Mathematisches Forschungsinstitut Oberwolfach

Report No. 13/2009

DOI: 10.4171/OWR/2009/13

Mini-Workshop: The Pisot Conjecture - From Substitution Dynamical Systems to Rauzy Fractals and Meyer Sets

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March 1st – March 7th, 2009

Abstract. This mini-workshop brought together researchers with diverse backgrounds and a common interest in facets of the Pisot conjecture, which relates certain properties of a substitution to dynamical properties of the associated subshift.

Mathematics Subject Classification (2000): 37B10, 28A80, 37B50, 52C23.

Introduction by the Organisers

A substitution is a non-erasing morphism of the free monoid. Subshifts generated by fixed points of substitutions are natural symbolic models for deterministic self-similar dynamical systems. The Pisot conjecture relates number theoretic properties of the substitution matrix to dynamical properties of the generated subshift. Explicitly, it states that the symbolic dynamical system of a unimodular Pisot substitution has pure point spectrum.

This conjecture has attracted a fair amount of attention. In fact, Pisot substitutions systems and the Pisot conjecture have numerous applications, for example to Diophantine approximation, equidistribution properties of toral translations and low discrepancy sequences, beta-shifts, multidimensional continued fraction expansions, generation or recognition of arithmetic discrete planes, or else effective construction of Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number. Furthermore, the conjecture is supported by numerical evidence since it can be reformulated in effective terms. Still, so far it has only been proved in the case of two symbols.

Primitive substitutions can not only be studied in the framework of symbolic dynamics but also in a higher dimensional geometric setting. There, one is interested in substitution-generated tilings and Delone sets. In this situation, there is an analogous version of the Pisot conjecture.

There exist several necessary and/or sufficient conditions for pure point spectrum for substitution dynamical systems. In fact, three related approaches to pure point spectrum have been developed in the last twenty years:

One approach is based on the notion of coincidence, introduced by Dekking, then by Host, in an unpublished paper, and lastly in greater generality by Arnoux and Ito and Hollander and Solomyak. This correspondence is especially apparent in the recent work of Barge and Kwapisz who showed that pure point spectrum is equivalent to what they call the geometric coincidence condition. This condition is algorithmically decidable.

A different approach relies on the geometric representation of substitution dynamical systems with pure point spectrum as translations on compact metric groups such as shown by the pioneering work of Rauzy in the 80's on the so-called Rauzy fractal. The Pisot conjecture can then be translated in tiling terms. The geometric coincidence can also be stated in this framework.

Finally, there is an approach connecting pure point spectrum with cut and project schemes and so-called Meyer sets. In a very recent work dealing with the higher dimensional case, Lee has shown that a primitive substitution Delone set has pure point spectrum if and only if it comes from a cut and project scheme. Thus, Lee's characterization links pure point spectrum and cut and project schemes within the framework of primitive substitutions. A crucial ingredient in her proof is a new coincidence condition for Delone sets generalizing all earlier conditions of this kind in the geometric setting. Another important ingredient is her recent work with Solomyak showing that pure point diffraction implies the Meyer property for primitive substitution systems, thereby answering a question of Lagarias. This is then combined with a new understanding of cut and project schemes in terms of topologies brought forward in recent work of Baake and Moody.

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Table of Contents

Abstracts

The Silver Mean Chain is a Regular Model Set

Michael Baake

The connection between substitution dynamical systems and model sets is at the heart of the Pisot substitution conjecture. One version of the latter says that the geometric point set realization of a primitive substitution on a finite alphabet, with a Pisot-Vijayaraghavan (PV) number as inflation multiplier and some mild additional irreducibility assumptions, gives rise to a continuous dynamical system (under the action of $\mathbb{R}$) with a pure point dynamical spectrum. One way to prove this claim is by showing the point set to be a regular model set. Here, we sketch the simplest case of this method, followed by some general comments.

The *silver mean substitution* ϱ on the binary alphabet $\{a, b\}$ is conveniently defined by $a \mapsto aba$ and $b \mapsto a$. The corresponding substitution matrix $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is primitive and has leading eigenvalue $s = 1 + \sqrt{2}$. The latter is a PV unit, with algebraic conjugate $s' = 1 - \sqrt{2}$. The Perron-Frobenius (PF) left and right eigenvectors are $(s, 1)$ and its transpose. Starting from the legal seed aa (which occurs in $\varrho^2(a)$ as a subword) and using $|$ as the central marker, one iterates as

$$a|a \xrightarrow{\varrho} aba|aba \xrightarrow{\varrho} abaaaba|abaaaba \xrightarrow{\varrho} \ldots \xrightarrow{\varrho} w = \varrho(w),$$

converging towards a reflection symmetric (or palindromic) bi-infinite fixed point w in the product topology. In this topology, the *discrete* hull is the compact set $\mathbb{X}(w) := \overline{\{S^j \mid j \in \mathbb{Z}\}}$, where S denotes the shift. Now, $(\mathbb{X}(w), \mathbb{Z})$ is a discrete dynamical system, which is known to be strictly ergodic by standard arguments [9]. In particular, it is repetitive (equivalent to minimal) and has uniform subword frequencies (equivalent to unique ergodicity).

To turn this into a geometric setting, let a and b denote intervals of lengths s and 1, respectively, which correspond to the entries of the PF eigenvector. An expanded version (by a factor of s) of the intervals can then precisely be dissected into the correct number of the original intervals. This way, the sequence w becomes a tiling of $\mathbb{R}$, and the left endpoints of the tiles (intervals) form a point set $\varLambda = \varLambda_a \dot{\cup} \varLambda_b$, where a and b refer to the tile types. By construction, we then have $\langle \varLambda - \varLambda \rangle_{\mathbb{Z}} = \mathbb{Z}[\sqrt{2}\,]$, which is the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{2}\,)$, where the algebraic conjugation map $'$ is defined by $\sqrt{2} \mapsto -\sqrt{2}$.

If we now define a natural $\mathbb{R}$-action by translation, we obtain the *continuous* hull as $\mathbb{X}(\varLambda) := \overline{\{t + \varLambda \mid t \in \mathbb{R}\}}$, where the closure is now in the local topology. Here, two locally finite point sets are close when, after a translation of at most ε, they agree on the centered (at 0) interval of length $2/\varepsilon$. The space $\mathbb{X}(\varLambda)$ is compact, and $(\mathbb{X}(\varLambda), \mathbb{R})$ is a continuous dynamical system that is once again strictly ergodic; compare [7] for a general exposition. One consequence for later is that $\varLambda$ is a repetitive point set of density $\frac{1}{2}$, the latter following from a simple calculation with the frequencies and the lengths of the tiles (as obtained from the PF eigenvectors).

The geometric counterpart of the fixed point condition for w consists of the two set valued equations

$$\begin{aligned} \varLambda_a &= s\varLambda_a \,\dot\cup\, \big(s\varLambda_a + (s+1)\big) \,\dot\cup\, s\varLambda_b, \\ \varLambda_b &= s\varLambda_a + s. \end{aligned}$$

Unfortunately, these equations have no nice structure concerning uniqueness of solutions. Under algebraic conjugation and taking closures, they turn into

$$\begin{aligned} W_a &= s'W_a \cup \big(s'W_a + (s'+1)\big) \cup s'W_b, \\ W_b &= s'W_a + s', \end{aligned}$$

where $W_a := \overline{\varLambda'_a}$, and analogously for W_b. Note that we lost disjointness of the unions on the right hand side due to taking closures. In return, since $|s'| < 1$, we now have a contractive iterative function systems (IFS), acting on pairs of compact subsets of $\mathbb{R}$. By Hutchinson's theorem [4], the IFS has a unique solution, which is simply given by the closed intervals

$$W_a = \big[\tfrac{\sqrt{2}-2}{2}, \tfrac{\sqrt{2}}{2}\big] \quad \text{and} \quad W_b = \big[-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}-2}{2}\big].$$

Using the shorthand $L = \mathbb{Z}[\sqrt{2}\,]$ from now on, we so far have the inclusions

$$\varLambda_a \subset \{x \in L \mid x' \in W_a\} \quad \text{and} \quad \varLambda_b \subset \{x \in L \mid x' \in W_b\},$$

which also imply $\varLambda = \varLambda_a \,\dot\cup\, \varLambda_b \in \{x \in L \mid x' \in W\}$, where we set $W = W_a \cup W_b = \big[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\big]$. Observing $L' = L$ and noticing that the endpoints of W_a and W_b are not in L, we actually get the slightly stronger relations

$$\varLambda_a \subset \{x \in L \mid x' \in W_a^\circ\} \quad \text{and} \quad \varLambda_b \subset \{x \in L \mid x' \in W_b^\circ\},$$

where A° denotes the interior of a set A.

The dense point set $L \subset \mathbb{R}$ gives rise to a planar lattice via its Minkowski embedding, which is $\mathcal{L} = \{(x, x') \mid x \in L\}$. This lattice is spanned by the vectors $(1,1)$ and $(\sqrt{2}, -\sqrt{2}\,)$, which form an orthogonal basis. The density of the lattice is now easily calculated as $\operatorname{dens}(\mathcal{L}) = \frac{1}{4}\sqrt{2}$. At this point, we have identified the *cut and project scheme* (CPS) of the silver mean chain as the diagram

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\;\pi\;} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\;\pi_{\text{int}}\;} & \mathbb{R} \\ \text{dense}\ \cup & & \cup & & \cup\ \text{dense} \\ \mathbb{Z}[\sqrt{2}\,] & \xleftarrow{\;1-1\;} & \mathcal{L} & \xrightarrow{\;1-1\;} & \mathbb{Z}[\sqrt{2}\,] \\ \| & & & & \| \\ L & & \xrightarrow{\qquad\qquad ' \qquad\qquad} & & L' \end{array}$$

which is a central part in the model set construction [5]. Note that algebraic conjugation $'$ plays the role of the $\star$-map, which is also discussed in the contributions of Frettlöh and Sing. A *model set* for this CPS is now of the form

$$\curlywedge(A) := \{x \in L \mid x' \in A\},$$

where A is any relatively compact set with non-empty interior, such as our sets W_a, W_b or W above. These sets are called *windows* for the CPS. A model set is called *regular* when the boundary of its window has measure 0. This is clearly the case for W_a and W_b, hence also for W.

Due to the uniform distribution property of $\bigl(\curlywedge(W)\bigr)'$ in W, compare [6], we can calculate the existing density of the projection set $\curlywedge(W^\circ)$ as

$$\operatorname{dens}\bigl(\curlywedge(W^\circ)\bigr) = \operatorname{dens}(\mathcal{L})\operatorname{vol}(W) = \tfrac{1}{2} = \operatorname{dens}(\varLambda).$$

We thus have shown that $\varLambda$ is a subset of $\curlywedge(W^\circ)$, with $\curlywedge(W^\circ)\setminus\varLambda$ being a set of density 0. Although this would already be enough to establish the Pisot substitution conjecture in this specific example, we can actually show more, namely $\curlywedge(W^\circ)=\varLambda$, via the following observation: Assume that $\curlywedge(W^\circ)$ contains a point in the complement of $\varLambda$, which then introduces a new distance, and hence a local patch that does not exist in $\varLambda$. However, by construction, $\curlywedge(W^\circ)$ is repetitive, so that this patch must reoccur with bounded gaps. As this implies $\operatorname{dens}\bigl(\curlywedge(W^\circ)\bigr) > \operatorname{dens}(\varLambda)$, we get a contradiction, hence the extra point cannot exist.

In general, the above approach can be used as well, also beyond the unimodular case, see [8] for a systematic account. One always finds that $\varLambda$ is a subset of a regular model set, but there is (so far) no general method to determine the volume of the window. In [3], this was possible by another lattice argument (the total window tiled internal space periodically). More generally, one needs some independent argument to control the windows (which are usually fractally shaped), which is still missing.

Most papers so far have concentrated on the unimodular case of the conjecture. Examples from constant length substitutions (such as the period doubling chain, or the chair tiling of the plane) show that the model set description is completely natural [2]. Further examples are given in Sing's contribution, or can be found in his thesis [8]. All equivalent formulations known for the unimodular conjecture possess counterparts in the general setting.

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Quasi-periodic tilings from non-Pisot unimodular matrices

Shunji Ito

Let $\sigma : \mathcal{A} = \{1, 2, \ldots, d\} \to \mathcal{A}^*$ be a substitution and L_σ be the matrix of σ.

Assumption

We assume the following conditions:

(1): $\exists N : L_\sigma^N > 0$ (primitive condition);
(2): $\det L_\sigma = \pm 1$ (unimodular condition);
(3): eigenvalues of L_σ satisfies $\lambda = \lambda_1 > 1 > \mid \lambda_2 \mid, \ldots, \mid \lambda_d \mid$ (Pisot condition);
(4): the characteristic polynomial $\Phi_\sigma(x)$ of L_σ is irreducible.

The above substitution is called the *unimodular Pisot substitution.*

Example The subsitution $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ is the unimodular Pisot substitution.

Under Assumption, let us denote the contractive (resp. expanding) L_σ-invariant plane by P_c (resp. P_e) and let us define the projection $\pi_c : \mathbb{R}^d = P_c \oplus P_e \to P_c$.

For $(\boldsymbol{x}, i^*) \in \mathbb{Z}^d \times \{1^*, \ldots, d^*\}$, we give the geometrical meaning as

$$\pi_c(\boldsymbol{x}, i^*) \quad := \quad \left\{ \pi_c \sum_{\substack{j = 1, \ldots, d, \\ j \neq i}} (\boldsymbol{x} + \mu \boldsymbol{e}_j) \;\middle|\; 0 \leq \mu \leq 1 \right\}$$

Using the notation $\sigma(i) = W_1^{(i)} W_2^{(i)} \ldots W_{l_i}^{(i)}$, let us define the tiling substitution σ^* by

$$\sigma^*(\pi_c(\boldsymbol{x}, i^*)) = L_\sigma^{-1}(\pi_c \boldsymbol{x}) + \sum_{j=1}^{d} \sum_{\binom{j}{k} : W_k^{(j)} = i} \pi_c\left(L_\sigma^{-1} f\left(P_k^{(j)}\right), j^*\right)$$

where $P_k^{(i)} = W_1^{(i)} W_2^{(i)} \ldots W_{k-1}^{(i)}$ and $f : \mathcal{A}^* \to \mathbb{Z}^d$ is the homomorphism given by $f(i) = \boldsymbol{e}_i,\ i = 1, 2, 3$.

On Example, $\sigma^*(\pi_c(\boldsymbol{0}, i^*))$ is given by

$$\begin{aligned} \sigma^*(\pi_c(\boldsymbol{0}, 1^*)) &= \sum_{i=1,2,3} \pi_c(\boldsymbol{0}, i^*) \\ \sigma^*(\pi_c(\boldsymbol{0}, 2^*)) &= \pi_c(\boldsymbol{e}_3, 1^*) \\ \sigma^*(\pi_c(\boldsymbol{0}, 3^*)) &= \pi_c(\boldsymbol{e}_3, 2^*) \end{aligned}$$

(see Figure 1).

Then we have the following theorem:

Theorem ([A-I], [I-R]) Let $\mathcal{U} = \sum_{i=1,2,\ldots,d} \pi_c(\boldsymbol{0}, i^*)$, then

(1): $\sigma^*(\mathcal{U}) \succ (\mathcal{U})$;

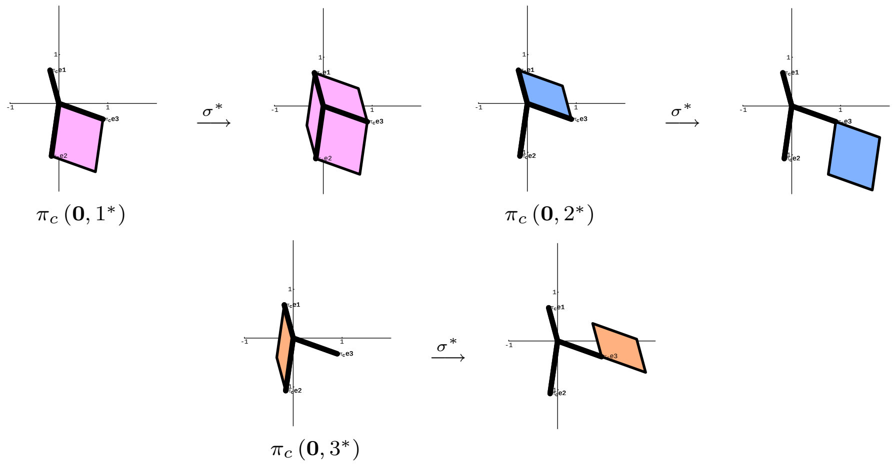

Figure 1: The tiling substitution σ^* on Example.

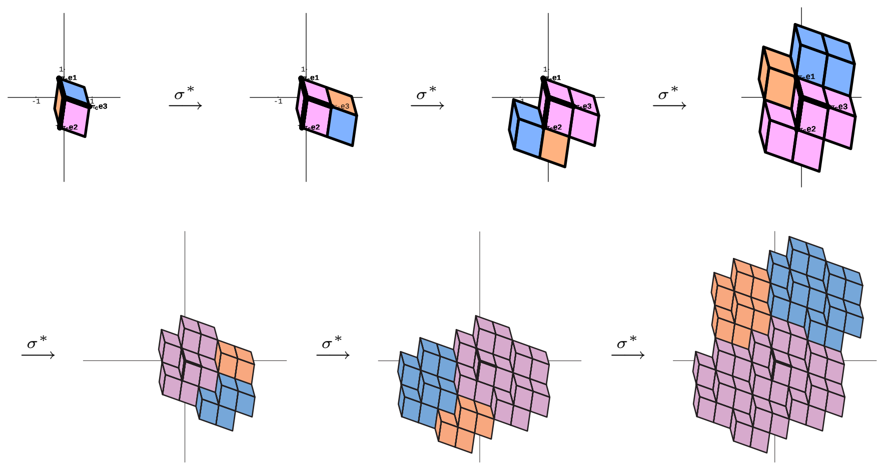

Figure 2: $\sigma^{*n}(\mathcal{U})$, $n = 0, 1, \ldots, 6$ on Example.

(2): if $d(\partial(\sigma^{*n}(\mathcal{U})), \mathbf{0}) \to \infty$ $(n \to \infty)$, then

$$\tau' := \{\pi_c(\boldsymbol{x}, j^*) \mid \pi_c(\boldsymbol{x}, j^*) \in {\sigma^*}^n \pi_c(\mathbf{0}, i^*) \text{ for some } n \text{ and } i^*\}$$

is a quasi-periodic polygonal tiling;

(3): let S be the stepped surface of P_c, that is,

$$S := \{(\boldsymbol{x}, j^*) \mid \langle \boldsymbol{x}, \boldsymbol{v}_1 \rangle \geq 0,\ \langle \boldsymbol{x} - \boldsymbol{e}_{j^*}, \boldsymbol{v}_1 \rangle < 0\},$$

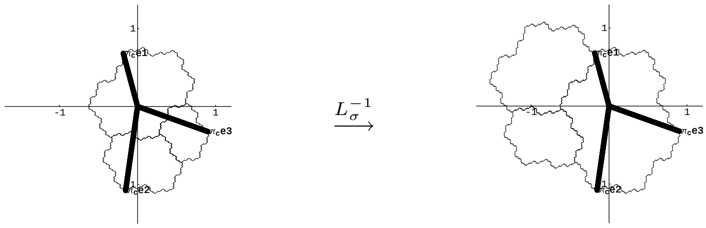

Figure 3: $\{X_i\}_{i=1,2,3}$ and $\{L_\sigma^{-1}X_i\}_{i=1,2,3}$ on Example.

then

$$\{\pi_c(\boldsymbol{x}, j^*) \mid (\boldsymbol{x}, j^*) \in S\} = \tau';$$

(4): let $X_i := \lim_{n\to\infty} {L_\sigma}^n {\sigma^*}^n (\pi_c(\mathbf{0}, i^*))$, then

$$\tau := \{\pi_c \boldsymbol{x} + X_j \mid \pi_c(\boldsymbol{x}, j^*) \in \tau'\}$$

is also a quasi-periodic tiling. Moreover, the following set equation holds:

$$L_\sigma^{-1} X_i = \bigcup_{j=1}^{d} \bigcup_{\binom{j}{k}: W_k^{(j)} = i} \pi_c \left(L_\sigma^{-1} f\left(P_k^{(j)}\right) + X_j \right) \quad \text{(disjoint)}.$$

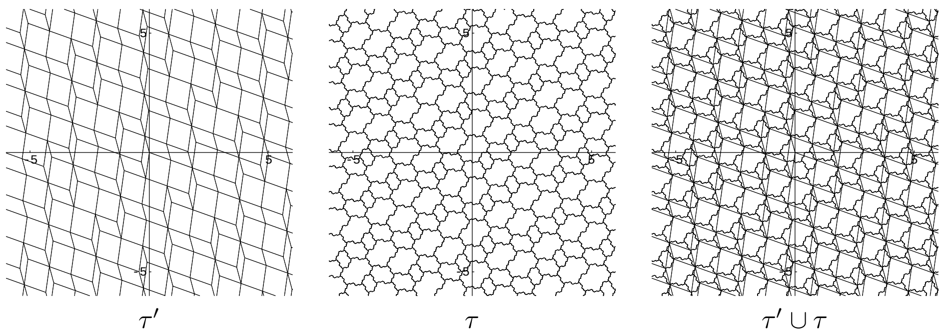

Figure 4: τ', τ, and $\tau' \cup \tau$ on Example.

Moreover, I have spent the time to discuss how we obtain the analogous quasi-periodic tilings under the unimodular non-Pisot assumption. The details can be found in the references.

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Rauzy Tilings and Generalized Substitutions

Valérie Berthé

Rauzy fractals were first introduced in [14] in the case of the Tribonacci substitution $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$, and then in [16], in the case of the β-numeration associated with the Tribonacci number (which is the Pisot root of X^3-X^2-X-1). One motivation for Rauzy's construction was to exhibit explicit factors of Pisot substitutive dynamical systems as rotations on compact abelian groups, and to deduce explicit bounded remainder sets and discrepancy estimates for Kronecker sequences.

Rauzy fractals can more generally be associated with Pisot substitutions (see the surveys [4, 7, 13]), as well as with Pisot β-shifts under the name of central tiles (see e.g. [1]). More precisely, Rauzy fractals are defined in the Pisot substitutive case as the closure of the projection π on the contracting plane of the incidence matrix $\mathbf{M}_\sigma$ of a Pisot substitution σ along its expanding direction of the images by the abelianization map of prefixes of a σ-periodic point, where the abelianization map is defined as $\mathbf{l}\colon \mathcal{A}^* \to \mathbb{N}^n$, $\mathbf{l}(W) \mapsto (|W|_k)_{k=1,\ldots,n} \in \mathbb{N}^n$. A different approach via graph-directed iterated function systems and generalized substitutions has been developed in [2, 3]: Rauzy fractals can be described as attractors of some graph-directed iterated function system. Generalized substitutions provide an algebraic way to describe this equation with respect to the substitution σ. This is on this last approach that we focus in the present introductory lecture.

Let us recall that a substitution is a non-erasing morphism of the free monoid. Generalized substitutions can be considered as multidimensional substitutions of non-constant length acting on multidimensional words (see e.g. [2, 3, 9]). This formalism due to Arnoux and Ito [2] was inspired by the geometrical formalism of [12], whose aim was to provide explicit Markov partitions for hyperbolic automorphisms of the torus associated with particular morphisms of the free group. They have already proved their efficiency for the construction of explicit Markov partitions, as well as for Diophantine approximation [11], and in the spectral study of Pisot substitutive dynamical systems [5, 13]. With any usual unimodular substitution σ can be associated a generalized substitution $E_1^*(\sigma)$ (a substitution is said unimodular if the determinant of its incidence matrix equals ± 1). The generalized substitution $E_1^*(\sigma)$ is defined as the dual map of a natural geometric realization of σ. It maps facets of unit cubes onto unions of facets of unit cubes.

One of the key properties of generalized substitutions is that they map arithmetic discrete planes onto arithmetic discrete planes. Arithmetic discrete planes are basic objects in discrete geometry: the arithmetic discrete plane $\mathcal{P}_{\alpha,\rho}$ of normal vector $\alpha \in \mathbb{R}_+^d \backslash \{\vec{0}\}$ and intercept $\rho \in \mathbb{R}$ is defined as the union of facets of unit cubes whose vertices belong to the set $\{\mathbf{x} \in \mathbb{Z}^d \mid 0 \leq \langle \mathbf{x} | \alpha \rangle + \rho < |\alpha|_1\}$. More precisely, $E_1^*(\sigma)(\mathcal{P}_{\alpha,\rho}) = \mathcal{P}_{{}^t\mathbf{M}_\sigma \alpha,\rho}$.

Let $\mathcal{U}$ denote the upper unit cube. It belongs to any discrete plane with intercept $\rho = 0$. Let σ be a Pisot unimodular substitution. The sets $E_1^*(\sigma)^n(\mathcal{U})$ provide larger and larger patches of the arithmetic discrete plane associated with the contracting space of $\mathbf{M}_\sigma$. For more details, see [2]. The fact that these patches cover the whole arithmetic discrete plane can be considered as an analogue of the so-called finiteness property for β-numerations: a Pisot number β is said to have the finiteness property if every $x \in \mathbb{Z}[1/\beta] \cap [0,1)$ has a finite β-expansion. For more details, see [7].

Furthermore, by renormalizing by $\mathbf{M}_\sigma^n$ the projection π of the sets $E_1^*(\sigma)^n(\mathcal{U})$ and by taking the limit with respect to the Hausdorff metric, one recovers the Rauzy fractal associated with σ. Note also that the Rauzy fractal is conjectured to tile the contracting plane of $\mathbf{M}_\sigma$ according to a self-replicating tiling which is given by the projection π of the arithmetic discrete plane. This is one of the equivalent statements of the Pisot conjecture.

Generalized substitutions thus provide a generation method for arithmetic discrete planes with parameter $\rho = 0$ for some algebraic parameters α. To generate nonalgebraic discrete planes, one can expand a given α with respect to a unimodular continued fraction algorithm. Brun's algorithm (also called modified Jacobi-Perron algorithm) is one of the most classical unimodular multi-dimensional continued fraction algorithms (see e.g. [15]). We then can translate the expansion produced by Brun's algorithm as a product of matrices in the formalism of generalized substitutions. A geometric version of Brun multidimensional continued fraction algorithm acting on discrete planes is thus given in [6] in terms of generalized substitutions. If one wants to describe an arithmetic discrete plane $\mathcal{P}_{\alpha,\rho}$ with nonzero ρ, we then need to involve a skew product of Brun's algorithm in order to also expand ρ: such a skew product will play the role of Ostrowski's skew product in the Sturmian case.

This geometric extension of the Brun algorithm is motivated by the discrete plane recognition problem: given a set of points in $\mathbb{Z}^d$, is there a naive arithmetic discrete plane that contains it? A strategy based on multidimensional continued fractions inspired by the one-dimensional Sturmian case is thus given in [6, 10].

Let us conclude with the following open question: how to associate a Rauzy fractal with a Brun expansion? If we know that the a.e. exponential convergence of Brun's algorithm gives us convergence toward a Rauzy fractal, can we use a generalized Perron Frobenius theorem to prove that its subtiles will be disjoint in measure? What about the tiling properties?

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Pisot Substitutions and Algebraic Number Theory

Bernd Sing

1. Philosophy. For us, Pisot substitutions yield geometrical objects, namely, aperiodic tilings and/or aperiodic point sets, that live in an algebraic number field. Everything else – like dynamical systems, diffraction etc. – comes later.

2. Point and Tile Substitutions. We consider the following examples:

- Silver mean substitution (see Michael Baake's presentation):

$$\sigma: \begin{array}{l} a \to aba \\ b \to a \end{array} \qquad \begin{tikzcd} a \arrow[loop left, "{0,s+1}"] \arrow[r, bend left, "s"] & b \arrow[l, bend left, "0"] \end{tikzcd}$$

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{ll} \text{eingenvalues:} & s = 1+\sqrt{2},\ s' = 1-\sqrt{2} \\ \text{eigenvectors:} & \begin{pmatrix} s \\ 1 \end{pmatrix}, \begin{pmatrix} s' \\ 1 \end{pmatrix} \end{array}$$

- a non-unimodular example (see [1, Section 6.10.2]):

$$\sigma: \begin{array}{l} a \to aaba \\ b \to aa \end{array} \qquad \begin{tikzcd} a \arrow[loop left, "{0,\frac{\lambda}{2},\lambda+1}"] \arrow[r, bend left, "\lambda"] & b \arrow[l, Rightarrow, bend left, "{0,\frac{\lambda}{2}}"] \end{tikzcd}$$

$$M = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{ll} \text{eingenvalues:} & \lambda = \frac{3+\sqrt{17}}{2},\ \lambda' = \frac{3-\sqrt{17}}{2} \\ \text{eigenvector:} & \begin{pmatrix} \lambda/2 \\ 1 \end{pmatrix}, \begin{pmatrix} \lambda'/2 \\ 1 \end{pmatrix} \end{array}$$

We recall silver mean here (also see Michael's Baake talk):
The "incoming arrows" in the substitution graph (variant of the prefix graph) tell us how we get left endpoints of the intervals in the tiling:

$$\begin{aligned} \Lambda_a &= s\cdot\Lambda_a \ \cup\ s\cdot\Lambda_a + s + 1 \ \cup\ s\cdot\Lambda_b \\ \Lambda_b &= s\cdot\Lambda_a + s \end{aligned} \tag{1}$$

The "outgoing arrows" tell us what we get from an interval/prototile (multiplying this with the inflation factor s yields the usual tile substitutions):

$$\begin{aligned} A_a &= \tfrac{1}{s}\cdot A_a \ \cup\ \tfrac{1}{s}\cdot(A_a + s + 1) \ \cup\ \tfrac{1}{s}\cdot(A_b + s) \\ A_b &= \tfrac{1}{s}\cdot A_a \end{aligned} \tag{2}$$

We use the *matrix function system* notation (also compare Jeong-Yup Lee's talk) to summarise these two equations as $\underline{\Lambda} = \Theta(\underline{\Lambda})$ and $\underline{A} = \Theta^{\#}(\underline{A})$. Note that all maps in $\Theta^{\#}$ are of the form $x \mapsto \frac{1}{s}(x+t)$ (where $s > 1$ is the inflation factor, t is some translation), and thus contractions with contraction factor $\frac{1}{s}$. But then[1] $\Theta^{\#}$ itself is a contraction (again with factor $\frac{1}{s}$) on $(\mathcal{K}\mathbb{R}, d_H)$, where $\mathcal{K}\mathbb{R}$ denotes the metric space of (nonempty) compact subsets of $\mathbb{R}$ equipped with the Hausdorff metric[2]. One calls $\Theta^{\#}$ a *(graph-directed) iterated function system (IFS)* and has by Hutchinson's argument (i.e., applying Banach's Fixed Point Theorem) that there is unique fixed point. This unique fixed "point" is actually a collection of compact sets $\underline{A} = (A_a, A_b)$ and (trivially) given by $A_a = [0, s]$ and $A_b = [0, 1]$.

In fact more is true, and we have the following **Theorem**[3]:
Assume the setting as before; in particular,

[1] Also compare [1, Chapter 4] and references therein on iterated function systems.

[2] More precisely, $\Theta^{\#}$ is actually a contraction on the product space $(\mathcal{K}\mathbb{R})^n$.

[3] Exact but somewhat technical formulations can be found in [1, Proposition 4.99 & Corollary 5.63, resp. Corollary 6.66]

- all maps in $\Theta^{\#}$ are of the form $f(x) = \frac{1}{\lambda}x + t$ with some translation t,
- for the Lebesgue measure μ on $\mathbb{R}$ we have $\mu(f(S)) = \frac{1}{|\lambda|} \cdot \mu(S)$ where $S \subset \mathbb{R}$,
- and $\lambda = |\lambda|$ is the PF-eigenvalue of the substitution matrix M.

Then, if one A_k has nonzero measure,

(1) all A_i have nonzero measure (by the primitivity of the IFS),
(2) unions on the right-hand side of the IFS in Eq. (2) are disjoint in measure (since the above factors of $|\lambda|$ cancel each other),
(3) the boundaries ∂A_i have zero measure, the sets A_i are perfect sets and are regularly closed. □

For the intervals A_i this is all trivial; however, if we replace s in Eq. (1) by its algebraic conjugate s', we get:

$$\begin{aligned} W_a &= s' \cdot W_a \;\cup\; s' \cdot W_a + s' + 1 \;\cup\; s' \cdot W_b \\ W_b &= s' \cdot W_a + s' \end{aligned} \tag{3}$$

Since $|s'| = \frac{1}{s}$, it is again an IFS and the previous theorem applies[4].

In the non-unimodular example, the intervals $A_a = [0, \frac{\lambda}{2}]$ and $A_b = [0,1]$ are the solution of a corresponding IFS $\Theta^{\#}$, and replacing λ by its algebraic conjugate λ' in the substitution Θ yields an IFS. However, we have $|\lambda'| = \mathbf{2} \cdot \frac{1}{\lambda}$ – there is an additional factor of 2 (which comes from the minimal polynomial $x^2 - 3x - \mathbf{2}$ for λ) and we cannot apply the stated theorem in the same way as for silver mean.

<u>3. Local Fields.</u> We observe that *everything* (i.e., all numbers in Θ, $\Theta^{\#}$, etc.) "lives" in an algebraic number field $K = \mathbb{Q}(\lambda)$. Recall that the (non-trivial) completions of an algebraic number field K are called *local fields*. Ostrowski's Theorem tells us that a local field is either $\mathbb{R}$ or $\mathbb{C}$ (Archimedean case) or a p-adic field $\mathbb{Q}_p$ or $\mathbb{Q}_{\mathfrak{p}}$ (non-Archimedean/ultrametric case).

The field of p-adic numbers, i.e., the p-adic completion[5] of $\mathbb{Q}$, is given by $\mathbb{Q}_p = \{\sum_{n=m}^{\infty} s_n \cdot p^n \mid m \in \mathbb{Z},\ s_n \in \{0,1,\ldots,p-1\}\}$. We write a p-adic number either as $s_m s_{m+1} \ldots s_{-1}.s_0 s_1 s_2 \ldots$ (if $m < 0$) or as $.0\ldots 0 s_m s_{m+1} \ldots$ (if $m \geq 0$); if $s_m \neq 0$, then the absolute value is given by p^{-m}.

We return to the non-unimodular example: 2-adically, the minimal polynomial of λ splits as follows: $x^2 - 3\,x - 2 \;=\; (x - .10110\ldots) \cdot (x - .01101\ldots)$. Thus, we can complete $\mathbb{Q}(\lambda)$ 2-adically (and obtain $\mathbb{Q}_2$ in either case) either by identifying λ with $.10110\ldots$ (with $|.10110\ldots|_2 = 1$) or by identifying λ with $.01101\ldots$ (with $|.01101\ldots|_2 = \frac{1}{2}$).

We now obtain an iterated function system that satisfies our theorem by diagonally embedding the substitution $\underline{\Lambda} = \Theta(\underline{\Lambda})$ into $\mathbb{R} \times \mathbb{Q}_2$ where in the first coordinate λ is interpreted as (the real number) $-0.56\ldots$ (i.e., λ') while in the

[4]The solution of the IFS in Eq. (3) is given by $W_a = \left[\frac{\sqrt{2}-2}{2}, \frac{\sqrt{2}}{2}\right]$ and $W_b = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}-2}{2}\right]$.

[5]Note that if we consider an algebraic number field $\mathbb{Q}(\lambda)$ (an extension of $\mathbb{Q}$), we might get as its completion some extension of $\mathbb{Q}_p$. Thus, we use the notation $\mathbb{Q}_{\mathfrak{p}}$ where $\mathfrak{p}$ is a prime ideal containing p in the following. For more on local fields, see [1, Chapter 3] and references therein.

second coordinate it is (the 2-adic number) .01101.... This yields an iterated function system $\underline{W} = \Theta^\star(\underline{W})$ on $\mathcal{K}(\mathbb{R} \times \mathbb{Q}_2)$ where for a set $S \subset \mathbb{R} \times \mathbb{Q}_2$ we have[6] $\mu(f(s)) = |\lambda'| \cdot |.01101\ldots|_2 \cdot \mu(S) = \frac{1}{|\lambda|} \cdot \mu(S)$. Then, we can apply our theorem and get "well-behaved" sets W_i.

4. Cut and Project Scheme. For a general[7] Pisot substitution with Pisot number $\lambda > 1$ of degree n the situation is as follows:

- there are $r-1$ real Galois conjugates of λ that are less than 1 in modulus,
- there are s pairs of complex conjugate Galois conjugates, also less than 1 in modulus (and $n = r + 2s$),
- there are[8] $\mathfrak{p}$-adic fields with $|\lambda|_\mathfrak{p} < 1$ only if $p|\det M$ (where p is contained in the prime ideal $\mathfrak{p}$).

We get a *cut and project scheme* with direct space $G = \mathbb{R}$ (the only local field where λ acts as an expansion), internal space $H = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \prod_{\mathfrak{p}:|\lambda|_\mathfrak{p}<1} \mathbb{Q}_\mathfrak{p}$ (where the diagonal embedding of λ acts as contraction) and a lattice $\tilde{\mathcal{L}}$ in $\mathbb{R} \times H$ that is the diagonal embedding of[9] $\mathcal{L} = \bigcup_{m=0}^\infty \frac{1}{\lambda^m}\langle \ell_1, \ldots, \ell_n\rangle_\mathbb{Z}$, where ℓ_i denotes the ith interval length and $\langle S\rangle_\mathbb{Z}$ the group generated by S.

All this yields a symmetric cut and project scheme $(\mathbb{R}, H, \tilde{\mathcal{L}})$ ($\pi_\mathbb{R}, \pi_H$ denote the canonical projections from $\mathbb{R} \times H$ onto $\mathbb{R}$ respectively H):

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\pi_\mathbb{R}} & \mathbb{R} \times H & \xrightarrow{\pi_H} & H = \mathbb{R}^{r-1} \times \mathbb{C}^s \times \prod_{\mathfrak{p}:|\lambda|_\mathfrak{p}<1} \mathbb{Q}_\mathfrak{p} \\
\text{dense} \cup & & \cup & & \cup \text{ dense} \\
\mathcal{L} & \xleftrightarrow{1-1} & \tilde{\mathcal{L}} & \xleftrightarrow{1-1} & \mathcal{L}^\star
\end{array}$$

Thus, the star-map $(\cdot)^\star$ denotes the diagonal embedding into the "contracting" internal space H, and similarly $\tilde{\cdot}$ the diagonal embedding into the product space $\mathbb{R} \times H$.

We also find a symmetric situation by looking at the IFSes (oval boxes) and substitutions (rectangular boxes) on "direct space" $\mathbb{R}$ and "internal space" H:

On $\mathbb{R}$: $\boxed{\underline{\Lambda} = \Theta(\underline{\Lambda})}$ $\boxed{\underline{A} = \Theta^\#(\underline{A})}$ On H: $\boxed{\underline{W} = \Theta^\star(\underline{W})}$ $\boxed{\underline{\Upsilon} = \Theta^{\#\star}(\underline{\Upsilon})}$

So, an IFS in one space corresponds to a substitution in the other space and vice versa; the unique compact solutions of the IFSes are the intervals A_i and the possible windows/"Rauzy fractals" W_i.

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[6]Note that the product over all absolute values of a nonzero number x in some number field equals 1.

[7]For this section compare [1, Chapter 6] and references therein.

[8]We only have to consider finitely many p-adic local fields of $\mathbb{Q}(\lambda)$ here: If p does not divide the constant term of the minimal polynomial of λ, which is also given $\pm\det M$, then the corresponding p- respectively $\mathfrak{p}$-adic value is 1, otherwise it is less than or equal to 1.

[9]In the unimodular case, where λ is an algebraic unit, this reduces to $\mathcal{L} = \langle \ell_1, \ldots, \ell_n\rangle_\mathbb{Z}$.

5. Pisot Conjectures. Before stating various equivalent formulations[10] of the *Pisot Substitution Conjecture*, we need to introduce the notion of a *model set*. Let $(G, H, \tilde{\mathcal{L}})$ be a cut and project scheme and assume that $S \subset H$ has nonempty interior and is relatively compact; then the following Delone subset of G is called a model set with window S: $\Lambda(S) = \{\pi_G(x) \mid x \in \tilde{\mathcal{L}}, \pi_H(x) \in S\}$. Since the cut and project scheme above is symmetric, we will write $\Lambda_G(S)$ to emphasise that this model set is a subset of G. If additionally the Haar measure of the boundary ∂S vanishes, we call the model set *regular*.

Pisot Substitution Conjecture I. $\underline{\Lambda}$ is a regular model set with windows $\underline{W}$ (up to boundary points of the sets W_i) – meaning that $\Lambda_i = \Lambda_G(W_i)$ for all i, up to points arising from the boundary of W_i.

By construction, $\underline{\Lambda} + \underline{A} = \{A_i + t \mid t \in \Lambda_i, 1 \leq i \leq n\}$ is a tiling of $\mathbb{R}$, and some reflection shows that we reformulate our conjecture as follows:

Pisot Substitution Conjecture II. The Pisot Substitution Conjecture holds iff $\Lambda_{\mathbb{R}}(\underline{W}) + \underline{A}$ is a tiling (again, up to boundary points of the sets W_is).

We note that $\Lambda_{\mathbb{R}}(\underline{W}) + \underline{A}$ is always a multi-covering of a.e.-constant covering degree. Using the symmetric structure of the cut and project scheme, we get:

Pisot Substitution Conjecture III. The Pisot Substitution Conjecture holds iff $\Lambda_H(\underline{A}) + \underline{W}$ is a tiling, where $A_i = [0, \ell_i[$ denotes the half-open interval (this makes the model set in H in question repetitive).

The final formulation makes (direct) use of the unique solutions of the IFSes involved:

Pisot Substitution Conjecture IV. The Pisot Substitution Conjecture holds iff $\bigcup_{i=1}^{n}(-A_i) \times W_i$ (and then also $\bigcup_{i=1}^{n} A_i \times (-W_i)$) is a fundamental domain of the lattice $\tilde{\mathcal{L}} \subset \mathbb{R} \times H$.

This, of course, is a nice way to write the torus that plays such an important role for the dynamical systems associated with these tilings. But we stop here.

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Pure Point Spectrum, Diffraction Spectrum and Higher Order Correlations

Daniel Lenz

(joint work with Michael Baake, Robert V. Moody)

Here we shortly review some elements presented in a talk and a problem session. We refer to the books [3, 16] and the survey [13] for further discussion and references concerning diffraction and aperiodic order.

[10] These equivalent formulations of the Pisot Substitution Conjecture are amongst those stated in [1, Theorem 6.116]. See there for further references.

The setting: Ergodic measures on point sets. Let E^d denote the d-dimensional Euclidean space. We denote by $\mathcal{D}$ the set of all uniformly discrete subsets of E^d whose points have distance 1 or bigger. (Of course, 1 could be replaced by any other positive number.) The set of all continuous compactly supported functions on E^d is denoted by $C_c(E^d)$. Each $\varphi \in C_c(E^d)$ induces a function

$$N_\varphi : \mathcal{D} \longrightarrow \text{Complex numbers}, \; N_\varphi(\Lambda) := \sum_{x\in\Lambda} \varphi(x).$$

These functions induce a topology on $\mathcal{D}$ viz the smallest topology making all of them continuous. In this topology, the space $\mathcal{D}$ is a compact topological space. It allows for a continuous action via translations given by

$$\alpha : E^d \times \mathcal{D} \longrightarrow \mathcal{D}, \; (t, \Lambda) \mapsto t + \Lambda.$$

We are interested in the ergodic probability measures on $\mathcal{D}$. Each such measure specifies a subset of $\mathcal{D}$ viz its support.

Correlations. Let μ be an ergodic probability measure on $\mathcal{D}$. Then, for each natural number n, there exists an measure $\gamma^{(n)}$ on $E^d \times \ldots \times E^d$ (n-factors) such that

$$\gamma^{(n+1)}(\Phi) = \lim_{R\to\infty} \frac{1}{|B_R|} \sum_{x,y_1,\ldots,y_n\in\Lambda} \Phi(-x + y_1, \ldots, -x + y_n)$$

for μ-almost every $\Lambda \in \mathcal{D}$ and every continuous function Φ with compact support on $E^d \times \ldots \times E^d$. Here, $|B_R|$ denotes the Lebesgue measure of a ball with radius R in E^d.

Diffraction spectrum and dynamical spectrum. A particular role is played by $\gamma := \gamma^{(2)}$. It is called *autocorrelation.* Its Fourier transform $\widehat{\gamma}$ is called the *diffraction measure.* This measure can be determined in a diffraction experiment. The set S of Bragg peaks is defined by

$$S := \{k \in E^d : \widehat{\gamma}(\{k\}) > 0\}.$$

The group of eigenvalues of $(\mathcal{D}, \alpha, \mu)$ is denoted by $E(\mu)$. It turns out that $\widehat{\gamma}$ is stronlgy linked to the spectrum of the dynamical system $(\mathcal{D}, \alpha, \mu)$. More precisely, the following holds.

Theorem. *$\widehat{\gamma}$ is a pure point measure if and only if $(\mathcal{D}, \alpha, \mu)$ has pure point dynamical spectrum. In this case the group $E(\mu)$ of eigenvalues of the dynamical systems is generated by the set S.*

Remarks. (a) The theorem has a long history. In the symbolic dynamics case it can be found in [17]. Starting with the work of Dworkin [6] the 'if' direction was shown in [9, 18] (see [7, 19] for related material as well). The equivalence was first shown in [11] (for point sets with further regularity properties) and then in increasing generality including our setting in the works [2, 8, 15]. The statement on the eigenvalues is implicit in [11]. It can be found explicitely in [2].

(b) In the non-pure point case, the diffraction spectrum is still contained in the dynamical spectrum.

The higher order correlations. The previous section has been concerned with the autocorrelation $\gamma = \gamma^{(2)}$. A systematic study of higher order correlations in our context seems to have only started recently in the work [4]. As discussed there (see [14] as well) the measure μ is determined by the $\gamma^{(n)}$, $n = 1, 2, \ldots$. In fact, it turns out that the measure μ is already determined by finitely many moments in the pure point case under a certain further assumption:

Theorem. *If $E(\mu) = S + \ldots + S$ (N-terms) then μ is already determined by its first $2N+1$ correlation functions. In particular, if the set of eigenvalues equals S, then μ is determined by its first three correlation.*

Remarks. (a) The theorem is proven in [14].

(b) The autocorrelation alone does not determine the measure in general as discussed e.g. in [1].

(c) A particular interesting case concerns measures μ coming from so called regular model sets. This has been studied in [5]. As shown there the measure is determined by its first three correlations if the internal space is Euclidean.

Connection to Pisot Conjecture and some open problems. The above results are formulated in the context of point sets. However, they also hold in the case of symbolic dynamics. In particular, pure point dynamical spectrum is equivalent to pure point diffraction. This means that the conclusion of the Pisot conjecture (viz pure point spectrum) can also be formulated as result on diffraction (viz pure point diffraction).

One version of the Pisot conjecture says that a Pisot substitution gives rise to a regular model set (with real internal space). By the results of [4] such model sets seem to be determined by their first three correlations. This suggests the following question:

Question. Is a Pisot substitution determined by its first three correlations?

Note that this question does not refer to point spectrum. The work [14] suggests that this may be related to the structure of a cycle associated to the eigenfunctions (see [5] as well).

More generally we may ask whether any primitive substitution is already determined by finitely many correlations.

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Pure Point Diffraction and Coincidence on Substitution Point Sets

Jeong-Yup Lee

We first discuss how to construct substitution tilings keeping the same substitution rules from substitution point sets. We introduce the notion of representability in substitution point sets and give an equivalent condition for a substitution point set to be representable for a substitution tiling. This connection was first shown in [4] with a sufficient condition and improved with an equivalent condition in [7]. Under this equivalent condition, we can easily switch from substitution point sets to substitution tilings.

Throughout the talk, we assume that primitive substitution point sets are in $\mathbb{R}^d$ and representable for substitution tilings.

We show a circle of equivalences relating pure point diffraction spectrum, overlap coincidence, algebraic coincidence, and inter model sets for a substitution point set whose union is a Meyer set. As it has been discussed in Lenz's talk, pure point diffraction spectrum and pure point dynamical spectrum are equivalent in quite a general setting(see [6, 2, 3]). The equivalence between pure point dynamical spectrum and overlap coincidence has been shown in [9, 7]. The equivalence between overlap coincidence and algebraic coincidence is proved in [5].

Furthermore it was proved in [8] that pure point diffractive substitution point sets are always Meyer sets. So we show that the assumption of a Meyer set in the above assumption is not necessary.

Finally, using the notions of overlap coincidence and matrix function system, we provide a computable algorithm to check pure point diffraction in substitution point sets. We show this algorithm with an explicit example(see [1]).

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Duals of Pisot Substitutions

Dirk Frettlöh

The nature of this talk is twofold: Distinct concepts of the dual substitution of a Pisot substitution are explained and shown to be equivalent, at least in the simplest cases; and it is shown how to apply one of these concepts to obtain partial results for the Pisot conjecture.

A symbolic substitution on words, like

$$\sigma: \quad 0 \to 010,\ 1 \to 01010$$

gives rise to a self-similar tiling of the line in a canonical way: just replace each letter in the biinfinite word by an interval (tile) of appropriate length. If the substitution matrix $S = (S_{ij})$, where S_{ij} counts the number of is in $\sigma(j)$, is primitive (that is, there is $k \geq 1$ such that $S^k > 0$), then the appropriate lengths are given by the left Perron-Frobenius eigenvector v of S. We will always require primitivity from here on. Scaling v such that $v = (1, v_2, \ldots, v_n)$ ensures that each tile length is element of the number field $\mathbb{Q}(\lambda)$, where λ is the Perron-Frobenius eigenvector of S. Thus the vertex set of any tiling by these tiles is a discrete subset of $\mathbb{Q}(\lambda)$, whenever 0 is a vertex. The symbolic substitution σ yields a tile substitution s, which maps tiles to finite tilings, and tilings of the line to tilings of the line. In particular, if we start with a biinfinite word fixed by σ (take for instance $\ldots ababaaba|abaababa \ldots$ for the example above), we obtain a tiling of the line which is fixed under s, a so called *self-similar* tiling.

Let *Galois-dual* denote the map from $\mathbb{Q}(\lambda) \to \mathbb{R}^{d-1}$, where d is the algebraic degree of λ, which is obtained by replacing λ by the vector of its $d-1$ algebraic conjugates. Taking the Galois-dual of the vertex set of a self-similar tiling as above, where λ is a Pisot number, yields a bounded set in $\mathbb{R}^{d-1}$. The closure of this set is known as the Rauzy fractal (in the theory of discrete dynamical systems), or the window (in the theory of aperiodic order), of σ.

As Thurston noted in 1989, the expanding self-similarity of the tiling in the direct space $\mathbb{R}$ yields a dual substitution in the internal space $\mathbb{R}^{d-1}$. To be precise, the Galois dual of the substitution is a contracting IFS (iterated function system) with the window as its solution. (Hutchinson theorem shows that each IFS has a unique compact nonempty solution.) This IFS can be blown up and yields a substitution for tilings in the internal space $\mathbb{R}^{d-1}$ [5], see also [6].

This concept has been made precise in several ways: One possibility is the star-dual $\sigma^\star$ of a substitution. This is a generalisation of the Galois dual described above, see [3] for details. If λ is an algebraic unit, the star-dual is the same as the Galois dual. An advantage of the star dual is that it is easily described in terms of digit set matrices: If $\mathcal{D}$ is the digit set matrix encoding the tile substitution, then $(\mathcal{D}^T)^\star$ encodes the dual IFS, thus the dual substitution.

Another way to formalise the dual substitution is the dual map $E_d^*(\sigma)$ acting on 'stepped surfaces', i.e., approximations of hyperplanes by unit cube faces, see [1]. Moreover, in the case where λ is a quadratic algebraic unit, one might regard σ as an endomorphism of the free group $F_2 = \langle a, b\rangle$. If σ is invertible, then σ^{-1} (or its square) yields a substitution on the alphabet $\{a, b^{-1}\}$.

Theorem 1 [2] *Let σ a primitive unimodular (invertible) substitution on two letters. Then $E_1^*(\sigma)$ and $\sigma^\star$ (and σ^{-1}) yield equivalent sequences.*

The term equivalent sequences, rather than equal sequences, is due to the fact that we compare combinatorial sequences (words) with geometric ones (tilings). The relation is obtained as discussed above, by identifying letters with intervals. Moreover, the dual tiling might use a different alphabet, so we might want to rename letters. For the case of more than two letters we refer to future work.

The following result is a simple consequence of the fact that any tiling which dual is in fact a proper tiling (rather than a multiple covering) fulfils the Pisot conjecture (cp. [4]).

Theorem 2 [3] *A tile substitution with digit set matrix $\mathcal{D}$, where*

$$(\mathcal{D}^T)^\star = cP^{-1}\mathcal{D}P + t$$

for some $c > 0, t \in \mathbb{R}^n, P$ a permutation matrix, fulfils the Pisot conjecture.

The proof is based on the fact that the tilings which digit set matrices fulfil the equation above are self-dual, i.e., they are equivalent to themselves. Hence the dual tilings are proper tilings and no multiple coverings. Unfortunately, whereas this theorem applies to a large class of 2-letter substitutions, there is no single example known beyond this case. For instance, the author is not aware of any self-dual tile substitution in dimension $d \geq 2$.

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Diophantine Properties of Interval Exchange Transformations

Michael Boshernitzan

(joint work with Jon Chaika)

Let $\mathbb{J} = [0, 1) = \mathbb{R}/\mathbb{Z}$ stand for the unit interval and $\mathbb{R}, \mathbb{Z}$ denote the sets of real numbers and of the integers, respectively. We present shrinking targets results for minimal IETs (interval exchange transformations) $(\mathbb{J}, T)$, $\mathbb{J} = [0, 1)$.

Denote by $\langle a \rangle = \min_{k \in \mathbb{Z}} |a - k|$ the distance from $a \in \mathbb{R}$ to the nearest integer.

For a map $T \colon \mathbb{J} \to \mathbb{J}$, consider two family of functions $\phi_\alpha \colon \mathbb{J}^2 \to [0, \infty]$ and $\phi_\alpha \colon \mathbb{J}^2 \to [0, \infty]$, $\alpha \in \mathbb{R}$, defined by the formulae

$$\phi_\alpha(x, y) = \liminf_{n \to \infty} n^\alpha \langle T^n(x) - y \rangle$$

and

$$\psi_\alpha(x, y) = \liminf_{n \to \infty} n^\alpha \langle T^n(x) - T^n(y) \rangle.$$

The main result of our talk is that, for a minimal IET $(\mathbb{J}, T)$ which is ergodic relative to a Borel probability measure μ, the equality

$$\phi_1(x, y) = \liminf_{n \to \infty} n \langle T^n(x) - y \rangle = 0, \tag{1}$$

holds for $\mu \times \mu$ almost all pairs $(x, y) \in \mathbb{J}^2$.

A special case of this result, for minimal 2-IETs (irrational rotations) is already known ([1] and [2]). (In this case, $\mu = \lambda$, the Lebesgue measure).

We show that the result in (1) is the best possible already for 2-IETs in so that the factor n in it cannot be replaced by a factor approaching infinity faster.

We also show that (1) fails for $\mu = \lambda$ for an explicitly constructed 4-IET which is minimal but not ergodic relative the the Lebesgue measure λ. The example is based on the work of the second author [3]. In this example we show that $\phi_1(x, y)$ can be infinite on a set of positive Lebesgue measure in $\mathbb{J}^2$, even though it vanishes on a set of measure 1/2 (at least). In general, for a minimal (not necessarily ergodic) r-IET, $\phi_1(x, y)$ must vanish on a set of positive measure.

The result (1) is constructed with the following one. For a "random" 3-IET we show that for all $\alpha > 0$ and Lebesgue almost all $x, y \in \mathbb{J}$ the equality

$$\psi_\alpha(x, y) = \liminf_{n\to\infty} n^\alpha \langle T^n(x) - T^n(y)\rangle = \infty \tag{2}$$

holds, even though $\psi_0(x, y) = \liminf_{n\to\infty} \langle T^n(x) - T^n(y)\rangle = 0$ (for Lebesgue almost all $x, y \in \mathbb{J}$) because a "random" IET is weakly mixing (see [4]) and hence $T \times T$ is ergodic.

We conclude the lecture with the following two open questions:

1. If an IET $(\mathbb{J}, T)$ is uniquely ergodic, does the inequality $\phi_1(x, y) < \infty$ need to hold *for all* $x, y \in \mathbb{J}$ (versus almost all as we prove)? (Note that the answer is affirmative if T is an r-IET with $r \le 3$).
2. Is it possible (for a minimal IET $(\mathbb{J}, T)$) to have $\phi_1(x, y) = 0$, *for all* $x, y \in \mathbb{J}$ (versus *almost all* as we prove)? (Note that the answer is "no" if T is an r-IET with $r \le 3$; the proof of this fact we currently have is not short).

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MLD Relations of Pisot Substitution Tilings

Franz Gähler

We consider substitutions σ on an alphabet of three letters, whose abelianisation matrix (substitution matrix) M is primitive and unimodular, has irreducible characteristic polynomial, and a leading (Perron-Frobenius, PF) eigenvalue which is a Pisot number. The words generated by such a substitution can be regarded as elements of a free group with three generators, and automorphisms of the free group give rise to transformations of words. Alternatively, we can work with a geometric realisation of the substitution, by letting it act on three intervals, whose lengths are chosen proportional to the components of the left eigenvector associated with the leading PF-eigenvalue λ of M. Each tile is then substituted with a sequence of tiles, whose total length is equal to λ times the original length.

Such a geometric realisation generates a tiling of the line, instead of a sequence of symbols, or a word in a free group.

The matrix M represents a linear mapping A of $\mathbb{R}^3$, expressed with respect to some basis $\{b_i\}$. As M is unimodular, this mapping is an automorphism of the lattice L generated by $\{b_i\}$. We choose the geometry of L such that the expanding and contracting eigenspaces of A are perpendicular to each other, so that A commutes with the orthogonal projections on these eigenspaces. This can be realised as follows. After appropriate rescaling, the tile lengths, being components of the PF-eigenvector, are contained in the algebraic field $Q(\lambda)$, and so are all coordinates of lattice points in the expanding eigenspace of V of A. The corresponding coordinates in the contracting eigenspace W can be chosen as the $d-1$ Galois conjugates of the coordinate in V. We then have a cut-and-project scheme defined by the lattice L, and the eigenspaces V and W of A:

$$
\begin{array}{ccccc}
V \cong \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}^3 & \xrightarrow{\pi_2} & W \cong \mathbb{R}^2 \\
\cup & & \cup & & \cup \\
\Lambda & & L & & \Omega
\end{array}
\tag{1}
$$

One of the formulations of the Pisot conjecture states that the vertex set Λ of a Pisot substitution tiling always is a model set, which means that there exists a window set $\Omega \subset W$ which is the closure of its interior, and which has boundary of measure zero, such that $\Lambda = \{\pi_1(x) \,|\, x \in L, \pi_2(x) \in \Omega\}$. Similarly, the subsets of the left end points of all tiles of a given type in a Pisot substitution tiling are model sets, too, with appropriate subwindows Ω_i. For all examples considered below, the Pisot conjecture can be shown to hold, even though a proof for the general case is still missing. For a more detailed description of Pisot substitution tilings and their associated cut-and-project schemes, we refer to [1].

The cut-and-project scheme (1) does not specify the window Ω yet. Substitutions having the same abelianisation matrix M (but differ in the order of the letters within a substituted word) give rise to the same cut-and-project scheme, but will have different windows in general. As we shall see below, even substitutions with different abelianisation matrices may belong to a common cut-and-project scheme.

In the following, we shall study relations between certain substitution tilings belonging to a common cut-and-project scheme. For this, besides the geometric realisation of a substitution tiling it is also useful to consider the substitution action as an automorphism of the free group with three generators. If for two substitutions σ_1 and σ_2 there exists a fixed word w in the group, such that $\sigma_1(g) = w^{-1}\sigma_2(g)w$ for every generator g of the group, then the two substitutions produce tilings wich are locally isomorphic (LI), meaning that all their finite subpatterns are the same. This can be seen as follows. One first observes that there exists some power of σ_1, such that σ_1^k has a bi-infinite fixed point, and that σ_1^k and σ_2^k are still conjugate in the same way, with a (longer) word w'. In a second step, one can then show that the fixed point of σ_1^k is also a fixed point of σ_2^k, which implies that the two substitutions generate the same tilings.

A more delicate relation is mutual local derivability (MLD) [2]. Two tilings are MLD, if one can be reconstructed from the other in a *local* way, and vice versa.

For this to work, the two tilings must first be brought to the appropriate relative scale and position. A good starting point is to consider two tilings belonging to a common cut-and-project scheme. In fact, two (model set) tilings are MLD if and only if the window of one can be constructed by finite unions and intersections of lattice translates of the window of the other, and vice versa. Looking at the windows can suggest an MLD relation, but for proving such a relation it is very helpful if one substitution can be written as a conjugate of the other, $\sigma_1 = \rho^{-1} \circ \sigma_2 \circ \rho$, where ρ is an outer automorphism of the free group (an inner automorphism would lead to an LI relation). Such an automorphism will make the transformation of one tiling into the other explicit. In the following, different phenomena arising in this context are illustrated with a number of examples. As a short-hand notation, we write the action of a substitution σ on a free group as the list of images of the generators, in our case a triple $[\sigma(a), \sigma(b), \sigma(c)]$.

As a first example, we consider the substitutions $\sigma_1 = [cb, c, cab]$ and $\sigma_1' = [bc, c, cba]$, which have the same abelianisation matrix. These two substitutions are conjugate by the free group automorphism $\rho_1 = [bab^{-1}, b, c]$, with inverse $\rho_1^{-1} = [b^{-1}ab, b, c]$. It is easily checked that indeed we have $\sigma_1 = \rho_1^{-1} \circ \sigma_1' \circ \rho_1$. In a word generated by σ_1, there is always a b to the right of an a. ρ_1 eats up that b, and adds a b to the left of the a instead, effectively replacing ab pairs by ba pairs. ρ_1^{-1} performs the opposite operation. This is obviously a local operation, no matter whether one works with words in a free group, with symbolic sequences, or with tilings. The LI classes of tilings generated by the two substitutions are MLD.

In the second example, we consider two substitutions with different abelianisation matrices, $\sigma_2 = [c, a, cab]$ and $\sigma_2' = [c, ca, cb]$. Again, there is a conjugating automorphism $\rho_2 = [a, a^{-1}b, c]$, with inverse $\rho_2^{-1} = [a, ab, c]$, so that $\sigma_2 = \rho_2^{-1} \circ \sigma_2' \circ \rho_2$. Here, in words produced by σ_2, all b are to the right of an a. ρ_2 eats up the a to the left of a b, effectively replacing all ab pairs by just one b. Other a (not to the left of a b) and all c are left as they are. Conversely, ρ_2^{-1} splits all b in a σ_2'-word into ab pairs. On the tiling level, this operation is local if and only if the length of an ab pair of tiles in the σ_2-tiling is the same as the length of a b tile in the σ_2'-tiling, whereas a and c tiles have the same length for both tilings. With appropriate global scalings, this is indeed the case. $\sigma_2 = \rho_2^{-1} \circ \sigma_2' \circ \rho_2$ implies that the two abelianisation matrices are conjugate in $GL_3(\mathbb{Z})$. In fact, the two substitutions have the same cut-and-project scheme, with the same lattice L. The only difference is, that the linear mapping A is expressed with respect to two different lattice bases, yielding two different matrix representations of A, and different tile lengths (which are the lengths of the projected basis vectors). It is therefore not surprising, that the length of tile b in the σ_2'-tiling is the sum of the lengths of the two tiles a and b of the σ_2-tiling. MLD relations can therefore arise also if the two abelianisation matrices are not equal, but conjugate in $GL_3(\mathbb{Z})$, because the two substitutions then share a common cut-and-project scheme. We emphasise, however, that this relation is local only for the tilings with properly sized tiles.

This pair of examples had been discussed in detail already in [3]. σ_2' is LI to the Rauzy or Tribonacci substitution.

Finally, as a third example, we consider a quartet of substitutions, all with the same abelianisation matrix M. These substitutions are $\sigma_A = [ca, ab, cab]$, $\sigma_B = [ac, ab, abc]$, $\sigma_C = [ca, ba, bac]$, and $\sigma_D = [ac, ab, bac]$. σ_A and σ_D are conjugate in a way already seen in the first example: $\sigma_D = \rho_3^{-1} \circ \sigma_A \circ \rho_3$, where $\rho_3 = [a, b, a^{-1}ca]$ simply replaces ac pairs by ca pairs. In order to discuss the relations to the other substitutions, we introduce the automorphisms $w_1 = [c, a, ab]$ and $w_2 = [c, a, ba]$. We then have $\sigma_A = w_1 \circ w_1 \circ w_2$, $\sigma_B = w_1 \circ w_2 \circ w_1$, and $\sigma_C = w_2 \circ w_1 \circ w_1$, so that $\sigma_A = w_1 \circ \sigma_B \circ w_1^{-1}$ and $\sigma_C = w_1^{-1} \circ \sigma_B \circ w_1$. The conjugating automorphism w_1 has an abelianisation matrix W which commutes with the common abelianisation matrix M of the substitutions, even though W is non-trivial. This is possible because M is equal to the third power of W. Therefore, M is conjugate to itself by some non-trivial mapping, which acts non-trivially on the lattice L, changing the scale of the tiling by the cubic root of the inflation factor λ of the substitution (or its inverse). Consequently, in order to be MLD, the tilings produced by σ_A, σ_B and σ_C must be at relative scales $\lambda^{-\frac{1}{3}}$, 1, and $\lambda^{\frac{1}{3}}$, respectively. The situtation is in fact similar to the second example, where the substitutions share a common cut-and-project scheme, but different lattice bases of L are used. Here, these different lattice bases still lead to the same abelianisation matrix M, but produce tiles of different sizes.

Acknowledgements. The author would like to thank Pierre Arnoux, Dirk Frettlöh, and Edmund Harriss for fruitful discussions.

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Schrödinger Operators Associated with Substitution Dynamical Systems

David Damanik

Let $\sigma : A \to A^*$ be a primitive substitution and let $u \in A^{\mathbb{N}}$ be a fixed point of some power of σ. Consider the associated hull

$$\Omega = \{\omega \in A^{\mathbb{Z}} : F_\omega = F_u\},$$

where F_s denotes the set of finite subwords of a one-sided or two-sided infinite sequence. It is known that there is a unique shift invariant probability measure ν on Ω.

We can define the associated Schrödinger operators as follows. Given a one-to-one function $f : A \to \mathbb{R}$, we let

$$V_\omega(n) = f(\omega_n), \quad \omega \in \Omega, \ n \in \mathbb{Z}.$$

This gives rise to a bounded self-adjoint operator H_ω on $\ell^2(\mathbb{Z})$,

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

We focus here on the problem of existence of eigenvalues and consider the set

$$\Omega_c = \{\omega \in \Omega : H_\omega \text{ has no eigenvalues}\}.$$

There is the following relative of the Pisot conjecture:

$$\text{Does } \sigma \text{ Pisot imply that } \Omega_c = \Omega?$$

One could also ask whether pure point dynamical spectrum implies $\Omega_c = \Omega$. There are only few results in this direction. The statement $\Omega_c = \Omega$ is known for all Sturmian substitutions [3] and the period doubling substitution [2].

There are combinatorial methods that enable one to prove that Ω_c is large in a certain sense. For example, if u contains arbitrarily long palindromes, then Ω_c is a dense G_δ set [5]; and if u contains a fractional power greater than three, then $\nu(\Omega_c) = 1$ [1]. On the other hand, the methods from [5] and [1] can never prove the full statement $\Omega_c = \Omega$ as there are always exceptional ω's that are not accessible [1, 4].

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Spectral Theory of Bijective Substitution Sequences

Natalie Priebe Frank

1. Introduction

The dynamical systems of bijective substitution sequences in $\mathbb{Z}^d$ have a mixed dynamical spectrum, while many of their factors may only have a discrete part. Using as examples the well-known Thue-Morse and period-doubling substitutions, we will show what happens to the continuous part of the dynamical spectra through the factoring process.

2. Constant-length substitutions in $\mathbb{Z}^d$

When defining a constant-length substitution in $\mathbb{Z}^d$, the first order of business is to decide on the size of the rectangular blocks that the substitution will use. So we choose positive integers $l_1, l_2, ..., l_d$ and define

$$B = B(l_1, ..., l_d) = 0, 1, ..., l_1 - 1 \times ... \times 0, 1, ..., l_d - 1 \subset \mathbb{Z}^d$$

The block B defines an empty set of spaces for the substitution to fill in, sort of a "wire frame" structure waiting to be decorated (or colored in) by letters from some finite alphabet $\mathcal{A}$. To decide how to color in each space $\vec{\jmath} \in B$, we next choose a map $p_{\vec{\jmath}} : \mathcal{A} \to \mathcal{A}$. This gives a *substitution* $\mathcal{S} = (p_{\vec{\jmath}})_{\vec{\jmath} \in B}$, which assigns to each $a \in \mathcal{A}$ a block of letters of size B. The substitution may be iterated; we call a *level-n block* a letter which has been substituted n times.

Example 1. *Let $B = B(2) = \{0, 1\}$ and let $\mathcal{A} = \{a, b\}$. The* period-doubling *substitution takes $a \to a\, b$ and $b \to a\, a$. In our notation, we see that the map p_0 takes both a and b to a, where p_1 is the map taking a to b and b to a.*

Example 2. *Again let $B = B(2) = \{0, 1\}$ and let $\mathcal{A} = \{a, b\}$. The* Thue-Morse *substitution takes $a \to a\, b$ and $b \to b\, a$. In our notation, we see that the map p_0 is the identity and p_1 is again the map taking a to b and b to a.*

For details, examples, and a spectral analysis of multidimensional constant-length substitution sequences, see [2].

A substitution is said to be *bijective* if for each $\vec{\jmath} \in B$, p_j is a bijection on $\mathcal{A}$. Notice that if a substitution is bijective, then there can never be coincidences in the sense of Dekking [1], and therefore will have a mixed dynamical spectrum.

2.1. **Substitution dynamical systems.** Once a substitution is decided upon, we define the *hull* X of the substitution as the space of all sequences in $\mathcal{A}^{\mathbb{Z}^d}$, all of whose subblocks appear somewhere in a level-n block. Translation by elements of $\mathbb{Z}^d$ give a multidimensional action that is known, when the substitution is primitive, to be uniquely ergodic with probability measure we will call μ.

The Thue-Morse and period-doubling substitution dynamical systems, denoted $(X_{TM}, \mathbb{Z}, \mu_{TM})$ and $(X_{PD}, \mathbb{Z}, \mu_{PD})$ respectively, are our main examples. It is known that the Thue-Morse system factors onto the period-doubling system. It is also known that the Thue-Morse system has a mixed dynamical spectrum while the period-doubling system has pure point spectrum. We will show what becomes of the continuous part of the Thue-Morse spectrum during the factoring process.

3. Spectral theory of substitution sequences in $\mathbb{Z}^d$

Consider the unitary $\mathbb{Z}^d$-action on a Hilbert space given by $U^{\vec{\jmath}} : L^2(X, \mu) \to L^2(X, \mu)$ with $U^{\vec{\jmath}}(f(\mathcal{T})) = f(\mathcal{T} - \vec{\jmath})$ for all $\vec{\jmath} \in \mathbb{Z}^d$. We can analyze the action of $\mathbb{Z}^d$ on X by consideration of the action of $U^{\vec{\jmath}}$ on $L^2(X, \mu)$. The *spectral coefficients* of an $L^2(X, \mu)$ function are given, for each $\vec{\jmath} \in \mathbb{Z}^d$, by

$$\hat{f}(\vec{\jmath}) = \langle U^{\vec{\jmath}} f, f \rangle = \int_X U^{\vec{\jmath}} f(\mathcal{T}) \overline{f(\mathcal{T})} d\mu(\mathcal{T}). \tag{1}$$

It is known that these coefficients form a positive definite sequence and that therefore there is a unique measure σ_f on the d-torus [3] with:

$$\hat{f}(\vec{\jmath}) = \int_{\mathbb{T}^d} z^{\vec{\jmath}} d\sigma_f(z), \tag{2}$$

where $z^{\vec{\jmath}} = z_1^{j_1} \cdot ... \cdot z_d^{j_d}$.

It is hard to visualize these measures, but we know that they must decompose relative to Lebesgue measure into pieces that are atomic (discrete), singular continuous, and absolutely continuous. It is much easier to consider functions in L^2 and draw conclusions based on their spectral coefficients only, as we do in the case of eigenfunctions below.

An *eigenvalue* of U is an $\vec{\alpha} \in \mathbb{R}^d$ such that there is an $f \in L^2(X, \mu)$ for which $U^{\vec{\jmath}}(f) = \exp(2\pi i \vec{\alpha} \cdot \vec{\jmath}) f$ for all $\vec{\jmath} \in \mathbb{Z}^d$. (Equivalently, $f(\mathcal{T} - \vec{\jmath}) = \exp(2\pi i \vec{\alpha} \cdot \vec{\jmath}) f(\mathcal{T})$ for all $\mathcal{T} \in X$. It is not hard to check that the spectral measure of an eigenfunction is an atomic measure. Thus we call the closure of the linear span of eigenfunctions $H_D \subseteq L^2(X, \mu)$ the *discrete spectrum* of U. A substitution is said to have *pure point spectrum* if $H_D = L^2(X, \mu)$.

3.1. **Odometer structure and eigenfunctions.** The underlying box $B = B(l_1, ..., l_d)$ provides a wire-frame structure of the level-n blocks of any sequence in the hull X as follows (see [2] for details). For each $n = 1, 2, ...$ we define a map $\mathcal{O}_n : X \to \mathbb{Z}^d$ by $\mathcal{O}_n(\mathcal{T})$ = the position of the level-$(n-1)$-block of $\mathcal{T}$ containing the origin inside its level-n block. Each $\mathcal{T} \in X$ has a coding by level-n blocks given by the sequence $\{\mathcal{O}_n(\mathcal{T})\}$. The action of translation by elements of $\mathbb{Z}^d$ acts as an odometer on the space of level-n codings.

Odometer actions are know to have pure point spectrum, and the substitution dynamical system factors onto the odometer action, thus inheriting its eigenfunctions. Under the (relatively mild) condition of "trivial height", the odometer system forms the maximal equicontinuous factor of the substitution system and so gives all the the eigenfunctions. In fact the eigenvalues must then take the form $\vec{\alpha} = \left(\frac{m_1}{l_1^{n_1}}, ..., \frac{m_d}{l_d^{n_d}}\right)$, where the l_i's remain the lengths of the substitution that define the block B.

Example 3. *Consider either the PD or the TM substitution, so that $l_1 = 2$, and let $\vec{\alpha} = 1/2$. We have the eigenfunction given by*

$$g(\mathcal{T}) = \begin{cases} 1 & \text{if } \mathcal{O}_1(\mathcal{T}) = 0 \\ -1 & \text{if } \mathcal{O}_1(\mathcal{T}) = 1 \end{cases}$$

(i.e. it is 1 if the origin is in the left-hand side of its level-1 block and it is -1 if it is in the right-hand side.) The reader should check that g is an eigenfunction with eigenvalue $1/2$.

An important thing to notice is that the eigenfunctions only "see" the odometer structure given by B, not the labellings the substitution has decided to include. Thus if a substitution is pure point spectrum, the odometers must "see" everything there is to know about the hull X.

3.2. **Continuous spectrum in bijective substitutions.** Given a bijective substitution of trivial height, it is easy to write down functions in the orthocomplement of H_D. Let $F : \mathcal{A} \to \{1, 2, ..., |\mathcal{A}|\}$, and define

$$f(\mathcal{T}) = \exp\left(2\pi i \frac{F(\mathcal{T}(\vec{0}))}{|\mathcal{A}|}\right)$$

Obviously f only cares about the symbol at the origin in any sequence.

The fact that this is orthogonal to each eigenfunction is proved in [2], but it is instructive to consider the specific case of the Thue-Morse substitution and the eigenfunction defined in our previous example.

Example 4. *For each $\mathcal{T} \in X_{TM}$, define*

$$f(\mathcal{T}) = \begin{cases} 1 & \text{if } \mathcal{T}(0) = a \\ -1 & \text{if } \mathcal{T}(0) = b \end{cases}$$

We can show that this function is orthogonal to the eigenfunction g constructed in Example 3. To do this, we write X_{TM} as the union of four sets, $X_{0,a}, X_{1,a}, X_{0,b}$, and $X_{1,b}$, where $\mathcal{T} \in X_{i,e}$ if $\mathcal{O}(\mathcal{T}) = i$ and $\mathcal{T}(0) = e$. It is not difficult to show these sets have equal measure; moreover the product $g(\mathcal{T})f(\mathcal{T})$ is constant on each set. Thus we compute

$$\begin{aligned} < g, f > &= \int_{X_{TM}} g(\mathcal{T})f(\mathcal{T})d\mu \\ &= \int_{X_{0,a}} g(\mathcal{T})f(\mathcal{T})d\mu + \int_{X_{1,a}} g(\mathcal{T})f(\mathcal{T})d\mu + \int_{X_{0,b}} g(\mathcal{T})f(\mathcal{T})d\mu \\ &\qquad + \int_{X_{1,b}} g(\mathcal{T})f(\mathcal{T})d\mu \\ &= \int_{X_{0,a}} 1 \cdot 1 d\mu + \int_{X_{1,a}} -1 \cdot 1 d\mu + \int_{X_{0,b}} 1 \cdot -1 d\mu + \int_{X_{1,b}} -1 \cdot -1 d\mu \\ &= .25 - .25 - .25 + .25 = 0 \end{aligned}$$

Because f is in fact orthogonal to all of the eigenfunctions, this means the eigenfunctions in $L^2(X_{TM})$ cannot "see" the color at the origin. Moreover we get a spectral decomposition $L^2(X_{TM}) = H_D \oplus \overline{Span(f)}$.

4. Conclusion

We know that the Thue-Morse system factors onto that of the period-doubling, so why does f have a continuous spectral measure for the Thue-Morse system and an atomic one for the period-doubling substitution? The answer is simply that for the period-doubling substitution, the odometer coding can tell you whether a sequence has a a or a b at the origin. To see this, notice that if $\mathcal{O}(\mathcal{T}) = 0$, meaning that the level-0 block containing the origin is in the left of its level-1 block, then $\mathcal{T}(0)$ must equal a and thus $f(\mathcal{T}) = 1$. If the coding of $\mathcal{T}$ begins by 1 0 then

$\mathcal{T}(0) = b$ and so $f(\mathcal{T}) = -1$. Indeed, the reader can check that if the coding begins with n 1's and then a 0, then $f(\mathcal{T}) = -1^n$.

In this way we see that the period-doubling substitution has not really altered the odometer at all, but the Thue-Morse substitution has.

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On the Boundary of Rauzy Fractals

Jörg M. Thuswaldner

Let σ be a unimodular Pisot substitution over the alphabet $A = \{1, \ldots, d\}$, *i.e.*, a substitution whose incidence matrix $\mathbf{M}$ has an irreducible characteristic polynomial whose dominant root is a Pisot unit. It is well-known (*cf. e.g.* [1]) that one can attach a tile with fractal boundary to each of these substitutions. One way to define this tile – which we will sketch below – runs via a graph directed iterated function system.

The *prefix suffix graph* associated to σ is defined as follows. Let

$$\mathcal{P} := \{(p, i, s) \in A^* \times A \times A^*;\ \exists\, j \in A,\ \sigma(j) = pis\}.$$

The *prefix-suffix graph of* σ is the graph Γ_σ with nodes in A and such that there is an edge labelled by $(p, i, s) \in \mathcal{P}$ from i to j if and only if $pis = \sigma(j)$.

Let π be the projection of $\mathbb{R}^d$ to the contractive hyperplane P of $\mathbf{M}$ (note that each conjugate of a Pisot number has modulus less than one) along the expanding eigenvector of $\mathbf{M}$. Moreover, denote by $\mathbf{l} : A^* \to \mathbb{R}^d$ the *abelianization map*. Then the *Rauzy fractal* $X = X_1 \cup \ldots \cup X_d$ associated to σ is defined as the unique compact non-empty solution of the graph directed iterated function system

$$X_i = \bigcup_{i \xrightarrow{(p,i,s)} j} \mathbf{M}X_j + \pi\mathbf{l}(p) \qquad (i \in A) \tag{1}$$

where the union is extended over all edges in Γ_σ leading away from i.

The *Pisot conjecture* asserts that the collection $\mathcal{I} := \{X_i + \gamma \mid (\gamma, i) \in S\}$ forms a tiling of the contractive hyperplane P. Here $S \subset P$ is a certain self-replicating and repetitive Delone set which can be defined in terms of σ. It is not hard to see that this conjecture is true if the intersections of the shape

$$(X_i + \gamma) \cap (X_j + \delta) \tag{2}$$

have measure zero provided that $(\gamma, i), (\delta, j) \in S$ are distinct. Indeed, in this case we even have that

$$\partial X_i = \bigcup_{(\gamma,j)\neq(0,i)} (X_i \cap (X_j + \gamma))$$

where the union is taken over all non-zero elements of S. Thus it is of interest to study the intersections in (2) (see [2]). Using (1) we get

$$X_i \cap (X_j + \gamma) = \bigcup_{\substack{\sigma(i_1)=p_1 i s_1\\ \sigma(j_1)=p_2 j s_2}} (\mathbf{M}X_{i_1} + \pi\mathbf{l}(p_1)) \cap (\mathbf{M}X_{j_1} + \pi\mathbf{l}(p_2) + \gamma).$$

We express each element of this decomposition as the image by $\mathbf{M}$ of a translated intersection of tiles

$$X_i \cap (X_j + \gamma) = \bigcup_{\substack{\sigma(i_1)=p_1 i s_1\\ \sigma(j_1)=p_2 j s_2}} \mathbf{M}\left(X_{i_1} \cap (X_{j_1} + \underbrace{\mathbf{M}^{-1}\pi\mathbf{l}(p_2) - \mathbf{M}^{-1}\pi\mathbf{l}(p_1) + \mathbf{M}^{-1}\gamma}_{=\gamma_1})\right) + \pi\mathbf{l}(p_1).$$

Thus the intersection between two tiles can be expressed as the union of intersections between other tiles. Set $B(i, \gamma, j) := X_i \cap (X_j + \gamma)$. Then we have

$$B(i, \gamma, j) = \bigcup_{\substack{\sigma(i_1)=p_1 i s_1, \sigma(j_1)=p_2 j s_2\\ \gamma_1 = \mathbf{M}^{-1}\pi\mathbf{l}(p_2) - \mathbf{M}^{-1}\pi\mathbf{l}(p_1) + \mathbf{M}^{-1}\gamma}} \mathbf{M}B(i_1, \gamma_1, j_1) + \pi\mathbf{l}(p_1).$$

It can be shown that it suffices to consider only finitely many constellations (i, γ, j). This enables one to define the intersections $B(i, \gamma, j)$ as the solutions of a graph directed iterated function system. The finite graph involved here is called the *boundary graph* of σ (*cf.* [2]).

This graph can be used to determine algorithmically whether a given unimodular Pisot substitution gives rise to a tiling or not. In other words, it permits to check the validity of the Pisot conjecture for any given example. Moreover, in [2] the boundary graph and related graphs have been used in order to describe topological properties of Rauzy fractals, like connectivity, homeomorphy to a disk or the fundamental group.

Acknowledgements. The author was supported by the Austrian Science Foundation (FWF), project S9610, which is part of the national research network FWF-S96 "Analytic combinatorics and probabilistic number theory".

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Constructing Substitution Rules

Edmund Harriss

(joint work with Jeroen Lamb, Pierre Arnoux, Shunji Ito, Maki Furukado)

In one dimension it is easy to construct a large family of substitution rules. One simply considers combinatorial substitution rules. With the assumption that the tiles are connected and the substitution rule is vertex hierarchic this gives all possible substitution rules on the line [Ken90]. However in two dimensions we must consider geometry in order to construct substitution rules. We will consider two methods of constructing substitution rules in dimension greater than one. The first takes the set of canonical projection tilings and characterises those that have a substitution rule [HL, Har04]. The second is to return to combinatorial structures to define the one dimensional (edge) structure of the tiling [AHIF, Ken96]. In certain cases the boundary produced by these one dimensional structures can be filled with two dimensional tiles to give a substitution rule [FIR06]. In this talk we give specific examples of these constructions, the general theory can be found in the papers cited.

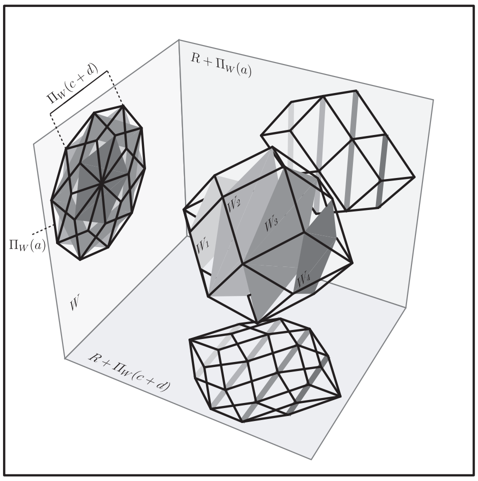

Figure 1: Window for the Penrose tiling.

1. The Penrose Tilng

A canonical projection tiling is constructed from a slice of lattice. Take the lattice $\mathbb{Z}^n$, a subspace V of $\mathbb{R}^n$ and a unit hypercube $\mathcal{H}$. Let Π_V, be a projection onto V and W the kernel of Π_V. We can consider the set of points $\Pi_V((V+\mathcal{H})\cap\mathbb{Z})$. This gives a discrete set of points. By considering the facets of the hypercube tiling of $\mathbb{R}^n$ of equal dimension to V we can project to obtain a tiling.

In the case of the Penrose tiling we begin with a matrix:

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

This has three eigenspaces, a plane V_1 on which it acts as multiplication by $\phi = \frac{1+\sqrt{5}}{2}$, a second plane W_1 on which it acts as multiplication by $-\phi^{-1}$ and a line R on which it acts as multiplication by 3. We now consider the canonical projection defined by V and $W+R$, as subspaces of $\mathbb{R}^5$, containing $\mathbb{Z}^5$. The projection Π_{V_1} commutes with the action of the matrix M_1. Similarly we have a projection to $W+R$, with V as kernel that commutes with M_1. The $\mathbb{Z}$-module $\Pi W_1 + R(\mathbb{Z}^5)$ gives densely filled planes parallel to W, arranged on a lattice in R. The intersection of these planes with the projection of the unit hypercube is shown in Figure 1.

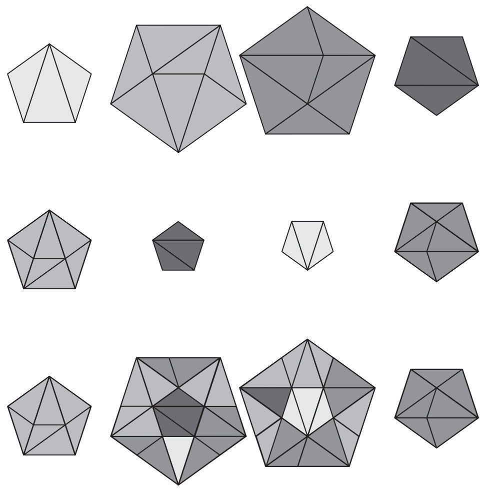

Figure 2: The windows for the Penrose tiling, the windows under the action of M_1 and the windows after applying the substitution rule.

In what follows we want to consider points in $\mathbb{Z}^5$, the projection of $\mathbb{Z}^5$ to (W_1+R) is a bijection, so we can consider this set without seeing five dimensions. However the projection of $\mathbb{Z}^5$ to V_1 is not a bijection. Furthermore, we are primarily interested in the tiling in V, thus we may consider points in $\mathbb{Z}^5$ that project to the same point in V_1 to be the same. This induces a cycle of order 5 in the direction of R. The action of M_1 on R can therefore be considered as multiplication by 3 mod 5. We now need to consider five densely filled planes in $W+R$, so we can just look at the four pentagons on which $\Pi_{W_1+R}(\mathcal{H})$ intersects them rather than the three dimensional shape. In the fifth plane the intersection is simply a point. There are some finer considerations of what happens on the boundary that we will gloss over here.

As the projections commute with M_1 applying this to the whole system just gives an expanded tiling in V_1, but this tiling can be generated by considering $\Pi_V((M_1\mathcal{H}+V)\cap\mathbb{Z}^5)$. Similarly we can consider the projection of this to $W+R$. When we apply M_1 to the four pentagons therefore they are shrunk and permuted round as shown in Figure 2.

We now want to relate what happens on the window to what happens in the tiling in V. We have a $\mathbb{Z}$-module of points in $W_1 + R$ and in V_1 with a bijection B between them given by lifting to $\mathbb{Z}$ (mod 5 in the direction of R). This bijection B is closely related to algebraic conjugation. We can partition the windows in $W_1 + R$ to give information about the patches around the corresponding point (by B) in V_1. The partition shown in Figure 2 allows us to consider tiles rather than points. We chose a direction in V and this partition gives the tile in this direction from the vertex (note that, as the tiles are rhombs, this defines a bijection between vertices and tiles). We now apply the matrix to get the set of subwindows. Each of the larger tiles is replaced by a patch of the original tiles. Consider the set of vertices T for a particular tile. The addition points will be $T + t_i$ for some t_i in $\Pi_{V_1}(\not\triangle)$. In the window the points $B(T)$ fill one of the partition regions. The points $B(T+t_i)$ therefore fill a translation of this region. To consider the image of this tiling under the Penrose substitution rule, therefore we simply apply the rule and consider the effect on the window. This is shown in Figure 2. As you can see the window shapes are taken back to themselves. The Penrose substitution rule therefore takes a Penrose tiling to a Penrose tiling.

Remarkably the existence of a substitution rule for a canonical projection tiling depends only on the existence of a suitable matrix to play the role of M_1. The suitable matrices and the proof that they characterise all canonical projection tilings with substitution rule is shown in [HL].

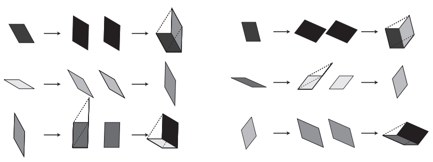

Figure 3: The substitutions generated by the morphisms σ_1 and σ_2 on their boundaries

2. The Nautilus and Conch tilings

Consider the morphisms $\sigma_1 := a \to b \to c \to d \to da^{-1}$ and $\sigma_2 := d \to c \to b \to a \to d^{-1}c$, that are an inverse pair of automorphisms of the free group with four generators [MKS66]. We can consider the substitution matrix for σ_1:

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

This has an expanding planar eigenspace V_2 and a contracting planar eigenspace W_2. Let Π_{W_2} be the projection to W_2 such that $\Pi_{W_2}(V_2) = 0$ and Π_{V_2} the projection to V_2 such that $\Pi_{V_2}(W_2) = 0$. We can now take the lattice generators $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ and project to V_2 and W_2. By labelling the vectors in V_2 and W_2, a, b, c and d we can now play combinatorial games in V_2 and W_2. In addition to the set of translations the $\mathbb{Z}$-modules generated by these vectors in V_2 and W_2 is invariant under the linear map induced by M_2 on V_2 and W_2. Moreover this linear map on V_2 take a to b, b to c, c to d and d to $d - a$.

We can therefore apply the linear map, that is expanding on V_2 and replace the new vectors using σ_1. We can now consider six parallelogram tiles in V_2 whose

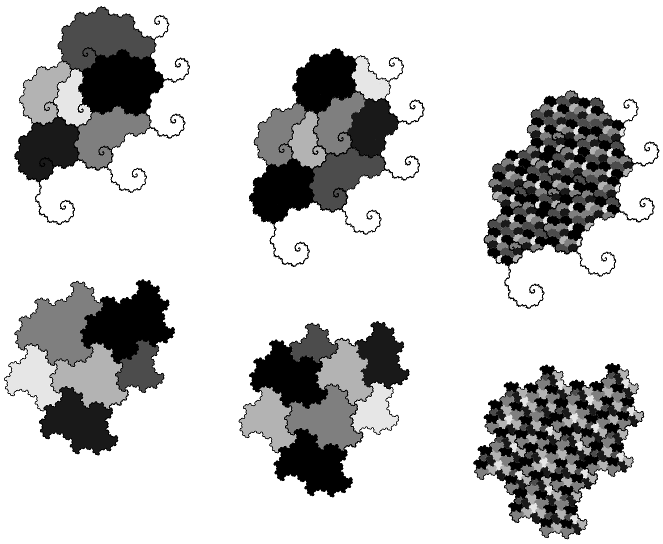

Figure 4: Nautilus and Conch tiles, and substitution. These shapes form windows for the substitution tilings generated by the rule in Figure 3.

boundaries are given by the words $aba^{-1}b^{-1}$, $bcb^{-1}c^{-1}$, and so on. Applying σ_1 to these words gives new words for example $\sigma_1(bdb^{-1}d^{-1}) = cda^{-1}c^{-1}ad^{-1}$. This gives a new set of tiles with more complex boundaries. If we take a patch of tiling with the original boundaries, first apply the linear map induced by M_2 and then replace the edges we obtain a new tiling with these new tiles. In good cases, such as this one the new tiles can be filled by a union of the original tiles. The morphisms σ_1 and σ_2 therefore give the substitution rules on the plane shown in Figure 3.

By iterating these substitution rules we obtain tilings of the plane by parallelograms. These tilings can be lifted to a broken plane of squares in four dimensions. The set of vertices of this broken plane can then be projected to W_2. As M_2 is contracting 0n W_2 this projection lies in a bounded region and in fact the closure of these points gives a window for the tiling. The windows for the two tilings are shown in Figure 4.

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Endomorphisms related to the boundaries of Rauzy fractals

Hiromi Ei

For the substitution on 3 letters:

$$\sigma : \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{cases},$$

the domain X_σ with fractal boundary, so called the Rauzy fractal, such that the substitution dynamical system related to σ is measurably conjugate to the domain exchange transformation on it was found in [5]. The construction of the Rauzy fractal X_σ is as follows. Let $w = w_1 w_2 \cdots = 1213121 \cdots$ be the fixed point of σ, M_σ be the incidence matrix, P be the contractive plane for M_σ and $\pi : \mathbb{R}^3 \to P$ be the projection along the eigenvector corresponding to the maximum eigenvalue which is Pisot number. Then the Rauzy fractal X_σ is given by the closure $\{\pi \sum_{i=1}^{n} e_{w_i} \mid n = 1, 2, \cdots\}$, where the e_i $(i = 1, 2, 3)$ is the canonical basis of $\mathbb{R}^3$.

Another way to construct the Rauzy fractal was introduced in [1] by using the tiling substitution $E_1^*(\sigma)$ with three prototiles which are parallelograms spanned by $\{\pi e_i, \pi e_j\}$ $(i, j = 1, 2, 3,\ i \neq j)$. On this framework, [6] gives the formulation of $E_2^*(\sigma)$ corresponding to the tiling substitution to construct the boundary ∂X_σ of the Rauzy fractal (See Figure 1.).

On the other hand, as we see in [4], by the generating fractal curve method of Dekking the automorphism θ on the free group of rank 3 provides the boundary ∂X_σ as follows:
Let θ be

$$\theta : \begin{cases} 1 \to 3 \\ 2 \to 3^{-1}1 \\ 3 \to 3^{-1}2 \end{cases},$$

B_n be the closed broken curve on P through the points

$$\{\pi \sum_{i=1}^{n} e_{b_i} \mid n = 1, 2, \cdots, l_n\} \cup \{o\},$$

where $b^{(n)} = b_1 b_2 \cdots b_{l_n} = \theta^n(1^{-1}32^{-1}13^{-1}2)$. The limit set $\lim_{n\to\infty} M_\sigma^n B_n$ in the sense of Hausdorff metric gives the boundary ∂X_σ.

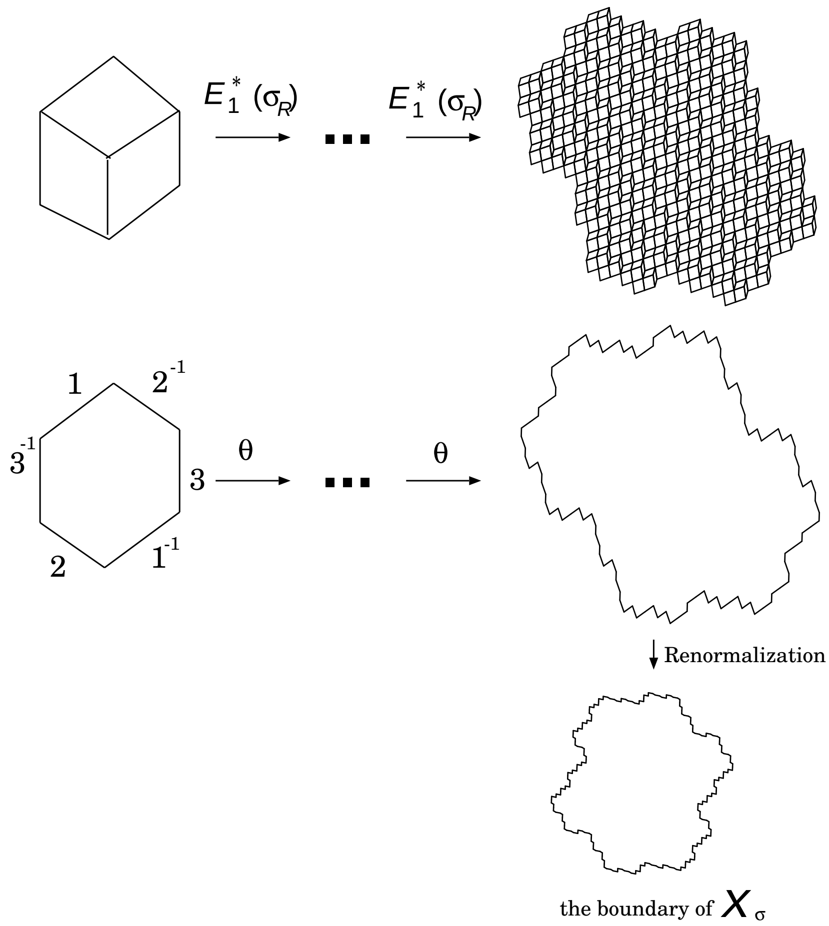

Figure 1: The Rauzy fractal X_σ and its boundary

For a substitution σ on d letters with Pisot, unimodular, primitive and irreducible conditions, we can define a generalized Rauzy fractal X_σ of σ in the same manner; and the aim of my talk is to show the following theorem describing the automorphisms θ generating the boundary:

Theorem. *(*[2]*) If a substitution σ is invertible as an endomorphism on the free group, the boundary ∂X_σ is given by the automorphism $\theta = \sigma^{-1}$.*

Remark.
- *In some reducible case, the theorem holds (*[3]*).*
- *On the example, cancellations occur in the word $b^{(n)}$. In fact $\theta(1^{-1}32^{-1}13^{-1}2) =$*

$3^{-1}3^{-1}21^{-1}332^{-1}\underline{33^{-1}}1$. *In such a case, we need the blocking method to control the cancellation ([4]).*

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