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Mini-Workshop: Support Varieties

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Abstract. The notion of support is a fundamental concept which provides a geometric approach for studying various algebraic structures. The prototype for this has been Quillen's description of the algebraic variety corresponding to the cohomology ring of a finite group, based on which Carlson introduced support varieties for modular representations. This has made it possible to apply methods of algebraic geometry to obtain representation theoretic information. Their work has inspired the development of analogous theories in various contexts, notably modules over commutative complete intersection rings, and over cocommutative Hopf algebras. The aim of this workshop has been to bring together experts from these fields and to stimulate interaction and exchange of ideas.

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Introduction by the Organisers

Let K be a field of characteristic p and G a finite group. Quillen gave a description of the cohomology ring $H^*(G, K)$ modulo nilpotent elements as an inverse limit of cohomology rings of elementary abelian p-subgroups of G. This has led to the work of Benson, Carlson and others on the theory of varieties for KG-modules, and in general to deep structural information about modular representations of finite groups. Inspired by this success similar theories have been developed in other contexts. This includes p-Lie algebras, finite group schemes, and complete intersection rings in commutative algebra. More recently, support varieties have been constructed for Lie superalgebras. Going in a different direction, Snashall and Solberg initiated the construction of support varieties for modules of more general finite dimensional algebras, via the Hochschild cohomology, with appropriate finite generation properties. Furthermore, generalising Jon Carlson's construction for group algebras in several contexts, rank varieties have been introduced and shown to be isomorphic to support varieties. Some work towards a unified approach has been done, in particular by Balmer, and then by Buan, Krause and Solberg. This workshop has brought together experts working on the various aspects of support in different areas, to review what is known, and to clarify unified concepts. The focus was on the following three aspects: the theory, computations and applications. Introductory surveys were given by Petter Bergh (support via central ring actions), Ivo Dell'Ambrogio (tensor triangular geometry) and Dan Nakano (applications of support varieties). Then there were 9 talks presenting recent developments in the subject. Three additional evening sessions completed the picture: Ralf Kroemer (one of the organisers of a parallel workshop on the history of category theory) presented a portrait of Samuel Eilenberg, who contributed towards the homological foundations for today's work on support varieties, Dan Nakano explained techniques for calculating support varieties, and a third evening was used for a problem session. The mix of participants from different areas and the relatively small size of the workshop provided an ideal atmosphere for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

Mini-Workshop: Support Varieties

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Support via central ring actions

Petter Andreas Bergh

The notions of central ring actions and support in triangulated categories have proved quite fruitful recently, cf. [1], [2], [3], [4]. They unify ideas and techniques from group cohomology, commutative ring theory (complete intersections) and Hochschild cohomology. This brief survey is an introduction to the basic concepts.

Let $\mathcal{T}$ be a triangulated category with suspension functor Σ. A subcategory of $\mathcal{T}$ is *thick* if it is a full triangulated subcategory closed under direct summands. Given an object $X \in \mathcal{T}$, we denote by $\operatorname{thick}_{\mathcal{T}}(X)$ the smallest thick subcategory of $\mathcal{T}$ containing X; this is the intersection of all thick subcategories containing X.

The *graded center* $Z^*(\mathcal{T})$ of $\mathcal{T}$ is a graded ring, whose degree n component $Z^n(\mathcal{T})$ (for $n \in \mathbb{Z}$) consists of the natural transformations $\operatorname{Id} \xrightarrow{f} \Sigma^n$ satisfying $f_{\Sigma X} = (-1)^n \Sigma f_X$ on the level of objects. For such a central element f and objects $X, Y \in \mathcal{T}$, consider the graded group $\operatorname{Hom}^*_{\mathcal{T}}(X,Y) = \oplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^i Y)$. The element f acts from the right on this graded group via the morphism $X \xrightarrow{f_X} \Sigma^n X$, and from the left via the morphism $Y \xrightarrow{f_Y} \Sigma^n Y$. Namely, given a morphism $g \in \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^m Y)$, the scalar product gf is the composition $X \xrightarrow{f_X} \Sigma^n X \xrightarrow{\Sigma^n g} \Sigma^{m+n} Y$, whereas fg is the composition $X \xrightarrow{g} \Sigma^m Y \xrightarrow{\Sigma^m f_Y} \Sigma^{m+n} Y$. However, since $\operatorname{Id} \xrightarrow{f} \Sigma^n$ is a natural transformation, the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & \Sigma^m Y \\ \Big\downarrow{\scriptstyle f_X} & & \Big\downarrow{\scriptstyle f_{\Sigma^m Y}} \\ \Sigma^n X & \xrightarrow{\Sigma^n g} & \Sigma^{m+n} Y \end{array}$$

commutes, and so since $f_{\Sigma^m Y}$ equals $(-1)^{mn} \Sigma^m f_Y$ we see that $gf = (-1)^{mn} fg$. Thus $Z^*(\mathcal{T})$ acts graded-commutatively on $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ for all objects X and Y in $\mathcal{T}$. For further details on the graded center and its action on the cohomology groups, see [5].

Now let $R = \oplus_{n=0}^{\infty} R_n$ be a positively graded ring which is graded-commutative, i.e. $rs = (-1)^{|r||s|} sr$ for all homogeneous elements $r, s \in R$. Then R *acts centrally* on $\mathcal{T}$ if there exists a homomorphism $R \to Z^*(\mathcal{T})$ of graded rings. Thus, for every object $X \in \mathcal{T}$, there is a homomorphism $R \xrightarrow{\varphi_X} \operatorname{Hom}^*_{\mathcal{T}}(X,X)$ satisfying the following: for every object Y and all homogeneous elements $r \in R$ and $g \in \operatorname{Hom}^*_{\mathcal{T}}(X,Y)$, the equality

$$g \cdot \varphi_X(r) = (-1)^{|r||g|} \varphi_Y(r) \cdot g$$

holds. In other words, the left and right R-module structures on $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ coincide up to sign.

Example. *Let k be a commutative ring, and let Λ, Γ, Δ be k-algebras which are projective as k-modules. Furthermore, let ${}_\Lambda B_\Delta, {}_\Lambda B'_\Delta, {}_\Delta M_\Gamma, {}_\Delta N_\Gamma$ be bimodules with B and B' both Δ-projective. Let $\eta \in \operatorname{Ext}^n_{\Lambda\otimes_k\Delta^{\mathrm{op}}}(B, B')$ and $\theta \in \operatorname{Ext}^m_{\Delta\otimes_k\Gamma^{\mathrm{op}}}(M, N)$ be homogeneous elements. Then $B \otimes_\Delta \theta$ and $B' \otimes_\Delta \theta$ are exact since B and B' are Δ-projective, whereas $\eta \otimes_\Delta M$ and $\eta \otimes_\Delta N$ are exact since the short exact sequences comprising η split as sequences of Δ-modules. It was proved in* [6] *that the equality*

$$(\eta \otimes_\Delta N) \circ (B \otimes_\Delta \theta) = (-1)^{mn}(B' \otimes_\Delta \theta) \circ (\eta \otimes_\Delta M)$$

holds, where both sides are elements of $\operatorname{Ext}^{m+n}_{\Lambda\otimes_k\Gamma^{\mathrm{op}}}(B \otimes_\Delta M, B' \otimes_\Delta N)$

Specializing to the case $\Lambda = \Gamma = \Delta = B = B' = M = N$, we see that the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda) = \oplus_{n=0}^{\infty} \operatorname{Ext}^n_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(\Lambda, \Lambda)$ *of Λ is graded commutative. Moreover, if $\Lambda = \Delta = B = B'$, $\Gamma = k$ and M, N are left Λ-modules, then for homogeneous elements $\eta \in \operatorname{HH}^*(\Lambda)$ and $\theta \in \operatorname{Ext}^*_\Lambda(M, N)$ we see that the equality*

$$(\eta \otimes_\Lambda N) \circ \theta = (-1)^{|\eta||\theta|}\theta \circ (\eta \otimes_\Lambda M)$$

holds. Consequently, given a k-algebra Λ which is k-projective, for every left Λ-module M there is a graded ring homomorphism

$$\operatorname{HH}^*(\Lambda) \xrightarrow{\varphi_M = -\otimes_\Lambda M} \operatorname{Ext}^*_\Lambda(M, M)$$

satisfying the following: for every left Λ-module N and all homogeneous elements $\eta \in \operatorname{HH}^*(\Lambda), \theta \in \operatorname{Ext}^*_\Lambda(M, N)$, the equality

$$\varphi_N(\eta) \cdot \theta = (-1)^{|\eta||\theta|}\theta \cdot \varphi_M(\eta)$$

holds. Extending to the derived category $D(\Lambda)$ of Λ-modules via stalk complexes, we see that $\operatorname{HH}^*(\Lambda)$ acts centrally on $D(\Lambda)$.

Returning to our triangulated category $\mathcal{T}$ and the graded-commutative ring R acting centrally, let X and Y be objects of $\mathcal{T}$. Then the R-module $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ is *eventually Noetherian*, denoted $\operatorname{Hom}^*_{\mathcal{T}}(X, Y) \in \operatorname{Noeth} R$, if there exists an integer n_0 such that the R-module $\operatorname{Hom}^{\geqq n_0}_{\mathcal{T}}(X, Y) = \oplus_{n=n_0}^{\infty} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ is Noetherian. If, in addition, the R_0-module $\operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ has finite length for $n \gg 0$, then we write $\operatorname{Hom}^*_{\mathcal{T}}(X, Y) \in \operatorname{Noeth}^{\mathrm{fl}} R$ and say that the R-module $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ is *eventually Noetherian of finite length*.

It is not difficult to see that if $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ belongs to $\operatorname{Noeth} R$, then it also belongs to $\operatorname{Noeth} R^{\mathrm{ev}}$, where R^{ev} is the commutative even subring $\oplus_{n=0}^{\infty} R_{2n}$ of R. Similarly, if $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ belongs to $\operatorname{Noeth}^{\mathrm{fl}} R$, then it also belongs to $\operatorname{Noeth}^{\mathrm{fl}} R^{\mathrm{ev}}$. In the latter case, the rate of growth of the sequence $(\ell_{R_0} \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n Y))$ is finite and coincides with the Krull dimension of $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ as an R^{ev}-module (cf. [4, Proposition 2.6].

The support of a pair of objects (with respect to R) is defined in terms of the homogeneous prime spectrum of R^{ev}. Denote by $\operatorname{Proj} R^{\mathrm{ev}}$ the set of homogeneous prime ideals of R^{ev} not containing $\oplus_{n=1}^{\infty} R_{2n}$. Given two objects X and Y of $\mathcal{T}$, we define the *support* of the ordered pair (X, Y) as

$$\operatorname{Supp}^+_R(X, Y) \overset{\text{def}}{=} \{\mathfrak{p} \in \operatorname{Proj} R^{\mathrm{ev}} \mid \operatorname{Hom}^*_{\mathcal{T}}(X, Y)_{\mathfrak{p}} \neq 0\}.$$

In the following theorem, we summarize some of the standard elementary properties of support sets (cf. [1]).

Theorem 1 (Properties of support)**.**

(1) $\operatorname{Supp}_R^+(X,Y) = \operatorname{Supp}_R^+ \operatorname{Hom}_{\mathcal{T}}^{\geqq n}(X,Y)$ *for all* $n \in \mathbb{Z}$.

(2) *If* $\operatorname{Hom}_{\mathcal{T}}^{\geqq n}(X,Y)$ *is a finitely generated* R*-module for some* n*, then*

$$\operatorname{Supp}_R^+(X,Y) = \{\mathfrak{p} \in \operatorname{Proj} R^{\mathrm{ev}} \mid \operatorname{Ann}_{R^{\mathrm{ev}}}\Big(\operatorname{Hom}_{\mathcal{T}}^{\geqq n}(X,Y)\Big) \subseteq \mathfrak{p}\}.$$

In particular, if $\operatorname{Hom}_{\mathcal{T}}^*(X,Y) \in \operatorname{Noeth} R$*, then* $\operatorname{Supp}_R^+(X,Y)$ *is a closed set in* $\operatorname{Proj} R^{\mathrm{ev}}$.

(3) *If* $\operatorname{Hom}_{\mathcal{T}}^*(X,Y) \in \operatorname{Noeth} R$*, then* $\operatorname{Supp}_R^+(X,Y)$ *is empty if and only if* $\operatorname{Hom}_{\mathcal{T}}^*(X,Y)$ *is eventually zero.*

(4) *Given a triangle*

$$Z' \to Z \to Z'' \to \Sigma Z'$$

in $\mathcal{T}$*, there are inclusions*

$$\begin{aligned} \operatorname{Supp}_R^+(X,Z) &\subseteq \operatorname{Supp}_R^+(X,Z') \cup \operatorname{Supp}_R^+(X,Z''), \\ \operatorname{Supp}_R^+(Z,Y) &\subseteq \operatorname{Supp}_R^+(Z',Y) \cup \operatorname{Supp}_R^+(Z'',Y). \end{aligned}$$

(5) *If* G *is an object in* $\mathcal{T}$ *with* $\operatorname{thick}_{\mathcal{T}}(G) = \mathcal{T}$*, then*

$$\operatorname{Supp}_R^+(X,G) = \operatorname{Supp}_R^+(X,X) = \operatorname{Supp}_R^+(G,X).$$

Properties (3) and (5) provide a criterion for a finite dimensional algebra to be Gorenstein. For such an algebra Λ with radical $\mathfrak{r}$, the thick subcategory of $D^b(\Lambda)$ generated by the stalk complex $\Lambda/\mathfrak{r}$ is the whole of $D^b(\Lambda)$.

Corollary 2. *Let* Λ *be a finite dimensional algebra with radical* $\mathfrak{r}$*. Suppose that* $\operatorname{Ext}_\Lambda^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r}) \in \operatorname{Noeth} R$ *for some graded-commutative ring* R *acting centrally on* $D^b(\Lambda)$ *(for example* $R = \operatorname{HH}^*(\Lambda)$*). Then for every finitely generated* Λ*-module* M*, the implications*

$$\operatorname{pd} M < \infty \quad\Leftrightarrow\quad \operatorname{Ext}_\Lambda^n(M,M) = 0 \text{ for } n \gg 0 \quad\Leftrightarrow\quad \operatorname{id} M < \infty$$

hold. In particular, Λ *is Gorenstein.*

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Support and rank varieties for quantum complete intersections

Petter Andreas Bergh
(joint work with Karin Erdmann)

Support varieties for modules over finite dimensional algebras were introduced in [6], using Hochschild cohomology. As shown in [5], when certain finiteness conditions hold, the theory is very similar to the theory of cohomological support varieties for modules over group algebras and commutative complete intersections.

Fix a field k. Let Λ be a finite dimensional k-algebra with radical $\mathfrak{r}$. The Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ is graded commutative, and for every left Λ-module M there is a homomorphism

$$\operatorname{HH}^*(\Lambda) \xrightarrow{\varphi_M = - \otimes_\Lambda M} \operatorname{Ext}^*_\Lambda(M, M)$$

of graded rings. The Hochschild cohomology ring acts graded-commutatively on cohomology groups; for any Λ-module N and homogeneous elements $\eta \in \operatorname{HH}^*(\Lambda)$ and $\theta \in \operatorname{Ext}^*_\Lambda(M, N)$, the equality

$$\varphi_N(\eta) \cdot \theta = (-1)^{|\eta||\theta|} \theta \cdot \varphi_M(\eta)$$

holds.

Definition. Given a commutative graded subalgebra $H \subseteq \operatorname{HH}^*(\Lambda)$, the *support variety* of an ordered pair (M, N) of Λ-modules, with respect to H, is

$$\operatorname{V}_H(M, N) \overset{\text{def}}{=} \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \operatorname{Ann}_H (\operatorname{Ext}^*_\Lambda(M, N)) \subseteq \mathfrak{m}\}.$$

The support variety $\operatorname{V}_H(M)$ of a module is defined to be $\operatorname{V}_H(M, M)$; it is not difficult to show that $\operatorname{V}_H(M)$ equals both $\operatorname{V}_H(M, \Lambda/\mathfrak{r})$ and $\operatorname{V}_H(\Lambda/\mathfrak{r}, M)$. The following theorem summarizes the most important properties. Recall that the complexity $\operatorname{cx} M$ of M is the rate of growth of its minimal projective resolution, whereas the plexity $\operatorname{px} M$ is the rate of growth of its minimal injective resolution.

Theorem 1 ([5]). *Suppose H is Noetherian and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated H-module.*

1. *For all Λ-modules M, N the H-module $\operatorname{Ext}^*_\Lambda(M, N)$ is finitely generated.*
2. *Λ is Gorenstein.*
3. *The equalities $\dim \operatorname{V}_H(M) = \operatorname{cx} M = \operatorname{px} M$ hold. In particular, a module has finite projective (injective) dimension if and only if its support variety is trivial.*
4. *$\dim \operatorname{V}_H(M) = 1$ if and only if M is eventually periodic.*
5. *If V is a homogeneous subvariety of $\operatorname{MaxSpec} H$, then there exists a Λ-module M with $\operatorname{V}_H(M) = V$.*
6. *Suppose Λ is selfinjective and $\operatorname{V}_H(M) = V_1 \cup V_2$ with V_1, V_2 homogeneous subvarieties such that $V_1 \cap V_2$ is trivial. Then $M = M_1 \oplus M_2$ with $\operatorname{V}_H(M_i) = V_i$.*

Suppose now that k is algebraically closed, and fix integers $c \geq 1$ and $a \geq 2$. Define an integer b by

$$b \stackrel{\text{def}}{=} \begin{cases} a/\gcd(a, \operatorname{char} k) & \text{if } \operatorname{char} k > 0 \\ a & \text{if } \operatorname{char} k = 0, \end{cases}$$

and let $q \in k$ be a primitive bth root of unity. Denote by A the *quantum complete intersection* defined by these data, that is, the algebra

$$A \stackrel{\text{def}}{=} k\langle x_1, \dots, x_c\rangle/(\{x_i^a\}_{i=1}^c, \{x_i x_j - q x_j x_i\}_{i<j}).$$

This local algebra is selfinjective of dimension a^c. Note that when $a = 2$ and $q = -1$, then A is the exterior algebra on a c-dimensional k-vector space.

It follows from [3] that there exists a polynomial subalgebra $H = k[\eta_1, \dots, \eta_c]$ of $\operatorname{HH}^*(A)$, with each η_i in degree two, such that the H-module $\operatorname{Ext}_A^*(k,k)$ is finitely generated. Thus the finiteness condition from Theorem 1 is satisfied, and so the support varieties with respect to H encode homological information on the A-modules. However, the algebra also has *rank varieties*. Given a c-tuple $\lambda = (\lambda_1, \dots, \lambda_c) \in k^c$, denote the element $\lambda_1 x_1 + \cdots + \lambda_c x_c \in A$ by u_λ.

Definition. The rank variety of an A-module M is

$$\operatorname{V}_A^r(M) \stackrel{\text{def}}{=} \{0\} \cup \{0 \neq \lambda \in k^c \mid M \text{ is not a projective } k[u_\lambda]\text{-module}\}.$$

The terminology reflects the fact that since $u_\lambda^a = 0$, the algebra $k[u_\lambda]$ is isomorphic to $k[x]/(x^a)$. Hence the condition that M is not $k[u_\lambda]$-projective is equivalent to the condition that the rank of the map $M \xrightarrow{\cdot u_\lambda} M$ be strictly less than $[(a-1)/a] \dim M$.

Thus there are two types of varieties for A-modules. Since we may identify the maximal ideals of H with points in k^c, a natural question arises: is the support variety of a module related to its rank variety? Indeed, for group algebras of elementary abelian p-groups it was conjectured by Carlson (cf. [4]) that the support variety of a module is isomorphic to its rank variety. This was subsequently proved by Avrunin and Scott in [1]. As shown in [2, Theorem 3.6], a similar result holds for our quantum complete intersection A.

Theorem 2. *Let $k^c \xrightarrow{F} k^c$ be the map of affine spaces given by $(\lambda_1, \dots, \lambda_c) \mapsto (\lambda_1^a, \dots, \lambda_c^a)$. Then $F(\operatorname{V}_A^r(M)) = \operatorname{V}_H(M)$ for every A-module M.*

Corollary 3. *For every A-module M, the dimension of the rank variety $\operatorname{V}_A^r(M)$ equals the complexity of M. Moreover, the module is periodic if and only if $\dim \operatorname{V}_A^r(M) = 1$.*

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Hochschild Cohomology and the Centre of the Derived Category

Ragnar-Olaf Buchweitz

1. It seems by now folklore that for any K–algebra A there is a natural homomorphism of graded commutative K–algebras

$$\chi : \mathrm{HH}^\bullet(A) \to Z^\bullet(D(A))$$

from the Hochschild cohomology $\mathrm{HH}^\bullet(A) = \mathrm{HH}^\bullet(A/K, A)$ of A over K to the *graded centre* of the derived category of A. See e.g. [2] for the general construction of this *characteristic homomorphism* in the broader context of analytic spaces.

2. Indeed, if A is projective over K, this homomorphism is already implicit in [3, p.346(5)] as we now explain. Namely, given a (right) A–module M, adding a copy of the free module A if necessary, we may assume that M is an A–*generator*. With $B = \mathrm{End}_A(M)$ its endomorphism ring, the (left) B–module M is then *projective* and satisfies $A \cong \mathrm{End}_{B^{\mathrm{op}}}(M)$.

Accordingly, the spectral sequence from (loc.cit.),

$$\mathrm{HH}^p(A/K, \mathrm{Ext}^q_{B^{\mathrm{op}}}(M, M)) \Longrightarrow \mathrm{Ext}^{p+q}_{B^{\mathrm{op}} \otimes_K A}(M, M)$$

degenerates, identifying $\mathrm{HH}^\bullet(A) \cong \mathrm{Ext}^\bullet_{B^{\mathrm{op}} \otimes_K A}(M, M)$.

Exchanging the roles of A and B results thus in a spectral sequence

$$\mathrm{HH}^p(B/K, \mathrm{Ext}^q_A(M, M)) \Longrightarrow \mathrm{HH}^{p+q}(A)$$

and the edge homomorphism

$$\mathrm{HH}^\bullet(A) \to \mathrm{HH}^0(B/K, \mathrm{Ext}^\bullet_A(M, M)) = \mathrm{Ext}^\bullet_A(M, M)^B \subseteq \mathrm{Ext}^\bullet_A(M, M)$$

is the composition of χ followed by the evaluation $\mathrm{ev}_M : Z^\bullet(D(A)) \to \mathrm{Ext}^\bullet_A(M, M)$, which homomorphism we denote $\chi_M = \mathrm{ev}_M \circ \chi$. It can thus as well be interpreted as the map induced on extension algebras by the forgetful functor along the algebra homomorphism $A \to B^{\mathrm{op}} \otimes_K A$.

3. While the theory of (homological) support varieties, as developed, say, for group rings, some further self-injective algebras, or commutative complete intersections, essentially uses the (radical) of the kernel of χ_M in some suitable (subring of) Hochschild cohomology, the spectral sequence above shows that there is much more to be considered: the spectral sequence is one of graded algebras, thus, $\mathrm{Ext}^\bullet_A(M, M)$ already "knows about" Hochschild cohomology, at least a graded version of it that arises from the natural filtration on the limit of said spectral sequence. Geometrically, this endows the classical support varieties with infinitesimal structure, akin to the normal cone of a subvariety in its ambient space.

4. Not much is known about χ in general, but simple examples show already that the homomorphism is generally neither injective nor surjective.

If we take the principal ideal domains $A = K[x]$, the polynomial ring over a field K, or $A = K[x]_{(x)}$, its localisation at the origin, then χ is easy to describe, as the module categories are hereditary and every complex in the derived category is formal.

Both Hochschild cohomology and the graded centre of the derived category of finitely generated modules are concentrated in degrees $0, 1$, with the degree zero component $\chi^0 : \mathrm{HH}^0(A) \cong A \to Z^0(D^b(\mathrm{mod}\, A))$ an isomorphism.

The first Hochschild cohomology group is given by $\mathrm{HH}^1(A) = \mathrm{Der}_K(A) \cong A\frac{\partial}{\partial x}$, the K–linear derivations of A, while $Z^1(D^b(\mathrm{mod}\, A)) \cong \prod_{\mathfrak{m}} k(\mathfrak{m})^{\mathbb{N}}$, the product indexed by all maximal ideals of A. The component labeled $(\mathfrak{m}, n)$ maps under $\mathrm{ev}_{A/\mathfrak{m}^{n+1}}$ isomorphically onto the socle of $\mathrm{Ext}^1_A(A/\mathfrak{m}^{n+1}, A/\mathfrak{m}^{n+1}) \cong A/\mathfrak{m}^{n+1}$. Just comparing cardinalities, it follows that χ^1 cannot be surjective.

If $f(x) \in A$ is any element, and $D = g(x)\frac{\partial}{\partial x}$ any derivation, then $\chi_{A/(f)}(D) \in \mathrm{Ext}^1_A(A/(f), A/(f))$ is represented, up to sign, by the morphism of complexes

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & A & \xrightarrow{f} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g\cdot f'} & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{-f} & A & \longrightarrow & 0 & & \end{array}$$

that is, $\chi_{A/(f)}(D) \equiv g \cdot f' \pmod f \in A/(f) \cong \mathrm{Ext}^1_A(A/(f), A/(f))$.

If $\mathfrak{m} = (\pi) \subset A$ is a maximal ideal, thus $\pi \neq 0$ irreducible, then $M = A/(\pi^{n+1})$ is indecomposable and $\chi_M(D) = (n+1)g \cdot \pi' \cdot \pi^n \pmod{\pi^{n+1}} \cong (n+1)g\pi' \pmod \pi \in k(\mathfrak{m})$ is the component of $\chi^1(D)$ in $Z^1(D^b(\mathrm{mod}\, A))$ at $(\mathfrak{m}, n)$.

It thus follows that χ is *injective* in case $A = K[x]$, as for every $g \neq 0$ we may find a separable irreducible polynomial π not dividing g. By contrast, for $A = K[x]_{(x)}$ there is a large kernel, $\ker \chi = \ker \chi^1 = \mathfrak{m}\, \mathrm{Der}_K(A) \cong (x)\frac{\partial}{\partial x}$.

5. Inspecting more closely the argument for $A = K[x]$, one proves as well that χ is *injective for any polynomial ring over a field.* In that case, $\mathrm{HH}^\bullet(A) = \mathrm{Hom}_A(\Omega^\bullet_{A/K}, A)$, and for each polyvectorfield $D \in \mathrm{HH}^\bullet(A)$ one finds a prime ideal $\mathfrak{p}$ such that $\chi_{A/\mathfrak{p}}(D) \neq 0$ in $\mathrm{Ext}^\bullet_A(A/\mathfrak{p}, A/\mathfrak{p})$. To verify non-vanishing of $\chi_{A/\mathfrak{p}}(D)$, it suffices to find a suitable differential form $\omega \in \Omega^\bullet_{A/K}$ so that the *Grothendieck residue* of $\chi_{A/\mathfrak{p}}(D)$ on ω is non-zero. This approach is inspired by [5].

6. Turning to a more conceptual perspective, we follow a suggestion by Dwyer, as related to the author by Iyengar: If A is any K–linear abelian category and E the category of K–linear endofunctors on A, then E is again K–linear abelian and the Yoneda Ext-algebra $\mathrm{Ext}^\bullet_{\mathsf{E}}(\mathrm{id}_{\mathsf{A}}, \mathrm{id}_{\mathsf{A}})$ might be considered the Hochschild cohomology of A. We offer the following results supporting this suggestion.

Theorem 4. *With notations as just introduced, we have*

(1) *The Yoneda Ext-algebra* $\mathrm{Ext}^\bullet_{\mathsf{E}}(\mathrm{id}_{\mathsf{A}}, \mathrm{id}_{\mathsf{A}})$ *is graded commutative.*

(2) *There is a natural homomorphism of graded commutative K–algebras*

$$\chi : \mathrm{Ext}^\bullet_{\mathsf{E}}(\mathrm{id}_{\mathsf{A}}, \mathrm{id}_{\mathsf{A}}) \to Z^\bullet(D(\mathsf{A}))$$

so that the composition $\mathrm{ev}_X \circ \chi : \mathrm{Ext}^\bullet_{\mathsf{E}}(\mathrm{id}_{\mathsf{A}}, \mathrm{id}_{\mathsf{A}}) \to \mathrm{Ext}^\bullet_{\mathsf{A}}(X, X)$ *is the natural evaluation map for any* $X \in \mathsf{A}$.

(3) *In case* A *is the category of (right) modules over a* K*–algebra* A*, then the characteristic homomorphism factors naturally as*

$$\chi : \mathrm{HH}^\bullet(A) \xrightarrow{\alpha} \mathrm{Ext}^\bullet_{\mathsf{E}}(\mathrm{id}_{\mathsf{A}}, \mathrm{id}_{\mathsf{A}}) \xrightarrow{\beta} \mathrm{Ext}^\bullet_{A^{\mathrm{op}}\otimes_K A}(A, A) \xrightarrow{\gamma} Z^\bullet(D(A))$$

with β *an* isomorphism *if* A *is* flat *over* K*, and both* α, β isomorphisms *when* A *is* projective *over* K.

The proof of these results uses ideas from [6, 7] for (1), while (2) is essentially elementary. Part (3) follows from a derived version of the classical Eilenberg-Watts Theorem, see [1, 4, 8].

7. This raises some natural questions:

(1) If A is the heart of a t-structure in the (K–linear) triangulated category $\mathcal{T}$, is there then already a characteristic homorphism from the Hochschild cohomology of the heart to the graded centre of $\mathcal{T}$?

(2) If so, is that homomorphism independent of the heart?

(3) Is the counterpart of (3) true in the geometric context for an analytic space or scheme X, with $D(A)$ replaced by $D(X)$, the abelian category $\mathsf{A} = (\mathsf{Q}-)\mathsf{Coh}(X)$ that of (quasi-)coherent sheaves, and $\mathrm{HH}^\bullet(X)$ as defined in [2]?

8. The second question has the following positive partial answer. Over a field K, if $\mathcal{T}$ is a (TR 5)–category, thus, closed under arbitrary direct sums, and $T \in \mathcal{T}$ is a *classical tilting object*, thus, a compact generator without (higher) self-extensions, then it is known that $\mathcal{T} \cong D(\mathrm{End}_{\mathcal{T}}(T))$, and the characteristic homomorphism from the Hochschild cohomology of $A = \mathrm{End}_{\mathcal{T}}(T)$ to the graded centre of $\mathcal{T}$ is, up to an isomorphism of homomorphisms of graded commutative algebras to that centre, independent of the tilting object. In that sense, the derived category of an algebra encodes already both the Hochschild cohomology as well as the characteristic homomorphism.

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Endotrivial module for groups and group schemes

Jon F. Carlson

As this lecture is presented at a workshop on "Support Varieties", it seems appropriate to survey one of the major applications of the theory of support varieties, namely, the proof of the classification of endotrivial modules for p-groups. I will also make some remarks on the efforts by me and Dan Nakano to classify the endotrivial modules for some finite group schemes. Throughout we assume some familiarity with the definitions and properties of support varieties. See [10] for a general reference.

We assume that k is a field of characteristic $p > 0$ and unless otherwise indicated, that G is a finite group whose order is divisible by p. Endotrivial modules were defined by Dade [11]. Originally, the definition was meant only to apply to the case that G is a p-group. The definition goes as follows.

Definition 1. *A kG-module is endotrivial if* $\mathrm{Hom}_k(M, M) \cong k \oplus (\mathrm{proj})$*, as kG-modules.*

Here $\oplus\,(\mathrm{proj})$ means the direct sum with some projective module. The definition says that M is endotrivial if and only if its endomorphism ring is trivial in the stable category of kG-modules modulo projectives. Note that $\mathrm{Hom}_k(M, M) \cong M^* \otimes M$. Consequently, we can form a group of endotrivial modules $T(G)$. The group has elements equivalence classes $[M]$ of endotrivial modules. The relation is that $[M] = [N]$ if $M \oplus P \cong N \oplus Q$ for some projective modules P and Q. The opreation in the group is give by $[M] + [N] = [M \otimes N]$.

The endotrivial modules are the building blocks of the endopermutation modules, modules whose stable k-endomorphism rings are permutation modules. Dade showed that in the case that G is a p-nilpotent group, the endopermutation modules are the sources (in the sense of J. A. Green's theory of vertices and sources) of the simple modules.

Two facts get us started. (1.) (See Dade [11]) If G is an abelian p-group, then $T(G) \cong \mathbb{Z}$ is generated by $\Omega(k)$, the shift of the trivial module. In other words, any indecomposable endotrivial module has the form $\Omega^n(k)$ for some n. (2.) A kG-module is endotrivial if and only if its restriction to every elementary abelian p-subgroup is endotrivial.

Dade's result suggested that it might be possible to classifiy the endotrivial modules over p-groups. After a period of approximately 25 years, the classification was completed by the writer and Jacques Thévenaz [7, 8, 9] drawing on the work of several people, notably [1]. The result was a theorem which says that except in a few circumstances, any endotrivial module is the direct sum of $\Omega^n(k)$ for some n and a projective module. One of the keys to the construction of the proof was the realization that some known exceptional endotrivial modules could be created by carving up syzygies of the trivial module. Specifically, if G is a quaternion group, then Dade had found some exotic endotrivial modules, but the reason for their existence remained something of a mystery. These modules can be created and analyzed using the theory of support varieties.

Suppose that $G = \langle x, y | x^4 = 1, x^2 = y^2 = (xy)^2\rangle$ is a quaternion group of order 8. Let k be a field of characteristic 2 which contains a primitive third root of 1. It is straightforward to calculate that the first two steps in a minimal projective kG-resolution of k have the form

$$0 \longrightarrow \Omega^2(k) \longrightarrow (kG)^2 \longrightarrow kG \xrightarrow{\ \varepsilon\ } k \longrightarrow 0,$$

and hence the dimension of $M = \Omega^2(k)$ is 9. Let $z = x^2$. Then $\langle z\rangle$ is the only nontrivial elementary abelian subgroup of G, and the restriction of M to $\langle z\rangle$ has the form $M_{\langle z\rangle} = k \oplus (k\langle z\rangle)^4$. The restriction of M to $\langle x\rangle$ or to either of the other two cyclic groups of order 4 is the direct sum of a trivial module and two copies of the free module.

Let $\overline{G} = \langle \overline{x}, \overline{y}\rangle = G/\langle z\rangle$. Let $Z = z-1$ be be the generator of the radical of $k\langle z\rangle$ so that $k\overline{G} = kG/Z(kG)$. The modules ZM and $M_0 = M/\{m \in M | ZM = 0\}$ are naturally $k\overline{G}$-modules. Moreover, multiplication by Z induces an isomorphism $M_0 \to ZM$. Let $V \subseteq V_{\overline{G}}(k) = k^2$ be the support variety of ZM as a $k\overline{G}$-module. There are two important things to notice about V.

(1) V is an $\mathbb{F}_2$-rational variety, meaning that it is defined by polynomials over $\mathbb{F}_2$, because $M = \Omega^2(k)$ is defined over $\mathbb{F}_2$.
(2) V does not contain any $\mathbb{F}_2$-rational points, because the $\mathbb{F}_2$-rational points correspond to the subgroups $\langle\overline{x}\rangle = \langle x\rangle/\langle z\rangle$, $\langle\overline{y}\rangle$ and $\langle\overline{xy}\rangle$ all of which act freely on ZM.

Now ZM is not a cyclic module as otherwise it would be free as a $k\overline{G}$-module. Because ZM has dimension 4, its radical has dimension 2, and the variety V is the zero set of a polynomial f of degree 2. By condition (1), we can assume that f has coefficients in $\mathbb{F}_2$, while condition (2) tells us that f is irreducible over $\mathbb{F}_2$. The only possibility is that f has the form $f(t) = t^2 + t + 1 = (t-\alpha)(t-\beta)$ where α and β are primitive cube roots of unity. All of this means that the variety of ZM is the union of two lines through the points $(1, \alpha)$ and $(1, \beta)$, which intersect in the zero point. Consequencly, the module ZM also decomposes as $ZM = L_1 \oplus L_2$ where the variety of L_1 is one of the lines and the variety of L_2 is the other line.

We now know that M has the form in the diagram:

$$\begin{array}{ccccc} L_1 & & \oplus & & L_2 \\ & \diagdown & & \diagup & \\ & & k & & \\ & \diagup & & \diagdown & \\ L_1 & & \oplus & & L_2 \end{array}$$

Moreover multiplication by Z takes the $L_1 \oplus L_2$ in the top isomorphically to the bottom of the diagram. To finish the construction, we simply take U to be the submodule generated by L_1 so that $M/U \cong L_2$, and let $N = U/L_2$. So N has a filtration with factors L_1 at the top, k in the middle and L_1 at the bottom. Moreover multiplication by Z takes the L_1 at the top isomorphically to the L_1 at the bottom. So the restriction of N to $\langle z \rangle$ is the direct sum of a copy of k and two copies of $k\langle z \rangle$. Hence it is an endotrivial module and N is an endotrivial kG-module. Note that we could do the same thing taking the submodule of M generated by L_2 and factoring L_1 from the bottom.

It turns out that a complete set of generators for the group of endotrivial modules can be constructed in this way. Moreover, we use this to prove that except when $p = 2$ and G is a quaternion or semi-dihedral group, $T(G)$ is torsion free. The steps in such a proof are basically the following.

1. First reduce to the case that G is an extra special p-group or an almost extraspecial p-group [7]. This means that the commutator subgroup and Frattini subgroup are a cyclic subgroup $\langle z \rangle$ of order p and the center of G is cyclic of order at most p^2. This allows us to use a method similar to the above analysis in the quaternion case.

2. By a similar method as above show that if there is a nontrivial torsion endotrivial module M then there is one such that $(z-1)^{p-1}M$ has a support variety over $G/\langle z \rangle$ which is a single line. Now show that if we tensor two of these with nonintersecting sets of lines, then the resulting module N has the property that its $G/\langle z \rangle$ support variety is the union of the sets of lines. So applying the automorphism group of G (which is something like a symplectic group), then we can get many different lines and by tensoring we get a very large indecomposable torsion endotrivial module.

3. Apply a theorem that shows that there is a bound on the dimension of any torsion endotrivial module [10]. A careful analysis of the bounds shows that the module constructed in (2) has dimension exceeding this bound [8]. Thus we have a contradiction.

Alperin [1] constructed a set of generators for a subgroup of finite index for the torsion free part of $T(G)$ and in [9] it is shown that these module generate the entire torsion free part. The method of the quaternion example can also be used to construct a complete set of generators for the torsion free part of $T(G)$ as well as the torsion part when G is a p-group. The details of the this analysis can be found in [3].

For groups other than p-groups, some work has been done for general situations [12, 13] and for simple groups [4, 5, 6]. Balmer [2] has made connections with the Picard group of the spectrum of the stable category. Nakano and the author have had some success extending the results to more general finite group schemes. For example, in the case of a p-restricted Lie algebra whose cohomology ring satisfies some mild conditions on dimension, it can be shown that the group of endotrivial modules is isomorphic to $\mathbb{Z}$ and generated by the class of $\Omega(k)$. It is interesting to note that one of the open problems in this case is whether the group of endotrivial modules is finitely generated. That is, for general group schemes it is not know if an indecomposable torsion endotrivial module must have bounded dimension as was true and essential in the above step 3 for finite groups.

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Notes from a problem session on support varieties

Xiao-Wu Chen

I. Known Results:

The following tables summarize our present knowledge about the support/rank varieties of certain classes of algebras. The first table collects results about (super) Hopf algebras; the second table treats the remaining classes of algebras.

(super) Hopf algebras	support variety $V(M)$	rank variety $V_r(M)$	known for which M?
kG, G a finite group	$\surd$	$\surd$ for $G = E$ elementary abelian	the case $G = \Sigma_d$ the symmetric group: $M =$ Young modules, signed Young modules, some Specht modules and some simple modules
restricted enveloping algebra $\mathrm{u}(\mathfrak{g})$: the interesting case is $\mathfrak{g} = \mathrm{Lie}(G)$, G an algebraic group	$\surd$	$\surd$	known for $H^0(\lambda)$; for tilting modules, conjectured by Humphreys; for GL_n, conjectured by Cooper, proven for $p = 2$ and other cases; for $L(\lambda)$, still open
finite group scheme	$\surd$	rank varieties = compute with "p-points"	for sporadic modules
small quantum groups $\mathrm{u}_\xi(\mathfrak{g}) \subseteq U_\xi(\mathfrak{g})$ for ξ root of unity and $\mathfrak{g}$ a complex semisimple Lie algebra: in most cases $H^{2*}(\mathrm{u}_\xi, k) \simeq k[\mathcal{N}_1]$		the existence of $V_r(M)$ is still open	$H^0(\lambda)$ and some tilting modules $(l > h)$
Lie super algebra $\mathfrak{g}$ over $\mathbb{C}$: $\mathfrak{g}$ classical or $\mathfrak{g} = W(n)$ or $S(n)$	$\surd$	$\surd$	simple modules for $\mathfrak{g} = \mathfrak{gl}(m\|n), W(n), S(n)$

Note: "$\surd$" means that the corresponding notion is well-defined and well-behaved.

(non-Hopf) algebras	support variety $V(M)$	rank variety $V_r(M)$	known for which M?
commutative complete intersection $A := \frac{k[x_1,\cdots,x_c]}{(f_1,\cdots,f_s)}$	$\surd$	$V_r(M) \subseteq k^s$ such that $(\lambda_1,\cdots,\lambda_s) \in V_r(M)$ iff the restriction $M\|_{\frac{k[x_1,\cdots,x_c]}{(\sum_{i=1}^s \lambda_i f_i)}}$ is not of finite projective dimension; $V(M) \simeq V_r(M)$	$M = k$, existence
quantum complete intersection $A^n_{\mathbf{q}} := \frac{k\langle x_1,\cdots,x_c\rangle}{(x_i^n, x_ix_j - q_{ij}x_jx_i)}$, where $\mathbf{q} = (q_{ij})$ satisfies $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$; special case: all $q_{ij=1}$, the truncated polynomial algebra	$\surd$ iff all q_{ij} are roots of unity	defined by usual formula when all $q_{ij} = q$; otherwise, slightly different	only known for Λ_{u_λ} where $u_\lambda = \sum_{i=1}^c \lambda_i x_i$; any other interesting modules M for $A^n_{\mathbf{q}}$?
Λ weakly symmetric $J^3 = 0$, Λ indecomposable	$\surd$ iff Λ is tame or of finite representation type; otherwise, no hope at all, since all indecomposables have projective resolutions of exponential growth		
reduced enveloping algebra $\mathrm{u}_\chi(\mathfrak{g}) := \frac{U(\mathfrak{g})}{(x^p - x^{[p]} - \chi(x),\, x\in\mathfrak{g})}$, where $\mathfrak{g}$ is a restricted Lie algebra, $\chi : \mathfrak{g} \longrightarrow k$ is a nonzero character	$\surd$		simple modules for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = W(1,1)$

(non-Hopf) algebras	support variety $V(M)$
Hecke algebra $H_q(n)$	$\surd$ for tame and finite representation type; in general still open
finite-dimensional preprojective algebras	$\surd$
$\Lambda = E(R)$, where R is a Koszul Artin-Schelter regular algebra which is a finitely generated module over its center	Λ is a finite-dimensional self-injective algebra satisfying (Fg), thus $\surd$
Λ self-injective of finite representation type, which is finite-dimensional over $k = \bar{k}$	Λ satisfies (Fg), thus $\surd$
Λ Gorenstein and Nakayama	Λ satisfies (Fg), thus $\surd$

II. Open Problems:

(1) Find a good definition of "rank varieties".

Counter example: Take $\Lambda := \frac{k[x_1,x_2]}{(x_1^2,x_2^2)}$, char$k \neq 2$, and define $\widetilde{V}_r(M) := \{0\} \cup \{\underline{\lambda} \in k^2 \mid M|_{k[u_\lambda]}$ is not projective$\}$. Then there exists a non-projective module M such that $\widetilde{V}_r(M) = \{0\}$.

(2) Given a path algebra kQ, find a reasonable notion of "support" for objects in $D^b(\text{mod}kQ)$. Hope to have: $\text{supp}(X) \subseteq \text{supp}(Y)$ iff $\text{Thick}(X) \subseteq \text{Thick}(Y)$.

Example: Take Q to be the Kronecker quiver and recall that $D^b(\text{mod}kQ) \simeq D^b(\text{coh}(\mathbb{P}^1))$. Thus the "support variety" of kQ should be $\mathbb{P}^1$ (replacing "Thick" by certain "$\otimes$-Thick").

In general, given a derived equivalence $D^b(\text{mod}\Lambda) \simeq D^b(\text{coh}(\mathbb{X}))$ between a finite-dimensional algebra Λ and a (graded) scheme $\mathbb{X}$, one may expect that certain "support variety" of Λ is $\mathbb{X}$; the key point might be to understand the meaning of the tensor structure on $D^b(\text{mod}\Lambda)$ inherited from $D^b(\text{coh}(\mathbb{X}))$.

(3) Find an algebra Λ such that there is "no hope" to classify the thick triangulated subcategories of $D^b(\text{mod}\Lambda)$.

(4) Find a self-injective algebra which has a simple module of complexity ≥ 3 but which is tame.

(5) Given a triangulated category $\mathcal{T}$, are there any distinguished objects which control the support/classification of thick subcategories? Compare this with the role of the injective objects in the classification of localizing subcategories of Grothendieck categories in the work of Gabriel.

A survey of tensor triangular geometry and applications

Ivo Dell'Ambrogio

We gave a mini-survey of Paul Balmer's geometric theory of tensor triangulated categories, or *tensor triangular geometry*, and applications. In the following, $\mathcal{K} = (\mathcal{K}, \otimes, 1)$ will denote a tensor triangulated category, i.e., a triangulated category $\mathcal{K}$ equipped with a tensor product (a symmetric monoidal structure) $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ with unit object 1, such that $a \otimes -$ and $- \otimes a$ are exact functors $\mathcal{K} \to \mathcal{K}$ for every object $a \in \mathcal{K}$. The main tool of tensor triangular geometry is the spectrum of a tensor triangulated category:

Definition 1 ([Ba05])**.** Let $\mathcal{K}$ be an essentially small $\otimes$-triangulated category. A *prime ideal* $\mathcal{P}$ of $\mathcal{K}$ is a proper (i.e., $\mathcal{P} \neq \mathcal{K}$) full triangulated subcategory $\mathcal{P} \subset \mathcal{K}$ which is: thick (i.e., $a \oplus b \in \mathcal{P} \Rightarrow a, b \in \mathcal{P}$), $\otimes$-ideal ($a \in \mathcal{P}, x \in \mathcal{K} \Rightarrow a \otimes x \in \mathcal{K}$) and prime ($a \otimes b \in \mathcal{P} \Rightarrow a \in \mathcal{P}$ or $b \in \mathcal{P}$). The *spectrum of* $\mathcal{K}$ is the set of its prime ideals:

$$\operatorname{Spc}(\mathcal{K}) := \{\mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is a prime ideal of } \mathcal{K}\}.$$

We give $\operatorname{Spc}(\mathcal{K})$ the topology determined by the following basis of closed subsets:

$$\operatorname{supp}(a) := \{\mathcal{P} \mid a \notin \mathcal{P}\} = \{\mathcal{P} \mid a \not\simeq 0 \text{ in } \mathcal{K}/\mathcal{P}\} \subseteq \operatorname{Spc}(\mathcal{K}) \quad (\text{for } a \in \mathcal{K}).$$

Remarks 2. (a) The space $\operatorname{Spc}(\mathcal{K})$ is always non-empty (if $\mathcal{K} \not\simeq 0$) and spectral, in the sense of Hochster [Ho69]: it is quasi-compact, it has an open basis of quasi-compact opens, and every irreducible closed subset has a unique generic point.

(b) $\operatorname{Spc}(\mathcal{K})$ is naturally equipped with a sheaf of rings $\mathcal{O}_{\mathcal{K}}$. The ringed space

$$\operatorname{Spec}(\mathcal{K}) := (\operatorname{Spc}(\mathcal{K}), \mathcal{O}_{\mathcal{K}})$$

is always a locally ringed space ([Ba09b]) and sometimes a scheme (cf. Ex. 5.a-c).

(c) Every monoidal exact functor $F : \mathcal{K} \to \mathcal{L}$ induces a continuous map $\operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$ by $\mathcal{P} \mapsto F^{-1}\mathcal{P}$. This defines a functor Spec from the category of $\otimes$-triangulated categories to that of locally ringed (spectral) spaces.

Universal property and classification. The *support* assignment

$$\operatorname{supp} : \operatorname{Ob}(\mathcal{K}) \to \operatorname{Closed}(\operatorname{Spc}(\mathcal{K})), \quad a \mapsto \operatorname{supp}(a)$$

is compatible with the $\otimes$-triangulated structure, and is the finest such:

Proposition 3 (Universal property of $(\operatorname{Spc}(\mathcal{K}), \operatorname{supp})$)**.** We have the following:

(1) $\operatorname{supp}(0) = \emptyset$ and $\operatorname{supp}(1) = \operatorname{Spc}(\mathcal{K})$
(2) $\operatorname{supp}(a \oplus b) = \operatorname{supp}(a) \cup \operatorname{supp}(b)$
(3) $\operatorname{supp}(T(a)) = \operatorname{supp}(a)$, where $T : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$ is the translation of $\mathcal{K}$
(4) $\operatorname{supp}(b) \subseteq \operatorname{supp}(a) \cup \operatorname{supp}(c)$ for every exact triangle $a \to b \to c \to T(a)$
(5) $\operatorname{supp}(a \otimes b) = \operatorname{supp}(a) \cap \operatorname{supp}(b)$.

Moreover, if (X, σ) is a pair where X is a topological space and σ is an assignment from objects of $\mathcal{K}$ to closed subsets of X satisfying (1)-(5) above (we say that (X, σ) is a *support datum*), then there exists a unique continuous map $f : X \to \operatorname{Spc}(\mathcal{K})$ such that $\sigma(a) = f^{-1}(\operatorname{supp}(a))$ for all objects $a \in \mathcal{K}$.

Theorem 4 (Classification [Ba05] [BKS07])**.** There is a bijection

$$\begin{array}{rcl} \{\text{radical thick } \otimes\text{-ideals of } \mathcal{K}\} & \simeq & \{\text{Thomason subsets of } \mathrm{Spc}(\mathcal{K})\} \\ \mathcal{J} & \mapsto & \mathrm{supp}(\mathcal{J}) := \cup_{a\in\mathcal{J}}\mathrm{supp}(a) \\ \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subseteq Y\} =: \mathcal{K}_Y & \leftarrow\!\shortmid & Y \end{array}$$

(a $\otimes$-ideal $\mathcal{J}$ is *radical* if $a^{\otimes n} \in \mathcal{J}$ for some $n \geq 1$ implies $a \in \mathcal{J}$, and a subset Y of the spectrum is *Thomason* if it is a union of closed subsets, each with quasi-compact open complement). Moreover, if (X, σ) is a support datum inducing the above bijection, then the canonical map $f : X \to \mathrm{Spc}(\mathcal{K})$ is a homeomorphism.

By exploiting existing classifications of $\otimes$-ideals, the Classification theorem can be used to provide concrete descriptions of the spectrum $\mathrm{Spc}(\mathcal{K})$ in examples ranging over the most disparate branches of mathematics.

Examples 5. (a) **(Algebraic geometry).** Let X be a quasi-compact and quasi-separated scheme, and let $\mathcal{K} := D^{\mathrm{perf}}(X)$ be its derived category of perfect complexes with $\otimes = \otimes^L_X$ and $1 = \mathcal{O}_X$. From Thomason's classification of thick tensor ideals [Th97] we deduce a natural isomorphism $\mathrm{Spec}(D^{\mathrm{perf}}(X)) \simeq X$ of schemes. Thus tensor triangular geometry generalizes algebraic geometry ([Ba02] [Ba05]).

(b) **(Commutative algebra)** As a special case of (a), if R is any commutative ring and $\mathcal{K} := K^b(R-\mathrm{proj})$ its bounded derived category of finitely generated projective modules, then $\mathrm{Spec}(K^b(R-\mathrm{proj})) \simeq \mathrm{Spec}(R)$ is the Zariski spectrum.

(c) **(Modular representation theory).** Let G be a finite group (or a finite group scheme), and let k be a field with $\mathrm{char}(k) > 0$. From the classification in [BCR97] (resp., in [FP07]) of the thick $\otimes$-ideals in the stable category $\mathcal{K} := kG-\mathrm{stab}$ of finite dimensional modules, with $\otimes = \otimes_k$ and $1 = k$, one deduces an isomorphism $\mathrm{Spec}(kG-\mathrm{stab}) \simeq \mathrm{Proj}(H^*(G,k))$ of projective varieties. Similarly, $\mathrm{Spec}(D^b(kG-\mathrm{mod})) \simeq \mathrm{Spec}^{\mathrm{h}}(H^*(G,k))$, the spectrum of homogeneous primes.

(d) **(Stable homotopy).** Let $\mathcal{K} := SH^{\mathrm{fin}}$ be the homotopy category of finite spectra (of topology), i.e., the stable homotopy category of finite based CW-complexes. The famous Thick Subcategory theorem of Hopkins and Smith [HS98] translates neatly into a description of $\mathrm{Spc}(SH^{\mathrm{fin}})$ in terms of the chromatic towers at all prime numbers ([Ba09b]). Note that the ringed space $\mathrm{Spec}(SH^{\mathrm{fin}})$ is *not* a scheme.

Remark 6. Other concrete classifications known so far are: The category of perfect complexes over a Deligne-Mumford stack [Kr08]; The category $\mathcal{K} = Boot_c$ of compact objects in the Bootstrap category of separable C*-algebras (the latter simply yields $\mathrm{Spec}(Boot_c) \simeq \mathrm{Spec}(\mathbb{Z})$ [De09]).

Hypothesis 7. *From now on, we assume that our tensor triangulated category $\mathcal{K}$ is* rigid*, i.e., that there is an equivalence $D : \mathcal{K}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{K}$ with $\mathrm{Hom}(a \otimes b, c) \simeq \mathrm{Hom}(a, D(b) \otimes c)$. Moreover, we assume that $\mathcal{K}$ is* idempotent complete*: if $e = e^2 : a \to a$ is an idempotent morphism in $\mathcal{K}$, then $a \simeq \mathrm{Ker}(e) \oplus \mathrm{Im}(e)$. Both are light hypotheses; e.g., they are satisfied by all categories in Example 5.*

Decomposition of objects. The support $\operatorname{supp}(a)$ can be used to decompose the object a in $\mathcal{K}$, or to test its indecomposability:

Theorem 8 ([Ba07])**.** Let $\mathcal{K}$ be a $\otimes$-triangulated category (see Hypothesis 7). Let $a \in \mathcal{K}$ be an object such that $\operatorname{supp}(a) = Y_1 \cup Y_2$, where Y_1 and Y_2 are disjoint Thomason subsets of $\operatorname{Spc}(\mathcal{K})$ (as in Thm. 4). Then there is a decomposition $a \simeq a_1 \oplus a_2$ in $\mathcal{K}$ with $\operatorname{supp}(a_i) = Y_i$ (for $i = 1, 2$).

In modular representation theory (Example 5.c), for instance, the latter result generalizes to finite group schemes a celebrated theorem of Carlson [Ca84], saying that the projective support variety of a finitely generated indecomposable module is connected. The corresponding statement, of course, is now available in all examples.

Topological filtrations and local-to-global spectral sequences. Given a reasonable notion of "dimension" for the closed subsets of $\operatorname{Spc}(\mathcal{K})$ (such as the usual Krull dimension, or minus the Krull codimension in $\operatorname{Spc}(\mathcal{K})$), one can produce filtrations of the category $\mathcal{K}$ of the form

$$0 \subseteq \mathcal{K}_{(-\infty)} \subseteq \cdots \subseteq \mathcal{K}_{(n-1)} \subseteq \mathcal{K}_{(n)} \subseteq \mathcal{K}_{(n+1)} \subseteq \cdots \subseteq \mathcal{K}_{(+\infty)} = \mathcal{K}$$

where $\mathcal{K}_{(n)} \subseteq \mathcal{K}$ is the subcategory of those objects whose support has dimension at most n ($n \in \mathbb{Z} \cup \{\pm\infty\}$). Every term in the filtration is a thick triangulated subcategory of the next one up, so the subquotients $\mathcal{K}_{(n)}/\mathcal{K}_{(n-1)}$ are again triangulated. Each has a decomposition into a sum of local terms. More precisely:

Theorem 9 ([Ba07])**.** Assume that the space $\operatorname{Spc}(\mathcal{K})$ is noetherian (i.e., every open subset is quasi-compact). Then the quotient functors $q_{\mathcal{P}} : \mathcal{K} \to \mathcal{K}/\mathcal{P}$ induce a fully faithful triangulated functor

$$\mathcal{K}_{(n)}/\mathcal{K}_{(n-1)} \longrightarrow \coprod_{\mathcal{P} \in \operatorname{Spc}(\mathcal{K}) \; s.t. \dim(\overline{\{\mathcal{P}\}})=n} (\mathcal{K}/\mathcal{P})_{(0)}$$

which moreover is cofinal (that is, essentially surjective up to direct summands).

In algebraic geometry, the above decomposition is well known for regular schemes and hides behind various local-to-global spectral sequences. Indeed, Theorem 9 becomes an essential ingredient in the following generalization to singular schemes of Quillen's [Qu73] classical construction of a local-to-global spectral sequence for the algebraic K-theory of regular schemes:

Theorem 10 ([Ba09a])**.** Let X be any (topologically) noetherian scheme of finite Krull dimension. Then there exists a cohomological spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(\mathcal{O}_{X,x} \text{ on } \{x\}) \quad \stackrel{n=p+q}{\Longrightarrow} \quad K_{-n}(X)$$

converging to the algebraic K-theory of X; the E_1-page contains Thomason's non-connective K-theory of the local ring $\mathcal{O}_{X,x}$ with support on the closed point x.

Gluing of morphisms and objects. To each quasi-compact open set $U \subseteq \mathrm{Spc}(\mathcal{K})$ we associate the (again, rigid and idempotent complete) $\otimes$-triangulated category $\mathcal{K}(U) := \widetilde{\mathcal{K}/\mathcal{K}_Y}$ obtained by idempotent completing (see [BS01]) the quotient of $\mathcal{K}$ by all objects supported on the complement $Y := \mathrm{Spc}(\mathcal{K}) \smallsetminus U$. Given a covering $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$, it is natural to ask if and how it is possible to glue information in $\mathcal{K}(U_i)$ $(i = 1, 2)$, compatible over $\mathcal{K}(U_1 \cap U_2)$, in order to provide information in $\mathcal{K}$. The "gluing technique" of Balmer-Favi [BF07] provides some general answers:

Theorem 11 (Mayer-Vietoris for morphisms)**.** There is a long exact sequence

$$\cdots \mathrm{Hom}_{12}(a, T^{-1}b) \xrightarrow{\partial} \mathrm{Hom}(a, b) \to \mathrm{Hom}_1(a, b) \oplus \mathrm{Hom}_2(a, b) \to \mathrm{Hom}_{12}(a, b) \xrightarrow{\partial} \cdots$$

of Hom groups for every two objects $a, b \in \mathcal{K}$ (here we use the short-hand notation $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{K}}$, $\mathrm{Hom}_i = \mathrm{Hom}_{\mathcal{K}(U_i)}$ and $\mathrm{Hom}_{12} = \mathrm{Hom}_{\mathcal{K}(U_1 \cap U_2)}$, and we keep writing a and b for the canonical images of a and b in the appropriate categories).

Theorem 12 (Gluing of two objects)**.** Given two objects $a_i \in \mathcal{K}(U_i)$ $(i = 1, 2)$ and an isomorphism $\sigma : a_1 \xrightarrow{\sim} a_2$ over $U_1 \cap U_2$, i.e., in $\mathcal{K}(U_1 \cap U_2)$, there exists an (up to isomorphism, unique) object $a \in \mathcal{K}$ mapping to a_i in $\mathcal{K}(U_i)$ $(i = 1, 2)$.

The Picard group. For any $\otimes$-triangulated category $\mathcal{K}$, define its *Picard group* $\mathrm{Pic}(\mathcal{K})$ to be the abelian group of $\otimes$-invertible objects (i.e., those $a \in \mathcal{K}$ such that there exists $b \in \mathcal{K}$ and an isomorphism $a \otimes b \simeq 1$), with $\otimes$ as group operation.

Examples 13. (a) For a scheme X, we have $\mathrm{Pic}(D^{\mathrm{perf}}(X)) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^{\ell}$, where ℓ is the number of connected components of X.

(b) For a finite group G and a field k, we recognise $\mathrm{Pic}(kG - \mathrm{stab})$ as the group of endotrivial kG-modules, usually denoted $T(G)$.

Theorem 12 supplies the connecting map δ used in the next result.

Theorem 14 (Mayer-Vietoris for Picard [BF07])**.** Let $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$ as above. There is a long exact sequence (extending to the left as in Theorem 11)

$$\begin{aligned} &\cdots \to \mathrm{Hom}_{\mathcal{K}(U_1 \cap U_2)}(1, T^{-1}1) \xrightarrow{1+\partial} \\ &\mathbb{G}_{\mathrm{m}}(\mathcal{K}) \to \mathbb{G}_{\mathrm{m}}(\mathcal{K}(U_1)) \oplus \mathbb{G}_{\mathrm{m}}(\mathcal{K}(U_2)) \to \mathbb{G}_{\mathrm{m}}(\mathcal{K}(U_1 \cap U_2)) \xrightarrow{\delta} \\ &\mathrm{Pic}(\mathcal{K}) \to \mathrm{Pic}(\mathcal{K}(U_1)) \oplus \mathrm{Pic}(\mathcal{K}(U_2)) \to \mathrm{Pic}(\mathcal{K}(U_1 \cap U_2)). \end{aligned}$$

Here $\mathbb{G}_{\mathrm{m}}(\mathcal{L}) := \mathrm{End}_{\mathcal{L}}(1)^{\times}$ denotes the automorphism group of the tensor unit 1 in a $\otimes$-triangulated category $\mathcal{L}$.

Applications of gluing to modular representation theory. The authors of [BBC08] compare the above gluing techniques with similar-minded uses of Rickard's idempotent modules ([Ri97]) in modular representation theory. Among other things, they provide a new proof for Alperin's computation ([Al01] [Ca06]) of the rank of the group $T(G)$ in terms of the number of conjugacy classes of maximal elementary abelian subgroups of G. They also show that the above gluing technique provides a subgroup of finite index inside $T(G)$. Further enquiry along these lines brings to light the following deep connection between algebraic geometry and modular representation theory:

Theorem 15 ([Ba08])**.** Let G be a finite group and k a field of positive characteristic. Then the gluing construction induces an isomorphism

$$\mathrm{Pic}\big(\mathrm{Proj}(H^*(G,k))\big) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} T(G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

which rationally identifies the Picard group of line bundles on the projective variety of G with the group of endotrivial kG-modules.

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The complexity and support varieties for some Specht modules of the symmetric group

David J. Hemmer

During the 2004-2005 academic year the VIGRE Algebra Research Group at the University of Georgia (UGA VIGRE) computed the complexities of certain Specht modules S^λ for the symmetric group Σ_d, using the computer algebra program Magma. The complexity of an indecomposable module does not exceed the p-rank of the defect group of its block. The UGA VIGRE Algebra Group conjectured that, generically, the complexity of a Specht module attains this maximal value; that it is smaller precisely when the Young diagram of λ is built out of $p \times p$ blocks. In our talk, we presented our recent proof of one direction of this conjecture. We prove that these Specht modules do indeed have less than maximal complexity. It remains open to show that the remaining Specht modules have maximal complexity.

It turns out that this class of partitions, which has not previously appeared in the literature, arises naturally as the solution to a question about the p-weight of partitions and branching. We first define this class precisely:

Definition A partition $\lambda \vdash d$ is $p \times p$ if $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_s^{a_s})$ where $p \mid \lambda_i$ and $p \mid a_i$ for all i.

Such λ can exist only if $p^2 \mid d$. Equivalently, λ is $p \times p$ if both λ and its transpose λ' are of the form $p\tau$. Also equivalently, the Young diagram of λ is built from $p \times p$ blocks.

Now suppose S^λ is in a block $B(\lambda)$ of weight w corresponding to a p-core $\tilde{\lambda} \vdash d - pw$. Then the defect group of $B(\lambda)$ is isomorphic to a Sylow p-subgroup of Σ_{pw} and has p-rank w. In particular, the maximum complexity of any module in the block $B(\lambda)$ is w. The UGA VIGRE Algebra Group made the following conjecture:

Conjecture 1 (UGA VIGRE [1].). *Let S^λ be in a block B of weight w. Then the complexity of S^λ is w if and only if λ is not $p \times p$.*

This conjecture was verified in [3] for the partition $\lambda = (p^p) \vdash p^2$ and in [2] for λ a hook partition, i.e. of the form $(a, 1^b)$. Conjecture 1 implies that almost every Specht module has maximal complexity among modules in its block. Indeed it would imply that if $p^2 \nmid d$, then all the Specht modules for Σ_d have this property.

[1] This conjecture and some discussion can be found at http://www.math.uga.edu/∼nakano/vigre/vigre.html

As far as we know the condition we call $p \times p$ has not appeared anywhere in the literature, and it seems quite mysterious. However we demonstrated that it arises very naturally from considering the weights of Σ_d blocks and the branching theorems. Specifically we proved:

Theorem 2. *Suppose $\lambda \vdash d$ has p-weight w. Then λ is $p \times p$ if and only if $w(\lambda_A) \leq w - 2$ for each removable node A of λ. In this case, $w(\lambda_A)$ is always equal to $w - 2$.*

Theorem 2 gives one direction of Conjecture 1 as a fairly immediate corollary.

Corollary 3. *Suppose $\lambda \vdash p^2 d$ is $p \times p$, and hence of weight $w = pd$. Then the complexity of S^λ is less than w.*

The proof of Theorem 2 is purely combinatorial, and uses the abacus combinatorics of James.

There are several obvious problems left unsolved.

Problem 4. *Resolve the other direction of Conjecture 1.*

Problem 5. *Suppose λ is $p \times p$ of weight w. Is the complexity of S^λ equal to $w - 1$, or can it be less than $w - 1$?*

Problem 6. *One can generalize the definition of $p \times p$. For example the first obvious generalization would be to require λ be $p^2 \times p^2$. Can one say anything interesting about these situations? Perhaps the complexity drops by even more in this case?*

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Stratifying modular representations of finite groups

Srikanth B. Iyengar

(joint work with Dave Benson, Henning Krause)

The work presented below arises from a study of homological properties of complexes over a noetherian (or even, Artinian) A. Let me explain by way of an example. Recall that a complex of A-modules is *perfect* if it is isomorphic in the derived category of A to a complex of the form $0 \to P^s \to \cdots \to P^t \to 0$, with each P^i a finitely generated projective A-module. Let $\mathsf{P}(A)$ denote the full subcategory of the derived category consisting of perfect complexes; it is even a subcategory of $\mathsf{D}^{\mathsf{f}}(A)$, the complexes of A-modules with finitely generated cohomology. The subcategory $\mathsf{P}(A)$ has two salient properties:

(a) If $M \bigoplus N$ is in $\mathsf{P}(A)$, then both M and N are in $\mathsf{P}(A)$;

(b) In any exact triangle $L \to M \to N \to$ in $\mathsf{D}^{\mathsf{f}}(A)$, if any two of $\{L, M, N\}$ are in $\mathsf{P}(A)$, then all three are in $\mathsf{P}(A)$.

In other words, $\mathsf{P}(A)$ is a *thick subcategory* of $\mathsf{D}^{\mathsf{f}}(A)$. Most homological conditions (e.g. finite Gorenstein dimension, in the sense of Auslander and Bridger), and some not-obviously-homological ones (e.g. finiteness of length of the homology module), define thick subcategories of $\mathsf{D}^{\mathsf{f}}(A)$. Another reason to care about thick subcategories is that they are precisely the kernels of exact functors on $\mathsf{D}^{\mathsf{f}}(A)$; see [10]. These considerations suggest that the following

Problem. *Classify the thick subcategories of* $\mathsf{D}^{\mathsf{f}}(A)$.

While I have tried to argue that this is a natural problem to consider, the investigation of such global questions in derived categories was pioneered by Mike Hopkins through his work on stable homotopy theory. Motivated by this, Hopkins [6], see also Neeman [9], proved that when A is a commutative noetherian ring, the thick subcategories $\mathsf{P}(A)$ are in bijection with specialization closed subsets of Spec A. This 'thick subcategory' theorem solves the classification problem stated above for regular rings, for then $\mathsf{P}(A) = \mathsf{D}^{\mathsf{f}}(A)$.

Hopkins' proof of the thick subcategory is via a nilpotence theorem for morphisms of perfect complexes. In [9] Neeman gave a different proof, based on a classification of the *localizing* subcategories—these are thick subcategory also closed under arbitrary direct sums—of $\mathsf{D}(A)$, the full derived category of A. There is also a third proof of the thick subcategory theorem, based on an idea of Dwyer and Greenlees from [5]; see [7] for details.

The crucial point in Neeman's deduction of the thick subcategory theorem for $\mathsf{P}(A)$ from the localizing subcategory theorem for $\mathsf{D}(A)$ is that the perfect complexes are precisely the compact objects in the derived category. Much of the proof for the result about $\mathsf{D}(A)$ can be carried over to a setting of triangulated categories with ring actions; see [3]. This suggests the possibility of classifying thick subcategories of other triangulated categories (for example, $\mathsf{D}^{\mathsf{f}}(A)$) by realizing it as the subcategory of compact objects in a suitable triangulated category, and classifying the localizing subcategories of the larger triangulated category.

In [1] Avramov, Buchweitz, Christensen, Piepmeyer and I use this approach to classify the thick categories of 'perfect' differential modules over a commutative noetherian ring, by classifying the localizing subcategories of the derived category of all differential modules.

Returning to the classification of thick subcategories of $\mathsf{D}^{\mathsf{f}}(A)$: Krause [8] has proved that for any noetherian ring A, the map which associates to a complex its injective resolution identifies $\mathsf{D}^{\mathsf{f}}(A)$ with the compact objects in $\mathsf{K}(\text{Inj } A)$, the homotopy category of complexes of injective A-modules. Thus one is lead to:

Problem. *Classify the localizing subcategories of* $\mathsf{K}(\text{Inj } A)$.

Benson, Krause, and I [4] solved this problem for the case where $A = kG$, the group algebra of a finite p-group G, over a field k of characteristic p. In my talk, I described the structure of our proof, and some of the key ideas in it. Among its many corollaries is a new proof of the classification of the thick subcategories of the stable module category of finite dimensional kG-modules, due to Benson, Carlson, and Rickard [2].

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Some explicit rank varieties for Specht modules

Kay Jin Lim

Some explicit rank varieties for Specht modules

Let k be an algebraically closed field of characteristic $p > 0$, G be a finite group and M be an indecomposable kG-module. Suppose that Q is a vertex of M and N is a kQ-source of M. Then we have both $M|N{\uparrow}^G$ and $N|M{\downarrow}_Q$. So the support variety $V_G(M)$ of M is $\mathrm{res}^*_{G,Q}\, V_Q(N)$. The complexity $c_G(M) = \dim V_G(M)$ of the module M is bounded above by the p-rank of the vertex Q and hence bounded above by the p-rank of a defect group of the block containing M. However, in practice, it is difficult to calculate vertices and sources.

Let E be an elementary abelian p-group of rank n with generators $g_1, \ldots, g_n$. For any point $\omega = (\omega_1, \ldots, \omega_n) \in k^n - \{0\}$, we write $u_\omega = 1 + \sum_{i=1}^n \omega_i(g_i - 1) \in kE$. The rank variety $V_E^\sharp(M)$ of a kE-module M is the set

$$\{0\} \cup \{0 \neq \omega \in k^n \mid M{\downarrow}_{\langle u_\omega \rangle} \text{ is not } k\langle u_\omega\rangle\text{-free}\}.$$

It is well-known that $V_E(M) \cong V_E^\sharp(M)$. If $\omega \in k^n$ is a generic point, then the Jordan type $[\omega](M)$ of $M{\downarrow}_{\langle u_\omega \rangle}$ is called the generic Jordan type of the module M [5]. If N is another kE-module, then $[\omega](M \oplus N) \cong [\omega](M) \oplus [\omega](N)$ (see 4.7 of [1]). The stable generic Jordan type of M is its generic Jordan type modulo all projective summands.

A partition $\mu = ({\mu_1}^{n_1}, \ldots, {\mu_s}^{n_s})$ is $p \times p$ if for each $1 \leq i \leq s$ both μ_i, n_i are divisible by p. We write $D_{\widetilde{\mu}}$ for a defect group of the block containing the Specht module S^μ corresponding to a partition μ. We are concerned with the VIGRE conjecture; namely, that the complexity of S^μ is the p-weight of μ if and only if μ is not $p \times p$.

Let E_s be the elementary abelian p-subgroup of the symmetric group $\mathfrak{S}_n$ generated by the p-cycles $((i-1)p+1, \ldots, ip)$ with $1 \leq i \leq s \leq n/p$. For any partition μ, we write m_μ for the p-weight of μ. For some cases where μ are not $p \times p$, we show that $S^\mu\downarrow_{E_{m_\mu}}$ is not generically free. The key tool is the following proposition.

Proposition 1. *A kE-module M is not generically free if and only if $V_E^\sharp(M) = V_E^\sharp(k)$.*

Theorem 2 (Abelian defect case)**.** *If $D_{\widetilde{\mu}}$ is abelian, i.e., $m_\mu < p$, then $V_{\mathfrak{S}_n}(S^\mu) = \operatorname{res}^*_{\mathfrak{S}_n, D_{\widetilde{\mu}}} V_{D_{\widetilde{\mu}}}(k)$. In particular, the complexity of the Specht module S^μ is the p-weight of the partition μ.*

This result implies that if $D_{\widetilde{\mu}}$ is abelian, then a vertex of the Specht module S^μ is precisely the defect group $D_{\widetilde{\mu}}$; namely, the elementary abelian p-group of rank m_μ.

Remark 3. It is not true that the variety of the Specht module $V_{\mathfrak{S}_n}(S^\mu)$ is $\operatorname{res}^*_{\mathfrak{S}_n, D_{\widetilde{\mu}}} V_{D_{\widetilde{\mu}}}(k)$ if and only if μ is $p \times p$.

Using Mackey's decomposition formula, we determine the stable generic Jordan type of signed permutation modules $M(\alpha|\beta)$ restricted to elementary abelian p-subgroups [4]. We specialize to the partitions $\alpha = (a)$ and $\beta = (b)$ with a, b non-negative integers. In the case where $a + b \not\equiv 0 (\text{mod } p)$, we have a direct sum decomposition $M((a)|(b)) \cong S^{(a,1^b)} \oplus S^{(a+1,1^{b-1})}$. In the case where $a + b = dp$, we show that the short exact sequence

$$0 \to S^{(a-1,1^{b+1})}\downarrow_{E_d} \to S^{(a,1^{b+1})}\downarrow_{E_d} \to S^{(a,1^b)}\downarrow_{E_d} \to 0$$

generically splits, i.e., $S^{(a,1^{b+1})}\downarrow_{\langle u_\omega \rangle} \cong S^{(a,1^b)}\downarrow_{\langle u_\omega \rangle} \oplus S^{(a-1,1^{b+1})}\downarrow_{\langle u_\omega \rangle}$ for a generic point $\omega \in k^d$. Using induction, we prove the following.

Theorem 4 (Hook partitions case)**.** *Let $\mu = (a, 1^b)$. Suppose that $a + b = dp + r$ and $b = sp + b_0$ with $0 \leq r, b_0 \leq p - 1$. The stable generic Jordan type of $S^\mu\downarrow_{E_{m_\mu}}$ is given as follows.*

(i) $1^{N(\mu;d-1)}$ *if* $0 \neq r \leq b_0$ *with* $N(\mu; d-1) = \binom{d-1}{s-1}\binom{p+r-1}{p+b_0} + \binom{d-1}{s}\binom{p+r-1}{b_0}$.
(ii) $1^{N(\mu;d)}$ *if* $b_0 < r$ *with* $N(\mu; d) = \binom{d}{s}\binom{r-1}{b_0}$.
(iii) $1^{\binom{d-1}{s}}$ *if* $r = 0$ *and* b_0 *is even.*
(iv) $(p-1)^{\binom{d-1}{s}}$ *if* $r = 0$ *and* b_0 *is odd.*

In all cases, the complexity of the Specht module S^μ is the p-weight of μ.

Let $\Gamma(dp)$ be the set of all partitions of dp with no more than p-parts with empty p-cores. In §5.2 of [3], we define a map $\Phi : \Gamma(dp) \to \Gamma(dp)$ with the following properties.

(i) Every part of $\Phi(\mu)$ is a multiple of p.
(ii) The map $\Phi(\mu) = \mu$ if and only if every part of μ is a multiple of p.

(iii) The stable generic Jordan types of $S^{\Phi(\mu)}\downarrow_{E_d}$ and $S^{\mu}\downarrow_{E_d}$ are either the same or complement to each other.

Theorem 5. *If $\mu \in \Gamma(dp)$ and the prime p satisfy one of the following conditions listed below, then the Specht module S^μ has complexity the p-weight of μ.*

(a) *The prime p is odd, $d = p$ and $\Phi(\mu)$ is a 2-part partition $(p^2 - mp, mp)$ of p^2 for some $1 \leq m < p/2$.*
(b) *The prime p is odd and $\Phi(\mu)$ is a 2-part partition $(dp - \varepsilon p, \varepsilon p)$ such that $n \not\equiv 2 (\mathrm{mod}\ p)$ and $\varepsilon \in \{1, 2\}$.*
(c) $\Phi(\mu) = (dp)$.
(d) *$p = 2$ and $\Phi(\mu)$ is the partition $(2d - 2, 2) \neq (2, 2)$ or $(2d - 4, 4) \neq (4, 4)$.*

Let $\lambda = (p^p)$. This is the smallest $p \times p$ partition that one can construct. One can verify the conjecture easily for the case $p = 2$. Suppose that p is odd. The complexity of the Specht module S^λ is $p - 1$. Note that there are two types of non-conjugate maximal elementary abelian p-subgroups E_p, F of $\mathfrak{S}_{p^2}$ which F has p-rank 2. One can see easily that $p - 1 = \dim V^\sharp_{E_p}(S^\lambda)$.

Theorem 6. *Suppose that p is an odd prime and $\lambda = (p^p)$. Let W be the union of all components of $V^\sharp_{E_p}(S^\lambda)$ of dimension $\dim V_{\mathfrak{S}_{p^2}}(S^\lambda) = p - 1$. The radical ideal corresponding to the variety $W \subseteq k^p$ is (f) where*

$$f(x_1, \ldots, x_p) = (x_1 \ldots x_p)^{p-1}\widetilde{f} + \sum_{i=1}^{p} {x_1}^{n(p-1)} \ldots \widehat{{x_i}^{n(p-1)}} \ldots {x_p}^{n(p-1)}$$

for some homogeneous polynomial $\widetilde{f} \in k[x_1, \ldots, x_p]^{(\mathbb{F}_p^\times)^p \rtimes \mathfrak{S}_p}$ and positive integer n where ${x_1}^{n(p-1)} \ldots \widehat{{x_i}^{n(p-1)}} \ldots {x_p}^{n(p-1)}$ is the product of all ${x_j}^{n(p-1)}$'s with $1 \leq j \neq i \leq p$. In particular, the degree of the projectivized rank variety $\overline{V^\sharp_{E_p}(S^\lambda)}$ is non-zero and divisible by $(p-1)^2$.

Remark 7. Carlson gave a degree bound for an arbitrary kE-module M where E is an elementary abelian p-group of rank n; namely if $r = \dim V^\sharp_E(M)$, then $\deg(\overline{V^\sharp_E(M)}) \leq \dim_k M/p^{n-r}$. In the case $p = 3$, indeed $V^\sharp_E(S^{(3^3)}) = ({x_1}^2{x_2}^2 + {x_2}^2{x_3}^2 + {x_1}^2{x_3}^2)$ as computed by Carlson using MAGMA. In this case, we have $\widetilde{f} = 0$ and $n = 1$.

Hemmer proved one direction of the VIGRE conjecture; namely, if a partition μ is $p \times p$, then the complexity of the Specht module S^μ is strictly less than the p-weight of μ [2]. The other direction of the conjecture is still wide open. We collect some questions and suggestions as follows.

Question 8. For all partitions in $\Gamma(dp)$, the map Φ suggests that one can only work with partitions such that each part is a multiple of p. Can one say something about Specht modules corresponding to these types of partitions?

Question 9. In Theorem 6, is the radical ideal corresponding to $V^\sharp_{E_p}(S^\lambda)$ the ideal (f)? If so, is $\widetilde{f}$ an "error term"? Is n always 1? Can one give a better upper bound for the degree of the projectivized rank variety $\overline{V^{\sharp}_{E_p}(S^\lambda)}$? Say $\deg(\overline{V^{\sharp}_{E_p}(S^\lambda)} < (\dim_k S^\lambda)/p^2$?

Suggestion 10 (David Benson)**.** Let E be an elementary abelian p-group of rank d, M be an kE-module and $\omega \in k^d$ be a generic point. If there is an element $m \in M$ such that $m \in \ker(u_\omega - 1)$ and $m \notin \operatorname{im}(u_\omega - 1)^{p-1}$, then M has a generic Jordan block of size j with $0 < j < p$, i.e., M is not generically free. Find such an element in $S^\mu{\downarrow}_{E_{m_\mu}}$ in the case where μ is not $p \times p$. Once one has done so, then one can conclude that the complexity of the Specht module S^μ is bounded below by m_μ. So the complexity is precisely m_μ.

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Complexity and Support Varieties: Lie Algebras and Lie Superalgebras

Daniel K. Nakano

Support varieties were first developed 30 years ago to study representation theory from a geometric viewpoint via the cohomology of an algebra. Through the cohomology one can reintroduce underlying geometry even though the algebra one starts with is a finite-dimensional vector space. Support varieties encode essential information about the general representation theory of an algebra and "have proved to be an indispensable tool in the arsenal of a modern representation theorist."[1] The purpose of my three talks were to (1) provide a historical perspective on support varieties, (2) survey important results in the development of the theory and (3) demonstrate how support varieties can be computed in concrete examples such as Lie algebras and Lie superalgebras.

Given a module M for an algebraic structure A (i.e., a group, quantum group, or Lie algebra), one can construct using cohomology operations a variety $\mathcal{V}_A(M)$ called the *support variety* of M that is contained in the spectrum of the cohomology ring. For example, let k be an algebraically closed field of characteristic $p > 0$. If $A = u(\mathfrak{gl}_n(k))$ (the restricted enveloping algebra of the Lie algebra of $n \times n$ matrices) and M is an A-module, then $\mathcal{V}_A(M)$ will be a conical subvariety of the set of $n \times n$ nilpotent matrices $\mathcal{N}$ (nullcone). The nullcone is a well-studied geometric object with beautiful combinatorial properties related to the associated root system and Weyl group. If M is a rational $G = \mathrm{GL}_n(k)$-module then the

[1]Mathematical Review: MR:2003b:20063, by Dmitry Rumynin

support variety of M will be G-invariant. This fact allows one to employ a host of techniques from the theory of nilpotent orbits for semisimple Lie algebras to compute support varieties for rational G-modules (cf. [FP]).

In 1987, Jantzen [Jan] conjectured that for a reductive algebraic group, G, the support varieties of the induced modules $H^0(\lambda)$ over the (restricted) Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ are given by the closure of certain Richardson orbits in the nullcone when the characteristic, p, of the underlying field is good. For $\mathrm{GL}_n(k)$, the Jantzen conjecture can be made quite concrete. In this case the nullcone $\mathcal{N}$ is the set of $n \times n$ nilpotent matrices. The induced modules $H^0(\lambda)$ are parametrized by dominant weights λ. For each dominant weight, the p-stabilizer $\Phi_{\lambda,p}$ is a subroot system of the root system of type A_{n-1}. The size of this subroot system naturally yields a partition $\sigma(\lambda)$ of n. Let $\sigma(\lambda)^t$ be the transposed partition and $x_{\sigma(\lambda)^t}$ be the nilpotent matrix having Jordan blocks of size corresponding to the parts of $\sigma(\lambda)^t$. Then

$$\mathcal{V}_{\mathfrak{gl}_n(k)}(H^0(\lambda)) = \overline{\mathrm{GL}_n(k) \cdot x_{\sigma(\lambda)^t}}$$

where the "$\cdot$" is the action by conjugation and the closure is taken in the Zariski topology of $\mathcal{N}$. Parshall, Vella and the author [NPV] discovered a strong connection between the support varieties of the induced modules and the infinitesimal induced modules. This relationship allowed us to prove the Jantzen conjecture in its full generality. Later the University of Georgia VIGRE Algebra Research Group [UGA1, UGA2, UGA3] in a series of papers computed the restricted nullcone for all Lie algebras arising from reductive algebraic groups and extended the computation of support varieties of $H^0(\lambda)$ for fields of bad characteristic. Other recent developments that were surveyed included the computation of support varieties for tilting modules. This included recent work of Cooper [C] who extended an earlier conjecture of Humphreys [H] for $G = \mathrm{GL}_n(k)$ to include all primes and all dominant weights. Cooper has verified the conjecture in the case when $p = 2$ which proved a conjecture made by Donkin [D].

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra over the complex numbers. The classical Lie superalgebras are the simple Lie superalgebras whose $\mathfrak{g}_{\bar{0}}$-component is a reductive Lie algebra. Let $G_{\bar{0}}$ be the reductive algebraic group such that $\mathrm{Lie}\, G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$. Consider the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-modules which are completely reducible as $\mathfrak{g}_{\bar{0}}$-module. The category $\mathcal{F}$ is self-injective and behaves like a module category for a finite dimensional cocommutative Hopf algebra.

In [BKN1], Boe, Kujawa and the author constructed "detecting" subalgebras of $\mathfrak{g}$ and showed that these subalgebras arise naturally using the invariant theory of reductive groups. Let $R = \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})$ be the relative cohomology for the Lie superalgebra $\mathfrak{g}$ relative to $\mathfrak{g}_{\bar{0}}$. We proved there exists a sub Lie superalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ such that

$$R \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{e}_{\bar{1}}^*)^W \cong \mathrm{H}^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}, \mathbb{C})^W$$

where W is a finite reflection group. This demonstrates that R is a finitely generated algebra. The finite generation of R allowed us to develop a theory of support varieties for modules over the Lie superalgebra.

Given a $\mathfrak{g}$-module M, we considered the support varieties $\mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(M)$ (which can be identified via a "rank variety" description) and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$. For basic simple classical Lie algebras, the "atypicality" of a block and a simple module (due to Kac and Serganova) are combinatorial invariants used to give a rough measure of the complications involved in the block structure. We conjectured in [BKN1] that for a simple $\mathfrak{g}$-module $L(\lambda)$ (of highest weight λ) that the atypicality equals $\dim \mathcal{V}_{(\mathfrak{e},\mathfrak{e}_{\bar{0}})}(L(\lambda))$. Using results of Serganova involving translation functors, we proved the conjecture when $\mathfrak{g} = \mathfrak{gl}(m|n)$ in [BKN2].

For arbitrary classical $\mathfrak{g}$, we recently proved in [BKN3] that the complexity of modules in $\mathcal{F}$ is always bounded by $\dim \mathfrak{g}_{\bar{1}}$. The relative cohomology ring is not large enough to detect the complexity. For example, the Krull dimension of the relative cohomology ring for $\mathfrak{gl}(1|1)$ is one while the complexity of the trivial module is two. An elusive problem has been to develop a theory of module varieties in this context which measures the rate of growth of projective resolutions.

My talks at this conference and this abstract are dedicated to the memory of my father Akira Nakano (1931-2009).

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Weakly symmetric algebras with radical cube zero and support varieties

Øyvind Solberg
(joint work with Karin Erdmann)

Throughout let Λ be a finite dimensional algebra over an algebraically closed field k with Jacobson radical $\mathfrak{r}$. The algebra Λ has a good theory of cohomological support varieties via the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ of Λ, if $\operatorname{HH}^*(\Lambda)$ is Noetherian and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated $\operatorname{HH}^*(\Lambda)$-module (see [4]. Denote this condition by **(Fg)**. The aim of this talk was to characterize when a weakly symmetric algebra Λ with $\mathfrak{r}^3 = (0)$ satisfies **(Fg)**.

For the algebras Λ we consider, it is well-known that $\Lambda \simeq kQ/I$ for some finite quiver Q and some ideal I in kQ, up to Morita equivalence. Furthermore there is a homomorphism of graded rings

$$\varphi_M \colon \operatorname{HH}^*(\Lambda) \to \operatorname{Ext}^*_\Lambda(M, M) = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(M, M)$$

for all Λ-modules M, with $\operatorname{Im}\varphi_M \subseteq Z_{\mathrm{gr}}(\operatorname{Ext}^*_\Lambda(M, M))$ (see [7, 8]). Here $Z_{\mathrm{gr}}(\operatorname{Ext}^*_\Lambda(M, M))$ denotes the graded centre of $\operatorname{Ext}^*_\Lambda(M, M)$.

Weakly symmetric algebras are selfinjective algebras where all indecomposable projective modules P have the property that $P/\mathfrak{r}P \simeq \operatorname{Soc}(P)$. All selfinjective algebras Λ of finite representation type are shown to be periodic algebras [3], meaning that $\Omega^n_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(\Lambda) \simeq \Lambda$ for some $n \geq 1$. It is easy to see that all periodic algebras Λ satisfy **(Fg)**. Furthermore, for selfinjective algebras Λ with radical $\mathfrak{r}^3 = (0)$ we have the following result.

Theorem 1 ([5, 6])**.** *Let Λ be a selfinjective algebra with radical cube zero. Then Λ is Koszul if and only if Λ is of infinite representation type.*

Hence in our study of weakly symmetric algebras Λ with $\mathfrak{r}^3 = (0)$, we can concentrate on infinite representation type and consequently Koszul algebras. For Koszul algebras Λ the homomorphism of graded rings from $\operatorname{HH}^*(\Lambda)$ to the Ext-algebra of the simple modules has an even nicer property than general algebras as the following result shows.

Theorem 2 ([2])**.** *Let $\Lambda = kQ/I$ be a Koszul algebra with degree zero part of the graded algebra Λ given by Λ_0. Let $E(\Lambda) = \oplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$. Then*

$$\operatorname{Im}\varphi_{\Lambda_0} = Z_{\mathrm{gr}}(E(\Lambda)).$$

This enables us to characterize when **(Fg)** holds for a finite dimensional Koszul algebra Λ over an algebraically closed field as follows.

Theorem 3. *Let $\Lambda = kQ/I$ be a finite dimensional algebra over an algebraically closed field k, and let $E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$.*

(a) *If Λ satisfies* **(Fg)***, then $Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$.*

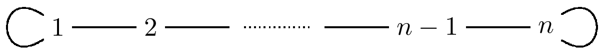
(b) *When Λ is Koszul, then the converse implication also holds, that is, if $Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$, then Λ satisfies* **(Fg)**.

Again let Λ be a symmetric algebra with $\mathfrak{r}^3 = (0)$. Denote by $\{S_1, \ldots, S_n\}$ all the non-isomorphic simple Λ-modules, and let E_Λ be the $n \times n$-matrix given by $(\dim_k \operatorname{Ext}^1_\Lambda(S_i, S_j))_{i,j}$. These algebras are classified in [1], and among other things the following is proved there.

Theorem 4 ([1])**.** *Let Λ be a finite dimensional indecomposable basic weakly symmetric algebra over an algebraically closed field k with $\mathfrak{r}^3 = (0)$. Then the matrix E_Λ is a symmetric matrix, and the eigenvalue λ of E_Λ with largest absolute value is positive.*

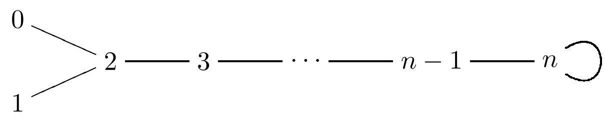
(a) *If $\lambda > 2$, then the dimensions of the modules in a minimal projective resolution of any finitely generated Λ-module has exponential growth.*
(b) *If $\lambda = 2$, then the dimensions of the modules in a minimal projective resolution of any finitely generated Λ-module are either bounded or grow linearly. The matrix E_Λ is the adjacency matrix of a Euclidean diagram $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$ for $n \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, or*

$\widetilde{\mathbb{Z}}_n$:



or

$\widetilde{D\mathbb{Z}}_n$:



(c) *If $\lambda < 2$, then the dimensions of the modules in a minimal projective resolution of any finitely generated Λ-module is bounded.*

The trichotomy in Theorem 4 corresponds to the division in wild, tame and finite representation type as pointed out in [1]. By [4, Theorem 2.5] the complexity of any finitely generated module over an algebra satisfying **(Fg)** is bounded above by the Krull dimension of the Hochschild cohomology ring, hence finite. It follows from this that a weakly symmetric algebra with radical cube zero only can satisfy **(Fg)** in case (b) and (c) in the above theorem. The above result gives the quiver of the algebra Λ, but since Λ is supposed to be weakly symmetric with $\mathfrak{r}^3 = (0)$, it is easy to write down the possible relations. In these relations one can introduce scalars from the field. Most of the times the results are independent of these scalars, except in the $\widetilde{\mathbb{A}}_n$ case, where it suffices to introduce one scalar q in one commutativity relation. In particular, we have the following.

Theorem 5. *Let Λ be a finite dimensional symmetric algebra over an algebraically closed field with radical cube zero. Then Λ satisfies* **(Fg)** *if and only if Λ is of finite representation type, Λ is of type $\widetilde{D}_n$ for $n \geq 4$, $\widetilde{Z}_n$, $\widetilde{D\mathbb{Z}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, or Λ is of type $\widetilde{\mathbb{A}}_n$ when q is a root of unity.*

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Support varieties – an axiomatic approach

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(joint work with Aslak B. Buan, Henning Krause, Nicole Snashall)

The talk of Petter A. Bergh on "Support via central ring actions" underlined that there is a theory of support varieties obtained from a homomorphism of graded rings $R \to Z^*(\mathcal{A})$ for a triangulated category $\mathcal{A} = (\mathcal{A}, \Sigma)$, where R is a graded commutative ring and $Z^*(\mathcal{A})$ is the graded centre of $\mathcal{A}$. Such ring actions have been considered in [1, 2, 3, 4]. The aim of this talk was threefold: (1) show that a tensor triangulated category acting on a triangulated category $\mathcal{A}$ provides a categorification of a central ring action of a graded ring on $\mathcal{A}$, (2) to show that some of the support varieties for triangulated categories studied in the literature come from a central ring action obtained from a tensor triangulated category acting, and thirdly, (3) under some additional assumptions we point out that this puts restrictions on what one can expect to classify through support varieties. For a discussion on the graded centre of a triangulated category $\mathcal{A}$ and its action on the graded Hom-sets in $\mathcal{A}$ see [5].

Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathfrak{e}, T)$ be a tensor triangulated category with tensor product $-\otimes -\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, tensor identity $\mathfrak{e}$ and suspension T. The starting point of a theory of support varieties in this setting is the following result. Note that there is no assumption on the tensor product being exact in either variable.

Theorem 1 ([7]). *Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathfrak{e}, T)$ be a tensor triangulated category. Then the graded endomorphism ring*

$$\operatorname{End}^*_{\mathcal{C}}(\mathfrak{e}) = \oplus_{i\geq 0} \operatorname{Hom}^i_{\mathcal{C}}(\mathfrak{e}, T^i(\mathfrak{e}))$$

is a graded commutative ring.

We give two examples illustrating this construction.

Example 1. *For a finite group (or a finite group scheme) G and a field k, the derived category $D^b(kG)$ of finitely generated left kG-modules under the tensor product $- \otimes_k -$ is a tensor triangulated category with tensor product identity $\mathfrak{e} = k$. Furthermore, the graded endomorphism ring of k in $D^b(kG)$ is given by $\operatorname{End}^*_{D^b(kG)}(k) \simeq H^*(G,k)$, the group cohomology ring.*

Example 2. *Let Λ be a finite dimensional algebra over a field k. Let $\mathcal{B}$ be the full triangulated subcategory of $D^-(\Lambda \otimes_k \Lambda^{\mathrm{op}})$ generated by all finitely generated bimodules which are projective as a left and as a right Λ-module. Then $\mathcal{B}$ is a tensor triangulated category with tensor product induced from $- \otimes_\Lambda -$ and with tensor identity given by Λ. The graded endomorphism ring $\operatorname{End}^*_{\mathcal{B}}(\Lambda)$ is given by the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda)$ of Λ.*

It is well-known that in these two settings the graded endomorphism rings of the tensor identity gives rise to a notion of support variety in both cases. Next we explain how this comes about as a consequence of one and the same construction.

Let $\mathcal{A} = (\mathcal{A}, \Sigma)$ be a triangulated category endowed with an action from $\mathcal{C}$ given by an additive bifunctor $- * - \colon \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ (see [6] for definition of an action, and make the additional adjustments for respecting the triangulated structure similar as in [7]).

For any $h\colon \mathfrak{e} \to T^q(\mathfrak{e})$ in $\operatorname{End}^*_{\mathcal{C}}(\mathfrak{e})$, define $\varphi(h)\colon 1_{\mathcal{A}} \to \Sigma^q$ as the composition

$$1_{\mathcal{A}} \simeq \mathfrak{e} * - \xrightarrow{h*1} T^p(\mathfrak{e}) * - \simeq \Sigma^p(\mathfrak{e} * -) \simeq \Sigma^q(-).$$

Then we can prove the following result.

Proposition 5. *Assigning to $h\colon \mathfrak{e} \to T^q(\mathfrak{e})$ in $\operatorname{End}^*_{\mathcal{C}}(\mathfrak{e})$ the composition*

$$1_{\mathcal{A}} \simeq \mathfrak{e} * - \xrightarrow{h*1} T^p(\mathfrak{e}) * - \simeq \Sigma^p(\mathfrak{e} * -) \simeq \Sigma^q(-)$$

gives rise to a homormophism of graded rings

$$\varphi\colon \operatorname{End}^*_{\mathcal{C}}(\mathfrak{e}) \to Z^*(\mathcal{A}).$$

The homomorphism of graded rings $\varphi\colon \operatorname{End}^*_{\mathcal{C}}(\mathfrak{e}) \to Z^*(\mathcal{A})$ induces a theory of support varieties by letting

$$V(a,b) = \operatorname{Supp}(\operatorname{Hom}^*_{\mathcal{A}}(a,b)) \subseteq \operatorname{Spec}(R),$$

where $\operatorname{Spec}(R)$ is the spectrum of graded prime ideals in R for a positively graded subalgebra R of $\operatorname{End}^*_{\mathcal{C}}(\mathfrak{e})$. It is then easy to see that the support varieties usually studied in the two above examples are induced through the general setup we have described.

There are often further structures floating around in an action of a tensor triangulated category on a triangulated category. Such a structure is the notion of a left function object for the action. One instance of such an object occurs in the following well-known isomorphism for group rings kG,

$$\operatorname{Hom}_{kG}(B \otimes_k A, C) \simeq \operatorname{Hom}_{kG}(A, \operatorname{Hom}_k(B,C)),$$

where A, B and C are finite dimensional kG-modules. Namely the function object is $\operatorname{Hom}_k(-,-)$. In general, an additive bifunctor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$ such that

(i) $F(x, -)\colon \mathcal{A} \to \mathcal{A}$ is a covariant functor for all x in $\mathcal{C}$,
(ii) $F(-, a)\colon \mathcal{C} \to \mathcal{A}$ is a contravariant functor for all a in $\mathcal{A}$,
(iii) there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(x * a, b) \simeq \mathrm{Hom}_{\mathcal{A}}(a, F(x, b))$$

for all x in $\mathcal{C}$ and all a and b in $\mathcal{A}$,

is a *left function object* for the action of $\mathcal{C}$ on $\mathcal{A}$.

Having this additional structure on the action of $\mathcal{C}$ on $\mathcal{A}$ forces the following stability condition of the support varieties defined above.

Proposition 6. *Assume that $\mathcal{C}$, $\mathcal{A}$ and F are as above. Then for x in $\mathcal{C}$ and a in $\mathcal{A}$ we have that*

$$V(x * a) \subseteq V(a)$$

and

$$V(F(x, a)) \subseteq V(a).$$

In studying all the localizations in a triangulated category $\mathcal{A}$, it is of interest to classify all thick subcategories of $\mathcal{A}$. Such a classification is possible using support varieties in some settings, however if the support varieties comes from a tensor triangulated category $\mathcal{C}$ acting on a triangulated category $\mathcal{A}$ with a left function object F, we see from the above that one only can except to classify thick tensor subcategories. By thick tensor subcategories $\mathcal{U}$ of $\mathcal{A}$ we mean thick subcategories $\mathcal{U}$ such that $\mathcal{C} * \mathcal{U} \subseteq \mathcal{U}$.

Finally we point out that the above can be viewed as a categorification of a central ring action of a graded ring on the triangulated category $\mathcal{A}$.

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Hochschild and ordinary cohomology rings of small categories

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These notes report some recent progress on the homological properties of category algebras. Let $\mathcal{C}$ be a small category and k a field. We define the category algebra $k\mathcal{C}$ to be a vector space with base elements the morphisms in $\mathcal{C}$ and the multiplication is given by composition of the base elements (if two morphisms are not composable then we ask their product to be zero). Since every group or poset can be regarded as a category, the notion of a category algebra generalizes that of a group algebra or an incidence algebra. We denote by $Vect_k$ the category of k-vector spaces. A representation of $\mathcal{C}$ is defined to be a covariant functor from $\mathcal{C}$ to $Vect_k$. All the representations of $\mathcal{C}$ form a category, namely the functor category $Vect_k^{\mathcal{C}}$. Let $k\mathcal{C}$-mod be the category of left $k\mathcal{C}$-modules. Mitchell [12] showed there exists a fully faithful functor

$$Vect_k^{\mathcal{C}} \to k\mathcal{C}-\text{mod}.$$

This means that there are certain $k\mathcal{C}$-modules carrying extra underlying functor structure. In our case, the modules we need to define and compute the Hochschild and ordinary cohomology rings are functors, as we shall see shortly. We comment that when $\text{Ob}\mathcal{C}$ is a finite set, the above functor becomes an equivalence, whence we can identify modules with functors.

A concept very similar to the category algebra was introduced by Gabriel in his thesis [2], but Gabriel only considered additive categories, and functors which are additive, so that his definition is a little different. The category algebra does appear in the work of Mitchell [4], but Mitchell did not give it this name. The notions which underlie the category algebra have been widely used since the 1960s, especially with questions to do with homological algebra, and more recently with the development of the theory of p-local finite groups [1]. The term "category algebra" was introduced by Webb (see e.g. [3, 8]).

As we mentioned above, we are interested in the $k\mathcal{C}$-modules which are functors. The underlying functor structure of such a module provides a new angle to describe the module structure. For instance, the trivial $k\mathcal{C}$-module $\underline{k}$ is defined as a constant functor which takes value k at each object in $\mathcal{C}$ and sends every morphism in $\mathcal{C}$ to the identity map on k. Let $\mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op}$. One can easily see that the category algebra $k\mathcal{C}^e$ is isomorphic to $(k\mathcal{C})^e$, the enveloping algebra of $k\mathcal{C}$. The $(k\mathcal{C})^e$-module (or equivalently the $k\mathcal{C}$-$k\mathcal{C}$-bimodule) $k\mathcal{C}$ thus becomes a $k\mathcal{C}^e$-module and moreover one can show it is a functor in $Vect_k^{\mathcal{C}^e}$.

The *Hochschild cohomology ring* of $\mathcal{C}$ is defined as

$$\mathrm{HH}^*(\mathrm{k}\mathcal{C}) = \mathrm{HH}^*(\mathrm{k}\mathcal{C}, \mathrm{k}\mathcal{C}) := \mathrm{Ext}^*_{\mathrm{k}\mathcal{C}^e}(\mathrm{k}\mathcal{C}, \mathrm{k}\mathcal{C}),$$

and the *ordinary cohomology ring* of $\mathcal{C}$ is defined by

$$\mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}}) := \mathrm{Ext}^*_{\mathrm{k}\mathcal{C}}(\underline{\mathrm{k}}, \underline{\mathrm{k}}).$$

We note that the ordinary cohomology ring is isomorphic to $\mathrm{H}^*(\mathrm{B}\mathcal{C}, \mathrm{k})$, where $B\mathcal{C}$ is the classifying space of the small category $\mathcal{C}$. These two rings are graded commutative and when $\mathcal{C}$ is a group or a poset, they are the usual Hochschild and ordinary cohomology rings of a group or a poset in the literature. Motivated by well known results on groups and posets, we prove the following statement.

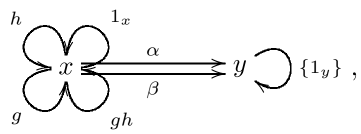
Theorem [9] *There exists a split surjective algebra homomorphism*

$$\mathrm{HH}^*(\mathrm{k}\mathcal{C}) \to \mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}}).$$

Moreover, we have $\mathrm{HH}^*(\mathrm{k}\mathcal{C}, \underline{\mathrm{k}}) \cong \mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}})$.

The proof is based mainly on functor cohomology theory, and Quillen's results on classifying spaces of small categories [5]. In fact, we first build one projective resolution of each of the $k\mathcal{C}^e$-module $k\mathcal{C}$ and the $k\mathcal{C}$-module $\underline{k}$ as a sequence of functors. Then using Quillen's results we may establish a connection between the resolutions, from which we deduce the above theorem.

Now we turn to the structure and computation of the two cohomology rings. By their definitions, both the Hochschild and ordinary cohomology ring can be computed by using projective resolutions. Alternatively, topological methods may be used and we want to emphasize that the isomorphism $\mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}}) \cong \mathrm{H}^*(\mathrm{B}\mathcal{C}, \mathrm{k})$ can dramatically simplify the calculations. Understanding the homotopy type of $B\mathcal{C}$ can be of great help to the computation of $\mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}})$, as we shall see in our example. Given the structure of $\mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}})$, along with our theorem, one can continue to describe the structure of $\mathrm{HH}^*(\mathrm{k}\mathcal{C})$. However, in the latter case, the representation theory of small categories is often needed since $\mathrm{H}^*(\mathcal{C}; \underline{\mathrm{k}})$ is only a summand of $\mathrm{HH}^*(\mathrm{k}\mathcal{C})$ and so purely topological method will not be enough. The following example [9] illustrates our method and furthermore is of great interest in its own right. Suppose char$k = 2$. We consider the category $\mathcal{E}_0$



where $g^2 = h^2 = 1_x, gh = hg, \alpha h = \beta g = \alpha$, and $\alpha g = \beta h = \beta$. The classifying space $B\mathcal{E}_0$ is homotopy equivalent to $(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty)/\mathbb{R}\mathrm{P}^\infty$ (where $\mathbb{R}\mathrm{P}^\infty \simeq \mathrm{B}\mathbb{Z}_2$). Using the long exact sequence for computing the relative cohomology $\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty, \mathbb{R}\mathrm{P}^\infty)$, we can obtain the structure of $\mathrm{H}^*((\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty)/\mathbb{R}\mathrm{P}^\infty, \mathrm{k})$. Consequently the mod-2 ordinary cohomology ring $\mathrm{H}^*(\mathcal{E}_0; \underline{\mathrm{k}})$, as computed by Aurélien Djament, Laurent Piriou and the author, is isomorphic to a subring of the polynomial ring $\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty, \mathrm{k}) \cong \mathrm{H}^*(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{k}) \cong \mathrm{k}[\mathrm{u}, \mathrm{v}]$, by removing all $u^n, n \geq 1$, and their scalar multiples. More explicitly $\mathrm{H}^0(\mathcal{E}_0; \underline{\mathrm{k}}) \cong \mathrm{k}$ and $\mathrm{H}^{*>0}(\mathcal{E}_0; \underline{\mathrm{k}}) \cong \mathrm{k}[\mathrm{u}, \mathrm{v}]\mathrm{v}$. This ring $\mathrm{H}^*(\mathcal{E}_0; \underline{\mathrm{k}})$ has no nilpotents and is not finitely generated. By our theorem, it implies that the Hochschild cohomology ring $\mathrm{HH}^*(\mathrm{k}\mathcal{E}_0)$ modulo nilpotents is not finitely generated either, which gives a counterexample to the finite generation conjecture in [7]. In fact one can explicitly calculate the structure of $HH^*(k\mathcal{E}_0)$ and show $HH^*(k\mathcal{E}_0)/\mathrm{Nil} \cong H^*(\mathcal{E}_0; \underline{k})$, where Nil is the ideal of nilpotents in $HH^*(k\mathcal{E}_0)$.

The category $\mathcal{E}_0$ can be modified to obtain infinitely generated cohomology rings of algebras over fields of odd characteristics. Fix a prime p. One can replace the automorphism group of x by $\mathbb{Z}_p \times \mathbb{Z}_p$ and set $\mathrm{Hom}(x, y)$ to be a set of p morphisms such that one of the $\mathbb{Z}_p$'s acts transitively and the other acts trivially on it. The same topological method as for the case $p = 2$ can be used to calculate the ordinary cohomology ring, which is not finitely generated modulo nilpotents. (Note that if $p > 2$, then the ordinary cohomology ring does have non-trivial nilpotents.) As a consequence, the Hochschild cohomology ring modulo nilpotents is not finitely generated either.

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