieties by 2×2 minors and the syzygies of subrings of the Cox ring. Overweight deformations were presented by Bernard Teissier. They provide a method to find simultaneous resolutions of singularities along toric degenerations. In the talk of Lutz Hille conditions for the existence of full, (strongly) exceptional sequences on rational surfaces were given. Each of these sequences leads to a "toric model" of this surface as well as of the sequence having the same Cartan matrix.

A central theme in toric geometry is to translate (algebro or symplectic) geometric facts or problems into the language of polytopes and combinatorics. This was especially done by Dusa McDuff in her detection of displaceable fibers of the moment map and in her extraction of topological information of the Hamiltonian group from the polytope. Sandra Di Rocco related toric fibrations to Cayley polytopes. This led to fertile discussions with the polytope people like Christian Haase and Benjamin Nill. Miguel Abreu expressed Kähler-Sasakian metrics on toric symplectic cones in terms of potentials on polyhedral cones. Mounir Nisse gave new relations between coamoebas and Newton polygons. Taras Panov defined the toric Kempf-Ness set and expressed its integral cohomology ring in terms of the fan. Victor Buchstaber presented a universal approach to deal with combinatorial invariants of polytopes.

Mirror symmetry did not play a big rôle as it did in the 2006 workshop. However, it was present in Bernd Siebert's talk about the tropical vertex. One of the most celebrated features of toric geometry, namely providing a test ground for general concepts and conjectures, came into the play when Tom Braden spoke about symplectic and Koszul duality. He presented this general approach for the special case of hypertoric varieties. Vladimir Baranovsky showed how to extend BGG to the Cox ring via utilizing A_∞ structures.

Many talks addressed the subject of lower-dimensional torus actions. While Dmitri Timashev uses valuations for describing those situations, Nathan Ilten reported on the description of divisors on T-varieties via piecewise linear functions on the coefficients on polyhedral divisors. Gavin Brown presented an example arising from 3-dimensional flips – this still waits for a description using one of the present theories. Yi Lin extended topological properties of moment maps to torus actions on generalized complex manifolds. Michèle Vergne computed the equivariant K-theory of certain complements of hyperplane arrangements. Valentina Kiritchenko considered compactifications of reductive groups and the associated action of the maximal torus. Here, the gap arising from the missing torus dimensions was filled by the Gelfand-Zetlin polytopes.

The workshop was closed on Friday night by an informal piano and cello recital by young Workshop participants Milena Hering, Laura Hinsch, and Benjamin Nill.

Workshop: Toric Geometry

Table of Contents

Abstracts

Overweight Deformations of Weighted Affine Toric Varieties

Bernard Teissier

I report on some recent progress in the program outlined in ([3], [4]) which tends to show that any excellent equicharacteristic germ of space X with an algebraically closed residue field k can be formally embedded in an affine space $\mathbb{A}^N(k)$ in such a way that it then has an embedded resolution by a toric modification of $\mathbb{A}^N(k)$.

The simplest example is the plane branch, say in characteristic zero, with equation $(U_2^2 - U_1^3)^2 - U_1^5U_2 = 0$. It cannot be resolved by any toric modification of the plane but if we embed it in $\mathbb{A}^3(k)$ as $U_2^2 - U_1^3 - U_3 = 0, U_3^2 - U_1^5U_2 = 0$ it is resolved by any toric modification of $\mathbb{A}^3(k)$ which resolves the monomial curve with equations $U_2^2 - U_1^3 = 0, U_3^2 - U_1^5U_2 = 0$, to which it specializes flatly by putting a parameter in front of U_3 in the first equation and which is the toric curve corresponding to the semigroup of values $\langle 4, 6, 13\rangle$ taken on the local algebra of the curve by its unique k-valuation.

The approach I use is to begin by studying the analogous problem for local uniformization. This means that given a valuation of the local excellent equicharacteristic ring of our germ X, we wish to embed it so that a toric modification of $\mathbb{A}^N(k)$ makes the strict transform of X regular at the point picked by the valuation. After reductions which are outlined in [3]) one is led to study the following situation:
Let k be an algebraically closed field and Φ a totally ordered abelian group of finite rational rank r.
A *weight* on the rings $k[U_1, \ldots, U_N]$ or $k[[U_1, \ldots, U_N]]$ is an homomorphism of groups $\lambda\colon \mathbb{Z}^N \to \Phi$ which is induced by an homomorphism of semigroups $\mathbb{N}^N \to \Phi_+$. It induces a weight on monomials by $w(U^m) = \lambda(m)$ and a monomial order by $U^m < U^n \leftrightarrow \lambda(m) < \lambda(n)$.

This rather general situation can be reduced to a more familiar one thanks to the following result (see [3], Proposition 4.12):
The positive semigroup of a totally ordered semigroup if finite rational rank r is the union of a nested sequence of free subsemigroups of rank r:

$$\cdots \mathbb{N}^r_{(h)} \subset \mathbb{N}^r_{(h+1)} \subset \cdots \subset \Phi_+,$$

the inclusions being semigroup maps.
As explained in *loc.cit.* this result can be viewed as an avatar of the Jacobi-Perron algorithm for approximating vectors in $\mathbb{R}^N$ by integral vectors.

So we may assume in the sequel that $\Phi = \mathbb{Z}^r$ with a total order, but there are good reasons to begin as I did.
Then denoting by e_i the i-th basis vector of $\mathbb{Z}^N$ and setting $\gamma_i = \lambda(e_i) \in \mathbb{Z}^r$ we may consider the semigroup Γ generated by $\gamma_1, \ldots, \gamma_N$ and the toric variety $X_0 = \mathrm{Spec}k[t^\Gamma]$ where $k[t^\Gamma]$ is the semigroup algebra with coefficients in k. It is the

closure of the orbit of the point $(1,1,\dots,1) \in \mathbb{A}^N(k)$ under the action of the torus k^{*r} determined by $(t, z_1, \dots, z_N) \mapsto (t^{\gamma_1} z_1, \dots, t^{\gamma_N} z_N)$ with $t = (t_1, \dots, t_r) \in k^{*r}$ and $t^{\gamma_j} = t_1^{\gamma_{j1}} \dots t_r^{\gamma_{jr}}$

Denoting by $\mathcal{L}$ the lattice which is the kernel of λ, we may choose a system of generators $(U^{m^\ell} - U^{n^\ell})_{\ell \in L}$ for $\mathcal{L}$, where all the exponents are non negative.
The ideal I_0 of $k[U_1, \dots, U_N]$ generated by the $(U^{m^\ell} - U^{n^\ell})_{\ell \in L}$ is a prime binomial ideal defining the embedding $X_0 \hookrightarrow \mathbb{A}^N(k)$. The Krull dimension of $k[t^\Gamma]$ is equal to r; it is the dimension of X_0. If we denote by H_ℓ the hyperplane of $\check{\mathbb{R}}^N$ which is dual to the vector $m^\ell - n^\ell \in \mathbb{R}^N$, it is shown in [2] that any regular fan Σ with support the first quadrant of $\check{\mathbb{R}}^N$ and compatible with the hyperplanes H_ℓ for $\ell \in L$ determines a toric modification $Z(\Sigma) \to \mathbb{A}^N(k)$ which gives an embedded resolution of the toric variety X_0.

We are going to show that *some of these toric modifications also resolve the singularities of some special deformations of X_0, called* overweight*, at the point picked by a valuation which is determined by the weight w.*

An *overweight deformation* of our affine toric variety X_0 is described by a deformation of its equations of the following form:

$$F_\ell = U^{m^\ell} - U^{n^\ell} + \sum_p c_p^\ell U^p \quad \text{with } w(U^p) > w(U^m) = w(U^n),\ \ell \in L.$$

Here the F_ℓ are elements of $k[[U_1, \dots, U_N]]$. We denote by I the ideal which they generate.
The initial forms of the series F_ℓ with respect to the monomial order determined by the weight w are the binomials $(U^{m^\ell} - U^{n^\ell})_{\ell \in L}$. We assume as part of the definition that the dimension of the ring $R = k[[U_1, \dots, U_N]]/I$ is equal to r, or equivalently that the $(U^{m^\ell} - U^{n^\ell})_{\ell \in L}$ generate the initial ideal of I with respect to w.
Then one can check that the ring R is endowed with a valuation defined in the following manner:
Define the order $\nu(x)$ of an element of R as the maximum weight of one of its representatives in $k[[U_1, \dots, U_N]]$ (this maximum exists if $x \neq 0$). This order defines a filtration of R whose associated graded ring is the quotient of $k[U_1, \dots, U_N]$ by the initial ideal of I. By our assumption it is $k[U_1, \dots, U_N]/I_0$ and an integral domain, so that ν is a valuation.

It is shown in [3] that any complete equicharacteristic local ring endowed with a rational valuation whose associated graded ring is finitely generated over k is obtained in this manner.

Now we want to find a fan subdividing $\check{\mathbb{R}}^N_+$, compatible with the hyperplanes H_ℓ, and such that the strict transform of each F_ℓ is a deformation of the strict transform of its initial form at the point determined by the valuation. This will ensure the nonsingularity at that point of the strict transform.

The idea is very simple to explain in the case where the value group Φ (or the order on $\mathbb{Z}^r$) is of rank one. For simplicity let us look at one equation $F = U^m - U^n + \sum_p c_p U^p$.
Let
$$E' = \langle \{p - n/c_p \neq 0\}, m - n\rangle \subset \mathbb{R}^N,$$
where as above $\langle a, b, ...\rangle$ denotes the cone generated by $a, b, ...$ Since there are infinitely many exponents p, the strictly convex cone E' may not be rational, but the power series ring being noetherian E' is contained in a rational cone E, also strictly convex.

Given a regular cone $\sigma = \langle a^1, \ldots, a^N\rangle \subset \check{\mathbb{R}}^N$, set $Z(\sigma) = \mathrm{Spec} k[\check{\sigma} \cap M]$.
The map $Z(\sigma) \to \mathbb{A}^N(k)$ is monomial and we write it as:
$$U_i \mapsto Y_1^{a_i^1} \ldots Y_N^{a_i^N},\ 1 \leq i \leq N.$$
Assuming that Φ is of (real) rank one, choose an ordered embedding $\Phi \subset \mathbb{R}$ and using it define the *weight vector*
$$\mathbf{w} = (w(U_1), \ldots, w(U_N)) \in \check{\mathbb{R}}^N.$$
The center of the valuation ν is in $Z(\sigma)$ if and only if the weights $w(Y_i)$, which are uniquely determined by the monomial map since the a^j form a basis, are all ≥ 0, which is equivalent to the condition that $\mathbf{w} \in \sigma$. Note that $\mathbf{w}$ is in the hyperplane H of $\check{\mathbb{R}}^N$ corresponding to the vector $m - n$ and that our overweight hypothesis says precisely that $\mathbf{w}$ lies in the interior of the intersection with H of the convex dual $\check{E}$ of E. Note that $\check{E}$ is of dimension N.
So if Σ is a regular subdivision of $\check{\mathbb{R}}^N_+$ which is compatible with H and $\check{E}$, it will contain a regular cone σ of dimension N whose intersection with H is of dimension $N-1$ and which contains $\mathbf{w}$. By compatibility with H the cone σ is entirely on one side of H so we may assume that the scalar products $\langle a^i, m-n\rangle$ which are not zero are all > 0. By compatibility with $\check{E}$ the cone σ is contained in $\check{E} \subseteq \check{E}'$, so that the $\langle a^i, p-n\rangle$ are all ≥ 0. In the corresponding chart $Z(\sigma)$ the transform of our equation F by the monomial map can then be written:
$$Y_1^{\langle a^1,n\rangle} \ldots Y_N^{\langle a^N,n\rangle}\big(Y_1^{\langle a^1,m-n\rangle} \ldots Y_N^{\langle a^N,m-n\rangle} - 1 + \sum_p c_p Y_1^{\langle a^1,p-n\rangle} \ldots Y_N^{\langle a^N,p-n\rangle}\big).$$
This shows that the strict transform of F, which is the quantity between parenthesis, is a deformation of the strict transform of its initial part, and this gives in this case the result we seek, since the initial part is obviously non singular.

The proof in the general case follows the same line but is somewhat less simple especially when the (real) rank of the value group is > 1.

One should note that unlike the case of plane branches or more generally of quasi-ordinary hypersurfaces (see [1], 5.3), one may have to choose a resolution of the toric variety adapted to the deformation and not just any resolution.

References

[1] P. González Pérez, *Toric embedded resolutions of quasi-ordinary hypersurfaces*, Annales Inst. Fourier (Grenoble) 53, (2003), 1819-1881.

[2] P. González Pérez et B. Teissier *Embedded resolutions of non necessarily normal affine toric varieties*, Comptes-rendus Acad. Sci. Paris, Ser.1, 334, (2002), 379-382.
[3] B. Teissier, *Valuations, deformations, and toric geometry*, Valuation Theory and its applications, Vol. II, Fields Inst. Commun. 33, AMS., Providence, RI., (2003), 361-459.
[4] B. Teissier, *Monomial ideals, binomial ideals, polynomial ideals*, Trends in Commutative Algebra, MSRI publications, Cambridge University Press (2004), 211-246.

Ring of Combinatorial Polytopes and Applications

Victor M. Buchstaber

A combinatorial polytope is a class of combinatorially equivalent convex polytopes. Denote by V_{2n} the abelian group generated by combinatorial n-dim polytopes in which addition is given by the disjoint union of polytopes and the zero is the empty set. Introduce a graded commutative associative differential ring $V=\sum_{n\geqslant 0} V_{2n}$ in which multiplication is given by the direct product of polytopes, the unit is $P^0=$ (point) and derivation dP is the disjoint union of all facets of P.

A convex n-dim polytope P^n is said to be simple if exactly n facets meet at each vertex of this polytope. The ring of simple polytopes $\mathcal{P}$ is a differential subring in V. For a simple polytope P^n the polynomial

$$f(P^n)=f(P^n)(\alpha,t)=\alpha^n+f_{1,n-1}t\alpha^{n-1}+\cdots+f_{n-1,1}t^{n-1}\alpha+f_{n,0}t^n.$$

is called an *enumerating polynomial* (a polynomial of faces, or an f-polynomial) of P^n, where $f_{n-k,k}$ is the number of its k-dimensional faces.

Theorem 1. Let $F\colon \mathcal{P}\longrightarrow \mathbb{Z}[\alpha,t]$ be a linear mapping such that

$$F(dP^n)=\frac{\partial}{\partial t}F(P^n),\quad F(P^n)\big|_{t=0}=\alpha^n.$$

Then F is the ring homomorphism and $F(P^n)=f(P^n)$.

A convex polytope is said to be k-simple if any of its k-dimensional faces is the intersection of exactly $n-k$ of its facets. The ring homomorphism $f\colon \mathcal{P}\to\mathbb{Z}[\alpha,t]$ can be extended to the ring homomorphism $f\colon V\to\mathbb{Z}[\alpha,t]$.

Theorem 2. Let $P\in V$. Then $f(dP^n)=\frac{\partial}{\partial t}f(P^n)+\delta(P^n)$, where $\delta(P^n)=\delta_2t^2\alpha^{n-3}+\ldots+\delta_{n-1}t^{n-1}$ and $\delta_k\geqslant 0,\ k=2,\ldots,n-1$. A convex polytope P^n is k-simple if and only if $\delta_{n-k-1}=0$; in this case $\delta_i=0$ for any $1\leqslant i\leqslant n-k-1$.

The correspondence $P\to\delta(P)$ defines an f-derivation $\delta\colon V\to\mathbb{Z}[\alpha,t]$, i.e. $\delta(P_1\times P_2)=\delta(P_1)f(P_2)+f(P_1)\delta(P_2)$.

Set $h(P^n)(\alpha,t)=h_0\alpha^n+h_1t\alpha^{n-1}+\ldots+h_nt^n=f(P^n)(\alpha-t,t)$.

Theorem 3 (Dehn–Sommerville relations)**.** Let P^n be a simple polytope. Then

$$f(P^n)(\alpha,t)=f(P^n)(-\alpha,\alpha+t)\ \text{ and }\ h(P^n)(\alpha,t)=h(P^n)(t,\alpha)$$

Set $V^1=\{P\in V\,:\,dP\in\mathcal{P}\}$. The group $\mathcal{S}$ generated by simplicial polytopes is a remarkable differential subgroup in V^1.

Using that $f(P^{n+1})(\alpha,0)=f(P^{n+1})(-\alpha,\alpha)$ for any polytope P we obtain from Theorems 2 and 3 an analog of Dehn–Sommerville relations for V^1.

Corollary 1. Let $P^{n+1} \in V^1$ then

$$f(P^{n+1})(\alpha,t) = f(P^{n+1})(-\alpha,\alpha+t) + \delta^1(P^{n+1})(\alpha,t),$$

where $\delta^1(P^2)=0$ and for $n \geqslant 2$

$$\delta^1(P^{n+1})(\alpha,t) = \sum_{k=2}^{n} \frac{\delta_k(P^{n+1})}{k+1}\left\{(-1)^{n-k}\left[(\alpha+t)^{k+1}-\alpha^{k+1}\right]-t^{k+1}\right\}\alpha^{n-k}.$$

For example: $2f_{2,1}=3f_{3,0}+\delta_2(P^3)$; $2f_{2,2}=2f_{1,3}+f_{3,1}+\frac{1}{3}\delta_2(P^4)$ and $f_{2,2}=f_{1,3}+f_{4,0}+\frac{1}{4}\delta_3(P^4)$. Here $\delta_k \geqslant 0$.

Let P^n be a simple polytope. We have for $a=\alpha+t,\ b=\alpha t,\ z=\frac{b}{a^2}$

$$h(\alpha,t)=\sum_{i=0}^{[n/2]} g_i b^i h(\Delta^{n-2i}) = \sum_{i=0}^{[n/2]} \gamma_i b^i h(I^{n-2i}) = a^n \sum_{i=0}^{[n/2]} \gamma_i z^i = a^n \gamma(z),$$

where $g_0=1$, $g_i=h_i-h_{i-1},\ i>0$, and $\Delta^k,\ I^k$ are k-dim simplices and cubes. It gives that the vector γ is transformed into the vector g by an integer lower triangular matrix with *nonnegative* coefficients and unit diagonal. We explicitly found (see [4]) this matrix and its inverse, which has *alternating* coefficients.

Corollary 2 (see [4])**.** (1). Let $\gamma_i(P^n) \geqslant 0$ for a given simple polytope P^n. Then

$$g_i(P^n) \geqslant g_i(I^n) = \Big(1-\frac{2i}{n+1}\Big)\binom{n+1}{i}.$$

(2). The condition $\gamma_i(P^n) \geqslant 0$ is equivalent to the condition

$$(-1)^i \sum_{j=0}^{i} (-1)^j \binom{n-i-j}{i-j} g_j(P^n) \geqslant 0.$$

In the case of flag-polytopes it gives a Charney–Davis form of known Gal conjecture.

The results obtained are applied to the description of *generating series with specially chosen coefficients* of the f-, h- and γ-polynomials of the following remarkable series of simple polytopes: simplices $\Delta=\{\Delta^n\}$, cubes $I=\{I^n\}$, associahedra $As=\{As^n=K_{n+2}\}$ (Stasheff polytopes), cyclohedra $Cy=\{Cy^n\}$ (Bott–Taubes polytopes), permutohedra $Pe=\{Pe^n\}$ and stellohedra $St=\{St^n\}$. We obtained differential equations that define these series and found their solutions (see [4]).

Corollary 3 (see [4, 3])**.** (1). The generating series of h-polynomials of the families Δ, I, Pe, and St are related by $I_h Pe_h = \Delta_h St_h$.

(2). The generating series Δ_h and Pe_h coincide with the exponential of formal group laws corresponding to the fundamental two-parameter Hirzebruch genera, which give the classical Todd genus for $t=0$.

A collection B of non-empty subsets of the set $[n+1]=\{1,\dots,n+1\}$ is called a *connected building set* on $[n+1]$ if:(1) $S,S' \in B$ and $S\cap S' \neq \emptyset \Rightarrow S\cup S' \in B$; (2) $\{i\}\in B$ for all $i\in[n+1]$; (3) $[n+1]\in B$ (see [5]).

Let $l_S = (l_{S,1}, \dots, l_{S,n})$ for $S \subset [n+1]$, $S \neq [n+1]$, where

$$l_{S,i} = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{if } i \notin S, \end{cases} \quad \text{if } (n+1) \notin S, \quad l_{S,i} = \begin{cases} 0, & \text{if } i \in S, \\ -1, & \text{if } i \notin S, \end{cases} \quad \text{if } (n+1) \in S,$$

and

$$a_S = \begin{cases} \mu(S), & \text{if } (n+1) \notin S, \\ \mu(n+1) - \mu(S), & \text{if } (n+1) \in S, \end{cases}$$

where $\mu(S)$ is the number of $S_l \in B$ such that $S_l \subseteq S$.

For a connected building set B consider

$$P(B) = \{x \in \mathbb{R}^n \ : \ \langle l_S, x\rangle \geqslant -a_S,\ S \in B,\ S \neq [n+1]\}.$$

Using the results from [5] we obtain that $P(B)$ is the simple polytope and one can describe the set F_B of all facets of $P(B)$
as the set $\{F_S,\ S \in B,\ S \neq [n+1]\}$.

We will consider the set F_B as the ordered set, where $F_S < F_{S'}$, if $|S| < |S'|$ or if $|S| = |S'|$, and lexicographically $S < S'$ as subsets in $[n+1]$.

Using this ordering, let us form the $(n \times m)$-matrix $L_{P(B)}$ whose column are the vectors $l_S \in \mathbb{R}^n$, where $m = |F_B|$. We have $L_{P(B)} = (I_n, \widetilde{L})$, where I_n is the unit matrix.

Theorem 4. (1). Moment-angle manifold $\mathcal{Z}_{P(B)}$ (see [1, 2]) is a complete intersection of real quadratic hypersurfaces in $\mathbb{R}^{2m}$

$$\mathcal{F}_j = \Big\{\big((x_1,y_1),\dots,(x_m,y_m)\big) : x_j^2 + y_j^2 - \sum_{k=1}^{n} l_{j,k}(x_k^2 + y_k^2) = a_j\Big\},\ j = n+1, \dots, m,$$

where $j = j(S)$ is the number in our ordering of F_S and $l_{j,k} = l_{S,k},\ a_j = a_S$.

(2). The $(m-n)$-dim subgroup $H = \big\{(e^{2\pi\sqrt{-1}\psi_1}, \dots, e^{2\pi\sqrt{-1}\psi_m}) \in \mathbb{T}^m\big\}$, where $\psi_k = -\sum_{j=n+1}^{m} l_{j,k}\varphi_j,\ k = 1, \dots, n;\ \psi_{n+q} = \varphi_q,\ q = 1, \dots, m-n$, acts freely on $\mathcal{Z}_{P(B)}$ and the orbit space $\mathcal{Z}_{P(B)}/H$ is a toric manifold $M^{2n}(B)$.

References

[1] V. M. Buchstaber, T. E. Panov, *Torus actions and their applications in topology and combinatorics.*, AMS, University Lecture Series, v. 24, Providence, RI, 2002.

[2] V. M. Buchstaber, T. E. Panov, N. Ray, *Spaces of polytopes and cobordism of quasitoric manifolds.*, Moscow Math. J., v. 7, N 2, 2007, 219–242; arXiv: Math AT/0609346.

[3] V. M. Buchstaber, *Face-polynomials of simple polytopes and two-parameter Todd genus.*, Russian Math. Surveys, v. 63, Issue 3, 2008, 554–556.

[4] V. M. Buchstaber, *Ring of simple polytopes and differential equations.*, Proceedings of the Steklov Institute of Mathematics, v. 263, 2008, 1–25.

[5] A. Postnikov, V. Reiner, L. Williams, *Faces of generalized permutohedra.*, arXiv: math/0609184 v2 [math.CO] 18 May 2007.

Some Aspects of Symplectic Toric Geometry

Dusa McDuff

In my talk I discussed two questions in symplectic toric geometry that I have recently been thinking about. Both concern the Hamiltonian group $\mathrm{Ham}(M,\omega)$, which is the group of Hamiltonian symplectomorphisms of the closed symplectic manifold (M,ω). Recall that each element $\phi \in \mathrm{Ham}(M,\omega)$ is the time 1 map of a Hamiltonian isotopy $\phi_t^H : M \to M, t \in [0,1]$, where the path $\phi_t^H, t \in [0,1]$, starting at $\phi_0 = id$ in Ham is generated by the family of functions $H_t : M \to \mathbb{R}$ according to the following recipe: for all $(x,t) \in M \times [0,1]$, the tangent vector X_H^t to the path $s \mapsto \phi_s(x)$ at the point $\phi_t(x)$ satisfies the equation $\omega(X_H^t, \cdot) = dH_t(\cdot)$.

1. Displacing toric fibers

Consider a symplectic toric manifold (M,ω,T) with moment map $\Phi : M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the dual of the Lie algebra $\mathfrak{t}$ of the (real) torus $T = (S^1)^n$. The image $\Phi(M) = \Delta$ is the (convex) moment polytope. The inverse image of each interior point $u \in \Delta$ is a smooth Lagrangian manifold L_u, that is $\omega|_{L_u} = 0$.

Question 1. Which toric fibers L_u are displaceable? i.e. are such that $\phi(L_u) \cap L_u = \emptyset$ for some $\phi \in \mathrm{Ham}(M,\omega)$.

Example. If (M,ω) is $\mathbb{C}P^n$ with its usual toric structure, the moment polytope is the simplex. Every fiber except that over the barycenter u_0 is displaceable by some unitary transformation. But L_{u_0}, the so-called Clifford torus, is not displaceable. This was first proved by Entov–Polterovich in [1] by analytic methods, using the existence of a Calabi quasimorphism. Their argument shows that for many toric manifolds (and in particular for $M = \mathbb{C}P^n$) at least one toric fiber is nondisplaceable.

In a recent series of papers [3, 4], Fukaya-Oh–Ohta–Ono construct Floer homology groups $HF^*(L_u)$ that vanish whenever L_u is displaceable. For example, they give examples in 2-dimensions of moment polytopes with a line segment of nondisplaceable fibers: see Figure 1.

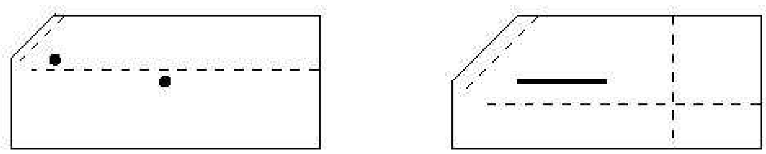

Figure 1. The dark dots and heavy lines show the points u for which the Floer homology of L_u is nonzero. The dotted lines show permissible directions of probes. It is easy to check that in these cases all the points whose Floer homology is trivial may be displaced by probes.

I am interested in the opposite problem of developing geometric methods that displace fibers. The most powerful method I have so far discovered (one that displaces all fibers that I know to be displaceable) is that of the *probe*. This is a line segment inserted into a moment polytope from the interior of a facet pointing in an integral direction, i.e. there is an integral basis for the ambient affine space containing Δ consisting of an integral vector parallel to the probe together with some other vectors parallel to the facet. It is easy to show that any point that is less than half way along such a probe is displaceable. This is enough to displace all the fibers in a two point blow up of $\mathbb{C}P^2$ for which the Floer homology groups constructed in Examples 4.7, 9.17 and 9.18 of [3] are zero.

Question 2. Some 2-dimensional polytopes have open subsets of points that cannot be displaced by a probe. Are these points really nondisplaceable? Can this be detected by some version of Floer homology? (The groups described in [3, 4] are nontrivial only on 1-dimensional subsets.)

There are many similar question in higher dimensions. In particular, Fano moment polytopes (with facets placed so that the corresponding symplectic form is in the class c_1^M) have a unique interior integral point, that algebraic geometers usually put at the origin. Entov–Polterovich gave a symplectic characterization of this point (they called it the *special point*) and showed in [2] that it cannot be displaced.

Question 3. Can every point in a Fano moment polytope except for this special point be displaced by a probe?

After discussions at the conference with Benjamin Nill, it seems that this question is closely related to the Ewald conjecture. These ideas will be discussed further in [5].

2. Polytopes with mass linear functions

This is joint work with Sue Tolman. The problem here is to understand the image of the group $\pi_1(T)$ in the fundamental group $\pi_1\big(\mathrm{Ham}(M,\omega)\big)$ of the Hamiltonian group. There is an intermediate group $\pi_1\big(\mathrm{Isom}_0(M,J_\Delta,\omega)\big)$, where $\mathrm{Isom}_0(M,J_\Delta,\omega)$ is the identity component of the group of isometries of the toric manifold (M,ω) with respect to the natural Kähler structure (J_Δ,ω) induced by thinking of (M,ω) as the quotient $\mathbb{C}^N//K$, for a suitable subtorus $K\subset T^N$. (Note: this Kähler structure is canonical (depending only on Δ) because we take N to be the number of nonempty facets of the moment polytope, i.e. there are no hidden facets.) Here are a few of our results:

- $\mathrm{Isom}_0(M,J_\Delta,\omega)$ is a maximal compact connected subgroup of $\mathrm{Ham}(M,\omega)$.
- For "generic" (M,ω), the homomorphism

$$\Psi:\pi_1(T)\to\pi_1\big(\mathrm{Ham}(M,\omega)\big)$$

is injective. (If it is noninjective, Δ either has a facet that meets all the others or it has a flat facet, i.e. a facet F with a product neighborhood $F\times I$.)

• Ψ has finite image if and only if (M, ω) is a product of projective spaces with the product toric structure.

• The kernel of the homomorphism $\pi_1(T) \to \pi_1(\mathrm{Isom}_0(M, J_\Delta, \omega))$ can be described by an explicit formula depending only on Δ.

• If $\dim M \leq 6$ then the homomorphism $\pi_1(\mathrm{Isom}_0(M, J_\Delta, \omega)) \to \pi_1\big(\mathrm{Ham}(M, \omega)\big)$ is injective provided that M is not a $\mathbb{C}P^2$-bundle over $\mathbb{C}P^1$. In a subsequent paper we classify all 8-dimensional manifolds for which this homomorphism is not injective.

Our analysis is based on studying *mass linear* functions, i.e. linear functions on the polytope Δ whose value at the center of gravity of Δ depends linearly on the positions of the facets. For more details see [6].

References

[1] M. Entov and L. Polterovich, Quasi-states and symplectic intersections, arXiv:math/0410338, *Comment. Math. Helv.* **81** (2006), 75–99.
[2] M. Entov and L. Polterovich, Rigid subsets of symplectic manifolds, arXiv:math/0704.0105.
[3] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, Hamiltonian Floer theory on compact toric manifolds I, arXiv-math:0802.1703.
[4] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, Hamiltonian Floer theory on compact toric manifolds II, Bulk deformation, arXiv-math:0810.5654.
[5] D. McDuff, Displacing Lagrangian toric fibers via probes, in preparation.
[6] D. McDuff and S. Tolman, Polytopes with mass linear functions, part I, arXiv:0807.0900.

Vanishing Theorems for Log Homogeneous Varieties

Michel Brion

This talk is based on the preprint [5]. The main motivation comes from the well-developed theory of complete intersections in toric varieties. In particular, the Hodge numbers of these complete intersections were determined by Danilov and Khovanskii, and their Hodge structure, by Batyrev, Cox and others (see [7, 2, 13]). This is made possible by the special features of toric geometry; two key ingredients are the triviality of the logarithmic tangent bundle $T_X(-\log D)$, where X is a complete nonsingular toric variety with boundary D, and the Bott–Danilov–Steenbrink vanishing theorem for Dolbeault cohomology: $H^i(X, L \otimes \Omega^j_X) = 0$ for any ample line bundle L on X and any $i \geq 1$, $j \geq 0$.

A natural problem is to generalise this theory to complete intersections in algebraic homogeneous spaces and their equivariant compactifications. As a first observation, the preceding two results also hold for abelian varieties and, more generally, for the "semi-abelic" varieties of Alexeev (see [1]), that is, toric fibrations over abelian varieties. In fact, for a complete nonsingular variety X and a divisor D with normal crossings on X, the triviality of $T_X(-\log D)$ is equivalent to X being semi-abelic with boundary D, by a result of Winkelmann (see [16]). Moreover, it is easy to see that semi-abelic varieties satisfy Bott vanishing.

The next case to consider after these "log parallelisable varieties" should be that of flag varieties. Here counter-examples to Bott vanishing exist for grassmannians and quadrics, as shown by work of Snow (see [15]). For example, any smooth quadric hypersurface X in $\mathbb{P}^{2m}$ satisfies $H^{m-1}\big(X, \Omega_X^m(1)\big) \neq 0$.

However, a vanishing theorem due to Broer asserts that $H^i(X, L \otimes \Omega_X^j) = 0$ for any nef line bundle L on a flag variety X, and all $i > j$ (see [6]). This also holds for nef line bundles on complete simplicial toric varieties, in view of a recent result of Mavlyutov (see [14]).

We have obtained generalisations of Broer's vanishing theorem to any "log homogeneous" variety, that is, to a complete nonsingular variety X having a divisor with normal crossings D such that $T_X(-\log D)$ is generated by its global sections. Then X contains only finitely many orbits of the connected automorphism group $\mathrm{Aut}^0(X, D)$, and these are the strata defined by D. The class of log homogeneous varieties, introduced and studied in [4], contains of course the log parallelisable varieties, and also the "wonderful (symmetric) varieties" of De Concini–Procesi and Luna (see [8, 12]). Log homogeneous varieties are closely related to spherical varieties; in particular, every spherical homogeneous space has a log homogeneous equivariant compactification (see [3]).

Our main result asserts that $H^i(X, L \otimes \Omega_X^j) = 0$ *for any nef (resp. ample) line bundle L on a log homogeneous variety X, and for any $i > j + q + r$ (resp. $i > j$).* Here q denotes the irregularity of X, i.e., the dimension of the Albanese variety, and r its rank, i.e., the codimension of any closed stratum (these are all isomorphic). Thus, $q + r = 0$ if and only if X is a flag variety; then we recover Broer's vanishing theorem.

We deduce our main result from the vanishing of the logarithmic Dolbeault cohomology groups, $H^i\big(X, L^{-1} \otimes \Omega_X^j(\log D)\big)$, for L nef and $i < j - c$, where $c \leq q + r$ is an explicit function of (X, D, L). The latter statement is optimal, but we do not know whether $H^i(X, L \otimes \Omega_X^j) = 0$ for L nef on log homogeneous X, and $i > j$.

Since the proof of these results is somewhat indirect, we first present it in the setting of flag varieties, and then sketch how to adapt it to log homogeneous varieties. For a flag variety $X = G/P$, the tangent bundle T_X is the quotient of the trivial bundle $X \times \mathfrak{g}$ (where $\mathfrak{g}$ denotes the Lie algebra of G) by the sub-bundle R_X of isotropy Lie subalgebras. Via a homological argument of "Koszul duality", Broer's vanishing theorem is equivalent to the assertion that $H^i(R_X, p^*L) = 0$ for all $i \geq 1$, where $p : R_X \to X$ denotes the structure map. But one checks that the canonical bundle of the nonsingular variety R_X is trivial, and the projection $f : R_X \to \mathfrak{g}$ is proper, surjective and generically finite. So the desired vanishing follows from the Grauert–Riemenschneider theorem.

For an arbitrary log homogeneous variety X with boundary D, we consider the algebraic group $G := \mathrm{Aut}^0(X, D)$, with Lie algebra $\mathfrak{g} := H^0\big(X, T_X(-\log D)\big)$. We may still define the "bundle of isotropy Lie subalgebras" R_X as the kernel of the (surjective) evaluation map from the trivial bundle $X \times \mathfrak{g}$ to $T_X(-\log D)$, and the resulting map $f : R_X \to \mathfrak{g}$. If G is linear, we show that the connected components of the general fibres of f are toric varieties of dimension $\leq r$. Moreover, any nef line bundle L on X is generated by its global sections. By a generalisation of the Grauert–Riemenschneider theorem due to Kollár (see [9, Cor. 6.11]), together with a vanishing theorem of Fujino for toric varieties (see [10]), it follows that $H^i(R_X, p^*L \otimes \omega_{R_X}) = 0$ for any $i > r$. Via homological duality arguments again, this is equivalent to the vanishing of $H^i\big(X, L^{-1} \otimes \Omega^j_X(\log D)\big)$ for any such L, and all $i < j - r$. In turn, this easily yields our main result, under the assumption that G is linear.

The case of an arbitrary algebraic group G may be reduced to the preceding setting, in view of some remarkable properties of the Albanese morphism of X: this is a homogeneous fibration, which induces a splitting of the logarithmic tangent bundle, and a decomposition of the ample cone.

The geometry of the morphism $f : R_X \to \mathfrak{g}$ bears a close analogy with that of the moment map $\phi : \Omega^1_X \to \mathfrak{g}^*$, studied in depth by Knop for a variety X equipped with an action of a connected reductive group G (see [11]). Our construction differs from that of Knop in general, but they coincide in the case where X is a $G \times G$-equivariant compactification of a connected reductive group G: one may then identify R_X with $\Omega^1_X(\log D)$, and f with the compactified moment map. As applications, we obtain very simple descriptions of the algebra of differential operators on X which preserve D, and of the bi-graded algebra $H^\bullet\big(X, \Omega^\bullet_X(\log D)\big)$. The structure of the latter algebra also follows from Deligne's description of the mixed Hodge structure on the cohomology of G, while the former seems to be new. It would be very interesting to describe these algebras for larger classes of log homogeneous varieties.

References

[1] V. Alexeev, *Complete moduli in the presence of semiabelian group actions*, Ann. of Math. (2) **155** (2002), no. 3, 611–708.

[2] V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), no. 2, 293–338.

[3] F. Bien and M. Brion, *Automorphisms and local rigidity of regular varieties*, Compositio Math. **104** (1996), 1–26.

[4] M. Brion, *Log homogeneous varieties*, Actas del XVI Coloquio Latinoamericano de Álgebra, 1–39, Revista Matemática Iberoamericana, Madrid, 2007; arXiv: math/0609669.

[5] M. Brion, *Vanishing theorems for Dolbeault cohomology of log homogeneous varieties*, arXiv:0812.2658.

[6] A. Broer, *Line bundles on the cotangent bundle of the flag variety*, Invent. math. **113** (1993), no. 1, 1–20.

[7] V. I. Danilov and A. G. Khovanskii, *Newton polyhedra and an algorithm for calculating Hodge–Deligne numbers*, Math. USSR-Izv. **29** (1987), no. 2, 279–298.

[8] C. de Concini and C. Procesi, *Complete Symmetric Varieties*, Lect. Notes in Math., Springer, **996** (1983), 1–44.

[9] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, DMV Seminar **20**, Birkhäuser, Basel, 1992.

[10] O. Fujino, *Multiplication map and vanishing theorems for toric varieties*, Math. Z. **257** (2007), no. 3, 631–641.

[11] F. Knop, *A Harish-Chandra homomorphism for reductive group actions*, Ann. of Math. (2) **140** (1994), no. 2, 253–288.

[12] D. Luna, *Variétés sphériques de type A*, Pub. Math. IHÉS **94** (2001), 161–226.
[13] A. V. Mavlyutov, *Cohomology of complete intersections in toric varieties*, Pacific J. Math. **191** (1999), no. 1, 133–144.
[14] A. V. Mavlyutov, *Cohomology of rational forms and a vanishing theorem for toric varieties*, J. reine angew. Math. **615** (2008), 45–58.
[15] D. Snow, *Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces*, Math. Ann. **276** (1986), no. 1, 159–176.
[16] J. Winkelmann, *On manifolds with trivial logarithmic tangent bundle*, Osaka J. Math. **41** (2004), no. 2, 473–484.

Flag Varieties and Gelfand-Zetlin Polytopes

Valentina Kiritchenko

I describe a relation between geometry of complete flag varieties and combinatorics of Gelfand–Zetlin polytopes. This is similar to the rich interplay between toric varieties and their Newton polytopes. For motivation, I will first recall some well-known results for toric varieties and then outline their partial extension to the setting where a toric variety is replaced by a *regular* compactification of an arbitrary reductive group.

Let X be a smooth complex toric variety of dimension n, and D a very ample divisor on X. Recall that with a pair (X, D) one can associate a convex lattice polytope $P_D \subset \mathbb{R}^n$ called the Newton polytope of X (e.g. P_D can be defined as the convex hull of all Laurent monomials occurring in the defining equation of D). Many geometric invariants of X can be computed explicitly in terms of the polytope P_D (see the list below). One of the key ingredients in such computations is a one-to-one correspondence between G–orbits in X and faces of P_D. This correspondence preserves dimensions and incidence relations.

- The self-intersection index D^n of the divisor D is equal to $n!$ times the volume of P_D [7].
- The Picard group of X is isomorphic to the group of virtual lattice polytopes analogous to P_D (i.e. having the same normal fan) modulo parallel translations.
- The Euler characteristic $\chi(D_1 \cap \ldots \cap D_m)$ of a complete intersection of hypersurfaces can be computed explicitly for any $m \leq n$ [7].
- There is an explicit description of the cohomology ring $H^*(X)$ by generators and relations [4]. In particular, there is the following formula for the intersection product of the divisor D with the G–orbit $\mathcal{O}_\Gamma$ corresponding to a face Γ.
$$D\overline{\mathcal{O}}_\Gamma = \sum_{\Delta \subset \Gamma} d(v, \Delta)\overline{\mathcal{O}}_\Delta,$$
where the sum is taken over the facets Δ of Γ. Here $v \in \Gamma \cap \mathbb{Z}^n \subset \mathbb{R}^n$ is any point with integer coordinates, and $d(v, \Delta)$ denotes the integral distance from v to the face Δ.

Consider now a more general case. Let G be an arbitrary connected complex reductive group of dimension n. Note that the left and right actions of G on itself are in general different so it makes sense to consider the action by the doubled group $G \times G$. Let X be a $G \times G$*–equivariant* compactification of G, that is, the group $G \times G$ acts on X with the open dense orbit isomorphic to G and on this orbit the action coincides with the action by left and right multiplications. As in the toric case, X will always consists of a finite number of $G \times G$–orbits. One way to construct such a compactification is to take a projectively faithful representation $\pi : G \to \mathrm{End}(V)$. Then the closure X_π of $\mathbb{P}(\pi(G))$ in the projective space $\mathbb{P}(\mathrm{End}(V))$ is a $G \times G$ equivariant compactification of G. In particular, when G is a complex torus all projective toric varieties can be obtained in this way.

An important class of $G \times G$–equivariant compactifications consists of *regular* compactifications introduced in [3]. These are the closest relatives of smooth toric varieties. In particular, the closures of all $G \times G$–orbits in a regular compactification are smooth and intersect each other transversally. Regular compactifications include all smooth toric varieties and wonderful compactifications of semisimple groups of adjoint type.

As in the toric case, with each very ample divisor D one can associate a convex lattice polytope $P_D \subset \mathbb{R}^k$. Here k is the rank of G, that is, the dimension of a maximal torus, and $\mathbb{Z}^k \subset \mathbb{R}^k$ is identified with the weight lattice of G. E.g. when $X = X_\pi$ and D is the divisor of hyperplane section then P_D is the weight polytope of π. There is a one-to-one correspondence between $G \times G$–orbits in X and orbits of the Weyl group of G acting on the faces of P_D. This correspondence preserves codimensions and incidence relations. In particular, vertices of P_D correspond to the closed orbits in X, which have dimension $n - k$ and are isomorphic to the product $G/B \times G/B$ of two flag varieties. Again there is a strong relation between geometry of X and combinatorics of P_D.

- Fix a fundamental Weyl chamber $\mathcal{D} \subset \mathbb{R}^k$. Then
$$D^n = n! \int_{P_D \cap \mathcal{D}} F(x)dx,$$
where F is a homogeneous polynomial function on $\mathbb{R}^k$ of degree $n-k$ that depends only on the group G and not on X and D [5, 2]. In particular, if G is a complex torus, then $F \equiv 1$.
- The Picard group of X is isomorphic to the group of virtual lattice polytopes analogous to P_D and invariant under the action of the Weyl group modulo parallel translations.
- The Euler characteristic $\chi(D_1 \cap \ldots \cap D_m)$ of a complete intersection of hypersurfaces can be computed explicitly for any $m \leq n$ [8, 9].

However, no description of $H^*(X)$ by generators and relations is known. In order to obtain such a description it might be useful to consider a bigger polytope $\widetilde{P}_D \subset \mathbb{R}^n$ that fibers over $P_D \cap \mathcal{D}$ with fibers equal to the product of two Gelfand-Zetlin polytopes. Such a polytope has been recently constructed in a much more general setting [6]. The bigger polytope $\widetilde{P}_D$ contains more information about the variety X. In particular, the self-intersection index D^n is equal to $n!$ times the volume of $\widetilde{P}_D$, exactly as in the toric case.

In a sense, a regular compactification X is made up of a toric variety (corresponding to the smaller polytope P_D) and the product of two flag varieties (corresponding to the product of two Gelfand-Zetlin polytopes). I hope that the relation between flag varieties and Gelfand-Zetlin polytopes will help to get new insights into geometry of regular compactifications of reductive groups.

I will now come to the main object of my talk. Let G be the group $GL_n(\mathbb{C})$, and $X = G/B$ the complete flag variety for G. Recall that with each strictly dominant weight λ of G one can associate the Gelfand-Zetlin polytope Q_λ so that the integral points inside and at the boundary of Q_λ parameterize a natural basis in the irreducible representation of G with the highest weight λ. The Gelfand-Zetlin polytope Q_λ is a convex polytope in $\mathbb{R}^d$ with vertices lying in the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Here $d = n(n-1)/2$ denotes the dimension of X.

I have constructed a correspondence between the Schubert cycles in X and some special faces of the Gelfand-Zetlin polytope [10]. Namely, an l-dimensional face Γ of the Gelfand-Zetlin polytope is assigned to each l-dimensional Schubert cycle Z using *Demazure modules* for a Borel subgroup in G. There are some degrees of freedom in the construction, namely, the same Schubert cycle can be represented by different faces (different choices of a face correspond to different choices of a Borel subgroup containing a given maximal torus). For some Schubert cycles, it is possible to choose a face Γ so that combinatorics of Γ captures geometry of D very well (let us call such faces *admissible*). In particular, admissible faces behave well with respect to the incidence relation between Schubert cycles. Then the classical Chevalley formula [BGG] for the intersection product of Z with the divisor D_λ on X corresponding to the weight λ has the following interpretation in terms of an admissible face Γ.

$$D_\lambda Z_\Gamma = \sum_{\Delta \subset \Gamma} d(v, \Delta) Z_\Delta,$$

where the sum is taken over the facets Δ of Γ (these correspond to the Schubert cycles Z_Δ of codimension one at the boundary of Z_Γ). Here v is a fixed vertex of the face Γ. Note that in this form the formula is completely analogous to the formula for toric varieties mentioned above and to the analogous formula for regular compactifications of reductive groups [9].

Many Schubert cycles can be represented by an admissible face, but not all of them. In particular, all Schubert cycles that degenerate to a single toric variety under the Caldero's construction [11] of toric degenerations of flag varieties can be represented by admissible faces. However, there are many other examples of Schubert cycles represented by admissible faces. E.g. for the flag variety of GL_3 all Schubert cycles can be represented by admissible faces (although one of the 2-dimensional Schubert cycles degenerates into the union of two toric subvarieties under the Caldero's construction). For GL_4, exactly two Schubert cycles can not be represented by an admissible face. These two cycles are the homology classes of Schubert cells whose closures in the flag variety are not smooth. I conjecture that all Schubert cycles defined by Schubert cells with smooth closures can be represented by an admissible face. In our joint work with Evgeny Smirnov, we are currently proving this conjecture.

References

[1] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, *Schubert cells, and the cohomology of the spaces G/P,* Russian Math. Surveys **28** (1973), no. 3, 1–26

[2] Michel Brion, *Groupe de Picard et nombres caracteristiques des varietes spheriques*, Duke Math J. **58** (1989), no.2 , 397-/424

[3] C. De Concini and C. Procesi, *Complete symmetric varieties II Intersection theory,* Advanced Studies in Pure Mathematics **6** (1985), Algebraic groups and related topics, 481–513

[4] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** no.2 (1978), 97–154

[5] B.Ya. Kazarnovskii, *Newton polyhedra and the Bezout formula for matrix-valued functions of finite-dimensional representations,* Functional Anal. Appl. **21** (1987), no. 4, 319–321

[6] Kiumars Kaveh, Askold Khovanskii, *Convex bodies and algebraic equations on affine varieties*, preprint arXiv:0804.4095v1[math.AG]

[7] A.G. Khovanskii, *Newton polyhedra, and the genus of complete intersections,* Functional Anal. Appl. **12** (1978), no. 1, 38–46

[8] Valentina Kiritchenko, *Chern classes for reductive groups and an adjunction formula*, Annales de l'Institut Fourier, **56** (2006), no. 3, 1225-1256

[9] Valentina Kiritchenko, *On intersection indices of subvarieties in reductive groups*, Moscow Mathematical Journal, **7** no.3 (2007), 489-505

[10] Valentina Kiritchenko, *Gelfand-Zetlin polytopes and geometry of flag varieties*, preprint http://guests.mpim-bonn.mpg.de/kirichen/prints.html

[11] Mikhail Kogan, Ezra Miller, *Toric degeneration of Schubert varieties and Gelfand-Tsetlin polytopes,* Adv. Math. **193** (2005), no. 1, 1–17

Classifying Polytopes via Toric Fibrations

Sandra Di Rocco

(joint work with A. Dickenstein and R. Piene)

A toric fibration, $f : X \to Y$ is a proper surjective map, with connected positive dimensional fibers, between a normal toric variety X and a normal variety Y.

A polarized toric fibration is a a toric fibration $f : X \to Y$ together with an ample equivariant line bundle, L, on X.

Standard examples of *toric polarized fibrations* are given by projective bundles. Let $L_0, \dots, L_k$ be ample equivariant line bundles on a toric variety X and let ξ be the tautological line bundle of the vector bundle $E = L_0 \oplus \dots \oplus L_k$. The projection $\pi : \mathbb{P}(E) \to X$ with the ample line bundle ξ_E is a toric polarized fibration. The polytope associated to the embedding $(\mathbb{P}(E), s\xi)$ is an example of a *Cayley polytopes.*

Given lattice polytopes $P_0, ..., P_k \subset \mathbb{R}^m$, let $e_1, ..., e_k$ be a basis of $\mathbb{Z}^k$ and $e_0 = 0$. The Cayley sum of $P_0, \dots, P_k$ of order s is defined as

$$[P_0 \star \dots \star P_k]^s = Conv((se_i, P_i)_{i=0,\dots,k}) \subset \mathbb{R}^{m+k}.$$

When the polytopes are strongly combinatorially equivalent, defining the inner normal fan Σ, the Cayley sum is called a *strict Cayley polytope* and it is denoted by $Cayley^s_\Sigma(P_0, \dots, P_k)$.

The polytope $\mathrm{Cayley}^s_\Sigma(P_0, \dots, P_k)$ defines the polarized toric fibration $(\mathbb{P}(L_0 \oplus \dots, L_k), s\xi)$ where L_i is the ample line bundle on the toric variety $X(\Sigma)$, given by the fan Σ, defined by the polytope P_i, see e.g. [2] for details.

This class of polytopes is particularly interesting because it can be described by imposing combinatorial and geometrical constraints.

If $P \subset \mathbb{R}^n$ is a smooth maximal dimensional polytope then the following statements are equivalent:

(1) There is a $\delta > 0$ and strictly combinatorially equivalent lattice polytopes $P_0, \dots, P_{\frac{n+\delta}{2}}$ defining the normal fan Σ such that P is affinely equivalent to $\mathrm{Cayley}^1_\Sigma(P_0, \dots, P_{\frac{n+\delta}{2}})$.
(2) The smooth toric embedding defined by P has dual defect δ, i.e. the dual variety has codimension equal to $\delta + 1$.
(3) P is a smooth $\mathbb{Q}$-normal polytope with $\mathrm{codeg}(P) \geq \frac{n+3}{2}$.
(4) $\sum_{faces\, \emptyset \neq F \subset P} (-1)^{codim(F)} (\dim(F) + 1)! Vol(F) = 0$.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) is proven in [3].
(1) $\Leftrightarrow$ (3) is the main result in [2].

In order to understand the equivalence (1) $\Leftrightarrow$ (3) the notions of codegree and $\mathbb{Q}$-normality need to be defined.

For a lattice polytope $P \subset \mathbb{R}^n$ the codegree is defined as

$$\deg(P) = n + 1 - \mathrm{codeg}(P), \text{ where } \mathrm{codeg}(P) = min_{\mathbb{Z}}\{t, (tP \cap \mathbb{Z}^n)^{int} \neq \emptyset\}.$$

In [1] Batyrev and Nill classified polytopes of degree 0 and 1 and found that all have the structure of a Cayley sum. This motivated them to pose the following question:

Given a positive integer d, does there exist an integer $N(d)$, depending on d, such that every polytope of degree d and dimension $\geq N(d)$ is affinely equivalent to a Cayley sum?

A first answer to this question was given in [4] where it is shown that $N(d) \leq \frac{d^2+19d-4}{2}$. The equivalence (1) $\Leftrightarrow$ (3) above shows that if P is smooth and $\mathbb{Q}$-normal then $N(d) = 2d + 1$ and the Cayley structure is strict.

The notion of $\mathbb{Q}$-normality is dictated by the geometrical methods used in the proof. Because strict Cayley polytopes with large fibers are in fact $\mathbb{Q}$-normal, as showed in [2], this assumption is in fact quite natural.

If $P = \cap H^+_{\rho_i, -a_i}$ then "by going r steps inwards" one defines $P^r = \cap H^+_{\rho_i, -a_i + r}$. We generalize the notion of codegree as

$$\mathrm{codeg}_{\mathbb{Q}}(P) = inf_{\mathbb{Q}}\{\frac{a}{b}, (aP)^b \neq \emptyset\}.$$

Moreover we say that a smooth polytope P is *r-spanned* at a vertex $m = \cap_1^n H_{\rho_i, -a_i}$ if the point $m(r) = \cap_1^n H_{\rho_i, -a_i - r} \in P^r$. We say that p is r-spanned if it is r-spanned at every vertex. The *nef value* of P is defined as

$$\tau(P) = inf_{\mathbb{Q}}\{\frac{a}{b}, aP \text{ is } b\text{-spanned}\}.$$

Finally we say that a smooth polytope P is $\mathbb{Q}$-normal if $\tau(P) = \mathrm{codeg}_{\mathbb{Q}}(P)$.

The proof of (1) $\Leftrightarrow$ (3) is mainly algebro-geometrical and can be outlined as follows:

- Let (X, L) be the polarized variety associated to the polytope P. We first show that $\tau(P) = \tau(L) = \inf_{\mathbb{R}}\{\tau, K_X + \tau L \text{ is nef }\}$.
- Using the description of $Pic(\mathbb{P}(L_0 \oplus \ldots, L_k))$ we prove that
$$\tau(\mathrm{Cayley}^1_\Sigma(P_0, \ldots, P_{\frac{n+\delta}{2}})) = \mathrm{codeg}_{\mathbb{Q}}(\mathrm{Cayley}^1_\Sigma(P_0, \ldots, P_{\frac{n+\delta}{2}})) = \frac{k+1}{s}$$
when $\dim(P_i) \geq \frac{k+1}{s}$ and the P_i are combinatorially equivalent. This proves (1) $\Rightarrow$ (3).
- Let $\phi_\tau : X \to Y$ be the nef-value morphism. Using the hypothesis that P is $\mathbb{Q}$-normal we prove that this morphism is not birational.
- Finally the hypothesis $\tau \geq \frac{n+1}{2}$ implies that this is a flat morphism, with connected fibers and in fact a contraction of an extremal ray of the Mori cone, which gives X the structure of a fibration, with all fibers isomorphic to linear projective spaces. This proves (3) $\Rightarrow$ (1).

References

[1] Victor V. Batyrev and Benjamin Nill. *Multiples of lattice polytopes without interior lattice points.* Moscow Math. J., 7(2):195–207, 2007.

[2] A. Dickenstein, S. Di Rocco and R. Piene *Classifying polytopes via toric fibrations*, preprint, 2008. arXiv 0809.3136v1.

[3] Sandra Di Rocco. *Projective duality of toric manifolds and defect polytopes.* Proc. London Math. Soc. (3), 93(1):85–104, 2006.

[4] Christian Haase, Benjamin Nill, and Sam Payne. *Cayley decompositions of lattice polytopes and upper bounds for h^*-polynomials.* Preprint, arXiv:0804.3667, 2008. To appear in Journal für die reine und angewandte Mathematik.

Hamiltonian Torus Actions on Generalized Complex Manifolds

Yi Lin

Generalized complex geometry was introduced by Hitchin [H02] and was further developed by his students Gualtieri and Cavalcanti. On the one hand, it unifies both symplectic and complex geometries, and so is well suited to the study of phenomena related to both, e.g., mirror symmetry. On the other hand, it provides a natural geometric framework for other structures and physical models of interest to string physicists.

Recently, there have been considerable interests in extending the techniques of reduction and quotient construction in symplectic and Poisson geometries into the realm of generalized complex geometry, cf. [BCG05], [LT05], [Hu05], [SX05] and [Va05]. In collaboration with Susan Tolman, I [LT05] extended the notion of

Hamiltonian actions and Marsden-Weinstein reduction in symplectic geometry to the realm of generalized complex and generalized Kähler geometries.

In the presence of a Lie group G acting on an η-twisted generalized complex manifold M, our construction involves a generalized moment map $\mu : M \to \mathfrak{g}^*$ and a moment 1-form $\alpha \in (\Omega^1(M) \otimes \mathfrak{g}^*)^G$ for which $\eta + \alpha$ is an equivariantly closed 3-form in the Cartan model; moreover, it induces naturally an extended action of G on $TM \oplus T^*M$. As a first application, we worked out very simple explicit construction of bi-Hermitian structures on CP^n, the blown-up of CP^n at arbitrarily many points, Hirzebruch surfaces, many other toric surfaces, and complex Grassmannians.

As Hamiltonian symplectic geometry is the mathematical theory underlying classical mechanics, Hamiltonian actions on generalized complex manifolds have also been found to be something very natural in physics. In particular, it has been shown by Kapustin and Tomasiello [KT06] that our mathematical notion of Hamiltonian actions on generalized Kähler manifolds corresponds exactly to the physical notion of general $(2, 2)$ gauged sigma models with three form fluxes.

To encode the cohomological information of extended group actions on $TM \oplus T^*M$, it is not surprising that we need some generalization of the usual equivariant cohomology theory. For the Hamiltonian action of a group G on a generalized complex manifold M, I [L07b] observed that the twisted equivariant cohomology $H_G(M, \eta + \alpha)$, obtained from the usual equivariant cohomology twisted by the three form $\eta + \alpha$, is the right equivariant cohomology theory which carries information for the extended action of G on $TM \oplus T^*M$. As an application of this equivariant cohomology theory, I [L07b] generalized the Duistermaat-Heckman theorem in symplectic geometry to generalized Calabi-Yau manifolds in the sense of Hitchin [H02]. On the other hand, using the generalized Hodge theory I proved a stronger version of the Equivariant formality theorem for Hamiltonian actions on generalized Kähler manifolds.

Y. Nitta [NY07b] proved in a recent interesting work that the generalized moment map for a compact Hamiltonian generalized complex T-space has very nice Morse-Bott properties [1]. Building on Nitta's result, T. Baird and I [BL08] established the Equivariant formality, Kirwan injectivity and surjectivity results for Hamiltonian torus actions on compact twisted generalized complex manifolds. More specifically, consider the Hamiltonian action of a torus T on a compact η-twisted generalized complex manifold with a generalized moment map $\mu : M \to \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^T(M) \otimes \mathfrak{t}^*$. The following results were proved in [BL08][2].

1) (**Equivariant formality**) The twisted equivariant cohomology $H_T(M; \eta + \alpha)$ is isomorphic to $H(M; \eta) \otimes S\mathfrak{t}^*$, where $H(M, \eta)$ is the usual de Rham cohomology group twisted by the closed three form η.

[1] Using this result, Nitta himself generalized the Atiyah-Guillemin-Sternberg convexity theorem to the setting of Hamiltonian generalized complex torus actions.

[2] Indeed, we proved these results in a slightly more general setting which for instance includes the Kirwan injectivity and surjectivity for ordinary equivariant cohomology as special cases.

2) (**Kirwan injectivity**) The localization map

$$i^* : H_T(M; \eta + \alpha) \to H_T(M^T; \eta + \alpha)$$

is an injection, where i^* is induced by the inclusion map $i : M^T \to M$ from the fixed point set M^T to M.

3) (**Kirwan Surjectivity**) For a regular value $a \in \mathfrak{t}^*$ of μ, the Kirwan map

$$H_T(M; \eta + \alpha) \to H(\mu^{-1}(a); \eta + \alpha)$$

is a surjection.

The Equivariant formality and the Kirwan package plays an important role in equivariant symplectic geometry. Their generalized complex analogues are also essential of our understanding of Hamiltonian torus actions in generalized complex geometry. For instance, as a first application of these results, T. Baird and I [BL08b] proved that if there is an effective Hamiltonian action of a k dimensional torus on an η-twisted generalized complex manifold and if the closed three form η is cohomologically non-trivial, then $k \leq n - 2$.

The notion of Hamiltonian generalized complex manifolds introduced in [LT05] is a direct generalization of that of Hamiltonian symplectic manifolds. It is a very natural question to ask to which extent Hamiltonian generalized complex manifolds are more general than Hamiltonian symplectic manifolds. To date, there are two different constructions of compact examples of Hamiltonian torus actions on (non-symplectic) generalized complex manifolds.

Resorting to the minimal coupling construction, I constructed first such examples in [L07b]. In the simplest case, suppose that we have the following initial data:

- a generalized complex manifold M;
- the sphere S^2 equipped with the area form and the action of S^1 by rotation.

Applying this construction, we get a fibration $X \to M$ with fiber S^2 together with a generalized complex structure on the total space X. The action of S^1 on each fiber S^2 extends to a global action on X which is Hamiltonian in the sense of [LT05].

More recently, Baird and I [BL08b] found a large class of interesting new examples of Hamiltonian generalized complex torus actions. For instance, by using a surgery technique, out of any given Hamiltonian symplectic T^n-space M^{2n} we obtain a Hamiltonian generalized complex T^{n-2}-space X^{2n} which has a cohomologically non-trivial twisting. It will be very interesting to use these examples to study the similarities and differences between Hamiltonian symplectic torus actions and Hamiltonian generalized complex ones.

References

[BCG05] H. Bursztyn, G. Cavalcanti, and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Adv. in Math. 211 (2007), no.2, 726–765.

[BL08] Tom Baird, Yi Lin, *Topology of generalized complex quotients*, 33 pages, Preprint, math.DG/0802.1341.

[BL08b] Tom Baird, Yi Lin, *Generalized complex Hamiltonian torus actions: basic properties and new examples*, work in preparation, 2009.
[Gua04] Marco Gualtieri, *Generalized complex geometry*, Dphil thesis, University of Oxford, 2005, math.DG/0411221.
[H02] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q.J. Math., **54** (3) 281-308, 2003
[Hu05] S. Hu, *Hamiltonian symmetries and reduction in generalized geometry*, to appear in Houston J. Maths, arxiv: math.DG/0509060.
[KT06] Anton Kapustin, Alessandro Tomasiello, *The general* $(2,2)$ *gauged sigma model with three-form flux*, Journal of High Energy, 11 (2007) 053 doi: 10.1088/1126-6708/2007/11/053, hep-th/0610210.
[LT05] Yi Lin, Susan Tolman, *Symmetries in generalized Kähler geometry*, Comm. in. Math. Physics 268 (2006) no. 1, 199-222.
[L07b] Yi Lin, *The Equivariant cohomology theory of twisted generalized complex manifolds* , Comm. in Math. Physics, 281 (2008) 469 - 497, math.DG/0704.2804.
[NY06] Yasufumi Nitta, *Reduction of generalized Calabi-Yau structures*, J. Math. Soc. Japan, Vol 59, no. 4 (2007), also available on math.DG/0611341.
[NY07b] Yasufumi Nitta, *Convexity properties for generalized moment maps I*, Preprint, math.DG/0710.3924.
[SX05] M. Stiénon and X. Ping, *Reduction of generalized complex structures*, J. Geom. Phys. 58 (2008), 105-121, math.DG/0509393.
[Va05] Izu, Vaisman, *Reduction and submanifolds of generalized complex manifolds*, Differential Geometry and its Applications, Volume 25, Issue 2, April 2007, Pages 147-166, math.DG/0511013.

Tropicalization of Subvarieties of Toric Varieties and Nonarchimedean Analytification

Sam Payne

In the first half of this talk I presented a framework for understanding tropicalizations of toric varieties as orbit spaces under nonarchimedean analogues of compact tori. In this setup, the tropicalization of a subvariety of a toric variety is just its image in this orbit space.

In the second half of this talk I discussed nonarchimedean analytification following Baker's account [1] of Berkovich's work [2], and its relation to tropicalizations of closed embeddings in toric varieties. All of the results and proofs presented work over any algebraically closed field with a nontrivial valuation, but I restricted to the case of a specific field of generalized power series for simplicity.

Let $K = \mathbb{C}((t^{\mathbb{R}}))$ be the field of generalized power series with complex coefficients and real exponents. A generalized power series $a \in K$ is a formal sum $a = \sum_{i \in \mathbb{R}} a_i t^i$ such that the support $\mathrm{supp}(a) = \{i \in \mathbb{R} \mid a_i \neq 0\}$ is well-ordered. Let $\nu : K^* \to \mathbb{R}$ be the valuation taking a generalized power series to its leading exponent, the minimum of its support set. Associated to this valuation we have a nonarchimedean norm $|\ |$ on K given by

$$|a| = \exp(-\nu(a)).$$

Inside of K, we have the valuation subring $R = \{a \mid \nu(a) \geq 0\}$, and the multiplicative group of invertible elements of R is $R^* = \{a \mid \nu(a) = 0\}$.

Since K and $\mathbb{C}$ are uncountable algebraically closed fields of characteristic zero, algebraic geometry over K and $\mathbb{C}$ are extremely similar. However, the metric geometry over K with respect to the nonarchimedean norm above feels like a world apart from the usual metric geometry over $\mathbb{C}$. Comparing these two geometries suggests the following analogy.

Nonarchimedean	Archimedean
K	$\mathbb{C}$
K^*	$\mathbb{C}^*$
ν	$-\log\mid\ \mid$
R	D^2
R^*	S^1

Here, D^2 is the unit disc of complex numbers of absolute value at most one, and S^1 is the unit circle of complex numbers of absolute value exactly one. In particular, $(R^*)^n$ may be thought of as analogous to the maximal compact torus in the algebraic group $(K^*)^n$. Given an algebraic variety X over K with an action of $(K^*)^n$ it is then natural to consider the geometry of the orbit space $X/(R^*)^n$. Tropical geometry may be thought of as the geometry of such orbit spaces of toric varieties, as follows.

Let X be the toric variety over K associated to a fan Δ in $\mathbb{R}^n$.

Example 1. Suppose X is the one-dimensional torus K^*. Two points in K^* differ by an element of R^* if and only if they have the same valuation. Therefore, the map taking the orbit of a point a in K^* to $\nu(a)$ gives a natural identification

$$K^*/R^* \xrightarrow{\sim} \mathbb{R}.$$

The traditional tropicalization map

$$(K^*)^n \xrightarrow{\text{Trop}} \mathbb{R}^n$$

taking a point $(a_1, \ldots, a_n)$ to its coordinatewise valuation $(\nu(a_1), \ldots, \nu(a_n))$ is naturally identified with the projection to the orbit space $(K^*)^n/(R^*)^n \cong \mathbb{R}^n$.

Example 2. Suppose X is the n-dimensional affine space K^n. Let $\mathbf{R}$ be the extended real line $\mathbb{R} \cup \infty$, with the topology of a half-open interval. Extend the valuation to a map $\nu : K \to \mathbf{R}$ by setting $\nu(0) = \infty$. Then the tropicalization map extends to a projection

$$K^n \to \mathbf{R}^n,$$

where $\mathbf{R}^n \cong K^n/(R^*)^n$ is the corresponding orbit space.

Example 3. Suppose X is smooth and complete. Then X is a union of affine spaces $\bigcup_\sigma K^n$ indexed by the maximal cones σ of Δ, so the orbit space $X/(R^*)^n$ is a corresponding union $\bigcup_\sigma \mathbf{R}^n$.

For any rational polyhedral cone σ in $\mathbb{R}^n$, let S_σ be the monoid $\sigma^\vee \cap \mathbb{Z}^n$ of lattice points in the dual cone. So $K[S_\sigma]$ is the coordinate ring of the corresponding affine toric variety U_σ. The extended real line $\mathbf{R}$ is a monoid under addition, and a point $x \in U_\sigma$ induces a monoid homomorphism from S_σ to $\mathbf{R}$ through the composition

$$K[S_\sigma] \xrightarrow{\mathrm{ev}_x} K \xrightarrow{\nu} \mathbf{R}.$$

It is then straightforward to check that there is a natural identification $U_\sigma/(R^*)^n \cong \mathrm{Hom}(S_\sigma, \mathbf{R})$. Now if τ is a face of σ, then S_σ is a submonoid of S_τ, giving an inclusion $\mathrm{Hom}(S_\tau, \mathbf{R}) \hookrightarrow \mathrm{Hom}(S_\sigma, \mathbf{R})$. If X is an arbitrary toric variety, the orbit space $X/(R^*)^n$ is the space

$$X/(R^*)^n \cong \bigcup_{\sigma \in \Delta} \mathrm{Hom}(S_\sigma, \mathbf{R})$$

obtained by gluing along these inclusions.

Let Y be an algebraic variety over K. Then we propose that the tropicalization $\mathrm{Trop}(Y, \iota)$ of an embedding $\iota : Y \hookrightarrow X$ should be defined as the image of Y in $X/(R^*)^n$. In the second half of this talk, I discussed how, when Y is quasiprojective, the analytification of Y is naturally homeomorphic to a suitable inverse limit of such tropicalizations of all embeddings of Y in toric varieties, following [3], and conjectured how this result might be extended to nonquasiprojective varieties.

References

[1] M. Baker, Potential theory on Berkovich spaces, lecture notes from 2007 Arizona Winter School, available at http://swc.math.arizona.edu/aws/07/notes.html

[2] V. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields, Amer. Math. Soc., Math. Surveys and Monographs **33**, (1990).

[3] S. Payne, Analytification is the limit of all tropicalizations, preprint 2008. arXiv:0805.1916

Bounds on Nef Cones from Toric Embeddings

Diane Maclagan

(joint work with Angela Gibney)

The goal of this project is to obtain bounds on the nef cone of projective variety from embeddings of the variety into suitably chosen toric varieties.

The nef cone of projective variety Y is a convex cone in the Nerón-Severi space $\mathrm{NS}(Y)_\mathbb{R}$ of divisors modulo numerical equivalence, which is a finite-dimensional vector space. The cone $\mathrm{nef}(Y)$ is the closure of the cone of ample divisors. It is dual to the Mori cone of curves:

$$\mathrm{nef}(Y) = \{[D] \in \mathrm{NS}(Y)_\mathbb{R} : [D] \cdot [C] \geq 0 \text{ for all curves } C \subset Y\}.$$

The nef cone thus contains information about projective embeddings of Y, and morphisms to other varieties (such as the fibrations discussed in the talk of Di Rocco). While the nef cone of Y is convex, it is not necessarily polyhedral.

Embeddings of Y into an auxiliary variety Z give some information about $\mathrm{nef}(Y)$. The most elementary, but least useful, example of this phenomenon is to take $Z = \mathbb{P}^N$ for some N. If $i\colon Y \to \mathbb{P}^N$ is an embedding, then $i^*(\mathcal{O}(1))$ is very ample on Y, and moreover $i^*(\mathrm{nef}(\mathbb{P}^N)) \subseteq \mathrm{nef}(Y)$.

It is more useful to apply this principle to varieties Z for which the rank of $\mathrm{NS}(Z)_\mathbb{R}$ is larger and i^* is surjective. In this project we focus on the case where Z is a toric variety, which has the advantage that the Nerón-Severi group and the nef cone are well-understood.

Let $X(\Sigma)$ be a toric variety defined by a fan $\Sigma \subseteq \mathbb{R}^n$. For simplicity of exposition we assume that $X(\Sigma)$ is smooth in what follows, though versions of these results hold with only minor modifications in the nonsmooth case. The rays $\Sigma(1)$ of Σ determine the torus invariant divisors $D_1, \ldots, D_{|\Sigma(1)|}$ of $X(\Sigma)$. We have $\mathrm{NS}(X(\Sigma))_\mathbb{R} \cong \mathbb{R}^{|\Sigma(1)|-n}$, which comes from the standard toric short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X(\Sigma)) \to 0,$$

where $M \cong \mathbb{Z}^n$ is the dual lattice to the lattice $N \subset \mathbb{R}^n$ containing the fan.

When $X(\Sigma)$ is smooth projective toric variety then we have the following three equivalent descriptions of the nef cone:

(1) $\mathrm{nef}(X(\Sigma))$ is the cone of globally generated divisors;

$$\mathrm{nef}(X(\Sigma)) = \bigcap_{\sigma \in \Sigma} \mathrm{pos}([D_i] : i \notin \sigma);$$

(2) $\mathrm{nef}(X(\Sigma))$ is the cone of divisors whose restriction to every T-invariant subvariety is effective;

(3) $\mathrm{nef}(X(\Sigma))$ is the cone of all divisors that nonnegatively intersect all T-invariant curves.

Each of these descriptions has an alternative description in terms of the convexity of piecewise linear functions on the fan Σ.

Note that if $i\colon Y \to X(\Sigma)$, then $i^*(\mathrm{nef}(X(\Sigma))) \subseteq \mathrm{nef}(Y)$. It is often the case that Y does not intersect every T-orbit of $X(\Sigma)$, so there is a toric subvariety $X(\Delta) \subsetneq X(\Sigma)$ containing Y. A simple example of this is given by the inclusion of $\mathbb{P}^1$ into $\mathbb{P}^2$ as the variety of the polynomial $x_0 + x_1 + x_2$, which misses the torus-fixed points of $\mathbb{P}^2$.

When Δ is not complete, we can still define variations of the previous cones, though they are no longer still all equal.

Definition. Let Δ be a (not necessarily complete) fan in $\mathbb{R}^n$.

(1) Let $\mathcal{G}(\Delta)$ be the cone in $\mathrm{Pic}(X(\Delta))_\mathbb{R}$ of globally generated divisors on $X(\Delta)$.

(2) Let $\mathcal{L}(\Delta)$ be the cone in $\mathrm{Pic}(X(\Delta))_\mathbb{R}$ of divisors whose restriction to every T-invariant subvariety is effective.

(3) If every maximal cone of Δ is d-dimensional, given $\mathbf{w} \in \mathbb{R}^{|\Delta(d)|}$ (which represents a class in $A_d(X(\Delta))$), let $\mathcal{U}(\Delta, \mathbf{w}) = \{[D] \in \mathrm{Pic}(X(\Delta))_\mathbb{R} : [D] \cdot [\overline{\mathcal{O}_\sigma}] \cdot [\mathbf{w}] \geq 0 \text{ for all } \sigma \in \Delta(d-1)\}$, where $\overline{\mathcal{O}_\sigma}$ is the closure of the T-orbit on $X(\Delta)$ indexed by σ.

We have $\mathcal{G}(\Delta) \subseteq \mathcal{L}(\Delta) \subseteq \mathcal{U}(\Delta, \mathbf{w})$, and both inclusions can be proper.

If C is a curve in Y with $C \cap T \neq \emptyset$, and $i \colon Y \to X(\Delta)$, then $[C] \cdot i^*([D_i])$ can be computed from the data of $\mathrm{Trop}(C \cap T)$. This gives rise to the definition of the intersection number of a divisor on $X(\Delta)$ and a tropical curve. The cone $\mathcal{L}(\Delta)$ equal to the cone of all divisors on $X(\Delta)$ which nonnegatively intersect all tropical curves. Since not every tropical curve is the tropicalization of a curve in Y, it is possible to have $i^*(\mathcal{L}(\Delta)) \subsetneq \mathrm{nef}(Y)$.

The motivation for the definition of $\mathcal{U}(\Delta, \mathbf{w})$ comes from letting $\mathbf{w}$ be the class of Y under a given embedding $i \colon Y \to X(\Delta)$. If the embedding is sufficiently nice (if the tropical variety of $Y \cap T$ is the support of the fan Δ [2]), then $Y \cap O_\sigma$ is a curve for all $\sigma \in \Delta(d)$, and $i^*(\mathcal{U}(\Delta, \mathbf{w}))$ is those divisors in $\mathrm{NS}(Y)_{\mathbb{R}}$ that intersect

Theorem (Gibney, Maclagan)**.** Let $i \colon Y \to X(\Delta)$ be an embedding of a projective variety Y into a toric variety $X(\Delta)$. Then

(1) $i^*(\mathcal{G}(\Delta)) = \bigcup_{\Delta \subseteq \Sigma, \Delta(1) = \Sigma(1)} i^*(\mathrm{nef}(X(\Sigma))) \subseteq \mathrm{nef}(Y)$;
(2) If i^* is surjective, then $\mathcal{L}(\Delta) \subseteq \mathrm{nef}(Y)$;
(3) If $\mathrm{Trop}(Y \cap T) = (\Delta, \mathbf{w})$ then $\mathrm{nef}(Y) \subseteq \mathcal{U}(\Delta, \mathbf{w})$.

When Y is a Mori Dream Space [3] (so its Cox ring is finitely generated; see the talk of Hausen), there is a natural toric variety $X(\Delta)$ into which Y embeds, which can be chosen so that Y intersects every torus orbit of $X(\Delta)$. Then $\mathrm{nef}(Y) = i^*(\mathcal{G}(\Delta))$. It is important to note that this equality depends on choosing this toric embedding carefully. For example, a generic del Pezzo surface $Y = \mathrm{Bl}_6(\mathbb{P}^2)$ embeds into a fourteen dimensional toric variety obtained from regarding Y as the tropical compactification of the line arrangement of all lines through six general points in the plane [4, Example 4.1]. For this toric embedding we have $i^*(\mathcal{L}(\Delta) \subsetneq \mathrm{nef}(Y) \subsetneq \mathcal{U}(\Delta, [Y])$, even though Y is a Mori Dream Space.

Our motivating example is the moduli space $\overline{M}_{0,n}$ of stable genus zero curves with n marked points. This is the Deligne-Mumford compactification of the $(n-3)$-dimensional moduli space $M_{0,n}$ parameterizing isomorphism classes of arrangements of n distinct points on $\mathbb{P}^1$. The boundary $\overline{M}_{0,n} \setminus M_{0,n}$ has a combinatorial stratification where the codimension-k strata consists of those stable curves with k nodes. The components of the codimension-$(n-4)$ strata are thus curves in $\overline{M}_{0,n}$, which are known as F-curves. Fulton conjectured that a divisor on $\overline{M}_{0,n}$ is nef if and only if it nonnegatively intersects every F-curve. This is known only for $n \leq 7$. There is a toric embedding $i \colon \overline{M}_{0,n} \to X(\Delta)$ (see [4], [1]) for which this conjecture is equivalent to the claim $\mathrm{nef}(\overline{M}_{0,n}) = \mathcal{U}(\Delta, [\overline{M}_{0,n}])$. For $n \leq 6$ we have $\mathcal{L}(\Delta) = \mathcal{U}(\Delta, [\overline{M}_{0,n}])$, which provides a simple conceptual proof of the F-conjecture for this range.

References

[1] Angela Gibney and Diane Maclagan. Equations for Chow and Hilbert quotients. arXiv:0707.1801, 2007.

[2] Paul Hacking. The homology of tropical varieties. *Collect. Math.*, 59(3):263–273, 2008.

[3] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[4] Jenia Tevelev. Compactifications of subvarieties of tori. *Amer. J. Math.*, 129(4):1087–1104, 2007.

Torus Invariant Divisors

Nathan Owen Ilten

(joint work with Lars Petersen, Hendrik Süß)

We shall consider a complete, normal $n+1$-dimensional variety X admitting an effective codimension-one torus action with good quotient. Such varieties are a special case of T-varieties as described in [1]. Analog to the case of a toric variety, we describe invariant Cartier divisors in terms of combinatorial data, and show how these data can be used to obtain further information such as global sections, intersection numbers, and criteria for ampleness and semi-ampleness. If furthermore X is a smooth surface, we calculate the arithmetic genus of curves on X as well as the Euler characteristic of semi-ample line bundles.

Each variety X as above can be described in terms of a *fansy divisor*. Let N be a lattice of rank n with dual lattice M and let Y be a smooth projective curve. A fansy divisor is simply a formal finite sum

$$\Xi = \sum_{P \in Y} \Xi_P \otimes P$$

where the Ξ_P are polyhedral subdivisions of $N \otimes \mathbb{Q}$ all with some common tailfan Σ. In this case, finite means that for all but a finite number of points in Y, $\Xi_P = \Sigma$. Starting with a fansy divisor Ξ, we can construct a complete, normal variety $X = \tilde{X}(\Xi)$ with effective codimension one T-action and good quotient Y as described in [2] and [3]. In fact, all complete, normal varieties with effective codimension one T-action and good quotient Y can be constructed this way.

Let $\mathrm{SF}(\Xi)$ be the group of all *support functions* consisting of finite formal sums

$$h = \sum_{P \in Y} h_P \otimes P,$$

where for each P, $h_P : \Xi_P \to \mathbb{Q}$ is a continuous piecewise affine function such that for $k \in \mathbb{N}$, $v \in N \otimes \mathbb{Q}$, if $kv \in N$ it follows that $kh_P(v) \in \mathbb{Z}$. Furthermore, for all $v \in N \otimes \mathbb{Q}$, the linear part $h^0(v) := \lim_{k\to\infty} h_P(kv)/k$ should be independent of P. That the above sum is finite means that $h_P = h^0$ for all but finitely many $P \in Y$.

We call a support function $h \in \mathrm{SF}(\Xi)$ *principal* if it has the form $h(v) = \sum_{P \in Y} (\alpha_P + \langle u, v\rangle) \otimes P$ where $u \in M$ and $\sum_{P\in Y} \alpha_P P = \mathrm{div}(f)$ for some $f \in K(Y)$. To such a principal support function h we can associate the T-invariant principal divisor $D_h := \mathrm{div}(f^{-1}\chi^{-u})$. For some general support function h we can find a T-invariant cover of X such that h is locally principal and thus associate to it a Cartier divisor D_h. This induces a group isomorphism between $\mathrm{SF}(\Xi)$ and T-invariant Cartier divisors on X.

To $h \in \mathrm{SF}(\Xi)$ we can associate the polytope

$$\Box_h = \left\{ u \in M \otimes \mathbb{Q} \mid \langle u, v \rangle \geq h^0(v) \ \forall \ v \in N \otimes \mathbb{Q} \right\}$$

as well as the function $h^* : \Box_h \to \mathrm{Div}_{\mathbb{Q}} Y$, where $h^* = \sum_{P \in Y} h^*_P \otimes P$ and for all $P \in Y$, $h^*_P(u) = \min_{\{v \text{ vertex in } \Xi_P\}} \langle u, v \rangle - h_P(v)$. Using this, we can write the global sections of $\mathcal{O}(D_h)$ as

$$\Gamma\left(X, \mathcal{O}(D_h)\right) = \bigoplus_{u \in \Box_h \cap M} \Gamma(Y, \mathcal{O}(h^*(u)) \cdot \chi^u.$$

There are also nice criteria for the ampleness and semi-ampleness of a divisor D_h. Indeed, D_h is ample if and only if h is strictly concave and $h^*(u)$ is ample for all vertices u of $\Box_h$. Likewise, D_h is semi-ample if and only if h is concave and $h^*(u)$ is semi-ample for all vertices u of $\Box_h$.

If D_h is a semi-ample divisor, then the self-intersection number D_h^{n+1} can also be calculated combinatorially. Indeed,

$$D_h^{n+1} = (n+1)! \int_{\Box_h} \sum_{P \in Y} h^*(u) du.$$

There is a similar formula for the intersection number of $n+1$ (possibly different) semi-ample divisors.

Now consider the case $n = 2$ and suppose additionally that X is smooth. For a support function $h \in \mathrm{SF}(\Xi)$ set

$$\operatorname{int} h = \sum_{P \in Y} \sum_{u \in \Box_h \cap M} \#\{a \in \mathbb{Z}_{\geq 0} | a < |h^*_P(u)|\} \operatorname{sgn}(h^*_P(u)).$$

If D_h is semi-ample, then we can calculate the arithmetic genus of any curve $C \in |D_h|$ by

$$g(C) = \operatorname{int} h + 1 + \operatorname{vol} \Box_h \cdot (g(Y) - 1).$$

By using the adjunction formula, this leads to the equation

$$\chi(X, \mathcal{O}(D_h)) = \sum_{u \in \Box_h \cap M} \chi(Y, \mathcal{O}(h^*(u))).$$

References

[1] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transformation Groups*, 13(2):215–242, 2008.

[2] Nathan Ilten and Hendrik Süß. AG codes from polyhedral divisors. arXiv:math/0811.2696v1, 2008.

[3] Lars Petersen and Hendrik Süß. Torus invariant divisors. arXiv:math/0811.0517v1, 2008.

The Tropical Vertex

Bernd Siebert
(joint work with Mark Gross, Rahul Pandharipande)

The global vector fields on the algebraic 2-torus $\mathbb{T}^2 = \operatorname{Spec}\Bbbk[x^{\pm 1}, y^{\pm 1}]$ are generated by $x\partial_x$, $y\partial y$.[1] Accordingly the connected component of the identity of $\mathrm{Aut}(\mathbb{T}^2)$ is just $\mathbb{G}_m^2/\Bbbk$. A much more interesting structure is obtained if one works over a complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ such as $\Bbbk[\![t]\!]$. Restricting to automorphisms preserving the logarithmic holomorphic volume form $\frac{dx}{x} \wedge \frac{dy}{y}$ and gauging out the rescaling automorphisms generated by $x\partial_x$, $y\partial y$ leads to an algebraic group over R that we call the *tropical vertex group* $\mathbb{V}_R$. This is an extremely interesting group, introduced by Kontsevich and Soibelman [4] in their construction of rigid analytic $K3$-surfaces. In [3] it turned out to be something like the engine of mirror symmetry, and from this work it can be expected to be related to virtually any object turning up in mirror symmetry. This may explain its appearance in wall crossing formulas for BPS-state counting [5], [7]. While [3] was a complex geometry ("B-model") application of the tropical vertex, we give here the first interpretation on the mirror side ("A-model") via curve counting.

Explicitly, write $M = \mathbb{Z}^2$ and $N = \mathrm{Hom}(\mathrm{M}, \mathbb{Z})$. Then the tropical vertex group $\mathbb{V}_R$ is the projective limit of the unipotent groups over the Artinian $\Bbbk$-algebra $R_k = R/\mathfrak{m}^k$ with generators

$$\theta : z^m \longmapsto f^{\langle n_0, m\rangle} z^m.$$

Here we used the identification $R_k[x^{\pm 1}, y^{\pm 1}] = R_k[M]$, $f \in R_k[M]$ is of the form

$$f = 1 + \sum_{d>0} a_d z^{d m_0}$$

with $a_d \in \mathfrak{m}$, $m_0 \in M \setminus \{0\}$, and $n_0 \in m_0^\perp \cap N$. So θ is the identity modulo $\mathfrak{m}$, and $\theta(z^{m_0}) = z^{m_0}$. The other monomials z^m are multiplied by a power of f, the exponent depending on the integral distance of m from $n_0^\perp$, the line through m_0. In particular, $\mathbb{V}_R$ is a pro-unipotent algebraic group over R.

An even smaller set of generators depends only on $m_0 = (a, b) \in M \setminus \{0\}$ and some $c \in \mathfrak{m}$ leading to

$$\theta_{a,b}(c) : \begin{cases} x \longmapsto (1 + c x^a y^b)^{-b} x \\ y \longmapsto (1 + c x^a y^b)^{a} y \end{cases}.$$

Ordering the generators by the slope $\mathbb{R}_{\geq 0} m_0$ counterclockwise leads t the notion of *ordered products* in $\mathbb{V}_R$. These can be graphically described as follows. A *ray* $\mathfrak{d}$ is a half-line in $\mathbb{R}^2$ together with an $f_{\mathfrak{d}} \in R$ as above with $\pm m_0$ on the half-line. We think of f as travelling along the ray, so the case $\mathfrak{d} = \mathbb{R}_{\geq 0} m_0$ is called *outgoing*, $\mathfrak{d} = -\mathbb{R}_{\geq 0} m_0$ incoming. A ray $\mathfrak{d}$ defines a generator $\theta_{\mathfrak{d}}$ uniquely by taking n_0 the primitive generator of $m_0^\perp \cap N$ that increases when passing the ray in counterclockwise direction. Note that generators pointing in the same direction

[1] We work over a field $\Bbbk$ of characteristic 0

commute. Thus any set of rays $\mathfrak{D} = \{\mathfrak{d}\}$ with only finitely many non-trivial rays modulo $\mathfrak{m}^k$ for any k, called a *scattering diagram*, defines an ordered product

$$\theta_{\mathfrak{D}} = \prod_{\mathfrak{d}\in\mathfrak{D}}^{\rightarrow} \theta_{\mathfrak{d}},$$

depending only on the choice of starting direction, and hence well-defined up to conjugation. It is an important observation of [4] that given any finite set of incoming rays there is an essentially unique way of inductively inserting outgoing rays in such a way that $\Theta_{\mathfrak{D}} = \mathrm{Id}$ for the scattering diagram $\mathfrak{D}$ thus obtained.

Our main result is an interpretation of the functions $f_{\mathfrak{d}}$, $\mathfrak{d} \in \mathfrak{D}$ an outgoing ray, for given incoming rays, in terms of relative genus 0 Gromov-Witten invariants on toric surfaces.

Theorem. Let $R = \Bbbk[\![s_1, \ldots, s_{\ell_1}, t_1, \ldots, t_{\ell_2}]\!]$ and assume $m_1, m_2 \in M \setminus \{0\}$ different and primitive. Let $\mathfrak{D}$ be the scattering diagram obtained from the incoming rays $(\mathfrak{d}_1, f_{\mathfrak{d}_1}), (\mathfrak{d}_2, f_{\mathfrak{d}_2})$ with

$$f_{\mathfrak{d}_1} = \prod_{\xi=1}^{\ell_1}(1 + s_\xi z^{m_1}), \quad f_{\mathfrak{d}_2} = \prod_{\xi=1}^{\ell_2}(1 + t_\xi z^{m_2}).$$

If $\mathfrak{d} \in \mathfrak{D}$ is an outgoing ray with $m_{\mathrm{out}} := m_0 \in \mathbb{Q}_{>0}m_1 + \mathbb{Q}_{>0}m_2$, then

$$\log f_{\mathfrak{d}} = \sum_{k=1}^{\infty} \sum_{\mathbf{P}=(\mathbf{P_1},\mathbf{P_2})} k \; N_{\mathbf{m}}[\mathbf{P}] \; s^{\mathbf{P}_1} \; t^{\mathbf{P}_2} \; z^{k m_{\mathrm{out}}},$$

where the sum is over all ordered partitions $\mathbf{P}_i$ of length ℓ_i satisfying

$$|\mathbf{P}_1| m_1 + |\mathbf{P}_2| m_2 = k m_{\mathrm{out}}.$$

In the theorem $N_{\mathbf{m}}(\mathbf{P})$ is a relative Gromov-Witten invariant related to the toric surface X with rays $-\mathbb{R}_{\geq 0}m_1$, $-\mathbb{R}_{\geq 0}m_2$, $\mathbb{R}_{\geq 0}m_{\mathrm{out}}$ as follows. Let D_1, D_2, D_{out} be the toric prime divisors of X. We implement a curve count with incidence of given multiplicities at ℓ_i points $x_{ij} \in D_i$ by taking the blow-up $\tilde{X}$ of X in ℓ_i different points in the big cell of D_i. For $\mathbf{P}_i = (p_{i1}, \ldots, p_{il_i})$ let $\beta \in H_2(X, \mathbb{Z})$ be the homology class of the strict transform of a curve in X intersecting D_i at x_{ij} with multiplicity p_{ij}. Let $\tilde{X}^o$ be the complement of the preimages of the 0-dimensional torus orbits of X. Then $N_{\mathbf{m}}(\mathbf{P})$ is the genus 0 Gromov-Witten of $\tilde{X}^o$ in class β relative to the preimage D_{out}^o of D_{out}, with full tangency at only one point of D_{out}^o. This is a virtual count of rational curves on X intersecting D_i only in the points x_{ij}, of order p_{ij}, and with D_{out} at only one point. While this is a virtual count on an open surface one can show that there is enough properness in the relevant moduli space of relative stable maps to define the virtual count.

Theorem generalizes in a straightforward manner to any number of incoming rays. However, the most general ordered product formulas also involve non-primitive m_i's. The solution for this case involves *orbifold* versions of the $N_{\mathbf{m}}(\mathbf{P})$, with the orbifold structure on the exceptional divisors of $\tilde{X}$.

References

[1] M. Gross, R. Pandharipande, B. Siebert: *The tropical vertex*, preprint 2009.
[2] M. Gross, B. Siebert: *Mirror symmetry via logarithmic degeneration data I*, J. Differential Geom. **72** (2006), 169–338.
[3] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, math.AG/0703822.
[4] M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, in: *The unity of mathematics* (P. Etingof, V. Retakh, I.M. Singer, eds.), 321–385, Progr. Math. 244, Birkhäuser 2006.
[5] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.
[6] T. Nishinou, B. Siebert: *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. **135** (2006), 1–51.
[7] M. Reineke, *Poisson automorphisms and quiver moduli*, arXiv:0804.3214.

On Cox Rings of (Some) K3-Surfaces

Jürgen Hausen
(joint work with Michela Artebani, Antonio Laface)

This is a report on a joint article with Michela Artebani and Antonio Laface, where we determine the Cox rings of certain K3-surfaces, see [1]. Recall that the Cox ring $\mathcal{R}(X)$ of a normal complete complex algebraic variety X with a free finitely generated divisor class group $\mathrm{Cl}(X)$ is the multigraded algebra

$$\mathcal{R}(X) \quad := \quad \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

A first basic problem is to decide if the Cox ring $\mathcal{R}(X)$ is finitely generated. A normal surface X has a finitely generated Cox ring if and only if its cones of effective and movable rational divisor classes both are polyhedral and every movable divisor on X has a basepoint free positive multiple. As a consequence, we obtain the following characterization.

Theorem. A K3-surface has finitely generated Cox ring if and only if its cone of effective rational divisor classes is polyhedral.

The second basic problem is to describe the Cox ring $\mathcal{R}(X)$ in terms of generators and relations. We first consider K3-surfaces X having Picard number $\varrho(X) = 2$. In this setting, if the effective cone is polyhedral, then it is known that its generators are of self intersection zero or minus two. For the case that both generators are of self-intersection zero, we obtain the following.

Theorem. Let X be a K3-surface with $\mathrm{Cl}(X) \cong \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, intersection form given by $w_1^2 = w_2^2 = 0$ and $w_1 \cdot w_2 = k \geq 3$, and assume that w_1, w_2 are effective.

(1) The effective cone of X is generated by w_1 and w_2 and it coincides with the semiample cone of X.

(2) The Cox ring $\mathcal{R}(X)$ is generated in degrees w_1, w_2 and $w_1 + w_2$, and one has

$$\dim\left(\mathcal{R}(X)_{w_i}\right) \;=\; 2, \qquad \dim\left(\mathcal{R}(X)_{w_1+w_2}\right) \;=\; k+2.$$

In particular, any minimal system of generators of $\mathcal{R}(X)$ has $k+2$ members.

(3) For $k = 3$, the Cox ring $\mathcal{R}(X)$ is of the form $\mathbb{C}[T_1, \dots, T_5]/\langle f \rangle$ and the degrees of the generators and the relation are given by

$$\deg(T_1) = \deg(T_2) = w_1, \qquad \deg(T_4) = \deg(T_5) = w_2,$$
$$\deg(T_3) = w_1 + w_2, \qquad \deg(f) = 3w_1 + 3w_2.$$

(4) For $k \geq 4$, any minimal ideal $\mathcal{I}(X)$ of relations of $\mathcal{R}(X)$ is generated in degree $2w_1 + 2w_2$, and we have

$$\dim\left(\mathcal{I}(X)_{2w_1+2w_2}\right) \;=\; \frac{k(k-3)}{2}.$$

The statements on the generators are directly obtained, and for the relations, we use the techniques developed in [4]. When at least one of the generators of the effective cone is a (-2)-curve, then the semiample cone is a proper subset of the effective cone. We show that in this case the number of degrees needed to generate the Cox ring can be arbitrarily big by giving a lower bound for this number in terms of the intersection form of $\mathrm{Cl}(X)$.

For the K3-surfaces X with Picard number $\varrho(X) \geq 3$, we use a different approach. A basic fact is that most K3-surfaces X with $\varrho(X) \geq 3$ and polyhedral effective cone admit a non-symplectic involution $\sigma \in \mathrm{Aut}(X)$, i.e., one has $\sigma^*\omega_X \neq \omega_X$. The associated quotient map $\pi\colon X \to Y$ is a double cover. If it is unramified then $Y := X/\langle\sigma\rangle$ is an Enriques surface, otherwise Y is a smooth rational surface. In the latter case, one may use known results and techniques to obtain the Cox ring of Y.

This observation suggests to study the behaviour of Cox rings under double coverings $\pi\colon X \to Y$. As it may be of independent interest, we consider more general, e.g., cyclic, coverings $\pi\colon X \to Y$ of arbitrary normal varieties X and Y. We relate finite generation of the Cox rings of X and Y to each other and provide generators and relations for the Cox ring of X in terms of π and the Cox ring of Y for the case that π induces an isomorphism on the level of divisor class groups. This enables us to compute Cox rings of K3-surfaces that are general double covers of $\mathbb{F}_0$ or of del Pezzo surfaces.

Besides $\mathbb{F}_0$ and the del Pezzo surfaces, other rational surfaces $Y = X/\langle\sigma\rangle$ can occur. For $2 \leq \varrho(X) \leq 5$, these turn out to be blow ups of the fourth Hirzebruch surface $\mathbb{F}_4$ in at most three general points, and we are in this setting if and only if the branch divisor of the covering $\pi\colon X \to Y$ has two components. Then, in order to determine the Cox ring of X, we have to solve two problems. Firstly, the computation of the Cox ring of Y. While blowing up one or two points gives a toric surface, the blow up of $\mathbb{F}_4$ in three general points is non-toric; we compute its Cox ring using the technique of toric ambient modifications developed in [3].

A further problem is that $\pi\colon X \to Y$ induces no longer an isomorphism on the divisor class groups. We obtain the following results in the case of Picard number $2 \le \varrho(X) \le 5$.

Theorem. Let X be a generic K3-surface with a non-symplectic involution and associated double cover $X \to Y$ and Picard number $2 \le \varrho(X) \le 5$. Then the Cox ring $\mathcal{R}(X)$ is given as follows.

(1) For $\varrho(X) = 2$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \dots, T_5]/\langle T_5^2 - f\rangle$ and the degree of T_i is the i-th column of

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \quad \text{if} \quad Y = \mathbb{F}_0,$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad \text{if} \quad Y = \mathbb{F}_1,$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 1 & 6 \end{bmatrix} \quad \text{if} \quad Y = \mathbb{F}_4.$$

(2) For $\varrho(X) = 3$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \dots, T_6]/\langle T_6^2 - f\rangle$ and the degree of T_i is the i-th column of

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \quad \text{if} \quad Y = \mathrm{Bl}_1(\mathbb{F}_0),$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & 1 & 5 \end{bmatrix} \quad \text{if} \quad Y = \mathrm{Bl}_1(\mathbb{F}_4).$$

(3) For $\varrho(X) = 4$ one has $\mathcal{R}(X) = \mathbb{C}[T_1, \dots, T_7]/\langle T_7^2 - f\rangle$ and the degree of T_i is the i-th column of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix} \quad \text{if} \quad Y = \mathrm{Bl}_2(\mathbb{F}_0),$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 4 \end{bmatrix} \quad \text{if} \quad Y = \mathrm{Bl}_2(\mathbb{F}_4).$$

(4) For $\varrho(X) = 5$ one has the following two cases.

(a) The surface Y is the blow up of $\mathbb{F}_0$ at three general points. Then the Cox ring $\mathcal{R}(X)$ of X is

$$\mathbb{C}[T_1, \dots, T_{11}]/\langle f_1, \dots, f_5,\, T_{11}^2 - g\rangle,$$

where $f_1, \dots, f_5, g \in \mathbb{C}[T_1, \dots, T_{10}]$ and $f_1, \dots, f_5$ are the Plücker relations in $T_1, \dots, T_{10}$. The degree of T_i is the i-th column of

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 1 \end{bmatrix}$$

(b) The surface Y is the blow up of $\mathbb{F}_4$ at three general points. Then the Cox ring $\mathcal{R}(X)$ of X is

$$\mathbb{C}[T_1, \dots, T_9]/\langle T_2T_5 + T_4T_6 + T_7T_8,\ T_9^2 - f\rangle,$$

where $f \in \mathbb{C}[T_1, \dots, T_8]$ is a prime polynomial and the degree of $T_i \in \mathcal{R}(X)$ is the i-th column of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 \end{bmatrix}$$

If $Y = X/\langle\sigma\rangle$ is a del Pezzo surface, then the Cox ring of Y is known by [2] and [4]. We obtain the following for the Cox ring of X.

Theorem. Let X be a generic K3-surface with a non-symplectic involution, associated double cover $\pi\colon X \to Y$ and intersection form $U(2) \oplus A_1^{k-2}$, where $5 \le k \le 9$. Then Y is a del Pezzo surface of Picard number k and

(1) the Cox ring $\mathcal{R}(X)$ is generated by the pull-backs of the (-1)-curves of Y, the section T defining the ramification divisor and, for $k = 7$, the pull-back of an irreducible section of $H^0(Y, -\mathcal{K}_Y)$,
(2) the ideal of relations of $\mathcal{R}(X)$ is generated by quadratic relations of degree $\pi^*(D)$, where $D^2 = 0$ and $D \cdot \mathcal{K}_Y = -2$, and the relation $T^2 - f$ in degree $-2\pi^*(K_Y)$, where f is the pullback of the canonical section of the branch divisor.

References

[1] M. Artebani, J. Hausen, A. Laface: *On Cox rings of K3-surfaces,* arXiv:0901369.
[2] V. Batyrev, O. Popov, *The Cox ring of a del Pezzo surface.* Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 85–103, Progr. Math., 226, Birkh?user Boston, Boston, MA, 2004.
[3] J. Hausen, *Cox rings and combinatorics II.* Moscow Math. J. 8 (2008), 711-757.
[4] A. Laface, M. Velasco: *Picard-graded Betti numbers and the defining ideals of Cox rings*, arXiv:0707.3251.

Exceptional Loci and Toric Models of $\overline{\mathrm{M}}_{0,n}$

Jenia Tevelev
(joint work with Ana-Maria Castravet)

Let X be a $\mathbb{Q}$-factorial projective variety. The basic gadgets encoding combinatorics of its birational geometry are the Mori cone $\overline{\mathrm{NE}}$ and the effective cone $\overline{\mathrm{Eff}}$. By Kleiman's criterion, the cone $\mathrm{Nef} \subset \overline{\mathrm{Eff}}$ dual to $\overline{\mathrm{NE}}$ is nothing but the nef cone (the closure of the ample cone). One has a polyhedral decomposition of Eff into cones called Mori chambers (Nef is one of them). If X is very nice (a so called Mori Dream Space) then this is a finite decomposition, all Mori chambers are rational polyhedral cones and correspond to birational contractions $X \dashrightarrow Y$, i.e. birational maps with $\mathbb{Q}$-factorial Y such that for any resolution $X \leftarrow W \rightarrow Y$, $\mathrm{Exc}(W \to X) \subset \mathrm{Exc}(W \to Y)$. If Y is toric then we call it a toric model of X. The most familiar examples of birational contractions are small modifications, for instance flops.

Let $\overline{\mathrm{M}}_{0,n}$ be the moduli space of stable rational curves with n marked points. $\overline{\mathrm{M}}_{0,n}$ is stratified by the topological type of a stable rational curve and so it has "natural" boundary effective divisors. For example, $\overline{\mathrm{M}}_{0,5}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in 4 points, and boundary divisors are the ten (-1)-curves. They generate $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,5}) = \overline{\mathrm{NE}}_1(\overline{\mathrm{M}}_{0,5})$. For $n = 6$, Keel and Vermeire showed that $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,6})$ is not generated by classes of boundary divisors. Up to symmetries, there is only one extra divisor for $\overline{\mathrm{M}}_{0,6}$, which we call the Keel–Vermeire divisor. There are at least two known geometric description of the Keel–Vermeire divisor, but they do not easily generalize to higher n. We introduce a new description, which does generalize and gives many new divisors for each n. We also construct very interesting toric models of $\overline{\mathrm{M}}_{0,n}$. Before our work, essentially the only known toric model of $\overline{\mathrm{M}}_{0,n}$ was the permutohedron toric variety studied by Losev and Manin.

To introduce our approach, let us give the following (very close) analogy, which we find useful. Suppose C is a general smooth curve of genus g. Let $W^1_{g+1} \subset \mathrm{Pic}^{g+1}$ be the locus of line bundles with $h^0(L) > 1$ and let G^1_{g+1} be its canonical blow-up parameterizing pencils of degree $g + 1$. By the Brill–Noether theory, $W^1_{g+1} = \mathrm{Pic}^{g+1}(C)$, G^1_{g+1} is smooth, and the morphism $G^1_{g+1} \to W^1_{g+1}$ has an exceptional divisor E contracted to the locus W^2_{g+1} (of codimension 3 in Pic^{g+1}). So we see immediately that E generates an edge of $\overline{\mathrm{Eff}}(G^1_{g+1})$. For example, if $C \subset \mathbb{P}^2$ is a smooth quartic curve (of genus 3) then G^1_4 is a blow-up of Pic^4 in one point $\omega_C \in \mathrm{Pic}^4$.

This is exactly what we do to produce exceptional loci on $\overline{\mathrm{M}}_{0,n}$, except that we take a very reducible hypergraph curve instead of a smooth curve. To continue with the previous example, let $C \subset \mathbb{P}^2$ be the most degenerate quartic curve, namely the union of 4 lines. Then $\mathrm{Pic}^4 \simeq \mathbb{G}_m^3$ (more precisely, we take line bundles with degree 1 on each line) and it has a canonical compactification, namely the compactified Jacobian $\overline{\mathrm{Pic}}$ of torsion-free sheaves on C semi-stable w.r.t. the canonical polarization. It is a toric variety and its polytope is a rhombic dodecahedron. $\overline{\text{Pic}}$ is not $\mathbb{Q}$-factorial, but if we blow-up its 6 singular points, we will get a $\mathbb{Q}$-factorial toric variety Y. Let $G^1_4(C)$ be the blow up of Y in a single point, namely the log-canonical line bundle $\omega_C \subset \text{Pic}$. Then it turns out that G^1_4 is a small modification of $\overline{\mathrm{M}}_{0,6}$, Y is its toric model, and an exceptional divisor over ω_C is a proper transform of the Keel-Vermeire divisor! To see how $\overline{\mathrm{M}}_{0,6}$ enters the scene, notice that $M_{0,6}$ naturally embeds in $G^1_4(C)$, namely each choice of 6 points in $\mathbb{P}^1$ gives a natural degree 4 map $C \to \mathbb{P}^1$ which sends nodes of C to prescribed 6 points.

Quite remarkably, tt turns out that almost any degeneration of a genus g curve to a union of rational curves (and the choice of $g+1$ of these components) gives rise to a new extremal divisor of $\overline{\mathrm{M}}_{g+3}$.

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of algebraic curves. Vol. I.*, Springer–Verlag, New York (1985) xvi+386 pp.

[2] D. Bayer, D. Eisenbud, *Graph curves*, Advances in Math. **86** (1991), No. 1, 1–40

[3] A. Gibney, S. Keel, I. Morrison, *Towards the ample cone of* $\overline{\mathrm{M}}_{g,n}$, Journal of Amer. Math. Soc., **2** (2001), 273?-294

[4] T. Oda, C.S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc., **253** (1979), 1–90

[5] P. Vermeire, *A counterexample to Fulton?s Conjecture*, J. of Algebra, **248**, (2002), 780–784

[6] A.-M. Castravet, J. Tevelev, *Exceptional Loci on* $\bar{M}_{0,n}$ *and Hypergraph Curves*, arXiv:0809.1699

Torus Actions of Complexity One

Dmitri A. Timashev

The theory of toric varieties is now classical [4]. The general philosophy behind this theory is that the classification and geometric properties of toric varieties are expressed in terms of combinatorial data from convex geometry (polyhedral cones, fans, polytopes, etc). A natural question arises: can this philosophy be extended to arbitrary torus actions on algebraic varieties?

Let $T = (\mathbb{C}^\times)^r$ be a complex algebraic torus acting on a complex algebraic variety X. The *complexity* $c(T,X)$ is the codimension of generic T-orbits in X. Without loss of generality we may assume that the action is effective, i.e., generic T-orbits have trivial stabilizers. Every torus action admits a rational section, i.e., X is birationally and T-equivariantly isomorphic to $T \times Z$ with T acting thereon by translations of the first factor. Obviously, $c(T,X) = \dim Z$.

Toric varieties are exactly those of complexity 0. In this talk we give a combinatorial description of torus actions of the second level of complexity, i.e., of complexity 1, in the language of convex geometry in the same spirit as for toric varieties. We restrict our consideration to *normal* T-varieties.

In the case of complexity 1, Z is an algebraic curve, which may be assumed smooth and projective. The curve Z is a birational invariant of the T-action, which determines its birational type. The problem which we address is to classify normal T-varieties X birationally and T-equivariantly isomorphic to $T \times Z$.

The starting point is the following theorem, due to H. Sumihiro:

Theorem 1. Let an algebraic torus T act on a normal variety X. Then every point $x \in X$ has a T-stable affine open neighborhood.

By this theorem, X is covered by finitely many T-stable affine charts X_i. Thus the classification of T-varieties of complexity 1 reduces to two problems: (1) Classify affine T-varieties of complexity 1; (2) Indicate how to patch them together.

First suppose that X is affine. It is determined by its coordinate algebra $\mathbb{C}[X]$. The latter is a finitely generated integrally closed algebra, whence

$$\mathbb{C}[X] = \{f \in \mathbb{C}(T \times Z) \mid v_D(f) \geq 0,\ \forall D \subset X\},$$

where D ranges over all prime divisors in X and v_D denotes the valuation of the field of rational functions $\mathbb{C}(T \times Z)$ corresponding to D, i.e., the order of a function along D. Furthermore, it is easy to see that the set of v_D corresponding to non-T-stable D does not depend on X and

$$\{f \mid v_D(f) \geq 0 \text{ for each non-}T\text{-stable } D\} = \mathbb{C}[T] \otimes \mathbb{C}(Z).$$

Hence X is uniquely determined by the set $\mathcal{V}(X)$ of the T-invariant valuations corresponding to the T-stable prime divisors in X.

Now we describe T-invariant discrete valuations of $\mathbb{C}(T \times Z)$ taking values in $\mathbb{Q}$ (*T-valuations* in short). They are completely determined by the restriction to the multiplicative group of T-eigenfunctions, which is isomorphic to $\Lambda \times \mathbb{C}(Z)$, where $\Lambda \simeq \mathbb{Z}^r$ is the weight lattice of T. Restricting a valuation to Λ and $\mathbb{C}(Z)$, in turn, we obtain a vector $\gamma \in \mathcal{H} := \mathrm{Hom}(\Lambda, \mathbb{Q}) \simeq \mathbb{Q}^r$ and a valuation hv_z of $\mathbb{C}(Z)$, where $h \in \mathbb{Q}_+$, $z \in Z$, and v_z is the order at z of a function on Z.

Proposition 1. The T-valuations of $\mathbb{C}(T \times Z)$ are in a 1–1 correspondence with the triples (γ, h, z), $\gamma \in \mathcal{H}$, $h \in \mathbb{Q}_+$, $z \in Z$, modulo the equivalence relation $(\gamma, 0, z_1) \equiv (\gamma, 0, z_2)$, $\forall z_1, z_2 \in Z$. Hence the set of T-valuations is identified with $\mathcal{V} = \bigcup_{z \in Z} \mathcal{V}_z$, where the half-spaces $\mathcal{V}_z = \mathcal{H} \times \mathbb{Q}_+$ are patched together along their common boundary hyperplane $\mathcal{H}$.

The set $\mathcal{V}$ is called a *hyperspace*.

By the above, X is uniquely determined by the set $\mathcal{V}(X) \subset \mathcal{V}$. Instead of $\mathcal{V}(X)$, it is more convenient to introduce another invariant determining X:

Definition 1. A *hypercone* in $\mathcal{V}$ is a union $\mathcal{C} = \bigcup_{z \in Z} \mathcal{C}_z$ of pointed polyhedral cones $\mathcal{C}_z \subset \mathcal{V}_z$ such that:

(1) $\mathcal{C}_z \cap \mathcal{H} =: \mathcal{K}$ does not depend on $z \in Z$;
(2) $\mathcal{C}_z = \mathcal{K} \times \mathbb{Q}_+$ for all but finitely many z;
(3) Let $\mathcal{P}_z$ be the convex hull of the projections to $\mathcal{H}$ of the vertices of $\mathcal{C}_z \cap (\mathcal{H} \times \{1\})$; then $\mathcal{P} := \sum_{z \in Z} \mathcal{P}_z \subset \mathcal{K} \setminus \{0\}$. $\mathcal{P}$ is called the *center* of $\mathcal{C}$. ($\mathcal{P} = \emptyset$ whenever $\mathcal{P}_z = \emptyset$, i.e., $\mathcal{C}_z \subset \mathcal{H}$ for some $z \in Z$.)

(4) For any face $\mathcal{K}_0 \subset \mathcal{K}$, $\mathcal{K}_0 \cap \mathcal{P} \neq \emptyset$, and any $\lambda \in \Lambda$, $\langle \lambda, \mathcal{K}_0 \rangle = 0$, $\langle \lambda, \mathcal{K} \rangle \geq 0$, put $\ell_z = \min\langle \lambda, \mathcal{P}_z \rangle$; then a multiple of $\sum_{z \in Z} \ell_z \cdot z$ is a principal divisor on Z.

Note. The most intricate condition (4) holds automatically if $Z = \mathbb{P}^1$, i.e., if X is rational, because $\sum_{z \in Z} \ell_z = 0$.

Theorem 2. The normal affine T-varieties X with a fixed birational equivariant isomorphism $X \approx T \times Z$ are in a 1–1 correspondence with the hypercones $\mathcal{C} \subset \mathcal{V}$. Here $\mathcal{C}$ consists of all T-valuations v having center on X. The T-stable prime divisors in X correspond to the primitive lattice generators of the edges of the $\mathcal{C}_z$'s not intersecting $\mathcal{P}$. $\mathbb{C}[X]$ is spanned by $f \in \Lambda \times \mathbb{C}(Z)$ such that $v(f) \geq 0$, $\forall v \in \mathcal{C}$.

This theorem is similar to the description of affine toric varieties via pointed polyhedral cones in $\mathcal{H}$. In the same spirit we classify T-stable subvarieties in X.

Definition 2. Let $\mathcal{C}$ be a hypercone in $\mathcal{V}$ with center $\mathcal{P}$. A *face* of $\mathcal{C}$ is either a face $\mathcal{C}'$ of some $\mathcal{C}_z$ $(z \in Z)$ such that $\mathcal{C}' \cap \mathcal{P} = \emptyset$ or a hypercone $\mathcal{C}' \subseteq \mathcal{C}$ such that each $\mathcal{C}'_z$ is a face of $\mathcal{C}_z$, $\mathcal{C}'_z \not\subset \mathcal{H}$, and $\mathcal{C}' \cap \mathcal{P} \neq \emptyset$.

In the case $\mathcal{P} \neq \emptyset$ we define the (relative) *interior* of $\mathcal{C}$ as the union of the interiors of $\mathcal{C}_z$, $z \in Z$, with the interior of $\mathcal{K} = \mathcal{C}_z \cap \mathcal{H}$.

Proposition 2. Suppose X is the affine T-variety corresponding to a hypercone $\mathcal{C}$. The T-stable irreducible closed subvarieties of X are in a 1–1 correspondence with the faces of $\mathcal{C}$. For such a subvariety $Y \subseteq X$, the interior of the respective face $\mathcal{C}_Y \subseteq \mathcal{C}$ is the set of all T-valuations having center Y on X. We have $Y \subseteq Y'$ if and only if $\mathcal{C}_{Y'}$ is a face of $\mathcal{C}_Y$.

Theorem 2 and Proposition 2 are used in the talk to obtain a classification of normal affine surfaces with an effective $\mathbb{C}^\times$-action and a qualitative description of orbits and invariants of this action in an easy way. This problem was addressed previously by H. Flenner and M. G. Zaidenberg [3].

Now we are ready to classify arbitrary (normal) T-varieties of complexity 1. As we have seen above, such a variety X is a union of T-stable open affine subvarieties X_i. These affine charts are determined by hypercones $\mathcal{C}_i$. The T-stable subvarieties $Y \subseteq X$ correspond to the faces of the $\mathcal{C}_i$'s. It follows from the valuative criterion of separation that the interiors of these faces are pairwise disjoint. Thus we arrive to the following:

Definition 3. A *hyperfan* in $\mathcal{V}$ is a collection $\mathcal{F}$ of cones and hypercones with non-empty centers such that:

(1) The interiors of all $\mathcal{C} \in \mathcal{F}$ are pairwise disjoint;
(2) If $\mathcal{C} \in \mathcal{F}$ and $\mathcal{C}'$ is a face of $\mathcal{C}$, then $\mathcal{C}' \in \mathcal{F}$;
(3) There exist finitely many hypercones $\mathcal{C}_i$ such that each $\mathcal{C} \in \mathcal{F}$ is a face of one of the $\mathcal{C}_i$'s and all faces of the $\mathcal{C}_i$'s belong to $\mathcal{F}$.

Theorem 3. The normal T-varieties X with a fixed birational equivariant isomorphism $X \approx T \times Z$ are in a 1–1 correspondence with the hyperfans $\mathcal{F}$ in $\mathcal{V}$. The T-stable irreducible closed subvarieties $Y \subseteq X$ are in an inclusion reversing 1–1 correspondence with $\mathcal{C} \in \mathcal{F}$.

This theorem is similar to the description of toric varieties via fans in $\mathcal{H}$.

This talk is based on the results of [6]. The theory of T-varieties of complexity 1 can be generalized to arbitrary reductive group actions of complexity 1 [7]. (Here the *complexity* is the codimension of generic orbits of a Borel subgroup.) On the other hand, K. Altmann, J. Hausen [1], and H. Süß [2] introduced a language of *polyhedral divisors* to classify T-actions on normal varieties (of arbitrary complexity). We compare their approach with ours. Finally, we mention the symplectic counterpart of the above classification problem: to describe the effective Hamiltonian actions of compact $(n-1)$-dimensional tori on $2n$-dimensional symplectic varieties. This problem was solved by Y. Karshon and S. Tolman [5].

References

[1] K. Altmann, J. Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. **334** (2006), 557–607.

[2] K. Altmann, J. Hausen, H. Süß, *Gluing affine torus actions via divisorial fans*, `arXiv:math.AG/0606772`.

[3] H. Flenner, M. Zaidenberg, *Normal affine surfaces with $\mathbb{C}^*$-actions*, Osaka J. Math. **40** (2003), 981–1009.

[4] W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies **131**, The William H. Roever Lectures in Geometry, Princeton University Press, Princeton, NJ, 1993.

[5] Y. Karshon, S. Tolman, *Complete invariants for Hamiltonian torus actions with two dimensional quotients*, J. Symplectic Geom. **2** (2003), no. 1, 25–82.

[6] D. A. Timashev, *Torus actions of complexity one*, Proc. Internat. Conf. on Toric Topology, Osaka, 2006 (M. Harada, Y. Karshon, M. Masuda, T. Panov, eds.), Contemp. Math., vol. 460 (2008), 349–364.

[7] D. A. Timashev, *Classification of G-varieties of complexity 1*, Math. USSR-Izv. **61** (1997), no. 2, 363–397.

Toric Kähler-Sasaki Geometry in Action-Angle Coordinates

Miguel Abreu

1. Introduction

In the same way that a contact manifold determines and is determined by a symplectic cone, a Sasaki manifold determines and is determined by a Kähler cone. Kähler-Sasaki geometry is the geometry of such a pair. This talk is given from the Kähler side of this pair. It presents the Burns-Guillemin-Lerman [4] and Martelli-Sparks-Yau [11] generalization to toric Kähler-Sasaki geometry of the action-angle coordinates approach to toric Kähler geometry [2]. It also shows how this approach can be used to relate a recent new family of Sasaki-Einstein metrics constructed by Gauntlett-Martelli-Sparks-Waldram in 2004 [7, 8], to an old family of extremal Kähler metrics constructed by Calabi in 1982 [5].

2. Toric Kähler-Sasaki Cones

Definition 2.1. A *symplectic cone* is a triple (M, ω, X), where (M, ω) is a connected symplectic manifold and $X \in \mathcal{X}(M)$ is a vector field generating a proper $\mathbb{R}$-action $\rho_t : M \to M$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^{2t}\omega$. A *Kähler-Sasaki cone* is a symplectic cone (M, ω, X) equipped with an ω-compatible complex structure J such that the *Reeb vector field* $K := JX$ is Kähler. Any such J will be called a *Sasaki complex structure* on the symplectic cone (M, ω, X).

A Kähler-Sasaki cone (M, ω, X, J), with Reeb vector field $K = JX$, is said to be:

- *regular* if K generates a *free* S^1-action.
- *quasi-regular* if K generates a *locally free* S^1-action.
- *irregular* if K generates an *effective* $\mathbb{R}$-action.

Define $r := \|K\| = \|X\| : M \to \mathbb{R}^+$. Then

- K is the *Hamiltonian* vector field of $-r^2/2$;
- X is the *gradient* vector field of $r^2/2$;
- the smooth manifold $N := \{r = 1\} \subset M$ is a *Sasaki* manifold.

The *Kähler reduction* of (M, ω, X, J) by the action of K, $B := N/K$, is

- a *smooth* Kähler manifold if the KS cone is *regular*.
- a Kähler *orbifold* if the KS cone is *quasi-regular*.
- only a Kähler *quasifold* if the KS cone is *irregular*.

Definition 2.2. A *toric symplectic cone* is a symplectic cone (M, ω, X) of dimension $2(n+1)$ equipped with an effective X-preserving $\mathbb{T}^{n+1}$-action, with moment map $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ such that $\mu(\rho_t(m)) = e^{2t}\rho_t(m)$, $\forall\, m \in M$, $t \in \mathbb{R}$. Its *moment cone* is defined to be the set $C := \mu(M) \cup \{0\} \subset \mathbb{R}^{n+1}$.

Definition 2.3 (Lerman)**.** A cone $C \subset \mathbb{R}^{n+1}$ is *good* if there exists a non-empty minimal set of primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ such that

(i) $C = \bigcap_{a=1}^d \{x \in \mathbb{R}^{n+1} \,:\, \ell_a(x) := \langle x, \nu_a\rangle \geq 0\}$.
(ii) any codimension-k face F of C, $1 \leq k \leq n$, is the intersection of exactly k facets whose set of normals can be completed to an integral base of $\mathbb{Z}^{n+1}$.

Theorem 2.4 (Banyaga-Molino, Boyer-Galicki, Lerman [10])**.** For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ with moment cone C.

Example 2.5. Let $P \subset \mathbb{R}^n$ be an *integral Delzant polytope*. Then, its standard cone

$$C := \{z(x, 1) \in \mathbb{R}^n \times \mathbb{R} \,:\, x \in P\,,\ z \geq 0\} \subset \mathbb{R}^{n+1}$$

is a good cone. Moreover, the toric symplectic manifold (B_P, ω_P, μ_P) is the $S^1 \cong \{\mathbf{1}\} \times S^1 \subset \mathbb{T}^{n+1}$ *symplectic reduction* of the toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ (at level one).

Definition 2.6. A *toric Kähler-Sasaki cone* is a toric symplectic cone (M, ω, X, μ) equipped with a *toric Sasaki complex structure* J.

3. Cone Action-Angle Coordinates

It follows from the classification theorem that any good toric symplectic cone has toric Sasaki complex structures. In fact, these can be written very explicitly in suitable *cone action-angle coordinates*, i.e. linear coordinates $(x, y) \in \breve{M} \equiv \mu^{-1}(\breve{C}) \cong \breve{C} \times \mathbb{T}^{n+1}$, $\breve{C} \equiv$ interior of C, such that

$$\omega|_{\breve{M}} = dx \wedge dy \ , \quad \mu(x,y) = x \quad \text{and} \quad X|_{\breve{M}} = 2x\frac{\partial}{\partial x} = 2\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i} \ ,$$

as

$$J = \begin{bmatrix} 0 & -S^{-1} \\ & \\ S & 0 \end{bmatrix} \quad \text{with} \quad S = S(x) = (s_{ij}(x)) = \left(\frac{\partial^2 s}{\partial x_i \partial x_j}\right) > 0$$

for some *symplectic potential* $s : \breve{C} \to \mathbb{R}$ of the form

$$s = s_C + s_b + h \, .$$

Here s_C is the *canonical* symplectic potential associated to the cone

$$s_C(x) := \frac{1}{2}\sum_{a=1}^{d} \ell_a(x) \log \ell_a(x) \, ,$$

s_b determines the Reeb vector field $K_s = (0, b)$, $b \in \breve{C}^*$, and is given by

$$s_b(x) := \frac{1}{2}\left(\langle x, b\rangle \log\langle x, b\rangle - \langle x, K_C\rangle \log\langle x, K_C\rangle\right) \ , \ K_C = \sum_{a=1}^{d} \nu_a \, ,$$

and $h : C \to \mathbb{R}$ is homogeneous of degree 1 and smooth on $C \setminus \{0\}$. See [2, 9, 4, 11].

The norm of the Reeb vector field is then given by $\|K_s\|^2 = \|(0,b)\|^2 = 2\langle x, b\rangle$. The *characteristic hyperplane* H_b and *polytope* P_b are defined as

$$H_b := \{x \in \mathbb{R}^{n+1} \ : \ \langle x, b\rangle = 1/2\} \quad \text{and} \quad P_b := H_b \cap C \, .$$

Note that $N := \mu^{-1}(H_b)$ is a toric Sasaki manifold and P_b is the moment polytope of $B = M//K_b$.

Definition 3.1. Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope and $C \subset \mathbb{R}^{n+1}$ its standard good cone. Given a symplectic potential $s : \breve{P} \to \mathbb{R}$, define its *Boothby-Wang* symplectic potential $\tilde{s} : \breve{C} \to \mathbb{R}$ by

$$\tilde{s}(x,z) := z\, s(x/z) + \frac{1}{2} z \log z \ , \ \forall\, x \in \breve{P} \ , \ z \in \mathbb{R}^+ \, .$$

Example 3.2. If $P = \bigcap_{a=1}^{d}\{x \in \mathbb{R}^n \ : \ \ell_a(x) := \langle x, \nu_a\rangle + \lambda_a \geq 0\}$ and

$$s(x) = \frac{1}{2}\sum_{a=1}^{d} \ell_a(x) \log \ell_a(x) - \frac{1}{2}\ell_\infty(x) \log \ell_\infty(x) \, ,$$

where $\ell_\infty(x) := \sum_a \ell_a(x) = \langle x, \nu_\infty\rangle + \lambda_\infty$, then

$$\tilde{s}(x,z) = s_C(x,z) + s_b(x,z) \quad \text{with} \quad b = (0, \dots, 0, 1) \, .$$

Proposition 3.3 (Calderbank-David-Gauduchon [6])**.** Symplectic potentials *restrict* naturally under toric symplectic reduction.

Proposition 3.4. Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \breve{P} \to \mathbb{R}$, let $\tilde{s} : \breve{C} \to \mathbb{R}$ be its Boothby-Wang symplectic potential. Then, s defines a toric Kähler-Einstein metric with $Sc \equiv 2n(n+1)$ iff $\tilde{s}$ defines a toric Ricci-flat Kähler metric. When this happens, the corresponding toric Sasaki metric is Einstein.

4. New Sasaki-Einstein from Old Kähler-Einstein

Calabi [5] constructed in 1982 a general 4-parameter family of $U(n)$-invariant extremal Kähler metrics, which he used to put extremal Kähler metrics on

$$H_m^n := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}\,,\ n, m \in \mathbb{N}\,,$$

in any possible cohomology class. In particular, when $n = 2$, on all Hirzebruch surfaces.

When written in action-angle coordinates, using symplectic potentials, Calabi's family can be seen to contain many other interesting Kähler metrics [1, 3].

In particular, it contains a 1-parameter family of Kähler-Einstein metrics directly related to the Sasaki-Einstein metrics constructed in 2004 by Gauntlett-Martelli-Sparks-Waldram [7, 8].

This is given by the 1-parameter family of symplectic potentials $s : (\mathbb{R}^+)^n \to \mathbb{R}$ of the form

$$s(x) = \frac{1}{2}\left(\sum_{a=1}^{n} x_a \log x_a + h(r)\right)\,,$$

where $r = x_1 + \cdots x_n$ and $h''(r) = -\dfrac{1}{r} + \dfrac{r^{n-1}}{r^n(1-r) - A}\,,\ 0 < A < n^n/(n+1)^{n+1}\,.$

These symplectic potentials can be seen to define Kähler-Einstein *singular* metrics with $Sc = 2n(n+1)$ on certain $H_m^n := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}$. For a countably infinite set of values for the variable parameter A, the corresponding singular Boothby-Wang cones are $GL(n+1)$ equivalent to good cones - precisely the ones corresponding to the Sasaki-Einstein metrics of Gauntlett-Martelli-Sparks-Waldram (at least when $n = 2$). Since symplectic potentials transform naturally under $GL(n+1)$ transformations, one gets this way an alternative explicit construction of these *smooth* Ricci flat Kähler-Sasaki metrics, with explicit formulas for the corresponding symplectic potentials.

References

[1] M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. **9** (1998), 641–651.

[2] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, in "Symplectic and Contact Topology: Interactions and Perspectives" (eds. Y.Eliashberg, B.Khesin and F.Lalonde), Fields Institute Communications 35, American Mathematical Society, 2003, pp. 1–24.

[3] M. Abreu, *$U(n)$-invariant extremal Kähler metrics in action-angle coordinates*, in preparation.
[4] D. Burns, V. Guillemin, E. Lerman, *Kaehler metrics on singular toric varieties*, math.DG/0501311.
[5] E. Calabi, *Extremal Kähler metrics*, in "Seminar on Differential Geometry" (ed. S.T.Yau), Annals of Math. Studies 102, Princeton Univ. Press, 1982, 259–290.
[6] D. Calderbank, L. David, P. Gauduchon, *The Guillemin formula and Kähler metrics on toric symplectic manifolds*, J. Symplectic Geom. **1** (2003), 767–784.
[7] J. Gauntlett, D. Martelli, J. Sparks, D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$*, Adv. Theor. Math. Phys. **8** (2004), 711–734.
[8] J. Gauntlett, D. Martelli, J. Sparks, D. Waldram, *A new infinite class of Sasaki-Einstein manifolds*, Adv. Theor. Math. Phys. **8** (2004), 987–1000.
[9] V. Guillemin, *Kähler structures on toric varieties*, J. Differential Geometry **40** (1994), 285–309.
[10] E. Lerman, *Contact toric manifolds*, J. Symplectic Geom. **1** (2003), 785–828.
[11] D. Martelli, J. Sparks, S.-T. Yau, *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*, Comm. Math. Phys. **268** (2006), 39–65.

Toric Kempf–Ness Sets

Taras E. Panov

In the theory of algebraic group actions on affine varieties, the concept of a Kempf–Ness set is used to replace the categorical quotient by the quotient with respect to a maximal compact subgroup. By making use of the recent achievements of "toric topology" we show that an appropriate notion of a Kempf–Ness set exists for a class of algebraic torus actions on quasiaffine varieties (coordinate subspace arrangement complements) arising in the Batyrev–Cox approach to toric varieties. We proceed by studying the cohomology of these "toric" Kempf–Ness sets. In the case of projective non-singular toric varieties the Kempf–Ness sets can be described as complete intersections of real quadrics in a complex space.

Classical setting. Let G be a algebraic group acting on an affine variety X, and $\mathbb{C}[X]$ the algebra of regular functions on X. The *categorical quotient* $X/\!\!/\,\mathrm{G}$ is the affine variety corresponding to the subalgebra $\mathbb{C}[X]^{\mathrm{G}}$ of G-invariant polynomial functions on X. It coincides with the topological quotient X/G only if all G-orbits are closed.

Let K be a maximal compact subgroup of G; then there exists a compact K-invariant subset $KN_X \subset X$ (*the Kempf–Ness set*) satisfying the following two properties:

- the composition $KN_X \hookrightarrow X \to X/\!\!/G$ is proper and induces a homeomorphism $KN_X/K \to X/\!\!/G$;
- there is a K-equivariant deformation retraction of X to KN_X.

Toric Kempf–Ness sets. Now let X_Σ be the toric variety corresponding to a fan Σ with m one-dimensional cones. By means of the Batyrev–Cox construction, X_Σ may be obtained as the categorical quotient $U(\Sigma)/\!\!/G$. Here $U(\Sigma) \subset \mathbb{C}^m$ is the complement to a certain coordinate subspace arrangement in $\mathbb{C}^m$ determined by Σ, and G is a subgroup in the algebraic torus $(\mathbb{C}^\times)^m$. The categorical quotient coincides with the topological one $U(\Sigma)/G$ iff Σ is simplicial.

As $U(\Sigma)$ is not an affine variety (it is only quasiaffine in general), the classical approach to construct a Kempf–Ness set does not work. However, a compact K-invariant subset in $U(\Sigma)$ with necessary properties still exists, at least for simplicial fans, and is known to toric topologists as the *moment-angle complex* [BP]. It is constructed as follows. Identify the set of one-dimensional cones of Σ with the index set $[m]$, and consider the unit polydisc

$$(\mathbb{D}^2)^m = \{(z_1,\dots,z_m) \in \mathbb{C}^m \colon |z_j| \le 1 \text{ for all } j\}.$$

Given a cone $\sigma \in \Sigma$, define

$$\mathcal{Z}(\sigma) = \{(z_1,\dots,z_m) \in (\mathbb{D}^2)^m \colon |z_j| = 1 \text{ if } j \notin \sigma\},$$

and

$$\mathcal{Z}(\Sigma) = \bigcup_{\sigma\in\Sigma} \mathcal{Z}(\sigma),$$

where the union is taken inside $(\mathbb{D}^2)^m$.

Theorem 1. Assume that Σ is a simplicial fan.

(a) If Σ is complete, then the composition $\mathcal{Z}(\Sigma) \hookrightarrow U(\Sigma) \to U(\Sigma)/G$ induces a homeomorphism $\mathcal{Z}(\Sigma)/K \to U(\Sigma)/G$.

(b) There is a $\mathbb{T}^m$-equivariant deformation retraction of $U(\Sigma)$ to $\mathcal{Z}(\Sigma)$.

We therefore refer to $\mathcal{Z}(\Sigma)$ as the *toric Kempf–Ness set.* For complete simplicial fans Σ the space $\mathcal{Z}(\Sigma)$ is known to be a manifold [BP].

For non-singular projective toric varieties X_Σ there is a symplectic version of the Batyrev–Cox construction (due to Kirwan, Guillemin–Sternberg). It identifies X_Σ with the symplectic reduction space $\mu^{-1}(a)/K$ for the Hamiltonian K-action on $\mathbb{C}^m$. (Here $\mu\colon \mathbb{C}^m \to \mathfrak{k} \cong \mathbb{R}^{m-n}$ is the moment map, and a is its regular value.) In this case the toric Kempf–Ness set $\mathcal{Z}(\Sigma)$ can be identified with the level set $\mu^{-1}(a)$ via a $\mathbb{T}^m$-equivariant homeomorphism. The latter level set is a complete intersection of real quadratic hypersurfaces [BPR].

Topology of toric Kempf–Ness sets. We are able to identify the cohomology (both ordinary and equivariant) of toric Kempf–Ness sets, using our previous results on moment-angle complexes [P].

Denote by $\mathcal{K}_\Sigma$ the simplicial complex on $[m]$ determined by Σ (whose simplices correspond to the cones), and $\mathbb{Z}[\mathcal{K}_\Sigma]$ its *Stanley–Reisner face ring.*

Theorem 2. For every simplicial fan Σ there are ring isomorphisms

$$H^*\big(\mathcal{Z}(\Sigma);\mathbb{Z}\big) \cong \operatorname{Tor}^*_{\mathbb{Z}[v_1,\dots,v_m]}\big(\mathbb{Z}[\mathcal{K}_\Sigma],\mathbb{Z}\big) \cong H\big[\Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[\mathcal{K}_\Sigma],d\big].$$

where the latter denotes the cohomology of a differential graded algebra with $\deg u_i = 1$, $\deg v_i = 2$, $du_i = v_i$, $dv_i = 0$ for $1 \le i \le m$.

Given a subset $I \subset [m]$, denote by $\mathcal{K}(I)$ the corresponding *full subcomplex* of $\mathcal{K}$, or the restriction of $\mathcal{K}$ to I. We also denote by $\widetilde{H}^i(\mathcal{K}(I))$ the ith reduced simplicial cohomology group of $\mathcal{K}(I)$ with integer coefficients. A theorem due to Hochster expresses the Tor-modules above in terms of full subcomplexes of $\mathcal{K}_\Sigma$, which leads to the following description of the cohomology of $\mathcal{Z}(\Sigma)$.

Theorem 3. $H^k\big(\mathcal{Z}(\Sigma)\big) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{k-|I|-1}\big(\mathcal{K}_\Sigma(I)\big)$.

There is also a description of the product in $H^*(\mathcal{Z}(\Sigma))$ in terms of full subcomplexes of $\mathcal{K}_\Sigma$.

Explicit cohomology calculations show that the toric Kempf–Ness set $\mathcal{Z}(\Sigma)$ may be quite complicated topologically even for simple fans.

References

[BP] Victor M. Buchstaber and Taras E. Panov. *Torus actions and their applications in topology and combinatorics,* University Lecture Series, vol. **24**, Amer. Math. Soc., Providence, RI, 2002.

[BPR] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds.* Moscow Math. J. **7** (2007), no. 2, 219–242; arXiv:math.AT/0609346.

[P] Taras Panov. *Toric Kempf–Ness sets.* Proc. Steklov Inst. Math., vol. **263** (2008); arXiv:math.AG/0603556.

Strongly Exceptional Sequences of Line Bundles on Toric Varieties

Lutz Hille

(joint work with Markus Perling)

1. Strongly Exceptional Sequences

In this note X is always a smooth and projective algebraic variety. We are mainly interested in toric surfaces. On X we consider sequences $\varepsilon = (L_1, \ldots, L_n)$ of line bundles satisfying the following conditions:

T1) $\mathrm{Ext}^l_X(L_i, L_i) = 0$ for all $l \neq 0$ (each L_i is exceptional),

T2) $\mathrm{Ext}^l_X(L_i, L_j) = 0$ for all l and all $j < i$ (the sequence is exceptional if T1) and T2) hold),

T3) $\mathrm{Ext}^l_X(L_i, L_j) = 0$ for all $l \neq 0$ (the sequence is strongly exceptional of T1), T2) and T3) hold),

T4) the bundles $L_i, i = 1, \ldots, n$ generate the derived category of coherent sheaves on X.

A full strongly exceptional sequence of line bundles on X induces an equivalence of triangulated categories between the bounded derived category of coherent sheaves on X and the bounded derived category of finitely generated modules

$$\mathbb{R}\,\mathrm{Hom}(\oplus L_i, -) : \mathcal{D}^b(\mathrm{Coh}(X)) \longrightarrow \mathcal{D}^b(\mathrm{mod}\,{-A}),$$

where A is the endomorphism algebra of $\oplus L_i$.

The standard example of a full, strongly exceptional sequence of line bundles is $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ on $\mathbb{P}^n$.

For an exceptional sequence of line bundles we define the Cartan matrix $C = (a_{i,j})$ with entries the Euler characteristic

$$a_{i,j} = \chi(L_i, L_j) = \sum_{l=1}^{d} (-1)^l \dim \operatorname{Ext}_X^l(L_i, L_j),$$

where $d = \dim X$. This matrix is upper triangular with one on the principal diagonal. For a strongly exceptional sequence the entries are just the dimension of homomorphism spaces $\dim \operatorname{Hom}_X(L_i, L_j)$. In Section 2 we consider polynomial equations in the entries. In Section 3 we classify full, strongly exceptional sequences of line bundles on toric surfaces. Moreover, we also prove some partial results for rational surfaces.

The existence of a full, (strongly) exceptional sequence on a variety X is a very strong condition: in particular, the classes $[L_i]$ form an integral basis of the Grothendieck group K_0 of vector bundles on X (it coincides with the Grothendieck groups of coherent sheaves). Consequently, $K_0 \simeq \mathbb{Z}^n$. For X toric, n is just the number of d-dimensional cones in the fan of X, or, in other words, the number of fixed points under the torus action. As a consequence, the length of a full, (strongly) exceptional sequence is the rank n of the Grothendieck group.

2. Polynomial Invariants

In this part we study polynomials that are invariant in the entries of the Cartan matrix. The first invariants work for any full, strongly exceptional sequence of coherent sheaves. The second type of invariants work only hold for full, strongly exceptional sequences of line bundles on surfaces. We note that all those invariants also work for exceptional sequences, if we replace the dimension of the homomorphism spaces by the corresponding Euler characteristic.

2.1. **Polynomial Invariants.**

We are interested in polynomials F in the ring $R_n = k[x_{i,j} \mid 1 \leq i < j \leq n]$ in $n(n-1)/2$ variables corresponding to the possible non-vanishing values $a_{i,j}$ above. Such a polynomial is called an *invariant* if its value $F(a_{i,j})$ does only depend on the underlying variety X and not on the chosen sequence. It is not obvious that such invariants exist, however we can construct several ones. Another interpretation is the following: one can show, that those invariants are invariants under a certain braid group action.

Example. If $n = 3$ then $x_{1,2}^2 + x_{2,3}^2 + x_{1,3}^2 - x_{1,2}x_{2,3}x_{1,3} = 0$. If one considers a strongly exceptional sequence of line bundles on $\mathbb{P}^2$ then $x_{1,2} = x_{2,3} = 3$ and $x_{1,3} = 6$ ([5]).

Further invariant polynomials can be obtained as follows. Consider C with entries $x_{i,j}$ in the upper part and 1 on the principal diagonal. We define a matrix

$\Phi := C^t C^{-1}$ (its negative is called Coxeter transformation). Then we define the Coxeter polynomial

$$\det(\Phi - t\,\mathrm{Id}) = \det(tC^t - C) = \sum_{i=0}^{n} D_i t^i \in R_n[t]$$

and its unique decomposition with polynomials $D_i \in R_n$. Since the Coxeter transformation corresponds to the Serre functor on the level of K_0, the eigenvalues of Φ do not depend on the choice of a full exceptional sequence. Moreover, $D_i = D_{n-i}$. As a consequence the polynomials D_i for $i = 1, \dots, \lfloor n/2 \rfloor$ are invariants, even algebraically independent. Conjecturally, the D_i for $i = 1, \dots, \lfloor n/2 \rfloor$ generate the ring of all invariants for n odd and for n even there is one additional generator, the Pfaffian of $C^t - C$. Finally, also the values $D_l(a_{i,j})$ are known (see also [1]):

Theorem 2.1. The polynomials D_i are polynomial invariants and $\det(\Phi - t\,\mathrm{Id}) = (t - (-1)^{\dim X})^n$.

The first invariant (up to a constant) is

$$F_1 = \sum_{1 \le i < j \le n} x_{i,j}^2 + \sum_I x_I,$$

where I runs over all subsets $I \subset \{1, 2, \dots, n\}$ of cardinality at least 3 and for $I = (i_1 < i_2 < \dots < i_r)$ we define $x_I := x_{i_1,i_2} x_{i_2,i_3} \dots x_{i_{r-1},i_r} x_{i_1,i_r}$.

2.2. **Polynomial Invariants for Surfaces.** The situation for full exceptional sequences of line bundles on surfaces is much simpler, in fact one can identify the possible sequences $(a_{i,j})$ with certain two-dimensional fans that are almost convex.

Let Σ be a fan with lattice points $v_i, i = 1, \dots, n$. The fan is convex in v_i if $v_{i+1} + b_i v_i + v_{i-1} = 0$ for some $b \geq -2$ (note that b_i is the selfintersection number of the divisor corresponding to the ray of v_i). A fan is *almost convex* if it is convex in all points with at most one exception.

Theorem 2.2. a) The elements $a_{i,j}$ satisfy $a_{i,j} = a_{i,l} + a_{l,j}$ for all $i < l < j$.
b) The elements $b_i := a_{i,i+1} - 2$ are the selfintersection numbers of an almost convex fan Σ with n rays (in cyclic order).

Part one of the theorem reduces the computation of the $a_{i,j}$ to the entries in the diagonal $a_{i,i+1}$ above the principal diagonal. Part b) of the result also classifies those entries. For exceptional sequences all those numbers can occur, just take the sequence in Section 3.

According to part a) in the theorem we can define the levelled length of a strongly exceptional sequence of line bundles on a surface. For, we define L_i to be incomparable to L_j if $\dim \mathrm{Hom}(L_i, L_j) = 0 = \dim \mathrm{Hom}(L_j, L_i)$. A strongly exceptional sequence has levelled length l if there are at most l subsequences consisting of pairwise incomparable line bundles.
From the classification of two-dimensional fans we obtain the maximal levelled length of a full, strongly exceptional sequence.

Corollary 2.3. The levelled length of a full strongly exceptional sequence is at most 6.

A surface X, where all exceptional line L bundles satisfy $H^0(X, L) \neq 0$ or $H^0(X, L^*) \neq 0$, cannot admit a full, strongly exceptional sequence of line bundles. In fact the counterexample in [3] is of this form.

3. Main Results

In this part we collect some of our main results concerning the existence and the construction of full, strongly exceptional sequences of line bundles on rational surfaces.

Theorem 3.1. On each rational surface there exists a full, exceptional sequence of line bundles.

For a toric surface one can construct such a sequence just as $(\mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(E_1 + E_2), \ldots, \mathcal{O}(E_1 + E_2 + \ldots + E_{n-1}))$, where the E_i are the torus invariant divisors on X in a cyclic order. Such a sequence is strongly exceptional if $E_i \geq -1$ for $i = 1, \ldots, n-1$. This, in particular, proves the existence on any Hirzebruch surface and on any toric Fano surface.

For each rational surface X (that is not $\mathbb{P}^2$) we define $r(X)$ to be the minimal r so that there exists a sequence of morphisms

$$X_r = X \longrightarrow X_{r-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 = \mathbb{F}_a$$

satisfying: X_0 is a Hirzebruch surface $\mathbb{F}_a$ and each morphism $X_i \longrightarrow X_{i-1}$ is the blow up of finitely many points on X_{i-1}. Moreover, $a(X)$ is the minimal number a that can occur in such a sequence of blow ups.

Theorem 3.2. Let X be a rational surface with $r(X) \leq 2$. Then X admits a full, strongly exceptional sequence of line bundles.

The proof of the theorem is constructive. The full exceptional sequences on Hirzebruch surfaces are classified. In each step one can explicitly construct a new sequence by inserting further line bundles in the given sequence. This process stops after two steps. For toric surface we can also prove the converse.

Theorem 3.3. Let X be a toric surface. There exists a full, strongly exceptional sequence of line bundles on X precisely when $r(X) \leq 2$.

The proof of this result is based on a recursive classification of all exceptional line bundles on X via blow ups. The crucial problem are small values of $a(X)$. In particular, for $a(X) \leq 2$ there exist exceptional line bundles that are difficult to handle.

Finally we consider full, cyclic strongly exceptional sequences: each sequence $(L_i, L_{i+1}, L_n, L_1 \otimes \omega_X^{-1}, \ldots, L_{i-1} \otimes \omega_X^{-1})$ is also full, strongly exceptional.

Theorem 3.4. If a rational surface X admits a full, cyclic strongly exceptional sequence, then $n \leq 9$.

In fact we can classify those sequences. The toric surface associated to such a sequence (as in Theorem 2.2) is then a toric almost Fano surface ($E_i \geq -2$ for each torus invariant divisor E_i). From the finite classification of those surfaces we obtain $n \leq 9$: they correspond precisely to the 16 2-dimensional reflexive polytopes.

References

[1] A. I. Bondal, *A symplectic groupoid of triangular bilinear forms and the braid group*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 4, 19–74; translation in Izv. Math. 68 (2004), no. 4, 659–708.

[2] Lutz Hille, *Exceptional sequences of line bundles on toric varieties*, Mathematisches Institut, Georg-August-Universität Göttingen: Seminars 2003/2004, 175–190, Universitätsdrucke Göttingen, Göttingen, 2004.

[3] Lutz Hille, Markus Perling, *A counterexample to King's conjecture*, Compos. Math. 142 (2006), no. 6, 1507–1521.

[4] Lutz Hille, Markus Perling, *Exceptional Sequences of Invertible Sheaves on Rational Surfaces*, arXiv:0810.1936.

[5] A. N. Rudakov, *Markov numbers and exceptional bundles on P^2*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), no. 1, 100–112, 240; translation in Math. USSR-Izv. 32 (1989), no. 1, 99–112.

Determinantal Equations

Gregory G. Smith

(joint work with Jessica Sidman)

Which projective varieties are cut out by the minors of a matrix of linear forms? Well-known examples of this phenomena include rational normal curves, Segre varieties, and quadratic Veronese embeddings of a projective space. Indeed, the homogeneous ideal for these varieties is given by the (2×2)-minors of a generic Hankel matrix, a generic matrix, and a generic symmetric matrix respectively. For these classic examples, the determinantal presentation leads to the following insights: an explicit description of the minimal graded free resolution of the homogeneous ideal of the variety, interesting connections between the combinatorics of minors and the geometry of the variety, and equations for the higher secant varieties. Somewhat surprisingly, Mumford [3] shows that a suitable multiple of every projective embedding is defined by the (2×2)-minors of a matrix of linear forms. To extend these insights to other projective varieties, we require a better understanding of which embeddings are defined by minors.

To address this problem, we need an appropriate source of matrices. Composition of linear series or equivalently multiplication in the Cox ring of the variety provides the traditional supply for such matrices. To be more explicit, observe that, if $X \subset \mathbb{P}^r$ is a scheme embedded by the complete linear series $|L|$ corresponding to a line bundle L on X, then $H^0(X, L)$ is the space of linear forms on $\mathbb{P}^r$. Factoring L as $L = L_1 \otimes L_2$ for some $L_1, L_2 \in \operatorname{Pic}(X)$ yields a natural multiplication map $\mu\colon H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2) = H^0(X, L)$. Choose ordered bases $y_1, \ldots, y_m \in H^0(X, L_1)$ and $z_1, \ldots, z_n \in H^0(X, L_2)$ and let $A := [\mu(y_i \otimes z_j)]$ be the associated $(m \times n)$-matrix of linear forms on $\mathbb{P}^r$. Since $\mathscr{O}_X$

is a sheaf of commutative rings, it follows that the (2×2)-minors of A vanish on X (see Proposition 6.10 in Eisenbud [1]). Numerous classic examples of this construction can be found in Room [4]. What conditions on the line bundles L_1 and L_2 guarantee that the homogeneous ideal of X is generated by the (2×2)-minors of the matrix A?

We answer this question by providing sufficient conditions involving the minimal free graded resolutions of modules arising from the line bundles L_1, L_2, and L. More precisely, if $L = L_1 \otimes L_2$ is very ample and the following conditions hold:

- the module $\bigoplus_{d \geq 0} H^0(X, L \otimes L^d)$ has a linear presentation with respect to the polynomial ring $\mathrm{Sym}\big(H^0(X, L)\big)$,
- the module $\bigoplus_{d \geq 0} H^0(X, L_i \otimes L_j^d)$ has a linear presentation with respect to the polynomial ring $\mathrm{Sym}\big(H^0(X, L_j)\big)$ for $i \neq j$,
- the module $\bigoplus_{d \geq 0} H^0(X, L_1^2 \otimes L_2^d)$ has a linear presentation with respect to the polynomial ring $\mathrm{Sym}\big(H^0(X, L_2)\big)$,

then the homogeneous ideal of X is defined by the (2×2)-minors of A. In other words, we say that the embedding $\varphi_{|L|} \colon X \hookrightarrow \mathbb{P}^r$ is *determinantal*. By combining this with a cohomological criterion for a linear presentation and multigraded Castelnuovo-Mumford regularity, we establish that every projective embedding of a scheme determined by the complete linear series of a sufficiently ample line bundle is cut out by the (2×2)-minors of a matrix of linear forms. Hence, given a projective scheme X, there exists a line bundle L_0 on X such that $\varphi_{|L|}$ is determinantal for all $L \in \mathrm{Pic}(X)$ for which $L \otimes L_0^{-1}$ is numerically effective (nef). Extending the work of Eisenbud-Koh-Stillman [2] for reduced irreducible curves, we also specify effective bounds for L_0 on products of projective spaces, Gorenstein toric varieties, and smooth n-folds.

Finally returning to our initial motivation, this work suggests that, for an embedding $X \subset \mathbb{P}^r$ given by the complete linear series of a sufficiently ample line bundle, the homogeneous ideal of the k^{th} secant variety $\mathrm{Sec}^k(X)$ is defined by the $(k+2) \times (k+2)$-minors of A. Assuming this is true, it would be interesting to have explicit bounds for "sufficiently ample" in this context.

References

[1] David Eisenbud, *The Geometry of Syzygies*, Graduate Texts in Mathematics 229, Springer-Verlag, New York, 2005.

[2] David Eisenbud, Jee Koh, and Michael Stillman, *Determinantal Equations for Curves of High Degree*, American Journal of Mathematics, **110** (1988), 513–539.

[3] David Mumford, *Varieties defined by quadratic equations*, in *Questions on algebraic varieties*, CIME III Ciclo, Varenna, 1969 (Edizioni Cremonese, Rome 1970), 29–100.

[4] Thomas Gerald Room, *The Geometry of Determinantal Loci*, Cambridge Univ. Press, 1938.

Equivariant *K*-Theory and Dahmen-Micchelli Difference Equations

Michèle Vergne
(joint work with C. De Concini, C. Procesi)

Let T be a torus, with Lie algebra $\mathfrak{t}$. Let $\Delta := [a_1, a_2, \ldots, a_N] \subset \mathfrak{t}^*$ be a list of non zero weights of T. Assume that Δ spans $\mathfrak{t}^*$.

We consider the representation space $V_\Delta = \oplus_{i=1}^N L_{a_i}$ where L_{a_i} is the line with action of T via the character e^{a_i}.

Let $\mathcal{H}$ be the set of hyperplane in $\mathfrak{t}^*$, spanned by subsets of Δ. The open subset

$$U = V_\Delta \setminus \cup_{W \in \mathcal{H}} V_W$$

of V_Δ is the open subset consisting of points with finite stabilizers. We determine the equivariant K-theory of U. First $K_T^*(U) = 0$ if $*$ is not equal to $\dim(T)$ mod 2.

Let

$$\hat{R}(T) = \{\Theta = \sum_\lambda m(\lambda) e^\lambda\}$$

be the set of formal characters. This is a $R(T)$-module.

If $W \in \mathcal{H}$, we denote by $D_W = \prod_{a_i \notin W}(1 - e^{a_i}) \in R(T)$. The Dahmen-Micchelli space is

$$DM(\Delta) = \{\Theta \in \hat{R}(T)\,;\, D_W \Theta = 0 \,\text{for all}\, W \in \mathcal{H}\}.$$

The space $DM(\Delta)$ is a free $\mathbb{Z}$-module of rank equal to the volume of the zonotope spanned by Δ.

The index theory of transversally elliptic operators gives a map

$$ind : K_T^{\dim G}(U) \to \hat{R}(T).$$

We prove that ind is an isomorphism on $DM(\Delta)$.

In particular, this gives a new proof of the Atiyah-Singer theorem describing a set of generators for the $R(T)$-module $K_T(U)$.

We deduce from this theorem a "Thom isomorphism" for the index of a transversally elliptic operator on a vector space with a T-action.

Simultaneous Toric Equations and the Existence of Diptych Varieties

Gavin Brown
(joint work with Miles Reid)

1. What is a diptych variety?

A diptych variety is a special kind of affine Gorenstein 6-fold V with an action of a 4-dimensional torus $\mathbb{T} = (\mathbb{C}^*)^4$. In particular, it is an affine T-variety, in the sense of Altmann and Hausen [1].

A diptych variety V is characterised by the following properties: V contains two toric 4-folds T_1 and T_2, and the toric structure of each is the restriction of the $\mathbb{T}$ action; T_1 and T_2 intersect in a cycle $S = S_1 \cup S_2 \cup S_3 \cup S_4$ of four affine toric surfaces meeting along 1-strata; in addition, we require that $S_1 \cong S_3 \cong \mathbb{C}^2$ rather than arbitrary affine surfaces. So there is a $\mathbb{T}$-invariant diagram

$$\begin{array}{ccc} S & \subset & T_2 \\ \cap & & \cap \\ T_1 & \subset & V. \end{array} \tag{1}$$

Thinking of the toric configuration $T_1 \supset S \subset T_2$ as combinatorial data, the aim is to recover V—or simply to prove that it exists—from this toric data.

It may be possible to make a more general definition of diptych variety (more than four surfaces? higher dimension? triptychs?), but our naive attempts do not seem to work, at least not in quite the same way.

The classification and construction of diptych varieties is in two steps: (1) classify possible toric configurations (under certain hypotheses), and (2) prove that, given such data, there exists a 6-fold V as above. The key point in our approach to (2) is to exploit the convexity of the monomial cones of T_1 and T_2 simultaneously—since the toric varieties have the same torus $\mathbb{T}$ acting, we can identify their monomial lattices. The proof attempts to build the equations of V from those of T_1 and T_2, preserving $\mathbb{T}$-equivariance. The convexity imposes conditions on monomials to appear in the equations of V, and, although we cannot say what the equations are, it is just enough to prove that they exist.

This work has many relations with that of other people. For example, V is a T-variety in the sense of Altmann, Hausen and others. The combinatorics are related to continued fractions and resemble Riemenschneider staircases for cyclic quotient singularities. Our application relates V to Mori flips of Type A: certain specialisations of V are the canonical covers of such flips, and the combinatorics of the toric geometry are like Mori billiards as they arise in Mori's alternative proof [4] of a special case of the existence of flips.

The name 'diptych' refers to the toric configuration, since it resembles a 4-dimensional display consisting of two panels T_1, T_2 meeting along a hinge C.

2. Where do diptych varieties come from?

Our motivation comes from 3-fold Mori flips. If $X \to Y \leftarrow X^+$ is a Mori flip (see [3] for definitions), then

$$A_Y = \operatorname{Spec} \oplus_{m\in\mathbb{Z}} H^0(Y, mK_Y)$$

is an affine Gorenstein 4-fold with a $\mathbb{C}^*$ action determined by the $\mathbb{Z}$-grading. The flip is recovered by the variation of GIT in the familiar way. (This is analytic near the flipping point $P \in Y$, although in practice we often model it on an affine Y.)

Kollár and Mori [2] classify flips according to the behaviors of two general surfaces in Y: the general elephant $E \subset |-K_Y|$ and the general hyperplane section $P \in H \subset Y$. They prove that $P \in E$ is an ADE singularity and classify the types of singularity $P \in H$ into several types.

A flip is said to be of *Type A* when $P \in E$ is an A_n singularity, in which case $P \in H$ is necessarily also of type A. So such a flip contains a pair of toric surfaces $E \subset Y \supset H$. These lift to the canonical $\mathbb{C}^*$ cover A_Y to 3-folds admitting three independent $\mathbb{C}^*$ actions, and so we make the following ansatz for such A_Y:

$$\widetilde{E} \subset A_Y \supset \widetilde{H}, \quad \text{for affine Gorenstein toric 3-folds } \widetilde{E} \text{ and } \widetilde{H}.$$

This configuration has moduli coming from the equations of the terminal singularities of X. The definition of diptych variety absorbs some some of this freedom into additonal parameters A, B, L, M which is why diagram (1) is in higher dimension.

To get from a diptych variety to a flip is easy: specialise A, B, L, M to two variables u, t (reducing the dimension to 4) and then divide by some 1-parameter subgroup of $\mathbb{T}$. (There are mild conditions on each of these choices.)

3. A toric 4-fold V_{AB}

Fix a matrix $\begin{pmatrix} r & a \\ b & s \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and a lattice $M = \mathbb{Z}^4$ with basis ξ, η, A, B. Set $x_0 = (A\xi^{-1})^r \eta^a$, $y_0 = \xi^b (B\eta^{-1})^s$, and define a convex cone

$$\sigma_{AB} = \langle \xi, \eta, A, B, x_0, y_0 \rangle .$$

This is the monomial cone of an affine Gorenstein toric variety V_{AB}. The 2-faces of σ_{AB} are all basic except $\langle x_0, \xi \rangle$ and $\langle y_0, \eta \rangle$. If α is the least residue of $a \mod r$, then $(x_0 \xi^{r-\alpha})^{1/r} = \xi^{a_k} A \in M$, where $a_k = \lceil a/r \rceil$, so this face is the monomial cone of the quotient surface $\frac{1}{r}(a, 1)$. Thus this cone has semi-group generators $x_0, x_1, \dots, x_k = \xi$, and the x_i satisfy relations determined by the coefficients of the Hirzebruch–Jung continued fraction expansion of a/r. Similarly, the face $\langle y_0, \eta \rangle$ is the monomial cone of $\frac{1}{s}(b, 1)$ with generators $y_0, y_1, \dots, y_\ell = \eta$.

Setting $A = B = 0$ in V_{AB} leaves a cycle of four surfaces S as above with monomials cones $\langle \xi, \eta \rangle$ for S_1, $\langle \eta, y_0 \rangle$ for S_2 and so on. This gives $V_{AB} \supset S$. Although the cycle S seems fairly symmetric, the toric variety V_{AB} is not: in the image of the monomial cone σ_{AB} under the projection $M \to \mathbb{Z}^2 = M/(A\mathbb{Z} + B\mathbb{Z})$, the span of all x_i and y_j omitting x_0, y_0 lies in a halfspace. in this sense, the surface S_3 lies at the 'big' end of V_{AB} and the surface S_1 lies at the 'little' end.

4. Combining the equations of two toric varieties

We repeat the toric construction for another matrix $\begin{pmatrix} r & g \\ h & s \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and another lattice to get a second toric 4-fold V_{LM}. Again it contains a cycle of toric surfaces. We want this to be the same cycle S but with the big and little ends reversed. Elementary considerations of quotient surfaces show that this occurs exactly when $a + h$ and $b + g$ are congruent to zero modulo both r and s. These Diophantine equations can be solved: the result is that the continued fraction expansion that determines S_4 is repeating of arbitrary length: $[d, e, d, e, \dots]$ for $d, e \geq 1$ (with additional conditions when the product de is small).

In turn, this exposes the birational geometry of V_{AB}: it admits a series of projections $V_{AB} \dashrightarrow V'_{AB} \dashrightarrow \cdots \dashrightarrow V^N_{AB}$ to a toric complete intersection whose equations have leading terms x_1y_0 and x_0y_1. At each stage, the projection corresponds to the elimination of either an x_i or y_j variable in a predictable way. The inverses of these projections can be controlled—they are so-called Kustin–Miller unprojections [5]—and this provides an inductive birational construction of V_{AB} starting from a complete intersection.

We apply this birational construction of V_{AB} to build a diptych 6-fold V. To start, we can deform $V_{AB}^{(N)}$ easily since it is a complete intersection: we simply add to its equations terms from the corresponding equations for V_{LM}. The resulting equations are still homogeneous for the torus $\mathbb{T}$. In fact, the opposing convexity occurring in the equations of V_{AB} and V_{LM}—the fact that they had opposite 'big' ends—implies there are no other monomials at all that can be added preserving homogeneity. Although this convexity does not continue to be quite so strong throughout the sequence of unprojections (V will have equations with terms involving all of A, B, L, M, so not appearing in the equations of either V_{AB} or V_{LM}), it is enough to be able to apply the same sequence of unprojections inductively starting with the 6-fold complete intersection. This boils down to checking that a prescribed divisor lies in the deformation after each unprojection, and so amounts to proving that certain monomials could not appear (preserving homogeneity) in the equations of the unprojection.

References

[1] Altmann, Klaus and Hausen, Jürgen, Polyhedral divisors and algebraic torus actions, Math. Ann. **334**:3 (2006), 557–607.

[2] Kollár, János and Mori, Shigefumi, Classification of three-dimensional flips, J. Amer. Math. Soc. **5**:3 (1992), 533–703.

[3] Kollár, János and Mori, Shigefumi, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, With the collaboration of C. H. Clemens and A. Corti, CUP 1998, viii+254pp.

[4] Mori, Shigefumi, On semistable extremal neighborhoods, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math. **35** (2002), 157–184.

[5] Papadakis, Stavros and Reid, Miles, Kustin-Miller unprojection without complexes, J. Algebraic Geom. **13**:3 (2004), 563–577.

On the Geometry of Coamoebas of Complex Algebraic Hypersurfaces

Mounir Nisse

Amoeba and coamoeba are a very fascinating notions in mathematics where the first terminology has been introduced by I. M. Gelfand, M M. Kapranov and A. V. Zelevinsky in their book (see [GKZ-94]) in 1994, and the second one by M. Passare and A. Tsikh in 2001. Amoebas (resp. coamoebas) have their spines, contours and tentacles (resp. spines, contours and extra-pieces), and they have many applications in real algebraic geometry , complex analysis, mirror symmetry, algebraic statistics and in several other areas (see [M1-00], [M2-02], [M3-04], [RST-05], [FPT-00], [PR1-04], and [PS-04]). Amoebas and coamoebas are linked in a natural way to the geometry of Newton polytopes, which can be seen in particular with the Viro patchworking principle (i.e., tropical localization) based on the combinatorics of subdivisions of convex lattice polytopes. The purpose of this talk is to describe the relations and the similarities which exist between amoebas and coamoebas of a complex algebraic hypersurfaces. Let $V \subset (\mathbb{C}^*)^n$ be a complex algebraic hypersurface defined by a polynomial f with Newton polytope Δ. The amoeba $\mathscr{A}$ of an algebraic set $V = \{f(z) = 0\}$ in the algebraic torus $(\mathbb{C}^*)^n$ is defined as its image under the mapping $\mathrm{Log} : (z_1, \dots, z_n) \mapsto (\log \mid z_1 \mid, \dots, \log \mid z_n \mid)$. The amoeba's complement has a finite number of convex connected components, corresponding to domains of convergence of the Laurent series expansions of the rational function $\frac{1}{f}$. We know that the spine Γ of the amoeba $\mathscr{A}$ has a structure of a tropical hypersurface in $\mathbb{R}^n$ (proved by M. Passare and H. Rullgård in 2000 [PR1-04], and independently by G. Mikhalkin in 2000). In addition the spine of the amoeba is dual to some coherent (i.e. convex) subdivision τ of the integer convex polytope Δ. It is shown by M. Forsberg, M. Passare and A Tsikh that the set of vertices of τ is in bijection with the set of complement components of $\mathscr{A}$ in $\mathbb{R}^n$ [FPT-00]. An amoeba is called *solid* if the number of its complement components in $\mathbb{R}^n$ is minimal i.e., equal to the number of the Newton polytope vertices.

The coamoeba $co\mathscr{A}$ of an algebraic set $V = \{f(z) = 0\}$ in $(\mathbb{C}^*)^n$ is defined as its image under the argument mapping $\mathrm{Arg} : (z_1, \dots, z_n) \mapsto (e^{i \arg(z_1)}, \dots, e^{i \arg(z_n)})$. It is shown in [N2-08] that the complement components of the closure in the flat torus of the coamoeba of a complex algebraic hypersurface defined by a polynomial f with Newton polytope Δ are convex and their number don't exceed $n!\,\mathrm{Vol}(\Delta)$. Using geometric properties of coamoebas we prove the famous Passare-Rullgård's conjecture:

Theorem 1 ([N1-07]). Let V_f be an algebraic hypersurface in $(\mathbb{C}^*)^n$ defined by a maximally sparse polynomial f (this means its only monomials are those of index in the vertices of its Newton polytope). Then the amoeba $\mathscr{A}_f$ of V_f is solid.

Theorem 2 ([N2-08]). Let V_f be an algebraic hypersurface in $(\mathbb{C}^*)^n$ defined by a polynomial f with Newton polytope Δ and we denote by $\mathrm{co}\mathscr{A}$ the image of V_f under the argument map Arg. Then we have:

(a) The interior of any connected component of $(S^1)^n \setminus co\mathscr{A}$ is a convex set,
(b) the number of connected components of $(S^1)^n \setminus \overline{co\mathscr{A}}$ is not greater than $n!\,\mathrm{Vol}(\Delta)$ where $\overline{co\mathscr{A}}$ is the closure of $co\mathscr{A}$ in the flat torus $(S^1)^n$.

Theorem 3 ([N3-08]). Let V be a complex algebraic hypersurface defined by a polynomial f and $co\mathscr{A}$ its coamoeba. Then there exists a continuous deformation of the coamoeba $co\mathscr{A}$ into the coamoeba $co\mathscr{A}_\infty$ of a complex tropical hypersurface V_∞, such that the closure in the torus of the two coamoebas have the same topology (i.e., homeomorphic).

This means that the coamoeba of a complex algebraic hypersurface has the same topology (i.e., homeomorphic) of the coamoeba of a complex tropical hypersurface; more precisely, their closure in the real torus have the same topology.

The coamoebas of a complex algebraic plane curves have a similar properties than their amoebas, and we have the following (see [MR-00] for the amoebas):

Theorem 4 ([N4-08])**.** Let V be a complex algebraic plane curve defined by a polynomial with Newton polygon Δ. Then the area counted with multiplicity of the coamoeba of V cannot exceed $2\pi^2\,\mathrm{Area}(\Delta)$, and we have the following equivalent statements:

(i) $\mathrm{Area}_{mult}(co\mathscr{A}) = 2\pi^2\,\mathrm{Area}(\Delta)$,

(ii) The curve V is real up to multiplication by a constant in $\mathbb{C}^*$, and its real part $\mathbb{R}V$ is a Harnack curve possibly with ordinary real isolated double points.

Let us give a brief description of our ideas without technical details. Let A be the support of the polynomial f. The main ingredients in the construction are a special deformation of the standard complex structure on $(\mathbb{C}^*)^n$, Viro's tropical localization, and Kapranov's theorem [K-00]. We construct a family of polynomials f_t with $0 < t \leq \frac{1}{e}$ such that $f_{\frac{1}{e}} = f$, and we consider the family of the J_t-holomorphic hypersurfaces $H_t(\{f_t(z) = 0\})$ where H_t is a self-diffeomorphism of $(\mathbb{C}^*)^n$ conserving the arguments. When t tends to zero, we obtain a complex tropical hypersurface V_∞, such that its coamoeba is a deformation of the coamoeba of V which conserve the same topology. In addition, using the subdivision τ of Δ dual to the spine of the amoeba of V and Kapranov's theorem [K-00], we have an algorithm giving an explicit description of the coamoeba of V_∞. In other word, the results are obtained by deformation of the complex structure on the hypersurface to a degenerate structure called complex tropical structure which is a piecewise-linear polyhedral complex in $\mathbb{R}^n$ supplied with some lifting to $(\mathbb{C}^*)^n$ (see G. Mikhalkin [M1-00] and [M2-02]).

We give an example of a polygon which can be never the Newton polygon of a real algebraic plane curve (I mean defined over $\mathbb{R}$) realizing the maximum number of complement components of the coamoeba, but this maximum can be realized by a complex plane curve.

References

[FPT-00] M. Forsberg, M. Passare and A. Tsikh, *Laurent determinants and arrangements of hyperplane amoebas*, Advances in Math. **151**, (2000), 45-70.

[GKZ-94] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Birkhäuser Boston 1994.

[K-00] M. M. Kapranov, *Amoebas over non-Archimedean fields*, Preprint 2000.

[M1-00] G. Mikhalkin, *Real algebraic curves, moment map and amoebas*, Ann.of Math. **151** (2000), 309-326.

[M2-02] G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*,Topology **43**, (2004), 1035-1065.

[M3-04] G. Mikhalkin, *Enumerative Tropical Algebraic Geometry In* $\mathbb{R}^2$, J. Amer. Math. Soc. **18**, (2005), 313-377.

[MR-00] G. Mikhalkin and Rullgärd, *Amoebas of maximal area*, Internat. Math. Res. Notices **9**, (2001), 441-451.

[N1-07] M. Nisse, *Maximally sparse polynomials have solid amoebas*, Preprint 2007, http://fr.arXiv.org/pdf/0704.2216

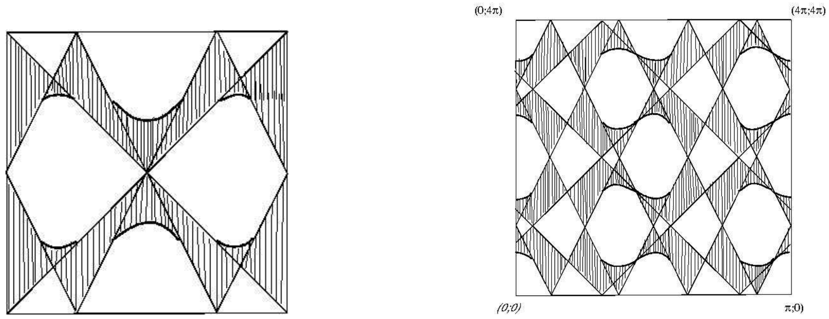

Figure 1. On the left the coamoeba of a real algebraic curve (with five complement components) and on the right the coamoeba of a complex algebraic curve with the same Newton polygon as the first one (with six complement components).

[N2-08] M. Nisse, *Coamoebas of Complex Algebraic Hypersurfaces*, Preprint (2008).

[N3-08] M. Nisse, *Amoebas and coamoebas, relationships and similarities*, Preprint (2008).

[N4-08] M. Nisse, , Preprint (2008), http://fr.arXiv.org/pdf/0805.2872

[PS-04] L. Pachter and B. Sturmfels, *Algebraic Statistics for Computational Biology*, Cambridge University Press, 2004.

[PR1-04] M. Passare and H. Rullgård, *Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope*, Duke Math. J. **121**, (2004), 481-507.

[RST-05] J. Richter-Gebert, B. Sturmfels et T. Theobald , *First steps in tropical geometry*, Idempotent mathematics and mathematical physics, Contemp. Math., **377**, (2005), 289-317 , Amer. Math. Soc., Providence, RI, 2005.

[V1-90] O. Viro, *Patchworking real algebraic varieties*, preprint: http://www.math.uu.se/ oleg; arXiv: AG/0611382

Symplectic Duality and Hypertoric Varieties

Tom Braden

(joint work with Anthony Licata, Nicholas Proudfoot and Ben Webster)

Symplectic duality is a (still largely conjectural) set of relations between certain pairs of symplectic algebraic varieties. More precisely, it involves data $(\mathfrak{M}, T, \eta, \xi)$, where fM is a smooth variety over $\mathbb{C}$ with a holomorphic symplectic form $\omega_{\mathbb{C}}$, T is an algebraic torus which acts with finite fixed-point set, η is a class in $H^2(\mathfrak{M};\mathbb{Z})$ and $\xi\colon \mathbb{C}^* \to T$ is a cocharacter of T. These are required to satisfy:

- The affinization map $\alpha\colon \mathfrak{M} \to \mathfrak{M}_0 = \operatorname{Spec}\Gamma(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}})$ is a resolution of singularities,
- η is the first Chern class of a line bundle on $\mathfrak{M}$ which is relatively ample for α, and
- the image $\xi(\mathbb{C}^*)$ has the same fixed point set as T.

Symplectic duality is a relation between pairs $(\mathfrak{M}, T, \eta, \xi)$, $(\mathfrak{M}^\vee, T^\vee, \eta^\vee, \xi^\vee)$ of data of this type. These pairs have appeared in mathematical physics, as Higgs branches of dual $N = 4$, $d = 3$ supersymmetric gauge theories [6] (classical mirror symmetry involves $d = 2$ theories). However, our conjectures go beyond what is currently predicted in the physics literature. Examples of symplectic dual pairs include the following.

(1) $\mathfrak{M} = T^*(G/B)$, the cotangent bundle to the flag variety of a semisimple algebraic group G. The torus T is a Cartan subgroup contained in B, the parameter η is given by a dominant weight, and ξ comes from a dominant coweight. The symmetric dual variety is $T^*({}^LG/{}^LB)$, where ${}^LG$ is the Langlands dual group.
(2) For $G = SL_n(\mathbb{C})$, a resolution of $\overline{O_\lambda} \cap N_\mu$ is dual to a resolution of $\overline{O_{\mu^t}} \cap N_{\lambda^t}$, where O_λ denotes the nilpotent orbit with Jordan form λ and N_λ is a transverse slice to O_λ.
(3) Certain pairs of Nakajima quiver varieties for the quiver of affine type A should be symplectic dual. As a special case, the Hilbert scheme of n points in $\mathbb{C}^2$ should be self-dual.
(4) A hypertoric variety defined by a hyperplane arrangement (see the discussion below) is symplectic dual to the hypertoric variety defined by the Gale dual arrangement.

There are a number of relations that should be satisfied by symplectic dual pairs. They are all known in case (1), and by our recent work [2, 3, 4], in case (4). Specifically, symplectic dual pairs should have:

(a) natural isomorphisms $H^2(\mathfrak{M};\mathbb{C}) \cong \operatorname{Lie} T^\vee$ and $H^2(\mathfrak{M}^\vee;\mathbb{C}) \cong \operatorname{Lie} T$, under which the parameter η is sent to $\xi^\vee$ and ξ is sent to $\eta^\vee$,
(b) a natural bijection between the fixed point sets $\mathfrak{M}^T$ and $(\mathfrak{M}^\vee)^{T^\vee}$, inducing a bijection between the components of the Lagrangian varieties $\mathcal{X} = \{x \in \mathfrak{M} \mid \lim_{t\to\infty} \xi(t)\cdot x \text{ exists}\}$ and $\mathcal{X}^\vee = \{y \in \mathfrak{M}^\vee \mid \lim_{t\to\infty} \xi^\vee(t)\cdot y \text{ exists}\}$,
(c) a natural order-reversing bijection between strata of the affine varieties $\mathfrak{M}_0$ and $\mathfrak{M}_0^\vee$ (they are Poisson varieties, and so have natural stratifications by Poisson leaves),
(d) a perfect pairing between compactly supported cohomology groups $H_c^*(\mathcal{X})$ and $H_c^*(\mathcal{X}^\vee)$ which induces perfect pairings on the associated graded pieces for the filtrations induced by the decomposition theorem applied to the resolutions $\mathfrak{M} \to \mathfrak{M}_0$ and $\mathfrak{M}^\vee \to \mathfrak{M}_0^\vee$ (note that these groups are nonzero in only one degree),
(e) a perfect pairing between the equivariant homology spaces $H_2^T(\mathfrak{M}) = \operatorname{Spec}(\operatorname{Sym}(H_T^2(\mathfrak{M}))$ and $H_2^{T^\vee}(\mathfrak{M}^\vee) = \operatorname{Spec}(\operatorname{Sym}(H^2_{T^\vee}(\mathfrak{M}^\vee))$, under which the images of the affine varieties $\operatorname{Spec}(H_T^*(\mathfrak{M}))$ and $\operatorname{Spec}(H^*_{T^\vee}(\mathfrak{M}^\vee))$ are orthogonal arrangements of subspaces,
(f) and finally, certain abelian categories associated to the data $(\mathfrak{M}, T, \eta, \xi)$ and $(\mathfrak{M}^\vee, T^\vee, \eta^\vee, \xi^\vee)$ should be Koszul dual. These are categories of modules over quantizations of the structure sheaves of $\mathfrak{M}$ and $\mathfrak{M}^\vee$, supported on the Lagrangians $\mathcal{X}$ and $\mathcal{X}^\vee$.

All of these properties are known in the case (1) of cotangent bundles to flag varieties. In this case, the category that appears in (f) is a regular block of Bernstein-Gelfand-Gelfand category $\mathcal{O}$, and Koszul duality was established by Beilinson, Ginzburg, and Soergel [1]. The linear duality (e) was first noticed by Goresky and MacPherson [7], where they established it in case (2) for $\mu = (n)$.

Hypertoric varieties

The papers [2, 3, 4] show that all of these aspects of symplectic duality also hold for hypertoric varieties, whose very explicit geometry makes direct calculations possible. A hypertoric variety $\mathfrak{M} = \mathfrak{M}(V, \eta)$ is determined by giving a linear subspace $V \subset \mathbb{R}^n$, defined over $\mathbb{Q}$, and a vector $\eta \in \mathbb{R}^n/V$ which lies in the image of the lattice $\mathbb{Z}^n$. The torus $(\mathbb{C}^*)^n$ acts on $T^*\mathbb{C}^n$ preserving the natural holomorphic symplectic form, with moment map

$$\Phi(\mathbf{z}, \mathbf{w}) = (z_1 w_1, \ldots, z_n w_n).$$

The hypertoric variety $\mathfrak{M}$ is defined to be the GIT quotient

$$\Phi^{-1}(V) /\!\!/_{\eta}\, G,$$

where $G \subset (\mathbb{C}^*)^n$ is the subtorus with Lie algebra $V^\perp \otimes_{\mathbb{R}} \mathbb{C}$. It carries an induced symplectic form which is preserved by the induced action of the quotient torus $T := (\mathbb{C}^*)^n/G$.

Just as a semiprojective toric variety can be described by its moment polyhedron, the resulting hypertoric variety is governed by the hyperplane arrangement in the affine space V_η (the linear space V translated away from the origin by η) whose hyperplanes are $H_i = \{x \in V_\eta \mid x_i = 0\}$, $i = 1, \ldots, n$. For instance, $\mathfrak{M}$ has at worst orbifold singularities if and only if the arrangement is simple, which happens for generic values of the parameter η. For each sign vector $\alpha \in \{\pm 1\}^n$, the toric variety X_α with moment polyhedron

$$\Delta_\alpha := \{x \in V_\eta \mid \alpha(i) x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$$

appears as a (possibly empty) T-invariant Lagrangian subvariety of $\mathfrak{M}$.

A covector $\xi \in V^*$ which takes integral values on $V \cap \mathbb{Z}$ defines a cocharacter $\mathbb{C}^* \to T$. For generic ξ it will have the same fixed point set as T, and the components of the Lagrangian $\mathcal{X}$ are just the toric varieties X_α for which the moment polyhedron Δ_α is nonempty and bounded above under the linear function ξ.

For such parameters V, η, ξ, the symplectic dual variety $\mathfrak{M}^\vee$ is defined by the Gale dual triple $(V^\perp, -\xi, -\eta)$. Properties (a) and (e) of symplectic dual varieties follow essentially by definition, using natural identifications $H^2(\mathfrak{M}; \mathbb{R}) \cong \mathbb{R}^n/V$ and $H^2_T(\mathfrak{M}; \mathbb{R}) \cong \mathbb{R}^n$. Properties (b) and (c) are easy consequences of combinatorial facts about Gale duality.

Property (f) (Koszul duality) is the subject of [3], which gives two presentations of an algebra in terms of the data (V, η, ξ), so that modules over this algebra give the required category $\mathcal{O}$. The first presentation is by generators and relations, while the second is geometric, putting a convolution product on the vector space

$$\bigoplus_{\alpha,\beta} H^*(X_\alpha \cap X_\beta)[-d_{\alpha\beta}],$$

where the sum is over α and β with $X_\alpha, X_\beta \subset \mathcal{X}$, and $d_{\alpha\beta}$ is the codimension of $\Delta_\alpha \cap \Delta_\beta$ in V_η. Combinatorial arguments show that this ring is Koszul, quasi-hereditary, and Koszul dual to the same ring for the Gale dual arrangement. A forthcoming paper [5] will show that modules over this ring are equivalent to modules over a quantization of $\mathcal{O}_{\mathfrak{M}}$ supported on $\mathcal{X}$.

We speculate that the Koszul duality property (f) is the most fundamental aspect of symplectic duality. To that end, we show [2] with Chris Phan that suitable pairs of Koszul dual algebras define orthogonal subspace arrangements as in (e), without reference to any variety. The paper [4] will treat the duality (d). Under appropriate hypotheses, it can be deduced from Koszul duality as well.

References

[1] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[2] Tom Braden, Anthony Licata, Christopher Phan, Nicholas Proudfoot, and Ben Webster, *Goresky-MacPherson duality and deformations of Koszul algebras*, In preparation.

[3] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Gale duality and Koszul duality*, preprint arXiv:0806.3256.

[4] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Cohomological symplectic duality*, In preparation.

[5] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Categorical symplectic duality*, In preparation.

[6] Jan de Boer, Kentaro Hori, Hirosi Ooguri, and Yaron Oz, *Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*, Nuclear Phys. B **493** (1997), no. 1-2, 101–147.

[7] Mark Goresky and Robert MacPherson, *On the spectrum of the equivariant cohomology ring*, preprint available at http://www.math.ias.edu/∼goresky/pdf/Spec6.pdf.

Coherent Sheaves on Toric Complete Intersections

Vladimir Baranovsky

Let V be a finite dimensional vector space over a field k of characteristic zero and $\mathbb{P}(V)$ the projective space of lines in V. By well-known results of Serre vector bundles on $\mathbb{P}(V)$, and more generally coherent sheaves, may be described via graded modules over the polynomial algebra $Sym^\bullet(V^*) =: S$.

By a result of Bernstein-Gelfand-Gelfand, cf. [BGG], modules over S are also related to modules over the exterior algebra $E = \Lambda^\bullet(V)$. It turns out that the two algebras are *derived equivalent*, i.e. one can identify complexes of finitely generated S-modules up to quasi-isomorphism and complexes of finitely generated E-modules up to quasi-isomorphism. This result has had many applications including description of parameter spaces of algebraic vector bundles on $\mathbb{P}(V)$.

Later Kapranov, cf. [Ka], have generalized the BGG correspondence to the case of a projective variety $X \subset \mathbb{P}(V)$ which is a complete intersection given by quadratic equations $W_1 = \ldots = W_m = 0$. In this case S should be replaced with the quotient $S_W = S/\langle W_1, \ldots, W_m \rangle$ by the ideal generated by the equations. Kapranov explains in *loc. cit.* that one should modify the algebra E by considering the vector space $E_W = \Lambda^\bullet(V) \otimes_k k[z_1, \ldots, z_m]$ with a product structure in which

$$v_1 v_2 + v_2 v_1 = \sum_{j=1}^{m} Q_j(v_1, v_2) z_j$$

where Q_j is the bilinear form corresponding to the quadratic equation W_j. Then the derived equivalence holds for S_W and E_W.

I consider a more general case when $Sym^\bullet(V^*)$ has a grading by a finitely generated abelian group A, as it happens for the Cox ring of a toric variety $\mathbb{P}$, and the equations $W_1 = \ldots = W_m = 0$ defining a complete intersection $X \subset \mathbb{P}$, are homogeneous in the A grading (but not necessarily the usual $\mathbb{Z}$-grading on the polynomial ring). One can still consider the ring E_W by taking Q_j which come from the quadratic parts of W_j but it is unreasonable to hope that E_W will contain the full information about sheaves on X (e.g. when all W_j have degree $d \geq 3$ in the usual grading, the product in E_W has no information about X at all).

Nevertheless it turns out, cf. [B1], [B2], that it is possible to repair the situation by introducing an A_∞-structure on E_W, i.e. a series of higher products $m_k : E_W^{\otimes k} \to E_W$ which satisfy a chain of identities generalizing associativity for m_2. The concept of a finitely generated module over E_W should be adjusted accordingly, and this gives a derived equivalence between complexes of finitely generated S_W-modules and those of E_W-modules. Roughly speaking, the parts of W_j which have degree d in the usual grading "are responsible for" the products m_d, $d \geq 2$.

In more detail: the original BGG correspondence uses the classical Koszul complex

$$\ldots \to \Lambda^2(V) \otimes_k Sym^\bullet(V) \to \Lambda^1(V) \otimes_k Sym^\bullet(V) \to Sym^\bullet(V) \to 0$$

to convert $\Lambda^\bullet(V)$-modules into $Sym^\bullet(V^*)$-modules. In Kapranov's version the quadratic equations W_j define a structure of a graded Lie algebra on a vector space $L = V \oplus U$ where U is the vector space spanned by $z_1, \ldots, z_m$, and V is placed in homological degree one while U is placed in homological degree two. Then then algebra E_W can be identified with the universal enveloping $U(L)$ of L and the Koszul complex turns into a differential on a tensor product $U(L) \otimes_k C(L)$ where $C(L)$ is the graded exterior algebra of L with the differential which encodes its Lie structure (the so-called Cartan-Chevalley-Eilenberg complex of L).

In the general case I consider the same vector space L but now the homogeneous components of W_j define a structure of a *homotopy* Lie algebra on L (also called an L_∞-algebra) which consists of a sequence of higher brackets $l_k : L^{\otimes k} \to L$ which are antisymmetric in the graded sense, and satisfy a chain of identities which generalize the classical Jacobi identity. It turns out, cf. [B2], that for such objects one can still define a universal enveloping $U(L)$, which is an A_∞-algebra. The differential of the Koszul complex generalizes to a differential on $C(L) \otimes_k U(L)$, which allows to convert (complexes of) S_W-modules into modules over $E_W := U(L)$.

To pass from A-graded modules over S_W to coherent sheaves on the toric complete intersection X, one has to "eliminate" the modules supported on the "irrelevant ideal" $V(B) \in S_W$. I discuss the results describing the image of such modules under the generalized BGG correspondence, as well as a closely related operation of "eliminating" complexes of projective modules, which gives a mathematical version of the physical Landau-Ginzburg B-model arising from the potential $z_1 W_1 + \ldots + z_m W_m$.

References

[B1] Baranovsky, V.: BGG correspondence for toric complete intersections, *Mosc. Math. J.* **7** (2007), no. 4, 581-599.

[B2] Baranovsky, V.: A universal enveloping for L_∞-algebras, *Math.Res. Lett.* **15**(2008).

[BGG] Bernstein, I. N.; Gelfand, I. M.; Gelfand, S. I.: Algebraic vector bundles on P^n and problems of linear algebra, *Functional Anal. Appl.* **12** (1978), no. 3, 212-214.

[Ka] Kapranov, M.: On the derived category and K-functor of coherent sheaves on intersections of quadrics, *Math. USSR-Izv.* **32** (1989), no. 1, 191-204.

Reporter: Lars Petersen

Participants

Prof. Dr. Klaus Altmann
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Michael Bailey
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Vladimir Baranovsky
Department of Mathematics
University of California at Irvine
Irvine , CA 92697-3875
USA

Prof. Dr. Victor V. Batyrev
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Rene Birkner
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Dr. Mark Blume
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Prof. Dr. Thomas C. Braden
Dept. of Mathematics & Statistics
University of Massachusetts
710 North Pleasant Street
Amherst , MA 01003-9305
USA

Dr. Michel Brion
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Gavin D. Brown
Institute of Mathematics,
Statistics & Actuarial Science
University of Kent
GB-Canterbury Kent CT2 7NF

Prof. Dr. Victor M. Buchstaber
Steklov Mathematical Institute
Russian Academy of Science
Gubkina 8
119 991 Moscow GSP-1
RUSSIA

Prof. Dr. Linda Chen
Dept. of Mathematics and Statistics
Swarthmore College
500 College Ave.
Swarthmore PA 19081
USA

Prof. Dr. Alastair Craw
Department of Mathematics
University of Glasgow
University Gardens
GB-Glasgow G12 8QW

Prof. Dr. Sandra Di Rocco
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
S-100 44 Stockholm

Prof. Dr. Alicia Dickenstein
Depto. de Matematica - FCEYN
Universidad de Buenos Aires
Ciudad Universitaria
Pabellon 1
Buenos Aires C 1428 EGA
ARGENTINA

Prof. Dr. Matthias Franz
Department of Mathematics
The University of Western Ontario
London ONT N6A 5B7
CANADA

Dr. Christian Haase
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Fatima Haddad
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Prof. Dr. Jürgen Hausen
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Dr. Milena Hering
Inst. for Math. and Applications
University of Minnesota
400 Lind Hall
207 Church Street S.E.
Minneapolis , MN 55455-0436
USA

Prof. Dr. Lutz Hille
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Laura Hinsch
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Andreas Hochenegger
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Nathan Ilten
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Prof. Dr. Yael Karshon
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Lars Kastner
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Prof. Dr. Liat Kessler
Department of Mathematics
MIT
Cambridge , MA 02139
USA

Dr. Valentina Kiritchenko
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Dr. Yi Lin
Department of Mathematical Sciences
Georgia Southern University
Statesboro GA 30460-8093
USA

Prof. Dr. Diane Maclagan
Mathematics Institute
Zeeman Building
University of Warwick
GB-Coventry CV4 7AL

Prof. Dr. Dusa McDuff
Department of Mathematics
Columbia University
2990 Broadway
New York , NY 10027
USA

Dr. Benjamin Nill
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Dr. Mounir Nisse
Université Paris 6
IMJ (UMR 7586)
Laboratoire Analyse Algébrique
175, rue du Chevaleret
F-75013 Paris

Dr. Taras E. Panov
Department of Geometry and Topology
Faculty of Mechanics and Mathematics
Moscow State University
Leninskie Gory
119992 Moscow
RUSSIA

Prof. Dr. Sam Payne
Department of Mathematics
Stanford University
Stanford , CA 94305-2125
USA

Dr. Markus Perling
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

Lars Petersen
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Prof. Dr. Ragni Piene
CMA
Department of Mathematics
University of Oslo
P.O.Box 1053 - Blindern
N-0316 Oslo

Prof. Dr. Martin Pinsonnault
Department of Mathematics
University of Western Ontario
Middlesex College
London , ON N6A 5B7
CANADA

Prof. Dr. Bernd Siebert
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg

Prof. Dr. Gregory G. Smith
Dept. of Mathematics & Statistics
Queen's University
Jeffery Hall
Kingston, Ontario K7L 3N6
CANADA

Prof. Dr. Alan Stapledon
Department of Mathematics
University of Michigan
East Hall, 525 E. University
Ann Arbor , MI 48109-1109
USA

Hendrik Süß
Technische Universität Cottbus
Postfach 101344
03013 Cottbus

Dr. Bernard Teissier
Equipe "Geometrie et dynamique"
Institut Mathematique de Jussieu
175 rue du Chevaleret
F-75013 Paris

Prof. Dr. Jenia Tevelev
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst MA 01003-9305
USA

Prof. Dr. Dmitri Timashev
Department of Higher Algebra
Faculty of Mechanics and Mathematics
Moscow State University
119991 Moscow
RUSSIA

Jaron Treutlein
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Prof. Dr. Miguel Tribolet de Abreu
Departamento de Matematica
Instituto Superior Tecnico
Avenida Rovisco Pais, 1
Lisboa 1049-001
PORTUGAL

Prof. Dr. Michele Vergne
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Robert Vollmert
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Dr. Frederik Witt
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg